A Study of Dispersive Evolution Equations with Nonlocal Nonlinearities

Hager Debech

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Preface

This master's thesis was written in the spring of 2015 to fulfill the Master's thesis requirements of the N5TeAM Master's Programme in Applied and Engineering Mathematics, Computational Mechanics study track. The partner universities attended were the Technical University of Denmark (DTU), and the Norwegian University of Science and Technology (NTNU). The thesis is being written at NTNU with Professor Mats Ehrnström and supervision by Associate Professor Christian Henriksen from DTU.

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Abstract

In this thesis we consider an equation of Whitham type,

$$u_t + Lu_x + u^2 = 0, \tag{1}$$

with the Fourier multiplier L given by

$$\mathscr{F}(Lf)(\xi) = \sqrt{\left(\frac{\tanh(\xi)}{\xi}\right)}\hat{f}(\xi).$$

We prove the local well-posedness for (1) in the Sobolev space $H^s(\mathbb{R})$ for $s > 1/2$ by using the Picard Lindelöf existence theorem to prove the existence of a unique solution. Additionally, we use the Crandall-Rabinowitz local bifurcation theorem to prove the existence of periodic traveling waves to our nonlinear-nonlocal equation.

Contents

Chapter 1

Introduction

The study of nonlinear waves started over a hundred years ago with the pioneering work of Stokes [2]. He derived approximate expressions for nonlinear periodic waves in infinitely deep water. In the latter half of the 18th century, Boussinesq(1871) and Rayleigh(1876), found approximate expressions for a wave propagating with unchanged shape and speed [2].

There has been a rich development of mathematical concepts and techniques to study the water-wave problem. This concerns the flow of an inviscid fluid described by the nonlinear Euler equations and equipped with a set of nonlinear conditions at the fluid surface [8]. Of particular interest to us is the theory of waves which propagate with unchanged shape and constant speed. One can distinguish between two types of such waves, solitary waves and traveling periodic waves. A solitary wave was observed in shallow water by John Scott Russell in 1844 [12] on the Edinburgh to Glasgow canal. He described his discovery to the British Association in his 'Report on Waves' [12] as follows:

"I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly grown along a narrow channel by a pair of horses, when the boat suddenly stopped, not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with

great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834 was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation..."

The scientific community did not support Russell's discovery at that time. Among the skeptics was George Airy [7], who strongly disagreed with Russell's findings because it directly contradicted with a theory of his own. The equations Airy developed for large shifts of mass, so the concept of everlasting waves was irrational to him. An other mathematician who opposed Russel's result was George Gabriel Stokes. He published a paper to explain that permanent translation waves could not exist [7].

By performing some laboratory experiments, Russell tried to generate solitary waves by dropping a weight at one end of a water channel. When the weight is removed, a long heap-shaped wave propagates down the channel. This resulted in two solitary waves. In general a wave in such a channel spreads out and ends as small ripples on the surface. However, under certain conditions a solitary water wave can be produced, and will travel without change in shape.

Russell further observed that the volume of water in the wave is equal to the volume of water displaced, and he could establish that the limiting long-wave speed, c_0, of the solitary wave is given by

$$c_0 = \sqrt{gh_0},$$

where h_0 is the depth of the canal and g is the gravitational acceleration.

In order to resolve the controversy about solitary water waves, Korteweg and de Vries [7] [12] [15] developed the partial differential equation (1.1) for the wave profile $\eta(x,t)$, with an amplitude a and a wavelength λ,

$$\eta_t + c_0\eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}c_0h_0^2\eta_{xxx} = 0, \qquad (1.1)$$

where it is required that the wave is long compared to the undisturbed depth of the fluid h_0 [8] and the depth ratio a/h_0 is small compared to the square of the depth to wavelength ratio λ/h_0 [3]. The function η describes the deflection of the fluid surface from the rest position at a point x at time t [8], and the model (1.1) consists of a nonlinear term and a dispersive term which allows the existence of both solitary-wave solutions and periodic traveling waves.

From an earlier specialization project (TMA4500), we derived the linear dispersion relation in the KdV equation (1.1) from a solution of the form $\cos(\xi x - \omega t)$, where ω is the frequency and $\xi = 2\pi/\lambda$. Then the linear phase velocity, $c(\xi)$, in (1.1) is defined by

$$c(\xi) = \frac{\omega}{\xi} = c_0 - \frac{1}{6}c_0 h_0^2 \xi^2 \tag{1.2}$$

which is a second-order approximation to the dispersion relation

$$c(\xi) = \sqrt{\frac{g\tanh(\xi h_0)}{\xi}} = c_0 - \frac{1}{6}c_0 h_0^2 \xi^2 + \mathcal{O}(h_0^2 \xi^4) \tag{1.3}$$

of the linearized full surface water-wave problem [3]. The fact that (1.2) is only a good approximation of (1.3) for very small ξ, inspired Whitham to propose what is now called the Whitham equation,

$$\eta_t + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + K_{h_0} * \eta_x = 0,$$

$$K_{h_0} = \mathscr{F}^{-1}\left(\sqrt{\frac{g\tanh(\xi h_0)}{\xi}}\right).$$

where his motivation for introducing the equation above was to have an equation with the function $\eta(x,t)$ describing the deflection of the surface from rest, and the exact form of the phase velocity (1.3) instead of an approximation. The Whitham equation is a nonlocal equation, thus making pointwise estimates is difficult.

Due to its interesting mathematical properties, the Whitham equation has been intensively studied in recent years. The paper [8] by Ehrntröm and Kalisch is concerned with the existence of periodic traveling waves to

the Whitham equation, while [11] considered the existence of a global bifurcation of periodic traveling-wave solutions. We refer to [10] for a proof of the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type. There have been several investigations of different types of the Whitham equation and this thesis is a study of a nonlinear-nonlocal equation of Whitham type,

$$u_t + Lu_x + u^2 = 0, \tag{1.4}$$

with the Fourier multiplier L given by

$$\mathscr{F}(Lf)(\xi) = \sqrt{\left(\frac{\tanh(\xi)}{\xi}\right)}\hat{f}(\xi).$$

The main novelty of the equation studied in this thesis compared to the one studied in [8, 11, 10], is that we consider a nonlinear-nonlocal equation with a different nonlinear term, which presents a new technical difficulty.

The first chapter is dedicated to a short presentation on the distribution theory and Sobolev spaces which are both a key tools in this thesis. We give a definition of the space $\mathscr{D}(\mathbb{R})$, the Schwartz space, and the Fourier transform. We then prove the invariance of the Schwartz space under the Fourier transform, and extend the Fourier transform from the Schwartz space to a unitary operator on $L_2(\mathbb{R}^n)$ by Plancherel's theorem. Distributions are introduced as a tool for finding solutions to partial differential equations. We define the space of tempered distributions $S'(\mathbb{R}^n)$ which is best suited for the use of the Fourier transform. Then distributional differentiation was introduced as a tool to the definition of the classical Sobolev spaces $W_p^k(\mathbb{R}^n)$, $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0$. In particular, when $p = 2$, we have the fractional Sobolev spaces. In view of the fact that the Fourier transform is an unitary operator on $L^2(\mathbb{R}^n)$, we prove that the Fourier transform maps weighted L_2 spaces unitarily onto $W_2^k(\mathbb{R}^n)$. Then we extend the parameter $k \in \mathbb{N}_0$ to $s \in \mathbb{R}$. The last subsection is devoted to prove the Sobolev embedding theorem. This theorem shows the existence of a linear and bounded map between the Sobolev spaces and spaces of bounded, continuous functions. Most of the stated results in this chapter are proved.

In chapter 2, we define the concept of Lipschitz continuity and contraction. With this foundation, we state and prove the Banach fixed-point theorem and the Picard Lindelöf existence theorem.

In chapter 3, using those theorems we prove the existence and uniqueness of a solution of our nonlinear-nonlocal equation (1.4). The local well-posedness for the Whitham equation was recently established [10] by using Kato's method. In this thesis, we use the Picard Lindelöf existence theorem and the Sobolev theory to obtain the existence of a unique solution. Then, to accomplish the proof of the local well-posedness for our equation, we prove whether this solution depends continuously on the initial data in the Sobolev space $H^s(\mathbb{R})$, $s > 1/2$.

In Chapter 4, we give a brief introduction to the local bifurcation theory. We use the Crandall-Rabinowitz local bifurcation theorem to prove the existence of periodic traveling waves to our nonlinear-nonlocal equation. The existence of such waves solutions of the Whitham equation has been proven before in [11, 8]. We, however, perform our local bifurcation analysis in periodic Sobolev spaces (as opposed to Hölder spaces), and consider a nonlocal nonlinearlity, something that is not the case in [11, 8]. To be precise, we prove the local bifurcation in the periodic Sobolev space $H^s_{even,2\pi}(\mathbb{R})$ for $s > 1/2$.

Chapter 2

Distribution Theory and Sobolev Spaces

When modeling reality by differential equations, we typically expect the solutions to be differentiable. Unfortunately, this is not always possible as some physical shapes in nature are not differentiable. This problem has been solved by introducing the concept of distributions, which is a generalization of functions, and the use of weak and distributional differentiation. The distribution theory admits functions as solutions to a differential equation even if they contain some non-differentiable points.

2.1 The Spaces $\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{D}'(\mathbb{R}^n)$

A distribution is defined to be a linear functional on a space of test functions. In this section we define distributions on the space denoted by $\mathscr{D}(\mathbb{R}^n)$, which is a class of test functions and distributions on the Schwartz space which consists of smooth, rapidly decreasing functions.

Notation 1
Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{N}_0^n = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \in \mathbb{N}_0, 1 \leq i \leq n\}$, where $n \in \mathbb{N}$, be the set of all multi-indices. For $\alpha \in \mathbb{N}_0^n$, we define the norm $|\alpha| = \sum_{i=1}^{n} \alpha_i$. For $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, derivatives and exponentiations are abbreviated by $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Definition 1 ([1])
Let Ω be a domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$, then

$$\mathscr{D}(\Omega) = \{\varphi \in C^\infty(\Omega) : \operatorname{supp} \varphi \text{ compact in } \Omega\}$$

We say that a sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathscr{D}(\Omega)$ converges in $\mathscr{D}(\Omega)$ to $\varphi \in \mathscr{D}(\Omega)$, $\varphi_j \longrightarrow \varphi$, if there is a compact set $K \subset \Omega$ with

$$\operatorname{supp} \varphi_j \subset K, \quad j \in \mathbb{N},$$

and uniform convergence for all derivatives,

$$D^\alpha \varphi_j \Longrightarrow D^\alpha \varphi, \text{ for all } \alpha \in \mathbb{N}_0^n.$$

The dual space of $\mathscr{D}(\mathbb{R}^n)$, denoted by $\mathscr{D}'(\mathbb{R}^n)$, is the collection of all complex-valued linear continuous functionals T over $\mathscr{D}(\mathbb{R}^n)$. The functionals $T \in \mathscr{D}(\mathbb{R}^n)$ are called distributions.

2.2   The Schwartz Space and Tempered Distributions

We say that a function φ is rapidly decreasing if there are constants M_N such that

$$|\varphi| \leq M_N |x|^{-N} \tag{2.1}$$

as $x \to \infty$ for $N = 1, 2, 3, \cdots$. Multiplying φ by any polynomial produces a function which still converges to zero as $x \to \infty$. A C^∞ function is in the Schwartz space if φ and all its partial derivatives are rapidly decreasing.

Definition 2
The Schwartz space is the linear space of rapidly decreasing smooth functions, denoted by $S(\mathbb{R}^n)$,

$$S(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{k,l} < \infty \text{ for all } k, l \in \mathbb{N}_0\}.$$

where

$$\|\varphi\|_{k,l} = \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{k/2} \sum_{|\alpha| \leq l} |D^\alpha \varphi(x)|$$

A sequence $\{\varphi_j\}_{j=1}^{\infty}$ in $S(\mathbb{R}^n)$ converges to $\varphi \in S(\mathbb{R}^n)$, if

$$\|\varphi_j - \varphi\|_{k,l} \longrightarrow 0 \quad \text{for } j \to \infty \text{ and all } k, l \in \mathbb{N}_0.$$

A function $\varphi \in C^\infty(\mathbb{R}^n)$ belongs to $S(\mathbb{R}^n)$ if and only if φ and all its partial derivatives are rapidly decreasing. This is why $S(\mathbb{R}^n)$ is called the Schwartz space of all rapidly decreasing infinitely differentiable functions in $\mathbb{R}^n$. Any function in $\mathscr{D}(\mathbb{R}^n)$ belongs to $S(\mathbb{R}^n)$ so that $\mathscr{D}(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, but $S(\mathbb{R}^n)$ is a larger class of functions and there exists $\varphi \in S(\mathbb{R}^n)$ which do not belong to $\mathscr{D}(\mathbb{R}^n)$. A typical example is the function $e^{-|x|^2}$ which does not have bounded support, hence does not belong to $\mathscr{D}(\mathbb{R}^n)$.

