NTNU – Trondheim
Norwegian University of
Science and Technology

A spectral method for fractional porous medium equations

Trygve Bærland

Master of Science in Physics and Mathematics
Submission date: June 2015
Supervisor: Espen Robstad Jakobsen, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Preface

With this report, I complete my master's degree in industrial mathematics at NTNU.

Tremendous thanks are due to my supervisor Espen Robstad Jakobsen for his constant support and help in my work throughout this semester. I have also greatly enjoyed our many mathematical discussions, which have never failed to be both interesting and enlightening.

Abstract

This master's thesis considers the fractional general porous medium equation; a nonlocal equation with nonlinear diffusivity. Properties of the nonlocal operator are derived. Existence of distributional solutions are proved, together with L^1-contraction and distance to the family of vanishing viscosity solutions. Then a Fourier Galerkin method with spectral vanishing viscosity (SVV) is proposed and shown to be convergent under suitable conditions to the distributional solution.

Lastly, numerical experiments for some important special cases of the problem are provided, together with convergence plots. This gives some information about when it is suitable to use SVV.

Sammendrag

Denne masteroppgaver betrakter den fraksjonelle, generelle porøse medierligningen; en ikke-lokal ligning på ikkelineær diffusjon. Egenskaper for den ikke-lokale operatoren blir utledet før eksistens av distribusjonsløsninger blir bevist. L^1-kontraksjon og avstand til "vanishing viscosity" løsninger blir også bevist for distribusjonsløsninger. Deretter foreslås en Fourier Galerkinmetode med spektral viskositet (SVV). Det blir bevist at metoden konvergerer under rimelige antakelser til distribusjonsløsningen.

Til slutt vises flere numeriske eksperimenter for noen av de viktigste spesialtilfellene av problemet. I tillegg fremvises noen konvergensplot, som peker på når det er tilrådelig å bruke SVV.

Contents

1 Introduction **2**

2 The fractional general porous medium equation **3**
2.1 The nonlocal operator . . . 3
2.2 Distributional solution and existence . . . 8
2.2.1 Properties of vanishing viscosity solutions . . . 9
2.2.2 Convergence to distributional solution . . . 19

3 Numerical formulation **22**
3.1 Fourier expansion . . . 23
3.2 Formulation of numerical methods . . . 28
3.2.1 Fourier Galerkin method . . . 28
3.2.2 Pseudospectral Fourier Galerkin . . . 29
3.2.3 Considering a Fourier collocation method . . . 29

4 Convergence of the Fourier spectral method **31**
4.1 Preliminaries and assumptions . . . 31
4.2 L^2-stability . . . 34
4.3 Energy estimate of derivatives . . . 35
4.4 L^∞-estimate . . . 39
4.5 BV estimate . . . 44
4.6 Time regularity estimate . . . 48
4.7 Convergence . . . 52

5 Implementation and numerical experiments **56**
5.1 Discretization and numerical solver . . . 56
5.2 Weakly degenerate . . . 57
5.3 Fast diffusion . . . 58
5.4 Strongly degenerate . . . 58
5.5 Approximating the initial data with a smoother kernel . . . 60
5.6 Verifying convergence . . . 63
5.6.1 Weakly degenerate . . . 66
5.6.2 Strongly degenerate with discontinuous initial data . . . 69

6 Concluding remarks **70**
6.1 Where to go from here . . . 70

A Gibb's phenomenon and handling of initial data **72**

B Proof of Lemma 2.9 **76**

C Software documentation **83**
C.1 Computing Fresnel integral . . . 83
C.2 Software documentation . . . 87

1 Introduction

Ever since Einstein showed in [9] the link between Brownian motion and the heat equation, the Laplace operator has played a significant role in modelling diffusive processes. To study an even wider range of processes than the heat equation could handle, the first generalization was to replace the laplacian by an elliptic operator with variable coefficients (for instance where diffusivity is dependent on direction) or even a nonlinear elliptic operator, where diffusivity is dependent on the magnitude of the solution at a point. For evolution equations of these types cf. e.g. [16, 27]. However, if the diffusion process should take long-range effects into account, the laplacian will have to be replaced by a nonlocal operator. Loosely speaking, this corresponds to changing the underlying process from Brownian motion to an another process. The most popular of these is the α-stable Lévy processes, which generate the fractional laplacian. This leads to evolutionary equations of the form

$$\frac{\partial u}{\partial t} = \mathcal{L}(u)$$

where $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in (0, 2)$ is the fractional laplacian.

In this project we will take the natural generalization of the fractional heat equation study the nonlinear problem

$$\frac{\partial u}{\partial t} = \mathcal{L}(\Phi(u)),$$

Since we are going to discuss and implement a numerical method for this equation, we need to restrict it somewhat by adding initial data and suitable boundary conditions. We will consider the following periodic problem:

$$\begin{cases} \frac{\partial u}{\partial t} &= \mathcal{L}(\Phi(u)), \quad \text{in } (0, 2\pi) \times (0, T) \\ u\Big|_{t=0} &= u_0, \end{cases} \tag{1.1}$$

which is a subclass of the problems studied by Stein-Olav Davidsen in his master's thesis [5]. However, a slight oversight in an energy estimate proved to have deep consequences, and the correcting of that argument lead to new insights regarding the stability of the convergence argument with regards to the fractionality parameter α. Therefore, the author has opted to disregard any convection terms in (1.1) in this project, to keep focus on the nonlinear diffusion.

The layout of the project is as follows: First, we will make sure we are on the same page when it comes to notation before we get more explicit about the nonlocal operator and derive some properties for it, we will also define in what sense we seek a solution of (1.1). Some time will also be spent on deriving existence of solutions as well as other properties of solutions. In section 3, we formulate a numerical method for (1.1), and in section 4 we prove convergence under suitable assumptions. Section 5 is devoted to some numerical experiments, getting to know the qualitative behaviour of solution of (1.1), and also numerically verifying the convergence established in section 4.

2 The fractional general porous medium equation

We are working on the spatial domain $(0,2\pi)$ and temporal domain $[0,T]$, where $T<\infty$. We will use Q_T to denote the space-time cylinder $(0,2\pi)\times[0,T]$. We will let $C^k_{per}(\mathbb{R})$ be the set of k times continuously differentiable functions on the real line that are 2π periodic. Its norm is the one inherited from $C^k(\mathbb{R})$.

For sake of brevity, the L^p-norms taken only in space we will use the shorthand $||\cdot||_p = ||\cdot||_{L^p((0,2\pi))}$. In the special case $p=2$ we will go even further and only write $||\cdot||$, which is justified in the central position this norm will take in the ensuing study.

Furthermore, we will use the notation

$$\langle f,g\rangle = \int_0^{2\pi} f(x)g(x)\mathrm{d}x$$

for the standard inner product on $L^2((0,2\pi))$. All other norms, e.g. those taken over Q_T will be made out explicit. Lastly, we will denote by $(\cdot)_+$ the function $\max\{\cdot,0\}$, and C will denote a generic positive constant. The reader should be aware that C may change value from line to line without this being made explicit in what's to follow.

With that out of the way we can direct our focus towards to the problem at hand, viz. 1.1. To begin this section, we will consider the nonlocal operator, and derive some useful properties. This will be done in a slightly more general setting that what is necessary, but attention will be paid to how the results work in the context of the fractional laplacian. Following the discussion on the nonlocal operator, we define in what sense we seek a solution of (1.1). We will then use a vanishing viscosity argument to show the existence of solutions in this sense, in addition to some properties they enjoy.

2.1 The nonlocal operator

We start off this section with a more general notion of the nonlocal operator than we will mainly be concerned with in this report. This section will be devoted to state some useful properties of the nonlocal operator, and pay attention to how these properties pertain to the fractional laplacian.

In the most general case we will consider here we take a nonnegative Radon measure, μ, and then define a nonlocal operator as

$$\mathcal{L}^\mu(u) = \int_{|y|>0} u(x+y)-u(x)-y\mathbf{1}_{|y|<1}\frac{\partial u}{\partial x}(x)\mathrm{d}\mu(y) \tag{2.1}$$

with $\mathbf{1}$ being the indicator function. We assume that the measure satisfies

$$\int_{\mathbb{R}} \min(|y|^2,1)\mathrm{d}\mu(y) < \infty. \tag{2.2}$$

Example 2.1. With the measure μ given by

$$d\mu(z) = \mathrm{d}\pi_\alpha(z) = \frac{c_\alpha \mathrm{d}z}{|z|^{1+\alpha}}, \quad \alpha\in(0,2), \tag{2.3}$$

and

$$c_\alpha = \frac{\alpha\Gamma\left(\frac{1+\alpha}{2}\right)}{2\pi^{1/2+\alpha}\Gamma\left(1-\frac{\alpha}{2}\right)}, \tag{2.4}$$

then $\mathcal{L}^{\pi_\alpha} = -(-\Delta)^{\frac{\alpha}{2}}$. The integrability condition is seen to hold for this measure by

$$\begin{aligned}\int_{\mathbb{R}} \min(|y|^2,1)\mathrm{d}\pi_\alpha(y) &= c_\alpha \int_{-1}^{1} |y|^{1-\alpha}\mathrm{d}y + 2c_\alpha \int_1^\infty |y|^{-1-\alpha}\mathrm{d}y \\ &= \frac{2c_\alpha}{2-\alpha} + \frac{2c_\alpha}{\alpha}\end{aligned}$$

In addition, notice that since the measure is symmetric we can equivalently define

$$-(-\Delta)^{\frac{\alpha}{2}}[u](x) = c_\alpha \int_{|y|>0} u(x+y) - u(x) - y\mathbf{1}_{|y|<r}\frac{\partial u}{\partial x}(x)\mathrm{d}\pi_\alpha(y), \tag{2.5}$$

where $r > 0$ is arbitrary.

Remark 1. With the definition (2.1), the integral is well-defined for any $\psi \in C^2_{per}(\mathbb{R})$. To see this we use the equality

$$\psi(x+y) - \psi(x) - y\frac{\partial\psi}{\partial x}(x) = y^2 \int_0^1 (1-\tau)\frac{\partial^2\psi}{\partial x^2}(x+\tau y)\mathrm{d}\tau,$$

which when put into (2.1) gives us that

$$\begin{aligned}\mathcal{L}^\mu[\psi(\cdot)](x) &= \int_{|y|<1} y^2 \int_0^1 (1-\tau)\frac{\partial^2\psi}{\partial x^2}(x+\tau y)\mathrm{d}\tau\mathrm{d}\mu(y) \\ &+ \int_{|y|>1} \psi(x+y) - \psi(x)\mathrm{d}\mu(y) \\ &\le \left|\left|\frac{\partial^2\psi}{\partial x^2}\right|\right|_\infty \int_{|y|<1} |y|^2\mathrm{d}\mu(y) \\ &+ 2\,||\psi||_\infty \int_{|y|>1} \mathrm{d}\mu(y),\end{aligned}$$

which is finite be the regularity of ψ together with the integrability assumption on the measure.

This fact we'll be used more or less implicitly throughout the report when we use Fubini's theorem to switch the integration over y with any other integration.

The first property of the nonlocal operator we will derive, can in some ways be viewed as an integration by parts formula for the nonlocal operator if the measure μ is symmetric. The result is summarised in the following Lemma.

Lemma 2.1. *For a symmetric Radon measure μ that can be written as $\mathrm{d}\mu(y) = m(y)dy$, where m is measurable, the identity*

$$\langle \mathcal{L}(u), v\rangle = -\frac{1}{2}\int_0^{2\pi} \int_{|y|>0} (u(x+y) - u(x))(v(x+y) - v(x))m(y)\mathrm{d}y\,\mathrm{d}x \tag{2.6}$$

for $u, v \in C^2_{per}(\mathbb{R})$.

Proof. For starters, the variable transformation $y = z - x$ show that

$$\begin{aligned}\langle \mathcal{L}(u), v\rangle &= \int_0^{2\pi}\int_{|z|>0}\left[u(z)-u(x)-(z-x)\mathbf{1}_{|z-x|<1}\frac{\partial u}{\partial x}(x)\right]v(x)m(z-x)\mathrm{d}z\,\mathrm{d}x\\ &= \int_0^{2\pi}\sum_{k\in\mathbb{Z}}\int_{2\pi k}^{2\pi(k+1)}\left[u(z)-u(x)-(z-x)\mathbf{1}_{|z-x|<1}\frac{\partial u}{\partial x}(x)\right]v(x)m(z-x)\mathrm{d}z\,\mathrm{d}x.\end{aligned}$$

Another translation of the z-variable and using the periodicity of u results in

$$\langle \mathcal{L}(u), v\rangle = \sum_{k\in\mathbb{Z}}\int_0^{2\pi}\int_0^{2\pi}\left[u(z)-u(x)-(z-x+2\pi k)\mathbf{1}_{|z-x+2\pi k|<1}\frac{\partial u}{\partial x}(x)\right]v(x)m(z-x+2\pi k)\mathrm{d}z\,\mathrm{d}x.$$

Since $u \in C^2_{per}(\mathbb{R})$ and v is bounded we can use Fubini's Theorem. Together with the symmetry of the measure, this yields

$$\begin{aligned}\langle \mathcal{L}(u), v\rangle &= \sum_{k\in\mathbb{Z}}\int_0^{2\pi}\int_0^{2\pi}\left[u(z)-u(x)-(z-x+2\pi k)\mathbf{1}_{|z-x+2\pi k|<1}\frac{\partial u}{\partial x}(x)\right]v(x)m(x-z-2\pi k)\mathrm{d}x\,\mathrm{d}z\\ &= \sum_{k\in\mathbb{Z}}\int_0^{2\pi}\int_{-2\pi k}^{2\pi(1-k)}\left[u(z)-u(x)-(z-x)\mathbf{1}_{|z-x|<1}\frac{\partial u}{\partial x}(x)\right]v(x)m(x-z)\mathrm{d}x\,\mathrm{d}z,\end{aligned}$$

where we in the last step did a translation in the x-variable. Summing over all k yields an integral in x over $|x| > 0$, which yields

$$\langle \mathcal{L}(u), v\rangle = -\int_0^{2\pi}\int_{|x|>0}\left[u(x)-u(z)-(x-z)\mathbf{1}_{|z-x|<1}\frac{\partial u}{\partial x}(x)\right]v(x)m(x-z)\mathrm{d}x\,\mathrm{d}z.$$

The last step we'll do in this line of reasoning is to recover the y-variable by the transformation $x = z + y$, resulting in

$$\langle \mathcal{L}(u), v\rangle = -\int_0^{2\pi}\int_{|y|>0}\left[u(z+y)-u(z)-y\mathbf{1}_{|y|<1}\frac{\partial u}{\partial x}(z+y)\right]v(z+y)m(y)\mathrm{d}y\,\mathrm{d}z.$$

Of course, the z can be interchanged with x, and so by interpolating this equality with the straightforward definition we have

$$\begin{aligned}\langle \mathcal{L}(u), v\rangle &= \frac{1}{2}\int_0^{2\pi}\int_{|y|>0}\left[u(x+y)-u(x)-y\mathbf{1}_{|y|<1}\frac{\partial u}{\partial x}(x)\right]v(x)m(y)\mathrm{d}y\,\mathrm{d}x\\ &\quad -\frac{1}{2}\int_0^{2\pi}\int_{|y|>0}\left[u(x+y)-u(x)-y\mathbf{1}_{|y|<1}\frac{\partial u}{\partial x}(x+y)\right]v(x+y)m(y)\mathrm{d}y\,\mathrm{d}x\\ &= -\frac{1}{2}\int_0^{2\pi}\int_{|y|>0}(u(x+y)-u(x))(v(x+y)-v(x))m(y)\mathrm{d}y\,\mathrm{d}x\\ &\quad +\frac{1}{2}\int_0^{2\pi}\int_{-1}^{1}y\left(\frac{\partial u}{\partial x}(x+y)v(x+y)-\frac{\partial u}{\partial x}(x)v(x)\right)m(y)\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

All that remains is to show that the latter of the right hand side terms is zero. This can be seen by using Fubini's Theorem as

$$\frac{1}{2}\int_0^{2\pi}\int_{-1}^{1} y\left(\frac{\partial u}{\partial x}(x+y)v(x+y) - \frac{\partial u}{\partial x}(x)v(x)\right)m(y)\mathrm{d}y\,\mathrm{d}x$$
$$= \frac{1}{2}\int_{-1}^{1} y\left(\int_0^{2\pi}\frac{\partial u}{\partial x}(x+y)v(x+y)dx - \int_0^{2\pi}\frac{\partial u}{\partial x}(x)v(x)\mathrm{d}x\right)m(y)\mathrm{d}y,$$

which is easily seen to be zero by a translation of the first inner integral together with using periodicity. □

Observation 1. We see that the form (2.6) is both symmetric and bilinear, making $\mathcal{L}$ a self-adjoint operator, at least in $C^2_{per}(\mathbb{R})$.

Remark 2. It is noteworthy to remark on the similarity between (2.6) and the identity

$$\langle \Delta u, v\rangle = -\left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right\rangle.$$

Consider now that a function u defined in $(0, 2\pi)$ lends itself to Fourier representation. That is, u may be written as

$$u(x) = \sum_{\xi\in\mathbb{Z}} \hat{u}_\xi e^{i\xi x}.$$

By the linearity of $\mathcal{L}^\mu$ we thus have that

$$\begin{aligned}\mathcal{L}^\mu[u](x) &= \sum_{\xi\in\mathbb{Z}} \hat{u}_\xi \mathcal{L}^\mu(e^{i\xi x})\\ &= \sum_{\xi\in\mathbb{Z}} \hat{u}_\xi \int_{|y|>0} e^{i\xi(x+y)} - e^{i\xi x} - i\xi y \mathbf{1}_{|y|<1} e^{i\xi x}\mathrm{d}\mu(y)\\ &= \sum_{\xi\in\mathbb{Z}} \underbrace{\left(\int_{|y|>0} e^{i\xi y} - 1 - i\xi y\mathbf{1}_{|y|<1}\mathrm{d}\mu(y)\right)}_{=:G^\mu(\xi)} \hat{u}_\xi e^{i\xi x}\\ &= \sum_{\xi\in\mathbb{Z}} G^\mu(\xi)\hat{u}_\xi e^{i\xi x}.\end{aligned}$$

And so we may see that the nonlocal operator acts as a Fourier multiplier (weighting of the frequencies). For μ symmetric we have the following result.

Lemma 2.2. *For a symmetric, nonnegative Radon measure μ the weighting coefficients $G^\mu(\xi)$ corresponding to the nonlocal operator $\mathcal{L}^\mu$ are real and nonpositive.*

Proof. By the definition, the imaginary part of $G^\mu(\xi)$ is

$$\mathrm{Im}(G^\mu(\xi)) = \int_{|y|>0} \sin(\xi y) - \xi y\mathbf{1}_{|y|<1}\mathrm{d}\mu(y),$$

and since the integrand is odd while the measure is even, this integral is zero.

Similarly the real part is

$$\mathrm{Re}(G^{\mu}(\xi)) = \int_{|y|>0} \cos(\xi y) - 1 \mathrm{d}\mu(y).$$

Here the integrand is even and nonpositive, whereas the measure is even and nonnegative, and so this integral is nonpositive. □

Example 2.2. Again, consider the case $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$. It can be shown that

$$G^{\pi_\alpha}(\xi) = -C_\alpha |\xi|^\alpha \tag{2.7}$$

(cf. e.g. [3]), where $C_\alpha = 2c_\alpha \alpha^{-1} \int_0^\infty x^{-\alpha} \sin(x) > 0$. Indeed, an equivalent definition of the fractional laplacian is via the Fourier transform relation

$$\mathcal{F}[\mathcal{L}[u]](\xi) = |\xi|^\alpha \mathcal{F}[u](\xi).$$

We'll end this section on nonlocal operators with the following interpolation estimate for the fractional laplacian.

Lemma 2.3 (Interpolation estimate for fractional laplacian)**.** *With* $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$ *there is for every* $\epsilon > 0$ *a constant* $C_\epsilon = K\epsilon^{-\alpha/(2-\alpha)}$, *where* K *only depends on* α, *so that*

$$||\mathcal{L}u|| \leq C_\epsilon ||u|| + \epsilon \left|\left| \frac{\partial^2 u}{\partial x^2} \right|\right| \tag{2.8}$$

holds for every $u \in H^2((0, 2\pi))$.

Proof. By Parseval's identity for Fourier series we have that

$$||\mathcal{L}u||^2 = 2\pi C_\alpha^2 \sum_{\xi\in\mathbb{Z}} |\xi|^{2\alpha} |\hat{u}_\xi|^2. \tag{2.9}$$

We'll make use of Young's inequality of the kind

$$xy \leq \epsilon x^p + C_\epsilon y^q \tag{2.10}$$

which holds for $x, y \in [0, \infty)$, $\epsilon > 0$ and $p \in (1, \infty)$. Here $\frac{1}{p} + \frac{1}{q} = 1$ and $C_\epsilon = C\epsilon^{1-q}$ is a constant depending only on p and ϵ that goes to infinity as $\epsilon \to 0$. Take now $\epsilon > 0$ and define $\tilde{\epsilon} = \frac{\epsilon^2}{2\pi C_\alpha^2}$. We then have when choosing $p = \frac{2}{\alpha}$ that

$$1 \cdot |\xi|^{2\alpha} \leq \tilde{\epsilon} |\xi|^4 + C_{\tilde{\epsilon}}.$$

With this choice for p we have $q = \frac{2}{2-\alpha}$, see then that $C_{\tilde{\epsilon}} = C\tilde{\epsilon}^{-\alpha/(2-\alpha)} = \hat{C}\epsilon^{-2\alpha/(2-\alpha)}$. Putting this back into (2.9) we get

$$\begin{aligned}
||\mathcal{L}u||^2 &\leq 2\pi C_\alpha^2 \sum_{\xi\in\mathbb{Z}} (\tilde{\epsilon}|\xi|^4 + C_{\tilde{\epsilon}}) |\hat{u}_\xi|^2 \\
&= C_\epsilon \sum_{\xi\in\mathbb{Z}} |\hat{u}_\xi|^2 + \epsilon^2 \sum_{\xi\in\mathbb{Z}} |\xi|^4 |\hat{u}_\xi|^2 \\
&= C_\epsilon ||u||^2 + \epsilon^2 \left|\left| \frac{\partial^2 u}{\partial x^2} \right|\right|^2 \\
&\leq \left(C_\epsilon ||u|| + \epsilon \left|\left| \frac{\partial^2 u}{\partial x^2} \right|\right| \right)^2.
\end{aligned}$$

Take note that the value of C_ϵ was redefined from line to line without this having any detrimental effect on the argument. And in particular in the last step $C_\epsilon = K\epsilon^{-\alpha/(2-\alpha)}$. The proof is completed after taking the square root on both sides of the inequality. □

This concludes our discussion of the more general nonlocal operator, so in the following $\mathcal{L}$ is always the fractional laplacian $-(-\Delta)^{\frac{\alpha}{2}}$, and $\mu = \pi_\alpha$.

2.2 Distributional solution and existence

Rather than looking for a strong solution (1.1), where the PDE is satisfied by u at every point, we will look for a solution in a weaker sense. This is of course very common when studying partial differential equations, as it it eases the regularity requirements on a solution. In this project we will consider distributional solutions, taken from [10].

Definition 2.1. A function $u \in L^\infty(Q_T)$ is an L^∞ **disitributional solution** of 1.1 if

a) For all $\psi \in C^\infty(\mathbb{R} \times [0,T])$ that is 2π-periodic in space and has compact support in $(0,T)$
$$\int_0^T \int_0^{2\pi} u(x,t)\frac{\partial \psi}{\partial t}(x,t) + \Phi(u(x,t))\mathcal{L}[\psi(\cdot,t)](x)\mathrm{d}x\,\mathrm{d}t = 0.$$

b) $u - u_0 \in L^1(Q_T)$.

c) The initial condition is imposed in the sense that
$$\operatorname{ess}\lim_{t\to 0^+} \int_0^{2\pi} |u(x,t) - u_0(x)|\mathrm{d}x = 0.$$

Remark 1. Uniqueness of distributional solutions can be proven for Φ continuous and nondecreasing and the initial data is bounded. However, the additional framework needed to prove uniqueness may distract a bit from our main focus. For the Cauchy problem, this is proven in [10] and the techniques used therein can most likely be extended to the periodic case we are stuying here.

To show existence, and the other properties of distributional solutions of (1.1) we will use the technique of vanishing viscosity. To do this we define the auxiliary problem

$$\begin{cases} \frac{\partial v_\epsilon}{\partial t} & = \mathcal{L}(\Phi(v_\epsilon)) + \epsilon\frac{\partial^2 v_\epsilon}{\partial x^2}, \quad (x,t) \in (0,2\pi)\times[0,T] \\ v_\epsilon(x,0) & = u_0(x), \quad x \in (0,2\pi) \\ v_\epsilon(0,t) & = v_\epsilon(2\pi,t), \quad t \in [0,T], \end{cases} \tag{2.11}$$

where $\epsilon > 0$. Showing existence and regularity of solutions to (2.11) lie beyond the scope of this project report. However, techniques for showing existence and regularity can be adapted from [14, App. B], and also see remark 2.6 given in [4].

What we will be doing in the remainder of this section is study some properties of solutions of (2.11) and finally using a compactness argument to show that the family of vanishing viscosity solutions converge to the unique distributional solution of (1.1).

2.2.1 Properties of vanishing viscosity solutions

On our way to using a compactness argument, the first property of vanishing viscosity solutions we establish is a variant of L^1-contraction. It is summarized in the following Lemma.

Lemma 2.4. *Assume Φ is continuous, nondecreasing and $\Phi(0)=0$. For a fixed $\epsilon>0$, let $v_1,v_2\in C^{2,1}(Q_T)$ be solutions of (2.11) with initial data $u_{0,1}$ and $u_{0,2}$ respectively. Then*

$$\int_0^{2\pi}(v_2(x,t)-v_1(x,t))_+\mathrm{d}x \le \int_0^{2\pi}(u_{0,2}(x)-u_{0,1}(x))_+\mathrm{d}x \tag{2.12}$$

holds for every $t\in[0,T]$.

Proof. Let $\operatorname{sgn}^+_\rho$ be a smooth approximation of sgn^+, which is defined as

$$\operatorname{sgn}^+(x)=\begin{cases}1, & \text{if } x>0\\ 0, & \text{otherwise.}\end{cases}$$

This we may do with the standard mollifier, ω, which is defined as

$$\omega(x)=\begin{cases}Ce^{-\frac{1}{1-|x|^2}}, & \text{for } |x|<1\\ 0, & \text{otherwise,}\end{cases}$$

where C is a normalization factor to ensure that $\int_{\mathbb{R}}\omega(x)\mathrm{d}x=1$. We then go on to define

$$\operatorname{sgn}^+_\rho(x)=\left(\operatorname{sgn}^+ * \frac{1}{\rho}\omega\left(\frac{\cdot}{\rho}\right)\right)(x) \tag{2.13}$$

for $\rho>0$. Take now the difference of (2.11) for v_2 and v_1. By multiplying this with $\operatorname{sgn}^+_\rho(v_2-v_1)$ and intergrating over $(0,2\pi)$ in space we get

$$\begin{aligned}\left\langle\frac{\partial}{\partial t}(v_2-v_1),\operatorname{sgn}^+_\rho(v_2-v_1)\right\rangle &= \left\langle\mathcal{L}(\Phi(v_2)-\Phi(v_1)),\operatorname{sgn}^+_\rho(v_2-v_1)\right\rangle\\ &\quad+\epsilon\left\langle\frac{\partial^2}{\partial x^2}(v_2-v_1),\operatorname{sgn}^+_\rho(v_2-v_1)\right\rangle.\end{aligned} \tag{2.14}$$

We have that

$$\lim_{\rho\to 0}\frac{\partial}{\partial t}(v_2-v_1)\operatorname{sgn}^+_\rho(v_2-v_1)=\frac{\partial}{\partial t}(v_2-v_1)_+,$$

pointwise almost everywhere. Together with the regularity of v_1 and v_2 we may then use the dominated convergence theorem to get that

$$\begin{aligned}\lim_{\rho\to 0}\left\langle\frac{\partial}{\partial t}(v_2-v_1),\operatorname{sgn}^+_\rho(v_2-v_1)\right\rangle &= \int_0^{2\pi}\frac{\partial}{\partial t}(v_2-v_1)_+\mathrm{d}x\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}(v_2-v_1)_+\mathrm{d}x.\end{aligned}$$

For the nonlocal term we make the claim that

$$\lim_{\rho\to 0}\left\langle \mathcal{L}(\Phi(v_2)-\Phi(v_1)),\operatorname{sgn}_\rho^+(v_2-v_1)\right\rangle \le 0. \tag{2.15}$$

To prove this, we start off by using Lemma 2.1 to find that

$$\begin{aligned}
&\left\langle \mathcal{L}(\Phi(v_2)-\Phi(v_1)),\operatorname{sgn}_\rho^+(v_2-v_1)\right\rangle \\
&= -\frac{1}{2}\int_0^{2\pi}\int_{|y|>0}\left[\Phi(v_2(x+y)-\Phi(v_1(x+y))-\Phi(v_2(x))+\Phi(v_1(x))\right] \\
&\times\left[\operatorname{sgn}_\rho^+(v_2(x+y)-v_1(x+y))-\operatorname{sgn}_\rho^+(v_2(x)-v_1(x))\right]\mathrm{d}\mu(y)\,\mathrm{d}x.
\end{aligned}$$

By the smoothness of the integrand, the innermost integral is finite, and since the integration in x is over a finite interval the whole integral is finite. Thus we use Fubini's Theorem to get

$$\begin{aligned}
&\left\langle \mathcal{L}(\Phi(v_2)-\Phi(v_1)),\operatorname{sgn}_\rho^+(v_2-v_1)\right\rangle \\
&= -\frac{1}{2}\int_{|y|>0}\int_0^{2\pi}\left[\Phi(v_2(x+y)-\Phi(v_1(x+y))-\Phi(v_2(x))+\Phi(v_1(x))\right] \\
&\times\left[\operatorname{sgn}_\rho^+(v_2(x+y)-v_1(x+y))-\operatorname{sgn}_\rho^+(v_2(x)-v_1(x))\right]\mathrm{d}x\,\mathrm{d}\mu(y) \\
&=: -\frac{1}{2}\int_{|y|>0} f_\rho(y)\mathrm{d}\mu(y).
\end{aligned}$$