The requirement that $\varphi \in \mathscr{D}(\mathbb{R}^n)$ has compact support allows distributions to grow arbitrarily rapidly as you approach infinity. Now we consider a smaller class of distributions, called tempered distributions, or slowly increasing distributions, and are denoted by $S'(\mathbb{R}^n)$. The space $S'(\mathbb{R}^n)$ consists of continuous linear functionals on $S(\mathbb{R}^n)$ which cannot grow as rapidly at infinity because of the weaker vanishing properties of the test functions.

Definition 3

The elements of $S'(\mathbb{R}^n)$ are called tempered distributions and are continuous linear functional on $S(\mathbb{R}^n)$, i.e. a mapping

$$T : S(\mathbb{R}^n) \to \mathbb{C}, \quad \varphi \in S(\mathbb{R}^n)$$

is a distribution if it satisfies the following

$$T(\lambda_1 \varphi_1 + \lambda_2 \varphi_2) = \lambda_1 T(\varphi_1) + \lambda_2 T(\varphi_2),$$
$$T(\varphi_j) \to T(\varphi) \text{ for } j \to \infty \text{ whenever } \varphi_j \to \varphi$$

for all $\varphi_1, \varphi_2 \in S(\mathbb{R}^n)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. We say that $S'(\mathbb{R}^n)$ is the continuous dual space of $S(\mathbb{R}^n)$.

While a distribution $T \in S'(\mathbb{R}^n)$ restricted to $\mathscr{D}(\mathbb{R}^n)$ is an element of $\mathscr{D}'(\mathbb{R}^n)$, $S'(\mathbb{R}^n) \subset \mathscr{D}(\mathbb{R}^n)$, it is not true that every distribution in $\mathscr{D}'(\mathbb{R}^n)$ corresponds to a tempered distribution. This means that any continuous function $f \in \mathbb{R}^n$ defines a distribution, but $\int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$ cannot converge for all $\varphi \in S(\mathbb{R}^n)$ if f grows too fast at infinity. For example, the function $e^{x^2} \in \mathbb{R}^1$ defines a distribution

$$T_{e^{x^2}} = \int_{-\infty}^{\infty} e^{x^2} \varphi(x)\,dx$$

which is bounded because the test function φ has a compact support. Whereas $e^{-x^2/2} \in S(\mathbb{R}^1)$ gives,

$$T_{e^{x^2}} = \int_{-\infty}^{\infty} e^{x^2} e^{-x^2/2}\,dx$$
$$= \int_{-\infty}^{\infty} e^{x^2/2}\,dx = +\infty$$

Hence e^{x^2} is not a tempered distribution. Furthermore, if a distribution $T \in S'(\mathbb{R}^n)$ defined by

$$T_f(\varphi) := \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx, \quad \varphi \in S(\mathbb{R}^n)$$

for some $f \in L_1^{loc}(\Omega)$, then T is said to be a regular distribution. The notation $f \in L_1^{loc}(\Omega)$ means that $f(x)$ is defined on a domain Ω for which $\int f(x)\varphi(x)\,dx$ is absolutely convergent for every $\varphi \in \mathscr{D}(\Omega)$. In this case we say that $f(x)$ is locally integrable.

Note that derivatives and multiplications with smooth functions can be extended from functions to distributions [1]. In order to establish distributional solutions of differential equations, we need to define the distributional derivative of a tempered distribution. Using integration by parts, the following definition agrees with the usual definition when the distribution is given by a differentiable function $f(x)$,

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)\varphi(x)\,dx = -\int_{\mathbb{R}^n} f(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx,$$

One verifies immediately that the operation above is continuous and linear thus it is a distribution. Since the test functions are infinitely differentiable,

we can through iterations obtain the distributional derivatives of all orders.

Definition 4

Let $\alpha \in \mathbb{N}_0^n$ and $T \in S'(\mathbb{R}^n)$. The derivative $D^\alpha T$ is defined by

$$(D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi), \tag{2.2}$$

where $\varphi \in S(\mathbb{R}^n)$.

Note that (2.2) is true for all distributions in $\mathscr{D}'(\mathbb{R}^n)$, whereas the multiplication of a tempered distribution by a smooth function requires some growth restriction at infinity.

Definition 5

Let $T \in S'(\mathbb{R}^n)$ and g be a smooth tempered function as defined in (2.1), then

$$(gT)(\varphi) = T(p\varphi)$$

where $\varphi \in S(\mathbb{R}^n)$.

The purpose of introducing tempered distributions is to define the Fourier transform, $\mathscr{F}$, of a tempered distribution as a tempered distribution.

2.3 The Fourier Transform

The Fourier transform is one of the most powerful operators in the theory of distributions and function spaces. The Schwartz space and the tempered distributions are best suited for this purpose. This is because if $\varphi \in S(\mathbb{R}^n)$ then $\mathscr{F}\varphi \in S(\mathbb{R}^n)$ while if $\varphi \in \mathscr{D}(\mathbb{R}^n)$ it may not be true that $\mathscr{F}\varphi \in \mathscr{D}(\mathbb{R}^n)$. It requires some steps to prove that the Fourier transform preserves the Schwartz space. We start this process by giving a definition of the Fourier transform and its inverse.

Definition 6

Let $\varphi \in S(\mathbb{R}^n)$. Then

$$(\mathscr{F}\varphi)(\xi) = \hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x) \exp(-i\xi x) \, dx, \quad \xi \in \mathbb{R}^n,$$

is the Fourier transform of φ, and

$$\varphi(x) = (\mathscr{F}^{-1}\hat{\varphi})(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \exp(ix\xi) \, dx, \quad \xi \in \mathbb{R}^n,$$

is the inverse Fourier transform of φ. Note that $x\xi$ in the exponential is the scalar product of $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$.

Then we will derive some important formulas in order to reach our goal of showing that $\mathscr{F}\varphi \in S(\mathbb{R}^n)$ whenever $\varphi \in S(\mathbb{R}^n)$.

Proposition 2

Let $\varphi \in S(\mathbb{R}^n)$. Then $\mathscr{F}\varphi \in S(\mathbb{R}^n)$ and $\mathscr{F}^{-1}\varphi \in S(\mathbb{R}^n)$ [1]. For every multi-index $\alpha \in \mathbb{N}_0$ and $\xi \in \mathbb{R}$ we have

$$i) D^\alpha(\mathscr{F}\varphi)(\xi) = (-i)^{|\alpha|}\mathscr{F}(x^\alpha \varphi(x))(\xi), \tag{2.3}$$

$$ii) \xi^\alpha \mathscr{F}(\varphi)(\xi) = i^{|\alpha|}\mathscr{F}(D^\alpha \varphi(x))(\xi), \tag{2.4}$$

where $x^\alpha \varphi \in S(\mathbb{R}^n)$ and $D^\alpha \varphi \in S(\mathbb{R}^n)$.

Proof. i)The function $e^{-ix\xi}\varphi(x)$ is infinitely differentiable and $D^\alpha \varphi(x)e^{-ix\xi}$ is integrable with respect to x. Hence, by the mean value theorem and Lebesgue's bounded convergence theorem, we have

$$\frac{\partial}{\partial \xi_k}(\mathscr{F}\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-i)x_k e^{-ix\xi}\varphi(x) \, dx$$
$$= (-i)\mathscr{F}(x_k\varphi(x))(\xi)$$

and iteration concludes the argument for (2.3).

ii) By assumption, all derivatives up to order $\alpha \in \mathbb{N}_0^n$ are sufficiently smooth and integrable. Keep in mind that

$$\xi_l(\mathscr{F}\varphi)(\xi) = (2\pi)^{-n/2} i \int_{\mathbb{R}^n} \frac{\partial}{\partial x_l} e^{-ix\xi}\varphi(x) \, dx$$

then integration by parts yields

$$\xi_l(\mathscr{F}\varphi)(\xi) = (-i)(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \frac{\partial}{\partial x_l} \varphi(x)\, dx$$
$$= (-i)\mathscr{F}\left(\frac{\partial\varphi(x)}{\partial x_l}\right)$$

and iteration leads to the desired result.

$\square$

Next, we show that the Fourier transform of a function $\varphi \in S(\mathbb{R}^n)$ is rapidly decreasing. In other words, $p(\xi)(\mathscr{F}\varphi)(\xi)$ remains bounded for any polynomial p. From the proposition above we have that

$$|\xi^\alpha(\mathscr{F}\varphi)(\xi)| = |\int_{\mathbb{R}^n} i^\alpha \mathscr{F}(D^\alpha \varphi)(\xi) e^{-ix\xi}\, dx|$$
$$\leq \int_{\mathbb{R}^n} |i^\alpha \mathscr{F}(D^\alpha \varphi)(\xi) e^{-ix\xi}|\, dx$$
$$= \int_{\mathbb{R}^n} |\mathscr{F}(D^\alpha \varphi)(\xi)|\, dx$$

where the integrand in the last equation is rapidly decreasing. Hence the integral is finite. Furthermore, all derivatives of $\mathscr{F}\varphi$ are rapidly decreasing since according to the proposition above, any derivative is the Fourier transform of a polynomial times φ. This leads to the desired result because a function in the Schwartz space multiplied by a polynomial is rapidly decreasing [1]. To summarize, we have that if $\varphi \in S(\mathbb{R}^n)$ then $\mathscr{F}\varphi \in S(\mathbb{R}^n)$. This argument ensures that the Fourier transform is defined for all $\varphi \in S(\mathbb{R}^n)$. Our next goal is to show that the Fourier transform preserves the Schwartz space.

Theorem 3
Let $\varphi \in S(\mathbb{R}^n)$, then
$$\varphi = \mathscr{F}^{-1}\mathscr{F}\varphi = \mathscr{F}\mathscr{F}^{-1}\varphi. \tag{2.5}$$

Furthermore, the Fourier transform $\mathscr{F}$ and the inverse Fourier transform $\mathscr{F}^{-1}$ map the Schwartz space one-to-one onto itself.

Proof. First we start with a preparation. By definition of the Fourier transform and the inverse Fourier transform, we obtain

$$\mathscr{F}^{-1}\mathscr{F}\varphi = \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-i(x-y)\xi) f(y) \, dy \, d\xi.$$

When we evaluate the absolute value of the integrand, we can eliminate the oscillating factor $\exp(-i(x-y)\xi)$ and the integral of what remains diverges. Therefore, we cannot use the Fubini theorem to interchange the order of integration. This problem can be solved by multiplying the integrand by a function $\psi(x) = \exp(-\frac{\epsilon^2|x|^2}{2})$ for $\epsilon > 0$, such that, for $\varphi, \psi \in S(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}_n} (\mathscr{F}\varphi)(\xi) \exp(ix\xi)\psi(\xi) \, d\xi = (2\pi)^{-n/2} \int_{\mathbb{R}_n} \int_{\mathbb{R}^n} \exp(-iy\xi)\varphi(y) \exp(ix\xi)\psi(\xi) \, dy \, d\xi,$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(y) \exp(-i(y-x)\xi)\psi(\xi) \, dy \, d\xi,$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(y) \exp(-i(y-x)\xi)\psi(\xi) \, d\xi \, dy,$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(y) \int_{\mathbb{R}^n} \exp(-i(y-x)\xi)\psi(\xi) \, d\xi \, dy,$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(y)(\mathscr{F}\psi)(y-x) \, dy,$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x+y)(\mathscr{F}\psi)(y) \, dy.$$

The multiplication by ψ gives a result which is no longer equal to φ but later we show that it converges to φ. According to [1, Proposition 2.40], the Fourier transform of $\psi(x)$ is given by

$$\begin{aligned}
(\mathscr{F}\psi)(y) &= \epsilon^{-1}\mathscr{F}\left(\exp\left(-\frac{|x|^2}{2}\right)\right)\left(\frac{y}{\epsilon}\right) \\
&= \frac{1}{\epsilon}\exp\left(-\frac{|y|^2}{2\epsilon^2}\right),
\end{aligned} \tag{2.6}$$

and by substitution

$$\int_{\mathbb{R}^n} \varphi(x+y)(\mathscr{F}\psi)(y) \, dy = \int_{\mathbb{R}^n} \varphi(x+\epsilon z) \exp\left(-\frac{|z|^2}{2}\right) \, dz,$$

where the transformations $y = \epsilon z$ and $dy = \epsilon\, dz$ were used. Now let $\epsilon \to 0$ and by Lebesgue's bounded convergence theorem and (2.6), we have

$$\int_{\mathbb{R}^n} (\mathscr{F}\varphi)(\xi) \exp(ix\xi)\, d\xi = \varphi(x) \int_{\mathbb{R}^n} \exp(-|z|^2/2)\, dz = (2\pi)^{n/2}\varphi(x)$$

which, compared to the definition of Fourier transform, gives that $\varphi = \mathscr{F}\mathscr{F}^{-1}\varphi$. A similar method can be used to prove the second equality in (2.5). We know that $\mathscr{F}\varphi \in S(\mathbb{R}^n)$ and $\mathscr{F}^{-1}\varphi \in S(\mathbb{R}^n)$ when $\varphi \in S(\mathbb{R}^n)$. Now consider $\psi = \mathscr{F}^{-1}\varphi$ then,

$$\varphi = \mathscr{F}\mathscr{F}^{-1}\varphi = \mathscr{F}\psi.$$

Thus $\mathscr{F}S(\mathbb{R}^n) = S(\mathbb{R}^n)$. Similarly for $\mathscr{F}^{-1}S(\mathbb{R}^n) = S(\mathbb{R}^n)$. Further we assume that $\mathscr{F}\varphi_1 = \mathscr{F}\varphi_2$, then

$$\begin{aligned} \varphi_1 &= \mathscr{F}^{-1}\mathscr{F}\varphi_1 \\ &= \mathscr{F}^{-1}\mathscr{F}\varphi_2 \\ &= \varphi_2 \end{aligned}$$

It follows that $\mathscr{F}$ and, similarly, $\mathscr{F}^{-1}$, are one-to-one mappings of $S(\mathbb{R}^n)$ onto itself. $\qquad\square$

Thus the Fourier transform is a continuous linear operator from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$. It is possible to extend the Fourier transform $\mathscr{F}$ to $L_2(\mathbb{R}^n)$ using the fact that the Schwartz space is densely embedded in $L_2(\mathbb{R}^n)$ [1]. The following theorem shows that the Fourier transform is an isometry from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ in the L_2 norm.

Theorem 4 (Plancherel's theorem)
The Fourier transform extends to a unitary operator on $L_2(\mathbb{R}^n)$. For $f, g \in L_2(\mathbb{R}^n)$,

$$\langle \hat{f}, \hat{g} \rangle_{L_2(\mathbb{R}^n)} = \langle f, g \rangle_{L_2(\mathbb{R}^n)} \tag{2.7}$$

Proof. Let $\varphi, \psi \in S(\mathbb{R}^n)$ and consider $\psi = \overline{\mathscr{F}\varphi}$, then

$$
\begin{aligned}
\|\mathscr{F}\varphi\|^2_{L_2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\mathscr{F}\varphi|^2 \, dx \\
&= \int_{\mathbb{R}^n} \mathscr{F}\varphi \overline{\mathscr{F}\varphi} \, dx = \int_{\mathbb{R}^n} \mathscr{F}\varphi\psi \, dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-ixy)\varphi(y) \, dy\psi(x) \, dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-ixy)\varphi(y)\psi(x) \, dx \, dy \qquad (2.8) \\
&= \int_{\mathbb{R}^n} \varphi\mathscr{F}\psi \, dx = \int_{\mathbb{R}^n} \varphi\mathscr{F}\left(\overline{\mathscr{F}\varphi}\right) dx \\
&= \int_{\mathbb{R}^n} \varphi\mathscr{F}(\mathscr{F}^{-1}\overline{\varphi}) \, dx = \int_{\mathbb{R}^n} \varphi\overline{\varphi} \, dx \\
&= \int_{\mathbb{R}^n} |\varphi|^2 \, dx = \|\varphi\|^2_{L_2(\mathbb{R}^n)}
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{\mathscr{F}\varphi} &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \overline{\exp(-ix\xi)\varphi(x)} \, dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(ix\xi)\overline{\varphi(x)} \, dx \\
&= \mathscr{F}^{-1}(\overline{\varphi})
\end{aligned}
$$

Now let f be an element of $L_2(\mathbb{R}^n)$. Since $S(\mathbb{R}^n)$ is dense in $L_2(\mathbb{R}^n)$ [1], there is a sequence $\{\varphi_k\}_{k\in\mathbb{N}} \in S(\mathbb{R}^n)$ such that

$$
\lim_{k\to\infty} \|f - \varphi_k\| = 0.
$$

Then $\{\varphi_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L_2(\mathbb{R}^n)$, and by using the result above (2.8), we have

$$
\|\hat{\varphi}_k - \hat{\varphi}_j\|^2_{L_2(\mathbb{R}^n)} = \|\varphi_k - \varphi_j\|^2_{L_2(\mathbb{R}^n)}
$$

which goes to zero as $k, j \to \infty$. Now since $L_2(\mathbb{R}^n)$ is complete, there exists an element $\hat{f} \in L_2(\mathbb{R}^n)$ such that $\mathscr{F}\varphi_k \to \hat{f}$ as $k \to \infty$. The last step to complete the proof is

$$\lim_{k,j\to\infty} \|\mathscr{F}\varphi_k - \mathscr{F}\varphi_j\|_{L_2(\mathbb{R}^n)}^2 = \left\|\hat{f} - \hat{g}\right\|_{L_2(\mathbb{R}^n)}^2$$
$$= \|f - g\|_{L_2(\mathbb{R}^n)}^2$$

Hence (2.7) holds also on $L_2(\mathbb{R}^n)$. $\qquad\square$

In light of Plancherel's theorem, it is possible to extend (2.5) to

$$\mathscr{F}\mathscr{F}^{-1} = \mathscr{F}^{-1}\mathscr{F} = id \text{ in } L_2(\mathbb{R}^n), \qquad (2.9)$$

meaning that $\mathscr{F}$ and $\mathscr{F}^{-1}$ are unitary operators on $L_2(\mathbb{R}^n)$.

2.4   Sobolev Spaces

We are now familiar with the distribution theory and the idea of finding distribution solutions. However, sometimes we are more interested in function solutions. This restriction can be solved by a technique for showing that certain distributions are in fact functions. This technique is called Sobolev theory.

We will interpret $f \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$ as a tempered distribution [1]. In particular, we may take derivatives of all orders $D^\alpha f \in S'(\mathbb{R}^n)$ where $\alpha \in \mathbb{N}_0^n$. In other words, there exists a $g \in L_p(\mathbb{R}^n)$ such that $g = D^\alpha f$ as a distribution [1]. Then, using (2.2), we obtain

$$\int_{\mathbb{R}^n} g(x)\varphi(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) D^\alpha \varphi(x)\, dx$$

for all test functions φ in the Schwartz space.

First we define the classical Sobolev spaces, and then, with the aid of the Fourier transform, we define fractional Sobolev spaces.

Definition 7
Let $k \in \mathbb{N}_0$ and $1 \le p < \infty$. Then

$$W_p^k(\mathbb{R}^n) = \{f \in L_p(\mathbb{R}^n) : D^\alpha f \in L_p(\mathbb{R}^n) \quad \text{for all } \alpha \in \mathbb{N}_0^n, |\alpha| \le k\}$$

are the classical Sobolev spaces.

The spaces defined above are Banach spaces equipped with the norm [1]

$$\|f\|_{W_p^k(\mathbb{R}^n)}^p = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L_p(\mathbb{R}^n)}^p \tag{2.10}$$

Furthermore, the Schwartz space $S(\mathbb{R}^n)$ and $D(\mathbb{R}^n)$ are dense in $W_p^k(\mathbb{R}^n)$ [1].

Proof. We choose to present only a plan of the proof, for more details we refer to [1]. 1. We need to check that (2.10) satisfies the conditions of a norm, i.e. positivity, homogeneity and triangle inequality.

2. We may use the fact that every Cauchy sequence in $W_p^k(\mathbb{R}^n)$ converges in $W_p^k(\mathbb{R}^n)$ to conclude that the classical Sobolev spaces are complete with respect to their norms. Hence $W_p^k(\mathbb{R}^n)$ is a Banach space.   □

The classical Sobolev spaces, $W_p^k(\mathbb{R}^n)$, consist of all $f \in L_p(\mathbb{R}^n)$ such that the distributional derivatives $D^\alpha f \in S'(\mathbb{R}^n)$ with $|\alpha| \leq k$ are regular and belong to $L_p(\mathbb{R}^n)$.

2.5   Fractional Sobolev Spaces on $\mathbb{R}^n$

As mentioned before, the classical Sobolev spaces are Banach spaces, which means that they are complete with respect to their norms. When $p = 2$ and $k \in \mathbb{N}_0$, the Sobolev space norm becomes

$$\|f\|_{W_2^k(\mathbb{R}^n)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L_2(\mathbb{R}^n)}$$
$$= \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |D^\alpha f|^2 \, dx$$
$$= \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} D^\alpha f \overline{D^\alpha f} \, dx$$

Therefore, when $p = 2$ the classical Sobolev spaces become Hilbert spaces equipped with the scalar product [1]

$$\langle f, g \rangle_{W_2^k(\mathbb{R}^n)} = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} D^\alpha f \overline{D^\alpha g} \, dx$$

Furthermore, we write

$$H^k(\mathbb{R}^n) = W_2^k(\mathbb{R}^n), \quad k \in \mathbb{N}_0$$

Note that $H^0(\mathbb{R}^n) = L_2(\mathbb{R}^n)$. Due to the fact that $H^k(\mathbb{R}^n)$ is a Hilbert space, it is easy to describe it in terms of Fourier transforms. This is a motivation to extend the parameter k to noninteger values and even negative values.

In order to characterize the space $W_2^k(\mathbb{R}^n)$ in terms of the Fourier transform, we need to consider the weighted $L_2(\mathbb{R}^n, \omega)$ space which is a Hilbert space where ω is a continuous positive function in $\mathbb{R}^n$ [1].

Definition 8

Let $n \in \mathbb{N}$ and

$$\omega_s(x) = (1 + |x|^2)^{s/2}. \quad s \in \mathbb{R}, x \in \mathbb{R} \tag{2.11}$$

then the weighted L^2 spaces are given by

$$L_2(\mathbb{R}^n, \omega) = \{f \in L_1^{loc}(\mathbb{R}^n) : \omega f \in L_2(\mathbb{R}^n)\}$$

and furnished with a scalar product

$$\langle f, g \rangle_{L_2(\mathbb{R}^n, \omega)} = \int_{\mathbb{R}^n} \omega(x) f(x) \overline{\omega(x) g(x)} \, dx$$

Furthermore, the Schwartz space and the space $D(\mathbb{R}^n)$ are dense in $L_2(\mathbb{R}^n, \omega_s)$ [1].

Additionally, given a function $f \in L_1^{loc}(\mathbb{R})$ then $f \mapsto \omega f$ maps $L_2(\mathbb{R}^n, \omega)$ unitarily onto $L_2(\mathbb{R}^n)$ [1].

The next theorem states that the Fourier transform $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ defined on $S'(\mathbb{R}^n)$ can be restricted to $W_2^k(\mathbb{R}^n)$ and to $L_2(\mathbb{R}^n, \omega_s)$. Due to this fact, one can replace the parameter $k \in \mathbb{N}_0$ with $s \in \mathbb{R}$.