We want to use the dominated convergence theorem to interchange the limit with both integrals, and the first step is to find a function that dominates $f_\rho(y)$ while still being integrable in the measure μ. We need to treat $|y|>1$ and $|y|<1$ separately. The case $|y|>1$ is the easier one, and we see that

$$\begin{aligned}
|f_\rho(y)| &\le \int_0^{2\pi}\left|\Phi(v_2(x+y)-\Phi(v_1(x+y))-\Phi(v_2(x))+\Phi(v_1(x))\right| \\
&\times\left|\operatorname{sgn}_\rho^+(v_2(x+y)-v_1(x+y))-\operatorname{sgn}_\rho^+(v_2(x)-v_1(x))\right|\mathrm{d}x \\
&\le 4\pi\left(||\Phi(v_2)||_\infty+||\Phi(v_1)||_\infty\right)<\infty,
\end{aligned}$$

using the boundedness of $\operatorname{sgn}_\rho^+$ together with the continuity of v_2 and v_1. Because the measure is singular at $y=0$, such a crude estimate will not do for $|y|<1$. Rather, in this case we expand the integrand and translate all terms so that the $\operatorname{sgn}_\rho^+$ terms do not depend on y. To be more explicit we get

$$\begin{aligned}
f_\rho(y) = \int_0^{2\pi} & (\Phi(v_2(x+y)) - \Phi(v_1(x+y)))\,\mathrm{sgn}_\rho^+(v_2(x+y) - v_1(x+y)) \\
& - (\Phi(v_2(x+y)) - \Phi(v_1(x+y)))\,\mathrm{sgn}_\rho^+(v_2(x) - v_1(x)) \\
& - (\Phi(v_2(x)) - \Phi(v_1(x)))\,\mathrm{sgn}_\rho^+(v_2(x+y) - v_1(x+y)) \\
& + (\Phi(v_2(x)) - \Phi(v_1(x)))\,\mathrm{sgn}_\rho^+(v_2(x) - v_1(x))\mathrm{d}x \\
& = \int_{-y}^{2\pi-y} (\Phi(v_2(x)) - \Phi(v_1(x)))\,\mathrm{sgn}_\rho^+(v_2(x) - v_1(x))\mathrm{d}x \\
& - \int_0^{2\pi} (\Phi(v_2(x+y)) - \Phi(v_1(x+y)))\,\mathrm{sgn}_\rho^+(v_2(x) - v_1(x))\mathrm{d}x \\
& - \int_{-y}^{2\pi-y} (\Phi(v_2(x-y)) - \Phi(v_1(x-y)))\,\mathrm{sgn}_\rho^+(v_2(x) - v_1(x))\mathrm{d}x \\
& + \int_0^{2\pi} (\Phi(v_2(x)) - \Phi(v_1(x)))\,\mathrm{sgn}_\rho^+(v_2(x) - v_1(x))\mathrm{d}x.
\end{aligned}$$

Since the integrand is periodic in all terms, the intervals we integrate over can be translated to a common interval. Using this and rearranging terms, we get

$$\begin{aligned}
f_\rho(y) = -\int_0^{2\pi} & \Big[\Big(\Phi(v_2(x+y)) - 2\Phi(v_2(x)) + \Phi(v_2(x-y))\Big) \\
& - \Big(\Phi(v_1(x+y)) - 2\Phi(v_1(x)) + \Phi(v_1(x-y))\Big)\Big] \\
& \times \mathrm{sgn}_\rho^+(v_2(x) - v_1(x))\mathrm{d}x.
\end{aligned}$$

For a general function $g \in C^2(\mathbb{R})$ we have that

$$\begin{aligned}
g(x+y) - 2g(x) + g(x-y) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} g(x+sy)\mathrm{d}s + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} g(x-sy)\mathrm{d}s \\
&= y\int_0^1 \frac{\mathrm{d}g}{\mathrm{d}x}(x+sy) - \frac{\mathrm{d}g}{\mathrm{d}x}(x-sy)\mathrm{d}s \\
&= y\int_0^1 \int_{-s}^{s} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}g}{\mathrm{d}x}(x+\tau y)\right)\mathrm{d}\tau\,\mathrm{d}s \\
&= y^2\int_0^1 \int_{-s}^{s} \frac{\mathrm{d}^2 g}{\mathrm{d}x^2}(x+\tau y)\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}$$

In our particular, this identity gives us the estimate

$$|f_\rho(y)| \le 2\pi|y|^2\left(\left\|\frac{\partial^2\Phi(v_2)}{\partial x^2}\right\|_\infty + \left\|\frac{\partial^2\Phi(v_1)}{\partial x^2}\right\|_\infty\right) < \infty.$$

Thus, there is a constant C, independent of ρ, so that

$$|f_\rho(y)| \le C\min\{|y|^2, 1\},$$

which is integrable in the measure μ. This enables us to use the dominated convergence theorem as

$$\lim_{\rho\to 0}\int_{|y|>0} f_\rho(y)\mathrm{d}\mu(y) = \int_{|y|>0} \lim_{\rho\to 0} f_\rho(y)\mathrm{d}\mu(y).$$

We can use the dominated convergence theorem again on $f_\rho(y)$ since the integrand is bounded. Since the integrand of $f_\rho(y)$ converges pointwise almost everywhere to

$$\begin{aligned}&\Big[\Phi(v_2(x+y)) - \Phi(v_1(x+y)) - \Phi(v_2(x)) + \Phi(v_1(x))\Big]\\ &\times \Big[\operatorname{sgn}^+(v_2(x+y) - v_1(x+y)) - \operatorname{sgn}^+(v_2(x) - v_1(x))\Big],\end{aligned}$$

the use of the dominated convergence theorem results in

$$\begin{aligned}\lim_{\rho\to 0} f_\rho(y) = \int_0^{2\pi} &\Big[\Phi(v_2(x+y)) - \Phi(v_1(x+y)) - \Phi(v_2(x)) + \Phi(v_1(x))\Big]\\ &\times \Big[\operatorname{sgn}^+(v_2(x+y) - v_1(x+y)) - \operatorname{sgn}^+(v_2(x) - v_1(x))\Big]\mathrm{d}x.\end{aligned}$$

We see that the integrand is nonzero if

$$\begin{aligned}v_2(x+y) &> v_1(x+y), \text{ and}\\ v_2(x) &\le v_1(x),\end{aligned}$$

and by the monotonicity of Φ this means that

$$\begin{aligned}\Phi(v_2(x+y)) &\ge \Phi(v_1(x+y)), \text{ and}\\ \Phi(v_2(x)) &\le \Phi(v_1(x)).\end{aligned}$$

By inspection we then find that the integrand is nonnegative. The other case, when

$$\begin{aligned}v_2(x+y) &\le v_1(x+y), \text{ and}\\ v_2(x) &> v_1(x)\end{aligned}$$

can be treated similarly. So we deduce that

$$\lim_{\rho\to 0} f_\rho(y) \ge 0,$$

for almost every y, which makes us finally arrive at the conclusion that

$$\begin{aligned}\lim_{\rho\to 0}\Big\langle \mathcal{L}(\Phi(v_2) - \Phi(v_1)), \operatorname{sgn}_\rho^+(v_2 - v_1)\Big\rangle &= -\frac{1}{2}\lim_{\rho\to 0}\int_{|y|>0} f_\rho(y)\mathrm{d}\mu(y)\\ &\le 0.\end{aligned}$$

With that out of the way, we move on to the viscous term of (2.14). Here we define

$$\eta_\rho(x) = \int_0^x \operatorname{sgn}_\rho^+(s)\,\mathrm{d}s, \tag{2.16}$$

which is a convex function since $\operatorname{sgn}_\rho^+ \ge 0$. We may now use the inequality

$$\begin{aligned}\eta_\rho'(v)\frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 \eta_\rho(v)}{\partial x^2} - \eta_\rho'(v)\left(\frac{\partial v}{\partial x}\right)^2\\ &\le \frac{\partial^2 \eta_\rho(v)}{\partial x^2}\end{aligned}$$

to get that

$$\epsilon\left\langle \frac{\partial^2}{\partial x^2}(v_2-v_1), \operatorname{sgn}^+_\rho(v_2-v_1)\right\rangle \leq \int_0^{2\pi} \frac{\partial^2}{\partial x^2}\eta_\rho(v_2-v_1)\mathrm{d}x$$
$$= 0$$

by the periodicity of v_2 and v_1. In total, we have from (2.14) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}(v_2(x,t)-v_1(x,t))_+\mathrm{d}x \leq 0.$$

Lastly, integrating in time from 0 to $t\in[0,T]$ results in

$$\int_0^{2\pi}(v_2(x,t)-v_1(x,t))_+\mathrm{d}x \leq \int_0^{2\pi}(u_{0,2}(x)-u_{0,1}(x))_+\mathrm{d}x.$$

□

This estimate immediately gives us a comparison principle.

Corollary 2.1. *With the same assumptions as in Lemma 2.4, assume further that $u_{0,2}\geq u_{0,1}$. Then $v_2\geq v_1$ in Q_T.*

Proof. By Lemma 2.4 and the assumption on the initial data

$$\int_0^{2\pi}(v_2(x,t)-v_1(x,t))_+\mathrm{d}x \leq 0$$

for every $t\in[0,T]$. Since $(\cdot)_+\geq 0$ and the continuity of v_2 and v_1, this means that

$$(v_2(x,t)-v_1(x,t))_+ = 0$$

everywhere, which proves the assertion. □

We will now put Lemma 2.4 to good use in proving both L^1-contractivity and L^∞ stability.

Lemma 2.5. *With the same notation and assumptions as Lemma 2.4*

$$||v_2(\cdot,t)-v_1(\cdot,t)||_1 \leq ||u_{0,2}-u_{0,1}||_1 \tag{2.17}$$

holds for every $t\in[0,T]$.

Proof. Using Lemma 2.4 for v_2-v_1 and v_1-v_2 yields

$$\int_0^{2\pi}(v_2(x,t)-v_1(x,t))_+\mathrm{d}x + \int_0^{2\pi}(v_1(x,t)-v_2(x,t))_+\mathrm{d}x$$
$$\leq \int_0^{2\pi}(u_{0,2}(x)-u_{0,1}(x))_+\mathrm{d}x + \int_0^{2\pi}(u_{0,1}(x)-u_{0,2}(x))_+\mathrm{d}x,$$

and by observing that for any $a,b\in\mathbb{R}$

$$|a-b| = (a-b)_+ + (b-a)_+,$$

we get

$$\int_0^{2\pi}|v_2(x,t)-v_1(x,t)|dx \leq \int_0^{2\pi}|u_{0,2}(x)-u_{0,1}(x)|\mathrm{d}x,$$

which completes the proof. □

Corollary 2.2. *(2.11) has at most one solution in* $C^{2,1}(Q_T)$.

Proof. Suppose that v_1 and v_2 are two distinct solutions of (2.11) with the same initial data u_0. Then by Lemma 2.5

$$||v_2(\cdot,t) - v_1(\cdot,t)||_1 = 0.$$

This implies that $v_2 = v_1$ almost everywhere in Q_T, and by continuity the equality is indeed everywhere. □

Lemma 2.6. *Let* $v_\epsilon \in C^{2,1}(Q_T)$ *be a solution of (4.42) and assume* $u_0 \in L^\infty((0,2\pi))$, *then*

$$||v_\epsilon(\cdot,t)||_\infty \leq ||u_0||_\infty, \tag{2.18}$$

for every $t \in [0,T]$.

Proof. By defining a function $f(x,t) = ||u_0||_\infty$, we find that f solves (2.11) with initial data constant equal to $||u_0||_\infty$, and so by Lemma 2.4

$$\int_0^{2\pi} (v_\epsilon(x,t) - ||u_0||_\infty)_+ \mathrm{d}x \leq \int_0^{2\pi} (u_0(x) - ||u_0||_\infty)_+ \mathrm{d}x = 0,$$

which implies that $v_\epsilon(x,t) \leq ||u_0||_\infty$.

Similarly $-f$ solves (2.11) with initial data $-||u_0||_\infty$ and so

$$\int_0^{2\pi} (-||u_0||_\infty - v_\epsilon(x,t))_+ \mathrm{d}x \leq \int_0^{2\pi} (-||u_0||_\infty - u_0(x))_+ \mathrm{d}x = 0, \tag{2.19}$$

which yields

$$v_\epsilon(x,t) \geq -||u_0||_\infty.$$

This together with the first estimate gives us that

$$|v_\epsilon(x,t)| \leq ||u_0||_\infty,$$

which proves the assertion. □

Remark 1. The result of Lemma 2.6 will do for our purposes since it gives a uniform bound on v_ϵ independent of ϵ. However, we should note that the technique used in the above proof can with minor changes yield the stronger result that

$$\inf_{y \in (0,2\pi)} u_0(y) \leq v_\epsilon(x,t) \leq \sup_{y \in (0,2\pi)} u_0(y).$$

Next we'll see that if in addition the initial data has bounded variation, then the vanishing viscosity solution also has bounded variation.

Lemma 2.7. *Assume that* u_0 *has bounded variation in addition to the assumptions of Lemma 2.4. Let* v_ϵ *be the solution to (2.11), then*

$$|v_\epsilon|_{BV} \leq |u_0|_{BV}, \tag{2.20}$$

holds for every $t \in [0,T]$.

Proof. Consider

$$\lambda(v_\epsilon,h) := \int_0^{2\pi} |v_\epsilon(x+h,t) - v_\epsilon(x,t)|\mathrm{d}x.$$

We have that $v_\epsilon(\cdot+h,\cdot)$ solves (2.11) with initial data $v_0 = u_0(\cdot+h)$, and so by Lemma 2.5

$$\lambda(v_\epsilon,h) \leq \int_0^{2\pi} |u_0(x+h) - u_0(x)|\mathrm{d}x.$$

Using Lemma A.1 in [14] we then have that

$$\begin{aligned}
|v_\epsilon|_{BV} &= \lim_{h\to 0} \frac{\lambda(v_\epsilon,h)}{|h|} \\
&\leq \lim_{h\to 0} \frac{1}{|h|}\int_0^{2\pi} |u_0(x+h) - u_0(x)|\mathrm{d}x \\
&\leq \lim_{h\to 0} \frac{|h|\cdot|u_0|_{BV}}{|h|} \\
&= |u_0|_{BV},
\end{aligned}$$

which completes the proof. □

The last estimate we need to enable the use of a compactness argument is a time regularity estimate. The following Lemma gives a sufficient estimate.

Lemma 2.8. *Let $v_\epsilon \in C^{2,1}(Q_T)$ be the solution of (2.11). With the same assumptions as in Lemma 2.7 there is a constant C so that*

$$||v_\epsilon(\cdot,t_2) - v_\epsilon(\cdot,t_1)||_1 \leq C|u_0|_{BV}\sqrt{|t_2-t_1|} \tag{2.21}$$

holds for every $t_2,t_1 \in [0,T]$.

Proof. We start off by defining $v_\epsilon^\rho(x,t) = v_\epsilon(\cdot,t) * \omega_\rho(x)$, with ω being the standard mollifier. We then have by the triangle inequality

$$\begin{aligned}
||v_\epsilon(\cdot,t_2) - v_\epsilon(\cdot,t_1)||_1 \leq\ & ||v_\epsilon(\cdot,t_2) - v_\epsilon^\rho(\cdot,t_2)||_1 \\
&+ ||v_\epsilon^\rho(\cdot,t_2) - v_\epsilon^\rho(\cdot,t_1)||_1 \\
&+ ||v_\epsilon^\rho(\cdot,t_1) - v_\epsilon(\cdot,t_1)||_1 .
\end{aligned} \tag{2.22}$$

For the first and last term on the right hand side of (2.22) we have that

$$\begin{aligned}
||v_\epsilon - v_\epsilon^\rho||_1 &= \int_0^{2\pi} \left| \int_{-\rho}^{\rho} (v_\epsilon(x) - v_\epsilon(x-y))\omega_\rho(y)\,\mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_0^{2\pi}\int_{-\rho}^{\rho} |v_\epsilon(x) - v_\epsilon(x-y)|\omega_\rho(y)\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}$$

and using Fubini's together with Lemma A.1 in [14] we get

$$\begin{aligned}
||v_\epsilon - v_\epsilon^\rho||_1 &\leq \int_{-\rho}^{\rho} \omega_\rho(y) \int_0^{2\pi} |v_\epsilon(x) - v_\epsilon(x-y)|\mathrm{d}x\,\mathrm{d}y \\
&\leq |v_\epsilon|_{BV} \int_{-\rho}^{\rho} |y|\omega_\rho(y)\mathrm{d}y \\
&\leq C|v_\epsilon|_{BV}\rho.
\end{aligned}$$

Putting this back into (2.22) and using Lemma 2.7 results in

$$||v_\epsilon(\cdot,t_2)-v_\epsilon(\cdot,t_1)||_1 \leq C|u_0|_{BV}\rho + ||v_\epsilon^\rho(\cdot,t_2)-v_\epsilon^\rho(\cdot,t_1)||_1 . \tag{2.23}$$

We need to estimate the remaining norm on the right hand side of (2.23), and to do so we see that

$$v_\epsilon^\rho(x,t_2)-v_\epsilon^\rho(x,t_1) = (t_2-t_1)\int_0^1 \frac{\partial v_\epsilon^\rho}{\partial t}(x,t_1+s(t_2-t_1))\mathrm{d}s,$$

which implies that

$$||v_\epsilon^\rho(\cdot,t_2)-v_\epsilon^\rho(\cdot,t_1)||_1 \leq |t_2-t_1|\int_0^{2\pi}\int_0^1 \left|\frac{\partial v_\epsilon^\rho}{\partial t}(x,t_2+s(t_2-t_1))\right|\mathrm{d}s\,\mathrm{d}x.$$

Fubini's Theorem then ensures us that

$$||v_\epsilon^\rho(\cdot,t_2)-v_\epsilon^\rho(\cdot,t_1)||_1 \leq |t_2-t_1|\int_0^1 \left|\left|\frac{\partial v_\epsilon^\rho}{\partial t}(\cdot,t_1+s(t_2-t_1))\right|\right|_1 \mathrm{d}s \tag{2.24}$$

Convolving (2.11) with $\hat{\omega}_\rho$ we find that

$$\begin{aligned}\left|\left|\frac{\partial v_\epsilon^\rho}{\partial t}\right|\right|_1 &\leq ||\mathcal{L}[\Phi(v_\epsilon)]*\omega_\rho||_1 \\ &\quad + \epsilon\left|\left|\frac{\partial^2 v_\epsilon^\rho}{\partial x^2}\right|\right|_1 .\end{aligned} \tag{2.25}$$

The nonlocal term of (2.25) is the trickiest to handle. Here we make the claim that there is a constant C so that

$$||\mathcal{L}[\Phi(v_\epsilon)]*\omega_\rho||_1 \leq C\left(\frac{1}{\rho}|v_\epsilon|_{BV} + ||v_\epsilon||\right) \tag{2.26}$$

To prove this claim we first notice that for a general φ

$$\begin{aligned}(\mathcal{L}(\varphi)*\omega_\rho)(x) &= \int_{\mathbb{R}}\left(\int_{|z|>0}\varphi(y+z)-\varphi(y)-z\mathbf{1}_{|z|<1}\frac{\partial\varphi}{\partial x}(y)\mathrm{d}\mu(z)\right)\omega_\rho(x-y)\mathrm{d}y \\ &= \int_{\mathbb{R}}\left(\int_{|z|<1}\varphi(y+z)-\varphi(y)-z\frac{\partial\varphi}{\partial x}(y)\mathrm{d}\mu(z)\right)\omega_\rho(x-y)\mathrm{d}y \\ &+ \int_{\mathbb{R}}\left(\int_{|z|>1}\varphi(y+z)-\varphi(y)\mathrm{d}\mu(z)\right)\omega_\rho(x-y)\mathrm{d}y.\end{aligned} \tag{2.27}$$

For the first of these terms we have that

$$\begin{aligned}\varphi(y+z)-\varphi(y)-z\frac{\partial\varphi}{\partial x}(y) &= \int_0^1\int_0^\theta z^2\frac{\partial^2\varphi}{\partial x^2}(y+\tau z)\mathrm{d}\tau\,\mathrm{d}\theta \\ &= \int_0^1\int_\tau^1 z^2\frac{\partial^2\varphi}{\partial x^2}(y+\tau z)\mathrm{d}\theta\,\mathrm{d}\tau \\ &= \int_0^1 z^2(1-\tau)\frac{\partial^2\varphi}{\partial x^2}(y+\tau z)\mathrm{d}\tau.\end{aligned}$$

Thus we get for the first term on the right hand side of (2.27), after using Fubini's and integration by parts, that

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\int_{|z|<1}\int_0^1 z^2(1-\tau)\frac{\partial^2\varphi}{\partial x^2}(y+\tau z)\mathrm{d}\tau\,\mathrm{d}\mu(z)\right)\omega_\rho(x-y)\mathrm{d}y\\
&=\int_{\mathbb{R}}\int_{|z|<1}\int_0^1 z^2(1-\tau)\frac{\partial\varphi}{\partial x}(y+\tau z)\frac{\partial\omega_\rho}{\partial x}(x-y)\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}y. \qquad (2.28)
\end{aligned}$$

Note here that the integration by parts was taken in the y-variable, and due to $\omega_\rho(x-y)$ this leads to no additional boundary terms, and no sign change. We now put this back into (2.27) and take the L^1-norm in space to get

$$\begin{aligned}
||\mathcal{L}(\varphi)*\omega_\rho||_1 \le &\underbrace{\int_0^{2\pi}\left|\int_{\mathbb{R}}\int_{|z|<1}\int_0^1 z^2(1-\tau)\frac{\partial\varphi}{\partial x}(y+\tau z)\frac{\partial\omega_\rho}{\partial x}(x-y)\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}y\right|\mathrm{d}x}_{=:I_1}\\
&+\underbrace{\int_0^{2\pi}\left|\int_{\mathbb{R}}\int_{|z|>1}\varphi(y+z)-\varphi(y)\mathrm{d}\mu(z)\omega_\rho(x-y)\mathrm{d}y\right|\mathrm{d}x}_{=:I_2}. \qquad (2.29)
\end{aligned}$$

Considering I_1, we have

$$I_1 \le \int_0^{2\pi}\int_{\mathbb{R}}\int_{|z|<1}\int_0^1 |z|^2(1-\tau)\left|\frac{\partial\varphi}{\partial x}(y+\tau z)\right|\cdot\left|\frac{\partial\omega_\rho}{\partial x}(x-y)\right|\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}y\,\mathrm{d}x,$$

and by the symmetry of convolutions

$$I_1 \le \int_0^{2\pi}\int_{\mathbb{R}}\int_{|z|<1}\int_0^1 |z|^2(1-\tau)\left|\frac{\partial\varphi}{\partial x}(x-y+\tau z)\right|\cdot\left|\frac{\partial\omega_\rho}{\partial x}(y)\right|\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}y\,\mathrm{d}x.$$

Now the only x-dependence is in $\frac{\partial\varphi}{\partial x}$, and so using Fubini's to make the x-integral the innermost we get

$$\begin{aligned}
I_1 &\le |\varphi|_{BV}\int_{\mathbb{R}}\int_{|z|<1}\int_0^1 |z|^2(1-\tau)\left|\frac{\partial\omega_\rho}{\partial x}(y)\right|\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}y.\\
&\le \frac{C}{\rho}|\varphi|_{BV}
\end{aligned}$$

Similarly for I_2 we may use the symmetry of convolutions in conjunction with Fubini's Theorem. This enables the estimates

$$\begin{aligned}
I_2 &\le \int_0^{2\pi}\int_{\mathbb{R}}\int_{|z|>1}|\varphi(x-y+z)|+|\varphi(x-y)|\mathrm{d}\mu(z)\omega_\rho(y)\,\mathrm{d}y\,\mathrm{d}x\\
&\le 2\,||\varphi||_1\int_{\mathbb{R}}\int_{|z|>1}\omega_\rho(y)\mathrm{d}\mu(z)\,\mathrm{d}y\\
&\le C\,||\varphi||_1.
\end{aligned}$$

When the estimates for I_1 and I_2 are put back into (2.27) we end up with

$$||\mathcal{L}(\varphi) * \omega_\rho||_1 \leq C\left(\frac{1}{\rho}|\varphi|_{BV} + ||\varphi||_1\right). \tag{2.30}$$

Putting in $\varphi = \Phi(v_\epsilon)$ we get

$$||\mathcal{L}[\Phi(v_\epsilon)] * \omega_\rho||_1 \leq C\left(\frac{1}{\rho}|\Phi(v_\epsilon)|_{BV} + ||\Phi(v_\epsilon)||_1\right). \tag{2.31}$$

To establish (2.26) we now use that Φ is locally Lipschitz and Lemma 2.6, so there is a constant L_Φ so that

$$\begin{aligned}|\Phi(v_\epsilon)| &= |\Phi(v_\epsilon) - \Phi(0)| \\ &\leq L_\Phi |v_\epsilon|,\end{aligned}$$

which means that

$$||\Phi(v_\epsilon)||_1 \leq L_\Phi \, ||v_\epsilon||_1 .$$

The BV term can be treated similarly. See that for any partition $0 = x_0 < \ldots < x_m = 2\pi$ we have

$$\begin{aligned}\sum_{k=0}^{m-1} |\Phi(v_\epsilon(x_{k+1})) - \Phi(v_\epsilon(x_k))| &\leq L_\Phi \sum_{k=0}^{m-1} |v_\epsilon(x_{k+1}) - v_\epsilon(x_k)| \\ &\leq L_\Phi |v_\epsilon|_{BV}.\end{aligned}$$

Taking the supremum over all such finite partitions we find that

$$|\Phi(v_\epsilon)|_{BV} \leq L_\Phi |v_\epsilon|_{BV}.$$

Putting this back into (2.31) results in the desired

$$||\mathcal{L}[\Phi(v_\epsilon)]||_1 \leq C\left(\frac{1}{\rho}|v_\epsilon|_{BV} + ||v_\epsilon||_1\right). \tag{2.32}$$

Using Lemma 2.5 and 2.7 this results in

$$||\mathcal{L}[\Phi(v_\epsilon)] * \omega_\rho||_1 \leq C\left(\frac{1}{\rho}|u_0|_{BV} + ||u_0||_1\right).$$

Moving on to the viscous term of (2.25) we use the rule of differentiating convolutions and Young's inequality, which yields

$$\begin{aligned}\left|\left|\frac{\partial^2 v_\epsilon^\rho}{\partial x^2}\right|\right|_1 &= \left|\left|\frac{\partial v_\epsilon}{\partial x} * \frac{\partial \omega_\rho}{\partial x}\right|\right|_1 \\ &\leq \left|\left|\frac{\partial v_\epsilon}{\partial x}\right|\right|_1 \left|\left|\frac{\partial \omega_\rho}{\partial x}\right|\right|_1 \\ &\leq C|u_0|_{BV}\frac{1}{\rho}.\end{aligned}$$

With these estimates (2.25) becomes

$$\left\|\frac{\partial v_\epsilon^\rho}{\partial t}\right\|_1 \le C\left(\frac{1}{\rho}+1\right). \tag{2.33}$$

Going further we put this back into (2.24) to get

$$||v_\epsilon^\rho(\cdot,t_2)-v_\epsilon^\rho(\cdot,t_1)||_1 \le C|t_2-t_1|\left(\frac{1}{\rho}+1\right), \tag{2.34}$$

which we put into (2.23) to finally arrive at

$$||v_\epsilon(\cdot,t_2)-v_\epsilon(\cdot,t_1)||_1 \le C\left(\rho+|t_2-t_1|\left(\frac{1}{\rho}+1\right)\right). \tag{2.35}$$

The proof is completed by setting $\rho=\sqrt{|t_2-t_1|}$. □

Before closing out this section on vanishing viscosity solutions, we want to estimate the difference between two vanishing viscosity solution with different viscosities. Although the following estimate is not optimal, it is sufficient for our purposes. The proof is rather long, and has therefore been relegated to appendix B, so as to not disrupt the continuity of the text too much.

Lemma 2.9. *Let $v_\epsilon, v_\delta \in C^{2,1}(Q_T)$ be solutions of (2.11) with viscosities ϵ and δ respectively. Assume further that u_0 has bounded variation. Then there is a constant C so that*

$$||v_\epsilon(\cdot,t)-v_\delta(\cdot,t)||_1 \le C|u_0|\sqrt{\epsilon+\delta}$$

for all $t\in[0,T]$.

2.2.2 Convergence to distributional solution

With the estimates proved in section 2.2.1, we are now in a position to use a compactness argument and show that v_ϵ converges to a distributional solution of (1.1). We will use Kolmogorov's compactness theorem, and for sake of the reader's ease of reference we state it in full now. It can also be found as Theorem A.8 in [14].

Theorem 2.1 (Theorem A.8 in [14])**.** *Let $u_\eta:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ be a family of functions such that for each positive T,*

$$|u_\eta(x,t)|\le C_T, \quad (x,t)\in\mathbb{R}^n\times[0,T]$$

for a constant C_T independent of η. Assume in addition for all compact $B\subset\mathbb{R}^n$ and for $t\in[0,T]$ that

$$\sup_{|\xi|\le|\rho|}\int_B |u_\eta(x+\xi,t)-u_\eta(x,t)|\mathrm{d}x \le \nu_{B,T}(|\rho|),$$

for a modulus of continuity ν. Furthermore, assume for s and t in $[0,T]$ that

$$\int_B |u_\eta(x,t)-u_\eta(x,s)|dx \le \omega_{B,T}(|t-s|) \text{ as } \eta\to 0$$

for some modulus of continuity ω_T. Then there exists a sequence $\eta_j \to 0$ such that for each $t \in [0,T]$ the function $\{u_{\eta_j}(t)\}$ converges to a function $u(t) \in L^1_{loc}(\mathbb{R}^n)$. The convergence is in $C([0,T]; L^1_{loc}(\mathbb{R}^n)$.

The main result of this section is summarized in the following Theorem.

Theorem 2.2. *Assume Φ is C^2, nondecreasing and $\Phi(0) = 0$. Assume further that $u_0 \in L^\infty((0,2\pi)) \cap BV((0,2\pi))$. Then there exists a unique distributional solution, u, to (1.1), and $||u||_{L^\infty(Q_T)} \leq ||u_0||_\infty$. Moreover, if v is a distributional solution of (1.1) with initial data v_0, then*

$$||u(\cdot,t) - v(\cdot,t)||_1 \leq ||u_0 - v_0||_1 \tag{2.36}$$

holds for all $t \in [0,T]$. In addition, if u_ϵ is a solution of (2.11), then there is a constant C, independent of ϵ, so that

$$||u(\cdot,t) - u_\epsilon(\cdot,t)||_1 \leq C|u_0|_{BV}\sqrt{\epsilon}. \tag{2.37}$$

Proof. Denote by $\{u_\epsilon\}_{\epsilon>0}$ the family of solution of (2.11). First we'll establish that u_ϵ has a limit point, and then show that this limit point is the unique distributional solution together with some properties.

Establishing limit point of u_ϵ: Again we want to use Kolmogorov's theorem. Lemma 2.6 $|u_\epsilon(x,t)| \leq ||u_0||_\infty$, and Lemma 2.7 ensures that the second assumption of Theorem 2.1 is satisfied. Time regularity is proved in Lemma 2.8. We may thus conclude that there is a subsequence $\{u_{\epsilon_j}\}_{j\in\mathbb{N}}$ that converges to some u in $C([0,T] : L^1((0,2\pi)))$. In addition, Lemma 2.9 implies that u_ϵ is Cauchy in $C([0,T] : L^1((0,2\pi)))$. Therefore u_ϵ converges to u, not just a subsequence of u_ϵ.