Theorem 5

The Fourier transform $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ generate unitary maps of $W_2^k(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n, \omega_k)$, and of $L_2(\mathbb{R}^n, \omega_k)$ onto $W_2^k(\mathbb{R}^n)$,

$$\mathscr{F} W_2^k(\mathbb{R}^n) = \mathscr{F}^{-1} W_2^k(\mathbb{R}^n) = L_2(\mathbb{R}^n, \omega_k),$$

Proof. We begin with a preparation. There exists two constants c_1 and c_2 such that

$$c_1(1 + |\xi|^2)^k \leq \sum_{|\alpha| \leq k} |\xi^\alpha|^2 \leq c_2(1 + |\xi|^2)^k, \tag{2.12}$$

then the functions $(1 + |\xi|^2)^k$ and $\sum_{|\alpha| \leq k} |\xi^\alpha|^2$ are of comparable size thus it is customary to use $(1 + |\xi|^2)^k$.

Now let $f \in W_2^k(\mathbb{R}^n)$ and evaluate the Sobolev space norm of f using (2.9), (2.10) and the counterpart of (2.4) for $f \in S'(\mathbb{R}^n)$ [1],

$$\begin{aligned}
\|f\|^2_{W_2^k(\mathbb{R}^n)} &= \sum_{|\alpha| \leq k} \|D^\alpha f\|^2_{L_2(\mathbb{R}^n} \\
&= \sum_{|\alpha| \leq k} \|\mathscr{F}(D^\alpha f)\|^2_{L_2(\mathbb{R}^n} \\
&= \sum_{|\alpha| \leq k} \left\| i^{|\alpha|} \xi^\alpha \mathscr{F}(f) \right\|^2_{L_2(\mathbb{R}^n} \\
&= \int_{\mathbb{R}^n} \left(\sum_{|\alpha| \leq k} |x^\alpha|^2 \right) |\mathscr{F} f(x)|^2 \, dx.
\end{aligned}$$

Using (2.12) gives that the Fourier transform is an isometric map from $W_2^k(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n, \omega_k)$. Now let $g \in L_2(\mathbb{R}^n, \omega_k)$ and $f = \mathscr{F}^{-1}g$ and by (2.9) and the the counterpart of (2.4) for $\mathscr{F}^{-1}f \in S'(\mathbb{R}^n)$ [1],

$$\begin{aligned}
D^\alpha f &= D^\alpha \mathscr{F}^{-1} g, \\
&= i^{|\alpha|} \mathscr{F}^{-1}(x^\alpha g) \in L_2(\mathbb{R}^n). \quad |\alpha| \leq k
\end{aligned}$$

Thus $f \in W_2^k(\mathbb{R}^n)$. The Fourier transform maps $L_2(\mathbb{R}^n, \omega_k)$ onto $W_2^k(\mathbb{R}^n)$ and since it is an isometric map, we conclude that the mapping is unitary. $\qquad \square$

In view of the fact that one can extend the definition of $W_2^k(\mathbb{R}^n)$ to $W_2^s(\mathbb{R}^n)$ where $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we can now define fractional Sobolev spaces.

Definition 9
Let $s \in \mathbb{R}$ and ω_s as in (2.11). Then

$$H^s(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : \mathscr{F} f \in L_2(\mathbb{R}^n, \omega_s) \}$$

are the fractional Sobolev spaces.

Since $f \mapsto \omega f$ maps $L_2(\mathbb{R}^n, \omega)$ unitarily onto $L_2(\mathbb{R}^n)$, then for positive $s \in \mathbb{R}$, we conclude $\mathscr{F}f \in L_2(\mathbb{R}^n, \omega_s)$ implies $\mathscr{F}f \in L_2(\mathbb{R}^n)$. Furthermore, by the identity (2.9), we find that all elements of the fractional Sobolev spaces are functions. Precisely this means that if $f \in L_2(\mathbb{R}^n, \omega_s)$, such that $\int_{\mathbb{R}^n}(1+|\xi|^2)^s|\mathscr{F}f(\xi)|^2 \, d\xi$ is finite and since $1 \leq (1+|\xi|^2)^s$, then $\int_{\mathbb{R}^n}|\mathscr{F}f(\xi)|^2 \, d\xi$ is finite as well which implies that $f \in L_2(\mathbb{R}^n)$. However, for negative $s \in \mathbb{R}^n$ then the situation is more complicated and more details are presented in [1]. For later use, we introduce the notation,

$$\langle\xi\rangle_s = (1+|\xi|^2)^{s/2}$$

2.6 Sobolev Embedding

Lastly we want to prove the existence of a linear and bounded map between the Sobolev spaces and spaces of bounded, continuous functions.

Definition 10
Let $BC^k(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, be the space of complex-valued, k-times differentiable functions furnished with a norm

$$\|f\|_{BC^k(\mathbb{R}^n)} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| < \infty$$

Theorem 6
Let $BC^l(\mathbb{R}^n)$ as defined above and $W_2^s(\mathbb{R}^n)$ be the Sobolev spaces. Then the embedding

$$id : W_2^s(\mathbb{R}^n) \hookrightarrow BC^l(\mathbb{R}^n) \tag{2.13}$$

where $l \in \mathbb{N}_0$ and $s > l + \frac{n}{2}$.

Note that the elements of $W_2^s(\mathbb{R}^n)$ are equivalence classes whereas $BC^l(\mathbb{R}^n)$ consists of functions. Therefore the identity (2.13) exists in the sense that for each equivalence class in $W_2^s(\mathbb{R}^n)$ there exists a representative function $f \in BC^l(\mathbb{R}^n)$.

Proof. As previously stated, the Schwartz space $S(\mathbb{R}^n)$ is dense in $W_2^s(\mathbb{R}^n)$ [1]. Due to the fact that both $BC^l(\mathbb{R}^n)$ and $W_2^s(\mathbb{R}^n)$ are Banach spaces, the proof is simplified to show that if there exist a constant $c \geq 0$, then

$$|D^\alpha \varphi(x)| \leq c \, \|\varphi\|_{W_2^s(\mathbb{R}^n)}, \tag{2.14}$$

for all multi-indices $\alpha \in \mathbb{N}_0$, with $|\alpha| \leq l$, $x \in \mathbb{R}^n$ and all $\varphi \in S(\mathbb{R}^n)$. This can be explained by the fact that a convergent sequence in $W_2^s(\mathbb{R}^n)$ is a Cauchy convergent sequence in $BC^l(\mathbb{R}^n)$, provided that (2.14) is satisfied. Using Hölder's inequality and equations (2.3), (2.8), we derive

$$
\begin{aligned}
|D^\alpha \varphi(x)| &= |D^\alpha (\mathscr{F} \mathscr{F}^{-1} \varphi)(x)| \\
&= |\mathscr{F}^{-1}(\xi^\alpha \mathscr{F}\varphi(\xi))(x)| \\
&= c | \int_{\mathbb{R}^n} e^{ix\xi} \xi^\alpha (\mathscr{F}\varphi)(\xi) \, d\xi | \\
&\leq \tilde{c} \int_{\mathbb{R}^n} \langle \xi \rangle^{l-s+s} |\mathscr{F}\varphi(\xi)| \, d\xi \\
&\leq \tilde{c} \left(\int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\mathscr{F}\varphi(\xi)|^2 \, d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} \langle \xi \rangle^{-2(s-l)} \, d\xi \right)^{1/2} \quad (2.15)
\end{aligned}
$$

which implies the desired result since the last integral converges due to $2(s - l) > n$ and the remain of (2.15) is an equivalent norm in $W_2^s(\mathbb{R}^n)$.

$\square$

Chapter 3

General Existence Theorems

In this chapter we consider the initial value problem,

$$\dot{x} = f(x, t),$$
$$x(t_0) = x_0 \tag{3.1}$$

where (t_0, x_0) is a fixed point in an open subset $I \times U \subset \mathbb{R} \times \mathbb{R}^n$, and $f \in C(I \times U, \mathbb{R}^n)$ is a continuous function on this subset. When solving differential equations, we are interested in proving the existence of solutions and their uniqueness. The following definition introduces the concept of Lipschitz continuity which guarantees the uniqueness of solutions.

Definition 11
A continuous function $f \in C(I \times U, \mathbb{R}^n)$ is said to be locally Lipschitz continuous with respect to $x \in U$ for any $(t_0, x_0) \in I \times U$ if there exists $\epsilon, L > 0$ [7] such that

$$|f(t, x) - f(t, y)| \leq L|x - y|,$$

for all $(t, x), (t, y) \in B_\epsilon(t_0, x_0)$. The notation $B_\epsilon(t_0, x_0)$ denotes a ball with radius ϵ centered at (t_0, x_0), and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Note that if the Lipschitz constant L does not depend on the point (t_0, x_0), then we say that the function f is uniformly Lipschitz continuous.

Definition 12

Let X be a set and $d : X \times X \to [0, \infty)$ a function such that

$$d(x, y) = d(y, x),$$

$$d(x, y) \leq d(x, z) + d(z, y),$$

$$d(x, y) = 0 \text{ if and only if } x = y.$$

We say (X, d) is a metric space and d is a metric on X.

3.1   The Banach Fixed-Point theorem

The aim of this section is to prove a theorem about existence and uniqueness of solutions of initial value problems. We begin by the definition of a contraction, then we state Banach fixed point theorem which guarantees the existence and uniqueness of fixed points.

Definition 13

Let (X, d) be metric space. A mapping $T : X \mapsto X$ is called a contraction if there exists $L < 1$ [7] such that

$$d(T(x), T(y)) \leq L d(x, y) \tag{3.2}$$

whenever $x, y \in X$. Furthermore, contractions are continuous.

Theorem 7

Let f be a contraction on a complete metric space (X, d). Then there exists a unique fixed point [7], x, of the mapping f, i. e.

$$f(x) = x.$$

Proof. We begin by showing the existence of a candidate for x. For this purpose, let $x_0 \in X$ and

$$x_1 := f(x_0), x_{n+1} := f(x_n) = f^{n+1}(x_0) \tag{3.3}$$

for $n \in \mathbb{N}$. For $n > m \geq n_0$ and by the triangle inequality we obtain,

$$d(x_n, x_m) \leq \sum_{k=m+1}^{n} d(x_k, x_{k-1})$$

then by substitution,

$$d(x_n, x_m) \leq \sum_{k=m+1}^{n} d(f^k(x_0), f^{k-1}(x_0)).$$

According to the definition of contractions, we have that

$$
\begin{aligned}
d(x_n, x_m) &\leq \sum_{k=m+1}^{n} L^{k-1} \, d(x_1, x_0) \\
&= d(x_1, x_0) L^m \sum_{k=0}^{n-m-1} L^k \\
&= d(x_1, x_0) L^m \frac{1 - L^{n-m}}{1 - L} \\
&\leq \frac{L^{n_0}}{1 - L} d(x_1, x_0) \to 0, \quad \text{as } n_0 \to \infty.
\end{aligned}
$$

where we used the geometric series to move from the second to the third line. This shows that the sequence $\{x_n\}_n$ is Cauchy and convergent as a consequence of the completeness of (X, d). Thus there exists $x \in X$ such that

$$x = \lim_{n \to \infty} x_n \tag{3.4}$$

The second step in this proof is to show that x is a fixed point for f. By (3.3), (3.4), (3.2) and the triangle inequality, we obtain

$$
\begin{aligned}
d(x, f(x)) &\leq d(x, x_n) + d(x_n, f(x_n)) + d(f(x_n), f(x)) \\
&\leq d(x, x_n) + d(x_n, x_{n+1}) + L d(x_n, x) \\
&\leq d(x, x_n) + L^n d(x_0, x_1) + L d(x_n, x)
\end{aligned}
$$

which goes to zero as $n \to \infty$, since $d(x_n, x) \to 0$ and $L < 1$. Therefore, x is

a fixed point for f. It remains to prove its uniqueness. Consider $y = f(y)$,

$$d(x, y) = d(f(x), f(y))$$
$$\leq Ld(x, y)$$

which is true if and only if $d(x, y) = 0$. Thus, by definition we have that $x = y$.

<div align="right">□</div>

3.2   The Picard–Lindelöf Theorem

Now we state our main theorem, named the Picard-Lindelöf theorem [7].

Theorem 8
Let $f : I \times U \mapsto \mathbb{R}^n$ be locally Lipschitz continuous with respect to its second variable and (t_0, x_0) a point in $I \times U$ determining the initial data. Then, for each $\mu > 0$, there exists $\epsilon > 0$ such that the initial-value problem has a unique solution $x \in C^1(B_\epsilon(t_0), B_\mu(x_0))$.