The limit u is a distributional solution: Take any $\psi \in C^\infty(\mathbb{R} \times [0,T])$ that is 2π-periodic and has compact support in the temporal variable. Using integration by parts, and that $\mathcal{L}$ is self-adjoint, we have for every $\epsilon > 0$ that u_ϵ satisfies

$$\int_0^T \int_0^{2\pi} u_\epsilon(x,t)\frac{\partial \psi}{\partial t}(x,t) + \Phi(u_\epsilon(x,t))\mathcal{L}[\psi(\cdot,t)](x) + \epsilon u_\epsilon(x,t)\frac{\partial^2 \psi}{\partial x^2}(x,t)\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.38}$$

By the convergence of u_ϵ to u we have that

$$||u_\epsilon - u||_{L^1(Q_T)} \leq T \sup_{t\in[0,T]} ||u_\epsilon(\cdot,t) - u(\cdot,t)||_1 \to 0,$$

as $\epsilon \to 0$. Similarly, using that Φ is locally Lipschitz, we get

$$\lim_{\epsilon\to 0} ||\Phi(u) - \Phi(u_\epsilon)||_{L^1(Q_T)} = 0.$$

For the viscous term we use the boundedness of u_ϵ as

$$\begin{aligned}\lim_{\epsilon\to 0} \epsilon \int_0^T \int_0^{2\pi} u_\epsilon(x,t)\frac{\partial^2 \psi}{\partial x^2}(x,t)\mathrm{d}x\,\mathrm{d}t &\leq \lim_{\epsilon\to 0} 2\pi T \epsilon\, ||u_0||_\infty \left|\left|\frac{\partial^2 \psi}{\partial x^2}\right|\right|_{L^\infty(Q_T)} \\ &= 0.\end{aligned}$$

It is a well-known fact that strong convergence implies weak convergence, so taking the limit $\epsilon \to 0$ in (2.38) we get

$$\int_0^T \int_0^{2\pi} u(x,t)\frac{\partial \psi}{\partial t}(x,t) + \Phi(u(x,t))\mathcal{L}[\psi(\cdot,t)](x) = 0. \tag{2.39}$$

That u satisfies the initial condition, we see by using the triangle inequality and Lemma 2.8, resulting in

$$\begin{aligned} ||u(\cdot,t) - u_0||_1 &\leq ||u(\cdot,t) - u_\epsilon(\cdot,t)||_1 + ||u_\epsilon(\cdot,t) - u_0||_1 \\ &\leq ||u(\cdot,t) - u_\epsilon(\cdot,t)||_1 + C\sqrt{t}, \end{aligned}$$

where C is a constant independent of ϵ, and so taking the limit $\epsilon \to 0$ we get

$$\lim_{t\to 0^+} ||u(\cdot,t) - u_0||_1 \leq \lim_{t\to 0^+} C\sqrt{t} = 0.$$

And so u is a distributional solution of (1.1).

L^1-contraction: Let v be a distributional solution (1.1) with initial data v_0, and is the limit of vanishing viscosity solution $\{v_\epsilon\}_{\epsilon>0}$. Using the triangle inequality

$$\begin{aligned} ||u(\cdot,t) - v(\cdot,t)||_1 \leq\ & ||u(\cdot,t) - u_\epsilon(\cdot,t)||_1 \\ &+ ||u_\epsilon(\cdot,t) - v_\epsilon(\cdot,t)||_1 \\ &+ ||v_\epsilon(\cdot,t) - v(\cdot,t)||_1 . \end{aligned}$$

Using Lemma 2.5 and taking the limit $\epsilon \to 0$ results in

$$||u(\cdot,t) - v(\cdot,t)||_1 \leq ||u_0 - v_0||_1 ,$$

which proves L^1-contractivity for distributional solutions.

Convergence rate of u_ϵ: Using the triangle inequality and Lemma 2.9 leads to

$$\begin{aligned} ||u(\cdot,t) - u_\epsilon(\cdot,t)||_1 &\leq ||u(\cdot,t) - v_\delta(\cdot,t)||_1 + ||u_\epsilon(\cdot,t) - u_\delta(\cdot,t)||_1 \\ &\leq ||u(\cdot,t) - v_\delta(\cdot,t)||_1 + C|u_0|_{BV}\sqrt{\epsilon+\delta}, \end{aligned}$$

where C is a constant independent of ϵ and δ. Taking $\delta \to 0$ we get

$$||u(\cdot,t) - u_\epsilon(\cdot,t)||_1 \leq C|u_0|_{BV}\sqrt{\epsilon}. \tag{2.40}$$

□

Remark 1. Note that strictly speaking, theorem 2.1 is not needed in this argument because of Lemma 2.9. However, we have incorporated it here to show that the existence of a distributional solution does not hinge on Lemma 2.9.

3 Numerical formulation

The main point of using spectral methods in a numerical code is that their order of convergence is directly related to the smoothness of the underlying exact solution of the problem one is studying. So if the solution is infinitely smooth, the numerical approximation will be satisfactory for a low number of degrees of freedom relative to other numerical formulations, like finite element methods. However, if the exact solution exhibits finite regularity the order of convergence of the spectral method will usually be truncated to a lower order. Therefore, before formulating a spectral method it is beneficial to know something of the regularity of the exact solution we are trying to approximate numerically.

As a representative example we will use $\Phi(u) = |u|^{m-1}u$, for $m \geq 1$. Then (1.1), with the fractional laplacian as nonlocal operator, is called the fractional porous medium equation (see e.g. [6]). It is also valuable to view this equation in conjunction with its local counterpart,

$$\frac{\partial u}{\partial t} = \Delta(|u|^{m-1}u), \tag{3.1}$$

the porous medium equation. More information on the pourous medium equation and some of its generalizations can be found in the monograph [27]. Existence, uniqueness and some regularity results for the fractional porous medium equation

$$\frac{\partial u}{\partial t} = -(-\Delta)^{\frac{\alpha}{2}}(|u|^{m-1}u), \tag{3.2}$$

was studied in amongst others [7]. One of the main difference between (3.1) and (3.2) is that for nonnegative initial data, the fractional porous medium equation behaves more like the heat equation in that the solution is instantly positive everywhere. This is in stark contrast to the local case (3.1), where if the initial data as compact support the solution will have a free boundary that moves at a finite speed (as is the case with the famous Barenblatt solution). Since the fractional laplacian recovers the standard laplacian in the limit $\alpha \to 2$, we can thus expect that the gradient of the solution will be less and less well-behaved for increasing α.

The phenomenon of solutions with compact support in the local case is in no way restricted only to the porous medium equation, and similar behaviour can be expected of any Φ if $\Phi' = 0$ somewhere in the range of the solution. If we want to make a robust and flexible code that can handle a wide range of Φ and $\alpha \in (0, 2)$, the risk of having a wildly varying gradient must in some way be addressed. To that end we will use stabilization technique well-suited to modal basis like trigonometric polynomials, viz. spectral vanishing viscosity (SVV).

This stabilization technique was first used by Tadmor in [24] and consist in adding a viscous term that acts only on the higher frequencies. The spectral vanishing viscosity was first used to recover spectral convergence for spectral methods on the Burgers' equation, whose solution may exhibit shock discontinuities in finite time. To be more precise the spectral vanishing viscosity method consists in adding a viscous term to (1.1) like

$$\frac{\partial u}{\partial t} = \mathcal{L}(\Phi(u)) + \epsilon_N \frac{\partial^2}{\partial x^2}(Q_N * u), \tag{3.3}$$

where Q_N is a symmetric viscosity kernel,

$$Q_N(x) = \sum_{|\xi|\leq N} \hat{Q}_\xi e^{i\xi x}, \tag{3.4}$$

and the coefficients $\hat{Q}_\xi$ satisfy

$$\begin{cases} \hat{Q}_\xi = 0, & |\xi| < m_N \\ 0 \leq \hat{Q}_\xi \leq 1, & |\xi| \geq 1 \\ \hat{Q}_{-\xi} = \hat{Q}_\xi, & |\xi| \leq N \text{ and} \\ \hat{Q}_{|\xi|} \leq \hat{Q}_{|\xi|+1}. \end{cases} \tag{3.5}$$

The main challenge with the spectral vanishing viscosity is calibrating the parameters ϵ_N and m_N so that $\epsilon_N \to 0$ and $m_N \to \infty$ at suitable rates as $N \to \infty$. This point will be more thoroughly discussed in section 4, where we want to establish convergence of the ensuing numerical formulation.

In the remainder of this section we will first consider some projections into Fourier space. These projections will figure prominently in the following where we derive a Fourier Galerkin method. Some space is also devoted to considering a Fourier collocation method, which from an implementation viewpoint will prove to be less feasible than the Galerkin methods.

3.1 Fourier expansion

Due to the representation of a nonlocal operator $\mathcal{L}$ as weighting of Fourier coefficients, a natural choice for the finite dimensional space to seek a numerical solution of (1.1) would seem to be

$$\mathcal{S}_N := \text{span}(\{e^{i\xi x}\}_{\xi=-N}^{N}). \tag{3.6}$$

With this choice of function space, the subsequent numerical methods, and the argument we will use to prove their convergence, is heavily reliant on knowledge about the relevant projections into $\mathcal{S}_N$. Therefore, it seems advisable to devote a section to studying the two most used projections into $\mathcal{S}_N$. This will be our focus in this section.

For the remainder of this section, let $f \in H^m((0,2\pi))$ with $m \geq 0$, which means it has the representation

$$f(x) = \sum_{\xi\in\mathbb{Z}} c_\xi e^{i\xi x}, \tag{3.7}$$

where

$$c_\xi = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-i\xi x}dx. \tag{3.8}$$

In addition, with $f \in H^m((0,2\pi))$ it can be shown that

$$\sum_{\xi\in\mathbb{Z}} (1+|\xi|^{2m})|c_\xi|^2 < \infty.$$

The first projection we'll consider is the truncation of the Fourier series, i.e. we define the projection S_N as

$$(S_N f)(x) = \sum_{|\xi|\leq N} c_\xi e^{i\xi x}. \tag{3.9}$$

The first property to be aware of, and one of the reasons why the space $\mathcal{S}_N$ figures so prominently in spectral methods, is that the projection error decreases with an order depending on the smoothness of f.

Lemma 3.1. *Assume* $f \in H^m((0,2\pi))$, *then*

$$||(I - S_N)f|| \leq N^{-m} \left|\left| \frac{\partial^s f}{\partial x} \right|\right|. \tag{3.10}$$

Proof. This result is easily derived from Parseval's identity as

$$\begin{aligned} ||(I - S_N)f|| &= \left(2\pi \sum_{|\xi|>N} |c_\xi|^2 \right)^{\frac{1}{2}} \\ &= \left(2\pi \sum_{|\xi|>N} \frac{1}{|\xi|^{2m}} |\xi|^{2m} |c_\xi|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{N^m} \left(2\pi \sum_{|\xi|>N} |\xi|^{2m} |c_\xi|^2 \right)^{\frac{1}{2}} \\ &\leq N^{-m} \left|\left| \frac{\partial^m f}{\partial x^m} \right|\right|. \end{aligned}$$

Note that here we have used that

$$\frac{\partial^r f}{\partial x^r} = \sum_{\xi \in \mathbb{Z}} (i\xi)^r c_\xi e^{i\xi x},$$

for $0 \leq r \leq m$. □

Remark 1. With $f \in C^\infty$, we see from (3.10) that the projection $S_N f$ converges to f faster than any polynomial order of N. This rapid rate of convergence is usually given the name of spectral (or exponential) convergence in the literature.

Remark 2. In a similar way it can be shown that for every $0 \leq r \leq m$ there is a C so that

$$\left|\left| \frac{\partial^r}{\partial x^r} (I - S_N) f \right|\right| \leq C N^{r-m} \left|\left| \frac{\partial^m f}{\partial x^m} \right|\right|, \tag{3.11}$$

cf. [1, section 5.1].

In numerical implementation the need to evaluate integrals such as (3.8) may arise, and doing so exaclty will in general not be feasible in an implementation. One will then need to resort to numerical integration to approximate c_ξ. If we use the trapezoidal rule together with the nodes $x_j = \frac{2\pi j}{2N+1}, j = 0, \ldots, 2N$ we get the approximate Fourier coefficients given by

$$\tilde{c}_\xi = \frac{1}{2N+1} \sum_{j=0}^{2N} f(x_j) e^{-i\xi x_j}, \tag{3.12}$$

and we define the projection I_N into $\mathcal{S}_N$ by

$$(I_N f)(x) = \sum_{|\xi|\le N} \tilde{c}_\xi e^{i\xi x}, \tag{3.13}$$

which enjoys the property of interpolating f at the nodes x_j for $j = 0,\dots,2N$ (cf. e.g. [12, sec. 8.1]). As we will see shortly, the operator I_N behaves asymptotically like the truncated Fourier series S_N, albeit with greater constants, but first there is a Lemma showing that I_N is just S_N with an additional "aliasing" error.

Lemma 3.2. *We have the decomposition*

$$I_N = S_N + A_N, \tag{3.14}$$

where

$$(A_N f)(x) = \sum_{|\xi|\le N} \left(\sum_{\substack{k\in\mathbb{Z}\\ k\ne 0}} c_{\xi+k(2N+1)} \right) e^{i\xi x}. \tag{3.15}$$

Moreover, $A_N f$ is orthogonal to $(I - S_N)f$, and so

$$||(I-I_N)f||^2 = ||(I-S_N)f||^2 + ||A_N f||^2. \tag{3.16}$$

Proof. By the Fourier representation of f

$$f(x_j) = \sum_{\xi\in\mathbb{Z}} c_\xi e^{i\xi x_j}, \quad j = 0,\dots,2N.$$

Take now any $k \in \mathbb{Z}$, and see that

$$\begin{aligned} e^{i(\xi+k(2N+1))x_j} &= e^{i(\xi x_j + 2\pi k j)} \\ &= e^{i\xi x_j}, \end{aligned}$$

and therefore

$$f(x_j) = \sum_{|\xi|\le N} \left(\sum_{k\in\mathbb{Z}} c_{\xi+k(2N+1)} \right) e^{i\xi x_j}. \tag{3.17}$$

We now multiply this by $\frac{e^{-inx_j}}{2N+1}$, with $|n| \le N$, and sum over j to get

$$\begin{aligned} \tilde{c}_n &= \sum_{|\xi|\le N} \left(\sum_{k\in\mathbb{Z}} c_{\xi+k(2N+1)} \right) \frac{1}{2N+1} \sum_{j=0}^{2N} e^{i(\xi-n)x_j} \\ &= \sum_{\substack{k\in\mathbb{Z}\\ k\ne 0}} c_{n+k(2N+1)}. \end{aligned}$$

The last step is justified in the evaluation of the sum as

$$\begin{aligned} \frac{1}{2N+1} \sum_{j=0}^{2N} e^{i(\xi-n)x_j} &= \frac{1}{2N+1} \sum_{j=0}^{2N} e^{2\pi i(\xi-n)\frac{j}{2N+1}} \\ &= \begin{cases} 1, & \text{if } \xi = n + (2N+1)k, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and since $-N \leq \xi, n \leq N$ the sum is different from zero only when $\xi = n$. Putting this expression in for $I_N f$ we find that

$$
\begin{aligned}
(I_N f)(x) &= \sum_{|\xi|\leq N} \left(\sum_{k\in\mathbb{Z}} c_{\xi+k(2N+1)} \right) e^{i\xi x} \\
&= \sum_{|\xi|\leq N} c_\xi e^{i\xi x} + \sum_{|\xi|\leq N} \left(\sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} c_{\xi+k(2N+1)} \right) e^{i\xi x} \\
&= S_N f + A_N f.
\end{aligned}
$$

Lastly we find that

$$
\begin{aligned}
(I - I_N) f &= (I - S_N) f - A_N f \\
&= \sum_{|\xi|>N} c_\xi e^{i\xi x} - \sum_{|\xi|\leq N} \left(\sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} c_{\xi+k(2N+1)} \right) e^{i\xi x},
\end{aligned}
$$

and we see that the first term contains no frequencies $\leq N$ whereas the second term only contains such terms. By the orthogonality of $\{e^{i\xi x}\}_{\xi\in\mathbb{Z}}$ and Parseval's identity we thus can conclude that

$$
||(I - I_N) f||^2 = ||(I - S_N) f||^2 + ||A_N f||^2. \tag{3.18}
$$

□

From Lemma 3.2 and 3.1 we see that for I_N to enjoy the same order of convergence as S_N we need quite a strong bound the aliasing error. Luckily this is case, and is summarized in the following lemma.

Lemma 3.3 (Lemma 3.1 in [23]). *Let $f \in H^m((0, 2\pi))$ with $m \geq 1$, then there is a constant C independent of f so that*

$$
||A_N f|| \leq C N^{-m} \left|\left| (I - S_N) \frac{\partial^m f}{\partial x^m} \right|\right|. \tag{3.19}
$$

Proof. First of all we have by Parseval's identity

$$
\begin{aligned}
||A_N f||^2 &= 2\pi \sum_{|\xi|\leq N} \left| \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} c_{\xi+k(2N+1)} \right|^2 \\
&\leq 2\pi \sum_{|\xi|\leq N} \left| \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} |c_{\xi+k(2N+1)}| \right|^2.
\end{aligned}
$$

For every $|\xi| \leq N$ and $k \in \mathbb{Z}$ the inequality

$$
|k| N \leq |\xi + k(2N+1)|
$$

holds, and so

$$||A_N f||^2 \le 2\pi \sum_{|\xi|\le N} \left| \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{|\xi + k(2N+1)|^m}{(|k|N)^m} |c_{\xi+k(2N+1)}| \right|^2 .$$

Use of Cauchy-Schwarz' inequality results in

$$\begin{aligned} ||A_N f||^2 &\le 2\pi \sum_{|\xi|\le N} \left(\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{1}{(|k|N)^{2m}} \right) \left(\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |\xi + k(2N+1)|^2 |c_{\xi+k(2N+1)}|^2 \right) \\ &= \frac{2\pi}{N^{2m}} \sum_{|\xi|\le N} \left(\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{1}{|k|^{2m}} \right) \left(\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |\xi + k(2N+1)|^{2m} |c_{\xi+k(2N+1)}|^2 \right) \\ &\le \frac{C}{N^{2m}} \sum_{|\xi|\le N} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |\xi + k(2N+1)|^{2m} |c_{\xi+k(2N+1)}|^2, \end{aligned} \tag{3.20}$$

since the sum $\sum_{k\neq 0} \frac{1}{|k|^{2m}} < \infty$ for $m \ge 1$. Lastly we notice that the double sum in (3.20) is the same as a single sum over all $|\xi| > N$, and thus we end up with

$$\begin{aligned} ||A_N f||^2 &\le \frac{C}{N^{2m}} \sum_{|\xi|>N} |\xi|^{2m} |c_\xi|^2 \\ &= \frac{C}{N^{2m}} \left|\left| (I - S_N) \frac{\partial^m f}{\partial x^m} \right|\right|^2 . \end{aligned}$$

Taking the square root of this inequality completes the proof. □

Remark 1. Lemma 3.1 in [23] is slightly more general and states the result for $f \in H^p((0,2\pi))$ with $p > \frac{1}{2}$, this corresponds to f being at least bounded and continuous (cf. [13, Thm. 3.32]). This seems reasonable since if f was allowed discontinuous then we would have little control on f outside of the discrete nodes we are doing the interpolation on.

Finally we are in a position where we can prove that the projection error of I_N behaves asymptotically the same as S_N. Since this spectral convergence is paramount in our subsequent proof of convergence, we will be able to show convergence when both using S_N and I_N in the numerical formulation.

Lemma 3.4. *Let $f \in H^m((0,2\pi))$, with $m \ge 1$, and let $0 \le r \le m$. Then there is a constant C so that*

$$\left|\left| \frac{\partial^r}{\partial^r} (I - I_N) f \right|\right| \le C N^{r-m} \left|\left| \frac{\partial^m f}{\partial x^m} \right|\right|. \tag{3.21}$$

Proof. By the decompostion of I_N in Lemma 3.2 the projection error is

$$\left\|\frac{\partial^r}{\partial x^r}(I-I_N)f\right\| \le \left\|\frac{\partial^r}{\partial x^r}(I-S_N)f\right\| + \left\|\frac{\partial^r}{\partial x^r}A_N f\right\|. \tag{3.22}$$

Now we use the spectral convergence of S_N and also realize that for trigonometric polynomials of degree $\le N$ we have $\left\|\frac{\partial^r}{\partial x^r}\cdot\right\| \le N^r\,||\cdot||$. This yields

$$\left\|\frac{\partial^r}{\partial x^r}(I-I_N)f\right\| \le CN^{r-m}\left\|\frac{\partial^m f}{\partial x^m}\right\| + N^r\,||A_N f||, \tag{3.23}$$

and the proof is completed by using Lemma 3.3 on the latter term. □

We close off this section by noting that although S_N and I_N enjoy the same spectral convergence, the interpolation operator I_N does not commute with differentiation or the nonlocal operator $\mathcal{L}^\mu$. This is in contrast with S_N which does commute with any operator that works as a weighting of frequencies ($\frac{\partial^r}{\partial x^r}$ and $\mathcal{L}^\mu$ being particular examples). Luckily, this property of S_N will not be required in the following convergence theory.

3.2 Formulation of numerical methods

3.2.1 Fourier Galerkin method

As is usual with Galerkin methods, we look for a solution of the type

$$u_N(x,t) = \sum_{|\xi|\le N} \hat{u}_\xi(t)e^{i\xi x}, \tag{3.24}$$

i.e. $u_N(\cdot,t) \in \mathcal{S}_N$ for every t, and we want u_N to satisfy (3.3) weakly in $\mathcal{S}_N$. That is,

$$\left\langle \frac{\partial u_N}{\partial t} - \mathcal{L}(\Phi(u_N)) - \epsilon_N\frac{\partial^2}{\partial x^2}(Q_N * u_N), \varphi \right\rangle = 0, \quad \forall \varphi \in \mathcal{S}_N. \tag{3.25}$$

Take now $\varphi(x) = \frac{e^{-i\xi x}}{2\pi}$, which when put in (3.25) results in

$$\frac{\partial \hat{u}_\xi}{\partial t} - G^\mu(\xi)\hat{\Phi}_\xi + \epsilon_N|\xi|^2\hat{Q}_\xi\hat{u}_\xi = 0, \tag{3.26}$$

where

$$\hat{\Phi}_\xi = \frac{1}{2\pi}\int_0^{2\pi}\Phi(u_N(x,t))e^{-i\xi x}dx,$$

i.e. the Fourier coefficients of $\Phi(u_N)$. Considering (3.26), we now multiply by $e^{i\xi x}$ and sum over all $|\xi| \le N$, which gives

$$\frac{\partial}{\partial t}\sum_{|\xi|\le N}\hat{u}_\xi e^{i\xi x} - \sum_{|\xi|\le N} G^\mu(\xi)\hat{\Phi}_\xi e^{i\xi x} + \epsilon_N\sum_{|\xi|\le N}|\xi|^2\hat{Q}_\xi\hat{u}_\xi e^{i\xi x} = 0. \tag{3.27}$$

We recognize this equation as

$$\frac{\partial u_N}{\partial t} = S_N\mathcal{L}(\Phi(u_N)) + \epsilon_N\frac{\partial^2}{\partial x^2}(Q_N * u_N), \tag{3.28}$$

and since S_N and $\mathcal{L}$ commute we arrive at the strong spectral vanishing viscosity method

$$\frac{\partial u_N}{\partial t} = \mathcal{L}(S_N\Phi(u_N)) + \epsilon_N\frac{\partial^2}{\partial x^2}(Q_N * u_N). \tag{3.29}$$

3.2.2 Pseudospectral Fourier Galerkin

If we take a moment to consider (3.29) from an implementational perspective, we see that even though we may have exact knowledge of the Fourier coefficients of u_N, that is no guarantee that we'll be able to solve the integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi(u_N)e^{i\xi x}dx$$

exactly for general Φ. Indeed, calculating $\hat{\Phi}_\xi$ exactly is only feasible for very particular Φ. To make for a flexible numerical code that can handle general Φ the most obvious thing to do seems to be to replace the S_N in (3.29) with the discrete operator I_N. This corresponds to approximating the coefficients $\hat{\Phi}_\xi$ using

$$\hat{\Phi}_\xi \approx \frac{1}{2N+1}\sum_{j=0}^{2N} \Phi(u_N(x_j))e^{i\xi x_j}. \tag{3.30}$$

With the decomposition of I_N from Lemma 3.2, we may define a general projection operator as

$$P_N = S_N + aA_N, \tag{3.31}$$

where a is 0 or 1, depending on whether we want to consider the truncated Fourier series or the discrete interpolation. With this more flexible projection defined we can consider the spectral method given by

$$\frac{\partial u_N}{\partial t} = \mathcal{L}(P_N\Phi(u_N)) + \epsilon_N \frac{\partial^2}{\partial x^2}(Q_N * u_N). \tag{3.32}$$

When $P_N = I_N$ is used, the resulting scheme is in the literature usually called a pseudospectral method, because of the introduction of an aliasing error.

To complete the method (3.32) we need a way to approximate u_0 in the space $\mathcal{S}_N$. The obvious choice would be to take $u_0^N := u_N(\cdot,0) = P_N u_0$, but here one needs to be careful. If u_0 contains jump discontinuities it is a well-known fact that $P_N u_0$ will exhibit oscillations in a neighbourhood of the discontinuities. First off all, this severly inhibits the rate of convergence for the method, but also control of the L^∞-norm and the total variation of $P_N u_0$. Further discussion of how this may be remedied is relegated to appendix A (see also [25]).

It is (3.32) that we will prove convergence for under suitable assumptions, and note that it contains both the Galerkin method and the pseuodospectral method as special cases.

3.2.3 Considering a Fourier collocation method

Another numerical formulation that's reasonable to consider in this context is the collocation method. Here we seek a numerical solution

$$\tilde{u}_N(x,t) = \sum_{|\xi|\leq N} \tilde{u}_\xi(t)e^{i\xi x} \tag{3.33}$$

that satisfies the strong form (3.3) exactly at the discrete points x_j for $j=0,\dots,2N$. That is, $\tilde{u}_N$ satisfies

$$\left(\frac{\partial \tilde{u}_N}{\partial t} - \mathcal{L}(\Phi(\tilde{u}_N)) - \epsilon_N \frac{\partial}{\partial x^2}(Q_N * \tilde{u}_N)\right)\Bigg|_{x=x_j} = 0, \quad j = 0,\dots 2N. \tag{3.34}$$

Now we'll do as in the Fourier Galerkin case and derive a PDE which $\tilde{u}_N$ solves, and to that end we multiply (3.34) by $\frac{e^{-i\xi x_j}}{2N+1}$ and sum over j to get

$$\begin{aligned}\frac{1}{2N+1}\sum_{j=0}^{2N}\frac{\partial \tilde{u}_N(x_j)}{\partial t}e^{-i\xi x_j} &= \frac{1}{2N+1}\sum_{j=0}^{2N}\mathcal{L}(\Phi(\tilde{u}_N(x_j)))e^{-i\xi x_j} \\ &\quad + \frac{\epsilon_N}{2N+1}\sum_{j=0}^{2N}\frac{\partial^2}{\partial x^2}(Q_N * \tilde{u}_N)(x_j)e^{i\xi x_j},\end{aligned} \tag{3.35}$$

which we recognize as

$$\left(I_N\frac{\partial \tilde{u}_N}{\partial t}\right)_\xi = (I_N\mathcal{L}(\Phi(\tilde{u}_N)))_\xi + \epsilon_N\left(I_N\frac{\partial^2}{\partial x^2}(Q_N * \tilde{u}_N)\right)_\xi. \tag{3.36}$$

First off, we recognize that I_N can be removed from the left hand side as well as from the spectral vanishing viscosity term since both these are trigonometric polynomials of degree $\leq N$ for all times. So when multiplying (3.36) by $e^{i\xi x}$ and summing over $|\xi| \leq N$ we get $\tilde{u}_N$ solves

$$\frac{\partial \tilde{u}_N}{\partial t} = I_N\mathcal{L}(\Phi(\tilde{u}_N)) + \epsilon_N\frac{\partial^2}{\partial x^2}(Q_N * \tilde{u}_N). \tag{3.37}$$

Since I_N does not commute with $\mathcal{L}$ this is not equivalent to the pesudospectral Galerkin method. Even worse is the fact that, as we encountered in (3.29), we need to calculate the exact Fourier coefficients of $\Phi(\tilde{u}_N)$, before using the relation

$$\mathcal{L}(\Phi(\tilde{u}_N)) = \sum_{\xi\in\mathbb{Z}} G^\mu(\xi)\tilde{\Phi}_\xi e^{i\xi x}$$

to get the exact point values of $\mathcal{L}(\Phi(\tilde{u}_N))$ at the nodes x_j, and finally using this point values to evaluate $I_N\mathcal{L}(\Phi(\tilde{u}_N))$. As already stated, having a numerical code that calculates $\tilde{\Phi}_\xi$ exactly for general Φ is overly ambitious. Of course, the most reasonable remedy would be to rather approximate $\tilde{\Phi}_\xi$ using I_N, but then we end up with the equation

$$\begin{aligned}\frac{\partial \tilde{u}_N}{\partial t} &= I_N\mathcal{L}(I_N\Phi(\tilde{u}_N)) + \epsilon_N\frac{\partial^2}{\partial x^2}(Q_N * \tilde{u}_N) \\ &= \mathcal{L}(I_N\Phi(\tilde{u}_N) + \epsilon_N\frac{\partial^2}{\partial x^2}(Q_N * \tilde{u}_N),\end{aligned}$$

i.e. the same method as the pseudospectral Galerkin method.

For this reason the collocation method will not be discussed further in this project.

4 Convergence of the Fourier spectral method

For linear problems, a Galerkin method may be proved to be convergent by the powerful Lax-Milgram Theorem (cf. [22, Lemma 3.1]), but in our case the form resulting from a Fourier Galerkin method would not be bilinear, and if we want to handle cases where Φ is degenerate, any hope of proving coercivity is ill-founded. However, spectral methods has the great advantage in the solutions being easily interpreted as a subset of some function space, and so various compactness arguments may be used. In [24, 18] E. Tadmor and Y. Maday were able to put compensated compactness (cf. [26]) to great use, but this type of compactness suits hyperbolic conservation laws well. This is because of the additional entropy condition that a solution needs to satisfy, and no such structure is easily found in parabolic equations. The choice of compactness argument to use should be informed by the structure of the PDE, and as an extension what types of a priori estimates one is able to derive. In [4] S. Cifani and E. R. Jakobsen were able to show L^∞-, total variation- and time estimates for the special case of (3.32) where $\Phi(u) = u$. We will in the following extend these estimates to more general Φ, which will enable the use Kolmogorov's compactness Theorem (Theorem 2.1), but more importantly it will lay the groundwork to show that u_N converges to the vanishing viscosity solution and as a consequence, to the distributional solution of (1.1).