Proof. The first step in this proof is to reformulate the initial value problem (3.1). Through integration, we obtain

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds \tag{3.5}$$

where the integration constant is defined by $x(t_0) = x_0$. While any continuously differentiable solution $x \in C^1(I, U)$ solves (3.1) also satisfies (3.5), a continuous solution of (3.5) needs to be of class C^1.

For clarity in this proof we define some constants. Consider $\delta > 0$ such that $[t_0 - \delta, t_0 + \delta] \subset I$ and let $\mu > 0$ be an arbitrary constant, then

$$R = [t_0 - \delta, t_0 + \delta] \times \overline{B_\mu(x_0)},$$

is the compact cylinder where f is defined. The bound

$$M = \max_{(t,x) \in R} |f(t, x)|.$$

exists since R is closed and bounded. Finally, let L be the Lipschitz constant of f in R,

$$\epsilon = \min\{\delta, \frac{\mu}{M}, \frac{1}{2L}\},$$
$$J = [t_0 - \epsilon, t_0 + \epsilon]$$

Further, we define

$$T(v)(t) = x_0 + \int_{t_0}^{t} f(s, v(s)) \, ds$$

for any function $v \in BC(J, \mathbb{R}^n)$ and $t \in J$. Then, we consider a closed subset of $BC(J, \mathbb{R}^n)$ defined by,

$$X = \{v \in BC(J, \mathbb{R}^n) : v(t_0) = x_0, \sup_{t \in J} |x_0 - v(t)| \leq \mu\}$$

and equipped with a metric

$$d(v_1, v_2) = \max_{t \in J} |v_1(t) - v_2(t)|,$$

thus (X, d) is a complete metric space. Next, we show that the operator T is well-defined, that is T maps X into X or equivalently,

$$|x_0 - T(v)(t)| = |\int_{t_0}^{t} f(s, v(s)) \, ds|, \quad v \in X,$$
$$\leq |t - t_0| \max_{t \in J} |f(t, v(t))|$$
$$\leq \epsilon M$$
$$\leq \mu,$$

with the requirement $\epsilon = \frac{\mu}{M}$. It remains to show that T is a contraction on X. Consider two elements $v_1, v_2 \in X$, then

$$|T(v_1)(t) - T(v_2)(t)| = |\int_{t_0}^{t} (f(s, v_1(s)) - f(s, v_2(s))) \, ds|$$
$$\leq \epsilon |f(s, v_1(s)) - f(s, v_2(s))|$$
$$\leq \epsilon L |v_1(t) - v_2(t)|$$
$$\leq \frac{1}{2} |v_1(t) - v_2(t)|.$$

To reach our goal, we take the maximum over all $t \in J$ and obtain

$$d(T(v_1), T(v_2)) \leq \frac{1}{2} d(v_1, v_2)$$

This allows us to apply the Banach fixed point theorem to conclude that there exists a unique solution $x \in BC(J, \overline{B_\mu(x_0)})$.

$\square$

Chapter 4

Sobolev Theory

A partial differential equation is called well-posed if there exists a unique solution which depends continuously on the initial data. The main goal of this chapter to study the local well-posedness for an equation of Whitham type

$$u_t + Lu_x + u^2 = 0 \tag{4.1}$$

which describes the evolution of a function u of time $t \in \mathbb{R}$ and space $x \in \mathbb{R}$. The operator L is the spatial Fourier multiplier operator defined by

$$\mathscr{F}(Lf(\xi)) = m(\xi)\mathscr{F}(f(\xi))$$

$$m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}}$$

Note that according to the convolution theorem, one can write L as a convolution with $K = \mathscr{F}^{-1}(m)$,

$$Lf = K * f$$

For later use, we calculate an upper bound of $m(\xi)$. Using L'Hôpital'sl rules we obtain

$$\lim_{\xi \to 0} \sqrt{\frac{\tanh(\xi)}{\xi}} = \lim_{\xi \to 0} \sqrt{\frac{\operatorname{sech}^2(\xi)}{1}} = 1$$

Then, since $\sqrt{\frac{\tanh(\xi)}{\xi}} < \sqrt{\xi}\sqrt{\frac{\tanh(\xi)}{\xi}}$ for $\sqrt{\xi} > 1$, $m(\xi)$'s upper limit is

$$\lim_{\xi \to \infty} \sqrt{\xi} \sqrt{\frac{\tanh(\xi)}{\xi}} = \lim_{\xi \to \infty} \sqrt{\frac{\sinh(\xi)}{\cosh(\xi)}}$$

$$= \lim_{\xi \to \infty} \sqrt{\frac{e^{\xi} - e^{-\xi}}{e^{\xi} + e^{-\xi}}}$$

$$= 1$$

Therefore, we obtain

$$\sqrt{\frac{\tanh(\xi)}{\xi}} \leq \min\left(1, \frac{1}{\sqrt{\xi}}\right)$$

$$\leq \frac{C}{1 + \sqrt{\xi}},$$

$$1 + \sqrt{\xi} \leq 2\max(1, \sqrt{\xi})$$

$$\leq \frac{2}{1 + \sqrt{\xi}},$$

$$\leq \frac{2}{(1 + \xi^2)^{1/4}}$$

where $\sqrt{a^2 + b^2} \leq a + b$. In the next section we will consider the equation of Whitham type (4.1) equipped with boundary conditions and discuss the well-posedness properties in Sobolev spaces H^s. We will prove that for $s > 1/2$ this equation is well-posed and its data-to-solution map is continuous.

4.1   Existence and Uniqueness of Solutions

We want to find out whether solutions of (4.1) exist using Sobolev theory. We are interested in finding solutions that do not grow too rapidly at infinity. Therefore, the method we use takes Fourier transforms in the x-variables only. This means for each fixed $t \geq 0$, we consider $u(x, t)$ as a function of x. Furthermore, u is bounded, thus it defines a tempered distribution. Since the Fourier transform extends to an isometric endomorphism on $L_2(\mathbb{R}^n)$, Schwartz space $S(\mathbb{R}^n)$, and the space of Schwartz distributions $S'(\mathbb{R}^n)$ [1], we may apply it to u and its derivatives. The differential equation becomes

$$\hat{u}_t - i\xi m(\xi)\hat{u} + \widehat{u^2} = 0, \tag{4.2}$$

on the Fourier side. In addition, we are interested in finding a solution that satisfies a boundary condition given by

$$u(x, 0) = u_0(x),$$

which, by fixing $\xi \in \mathbb{R}$, becomes

$$
\begin{aligned}
\mathscr{F}u(\xi, 0) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} u(x, 0) \, dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} u_0(x) \, dx \\
&= \hat{u}_0(\xi) \tag{4.3}
\end{aligned}
$$

One verifies immediately that the partial differential equation (4.1) has become a family of ordinary differential equation since only derivative with respect to t is involved. For each ξ, there is one differential equation with initial condition (4.3). Now for fixed ξ, the method we use to solve this ordinary differential equation is the integrating factor method. That is, one multiplies the equation by the integrating factor, $e^{-i\xi m(\xi)t}$,

$$\hat{u}_t e^{-i\xi m(\xi)t} - i\xi m(\xi)\hat{u}e^{-i\xi m(\xi)t} + \widehat{u^2}e^{-i\xi m(\xi)t} = 0 \cdot e^{-i\xi m(\xi)t},$$

such that (4.2) becomes equivalent to

$$\left(\hat{u}e^{-i\xi m(\xi)t}\right)' = -\widehat{u^2}e^{-i\xi m(\xi)t}$$

whereby integrating both sides yields

$$
\int_0^t \left(\hat{u}e^{-i\xi m(\xi)s}\right)' \, ds = -\int_0^t \widehat{u^2}e^{-i\xi m(\xi)s} \, ds
$$

$$
\hat{u}(t)e^{-i\xi m(\xi)t} - \hat{u}(0)e^{-i\xi m(\xi)\cdot 0} = -\int_0^t \widehat{u^2}e^{-i\xi m(\xi)s} \, ds.
$$

In order to obtain the well-posedness result for (4.1) by the Picard-Lindelöf existence theorem, we rearrange the formula above in a form suitable for our purposes.

$$\hat{u}(t) = \hat{u}_0 e^{i\xi m(\xi)t} - \int_0^t \widehat{u^2} e^{i\xi m(\xi)(t-s)} \, ds$$

$$= \mathscr{F}\left(u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right)\right) - \int_0^t \widehat{u^2} e^{i\xi m(\xi)(t-s)} \, ds \qquad (4.4)$$

It remains only to return the equation to physical space. By applying the inverse Fourier transform to (4.4) we thus obtain that

$$\mathscr{F}^{-1}\left(\hat{u}(t)\right) = \mathscr{F}^{-1}\mathscr{F}\left(u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right)\right) - \mathscr{F}^{-1}\left(\int_0^t \widehat{u^2} e^{i\xi m(\xi)(t-s)} \, ds\right)$$

$$u(t) = u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) - \int_0^t \mathscr{F}^{-1}\mathscr{F}\left(u^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right)\right) \, ds$$

$$= u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) - \int_0^t u^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) \, ds \qquad (4.5)$$

The equation (4.2) is equivalent to the integral equation (4.5), but now we are able to use the Picard-Lindelöf theorem since the initial value problem is well formulated for its use.

Let the map T be defined as follows

$$T(u)(t) = u(0) * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) - \int_0^t u^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) \, ds,$$

and satisfying $u(0) = u_0$. Then our aim will be to prove that for fixed t, $u \to Tu$ is a contraction with respect to H^s norm.

Lemma 9

Let $s > 1/2$. For fixed T_0, where $0 \leq t \leq T_0$, if $\|\varphi\|_{H^s(\mathbb{R})} < \delta$, then T is a contraction on the closed ball $B_\delta(0) \subset H^s(\mathbb{R})$.

To achieve this we will need to make some assumptions and observations. The fact that T must be well-defined, means that T maps $B_\delta(0)$ into itself. To simplify the proof, we start by showing that the mapping $B_\delta(0) \to B_\delta(0)$ defined by $u \mapsto \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) * u$ is continuous.

Proof. Let $u \in B_\delta(0)$ then we evaluate the Sobolev norm

$$\left\|\mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) * u\right\|_{H^s(\mathbb{R})} = \left\|\langle\xi\rangle_s \mathscr{F}\left(\mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) * u\right)\right\|_{L_2(\mathbb{R})}$$
$$= \int_{\mathbb{R}} |\langle\xi\rangle_s e^{i\xi m(\xi)t} u|^2 \, dx$$
$$= \int_{\mathbb{R}} |\langle\xi\rangle_s|^2 |u|^2 \, dx$$
$$= \|u\|_{H^s(\mathbb{R})}$$

This proves that $\left(\mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) *\right)$ is an isometric map from $B_\delta(0)$ to $B_\delta(0)$. $\qquad\square$

When for fixed T_0, $0 \leq t \leq T_0$, the map Tu is a contraction on the fractional Sobolev space $H^s(\mathbb{R})$, existence and uniqueness then follows directly by application of the Picard-Lindelöf existence theorem. The next step to prove that T maps $B_\delta(0)$ into itself, is to show that $\|Tu\|_{H^s(\mathbb{R})}$ is bounded.