The layout for the rest of this section is as follows: We start by stating the assumptions we will be working with. Then follows derivations of L^2-estimates for u_N and its derivatives. In the proof of the latter, a noteworthy sublety will emerge concerning the estimate's dependance on α. We shall see that this is in accordance with the heuristic that solutions are less well-behaved, or are closer to finitely regular, for α close to 2. We will then use these L^2-estimates to prove boundedness first in L^∞, then in total variation before deriving a time estimate. To close out the section, we bring all these estimates together to show that our numerical solutions u_N indeed converge to the distributional solution of (1.1).

4.1 Preliminaries and assumptions

For the spectral viscosity parameters (ϵ_N, m_N) we assume the following.

Assumption 4.1 (Spectral viscosity parameters)**.**

$$\epsilon_N \propto N^{-\theta}, \quad 0 < \theta < \frac{2}{\nu}$$
$$m_N \propto N^{\frac{\theta}{2}} (\log N)^{-\frac{1}{2}},$$

where $\nu = \frac{2+\alpha}{2-\alpha}$.

For the nonlinearity Φ we assume:

Assumption 4.2 (The nonlinearity Φ)**.** *For Φ we assume that $\Phi(0)$, is nondecreasing, that is,*

$$\Phi'(u) \geq 0, \quad \forall u \in \mathbb{R},$$

and is locally Lipschitz continuous. In addition we assume that $\Phi \in C^s(\mathbb{R})$ *where* s *satisfies*

$$s \geq \frac{4}{1-\theta\frac{\nu}{2}}.$$

Here ν *and* θ *are the same as in assumption 4.1.*

And lastly we need assumptions on the initial data.

Assumption 4.3 (The initial data u_0)**.** *We assume the initial data is integrable, bounded and of bounded variation, i.e.*

$$u_0 \in L^1((0,2\pi)) \cap L^\infty((0,2\pi)) \cap BV((0,2\pi)).$$

In addition, we assume there is a constant C *so that*

$$\epsilon_N^{r\nu} \left|\left| \frac{\partial^r u_0}{\partial x^r} \right|\right|^2 \leq C$$

holds for all integers $r \geq 0$*. Here* ϵ_N *and* ν *are the same as in assumption 4.1.*

Remark 1. With assumption 4.3, u_0 is sufficiently smooth to avoid any Gibb's phenomenon, and therefore the approximation of initial data $u_0^N = P_N u_0$ is what we will consider in the following convergence theory. Note then that by orthogonality and Lemma 3.4

$$\begin{aligned}
\left|\left| \frac{\partial^r u_0^N}{\partial x^r} \right|\right|^2 &\leq \left|\left| \frac{\partial^r u_0}{\partial x^r} \right|\right|^2 + \left|\left| \frac{\partial^r}{\partial x^r}(I-P_N)u_0 \right|\right|^2 \\
&\leq C\epsilon_N^{-r\nu} + C\left|\left| \frac{\partial^r u_0}{\partial x^r} \right|\right|^2 \\
&\leq C\epsilon_N^{-r\nu}.
\end{aligned}$$

Therefore, for each N, u_0^N satisfies assumption 4.3.

Furthermore, by the assumption u_0 is C^∞, and by the L^∞ estimate (5.1.17) in [1] we have

$$\begin{aligned}
\left|\left| u_0^N \right|\right|_\infty &\leq ||u_0||_\infty + ||(I-P_N)u_0||_\infty \\
&\leq ||u_0||_\infty + C\log(N)N^{-1} \left|\left| \frac{\partial u_0}{\partial x} \right|\right|_\infty < \infty,
\end{aligned}$$

so u_0^N is uniformly bounded. We may do similarly for the bounded variation: By Cauchy-Schwarz' inequality

$$\begin{aligned}
\left|\left| \frac{\partial u_0^N}{\partial x} \right|\right|_1 &\leq C \left|\left| \frac{\partial u_0^N}{\partial x} \right|\right| \\
&\leq C\left(\left|\left| \frac{\partial}{\partial x}(I-P_N)u_0 \right|\right| + \left|\left| \frac{\partial u_0}{\partial x} \right|\right| \right) \\
&\leq C \left|\left| \frac{\partial u_0}{\partial x} \right|\right| < \infty,
\end{aligned}$$

which means that u_0^N is also uniformly bounded in total variation.

Remark 2. There is also the possibility of using $u_0^N = K_N u_0$, where K_N is either the Fejer- or de la Vallée Poussin approximation, as initial data (cf. appendix A). In this case u_0^N would also satisfy assumption 4.3: By Young's inequality for convolutions

$$\left|\left|u_0^N\right|\right|_p = ||K_N * u_0||_p \le ||K_N||_1 \, ||u_0||_p ,$$

for any $p \ge 1$. In addition the kernel K_N is uniformly bounded in $L^1((0,2\pi))$. Setting $p = \infty$ gives that u_0^N is uniformly bounded.

Using differentiation for convolutions as

$$\begin{aligned}\left|\left|\frac{\partial^r u_0^N}{\partial x^r}\right|\right|^2 &\le ||K_N||_1^2 \left|\left|\frac{\partial^r u_0}{\partial x^r}\right|\right|^2 \\ &\le C\epsilon_N^{-r\nu}.\end{aligned}$$

This together with Lemma A.1, gives that also u_0^N satisfies assumption 4.3.

However, with this way of approximating the initial data, we lack a spectral estimate of the type in Lemma 3.4 in the L^2-norm. We need this type of estimate in the following convergence theory when we derive a rate of convergence for the numerical method.

In addition to the assumption, we will also have great use of the following result regarding the residual of the SVV operator, $R_N := I - Q_N$.

Lemma 4.1. *For $0 \le s \le 2$,*

$$\left|\left|\frac{\partial^s R_N}{\partial x^s}\right|\right|_{L^1(\Omega)} \le C m_N^s \log N. \tag{4.1}$$

Moreover, for $0 \le r \le s \le 2$, if $c_N = C\epsilon_N m_N^2 \log N \le \hat{C}$, then for all $p \ge 1$, $\phi \in L^p(\Omega)$

$$\epsilon_N \left|\left|\frac{\partial^2}{\partial x^2}(R_N * \phi)\right|\right|_{L^p(\Omega)} \le c_N \, ||\phi||_{L^p(\Omega)} . \tag{4.2}$$

Proof. Confer [2], Lemma 3.1 and Corollary 3.2 for a proof. □

Lemma 4.2 (Gagliardo-Nirenberg L^2 bound for a function $g(u)$)**.** *Assume that $g \in C^s$, then there exists a constant $\mathcal{K}_s$ so that*

$$\left|\left|\frac{\partial^s g(u)}{\partial x^s}\right|\right| \le \mathcal{K}_s \left|\left|\frac{\partial^s u}{\partial x^s}\right|\right|, \quad \mathcal{K}_s \le C \sum_{k=1}^{s} |g|_{C^k} \, ||u||_{L^\infty(\Omega)}^{k-1} \tag{4.3}$$

Proof. This is Theorem 7.1 [2], and a proof can be found there. □

Lemma 4.3. *Assume that $g \in C^s$ and let u_N be a trigonometric polynomial of degree $\le N$, then for every $0 \le r \le s$*

$$\left|\left|\frac{\partial^r}{\partial x^r}(I - P_N) g(u_N)\right|\right| \le \frac{\mathcal{K}_s}{N^{s-r}} \left|\left|\frac{\partial^s u_N}{\partial x^s}\right|\right|, \tag{4.4}$$

where $\mathcal{K}_s$ is asymptotically the same as in Lemma 4.2 with regards to g and u_N.

Proof. From (5.1.10) in [1] or Lemma 3.4 we have that

$$\left|\left|\frac{\partial^r}{\partial x^r}(I - P_N) g(u_N)\right|\right| \le C N^{r-s} \left|\left|\frac{\partial^s g(u_N)}{\partial x^s}\right|\right|,$$

and so use of Lemma 4.2 completes the proof. □

4.2 L^2-stability

We rewrite (3.32) as

$$\frac{\partial_t u_N}{\partial t} - \mathcal{L}(\Phi(u_N)) - \epsilon_N \Delta u_N = -\epsilon_N \Delta(R_N * u_N) - \mathcal{L}((I-P_N)\Phi(u_N)). \tag{4.5}$$

Equation (4.5) may be interpreted as (1.1) with the addition of a small diffusive part on the left hand side. On the right hand side we see that some error to the true (1.1) is introduced by the projection of $\Phi(u_N)$ and the residual of the spectral viscosity operator.

Our first estimate of the numerical solution of (3.32) we summarise in the following Lemma:

Lemma 4.4. *Assume that assumptions 4.1 through 4.3 hold. For a solution of (3.29) there is a constant $\mathcal{B}_0$, proportional to $||u_N||^2_{L^\infty(Q_T)}$, so that*

$$||u||^2(t) + 2\epsilon_N \left|\left|\frac{\partial u_N}{\partial x}\right|\right|^2_{L^2(Q_T)} \leq \mathcal{B}_0 + C\,||u_0||^2 \tag{4.6}$$

holds.

Proof. After multiplying (4.5) by u_N and integrating in space over $(0, 2\pi)$ we have

$$\begin{aligned}
&\left\langle \frac{\partial u_N}{\partial t}, u_N \right\rangle - \left\langle \frac{\partial^2 \Phi(u_N)}{\partial x^2}, u_N \right\rangle - \epsilon_N \left\langle \frac{\partial^2 u_N}{\partial x^2}, u_N \right\rangle \\
&= -\epsilon_N \left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), u_N \right\rangle - \left\langle \frac{\partial^2}{\partial x^2}(I-P_N)\Phi(u_N), u_N \right\rangle.
\end{aligned} \tag{4.7}$$

To ease the exposition somewhat, we'll consider (4.7) term by term.

i) The term involving the temporal derivative can be seen to equal

$$\left\langle \frac{\partial u_N}{\partial t}, u_N \right\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u_N||^2.$$

ii) For the nonlocal part we have by Lemma 2.1 that

$$\begin{aligned}
-\langle \mathcal{L}(\Phi(u_N)), u_N \rangle &= \frac{1}{2}\int_0^{2\pi}\int_{|y|>0} (\Phi(u_N(x+y)) - \Phi(u_N(x))(u_N(x+y) - u_N(x))\mathrm{d}\mu(y)\,\mathrm{d}x \\
&\geq 0
\end{aligned}$$

This nonnegativity is deduced from the assumed monotonicity of Φ: We have that $u_N(x+y) \gtreqless u_N(x)$ implies that $\Phi(u_N(x+y)) \gtreqless \Phi(u_N(x))$ and so the integrand is always nonnegative.

iii) We also use integration by parts on the viscous term to get

$$\begin{aligned}
-\epsilon_N \left\langle \frac{\partial^2 u_N}{\partial x^2}, u_N \right\rangle &= \epsilon_N \left[-u_N \frac{\partial u_N}{\partial x}\Bigg|_{x=0}^{2\pi} + \left\langle \frac{\partial u_N}{\partial x}, \frac{\partial u_N}{\partial x} \right\rangle \right] \\
&= \epsilon_N \left|\left|\frac{\partial u_N}{\partial x}\right|\right|^2,
\end{aligned}$$

where again we have used the periodicity to get rid of the boundary terms.

iv) Now to the residual of the spectral viscosity operator, where by Cauchy-Schwarz' inequality

$$-\epsilon_N \left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), u_N \right\rangle \leq \epsilon_N \|u_N\| \left\| \frac{\partial^2}{\partial x^2}(R_N * u_N) \right\|$$

Using Lemma 4.1 we get

$$-\epsilon_N \left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), u_N \right\rangle \leq c_N \|u_N\|_2,$$

where $c_N := C\epsilon_N m_N^2 \log(N)$, which by assumption 4.1 is uniformly bounded.

v) Lastly we consider

$$\langle \mathcal{L}((I-P_N)\Phi(u_N)), u_N \rangle. \tag{4.8}$$

Notice that for any $\phi \in L^2((0,2\pi))$

$$(I-S_N)\phi(x) = \sum_{|\xi|>N} \hat{\phi}_\xi e^{i\xi x},$$

and in particular has no terms with $e^{i\xi x}$ where $|\xi| \leq N$. It is easy to verify that this is true also for the spatial derivatives of ϕ. So the inner product above involves the product between a trigonometric polynomial of degree $\leq N$ and a function that involves no terms where $|\xi| \leq N$. Seeing as $\{e^{i\xi x}\}_{\xi\in\mathbb{Z}}$ is an orthogonal sequence in $L^2((0,2\pi))$ we conclude that the above inner product must be zero.

Combining all of the above into (4.7) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|^2(t) + \epsilon_N \left\| \frac{\partial u_N}{\partial x} \right\|^2 (t) \leq c_N \|u_N\|^2(t). \tag{4.9}$$

Now we integrate in time from 0 to T to get

$$\|u_N\|^2 + 2\epsilon_N \left\| \frac{\partial u_N}{\partial x} \right\|^2_{L^2(Q_T)} \leq \left\| u_0^N \right\|^2 + c_N \|u_N\|^2_{L^2(Q_T)}. \tag{4.10}$$

All that remains to do now is to employ that $c_N \|u_N\|^2_{L^2(Q_T)} \leq C \|u_N\|^2_{L^\infty(Q_T)} =: \mathcal{B}_0$, and that $\left\| u_0^N \right\| \leq C \|u_0\|$, to get the result. □

4.3 Energy estimate of derivatives

As a continuation of Lemma 4.4, we will now prove a similar energy estimate for the spatial derivatives of u_N.

Lemma 4.5. *Assume that assumptions 4.1 through 4.3 are satisfied. For a solution u_N of (3.29) and integer $r \geq 1$ there is a constant $\mathcal{B}_r \leq C\mathcal{B}_0 \left(\prod_{\substack{k=2 \\ k+r \text{ odd}}}^{r+1} \mathcal{K}_k \right)^{4/(2-\alpha)}$ so that*

$$\epsilon_N^{r\nu} \left\| \frac{\partial^r u_N}{\partial x^{2r}} \right\|^2 (t) + \epsilon_N^{r\nu+1} \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\|^2_{L^2(Q_T)} \leq \mathcal{B}_r + \epsilon_N^{r\nu} C \left\| \frac{\partial^r u_0}{\partial x^r} \right\|^2 \tag{4.11}$$

holds, where we recall that $\nu = \frac{2+\alpha}{2-\alpha}$.

Proof. Now we multiply (4.5) by $\frac{\partial^{2r} u_N}{\partial x^{2r}}$ and integrate in space over $(0, 2\pi)$. This leaves us with

$$\left\langle \frac{\partial u_N}{\partial t}, \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle - \left\langle \mathcal{L}[\Phi(u_N)], \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle - \epsilon_N \left\langle \frac{\partial^2 u_N}{\partial x^2}, \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle$$
$$= -\left\langle \mathcal{L}((I - P_N)\Phi(u_N)), \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle - \epsilon_N \left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle. \tag{4.12}$$

Again, it's seems reasonable to consider (4.12) term by term.

i) We have that

$$\left\langle \frac{\partial u_N}{\partial t}, \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle = (-1)^r \left\langle \frac{\partial}{\partial t} \frac{\partial^r u_N}{\partial x^r}, \frac{\partial^r u_N}{\partial x^r} \right\rangle$$
$$= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \frac{\partial^r u_N}{\partial x^r} \right\|^2,$$

after repeated use of integration by parts, and periodicity to get rid of boundary terms.

ii) Using integration by parts $r - 1$ times on the viscous term renders

$$-\epsilon_N \left\langle \frac{\partial^2 u_N}{\partial x^2}, \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle = -\epsilon_N (-1)^{r-1} \left\langle \frac{\partial^{r+1} u_N}{\partial x^{r+1}}, \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\rangle$$
$$= (-1)^r \epsilon_N \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\|^2.$$

iii) By the same argumentation as in the proof of Lemma 4.4 the term involving $(I - P_N)\Phi(u_N)$ is zero.

To summarize thus far, we have, after multiplying by $(-1)^r$,

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \frac{\partial^r u_N}{\partial x^r} \right\|^2 + \epsilon_N \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\| = (-1)^r \left\langle \mathcal{L}(\Phi(u_N)), \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle - \epsilon_N \left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle. \tag{4.13}$$

Considering the first term on the right hand side, we yet again use integration by parts and Cauchy-Schwarz' to get

$$(-1)^r \left\langle \mathcal{L}(\Phi(u_N)), \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle = -\left\langle \mathcal{L}\left(\frac{\partial^{r-1} \Phi(u_N)}{\partial x^{r-1}} \right), \frac{\partial u_N^{r+1}}{\partial x^{r+1}} \right\rangle$$
$$\leq \left\| \mathcal{L}\left(\frac{\partial^{r-1} \Phi(u_N)}{\partial x^{r-1}} \right) \right\| \cdot \left\| \frac{\partial u_N^{r+1}}{\partial x^{r+1}} \right\|.$$

Now, by the interpolation estimate in Lemma 2.3 and the Gagliardo-Nirenberg estimate in Lemma 4.2 we have for any $\epsilon > 0$

$$\left\| \mathcal{L}\left(\frac{\partial^{r-1} \Phi(u_N)}{\partial x^{r-1}} \right) \right\| \leq \epsilon \left\| \frac{\partial^{r+1} \Phi(u_N)}{\partial x^{r+1}} \right\| + C_\epsilon \left\| \frac{\partial^{r-1} \Phi(u_N)}{\partial x^{r-1}} \right\|$$
$$\leq \epsilon \mathcal{K}_{r+1} \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\| + C_\epsilon \mathcal{K}_{r-1} \left\| \frac{\partial^{r-1} u_N}{\partial x^{r-1}} \right\|.$$

We do similarly with the term involving R_N, i.e. after repeated integration by parts and Cauchy-Schwarz'

$$-\epsilon_N \left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), \frac{\partial^{2r} u_N}{\partial x^{2r}} \right\rangle = (-1)^r \epsilon_N \left\langle \frac{\partial^2}{\partial x^2}\left(R_N * \frac{\partial^{r-1} u_N}{\partial x^{r-1}}\right), \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\rangle$$
$$\leq \epsilon_N \left\| \frac{\partial^2}{\partial x^2}\left(R_N * \frac{\partial^{r-1} u_N}{\partial x^{r-1}}\right) \right\| \cdot \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\|,$$

and by Lemma 4.1

$$\epsilon_N \left\| \frac{\partial^2}{\partial x^2}\left(R_N * \frac{\partial^{r-1} u_N}{\partial x^{r-1}}\right) \right\| \leq c_N \left\| \frac{\partial^{r-1} u_N}{\partial x^{r-1}} \right\|.$$

After putting these two estimates back into (4.13) we get

$$\frac{1}{2}\frac{d}{dt}\left\| \frac{\partial^r u_N}{\partial x^r} \right\|^2 + \epsilon_N \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\| \leq \epsilon \mathcal{K}_{r+1} \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\|^2$$
$$+ (c_N + C_\epsilon \mathcal{K}_{r-1}) \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\| \cdot \left\| \frac{\partial^{r-1} u_N}{\partial x^{r-1}} \right\|. \tag{4.14}$$

By Young's inequality we have that for any $\tilde{\epsilon} > 0$

$$\left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\| \cdot \left\| \frac{\partial^{r-1} u_N}{\partial x^{r-1}} \right\| \leq \frac{\tilde{\epsilon}}{2} \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\|^2 + \frac{1}{2\tilde{\epsilon}} \left\| \frac{\partial^{r-1} u_N}{\partial x^{r-1}} \right\|^2,$$

and the name of the game is now to calibrate the free parameters ϵ and $\tilde{\epsilon}$ so that when put back into (4.14) yields something worthwhile. To be more specific we want the norm of $\frac{\partial^{r+1} u_N}{\partial x^{r+1}}$ to have a positive coefficient on the left hand side of (4.14), and so leaving the norm of $\frac{\partial^{r-1} u_N}{\partial x^{r-1}}$ alone on the right hand side. See now that the particular choice of

$$\epsilon = \frac{\epsilon_N}{4} \cdot \frac{1}{\mathcal{K}_{r+1}}, \quad \tilde{\epsilon} = \frac{\epsilon_N}{4} \cdot \frac{2}{c_N + C_\epsilon \mathcal{K}_{r-1}}$$

yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \frac{\partial^r u_N}{\partial x^r} \right\|^2 + \frac{\epsilon_N}{2} \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\| \leq \frac{(c_N + C_\epsilon \mathcal{K}_{r-1})^2}{\epsilon_N} \left\| \frac{\partial^{r-1} u_N}{\partial x^{r-1}} \right\|^2, \tag{4.15}$$

and after temporal integration we are left with

$$\left\| \frac{\partial^r u_N}{\partial x^r} \right\|^2 (t) + \epsilon_N \left\| \frac{\partial^{r+1} u_N}{\partial x^{r+1}} \right\|^2_{L^2(Q_t)} \leq \frac{(c_N + C_\epsilon \mathcal{K}_{r-1})^2}{\epsilon_N} \left\| \frac{\partial^{r-1} u_N}{\partial x^{r-1}} \right\|^2_{L^2(Q_t)} + \left\| \frac{\partial^r u_0^N}{\partial x^r} \right\|^2. \tag{4.16}$$

Now some attention should be put on the coefficient on the right hand side. By assumption we have that c_N is bounded, and by Lemma 2.3 $C_\epsilon = C\left(\frac{\mathcal{K}_{r+1}}{\epsilon_N}\right)^{\alpha/(2-\alpha)}$. And so

$$\begin{aligned}\frac{(c_N+C_\epsilon\mathcal{K}_{r-1})^2}{\epsilon_N} &\le \frac{C\mathcal{K}_{r+1}^{4/(2-\alpha)}}{\epsilon_N}(1+\epsilon_N^{-2\alpha/(2-\alpha)})\\ &\le C\mathcal{K}_{r+1}^{4/(2-\alpha)}\left(\epsilon_N^{-1}+\epsilon_N^{-1+\frac{2\alpha}{2-\alpha}}\right)\\ &\le C\mathcal{K}_{r+1}^{4/(2-\alpha)}\epsilon_N^{-\frac{2+\alpha}{2-\alpha}},\end{aligned}$$

where we have used the inequality $(x+y)^2 \le 2(x^2+y^2)$, and that $\frac{2+\alpha}{2-\alpha} > 1$ for $\alpha \in (0,2)$. We have also used for simplicity that $\mathcal{K}_{r-1} \le \mathcal{K}_{r+1}$, which of course can be done in its construction in Lemma 4.2. In addition we should now recognize that $\frac{2+\alpha}{2-\alpha} = \nu$, So we end up with the recurrence relation

$$\left\|\frac{\partial^r u_N}{\partial x^r}\right\|^2(t)+\epsilon_N\left\|\frac{\partial^{r+1}u_N}{\partial x^{r+1}}\right\|^2_{L^2(Q_t)} \le C\mathcal{K}_{r+1}^{4/(2-\alpha)}\epsilon_N^{-\nu}\left\|\frac{\partial^{r-1}u_N}{\partial x^{r-1}}\right\|^2_{L^2(Q_t)}+\left\|\frac{\partial^r u_0}{\partial x^r}\right\|^2. \quad (4.17)$$

We'll now use (4.17) together with Lemma 4.4 to show that there is a constant $\mathcal{B}_r$ so that

$$\left\|\frac{\partial^{r+1}u_N}{\partial x^{r+1}}\right\|^2_{L^2(Q_T)} \le \mathcal{B}_r\epsilon_N^{-(r\nu+1)}. \quad (4.18)$$

The case $r=0$ is evident from Lemma 4.4, and for $r=1$ we have by (4.17) and the assumption on u_0 that

$$\begin{aligned}\epsilon_N\left\|\frac{\partial^2 u_N}{\partial x^2}\right\|^2_{L^2(Q_T)} &\le C\mathcal{K}_2^{4/(2-\alpha)}\mathcal{B}_0\epsilon_N^{-\nu}+C\epsilon_N^{-\nu}\\ &\le \mathcal{B}_1\epsilon_N^{-\nu}.\end{aligned}$$

By induction we now have that for a general r that

$$\begin{aligned}\epsilon_N\left\|\partial^{r+1}u_N\right\|^2_{L^2(Q_T)} &\le C\mathcal{B}_{r-2}\mathcal{K}_{r+1}^{4/(2-\alpha)}\epsilon_N^{-(\nu+r\nu-2\nu+1)}+C\epsilon_N^{-r\nu}\\ &\le \mathcal{B}_r(\epsilon_N^{-(r\nu+(1-\nu))}+\epsilon_N^{-r\nu})\\ &\le \mathcal{B}_r\epsilon_N^{-r\nu},\end{aligned}$$

where the last step comes from the observation that $1-\nu<0$. Putting this back into (4.17) we finally get

$$\left\|\frac{\partial^r u_N}{\partial x^r}\right\|^2(t)+\epsilon_N\left\|\frac{\partial^{r+1}u_N}{\partial x^{r+1}}\right\|^2_{L^2(Q_t)} \le \mathcal{B}_r\epsilon_N^{-r\nu}+\left\|\frac{\partial^r u_0}{\partial x^r}\right\|^2, \quad (4.19)$$

and we are done with the proof after multiplying by $\epsilon_N^{r\nu}$. Notice also that the constants satisfy $\mathcal{B}_r = C\mathcal{B}_{r-2}\mathcal{K}_{r+1}^{4/(2-\alpha)}$, and so fulfills the relation stated in the Lemma. To see this, we first note that

$$\mathcal{B}_1 \le C\mathcal{B}_0\mathcal{K}_2^{4/(2-\alpha)},$$

and

$$\mathcal{B}_2 \leq C\mathcal{B}_0\mathcal{K}_3^{4/(2-\alpha)},$$

so the relation

$$\mathcal{B}_r \leq C\mathcal{B}_0 \left(\prod_{\substack{k=2 \\ k+r \text{ odd}}}^{r+1} \mathcal{K}_k \right)^{\frac{4}{2-\alpha}}$$

holds for $r = 1, 2$. For $r > 2$ we use induction, so suppose it holds for $r \leq s$, and consider $r = s+1$, then

$$\begin{aligned} \mathcal{B}_{s+1} &\leq C\mathcal{B}_{s-1}\mathcal{K}_{s+2}^{4/(2-\alpha)} \\ &\leq C\mathcal{B}_0 \left(\prod_{\substack{k=2 \\ k+(s-1) \text{ odd}}}^{s} \mathcal{K}_k \right)^{\frac{4}{2-\alpha}} \mathcal{K}_{s+2}^{4/(2-\alpha)} \\ &\leq C\mathcal{B}_0 \left(\prod_{\substack{k=2 \\ k+(s+1) \text{ odd}}}^{s+2} \mathcal{K}_k \right)^{\frac{4}{2-\alpha}}. \end{aligned}$$

The last step follows since if $k+(s-1)$ is odd, then so is $k+(s+1)$. In addition $(s+1)+(s+2)$ is odd, and so the additional factor can be added to the product. Therefore, the relation also holds for $r = s+1$. □

The proof of Lemma 4.5 is admittedly quite unruly, and a bit persnickety, but the usefulness of the result will emerge in the next section where we'll derive some L^∞-bounds.

4.4 L^∞-estimate

Lemma 4.6. *Let assumptions 4.1 through 4.3 hold, and let u_N be a solution of (3.29), then for each nonnegative integer r there is a constant C so that*

$$\left\| \frac{\partial^r}{\partial x^r}(I - P_N)\Phi(u_N) \right\|_\infty \leq C\mathcal{B}_s N^{r+2-s(1-\theta\frac{\nu}{2})} \tag{4.20}$$

holds for all times. Here s is as in assumption 4.2.

Proof. By Theorem 6, in section 5.6 of [11] (a generalization of Morrey's inequality), we have for any $\varphi \in H^{r+[\frac{1}{2}]+1}((0,2\pi)) = H^{r+2}((0,2\pi))$ that

$$\left\| \frac{\partial^r \varphi}{\partial x^r} \right\|_\infty \leq C \left\| \varphi \right\|_{H^{r+2}((0,2\pi))},$$

and again using (5.1.10) of [1] this yields

$$\left\| \frac{\partial^r \varphi}{\partial x^r} \right\|_\infty \leq C \left\| \frac{\partial^{r+2} \varphi}{\partial x^{r+2}} \right\|. \tag{4.21}$$

This leads us to deduce that

$$\left\|\frac{\partial^r}{\partial x^r}(I-P_N)\Phi(u_N)\right\|_\infty \leq C\left\|\frac{\partial^{r+2}}{\partial x^{r+2}}(I-P_N)\Phi(u_N)\right\|, \tag{4.22}$$

and by Lemma 4.3 this results in

$$\left\|\frac{\partial^r}{\partial x^r}(I-P_N)\Phi(u_N)\right\|_\infty \leq \frac{C}{N^{s-r-2}}\left\|\frac{\partial^s u_N}{\partial x^s}\right\|. \tag{4.23}$$

It's at this point Lemma 4.5 can be put to use, which together with the assumption on u_0 leads to

$$\left\|\frac{\partial^s u_N}{\partial x^s}\right\| \leq C\mathcal{B}_s\epsilon_N^{-s\nu/2}. \tag{4.24}$$

And so putting this into (4.23) together with $\epsilon_N \propto N^{-\theta}$ yields for $r=2$

$$\left\|\frac{\partial^r}{\partial x^r}(I-S_N)\Phi(u_N)\right\|_\infty \leq C\mathcal{B}_s N^{r+2-s(1-\theta\frac{\nu}{2})}. \tag{4.25}$$

□

This result will now now be put to immediate use to get a similar bound for the projection error on the nonlocal operator.

Lemma 4.7. *Let u_N be the solution to (3.32), and let $\Phi \in C^s(\mathbb{R})$, then*

$$||\mathcal{L}(I-P_N)\Phi(u_N)||_\infty \leq C\mathcal{B}_s N^{2+\alpha-s(1-\theta\frac{\nu}{2})}. \tag{4.26}$$

Observation 1. Before proving the Lemma, it is worth noticing that Lemma 4.7 interpolates Lemma 4.6 in that they coincide when $\alpha \to 0$ or $\alpha \to 2$.

Proof of Lemma 4.7. We consider the fractional laplacian on the form given in (2.5), and for sake of brevity we write $\varphi = (I-P_N)\Phi(u_N)$, then

$$||\mathcal{L}(u)||_\infty = \left\|\int_{|y|>0}\varphi(x+y)-\varphi(x)-y\mathbf{1}_{|y|<r}\frac{\partial\varphi}{\partial x}(x)\mathrm{d}\mu(z)\right\|_\infty. \tag{4.27}$$

Now a natural distinction comes into place for $0<\alpha<1$ and $1\leq\alpha<2$ since in the former case the term involving $\frac{\partial\varphi}{\partial x}$ is integrable and the integral is zero from the symmetry of the measure. So what we'll now do is consider both cases and derive an intermediate estimate in φ and its derivatives.