Proof. Given $u_0 \in B_\delta(0)$, then for fixed T_0, where $t \in [0, T_0]$ we have

$$\|Tu\|_{H^s(\mathbb{R})} = \left\|u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) - \int_0^t u^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) \, ds\right\|_{H^s(\mathbb{R})}$$
$$= \left\|\langle\xi\rangle_s \mathscr{F}\left(u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right)\right) - \right.$$
$$\left. - \langle\xi\rangle_s \mathscr{F}\left(\int_0^t u^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) \, ds\right)\right\|_{L_2(\mathbb{R})}$$
$$\leq \left\|\langle\xi\rangle_s \mathscr{F}\left(u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right)\right)\right\|_{L_2(\mathbb{R})} + $$
$$+ \left\|\langle\xi\rangle_s \mathscr{F}\left(\int_0^t u^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) \, ds\right)\right\|_{L_2(\mathbb{R})}$$
$$\leq \left\|\langle\xi\rangle_s \mathscr{F} u_0\right\|_{L_2(\mathbb{R})} + \int_0^t \left\|\langle\xi\rangle_s \mathscr{F}\left(u^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right)\right)\right\|_{L_2(\mathbb{R})} \, ds$$
$$= \left\|\langle\xi\rangle_s \mathscr{F} u_0\right\|_{L_2(\mathbb{R})} + \int_0^t \left\|\langle\xi\rangle_s \mathscr{F}(u^2) e^{i\xi m(\xi)(t-s)}\right\|_{L_2(\mathbb{R})} \, ds$$

$$= \|\langle\xi\rangle_s \mathscr{F} u_0\|_{L_2(\mathbb{R})} + \int_0^t \|u^2\|_{H^s(\mathbb{R})}\, ds$$

$$\leq \|u_0\|_{H^s(\mathbb{R})} + T_0 \|u^2\|_{H^s(\mathbb{R})}$$

$$\leq \|u_0\|_{H^s(\mathbb{R})} + T_0 \|u\|_{H^s(\mathbb{R})} \|u\|_{H^s(\mathbb{R})}$$

$$< \infty$$

Therefore, when $u_0, u \in B_\delta(0)$, it follows that $Tu \in H^s(\mathbb{R})$. $\qquad\square$

We are now in a position to prove Lemma 9. Recall that a map T is called a contraction if

$$\|Tu_1 - Tu_2\|_{H^s(\mathbb{R})} \leq A \|u_1(t) - u_2(t)\|_{H^s(\mathbb{R})}$$

where $u_1, u_2 \in B_\delta(0)$ and $A < 1$.

Proof. Again T_0 is fixed, $0 \leq t \leq T_0$ and $\|u\|_{H^s(\mathbb{R})} < \delta$. Let $u_1, u_2 \in B_\delta(0)$, then

$$\|Tu_1 - Tu_2\|_{H^s(\mathbb{R})} =$$

$$= \|u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) - \int_0^t u_1^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right)\, ds -$$

$$- u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) + \int_0^t u_2^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right)\, ds\|_{H^s(\mathbb{R})}$$

$$= \left\|\int_0^t u_2^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right)\, ds - \int_0^t u_1^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right)\, ds\right\|_{H^s(\mathbb{R})}$$

$$= \left\|\int_0^t u_2^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) - u_1^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right)\, ds\right\|_{H^s(\mathbb{R})}$$

$$= \left\|\int_0^t \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) * \left[(u_2 - u_1)(u_2 + u_1)\right]\, ds\right\|_{H^s(\mathbb{R})}$$

$$\leq \int_0^t \left\|\mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) * \left[(u_2 - u_1)(u_2 + u_1)\right]\right\|_{H^s(\mathbb{R})}\, ds$$

$$= \int_0^t \|(u_2 - u_1)(u_2 + u_1)\|_{H^s(\mathbb{R})}\, ds$$

$$\leq \int_0^t \|u_2 - u_1\|_{H^s(\mathbb{R})} \|u_2 + u_1\|_{H^s(\mathbb{R})}\, ds$$

$$\leq 2\delta \int_0^t \|u_2 - u_1\|_{H^s(\mathbb{R})} \, ds$$

$$\leq 2\delta T_0 \|u_2 - u_1\|_{H^s(\mathbb{R})}$$

Where by assumption,

$$\|u_2 + u_1\|_{H^s(\mathbb{R})} \leq \|u_2\|_{H^s(\mathbb{R})} + \|u_1\|_{H^s(\mathbb{R})}$$
$$\leq 2\delta$$

We choose that $T_0 = \frac{1}{4\delta}$ and we obtain

$$\|Tu_1 - Tu_2\|_{H^s(\mathbb{R})} \leq \frac{1}{2} \|u_2(t) - u_1(t)\|_{H^s(\mathbb{R})}$$

$\square$

According to the Picard-Lindölef existence theorem, for a fixed T_0 there exists a unique $u \in B_\delta(0) \subset H^s(\mathbb{R})$ such that $Tu = u$.

4.2 Sobolev Solutions of a Dispersive Equation with a Nonlocal Nonlinearity

Here and elsewhere, we use $C^j(\mathbb{R})$ to indicate the space of j-times continuously differentiable functions, such that $\sup_{x \in \mathbb{R}} |f^k(x)| < \infty$, for $k = 0, 1, \ldots, j$. We say that $u \in C^j([0, T), H^s)$ when the mapping $t \in [0, T) \to u$ is j-times continuously differentiable from $[0, T)$ to H^s.

Theorem 10
Let $s > \frac{1}{2}$. Given $u_0 \in H^s(\mathbb{R})$ there exists a unique solution u to (4.1) in the class $u \in C([0, T_0), H^s(\mathbb{R})) \cap C^1([0, T_0), L_2(\mathbb{R}))$ of the nonlinear Whitham equation (4.1) with $u(x, 0) = u_0(0)$

We require that $H^s(\mathbb{R})$ with $s > 1/2$ since H^s is a Banach algebra for $s > 1/2$, i.e. for any $x, y \in H^s$

$$\|xy\| \leq \|x\| \, \|y\| \, .$$

Furthermore, we can use the embedding of Sobolev spaces.

The proof of Theorem 10 is not entirely straightforward. In the process, we will show that

$$
t \mapsto \langle \xi \rangle_s \mathscr{F} \left(u_0 * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)t} \right) - \int_0^t u^2 * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)(t-s)} \right) ds \right)
$$

is continuous $[0, T_0) \hookrightarrow H^s(\mathbb{R})$ and the derivative of (4.5) with respect to t is continuous $[0, T_0) \hookrightarrow L_2(\mathbb{R})$.

Proof. Let $u_0 \in H^s(\mathbb{R})$. In order to show that $u \in C([0, T_0), H^s(\mathbb{R}))$, suppose that a sequence t_n converges to t as $n \to \infty$. Then

$$
\| u(t_n, x) - u(t, x) \|_{H^s(\mathbb{R})} =
$$

$$
= \left\| u_0 * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)t_n} \right) - \int_0^{t_n} u^2(s, x) * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)(t_n-s)} \right) ds - \right.
$$

$$
\left. - u_0 * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)t} \right) + \int_0^t u^2(s, x) * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)(t-s)} \right) ds \right\|_{H^s(\mathbb{R})}
$$

$$
\leq \left\| u_0 * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)t_n} \right) - u_0 * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)t} \right) \right\|_{H^s(\mathbb{R})} +
$$

$$
+ \left\| \int_0^t u^2 * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)(t-s)} \right) ds - \int_0^{t_n} u^2 * \mathscr{F}^{-1} \left(e^{i\xi m(\xi)(t_n-s)} \right) ds \right\|_{H^s(\mathbb{R})}
$$

$$
= \left\| \langle \xi \rangle_s \hat{u}_0 \left(e^{i\xi m(\xi)t_n} - e^{i\xi m(\xi)t} \right) \right\|_{L_2(\mathbb{R})} +
$$

$$
+ \left\| \langle \xi \rangle_s \int_0^t \widehat{u^2} e^{i\xi m(\xi)(t-s)} ds - \langle \xi \rangle_s \int_0^{t_n} \widehat{u^2} e^{i\xi m(\xi)(t_n-s)} ds \right\|_{L_2(\mathbb{R})} \tag{4.6}
$$

Since $e^{i\xi m(\xi)t_n} \to e^{i\xi m(\xi)t}$ pointwise as $t_n \to t$, and since there is a uniform integrable bound, we may apply the dominated convergence theorem on the first term to conclude that,

$$
\left\| \langle \xi \rangle_s \hat{u}_0 \left(e^{i\xi m(\xi)t_n} - e^{i\xi m(\xi)t} \right) \right\|_{L_2(\mathbb{R})}^2 = \int_{\mathbb{R}} | \langle \xi \rangle_s \hat{u}_0 \left(e^{i\xi m(\xi)t_n} - e^{i\xi m(\xi)t} \right) |^2 dx
$$

$$
\leq 4 \int_{\mathbb{R}} | \langle \xi \rangle_s \hat{u}_0 |^2 dx
$$

$$
= 4 \| u_0 \|_{H^s(\mathbb{R})}
$$

where we have used the inequality $|a - b|^2 \leq ||a| + |b||^2$ going from the first to the second line. Next we rewrite the second term in (4.6),

$$
\left\| \langle \xi \rangle_s \int_0^t \widehat{u^2} e^{i\xi m(\xi)(t-s)} \, ds - \langle \xi \rangle_s \int_0^{t_n} \widehat{u^2} e^{i\xi m(\xi)(t_n-s)} \, ds \right\|_{L_2(\mathbb{R})} =
$$

$$
= \left\| \langle \xi \rangle_s \left(e^{i\xi m(\xi)t} \int_0^t \widehat{u^2} e^{-i\xi m(\xi)s} \, ds - e^{i\xi m(\xi)t_n} \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right) \right\|_{L_2(\mathbb{R})}
$$

$$
= \| \langle \xi \rangle_s \left(e^{i\xi m(\xi)t} \int_0^t \widehat{u^2} e^{-i\xi m(\xi)s} \, ds - e^{i\xi m(\xi)t} \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right) +
$$

$$
+ \langle \xi \rangle_s \left(e^{i\xi m(\xi)t} \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds - e^{i\xi m(\xi)t_n} \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right) \|_{L_2(\mathbb{R})}
$$

$$
= \| \langle \xi \rangle_s e^{i\xi m(\xi)t} \left(\int_0^t \widehat{u^2} e^{-i\xi m(\xi)s} \, ds - \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right) +
$$

$$
+ \langle \xi \rangle_s \left(e^{i\xi m(\xi)t} - e^{i\xi m(\xi)t_n} \right) \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \|_{L_2(\mathbb{R})}
$$

$$
\leq \left\| \langle \xi \rangle_s e^{i\xi m(\xi)t} \left(\int_0^t \widehat{u^2} e^{-i\xi m(\xi)s} \, ds - \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right) \right\|_{L_2} +
$$

$$
+ \left\| \langle \xi \rangle_s \left(e^{i\xi m(\xi)t} - e^{i\xi m(\xi)t_n} \right) \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right\|_{L_2}. \tag{4.7}
$$

By the mean value theorem, the first term in (4.7) becomes

$$
\left\| e^{i\xi m(\xi)t} \left(\int_0^t \widehat{u^2} e^{-i\xi m(\xi)s} \, ds - \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right) \right\|_{L_2(\mathbb{R})} =
$$

$$
= \left\| \int_0^t \widehat{u^2} e^{-i\xi m(\xi)s} \, ds - \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right\|_{L_2(\mathbb{R})}
$$

$$
= \left\| \widehat{u^2} e^{-i\xi m(\xi)\tilde{t}} (t_n - t) \right\|_{L_2(\mathbb{R})}
$$

$$
= \left\| \widehat{u^2} (t_n - t) \right\|_{L_2(\mathbb{R})},
$$

which converges to zero as t_n goes to t. Again Since $e^{i\xi m(\xi)t_n} \to e^{i\xi m(\xi)t}$ pointwise as $t_n \to t$, and since there is a uniform integrable bound, we may apply the dominated convergence theorem to conclude that,

$$\left\| \langle \xi \rangle_s \left(e^{i\xi m(\xi)t} - e^{i\xi m(\xi)t_n} \right) \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right\|_{L_2(\mathbb{R})} \leq$$

$$\leq 4 \left\| \langle \xi \rangle_s \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right\|_{L_2(\mathbb{R})}$$

$$\leq 4 \int_0^{t_n} \left\| \langle \xi \rangle_s \widehat{u^2} e^{-i\xi m(\xi)s} \right\|_{L_2(\mathbb{R})} ds$$

$$= 4 \int_0^{t_n} \left\| \langle \xi \rangle_s \widehat{u^2} \right\|_{L_2(\mathbb{R})} ds$$

$$\leq 4T_0 \left\| u^2 \right\|_{H^s(\mathbb{R})}$$

Therefore, by dominated convergence theorem we obtain that

$$\left\| \langle \xi \rangle_s \left(e^{i\xi m(\xi)t} - e^{i\xi m(\xi)t_n} \right) \int_0^{t_n} \widehat{u^2} e^{-i\xi m(\xi)s} \, ds \right\|_{L_2(\mathbb{R})} \to 0$$

as $t_n \to t$.