$0<\alpha<1$: See that from the identity

$$\begin{aligned}\varphi(x+y)-\varphi(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi(x+\tau y)\mathrm{d}\tau \\ &= \int_0^1 y\frac{\partial\varphi}{\partial x}(x+\tau y)\mathrm{d}\tau,\end{aligned}$$

we can write for every $r>0$

$$\mathcal{L}[\varphi](x)=\int_{|y|\le r} y\int_0^1 \frac{\partial\varphi}{\partial x}(x+\tau y)\mathrm{d}\tau\mathrm{d}\mu(y)+\int_{|y|>r}\varphi(x+y)-\varphi(x)d\mu(y),$$

and so

$$||\mathcal{L}\varphi||_\infty \le \left|\left|\frac{\partial\varphi}{\partial x}\right|\right|_\infty \int_{|y|\le r}|y|\mathrm{d}\mu(y)+2\,||\varphi||_\infty\int_{|y|>r}\mathrm{d}\mu(y). \tag{4.28}$$

We have that

$$\int_{|y|\le r}|y|\mathrm{d}\mu(y)=2c_\alpha\int_0^r y^{-\alpha}\mathrm{d}y=\frac{2c_\alpha r^{1-\alpha}}{1-\alpha},$$
$$\int_{|y|>r}\mathrm{d}\mu(y)=2c_\alpha\int_r^\infty y^{-1-\alpha}\mathrm{d}y=\frac{2c_\alpha r^{-\alpha}}{\alpha},$$

yielding

$$||\mathcal{L}\varphi||_\infty\le 2c_\alpha\left(\frac{r^{1-\alpha}}{1-\alpha}\left|\left|\frac{\partial\varphi}{\partial x}\right|\right|_\infty+2\frac{r^{-\alpha}}{\alpha}\,||\varphi||_\infty\right). \tag{4.29}$$

The point of keeping the free parameter, r, was to minimize the right hand side. It is easy to check that the r yielding the best estimate is

$$r=\frac{2\,||\varphi||_\infty}{\left|\left|\frac{\partial\varphi}{\partial x}\right|\right|_\infty},$$

which results in the interpolation-like estimate

$$||\mathcal{L}\varphi||_\infty\le\frac{2^{2-\alpha}c_\alpha}{\alpha(1-\alpha)}\,||\varphi||_\infty^{1-\alpha}\left|\left|\frac{\partial\varphi}{\partial x}\right|\right|_\infty^\alpha. \tag{4.30}$$

Finally we use Lemma 4.6 with $r=0,2$ and s the smoothness of Φ to get

$$\begin{aligned}||\mathcal{L}\varphi||_\infty &\le C\mathcal{B}_s N^{(1-\alpha)(2-s(1-\theta\frac{\nu}{2}))+\alpha(3-s(1-\theta\frac{\nu}{2}))}\\ &= C\mathcal{B}_s N^{2+\alpha-s(1-\theta\frac{\nu}{2})}.\end{aligned}$$

$1\le\alpha<2$: We use a similar treatment in this case, but now the term involving the derivative of φ can't be ignored due to the fact that y is not integrable in any neightbour of the origin with this measure. To remedy this we use the identity

$$\begin{aligned}\varphi(x+y)-\varphi(x)-y\frac{\partial\varphi}{\partial x}(x)&=\int_0^1 y\left(\frac{\partial\varphi}{\partial x}(x+\tau y)-\frac{\partial\varphi}{\partial x}(x)\right)\mathrm{d}\tau\\ &=\int_0^1 y\int_0^\tau\frac{d}{d\theta}\frac{\partial\varphi}{\partial x}(x+\theta y)\mathrm{d}\theta\mathrm{d}\tau\\ &=y^2\int_0^1\int_0^\tau\frac{\partial^2\varphi}{\partial x^2}(x+\theta y)\mathrm{d}\theta\mathrm{d}\tau.\end{aligned}$$

So with (4.27) in mind we thus have

$$||\mathcal{L}\varphi||_\infty\le\left|\left|\frac{\partial^2\varphi}{\partial x^2}\right|\right|_\infty\int_{|y|\le r}\frac{|y|^2}{2}\mathrm{d}\mu(y)+2\,||\varphi||_\infty\int_{|y|>r}\mathrm{d}\mu(y). \tag{4.31}$$

Doing as we did in the previous case, we have with

$$\int_{|y|\le r} |y|^2 \mathrm{d}\mu(y) = 2c_\alpha \int_0^r y^{1-\alpha}\mathrm{d}y = \frac{2c_\alpha r^{2-\alpha}}{2-\alpha}$$

that

$$||\mathcal{L}\varphi||_\infty \le c_\alpha \left(\frac{r^{2-\alpha}}{2-\alpha} \left|\left| \frac{\partial^2 \varphi}{\partial x^2} \right|\right|_\infty + 4\frac{r^{-\alpha}}{\alpha} ||\varphi||_\infty \right). \tag{4.32}$$

We find the r minimizing the right hand side of (4.32) to be

$$r = 2\left(\frac{||\varphi||_\infty}{\left|\left|\frac{\partial^2\varphi}{\partial x^2}\right|\right|_\infty} \right)^{1/2},$$

which when put into (4.32) results in the estimate

$$||\mathcal{L}\varphi||_\infty \le \frac{2^{3-\alpha}c_\alpha}{\alpha(2-\alpha)} ||\varphi||_\infty^{1-\alpha/2} \left|\left| \frac{\partial^2\varphi}{\partial x^2}\right|\right|_\infty^{\alpha/2}. \tag{4.33}$$

Again using Lemma 4.6 with $r = 0, 2$ finally yields

$$\begin{aligned} ||\mathcal{L}\varphi||_\infty &\le C\mathcal{B}_s N^{(1-\alpha/2)(2-s(1-\theta\frac{\nu}{2}))+\alpha/2(4-s(1-\theta\frac{\nu}{2})} \\ &= C\mathcal{B}_s N^{2+\alpha-s(1-\theta\frac{\nu}{2})}. \end{aligned}$$

□

One may be justified in asking why we've paid so much attention to keeping the constants $\mathcal{B}_r$ separated from other constants that are depending on e.g. α. The reason for this is that $\mathcal{B}_r$ depends on the supremum norm of u_N, and so is as of yet not known to be bounded. This will be remedied in the Lemma to follow, and as such this result will be a culmination of all the work put in from Lemma 4.4 and up to this point. The boundedness in supremum norm is also one of the pillars on which our compactness argument will stand, making it a bit easier to appreciate the results proven thus far.

Lemma 4.8. *Let u_N be the solution of (3.32), and take assumptions 4.1 through 4.3 to hold. Then*

$$||u_N(\cdot,t)||_\infty \le C\,||u_0||_\infty \tag{4.34}$$

for $t < C\ln(N)$.

Proof. We do as in the proof of Lemma 5.1 in [4], and multiply (4.5) with pu_N^{p-1}, where p is an even integer, and integrate over $(0, 2\pi)$ in space to get

$$\begin{aligned} &\left\langle \frac{\partial u_N}{\partial t}, pu_N^{p-1}\right\rangle - \left\langle \mathcal{L}\Phi(u_N), pu_N^{p-1}\right\rangle - \epsilon_N \left\langle \frac{\partial^2 u_N}{\partial x^2}, pu_N^{p-1}\right\rangle \\ &= -\epsilon_N \left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), pu_N^{p-1}\right\rangle - \left\langle \mathcal{L}(I-P_N)\Phi(u_N), pu_N^{p-1}\right\rangle, \end{aligned} \tag{4.35}$$

and as usual it would seem best to consider this term by term.

i) For the first term

$$\left\langle \frac{\partial u_N}{\partial t}, p u_N^{p-1} \right\rangle = \int_0^{2\pi} \frac{\partial}{\partial t} u_N^p dx$$
$$= \frac{\mathrm{d}}{\mathrm{d}t} ||u_N||_p^p$$
$$= p\, ||u_N||_p^{p-1} \frac{d}{dt} ||u_N||_p,$$

which holds only for p even since then $u_N^p = |u_N|^p$.

ii) For the nonlocal term on the left hand side we use Lemma 2.1 as

$$-\left\langle \mathcal{L}\Phi(u_N), p u_N^{p-1} \right\rangle = \frac{p}{2} \int_0^{2\pi} \int_{|y|>0} (\Phi(u_N(x+y)) - \Phi(u_N(x))) \cdot (u_N^{p-1}(x+y) - u_N^{p-1}(x)) \mathrm{d}\pi_\alpha(y)\, \mathrm{d}x.$$

Since both $\Phi(u_N)$ and u_N^{p-1} are nondecreasing functions in u_N the integrand, and so the whole integral, is nonnegative. Again see that this only holds when p even.

iii) The viscous term on the left hand side is also nonnegative. This can be seen when using integration by parts as

$$-\epsilon_N \left\langle \frac{\partial^2 u_N}{\partial x^2}, p u_N^{p-1} \right\rangle = \int_0^{2\pi} \frac{\partial u_N}{\partial x} \frac{\partial}{\partial x}(p u_N^{p-1}) \mathrm{d}x$$
$$= p(p-1) \int_0^{2\pi} \left(\frac{\partial u_N}{\partial x} \right)^2 u_N^{p-2} \mathrm{d}x \geq 0.$$

iv) Moving on to the right hand side we use Hölder's inequality to get

$$-\epsilon_N \left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), p u_N^{p-1} \right\rangle \leq p \epsilon_N \left|\left| \frac{\partial^2}{\partial x^2}(R_N * u_N) \right|\right|_p \cdot \left|\left| u_N^{p-1} \right|\right|_{\frac{p}{p-1}},$$

and we note that $\left|\left| u_N^{p-1} \right|\right|_{\frac{p}{p-1}} = ||u_N||_p^{p-1}$.

v) A similar use of Hölder's inequality on the projection error for the nonlocal term yields

$$-\left\langle \mathcal{L}(I - P_N)\Phi(u_N), p u_N^{p-1} \right\rangle \leq p\, ||\mathcal{L}(I - S_N)\Phi(u_N)||_p\, ||u_N||_p^{p-1}.$$

Putting all this together we get from (4.35) that

$$p\, ||u_N||_p^{p-1} \frac{\mathrm{d}}{\mathrm{d}t} ||u_N||_p \leq p\, ||u_N||_p^{p-1} \left(||\mathcal{L}(I - P_N)\Phi(u_N)||_p + \epsilon_N \left|\left| \frac{\partial^2}{\partial x^2}(R_N * u_N) \right|\right|_p \right).$$

Dividing by $p\, ||u_N||_p^{p-1}$ on both sides and sending $p \to \infty$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t} ||u_N||_\infty \leq ||\mathcal{L}(I - P_N)\Phi(u_N)||_\infty + \epsilon_N \left|\left| \frac{\partial^2}{\partial x^2}(R_N * u_N) \right|\right|_\infty. \tag{4.36}$$

Using Lemmas 4.7 and 4.1 on the right hand side results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|_\infty \le C\mathcal{B}_s N^{2+\alpha-s(1-\theta\frac{\nu}{2})} + c_N \|u_N\|_\infty . \tag{4.37}$$

With the way $\mathcal{B}_s$ was constructed in Lemma 4.5, we have $\mathcal{B}_s \le C\left(\|u_N\|_\infty^{\frac{s^2}{4}}\right)^{\frac{4}{2-\alpha}} = C\|u_N\|_\infty^{\frac{s^2}{2-\alpha}}$ (see the differences and similarities this has to the corresponding constant in [4]). Now we go on to define $y(t) = \|u_N\|_\infty e^{-c_N t}$, which then results in the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) \le \underbrace{CN^{2+\alpha-s(1-\theta\frac{\nu}{2})}}_{:=h(N)} y(t)^{\frac{s^2}{2-\alpha}} e^{c_N(\frac{s^2}{2-\alpha}-1)t}. \tag{4.38}$$

Using separation of variables and integrating in time gives us

$$\frac{1}{1-\frac{s^2}{2-\alpha}}\left(y(t)^{1-\frac{s^2}{2-\alpha}} - y(0)^{1-\frac{s^2}{2-\alpha}}\right) \le \frac{h(N)}{c_N}\frac{1}{\frac{s^2}{2-\alpha}-1}\left(e^{c_N(\frac{s^2}{2-\alpha}-1)t}-1\right),$$

and solving for $y(t)$ yields the estimate

$$y(t) \le y(0)\left(1-\frac{h(N)}{c_N}\left(e^{c_N(\frac{s^2}{2-\alpha}-1)t}-1\right)y(0)^{\frac{s^2}{2-\alpha}}\right)^{\frac{1}{1-\frac{s^2}{2-\alpha}}}. \tag{4.39}$$

Here it is worthwhile to note that this inequality also holds when $\frac{s^2}{2-\alpha} > 1$. When putting back in the expression for $y(t)$ we get from (4.39) that

$$\|u_N\|_\infty \le e^{c_N t}\left(1-\frac{h(N)}{c_N}\left(e^{c_N(\frac{s^2}{2-\alpha}-1)t}-1\right)\left\|u_0^N\right\|_\infty^{\frac{s^2}{2-\alpha}}\right)^{\frac{1}{1-\frac{s^2}{2-\alpha}}}\left\|u_0^N\right\|_\infty . \tag{4.40}$$

Since $\left\|u_0^N\right\|_\infty \le C\|u_0\|_\infty$ and by the assumptions on s and c_N we can thus conclude that $\|u_N\|_\infty$ is uniformly bounded, independent of N for $t < C\ln(N)$, where C is some constant. ☐

4.5 BV estimate

Lemma 4.9. *Let u_N be a solution of (3.32), then for finite times the estimate*

$$|u_N|_{BV} \le e^{c_N t}\left(|u_0|_{BV} + CN^{r-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}(1-\nu)}\right) \tag{4.41}$$

holds. Here $r = 2$ if $\alpha \in (0,1]$ and $r = 3$ if $\alpha \in (1,2)$.

Proof. We will closely follow the proof of Lemma 5.2 in [4], with of course a slight modification required for the nonlinear term. So we differentiate (4.5) in the spatial variable to get

$$\frac{\partial}{\partial t}\left(\frac{\partial u_N}{\partial x}\right) - \mathcal{L}\left(\frac{\partial \Phi(u_N)}{\partial x}\right) - \epsilon_N\frac{\partial^3 u_N}{\partial x^3} = -\mathcal{L}\left(\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right) - \epsilon_N\frac{\partial^2}{\partial x^2}\left(R_N * \frac{\partial u_N}{\partial x}\right). \tag{4.42}$$

Let now sgn_ρ be the standard mollification of the sign function with parameter ρ. That is,

$$\text{sgn}_\rho(x) = \left(\text{sgn} * \frac{1}{\rho}\omega\left(\frac{\cdot}{\rho}\right)\right)(x),$$

where $\omega(x) = Ce^{-\frac{1}{1-|x|^2}}$ for $|x| < 1$ and zero otherwise. Again, the constant is a normalization factor to ensure that the integral of ω is 1. We now multiply (4.42) by $\text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)$ and integrate in space to get

$$\begin{aligned}&\left\langle \frac{\partial}{\partial t}\left(\frac{\partial u_N}{\partial x}\right), \text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle - \left\langle \mathcal{L}\left(\frac{\partial \Phi(u_N)}{\partial x}\right), \text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle - \epsilon_N\left\langle \frac{\partial^3 u_N}{\partial x^3}, \text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle \\ &= -\left\langle \mathcal{L}\left(\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right), \text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle - \epsilon_N\left\langle \frac{\partial^2}{\partial x^2}\left(R_N * \frac{\partial u_N}{\partial x}\right), \text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle. \end{aligned} \tag{4.43}$$

As has become our modus operandi at this point, we consider (4.43) term by term.

i) Using Theorem 7 in Appendix C of [11] we have that

$$\begin{aligned}\lim_{\rho\to 0}\frac{\partial}{\partial t}\left(\frac{\partial u_N}{\partial x}\right)\text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial u_N}{\partial x}\right)\text{sgn}\left(\frac{\partial u_N}{\partial x}\right) \\ &= \frac{\partial}{\partial t}\left|\frac{\partial u_N}{\partial x}\right|,\end{aligned}$$

where the convergence is pointwise. In addition the convergence is monotonic. Whether it's increasing or decreasing depends on the sign of $\frac{\partial u_N}{\partial x}$ and its temporal derivate. And so splitting $(0, 2\pi)$ into the subsets

$$B_\pm = \left\{x \in (0,2\pi) : \frac{\partial}{\partial t}\left(\frac{\partial u_N}{\partial x}\right)\text{sgn}\left(\frac{\partial u_N}{\partial x}\right) \gtrless 0\right\},$$

and use the monotone convergence theorem on each of these. Then

$$\begin{aligned}\lim_{\rho\to 0}\left\langle \frac{\partial}{\partial t}\left(\frac{\partial u_N}{\partial x}\right), \text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle &= \int_0^{2\pi}\frac{\partial}{\partial t}\left|\frac{\partial u_N}{\partial x}\right|\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial u_N}{\partial x}\right\|_1.\end{aligned}$$

ii) For the nonlocal term we use a similar line of reasoning as that in the proof of Lemma 2.4 to find that

$$\begin{aligned}&\lim_{\rho\to 0} -\left\langle \mathcal{L}\left(\frac{\partial \Phi(u_N)}{\partial x}\right), \text{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle \\ &= \frac{1}{2}\int_{|y|>0}\int_0^{2\pi}\left(\frac{\partial \Phi(u_N)}{\partial x}(x+y) - \frac{\partial \Phi(u_N)}{\partial x}(x)\right)\left(\text{sgn}\left(\frac{\partial u_N}{\partial x}(x+y)\right) - \text{sgn}\left(\frac{\partial u_N}{\partial x}(x)\right)\right)\mathrm{d}x\,\mathrm{d}\pi_\alpha(y)\end{aligned}$$

Expanding the integrand results in

$$\left(\frac{\partial\Phi(u_N)}{\partial x}(x+y)-\frac{\partial\Phi(u_N)}{\partial x}(x)\right)\left(\operatorname{sgn}\left(\frac{\partial u_N}{\partial x}(x+y)\right)-\operatorname{sgn}\left(\frac{\partial u_N}{\partial x}(x)\right)\right)$$
$$=\Phi'(u_N(x+y))\left(\left|\frac{\partial u_N}{\partial x}(x+y)\right|-\frac{\partial u_N}{\partial x}(x+y)\operatorname{sgn}\left(\frac{\partial u_N}{\partial x}(x)\right)\right)$$
$$+\Phi'(u_N(x))\left(\left|\frac{\partial u_N}{\partial x}(x)\right|-\frac{\partial u_N}{\partial x}(x)\operatorname{sgn}\left(\frac{\partial u_N}{\partial x}(x+y)\right)\right),$$

which we see is nonnegative because $\Phi' \geq 0$, and so the nonlocal term on the left hand side of (4.43) is nonnegative.

iii) Define now

$$\eta_\rho(u) = \int_0^u \operatorname{sgn}_\rho(s)ds,$$

which is a convex function since sgn_ρ is nondecreasing. For any sufficiently smooth convex function we have the useful inequality

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\eta_\rho\left(\frac{\partial u_N}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\eta_\rho'\left(\frac{\partial u_N}{\partial x}\right)\frac{\partial^2 u_N}{\partial x^2}\right)\\ &= \eta_\rho''\left(\frac{\partial u_N}{\partial x}\right)\left(\frac{\partial^2 u_N}{\partial x^2}\right)^2+\eta_\rho'\left(\frac{\partial u_N}{\partial x}\right)\frac{\partial^3 u_N}{\partial x^3}\\ &\geq \eta_\rho'\left(\frac{\partial u_N}{\partial x}\right)\frac{\partial^3 u_N}{\partial x^3},\end{aligned}$$

because $\eta_\rho'' \geq 0$. This computation is justified in our case because both u_N and η_ρ are C^∞ functions.

And so we get

$$\begin{aligned}\epsilon_N\left\langle\frac{\partial^3 u_N}{\partial x^3},\operatorname{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle &\geq -\int_0^{2\pi}\frac{\partial^2}{\partial x^2}\eta_\rho\left(\frac{\partial u_N}{\partial x}\right)\mathrm{d}x\\ &= \frac{\partial}{\partial x}\eta_\rho\left(\frac{\partial u_N}{\partial x}\right)\Bigg|_{x=0}^{2\pi}\\ &= 0,\end{aligned}$$

by the periodicity of u_N.

iv) For the first of the right hand side terms in (4.43) we use that $\left|\operatorname{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right| \leq 1$, to get the estimate

$$\begin{aligned}-\left\langle\mathcal{L}\left(\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right),\operatorname{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle &\leq \left\|\mathcal{L}\left(\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right)\right\|_1\\ &= \sqrt{2\pi}\left\|\mathcal{L}\left(\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right)\right\|,\end{aligned}$$

where the last step is achieved using Cauchy-Schwarz' inequality.

v) We do similarly for the latter right hand side term of (4.43), but now use Lemma 4.1 instead of Cauchy-Schwarz' to get

$$-\epsilon_N \left\langle \frac{\partial^2}{\partial x^2}\left(R_N * \frac{\partial u_N}{\partial x}\right), \mathrm{sgn}_\rho\left(\frac{\partial u_N}{\partial x}\right)\right\rangle \le c_N \left\|\frac{\partial u_N}{\partial x}\right\|_1 .$$

Putting all this back into (4.43) results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial u_N}{\partial x}\right\|_1 \le c_N \left\|\frac{\partial u_N}{\partial x}\right\|_1 + C\left\|\mathcal{L}\left(\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right)\right\|. \tag{4.44}$$

Turning our attention to the projection error for the nonlocal term, we get by Lemma 3.2 that

$$\left\|\mathcal{L}\left(\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right)\right\| \le \left\|\mathcal{L}(I-S_N)\frac{\partial \Phi(u_N)}{\partial x}\right\| + a\left\|\mathcal{L}\frac{\partial A_N\Phi(u_N)}{\partial x}\right\|, \tag{4.45}$$

where we recall that a is either 0 or 1 depending on whether we want to consider the Galerkin- or the pseudospectral formulation.

For the first part of (4.45)we get by Parseval's identity

$$\begin{aligned}\left\|\mathcal{L}(I-S_N)\frac{\partial \Phi(u_N)}{\partial x}\right\|^2 &= 2\pi C_\alpha^2 \sum_{|\xi|>N} |\xi|^{2(\alpha+1)}|\hat{\Phi}_\xi|^2 \\ &\le C\begin{cases}\sum_{|\xi|>N}|\xi|^4|\hat{\Phi}_\xi|^2, & \alpha\in(0,1],\\ \sum_{|\xi|>N}|\xi|^6|\hat{\Phi}_\xi|^2, & \alpha\in(1,2),\end{cases}\end{aligned}$$

and so again by Parseval's identity we thus have

$$\left\|\mathcal{L}(I-S_N)\frac{\partial \Phi(u_N)}{\partial x}\right\| \le C\begin{cases}\left\|\frac{\partial^2}{\partial x^2}(I-S_N)\Phi(u_N)\right\|, & \alpha\in(0,1]\\ \left\|\frac{\partial^3}{\partial x^3}(I-S_N)\Phi(u_N)\right\|, & \alpha\in(1,2).\end{cases} \tag{4.46}$$

After use of Lemma 4.2 this becomes

$$\left\|\mathcal{L}(I-S_N)\frac{\partial \Phi(u_N)}{\partial x}\right\| \le CN^{r-s}\left\|\frac{\partial^s u_N}{\partial x^s}\right\|, \tag{4.47}$$

where $r=2,3$, depending on the given α. Notice that we are no longer so concerned with the constant's depence on $||u_N||_{L^\infty}$ because of Lemma 4.8. We may do similarly to the aliasing term in (4.45) by seeing that

$$\begin{aligned}\left\|\mathcal{L}\frac{\partial A_N\Phi(u_N)}{\partial x}\right\|^2 &= 2\pi C_\alpha^2 \sum_{|\xi|\le N}|\xi|^{2(\alpha+1)}|(A_N\Phi(u_N))_\xi|^2 \\ &\le CN^{2r}\sum_{|\xi|\le N}|(A_N\Phi(u_N))_\xi|^2 \\ &= CN^{2r}\,||A_N\Phi(u_N)||^2,\end{aligned}$$

and so by using Lemma 3.3 and 4.3 we find that

$$\left\|\mathcal{L}\frac{\partial A_N\Phi(u_N)}{\partial x}\right\| \leq CN^{r-s}\left\|\frac{\partial^s u_N}{\partial x^s}\right\|.$$

We put this back into (4.44) and use separation of variables to arrive at

$$\left\|\frac{\partial u_N}{\partial x}\right\|_1 \leq e^{c_N t}\left(\left\|\frac{\partial u_0}{\partial x}\right\|_1 + CN^{r-s}\left\|\frac{\partial^s u_N}{\partial x^2}\right\|_{L^2(Q_T)}\right). \tag{4.48}$$

Finally, by the energy estimate in Lemma 4.5

$$\left\|\frac{\partial^s u_N}{\partial x^s}\right\|_{L^2(Q_T)} \leq C\epsilon_N^{-((s-1)\frac{\nu}{2}+\frac{1}{2})} \leq CN^{\theta(s\frac{\nu}{2}+\frac{1-\nu}{2})},$$

we have

$$\left\|\frac{\partial u_N}{\partial x}\right\|_1 \leq e^{c_N t}\left(\left\|\frac{\partial u_0^N}{\partial x}\right\|_1 + CN^{r-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}(1-\nu)}\right). \tag{4.49}$$

By the assumptions on s, and θ this expression is bounded uniformly, independent of N for finite times. Lastly, we have in one spatial dimension that the total variation of u_N is bounded by

$$|u_N|_{BV} \leq \left\|\frac{\partial u_N}{\partial x}\right\|_1, \tag{4.50}$$

and the result follows. □

4.6 Time regularity estimate

Lemma 4.10. *Let u_N be the solution of (3.32), then there is a constant C so that*

$$||u_N(\cdot,t_2) - u_N(\cdot,t_1)||_1 \leq C\sqrt{|t_2 - t_1|} \tag{4.51}$$

holds for all $t_1, t_2 \in [0,T]$.

Proof. The proof will go along very similar lines as that in [4], and so we start by defining the standard mollification of u_N as

$$u_N^\epsilon(x,t) = (u_N(\cdot,t) * \omega_\epsilon)(x), \tag{4.52}$$

where ω_ϵ is the mollifier as the one we used in the proof of Lemma 4.9. Take now $t_1, t_2 \in [0,T]$, then a simple use of the triangle inequality reveals that

$$\begin{aligned} ||u_N(\cdot,t_2) - u_N(\cdot,t_1)||_1 \leq{}& ||u_N(\cdot,t_2) - u_N^\epsilon(\cdot,t_2)||_1 \\ &+ ||u_N^\epsilon(\cdot,t_2) - u_N^\epsilon(\cdot,t_1)||_1 \\ &+ ||u_N^\epsilon(\cdot,t_1) - u_N(\cdot,t_1)||_1. \end{aligned} \tag{4.53}$$

For the first and last term on the right hand side we estimate as follows: By definition

$$\begin{aligned}
||u_N(\cdot,t) - u_N^\epsilon(\cdot,t)||_1 &= \int_0^{2\pi} \left| u_N(x,t) - \int_{\mathbb{R}} u_N(x-y)\omega_\epsilon(y)\mathrm{d}y \right| \mathrm{d}x \\
&= \int_0^{2\pi} \left| \int_{\mathbb{R}} (u_N(x,t) - u_N(x-y,t))\omega_\epsilon(y)\mathrm{d}y \right| \mathrm{d}x \\
&= \int_0^{2\pi} \left| \int_{\mathbb{R}} \left(\int_0^1 y \frac{\partial u_N}{\partial x}(x-sy)\mathrm{d}s \right) \omega_\epsilon(y)\mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_0^{2\pi} \int_{\mathbb{R}} \int_0^1 |y| \cdot \left| \frac{\partial u_N}{\partial x}(x-sy) \right| \omega_\epsilon(y)\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

By employing Fubini's theorem to take the integral over x to be the innermost integral, we get

$$\begin{aligned}
||u_N(\cdot,t) - u_N^\epsilon(\cdot,t)||_1 &\leq \int_{\mathbb{R}} \int_0^1 |y|\omega_\epsilon(y) \int_0^{2\pi} \left| \frac{\partial u_N}{\partial x}(x-sy) \right| \mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y \\
&\leq |u_N|_{BV} \int_{\mathbb{R}} |y|\omega_\epsilon(y)\mathrm{d}y.
\end{aligned}$$

The support of ω_ϵ is contained in $[-\epsilon,\epsilon]$ and so

$$||u_N(\cdot,t) - u_N^\epsilon(\cdot,t)||_1 \leq \epsilon |u_N|_{BV}. \tag{4.54}$$

What remains now is to estimate the term

$$||u_N^\epsilon(\cdot,t_2) - u_N^\epsilon(\cdot,t_1)||_1,$$

and to that end we express the difference as an integral, now in time. I.e.

$$\begin{aligned}
|u_N^\epsilon(\cdot,t_2) - u_N^\epsilon(\cdot,t_1)| &= \left| (t_2-t_1) \int_0^1 \frac{u_N^\epsilon}{\partial t}(t_1+\tau(t_2-t_1))\mathrm{d}\tau \right| \\
&\leq |t_2-t_1| \int_0^1 \left| \frac{\partial u_N^\epsilon}{\partial t}(t_1+\tau(t_2-t_1)) \right| \mathrm{d}\tau,
\end{aligned}$$

and so we find that

$$||u_N^\epsilon(\cdot,t_2) - u_N^\epsilon(\cdot,t_1)||_1 \leq |t_2-t_1| \int_0^1 \left\| \frac{\partial u_N^\epsilon}{\partial t}(\cdot,t_1+\tau(t_2-t_1)) \right\|_1 \mathrm{d}\tau. \tag{4.55}$$

It would seem evident from (4.55) that we now need to estimate the L^1-norm of the temporal derivative of u_N^ϵ. We do as in [4], and convolve (4.5) with ω_ϵ before taking the absolute value and integrating over $(0,2\pi)$ in space. This results in

$$\begin{aligned}
\left\| \frac{\partial u_N}{\partial t} * \omega_\epsilon \right\|_1 = \left\| \frac{\partial u_N^\epsilon}{\partial t} \right\|_1 & \\
\leq \epsilon_N \left\| \frac{\partial^2 u_N}{\partial x^2} * \omega_\epsilon \right\|_1 &+ \epsilon_N \left\| \frac{\partial^2}{\partial x^2}(R_N * u_N) * \omega_\epsilon \right\|_1 \\
+ ||\mathcal{L}(\Phi(u_N)) * \omega_\epsilon||_1 &+ ||\mathcal{L}((I-P_N)\Phi(u_N)) * \omega_\epsilon||_1,
\end{aligned} \tag{4.56}$$

and as usual we will for sake of clarity consider (4.56) term by term.

i) By Young's inequality for convolutions

$$\begin{aligned}\left|\left|\frac{\partial^2 u_N}{\partial x^2} * \omega_\epsilon\right|\right|_1 &= \left|\left|\frac{\partial u_N}{\partial x} * \frac{\partial \omega_\epsilon}{\partial x}\right|\right|_1 \\ &\leq \left|\left|\frac{\partial u_N}{\partial x}\right|\right|_1 \left|\left|\frac{\partial \omega_\epsilon}{\partial x}\right|\right|_1 \\ &\leq \frac{C}{\epsilon}|u_N|_{BV}.\end{aligned}$$

ii) We may similarly use Young's inequality on the term involving the residual of the spectral viscosity operator. Then

$$\begin{aligned}\epsilon_N \left|\left|\frac{\partial^2}{\partial x^2}(R_N * u_N) * \omega_\epsilon\right|\right|_1 &\leq \epsilon_N \left|\left|\frac{\partial^2}{\partial x^2}(R_N * u_N)\right|\right|_1 ||\omega_\epsilon||_1 \\ &\leq c_N\, ||u_N||_1,\end{aligned}$$

where the last step is from Lemma 4.1.

iii) The terms involving the fractional laplacian are by far the trickiest to handle here, but we can argue as we did in the proof of Lemma 2.8 to get that

$$||\mathcal{L}[\varphi] * \omega_\epsilon||_1 \leq C\left(\frac{1}{\epsilon}|\varphi|_{BV} + ||\varphi||_1\right) \tag{4.57}$$

Setting $\varphi = \Phi(u_N)$ in (4.57), we see that we need to estimate the total variation and L^1-norm of $\Phi(u_N)$. For the total variation we have

$$|\Phi(u_N)|_{BV} = \int_0^{2\pi} \Phi'(u_N)\left|\frac{\partial u_N}{\partial x}\right| \mathrm{d}x. \tag{4.58}$$

By Lemma 4.8 we know that for finite times u_N is uniformly bounded, independent of N. Say that u_N is bounded by C. Since by assumption Φ is locally Lipschitz continuous there is a constant $L_\Phi < \infty$ so that $\sup_{|u|\leq C} \Phi'(u) \leq L_\Phi$. Thus we get the estimate

$$|\Phi(u_N)|_{BV} \leq L_\Phi |u_N|_{BV}. \tag{4.59}$$

We can do similarly for the L^1-norm of $\Phi(u_N)$, see that by the assumption that $\Phi(0) = 0$ we have

$$\begin{aligned}||\Phi(u_N)||_1 &= \int_0^{2\pi} |\Phi(u_N)| \mathrm{d}x \\ &= \int_0^{2\pi} |\Phi(u_N) - \Phi(0)| \mathrm{d}x \\ &\leq L_\Phi \int_0^{2\pi} |u_N| \mathrm{d}x \\ &= L_\Phi\, ||u_N||_1.\end{aligned} \tag{4.60}$$

In conclusion, we get by (4.57) that

$$||\mathcal{L}(\Phi(u_N)) * \omega_\epsilon||_1 \leq C\left(\frac{1}{\epsilon}|u_N|_{BV} + ||u_N||_1\right). \tag{4.61}$$

iv) The hard work in considering the nonlocal operator is done, and we may now use (4.57) with $\varphi = (I - S_N)\Phi(u_N)$, which results in

$$||\mathcal{L}((I-P_N)\Phi(u_N)) * \omega_\epsilon||_1 \leq C\left(\frac{1}{\epsilon}|(I-P_N)\Phi(u_N)|_{BV} + ||(I-P_N)\Phi(u_N)||_1\right).$$

Since we're on a bounded domain, we have by Cauchy-Schwarz' inequality that $||\cdot||_1 \leq \sqrt{2\pi}\,||\cdot||$, and so

$$||\mathcal{L}((I-S_N)\Phi(u_N)) * \omega_\epsilon||_1 \leq C\left(\frac{1}{\epsilon}\left|\left|\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right|\right| + ||(I-S_N)\Phi(u_N)||\right). \tag{4.62}$$

By Lemma 4.2 and 4.5 the projection errors can be estimated as

$$\begin{aligned} ||(I-P_N)\Phi(u_N)|| &\leq CN^{-s(1-\theta\frac{\nu}{2})}, \text{ and} \\ \left|\left|\frac{\partial}{\partial x}(I-P_N)\Phi(u_N)\right|\right| &\leq CN^{1-s(1-\theta\frac{\nu}{2})}, \end{aligned}$$

where s is the smoothness of Φ. Finally, by the assumptions on s and θ, we end up with

$$||\mathcal{L}((I-P_N)\Phi(u_N)) * \omega_\epsilon||_1 \leq C\left(\frac{1}{\epsilon}N^{-3} + N^{-4}\right). \tag{4.63}$$

With estimates for all terms on the right hand side of (4.56) in place the resulting inequality becomes

$$\left|\left|\frac{\partial u_N^\epsilon}{\partial t}\right|\right|_1 \leq C\left(\frac{1}{\epsilon}|u_N|_{BV} + ||u_N||_1\right) + C\left(\frac{1}{\epsilon}N^{-3} + N^{-4}\right). \tag{4.64}$$

By Lemma 4.9 $|u_N|_{BV}$ is uniformly bounded, and by Lemma 4.4 together with $||\cdot||_{L^1} \leq \sqrt{2\pi}\,||\cdot||$, $||u_N||_{L^1}$ is uniformly bounded, and so

$$\left|\left|\frac{\partial u_N^\epsilon}{\partial t}\right|\right|_1 \leq C\left(\frac{1}{\epsilon} + 1\right). \tag{4.65}$$

Putting this back into (4.55) results in

$$||u_N^\epsilon(\cdot,t_2) - u_N^\epsilon(\cdot,t_1)||_1 \leq C|t_2 - t_1|\left(\frac{1}{\epsilon} + 1\right), \tag{4.66}$$

and finally we put this together with (4.54) back into (4.53) to end up with

$$||u_N(\cdot,t_2) - u_N(\cdot,t_1)||_1 \leq C\left(\epsilon + |t_2 - t_1|\left(\frac{1}{\epsilon} + 1\right)\right). \tag{4.67}$$

The proof is completed by taking $\epsilon = \sqrt{|t_2 - t_1|}$. □

4.7 Convergence

Finally all the a priori estimates we need are established, and so we are in a position to use the Kolmogorov's compactness (Theorem 2.1).

Theorem 4.1. *Under assumptions 4.1 through 4.3, the family of numerical solutions* $\{u_N\}_N$ *has a subsequence* $\{u_{N_j}\}_j$ *that converges in* $C([0,T]:L^1((0,2\pi)))$ *to a* $u \in C([0,T]: L^1((0,2\pi))) \cap L^\infty(Q_T) \cap L^\infty([0,T]:BV((0,2\pi)))$.

Proof. The uniform boundedness for u_N was established in Lemma 4.8, and temporal stability is given by Lemma 4.10 with modulus of continuity $\omega_T(|t-s|) = \sqrt{|t-s|}$. The last assumption that needs to be fulfilled for us to be able to use Theorem 2.1 is the shift stability in the spatial variable. By Lemma A.1 in [14] we have

$$\int_0^{2\pi} |u_N(x+\xi) - u_N(x)| \mathrm{d}x \leq |\xi| \cdot |u_N|_{BV},$$

and by Lemma 4.9 we thus get that this is bounded with modulus of continuity $\nu_T(|\rho|) = C|\rho|$, for some constant C. This leaves all assumptions in Theorem 2.1 fulfilled, and use of said Theorem yields the result. □

It remains to be established that u_N converges towards a distributional solution of (1.1), to do this we'll take an intermediate route, and first show that u_N is in some sense close to the strong vanishing viscosity solution of (2.11).

Theorem 4.2. *Let assumptions 4.1 through 4.3 hold. Let* u_N *be the solution of (3.32) with initial data* $u_0^N = P_N u_0$ *and let* v_{ϵ_N} *be the solution of (2.11), with viscosity parameter* ϵ_N. *Then there is a constant* C *so that*

$$||u_N - v_{\epsilon_N}||_1 \leq C\sqrt{\epsilon_N}\left(1 + N^{r-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}} + N^{-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}}\right), \tag{4.68}$$

where $r=1$ *if* $\alpha \in (0,1]$ *and* $r=2$ *if* $\alpha \in (1,2)$.

Proof. Taking the difference between (3.32) and (**??**) results in

$$\begin{aligned}
&\frac{\partial}{\partial t}(u_N - v_{\epsilon_N}) - \mathcal{L}(\Phi(u_N) - \Phi(v_{\epsilon_N})) - \epsilon_N \frac{\partial^2}{\partial x^2}(u_N - v_{\epsilon_N}) \\
&= -\epsilon_N \frac{\partial^2}{\partial x^2}(R_N * u_N) - \mathcal{L}(I - P_N)\Phi(u_N).
\end{aligned} \tag{4.69}$$

In the spirit of the proof of Lemma 4.9, we multiply by $\operatorname{sgn}_\rho(u_N - v_{\epsilon_N})$ and integrate over $(0,2\pi)$ in space before letting $\rho \to 0$. Let's yet again do as before and consider (4.69) term by term.

i) Arguing as in the proof Lemma 4.9, we find by the monotone convergence Theorem that

$$\begin{aligned}
\lim_{\rho\to 0}\left\langle \frac{\partial}{\partial t}(u_N - v_{\epsilon_N}), \operatorname{sgn}_\rho(u_N - v_{\epsilon_N})\right\rangle &= \int_0^{2\pi} \frac{\partial}{\partial t}(u_N - v_{\epsilon_N}) \cdot \operatorname{sgn}(u_N - v_{\epsilon_N}) \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} ||u_N - v_{\epsilon_N}||_1 .
\end{aligned}$$

ii) For the nonlocal term we argue as we did in the proof of Lemma 2.4 to get that

$$
\begin{aligned}
&-\lim_{\rho\to 0}\left\langle \mathcal{L}(\Phi(u_N)-\Phi(v_{\epsilon_N})), \operatorname{sgn}_\rho(u_N-v_{\epsilon_N})\right\rangle \\
&=\frac{1}{2}\int_0^{2\pi}\int_{|y|>0}\{\Phi(u_N(x+y))-\Phi(v_{\epsilon_N}(x+y))+\Phi(v_{\epsilon_N}(x))-\Phi(u_N(x))\} \\
&\times\{\operatorname{sgn}(u_N(x+y)-v_{\epsilon_N}(x+y))-\operatorname{sgn}(u_N(x)-v_{\epsilon_N}(x))\}\,\mathrm{d}\pi_\alpha(y)\,\mathrm{d}x.
\end{aligned}
$$

Considering the integrand, it is zero where $u_N - v_{\epsilon_N}$ has the same sign at $x+y$ and x. So suppose

$$
\begin{cases} u_N(x+y) & \geq v_{\epsilon_N}(x+y), \text{ and} \\ u_N(x) & \leq v_{\epsilon_N}(x), \end{cases}
$$

where at least one of the inequalities are strict. Then we have by the monotonicity of Φ that

$$
\begin{cases} \Phi(u_N(x+y)) & \geq \Phi(v_{\epsilon_N}(x+y)), \text{ and} \\ \Phi(u_N(x) & \leq \Phi(v_{\epsilon_N}(x)), \end{cases}
$$

and by inspection we find that the integrand is nonnegative in this case. The opposite case can be treated similarly, and we conclude that

$$
-\lim_{\rho\to 0}\langle \mathcal{L}(\Phi(u_N)-\Phi(v_{\epsilon_N})), \operatorname{sgn}(u_N-v_{\epsilon_N})\rangle \geq 0.
$$

iii) With the definition

$$
\eta_\rho(u)=\int_0^u \operatorname{sgn}_\rho(s)\mathrm{d}s,
$$

we may argue as in the proof of Lemma 4.9 to find that

$$
-\epsilon_N\left\langle \frac{\partial^2}{\partial x^2}(u_N-v_{\epsilon_N}), \operatorname{sgn}_\rho(u_N-v_{\epsilon_N})\right\rangle \geq 0
$$

for every $\rho > 0$.

iv) Since $|\operatorname{sgn}_\rho(u_N - v_{\epsilon_N})| \leq 1$ we have

$$
\begin{aligned}
-\epsilon_N\left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), \operatorname{sgn}_\rho(u_N-v_{\epsilon_N})\right\rangle &\leq \epsilon_N\left\|\frac{\partial^2}{\partial x^2}(R_N*u_N)\right\|_1 \\
&\leq \epsilon_N\left\|\frac{\partial R_N}{\partial x}\right\|_1\left\|\frac{\partial u_N}{\partial x}\right\|_1,
\end{aligned}
$$

and by Lemma 4.1

$$
\begin{aligned}
-\epsilon_N\left\langle \frac{\partial^2}{\partial x^2}(R_N * u_N), \operatorname{sgn}_\rho(u_N-v_{\epsilon_N})\right\rangle &\leq C\epsilon_N m_N\log(N) \\
&\leq C\sqrt{\epsilon_N}.
\end{aligned}
$$

v) For the last term of (4.69) we have

$$
\begin{aligned}
-\left\langle \mathcal{L}(I-P_N)\Phi(u_N), \operatorname{sgn}_\rho(u_N - v_{\epsilon_N})\right\rangle &\leq ||\mathcal{L}(I-P_N)\Phi(u_N)||_1 \\
&\leq C\, ||\mathcal{L}(I-P_N)\Phi(u_N)||.
\end{aligned}
$$

In toto,

$$
\frac{d}{dt}||u_N - v_{\epsilon_N}||_1 \leq C(\sqrt{\epsilon_N} + ||\mathcal{L}(I-P_N)\Phi(u_N)||), \tag{4.70}
$$

and after integrating in time we find that

$$
||u_N - v_{\epsilon_N}||_1(t) \leq C\left(\sqrt{\epsilon_N} + ||\mathcal{L}(I-P_N)\Phi(u_N)||_{L^2(Q_T)}\right) + \left|\left|u_0^N - u_0\right|\right|. \tag{4.71}
$$

We can estimate the projection error on the nonlocal term in a similar manner as in the proof of Lemma 4.9, but now with one less derivative, to get

$$
||\mathcal{L}(I-P_N)\Phi(u_N)||_{L^2(Q_T)} \leq CN^{r-s}\left|\left|\frac{\partial^s u_N}{\partial x^s}\right|\right|_{L^2(Q_T)},
$$

where $r=1$ if $\alpha \in (0,1]$ and $r=2$ if $\alpha \in (1,2)$. Using Lemma 4.5 together with the assumption on ϵ_N

$$
\begin{aligned}
||\mathcal{L}(I-P_N)\Phi(u_N)||_{L^2(Q_T)} &\leq CN^{r-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}(1-\nu)} \\
&\leq CN^{r-s(1-\theta\frac{\nu}{2})},
\end{aligned}
$$

where the last step is justified in the observation that $1-\nu<0$. For the error in initial data, we have

$$
\left|\left|u_0^N - u_0\right|\right| \leq CN^{-s}\left|\left|\frac{\partial^s u_0}{\partial x^s}\right|\right|,
$$

and by the assumption on the initial data

$$
\begin{aligned}
\left|\left|u_0^N - u_0\right|\right| &\leq CN^{-s}\epsilon_N^{-s\frac{\nu}{2}} \\
&\leq CN^{-s(1-\theta\frac{\nu}{2})}.
\end{aligned}
$$

Put back into (4.71), all this finally yields

$$
\begin{aligned}
||u_N - v_{\epsilon_N}||_1 &\leq C\left(\sqrt{\epsilon_N} + N^{r-s(1-\theta\frac{\nu}{2})} + N^{-s(1-\theta\frac{\nu}{2})}\right) \\
&\leq C\sqrt{\epsilon_N}\left(1 + N^{r-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}} + N^{-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}}\right).
\end{aligned} \tag{4.72}
$$

□

Finally, Theorem 4.2 together with Theorem 2.2 gives us that u_N does indeed converge to distributional solution of (1.1), as summarized in the following corollary.

Corollary 4.1. *Let assumptions (4.1) through (4.3) hold. Then the solutions, u_N, of the spectral vanishing viscosity method (3.32) converge to the unique distributional solution, u, of (1.1). Moreover, there is a a constant C, independent of N, so that*

$$||u_N(\cdot,t) - u(\cdot,t)||_1 \leq C\sqrt{\epsilon_N}\left(1 + N^{r-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}} + N^{-s(1-\theta\frac{\nu}{2})+\frac{\theta}{2}}\right), \tag{4.73}$$

holds for all $t \in [0,T]$.

Proof. The estimate (4.73) is established by using theorems 4.2 and 2.2 together with the triangle inequality. By the assumptions, the right hand side of (4.73) goes to 0 as $N \to \infty$, and so

$$\lim_{N\to\infty} ||u_N(\cdot,t) - u(\cdot,t)||_1 = 0,$$

which proves that u_N converges to u in $C([0,T] : L^1((0,2\pi)))$. □

5 Implementation and numerical experiments

With convergence of the spectral vanishing viscosity method (3.32) established in the previous section, this section will be devoted to giving directions on how to implement the method, and also provide some numerical examples. These examples will then give an insight into the behaviour of solutions to (1.1), but also provide some guidelines on when spectral vanishing viscosity should be applied. As with most stabilizing methods, the SVV adds some artificial diffusion which may have the disadvantage of smearing out the numerical too much, thus not capturing some characteristics of the exact solution in a satisfactory manner. This is not unlike the question of approximating the initial data, as discussed in appendix A.

5.1 Discretization and numerical solver

In physcial space, we use in this project the equidistant nodes

$$x_j = \frac{2\pi j}{N}, \quad j = 0, \dots, N-1,$$

for an even $N \in \mathbb{N}$. With this nodal set, it is convention in a Fourier method to have the finite-dimensional function space to be

$$\mathcal{S}_N = \{e^{i\xi x}\}_{\xi=-N/2}^{N/2-1}.$$

The reason for this is that in a flexible numerical code all integrals of the type

$$\hat{f}_\xi = \int_0^{2\pi} f(x) e^{-i\xi x} \mathrm{d}x$$

will have to be approximated using numerical quadrature, where the discrete Fourier transform

$$\hat{f}_\xi \approx \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) e^{-i\xi x_j},$$

is by far the most common. With this approximation $\hat{f}_{N/2} = \hat{f}_{-N/2}$, and so no information is gained in adding the $N/2$ frequency to $\mathcal{S}_N$.

Our numerical method now seeks a function of the form

$$u_N(x,t) = \sum_{\xi=-N/2}^{N/2-1} \hat{u}_\xi(t) e^{i\xi x}$$

that satisfies

$$\begin{cases} \frac{\partial u_N}{\partial t} & = \mathcal{L}(P_N \Phi(u_N)) + \epsilon_N \frac{\partial^2}{\partial x^2}(Q_N * u_N), \quad \text{for } (x,t) \in Q_T \\ u_N(\cdot,0) & = P_N u_0, \end{cases} \tag{5.1}$$

where we recall that P_N could be either S_N or I_N, but in the following we will always use $P_N = I_N$, which makes for a numerical code that can handle a wide range of problems.

If we now take $-N/2 \leq \xi \leq N/2-1$, and multiply (5.1) by $\frac{e^{-i\xi x}}{2\pi}$ and integrate over $(0,2\pi)$ in space, we find that

$$\begin{cases} \frac{\partial \hat{u}_\xi}{\partial t} &= -C_\alpha |\xi|^\alpha \hat{\Phi}_\xi - \epsilon_N |\xi|^2 \hat{Q}_\xi \hat{u}_\xi \\ \hat{u}_\xi(0) &= (P_N u_0)_\xi. \end{cases} \tag{5.2}$$

It's important to notice that (5.2) provides for $-N/2 \leq \xi \leq N/2-1$ a system of ordinary differential equations that can be solved numerically using a time integration scheme of your choosing. In this project, we have opted to use the explicit fourth order Runge-Kutta method for the time integration.

To make the method (5.2) prepared for a numerical implementation we need to explicitly address the choice of the SVV parameters ϵ_N, m_N and $\hat{Q}_\xi$. We will here follow [18] and assumption 4.1 as

$$\begin{aligned} \epsilon_N &= CN^{-\theta} \\ m_N &= N^{\frac{\theta}{2}} \log_2(N)^{-\frac{1}{2}} \\ \hat{Q}_\xi &= \begin{cases} 0, & |\xi| < m_N \\ \frac{|\xi|-m_N}{m_N}, & m_N \leq 2m_N \\ 1, & |\xi| > 2m_N, \end{cases} \end{aligned}$$

where θ and C needs be determined on a more ad-hoc basis. However, throughout the following numerical experiments $\theta = 0.5$, and $C = 0.05$ has done the job. Note also the resemblance between the SVV kernel Q_N and the de la Vallée Poussin kernel V_N as discussed in appendix A. Indeed, we have that $Q_N = I - V_{m_N}$.

Before moving on to the numerical examples, we note that also the coefficient for the fractional laplacian, C_α, needs to be calculated. We recall that

$$C_\alpha = \frac{2c_\alpha}{\alpha} \int_0^\infty x^{-\alpha} \sin(x) \, \mathrm{d}x,$$

where

$$c_\alpha = \frac{\alpha \Gamma\left(\frac{1+\alpha}{2}\right)}{2\pi^{\frac{1}{2}+\alpha} \Gamma\left(1 - \frac{\alpha}{2}\right)}.$$

For $\alpha \in (0,1]$, we have that

$$\int_0^\infty x^{-\alpha} \sin(x) \, \mathrm{d}x = \begin{cases} \Gamma(1-\alpha) \sin\left(\frac{\pi(1-\alpha)}{2}\right) & \alpha \in (0,1) \\ \frac{\pi}{2} & \alpha = 1 \end{cases}$$

(cf. [4]), but for $\alpha \in (1,2)$, the integral has to be approximated using quadrature. A further discussion of this can be found in appendix C.

5.2 Weakly degenerate

As our first numerical example, we consider the fractional porous medium equation with discontinuous initial data. In this case we then have

$$\Phi(u) = u|u|^{m-1},$$

where $m > 1$, and

$$u_0(x) = \begin{cases} 1, & \text{for } |x-\pi| < \frac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

This choice of Φ is degenerate for $u = 0$, and the local case is extensively studied in [27]. The Cauchy problem for the fractional porous medium equation was studied in [6, 7].

In figure 5.1, we see the results of numerical simulations for some values of α. What we are able to notice is that because of the smoothness of the underlying solution, the simulation without SVV seems fairly stable, and importantly the amount of SVV added in the other simulations are not enough to overly smear the solutions. Another important feature to notice from figure 5.1, is that for increasing α, the diffusion seems to slow down, and the areas of rapid change becomes smaller. This is in accordance with the fact that $\mathcal{L}$ approaches the regular laplacian (or a scaled version of it) as $\alpha \to 2$, together with what we may know of solutions to the porous medium equation (cf. [27]). Indeed for α close to 2, the gradient of the solution comes close to discontinuous. This will yield numerical oscillations if SVV is not incorporated, as is shown in figure 5.2a, which the SVV clearly helps to stabilize as we can see in figure 5.2b.

5.3 Fast diffusion

We will again consider $\Phi(u) = u|u|^{m-1}$, but now with $0 < m < 1$. In the local case, this equation has been dubbed the fast diffusion equation, so it seems fitting that we call it the fractional fast diffusion equation in this setting. Although the degeneracy at $u = 0$ is not present in the fast diffusion case, we no longer have a locally Lipschitz nonlinearity. The results of some numerical simulation, with and without SVV, are summarized in figure 5.3. Worth noting here is that even though the SVV approximation is fairly successful at retaining the overall structure of the solution, it is the least so for the lowest value of α. This observation is in agreement with assumption 4.1, where smaller values for α allows for greater values of θ, which in turn decreases the value of ϵ_N.

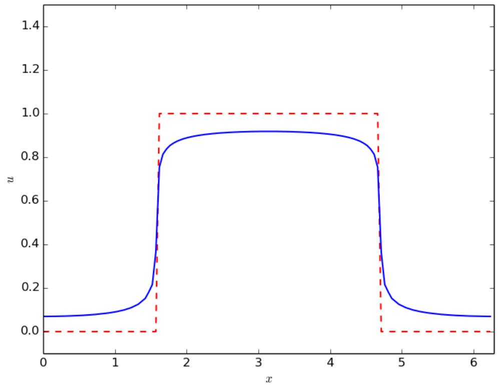
Although not clearly visible in figure 5.3, the tails of the solution are not lost for α closer to 2, which is in contrast to our previous discussion of the fractional porous medium equation and figure 5.2. This suggests that the fast diffusion case is similar to the heat equation in that the speed of propagation is infinite also in the limit $\alpha \to 2$, which lends the name fast diffusion some creedence.

In the analysis of the fractional fast diffusion equation, the literature makes a clear distinction for when $m > m_* := \frac{(d-\alpha)_+}{d}$, where d is the number of spatial dimension. As can be seen from fig 5.3a and 5.3b the implementation seems robust enough to also handle the case when $m < m_*$.

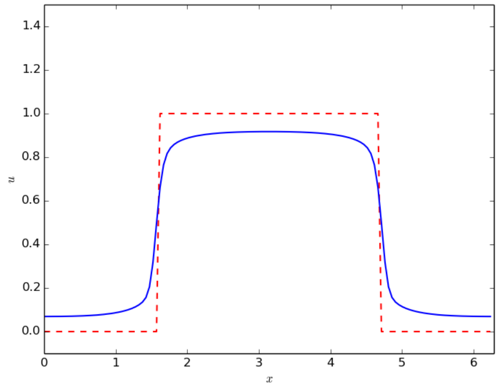
5.4 Strongly degenerate

As the last class of problems we'll consider is when the nonlinearity Φ is strongly degenerate, i.e. when $\Phi' = 0$ on a nontrivial interval. As a representative example we use

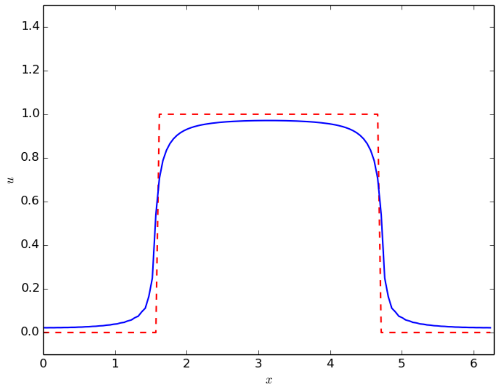
$$\Phi(u) = \max\{u, 0\}.$$



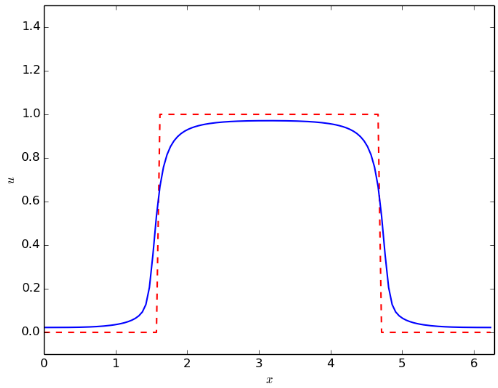
(a) $\alpha = 0.5$, without SVV.



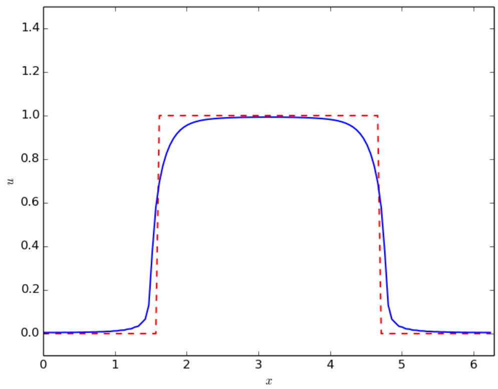
(b) $\alpha = 0.5$, with SVV.



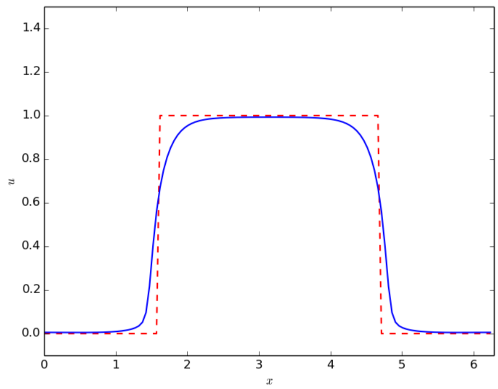
(c) $\alpha = 1.0$, without SVV.



(d) $\alpha = 1.0$, with SVV.

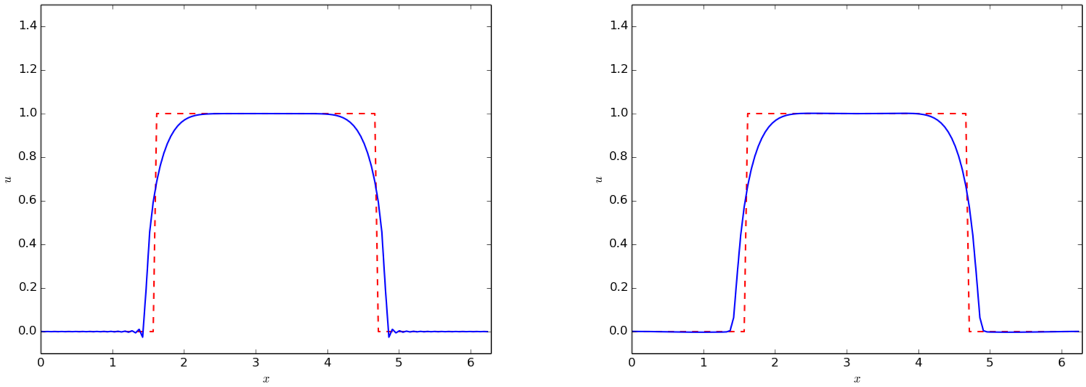


(e) $\alpha = 1.5$, without SVV.



(f) $\alpha = 1.5$, with SVV.

Figure 5.1: Numerical solution of the fractional porous medium equation for selected values of α. All simulation are with $N = 128$, $T = 0.5$ and $m = 3$. The dotted line is the initial data.



(a) $\alpha = 1.999$, without SVV. (b) $\alpha = 1.999$, with SVV.

Figure 5.2: Numerical solution of the fractional porous medium equation for $\alpha = 1.999$. The simulation are with $N = 128$, $T = 0.5$ and $m = 3$. The dotted line is the initial data.

In the local case, this equation is often called Stefan's problem, and may be used to model phase transitions, like the melting of an ice cube in a glass of water. So in the nonlocal case we will in this project dub this equation the fractional Stefan problem.

To better understand the qualitative behaviour of solutions to this problem, we will take as initial data

$$u_0(x) = \operatorname{sgn}(\pi - x), \tag{5.3}$$

since then a fair amount of the initial data lies within the degeneracy of Φ.

The results of the numerical simulations are summarized in figure 5.4. As has already been made clear in the previous examples, solutions are smoother for low values of α, but here already for $\alpha = 1$, we begin to see oscillations close to where the gradient changes rapidly. Again, these oscillations were not present when SVV was added.