Lastly we show that $u \in C^1([0, T_0), L_2(\mathbb{R}))$. Note that the operator $L\partial_x$ maps a function $u \in H^s(\mathbb{R})$ from $H^s(\mathbb{R})$ to $L_2(\mathbb{R})$,

$$\|L\partial_x u\|_{L_2(\mathbb{R})}^2 = \|\mathscr{F}(L\partial_x u)\|_{L_2(\mathbb{R})}^2$$

$$= \int_{\mathbb{R}} |m(\xi)\xi \hat{u}|^2 \, d\xi$$

$$\leq \int_{\mathbb{R}} |m(\xi)|^2 |\xi^2 + 1||\hat{u}|^2 \, d\xi$$

$$\leq \int_{\mathbb{R}} |\frac{C}{(\xi^2 + 1)^{1/4}}|^2 |\xi^2 + 1|^2 |\hat{u}|^2 \, d\xi$$

$$= \int_{\mathbb{R}} \tilde{C} \langle \xi \rangle_{1/2} |\hat{u}|^2 \, d\xi < \infty$$

This allows us to derive the derivative of u with respect to t from (4.1), and its continuity from $[0, T_0)$ to $L_2(\mathbb{R})$

$$\|u_t(t,x) - u_t(t_n,x)\|_{L_2(\mathbb{R})} = \left\|-Lu_x(t,x) - u^2(t,x) + Lu_x(t_n,x) + u^2(t_n,x)\right\|_{L_2(\mathbb{R})}$$

$$\leq \|Lu_x(t_n,x) - Lu_x(t,x)\|_{L_2(\mathbb{R})} +$$
$$= \left\|u^2(t_n,x) - u^2(t,x)\right\|_{L_2(\mathbb{R})}$$
$$= \|L\partial_x(u(t_n,x) - u(t,x))\|_{L_2(\mathbb{R})} +$$
$$+ \|(u(t_n,x) - u(t,x))(u(t_n,x) + u(t,x))\|_{L_2(\mathbb{R})}$$

where

$$\|L\partial_x(u(t_n,x) - u(t,x))\|_{L_2(\mathbb{R})} = \int_{\mathbb{R}} |L\partial_x(u(t_n,x) - u(t,x))|^2 \, dx$$

$$= \int_{\mathbb{R}} |i\xi m(\xi)(\hat{u}(t_n,x) - \hat{u}(t,x))|^2 \, dx$$

$$\leq \int_{\mathbb{R}} |\xi^2 + 1| \frac{C}{(\xi^2 + 1)^{1/4}} |^2 |\hat{u}(t_n,x) - \hat{u}(t,x)|^2 \, dx$$

$$= \tilde{C} \int_{\mathbb{R}} |\xi^2 + 1|^{1/2} |\hat{u}(t_n,x) - \hat{u}(t,x)|^2 \, dx$$

Hence for $u \in H^{1/2}(\mathbb{R})$, $\|L\partial_x(u(t_n,x) - u(t,x))\|_{L_2(\mathbb{R})}$ converges to zero as $t_n \to t$. This leads to the following,

$$\|u_t(t,x) - u_t(t_n,x)\|_{L_2(\mathbb{R})} \leq \tilde{C} \|\hat{u}(t_n,x) - \hat{u}(t,x)\|_{H^{1/2}(\mathbb{R})} +$$
$$+ \|(u(t_n,x) - u(t,x))(u(t_n,x) + u(t,x))\|_{L_2(\mathbb{R})}$$
$$\leq \tilde{C} \|\hat{u}(t_n,x) - \hat{u}(t,x)\|_{H^{1/2}(\mathbb{R})} +$$
$$+ 2\delta \|(u(t_n,x) - u(t,x))\|_{L_2(\mathbb{R})}$$

$\square$

4.3   Continuous dependence

The solution to (4.1) depends continuously on the initial data, i.e., the map $u_0 \mapsto u$ is continuous from $H^s(\mathbb{R})$ to $C([0,T_0), H^s(\mathbb{R})) \cap C^1([0,T_0), L_2(\mathbb{R}))$. Consider two solutions $u, v \in H^s(\mathbb{R}^n)$, then

$$\|u(t,x) - v(t,x)\|_{H^s(\mathbb{R})} =$$

$$= \left\| u_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) - \int_0^t u^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) \, ds - \right.$$

$$\left. - v_0 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) + \int_0^t v^2 * \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) \, ds \right\|_{H^s(\mathbb{R})}$$

$$= \left\| \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) * (u_0 - v_0) + \int_0^t \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) * (v^2 - u^2) \, ds \right\|_{H^s(\mathbb{R})}$$

$$\leq \left\| \mathscr{F}^{-1}\left(e^{i\xi m(\xi)t}\right) * (u_0 - v_0) \right\|_{H^s(\mathbb{R})} + \left\| \int_0^t \mathscr{F}^{-1}\left(e^{i\xi m(\xi)(t-s)}\right) * (v^2 - u^2) \, ds \right\|_{H^s(\mathbb{R})}$$

$$= \left\| \langle\xi\rangle_s e^{i\xi m(\xi)t}(\hat{u}_0 - \hat{v}_0) \right\|_{L_2(\mathbb{R})} + \left\| \langle\xi\rangle_s \int_0^t e^{i\xi m(\xi)(t-s)}(\hat{v} - \hat{u})(\hat{v} + \hat{u}) \, ds \right\|_{L_2(\mathbb{R})}$$

$$\leq \left\| \langle\xi\rangle_s (\hat{u}_0 - \hat{v}_0) \right\|_{L_2(\mathbb{R})} + \int_0^t \left\| \langle\xi\rangle_s e^{i\xi m(\xi)(t-s)}(\hat{v} - \hat{u})(\hat{v} + \hat{u}) \right\|_{L_2(\mathbb{R})} \, ds$$

$$\leq \|u_0 - v_0)\|_{H^s(\mathbb{R})} + 2\delta T_0 \|(v - u)\|_{H^s(\mathbb{R})}$$

Recall that $T_0 = \frac{1}{4\delta}$, then we obtain

$$\|u - v\|_{H^s(\mathbb{R})} \leq \|u_0 - v_0\|_{H^s(\mathbb{R})} + \frac{1}{2}\|u - v\|_{H^s(\mathbb{R})} \leq 2\|u_0 - v_0\|_{H^s(\mathbb{R})}$$

To summarize this chapter, we have shown that if $s > 1/2$ and the initial data $u_0 \in H^s(\mathbb{R})$, there exists $T_0 > 0$ and a solution $u \in C([0,T_0), H^s(\mathbb{R}))$ to the initial value problem for the equation of Whitham type (4.1). This solution u is unique in the space $u \in C([0,T_0), H^s(\mathbb{R}))$ and depends continuously on the initial data u_0. The data-to-solution map $u(0) \to u(t)$ is continuous.

Chapter 5

Local Bifurcation Theory

The object of this chapter is to state and prove the local bifurcation theory for our equation. This theory is a technique used to study nonlinear equations which depend on a number of parameters. What is interesting is that for certain values of these parameters, more than one solution may exist. As explained in [4], the local bifurcation occurs when small changes of parameters cause radical changes locally to the solution set.

5.1 Preliminaries

Consider a map $F : X \times \mathbb{R} \mapsto Y$, where X and Y are Banach spaces. Then bifurcation theory is related to the study of equations defined by

$$F(x, \lambda) = 0, \tag{5.1}$$

where x is an element of X, and λ is a parameter in $\mathbb{R}$.

We assume the existence of a solution, (x_0, λ_0) of (5.1) in $X \times \mathbb{R}$. Then, if the function F is C^1 on a neighborhood of the point (x_0, λ_0), one can expand $F(x, \lambda)$ in a Taylor series at (x_0, λ_0),

$$F(x, \lambda) = F_x(x_0, \lambda_0)x + \lambda F_{x\lambda}(x_0, \lambda_0)x + o(\|x\|, |\lambda|)$$

and if the partial derivative with respect to x, $F_x(x_0, \lambda_0)$ is invertible, we can apply the Implicit Function Theorem [5] to obtain a unique solution curve $\lambda \mapsto (x(\lambda), \lambda)$ in a neighborhood of (x_0, λ_0), and thus no bifurcation.

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A necessary condition for the existence of a bifurcation is the failure of the Implicit Function Theorem [4], which means non-invertibility of the partial derivative of F. Suppose now that $F(0, \lambda) = 0$ satisfies (5.1) for all $\lambda \in \mathbb{R}$. Then, one can normalize the curve of solutions $s \mapsto (\xi(s), \mu(s))$ to the so-called trivial solution line $\{(0, \lambda) | \lambda \in \mathbb{R}\}$, by making the change of variables $(x, \lambda) \mapsto (\xi(\lambda) + x, \mu(\lambda))$.

Definition 14 ([5])
We define $(0, \lambda_0)$ as a bifurcation point for (5.1) with respect to the trivial solution line, if every neighborhood of $(0, \lambda_0)$ in $X \times \mathbb{R}$-space contains non-trivial solutions, i.e. the variable $x \in X$ of the function $F(x, \lambda)$ in (5.1) is not zero.

The local bifurcation theory describes the bifurcation in a neighborhood of a bifurcation point, and by Crandall-Rabinowitz theorem one can establish the sufficient conditions for its existence. Before we state this theorem we need to define the concept of Fredholm operators. Those are useful tools since non-invertibility of $D_x F(0, \lambda_0)$ can be difficult to work with.

Definition 15
Let $F : X \to Y$ be a bounded linear operator, where X and Y are Banach spaces. Then F is a Fredholm operator if it satisfies

$$i) \dim(\ker T) < \infty$$
$$ii) \operatorname{codim}(\operatorname{ran} T) < \infty$$
$$iii) \operatorname{ran} T \text{ is closed in } Y$$

The integer, $\dim(\ker F(x)) - \operatorname{codim}(\operatorname{ran} F(x))$, *is called the Fredholm index of $F(x)$.*

As remarked in [4], the third condition in the definition above is customary to include but redundant as it is a consequence of the first and the second conditions.

Now suppose we have a solution curve of (5.1) through $(0, \lambda_0)$, then one can prove the intersection of a second solution curve at $(0, \lambda_0)$ if the conditions of the following theorem are met.

Theorem 11

Let X, Y be Banach spaces over a field $\mathbb{R}$, and $F \in C^2(U \times I, Y)$, where $U \times I \subseteq X \times \mathbb{R}$ is an open neighborhood of $(0, \lambda_0)$. Furthermore, assume that $F(0, \lambda) = 0$ for all $\lambda \in I$, and that

i) the operator $D_x F(0, \lambda_0)$ is a Fredholm operator of index zero,

ii) given some $0 \neq v_0 \in \ker(D_x F(0, \lambda_0))$, then $\ker(D_x F(0, \lambda_0)) = span\{v_0\}$

iii) $D^2_{x\lambda} F(0, \lambda_0)(v_0, 1)$ is not an element of the range of $D_x F(0, \lambda_0)$

Then there is a nontrivial continuously differentiable curve through $(0, \lambda_0)$,

$$\{(x(s), \lambda(s)) | (x(0), \lambda(0)) = (0, \lambda_0)\}$$

such that

$$F(x(s), \lambda(s)) = 0,$$

thus $(0, \lambda_0)$ is a bifurcation point.

For a proof we refer to [4].

5.2 Local Bifurcation Theory

In this section, the Crandall-Rabinowitz local bifurcation theorem is applied in order to establish the existence of periodic traveling waves for the equation of Whitham type (4.1). By traveling waves, we mean waves characterized by a constant speed and shape. In this context, periodic waves means waves that have velocity with finite minimal period. Suppose $\eta(x, t) = \varphi(x - ct)$, with $c > 0$, is a right-going traveling wave with a propagation speed, c. By substitution of this form in (4.1) we obtain

$$-c\varphi' + \varphi^2 + L * \varphi' = 0$$

which by integrating becomes

$$-c\varphi + \int_0^x (\varphi(s))^2 \, ds + L * \varphi = B, \qquad (5.2)$$

for some real constant B. For simplicity, we choose B to be zero. We will consider the equation (4.1) where u is a periodic function. We first define the periodic Sobolev space $H^s_{even,2\pi}$ for $s \in \mathbb{R}$ which consists of all even, 2π-periodic distributions $u \in S'(\mathbb{R})$.