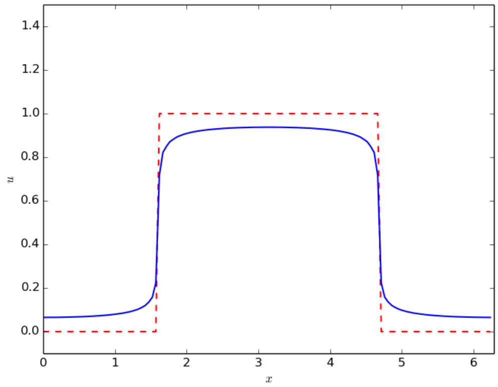
To conclude this discussion on the fractional Stefan's problem, we will see that it is not necessary for the initial data to be discontinuous for oscillations in the numerical solution to emerge. We continue to consider $\Phi(u) = \max\{u, 0\}$, but now we take the initial data to be

$$u_0(x) = \begin{cases} \exp\left\{1 - \frac{1}{1-|x-\pi|^2}\right\} - \frac{1}{2} & \text{for } |x - \pi| < 1 \\ -\frac{1}{2} & \text{otherwise,} \end{cases}$$

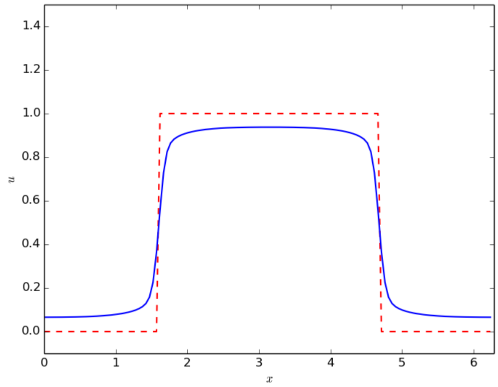
together with the somewhat extreme parameters $\alpha = 1.99$ and $T = 15$. The results are depicted in figure 5.5. As we can see, the smooth initial data does not hinder the emergence of oscillations about the discontinuities, and again the adding of SVV smooths out the oscillations in what may be called a satisfactory manner.

5.5 Approximating the initial data with a smoother kernel

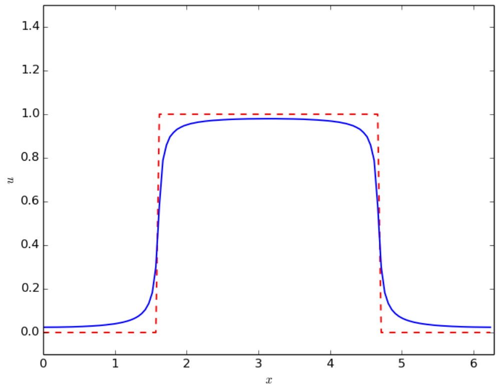
In all the above numerical examples we have used $u_0^N = I_N u_0$, but due to Gibb's oscilllations this may not be the soundest strategy for approximating the initial data. Especially



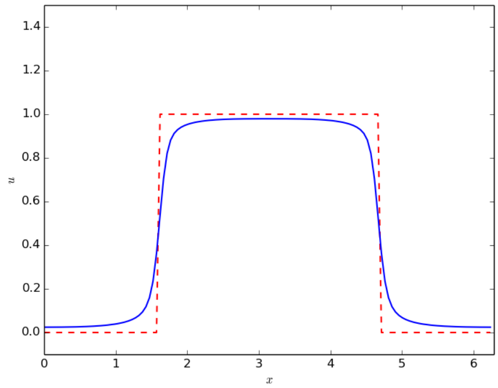
(a) $\alpha = 0.5$, without SVV.



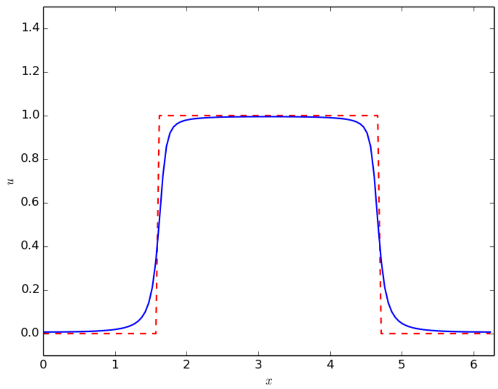
(b) $\alpha = 0.5$, with SVV.



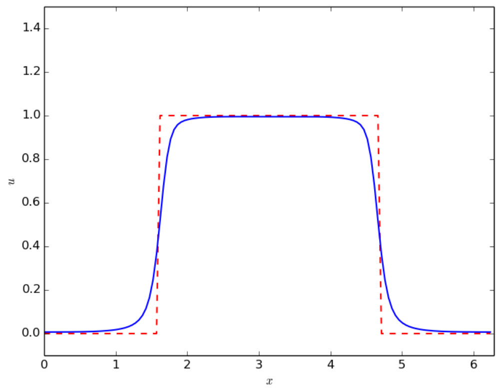
(c) $\alpha = 1.0$, without SVV.



(d) $\alpha = 1.0$, with SVV.

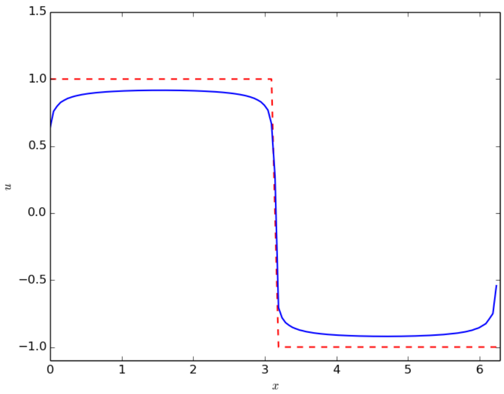


(e) $\alpha = 1.5$, without SVV.

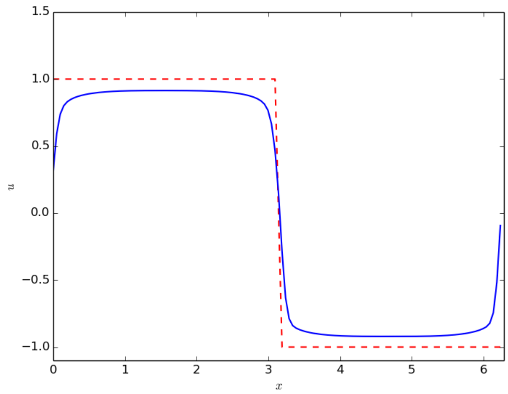


(f) $\alpha = 1.5$, with SVV.

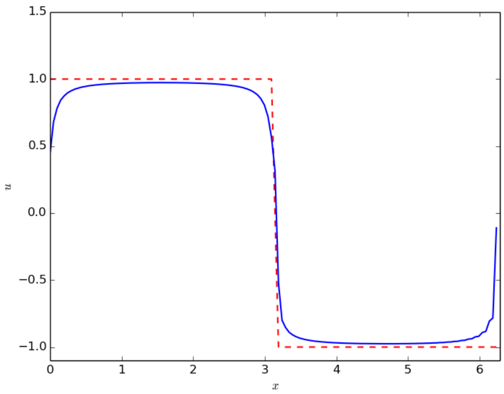
Figure 5.3: Numerical solution of the fractional fast diffusion equation for selected values of α. All simulation are with $N = 128$, $T = 0.5$ and $m = \frac{2}{5}$. The dotted line is the initial data.



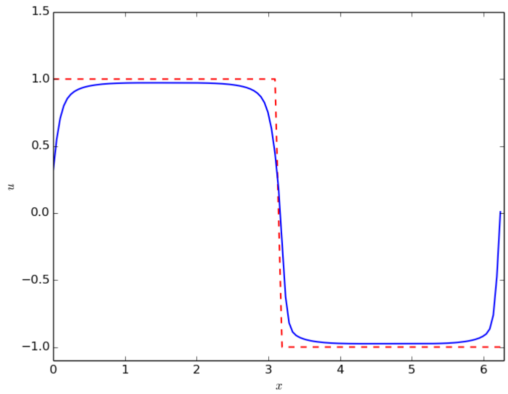
(a) $\alpha = 0.5$, without SVV.



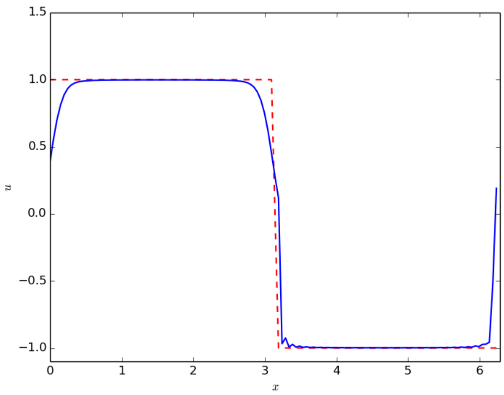
(b) $\alpha = 0.5$, with SVV.



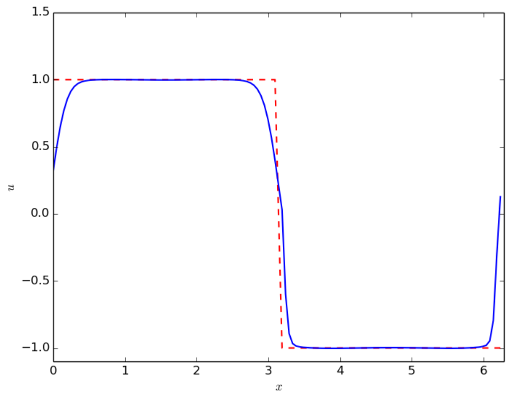
(c) $\alpha = 1.0$, without SVV.



(d) $\alpha = 1.0$, with SVV.

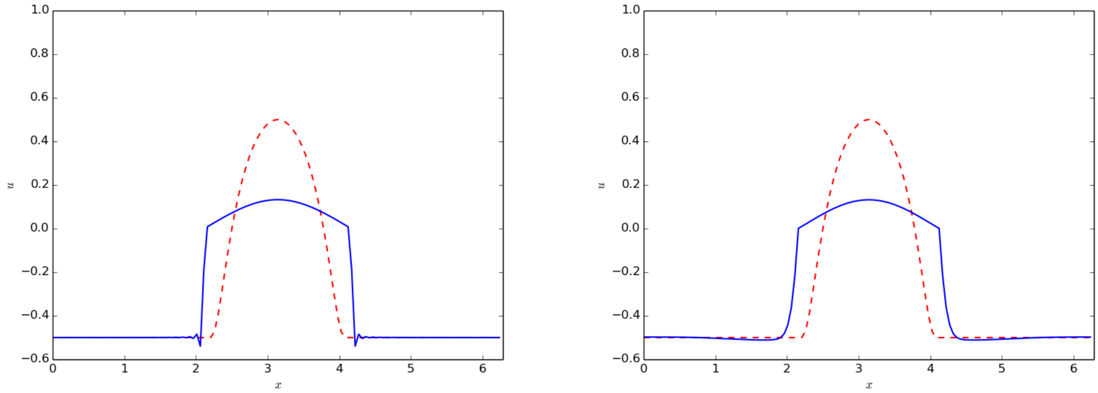


(e) $\alpha = 1.8$, without SVV.



(f) $\alpha = 1.8$, with SVV.

Figure 5.4: Numerical solution of the fractional Stefan problem for selected values of α. All simulation are with $N = 128$ and $T = 0.5$. The dotted line is the initial data.



(a) $\alpha = 1.99$, without SVV.

(b) $\alpha = 1.99$, with SVV.

Figure 5.5: Numerical solution of the fractional Stefan problem with smooth initial data. Both simulation are with $N = 128$ and $T = 15$. The dotted line is the initial data.

when the initial data is discontinuous, as has been the case in all our examples so far. The oscillations we have seen so far may just be Gibb's oscillations from the initial data that haven't had the time to diffuse away, rather than an artifact from the numerical scheme.

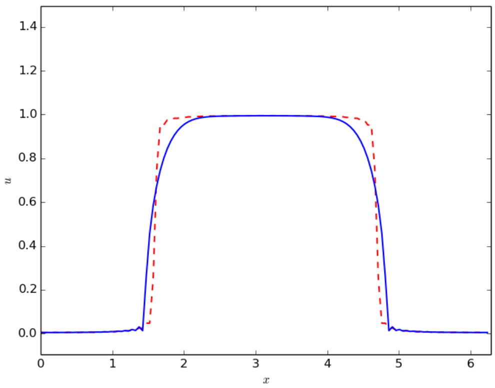
To address this issue, we review the cases where we encountered oscillations. In figure 5.6 the results from the case of fractional porous medium equation with a very high fractionality parameter. In figures 5.6a and 5.6b we have approximated the initial data using the discrete Fejer kernel. As we can see from these figures, the oscillations are still present without SVV, and the addition of SVV smooths out the oscillations without overly smearing the solution. In figures 5.6c and 5.6d we see the same effect also when approximating the initial data using the discrete de la Vallée Poussin Kernel.

Going back to fractional Stefan's problem ($\Phi(u) = \max\{u, 0\}$), we get a similar behaviour. Confer figure 5.7. As we can see from figure 5.7a and 5.7b, even though the Fejer approximation of the initial data does well to smooth out the osillation around the discontinuity, an increase in α and for larger times, the oscillations are still present, which the SVV is able to handle. A similar story is told in figure 5.7c and 5.7d when the initial data is approximated using de la Vallée Poussin kernel.

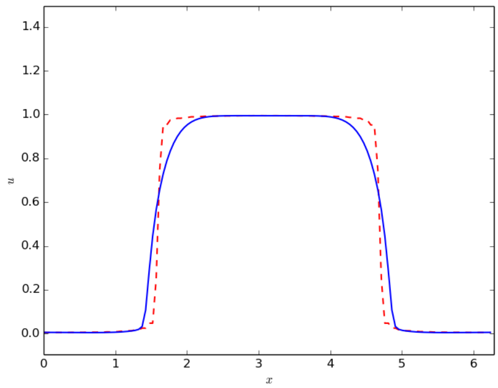
As a conclusion, we have seen that the oscillations that arise for degenerate problems when using no stabilization technique can not be attributed entirely to using the discrete interpolant I_N for approximating the initial data. Consequently, stabilization methods like SVV will have to be incorporated to create a numerical code that can handle as general a problem as possible.

5.6 Verifying convergence

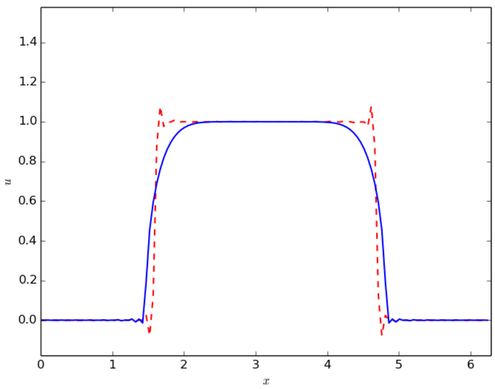
The previous numerical experiments hints at when SVV should be used, and to what extent. They also somewhat affirms that the proposed numerical method is good, in that the numerical solutions are well-behaved and do not stray far from what we might expect the solutions to look like. However, this does not constitute a sufficient confirmation



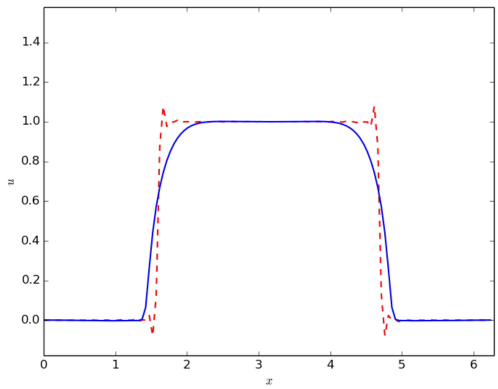
(a) $u_0^N = F_N u_0$, without SVV.



(b) $u_0^N = F_N u_0$, with SVV.

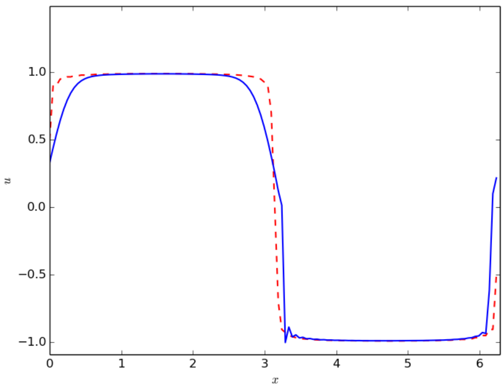


(c) $u_0^N = V_N u_0$, without SVV.

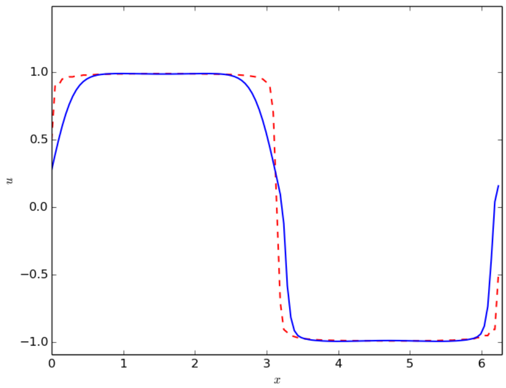


(d) $u_0^N = V_N u_0$, with SVV.

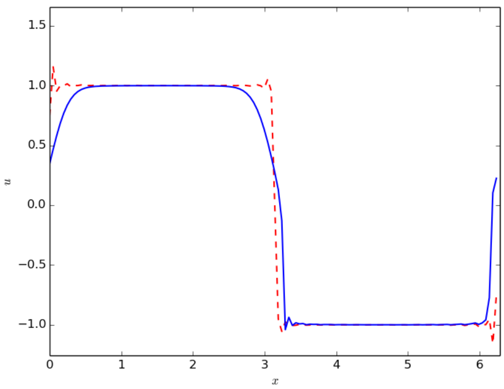
Figure 5.6: Numerical solution of the fractional porous medium equation for $\alpha = 1.999$, $m = 3$, and different ways to approximate the initial data. All simulation are with $N = 128$ and $T = 0.5$. The dotted line is the initial data.



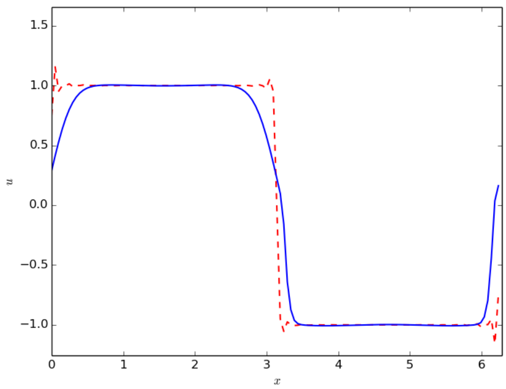
(a) $u_0^N = F_N u_0$, without SVV.



(b) $u_0^N = F_N u_0$, with SVV.



(c) $u_0^N = V_N u_0$, without SVV.



(d) $u_0^N = V_N u_0$, with SVV.

Figure 5.7: Numerical solution of the fractional Stefan's problem for $\alpha = 1.9$, and different ways to approximate the initial data. All simulation are with $N = 128$ and $T = 1$. The dotted line is the initial data.

that the implemented numerical method is correct, and to lay any remaining unease to rest we need to quantify the error in some way. Thus, a numerical verification of the implementation is of significant importance if we are to have solid trust in the preceding numerical experiments.

First off, we need a way to quantify the distance between functions periodic functions. Luckily, Parseval's identity provides us with a way of doing this in a way that is well-suited for a numerical method based on a Fourier basis. Let $u, v \in L^2((0,2\pi))$ be 2π-periodic, then we have that

$$||u - v|| = \sqrt{2\pi \sum_{\xi \in \mathbb{Z}} |\hat{u}_\xi - \hat{v}_\xi|^2},$$

where $\hat{u}_\xi$ and $\hat{v}_\xi$ are the Fourier coefficients of u and v respectively.

Next, we need a correct solution to compare the numerical results to. This is a bit trickier, since (1.1) has no known explicit, exact solutions. As a proxy we will in the subsequent use the numerical solution with a large number of degrees of freedom, N, as the exact solution.

Because (1.1) poses a variety of problems, and the solutions may have different regularity properties depending on u_0, α and Φ, we shall study several cases and see where SVV enhances convergence, but also where it to some degree slows it down. In addition to showing that the Fourier spectral method we have implemented converge, this will also inform when SVV should be used.

5.6.1 Weakly degenerate

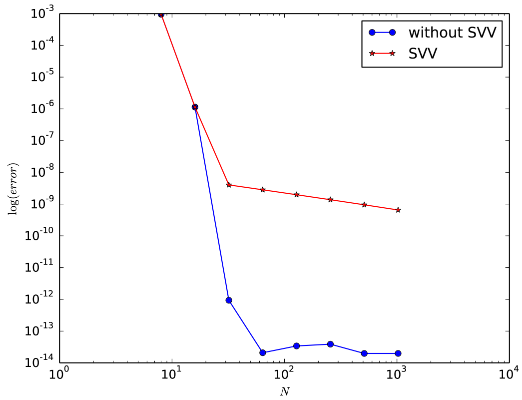
As the first case to consider, we use the fractional porous medium. That is, $\Phi(u) = u|u|^{m-1}$, with $m > 1$. For this case, it is of interest to consider both when we have smooth initial data away from the degeneracy $u = 0$, and when the initial data is discontinuous. For the former, we use as initial data

$$u_0(x) = \sin(\cos(x)) + 1,$$

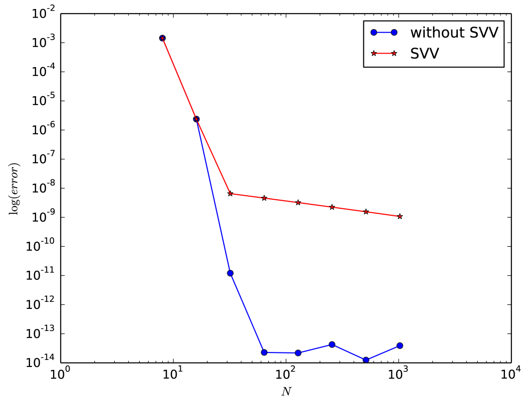
and the results for various α are shown in figure 5.8. As we can see, for all α the numerical method without SVV drops down to machine precision quite fast, whereas the inclusion of SVV makes the method have a slower rate of convergence. This may be somewhat surprising and can be to some extent chalked up to the need to calibrate the SVV parameters more than what has been done in this project. However, with this choice of Φ and u_0 (1.1) constitutes a uniformly parabolic problem in the sense that $0 < c \leq \Phi' \leq C$ for some constants c and C in the relevant domain of Φ. In the local case solutions will be infinitely regular, and so it is reasonable to believe that the fractional problem also will have a high order of regularity, which would diminish the need for any stabilization technique.

Going back to the initial data we used for the fractional porous medium equation in the numerical experiments, we also consider

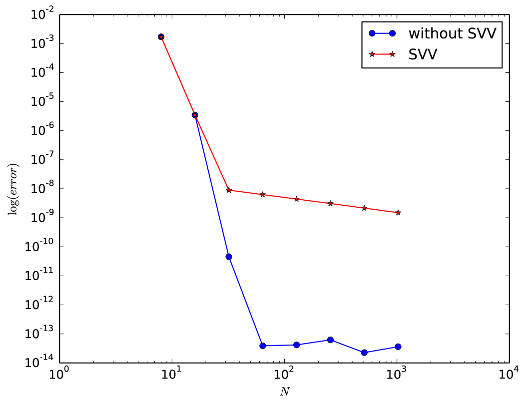
$$u_0(x) = \begin{cases} 1 & \text{for } |x - \pi| < \frac{\pi}{2} \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$



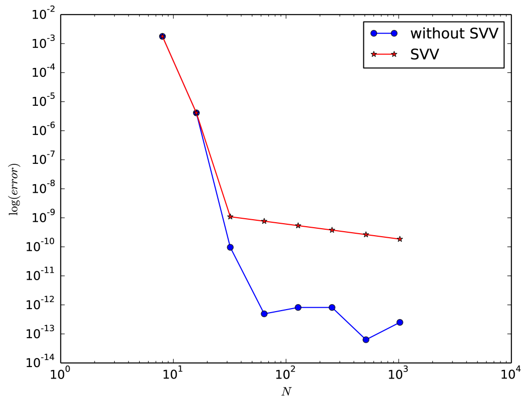
(a) $\alpha = 0.5$.



(b) $\alpha = 1$.

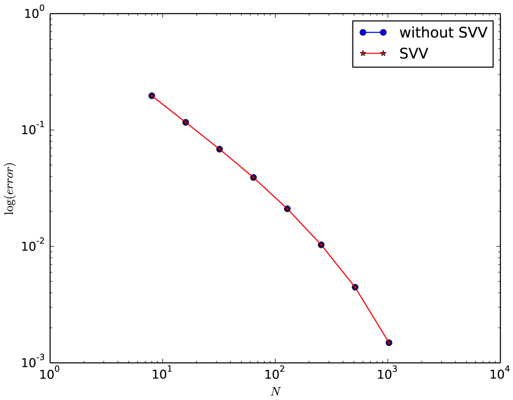


(c) $\alpha = 1.5$.

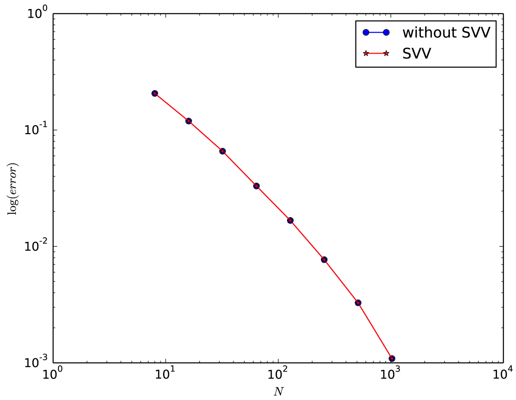


(d) $\alpha = 1.999$

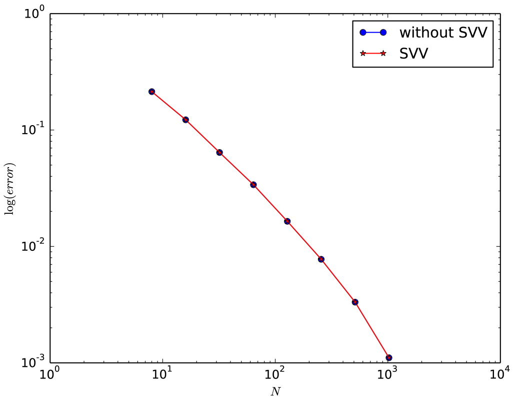
Figure 5.8: Numerical error for the fractional porous medium equation with $m = 2$, selected values for α and smooth initial data away from the degeneracy.



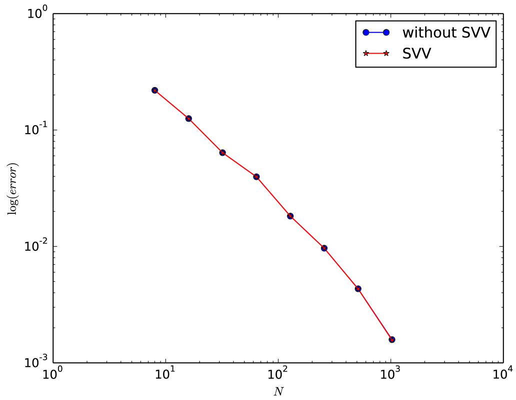
(a) $\alpha = 0.5$.



(b) $\alpha = 1$.



(c) $\alpha = 1.5$.

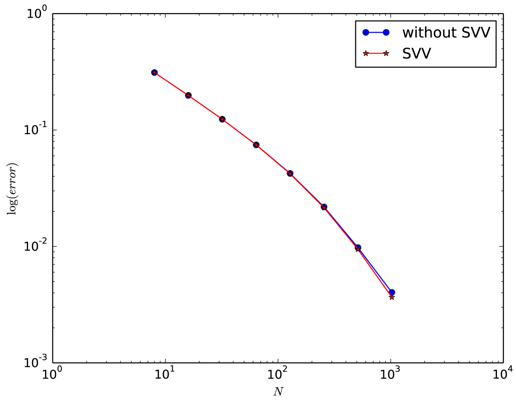


(d) $\alpha = 1.999$

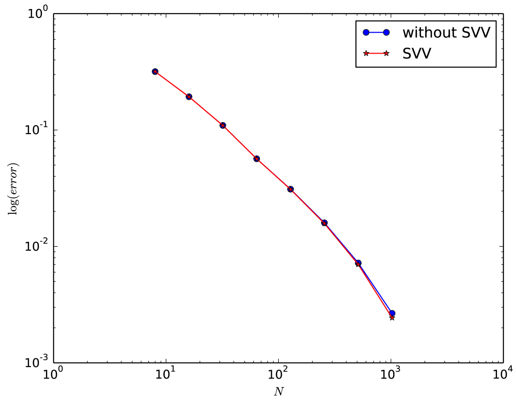
Figure 5.9: Numerical error for the fractional porous medium equation with $m = 2$, selected values for α and discontinuous initial data.

and the results are shown in figure 5.9. Here we see that both with and without SVV we have spectral convergence. Importantly, the addition of SVV does not take anything away from the convergence of the method. That the errors depicted in figure 5.9 have a much lower order of accuracy than what is shown in figure 5.8 can be explained by the fact that the approximation of the discontinuous initial will contribute to a larger error than the approximation of smooth initial data. It is also worth noting that already for the lower values of N the error curves in figure 5.9 show a significant downward slope. This suggests that the underlying solution is at least continuous.

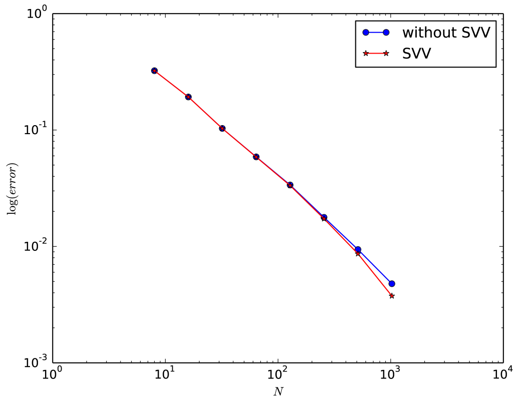
As we saw in the numerical experiments, for values of α close to 2, mild oscillations may occur, and seeing as SVV was both successful in removing these oscillations while also retaining the order of convergence, SVV seems like a quite reasonable stabilization technique to use in this case.



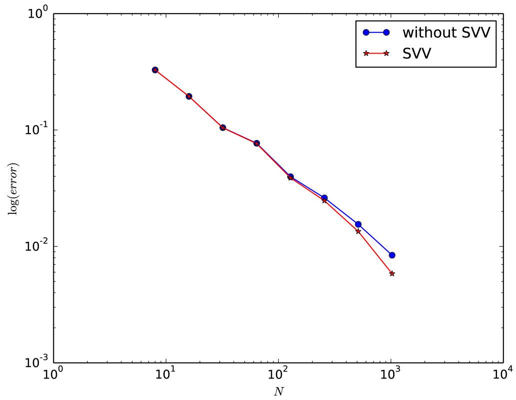
(a) $\alpha = 0.5$.



(b) $\alpha = 1$.



(c) $\alpha = 1.5$.



(d) $\alpha = 1.9$

Figure 5.10: Numerical error for the fractional Stefan's problem, selected values for α and discontinuous initial data.

5.6.2 Strongly degenerate with discontinuous initial data

Lastly we will consider the case that brought on the most consistent oscillations, viz. the strongly degenerate problem with

$$\Phi(u) = \max\{u, 0\},$$

and initial data

$$u_0(x) = \text{sgn}(\pi - x).$$

The results of this case is shown in figure 5.10, and we see that here the addition of SVV even improves on the convergence rate for all values of α. This effect is most prominent for α closer to 2, and for large N. Especially in the case $\alpha = 1.9$ (figure 5.10d) we see the concave error curve for large N that is the hallmark of spectral convergence.