Definition 16 ([9])
Let $L^2_{even,2\pi}$ be the space of even, 2π-periodic, locally square-integrable functions. Then by $H^s_{even,2\pi}$ we denote

$$H^s_{even,2\pi} = \{ u \in L^2_{2\pi} : \|u\|^s_{H_{even,2\pi}} = \left(\sum_{k \geq 0} \left(1 + |k|^2\right)^s |\hat{u}_k|^2 \right) < \infty \}$$

with inner product

$$\langle u, v \rangle_{H^s_{even,2\pi}} = \sum_{k \geq 0} (1 + |k|^2)^s \hat{u}(k) \overline{\hat{v}(k)}.$$

where one has the continuous embedding $H^s_{even,2\pi} \hookrightarrow BC(\mathbb{R})$ for all $s > \frac{1}{2}$.

The even 2π-periodic function $u \in H^s_{even,2\pi}$ is uniquely determined by its Fourier coefficient such that

$$u(x) = \frac{1}{2\sqrt{2\pi}} \hat{u}_0 + \frac{1}{\sqrt{2\pi}} \sum_{k \geq 1} \hat{u}_k \cos(kx)$$

$$= \frac{1}{\sqrt{2\pi}} \sideset{}{'}\sum_{k \geq 0} \hat{u}_k \cos(kx) \qquad (5.3)$$

where the prime indicates that the first term of the sum is multiplied by $1/2$. As explained in [10], the fact that $H^s_{even,2\pi} \subset S'$ implies that the Fourier transform $\mathscr{F} u \in S'$ is well-defined for all $u \in H^s_{even,2\pi}$. However, it is also an element of the sequence space l^s for $s \geq 0$. According to Parseval's theorem, for $u \in H^s_{even,2\pi}$ we have that

$$\|\hat{u}\|^2_{l^2} = \|u\|_{L_2((-\pi,\pi))},$$

and

$$\langle \hat{u}, \hat{v} \rangle_{l^2} = \sum_{k \geq 0} \hat{u}(k)\overline{\hat{v}(k)} = \int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx = \langle u, v \rangle_{L^2((-\pi,\pi))},$$

for all $u, v \in H^s_{even,2\pi}$. We conclude that all Fourier integral formulas defined on the line are equivalent to Fourier series in the periodic case [10]. Now let us see how the convolution operator L in (4.1) acts on periodic functions. Let $u \in H^s_{even,2\pi}$ be a periodic, even function. Then,

$$L * u(x) = \int_{-\infty}^{\infty} L(x-y)u(y)\,dy$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} L(x-y+2\pi k)u(y)\,dy$$

$$= \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} L(x-y+2\pi k) \right) u(y)\,dy$$

Later we will prove that the operator L is a bounded linear map $L : H^s_{even,2\pi} \to H^{s+1/2}_{even,2\pi}$. This fact implies that the integral above is finite and by definition the product

$$A(x) = \sum_{k=-\infty}^{\infty} L(x-y+2\pi k)$$

can be approximated by its Fourier series. Then,

$$A(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \geq 0}' \hat{A}_k \cos(kx)$$

and using the definition of Fourier transform, we obtain the Fourier coefficients $\hat{A}_k$

$$\hat{A}_k = \int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} L(x+2\pi j)\exp(-ixk)\,dx$$

$$= \sum_{j=-\infty}^{\infty} \int_{-\pi}^{\pi} L(x+2\pi j)\exp(-i(x+2\pi j)k)\,dx$$

$$= \int_{-\infty}^{\infty} L(x)\exp(-ixk)\,dx$$

$$= \hat{L}(k).$$

Hence, we have

$$L * u(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \geq 0}{}' \hat{u}_k \hat{A}_k \cos(kx)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k \geq 0}{}' \hat{u}_k \hat{L}(k) \cos(kx) \tag{5.4}$$

We refer to [9] for more information on the periodic Sobolev spaces. Before we introduce the local bifurcation theorem we state the following proposition

Proposition 12

Let the operator L be the Fourier multiplier in (4.1). Then L is a bounded linear map from $H^s_{even,2\pi}(\mathbb{R})$ to $H^{s+1/2}_{even,2\pi}(\mathbb{R})$.

Proof. We perform the proof in $H^s(\mathbb{R})$ which implies the same result in the periodic Sobolev space $H^s_{even,2\pi}(\mathbb{R})$. Let $\varphi \in H^s_{even,2\pi}$ then

$$\begin{aligned}
\|L\varphi\|_{H^{s+1/2}} &= \left\| \langle \xi \rangle_{s+1/2} \widehat{L\varphi} \right\|_{L_2(\mathbb{R})} \\
&= \left\| (1 + |\xi|^2)^{\frac{s+1/2}{2}} m(\xi) \hat{\varphi}(\xi) \right\|_{L_2(\mathbb{R})} \\
&\leq \left\| (1 + |\xi|^2)^{\frac{s+1/2}{2}} \frac{C}{(1 + |\xi|^2)^{1/4}} \hat{\varphi}(\xi) \right\|_{L_2(\mathbb{R})} \\
&= |C| \left\| (1 + |\xi|^{s/2} \hat{\varphi}(\xi) \right\|_{L_2(\mathbb{R})} \\
&= |C| \|\varphi\|_{H^s} .
\end{aligned}$$

$\square$

First, we use the Crandall–Rabinowitz theorem to establish the existence of a curve bifurcating from the trivial solutions.

Theorem 13

Let $H^s_{even,2\pi}(\mathbb{R})$ be the space of periodic, even functions with Fourier series representation (5.3). Let $s > \frac{1}{2}$ and the parameter $c > 0$. The solutions in $H^s_{even,\pi}(\mathbb{R})$ of (4.1), coincide with the kernel of an analytic operator $F : H^s_{even,2\pi} \times \mathbb{R} \to H^s_{even,2\pi}$,

$$F(\varphi, c) = -c\varphi + L\varphi + \int_0^x (\varphi(s))^2 \, ds \tag{5.5}$$

where L is bounded, linear map $L : H^s_{even,2\pi} \to H^{s+\frac{1}{2}}_{even,2\pi}$. Then for $c_0 = m(k)$, where $k \in \mathbb{Z}_+/\{0\}$, we have that $D_\varphi F(0, c_0)$ is Fredholm of index 0 and the conditions of Crandall-Rabinowitz theorem hold.

Proof. The space $H^s_{even,2\pi}$ is a Banach algebra for all $s > 1/2$. Consider the operator F in (5.5) then we have

$$
\begin{aligned}
\|F\|_{H^s_{even,2\pi}} &= \left\| -c\varphi + L\varphi + \int_0^x (\varphi(s))^2 \, ds \right\|_{H^s_{even,2\pi}} \\
&\leq \|c\varphi\|_{H^s_{even,2\pi}} + \|L\varphi\|_{H^s_{even,2\pi}} + \left\| \int_0^x (\varphi(s))^2 \, ds \right\|_{H^s_{even,2\pi}} \\
&\leq |c| \, \|\varphi\|_{H^s_{even,2\pi}} + \|L\varphi\|_{H^s_{even,2\pi}} + \int_0^x \left\| (\varphi(s))^2 \right\|_{H^s_{even,2\pi}} \, ds \\
&\leq |c| \, \|\varphi\|_{H^s_{even,2\pi}} + \|L\varphi\|_{H^s_{even,2\pi}} + \pi \, \|(\varphi(s))\|_{H^s_{even,2\pi}} \, \|(\varphi(s))\|_{H^s_{even,2\pi}},
\end{aligned}
$$

thus $F : H^s_{even,2\pi} \times \mathbb{R} \to H^s_{even,2\pi}$. The operator F is nonlinear in $\varphi \in H^s_{even,2\pi}$. However, the Fréchet derivative of F at (φ_0, c_0) given by

$$
\begin{aligned}
D_\varphi F(\varphi_0, c_0)[\varphi] &= \frac{d}{dt} F(\varphi_0 + \varphi t, c_0)|_{t=0} \\
&= [-c_0 + L + 2 \int_0^x \varphi_0(s) \, ds]\varphi \\
&= -c_0\varphi + L\varphi + 2 \int_0^x \varphi_0(s)\varphi(s) \, ds
\end{aligned}
$$

is linear operator on $H^s_{even,2\pi}$. Then we have that $D_\varphi F(0, c_0)[\varphi] = (c_0 - L)\varphi$ is a bounded linear operator on $H^s_{even,2\pi}$. In order to prove that the linearization $D_\varphi F(0, c_0)$ is Fredholm with index 0, we need to show that its kernel and its range are of dimension one. We substitute (5.3) and (5.4) in the kernel of $D_\varphi F(0, c_0)$ and we obtain

$$
(-c_0 + L)\varphi = 0
$$

$$
\sum_{k \geq 0}' \hat{\varphi}_k (m(k) - c_0) \cos(kx) = 0 \tag{5.6}
$$

One immediately verifies that there exists solutions of (**??**) given by

$$
\hat{\varphi}_k = 0
$$

for all k, but those solutions implies the Implicit Function Theorem, thus no bifurcation. Further, (5.6) holds if $c_0 = m(k)$ whenever $\hat{\varphi}_k = 0$. Since $m(k)$ is strictly decreasing in k, there can be at most one $\hat{\varphi}_{k_0} \neq 0$ for this to hold; and for that k_0, $c_0 = m(k_0)$. Then, $\cos(kx)$ will be the null space of $D_\varphi F(0, c_0)$ since it vanishes at $k = k_0$, thus $\ker(D_\varphi F(0, c_0)) = \operatorname{span} \cos(kx)$. Therefore, $\dim(\ker(D_\varphi F(0, c_0))) = 1$ since $\cos(kx)$ vanishes only at $k = k_0$. In order to identify the range of $D_\varphi F(0, c_0)$ we determine for which $u \in H^s_{even, 2\pi}$ one can solve

$$D_\varphi F(0, c_0)[\varphi] = u,$$

by applying the Fourier transform,

$$\sum_{k \geq 0} \hat{\varphi}_k (m(k) - c_0) \cos(kx) = \sum_{k \geq 0} \hat{u}_k \cos(kx). \tag{5.7}$$

If $m(k) - c_0$ is non-zero for some $k \geq 0$, then we can solve for $\hat{\varphi}_k$. Since it vanishes at $k = k_0$, we have $u_{k_0} = 0$. then if $u_{k_0} \neq 0$, there exists no φ which satisfy (5.7). Therefore, we can solve for φ if and only if u is orthogonal to $\cos(k_0 x)$. Hence, the range of $D_\varphi F(0, c_0) = \operatorname{span} \cos(kx)^\perp$ and we have that $\dim(\operatorname{span} \cos(kx)) = 1$. We have now proved that $D_\varphi F(0, c_0)$ is Fredholm with index 0. Lastly, we need to prove the transcritical condition of the Crandall-Rabinowitz theorem. This is done by verifying that $D^2_{\varphi c} F(0, c_0) \cos(kx)$ is not an element of $\operatorname{ran}(D_\varphi F(0, c_0))$. Taking the derivative of $D_\varphi F(\varphi, c)$ with respect to c, and evaluating the result at $(0, c_0)$ we obtain,

$$D^2_{\varphi c} F(0, c_0) = -1 \neq 0$$

thus, $D^2_{\varphi c} F(0, c_0) \cos(kx)$ is not an element of $\operatorname{ran}(D_\varphi F(0, c_0))$. Then we conclude that the so-called transcritical condition of the Crandall-Rabinowitz theorem is satisfied. Therefore, there exists a nontrivial continuously differentiable curve through $(0, c_0)$,

$$\{\varphi(s), c(s)) | \epsilon > 0, |s| < \epsilon, (\varphi(0), c(0)) = (0, c_0)\}$$

such that $\varphi(s = 0) = 0$ is the null solution of (5.2), and $\{\varphi(s)\}_s$ are nontrivial solutions with wave speeds $\{c(s)\}_s$.   □

We conclude that for every $k \in \mathbb{Z}_+/\{0\}$, there exists a curve $(\varphi(s), c(s))$ bifurcating in the direction of $\cos(kx)$. Those solutions are traveling, 2π-periodic solutions. In addition to the solutions constructed in Theorem 12, there is a curve of trivial, that is, constant solutions bifurcating at $c = m(0) = 1$.

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