6 Concluding remarks

With the results and discussion in section 5, we can note that SVV should not be applied in all cases. Some knowledge of the exact solution goes a long way to knowing when SVV, and how much, should be used. First off, we have seen that if Φ has no degeneracies, SVV is not needed as it would take away from the convergence. If you have discontinuous problem you will be better off approximating the initial data with the Fejer- or de la Vallée Poussin kernel in this case. This is not so surprising, since any finite regularity that might warrant the need for stabilization in a numerical scheme will most likely arise from Φ being degenerate. This reasoning is validating in the numerical experiments where we saw that both for weakly- and strongly degenerate problems, oscillations could arise. These oscillations where mild enough to not ruin convergence completely, but adding SVV in these cases both removed oscillations and in some cases improved on the convergence.

The need to use SVV to stabilize numerical solutions also increased with α close to 2, which is in agreement with the fact that $\mathcal{L}$ converges to the laplacian as $\alpha \to 2$ and solutions of the local variant of (1.1) can have finite regularity when Φ is degenerate.

This knowledge becomes additionally valuable if (1.1) is augmented with a convective term, a natural generalization. For $\alpha \in (0, 1)$ the dissipation is too weak to hinder the onset of shock discontinuities (cf. [8, 15]) in the case of a linear diffusive term, and similar behaviour is to be expected in the nonlinear case. With the possibility of shock discontinuities to form for $\alpha < 1$, stabilization with SVV will be required both for low and high values of α.

6.1 Where to go from here

As already stated, a natural extension of the discussion in this project is to add a convective term. Most of the analysis done in section 4 can be extended to this case, but extra care will be needed especially the energy estimate for the derivatives of u_N (Lemma 4.5).

Another extension of the problem considered herein is to let the nonlocal operator not only be the fractional laplacian, but defined by other symmetric- and asymmetric Radon measures. Again, the crux of the difficulty will lie in the generalization of Lemma 4.5, and in particular, the interpolation estimate of Lemma 2.3 that will need to be extended to more general measures.

Although (1.1) was in this project only considered in one spatial dimension, the extension to two- or three dimension would be interesting. The necessary modifications of the analysis in section 4 would most likely follow the notation and techniques that were used in [4].

In addition, the use of a Fourier basis in a spectral method is only advisable in the case of periodic boundary conditions. In applications, problems with essential- and natural boundary conditions like Dirichlet and Neumann conditions are common. So a basis other than Fourier would be required for a spectral method, e.g. a polynomial basis. Also, if SVV is to be used in this setting, a modal basis (as opposed to nodal) should be used. Further, to keep the matrices of the resulting numerical scheme sparse the polynomial basis should in some sense take the bilinear form of the nonlocal operator into account (Lemma 2.1). For instance, in the local case the "boundary adapted bubble functions" of [1, sec. 2.3.3], which are orthogonal in $H_0^1((-1,1))$, can be used. See also [17]. The use of a Legendre basis on Burgers' equation with SVV was studied in [19]. Other than that, the author has not been able to find SVV used with polynomial bases. However, successful use of a polynomial basis, especially on a bounded domain with homogeneous Dirichlet boundary conditions, is an important stepping stone to be able to use spectral element methods on (1.1). I.e. dividing the global domain into smaller subdomains, akin to finite element methods. This extension would let us solve (1.1) with spectral methods on more irregular domains, e.g. domains with holes.

A Gibb's phenomenon and handling of initial data

To complete the method (3.32) we need a way to project the initial data u_0 into the discrete function space $\mathcal{S}_N$. The obvious candidates are S_N or I_N. However, if u_0 contains discontinuities, then $P_N u_0$ will have oscillations about the discontinuities, which is widely known as Gibb's phenomenon (cf. [1, Sec. 2.1.4]). Loosely speaking, these oscillations comes from the fact that

$$S_N u_0 = D_N * u_0 \tag{A.1}$$

, where D_N is the Dirichlet kernel, and contains oscillations and rapidly changes sign about the origin.

Gibb's phenomenon has the detrimental effect that even though u_0 is bounded, this is no guarantee for $P_N u_0$ to be uniformly bounded. The same holds for the total variation, i.e. even if u_0 has bounded variation this does not imply that the total variation of $P_N u_0$ is uniformly bounded. So the Gibb's phenomenon poses an issue that should be adressed in a numerical implementation.

The usual way to tackle this problem stems from the observation that Gibb's phenomenon gives high frequency oscillations, and so a dampening of the higher frequencies will hopefully yields approximations that are better behaved. So assume now that the initial data u_0 lends itself to the Fourier representation

$$u_0(x) = \sum_{\xi \in \mathbb{Z}} u_{0,\xi} e^{i\xi x}.$$

The first such dampening of higher frequencies we'll consider is the use of the Fejer kernel, which operates on u_0 as

$$F_N u_0 = \sum_{|\xi| \leq N} \left(1 - \frac{|\xi|}{N+1}\right) u_{0,\xi} e^{i\xi x}. \tag{A.2}$$

We see that F_N acts as a linear dampening of the frequencies of u_0. Since all information of u_0 lies in the coefficients $u_{0,\xi}$, one can predict that a heavy dampening of the frequencies will lead to losing some of the features of u_0. Indeed, as we shall see shortly, the Fejer kernel leads to quite heavy smearing of discontinuities.

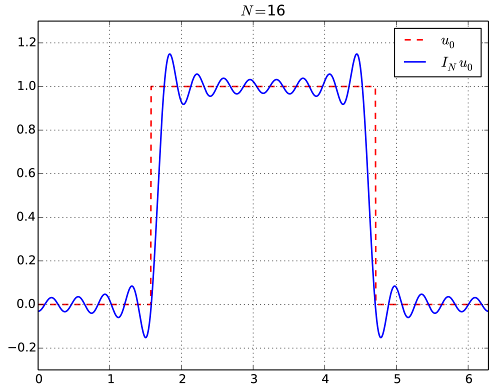
To retain as much of the structure of u_0 as possible, another smoothening operator is now proposed. This is the so-called de la Vallée Poussin kernel, V_N, which acts on u_0 as

$$V_N u_0 = \sum_{|\xi| \leq \left[\frac{N}{2}\right]} u_{0,\xi} e^{i\xi x} + \sum_{\left[\frac{N}{2}\right] < |\xi| \leq N} \left(2 - \frac{|\xi|}{\left[\frac{N}{2}\right]}\right) u_{0,\xi} e^{i\xi x}, \tag{A.3}$$

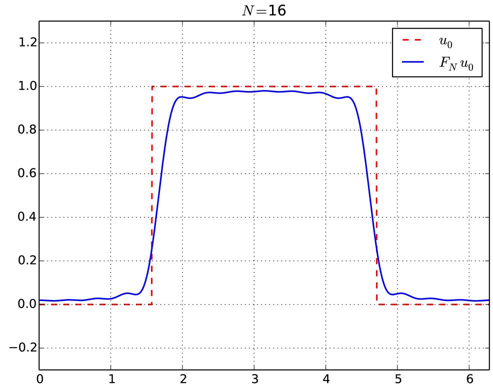
where $\left[\frac{N}{2}\right]$ is the smallest integer greater than or equal to $\frac{N}{2}$. See that V_N does no dampening on the lower frequencies, in contrast to the Fejer kernel.

Example A.1. To summarize what we have so far, and try to justify some of the heuristics in the discussion, let's consider u_0 defined on $[0, 2\pi)$ and periodic as

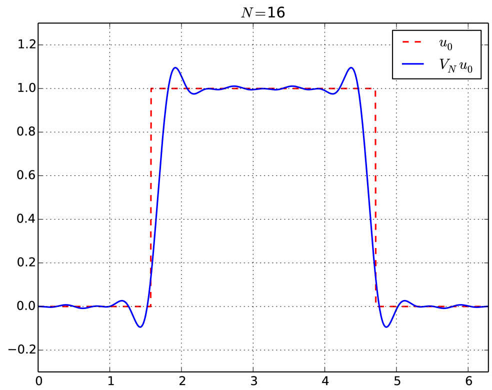
$$u_0(x) = \begin{cases} 1, & \text{if } x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ 0, & \text{otherwise.} \end{cases} \tag{A.4}$$



(a) $I_N u_0$ plotted with u_0.



(b) The discrete version of $F_N u_0$ plotted with u_0



(c) The discrete version of $V_N u_0$ plotted with u_0

Figure A.1: The various way of approximating the initial data. Here with $N = 16$ and u_0 as given in example A.1.

Consult figure A.1 to see how the discrete versions of the various approximations to u_0. In figure A.1a we see the Gibb's phenomenon clearly present, and will become more pronounced as N increases. From figure A.1b we see that the Fejer kernel extinguishes the high frequency oscillations, but this happens at a loss of smearing out u_0. Lastly, in figure A.1c we see a compromise between the two in $V_N u_0$, where some oscillations are still present to keep the some of the structure of u_0.

Of course, one need not stop here when considering frequency-dampening operators. In general we may define an operator as

$$K_N u_0 = \sum_{|\xi|\leq N} \sigma_\xi u_{0,\xi} e^{i\xi x}, \tag{A.5}$$

where the σ_ξ is an even sequence in ξ, $\sigma_0 = 1$, and $\sigma_{|\xi|}$ is nonincreasing in ξ. The reader is referred to [1, Sec. 2.1.4] for more examples.

To get back to why the approximations $F_N u_0$ and $V_N u_0$ are needed, we recall that the more obvious way of approximating the initial data lead to potenial Gibb's oscillations which made us lose control of both boundedness and total variation in the approximation. To see that the operator F_N and V_N regain this control it's useful to change perspective somewhat. Since both operators are acting componentwise on u_0, they can be expressed as convolutions with some kernel. For the Fejer kernel we have $F_N u_0(x) = F_N * u_0$, where

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin\left((N+1)\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)^2. \tag{A.6}$$

Notice that $F_N \geq 0$, which is in correspondence with $F_N u_0$ not exhibiting Gibb's oscillations. The analogous de la Vallée Poussin kernel is

$$V_N(x) = \frac{2}{N}\left(\frac{\sin\left(N\frac{x}{2}\right)^2 - \sin\left(\frac{N}{2}\cdot\frac{x}{2}\right)^2}{\sin\left(\frac{x}{2}\right)^2}\right) \tag{A.7}$$

(cf. [20]).

Assume now that u_0 is bounded, then Young's inequality for convolutions yields

$$||K_N * u_0||_\infty \leq ||K_N||_1 \cdot ||u_0||_\infty,$$

where K_N is a placeholder for either F_N or V_N. And so if K_N is uniformly bounded in $L^1((0,2\pi))$ we can control the boundedness of $K_N u_0$. For the Fejer kernel $||F_N||_1 = 1$, and for the de la Vallée Poussin kernel $||V_N||_1 = \frac{1}{3} + \frac{2\sqrt{3}}{\pi}$ (cf. [20, Cor. 1.3]).

The following lemma shows that also total variation is under control when using F_N or V_N, making both approximations feasible in our subsequent compactness argument.

Lemma A.1. *Assume u_0 has bounded variation, and that $\{K_N\}_{N\in\mathbb{Z}}$ is a family of kernels uniformly bounded in $L^1((0,2\pi))$. Then the total variation of $K_N * u_0$ is uniformly bounded by the estimate*

$$|K_N * u_0|_{BV} \leq \frac{1}{\pi}|u_0|_{BV}\,||K_N||_1. \tag{A.8}$$

Proof. The definition of convolutions of periodic functions gives us that

$$(K_N * u_0)(x) = \frac{1}{2\pi}\int_0^{2\pi} K_N(y)u_0(x-y)\mathrm{d}y.$$

Let now $0 = x_0 < x_1 < \ldots < x_m = 2\pi$ be a partition of $[0,2\pi]$, when then have

$$\begin{aligned}
\sum_{i=0}^{m-1} |(K_N * u_0)(x_{i+1}) - (K_N * u_0)(x_i)| &= \frac{1}{2\pi}\sum_{i=0}^{m-1}\left|\int_0^{2\pi} K_N(y)(u_0(x_{i+1}-y) - u_0(x_i-y))\mathrm{d}y\right| \\
&\le \frac{1}{2\pi}\sum_{i=0}^{m-1}\int_0^{2\pi} |K_N(y)|\cdot|u_0(x_{i+1}-y) - u_0(x_i-y)|\mathrm{d}y \\
&= \frac{1}{2\pi}\int_0^{2\pi} |K_N(y)|\left(\sum_{i=0}^{m-1}|u_0(x_{i+1}-y) - u_0(x_i-y)|\right)\mathrm{d}y \\
&\le \frac{|u_0|_{BV}}{\pi}\int_0^{2\pi}|K_N(y)|\mathrm{d}y \\
&= \frac{1}{\pi}|u_0|_{BV}\,||K_N||_1\,,
\end{aligned}$$

where the factor 2 comes from taking into account the possible jump between of $u_0(0^+)$ and $u_0(2\pi^-)$. Take now the supremum over all such finite partitions results in

$$|K_N * u_0|_{BV} \le \frac{1}{\pi}|u_0|_{BV}\,||K_N||_1\,. \tag{A.9}$$

□

B Proof of Lemma 2.9

Proof. The proof takes inspiration from the proof of Theorem 2.4 in appendix B of [4], which in turn is inspired by the celebrated Kruzkov's doubling of variables argument (cf. e.g. [14, Ch. 2]).

Establishing an entropy relation: Let η be a smooth convex function in one variable. For $v_\epsilon = v_\epsilon(x,t)$, multiply (2.11) by $\eta'(v_\epsilon)$ to get

$$\eta'(v_\epsilon)\frac{\partial v_\epsilon}{\partial t} = \eta'(v_\epsilon)\mathcal{L}[\Phi(v_\epsilon)] + \epsilon\eta'(v-_\epsilon)\frac{\partial^2 v_\epsilon}{\partial x^2}. \tag{B.1}$$

First, we notice that

$$\eta'(v_\epsilon)\frac{\partial v_\epsilon}{\partial t} = \frac{\partial \eta(v_\epsilon)}{\partial t},$$

but also that

$$\begin{aligned}\eta'(v_\epsilon)\frac{\partial^2 v_\epsilon}{\partial x^2} &= \frac{\partial^2 \eta(v_\epsilon)}{\partial x^2} - \eta''(v_\epsilon)\left(\frac{\partial v_\epsilon}{\partial x}\right)^2 \\ &\leq \frac{\partial^2 \eta(v_\epsilon)}{\partial x^2}\end{aligned}$$

by the convexity of η, and so (B.1) becomes

$$\frac{\partial \eta(v_\epsilon)}{\partial t} \leq \eta'(v_\epsilon)\mathcal{L}[\Phi(v_\epsilon)] + \epsilon\frac{\partial^2 \eta(v_\epsilon)}{\partial x^2}.$$

We now multiply this by a nonnegative test function $\psi \in C^\infty(\mathbb{R}\times[0,T])$ that is 2π-periodic in space and has compact support in time. Integrating over Q_T and using integration by parts appropriately leads to

$$\iint_{Q_T} \eta(v_\epsilon)\frac{\partial \psi}{\partial t} + \eta'(v_\epsilon)\mathcal{L}[\Phi(v_\epsilon)]\psi + \epsilon\eta(v_\epsilon)\frac{\partial^2 \psi}{\partial x^2}\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{B.2}$$

Now, take any $k \in \mathbb{R}$, and define $\eta_\gamma(v_\epsilon,k) = ((v_\epsilon - k)^2 + \gamma^2)^{\frac{1}{2}}$. We take the limit $\gamma \to 0$ and use the dominated convergence theorem on each term to get

$$\iint_{Q_T} \eta(v_\epsilon,k)\frac{\partial \psi}{\partial t} + \eta'(v_\epsilon,k)\mathcal{L}[\Phi(v_\epsilon)]\psi + \epsilon\eta(v_\epsilon,k)\frac{\partial^2 \psi}{\partial x^2}\mathrm{d}x\,\mathrm{d}t \geq 0, \tag{B.3}$$

with $\eta(v_\epsilon,k) = |v_\epsilon - k|$ and $\eta'(v_\epsilon,k) = \mathrm{sgn}(v_\epsilon - k)$.

Consider $v_\delta = v_\delta(y,s)$, and we get by the exact same approach that

$$\iint_{Q_T} \eta(v_\delta,k)\frac{\partial \psi}{\partial t} + \eta'(v_\delta,k)\mathcal{L}[\Phi(v_\delta)]\psi + \delta\eta(v_\delta,k)\frac{\partial^2 \psi}{\partial y^2}\mathrm{d}y\,\mathrm{d}s \geq 0, \tag{B.4}$$

for every $k \in \mathbb{R}$ and nonnegative $\psi \in C^\infty(\mathbb{R}\times[0,T])$ that's 2π-periodic in space and with compact support in time.

The doubling of variables: In (B.3) we take $k = v_\delta(y,s)$ and integrate over y and s. Similarly we let in (B.4) $k = v_\epsilon(x,t)$ and integrate over x and t. Summing these together, and we get

$$\begin{aligned}\iint_{Q_T}\iint_{Q_T} &\eta(v_\epsilon(x,t),v_\delta(y,s))\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)\psi(x,y,t,s)\\ &+\eta'(v_\epsilon(x,t),v_\delta(y,s))\,(\mathcal{L}[\Phi(v_\epsilon(\cdot,t)](x)-\mathcal{L}[\Phi(v_\delta(\cdot,s)](y))\,\psi(x,y,t,s)\\ &+\eta(v_\epsilon(x,t),v_\delta(y,s))\left(\epsilon\frac{\partial^2}{\partial x^2}+\delta\frac{\partial^2}{\partial y^2}\right)\psi(x,y,t,s)\mathrm{d}w \geq 0,\end{aligned} \tag{B.5}$$

with $\mathrm{d}w = \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$, and $\psi = \psi(x,y,t,s)$ is 2π-periodic and infintely regular in x and y and has compact support t and s.

To continue, we notice

$$\begin{aligned}\mathcal{L}[\Phi(v_\epsilon(\cdot,t)](x)-\mathcal{L}[\Phi(v_\delta(\cdot,s)](y) = \int_{|z|>0} &\Phi(v_\epsilon(x+z,t))-\Phi(v_\delta(y+z,s))\\ &-\Phi(v_\epsilon(x,t))+\Phi(v_\delta(y,s))\\ &-z\mathbf{1}_{|z|<1}\left(\frac{\partial\Phi(v_\epsilon)}{\partial x}(x,t)-\frac{\partial\Phi(v_\delta)}{\partial y}(y,s)\right)\mathrm{d}\mu(z)\\ =\ &\tilde{\mathcal{L}}[\Phi(v_\epsilon(\cdot,t))-\Phi(v_\delta(\cdot,s))](x,y),\end{aligned}$$

where we define

$$\tilde{\mathcal{L}}[\varphi(\cdot,\cdot)](x,y) := \int_{|z|>0}\varphi(x+z,y+z)-\varphi(x,y)-z\mathbf{1}_{|z|<1}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\varphi(x,y)\mathrm{d}\mu(z).$$

We then have that

$$\eta'(v_\epsilon(x,t),v_\delta(y,s))\tilde{\mathcal{L}}[\Phi(v_\epsilon(\cdot,t))-\Phi(v_\delta(\cdot,s))](x,y) \leq \tilde{\mathcal{L}}[\eta(\Phi(v_\epsilon(\cdot,t)),\Phi(v_\delta(\cdot,s)))](x,y). \tag{B.6}$$

To see this, we use

$$\operatorname{sgn}(v_\epsilon(x,t)-v_\delta(y,s))(\Phi(v_\epsilon(x,t))-\Phi(v_\delta(y,s))) = |\Phi(v_\epsilon(x,t))-\Phi(v_\delta(y,s))|$$

to get that

$$\begin{aligned}\eta'(v_\epsilon(x,t),v_\delta(y,s))\tilde{\mathcal{L}}[\Phi(v_\epsilon(\cdot,t))-\Phi(v_\delta(\cdot,s))](x,y) \leq \int_{|z|>0} &|\Phi(v_\epsilon(x+z,t))-\Phi(v_\delta(y+z,s))|\\ -|\Phi(v_\epsilon(x,t))-\Phi(v_\delta(y,s))|&\\ -z\mathbf{1}_{|z|<1}\operatorname{sgn}(v_\epsilon(x,t)-v_\delta(y,s))&\left(\frac{\partial\Phi(v_\epsilon)}{\partial x}(x,t)-\frac{\partial\Phi(v_\delta)}{\partial y}(y,s)\right)\mathrm{d}\mu(z).\end{aligned}$$

For the last term we use that $\operatorname{sgn}(v_\epsilon(x,t)-v_\delta(y,s)) = \operatorname{sgn}(\Phi(v_\epsilon(x,t))-\Phi(v_\delta(y,s))$ except possibly if $\Phi'(s)=0$ for all s between $v_\epsilon(x,t)$ and $v_\delta(y,s)$, but then $\frac{\partial\Phi(v_\epsilon)}{\partial x}(x,t) = \frac{\partial\Phi(v_\delta)}{\partial y}(y,s) = 0$. And so (B.6) is justified.

Next we show that $\tilde{\mathcal{L}}$ is self-adjoint. See that by Fubini's theorem

$$\iint_{Q_T}\iint_{Q_T}\tilde{\mathcal{L}}[\eta(\Phi(v_\epsilon(\cdot,t)),\Phi(v_\delta(\cdot,s))](x,y)\psi(x,y,t,s)\mathrm{d}w$$
$$=\int_{|z|>0}\iint_{Q_T}\iint_{Q_T}\eta(\Phi(v_\epsilon(x+z,t)),\Phi(v_\delta(y+z,s)))\psi(x,y,t,s)\mathrm{d}w$$
$$-\iint_{Q_T}\iint_{Q_T}\eta(\Phi(v_\epsilon(x,t)),\Phi(v_\delta(y,s)))\psi(x,y,t,s)\mathrm{d}w$$
$$-z\mathbf{1}_{|z|<1}\iint_{Q_T}\iint_{Q_T}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\eta(\Phi(v_\epsilon(x,t)),\Phi(v_\delta(y,s)))\psi(x,y,t,s)\mathrm{d}w\,\mathrm{d}\mu(z).$$

Using the variable transformation $(x,y)\mapsto(x-z,y-z)$ and periodicity to translate the domain of integration back to $(0,2\pi)$ on the first term, and integration by parts on the last, we find that

$$\iint_{Q_T}\iint_{Q_T}\tilde{\mathcal{L}}[\eta(\Phi(v_\epsilon(\cdot,t)),\Phi(v_\delta(\cdot,s))](x,y)\psi(x,y,t,s)\mathrm{d}w$$
$$=\int_{|z|>0}\iint_{Q_T}\iint_{Q_T}\eta(\Phi(v_\epsilon(x,t)),\Phi(v_\delta(y,s)))\psi(x-z,y-z,t,s)\mathrm{d}w$$
$$-\iint_{Q_T}\iint_{Q_T}\eta(\Phi(v_\epsilon(x,t)),\Phi(v_\delta(y,s)))\psi(x,y,t,s)\mathrm{d}w$$
$$+z\mathbf{1}_{|z|<1}\iint_{Q_T}\iint_{Q_T}\eta(\Phi(v_\epsilon(x,t)),\Phi(v_\delta(y,s)))\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\psi(x,y,t,s)\mathrm{d}w\,\mathrm{d}\mu(z).$$

Finally, using that the fractional laplacian measure is symmetric together with Fubini's theorem again, we get

$$\iint_{Q_T}\iint_{Q_T}\tilde{\mathcal{L}}[\eta(\Phi(v_\epsilon(\cdot,t)),\Phi(v_\delta(\cdot,s))](x,y)\psi(x,y,t,s)\mathrm{d}w$$
$$=\iint_{Q_T}\iint_{Q_T}\eta(\Phi(v_\epsilon(x,t),\Phi(v_\delta(y,s))$$
$$\times\left(\int_{|z|>0}\psi(x+z,y+z,t,s)-\psi(x,y,t,s)-z\mathbf{1}_{|z|<1}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\psi(x,y,t,s)d\mu(z)\right)\mathrm{d}w$$
$$=\iint_{Q_T}\iint_{Q_T}\eta(\Phi(v_\epsilon(x,t),\Phi(v_\delta(y,s))\tilde{\mathcal{L}}[\psi(\cdot,\cdot,t,s)](x,y)\mathrm{d}w.$$

We put this back into (B.5) to arrive at

$$\iint_{Q_T}\iint_{Q_T}\eta(v_\epsilon(x,t),v_\delta(y,s))\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)\psi(x,y,t,s)$$
$$+\eta(\Phi(v_\epsilon(x,t)),\Phi(v_\delta(y,s)))\tilde{\mathcal{L}}[\psi(\cdot,\cdot,t,s)](x,y)$$
$$+\eta(v_\epsilon(x,t),v_\delta(y,s))\left(\epsilon\frac{\partial^2}{\partial x^2}+\delta\frac{\partial^2}{\partial y^2}\right)\psi(x,y,t,s)\mathrm{d}w\geq 0. \tag{B.7}$$

Choice of test function: Let now ω be the standard mollifier, We also need the periodic variant of ω_ρ, which we define as

$$\hat{\omega}_\rho(x)=\sum_{k\in\mathbb{Z}}\omega_\rho(x+2\pi k).$$

We then take as a test function

$$\psi(x,y,t,s)=\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)\phi(t),$$

where $\rho,\gamma>0$, and ϕ is a C^∞ function with compact support $(0,T)$, and is to be determined later on. Before putting our choice for ψ into (B.7), we notice by direct calculation that

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)\psi(x,y,t,s)=\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)\phi'(t)\text{ and}\\&\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\psi(x,y,t,s)=0.\end{aligned}$$

From the latter of these identities and that $\psi(x+z,y+z,t,s)=\psi(x,y,t,s)$ we also have that $\tilde{\mathcal{L}}[\psi(\cdot,\cdot,t,s)](x,y)=0$. With this in mind, we get from (B.7) that

$$\begin{aligned}&-\iint_{Q_T}\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(y,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)\phi'(t)\mathrm{d}w\\&\le\iint_{Q_T}\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(y,s)|\left(\epsilon\frac{\partial^2}{\partial x^2}+\delta\frac{\partial^2}{\partial y^2}\right)\psi(x,y,t,s)\mathrm{d}w.\end{aligned}\tag{B.8}$$

Estimating the right hand side of (B.8): Using integration by parts, we get

$$\begin{aligned}&\epsilon\iint_{Q_T}\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(y,s)|\frac{\partial^2\psi}{\partial x^2}\mathrm{d}w\\&\le\epsilon\iint_{Q_T}\iint_{Q_T}\left|\frac{\partial u}{\partial x}(x,t)\right|\cdot\left|\frac{1}{2}\hat{\omega}_\rho'\left(\frac{x-y}{2}\right)\right|\omega_\gamma\left(\frac{t-s}{2}\right)\phi(t)\mathrm{d}w,\end{aligned}$$

and if we now take the integration first over y and s, then

$$\begin{aligned}&\epsilon\iint_{Q_T}\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(y,s)|\frac{\partial^2\psi}{\partial x^2}\mathrm{d}w\\&\le\epsilon\frac{C}{\rho}\iint_{Q_T}\left|\frac{\partial u}{\partial x}(x,t)\right|\int_0^T\omega_\gamma\left(\frac{t-s}{2}\right)\phi(t)\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\&\le C\int_0^T phi(t)\mathrm{d}t|v_\epsilon|_{BV}\frac{\epsilon}{\rho}\\&\le C\int_0^T phi(t)\mathrm{d}t|u_0|_{BV}\frac{\epsilon}{\rho},\end{aligned}$$

where the last step is justified by Lemma 2.7. We estimate the other term on the right hand side of (B.8) in a similar manner, and so we get

$$\iint_{Q_T}\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(y,s)|\left(\epsilon\frac{\partial^2}{\partial x^2}+\delta\frac{\partial^2}{\partial y^2}\right)\psi(x,y,t,s)dw\le C\int_0^T phi(t)\mathrm{d}t|u_0|_{BV}\frac{\epsilon+\delta}{\rho}.\tag{B.9}$$

Estimating the left hand side of (B.8): With the inequality $|a|-|b| \leq |a-b|$, we have that

$$\begin{aligned}&|v_\epsilon(x,t)-v_\delta(y,s)|\phi'(t)-|v_\epsilon(x,t)-v_\delta(x,t)|\phi'(t)\\&\leq |v_\delta(x,t)-v_\delta(y,s)|\cdot|\phi'(t)|\\&\leq |v_\delta(x,t)-v_\delta(x,s)|\cdot|\phi'(t)|+|v_\delta(x,s)-v_\delta(y,s)|\cdot|\phi'(t)|.\end{aligned}$$

Rearranging terms, we find that

$$\begin{aligned}-|v_\epsilon(x,t)-v_\delta(y,s)|\phi'(t)\geq &-|v_\epsilon(x,t)-v_\delta(x,t)|\phi'(t)\\&-|v_\delta(x,t)-v_\delta(x,s)|\cdot|\phi'(t)|\\&-|v_\delta(x,s)-v_\delta(y,s)|\cdot|\phi'(t)|.\end{aligned}$$

Since ω_γ and $\hat{\omega}_\rho$ are nonnegative functions, we get for the left hand side of (B.8) that

$$\begin{aligned}&-\iint_{Q_T}\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(y,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)\phi'(t)\mathrm{d}w\\&\geq-\iint_{Q_T}\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(x,t)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)\phi'(t)\mathrm{d}w\\&-\iint_{Q_T}\iint_{Q_T}|v_\delta(x,t)-v_\delta(x,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)|\phi'(t)|\mathrm{d}w\\&-\iint_{Q_T}\iint_{Q_T}|v_\delta(x,s)-v_\delta(y,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)|\phi'(t)|\mathrm{d}w.\end{aligned}\tag{B.10}$$

For the second term on the right hand side of (B.10) all factors in the integrand are bounded, so we may use Fubini's theorem and integrate first over y to get

$$\begin{aligned}&\iint_{Q_T}\iint_{Q_T}|v_\delta(x,t)-v_\delta(x,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)|\phi'(t)|\mathrm{d}w\\&=\iint_{Q_T}\int_0^T|v_\delta(x,t)-v_\delta(x,s)|\omega_\gamma\left(\frac{t-s}{2}\right)|\phi'(t)|\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

If we next take the integration over x and use the time estimate of Lemma 2.8 we end up with

$$\begin{aligned}&\left|\iint_{Q_T}\iint_{Q_T}|v_\delta(x,t)-v_\delta(x,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)|\phi'(t)|\mathrm{d}w\right|\\&\leq C\int_0^T\int_0^T\sqrt{|t-s|}\omega_\gamma\left(\frac{t-s}{2}\right)|\phi'(t)|\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Using now that as $\gamma\to 0$, ω_γ approaches Dirac's delta yields

$$\iint_{Q_T}\iint_{Q_T}|v_\delta(x,t)-v_\delta(x,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)|\phi'(t)|\mathrm{d}w\to 0,$$

as $\gamma\to 0$.

For the third term on the right hand side of (B.10), notice that

$$\iint_{Q_T}\iint_{Q_T}|v_\delta(x,s)-v_\delta(y,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)|\phi'(t)|dw \le C|u_0|_{BV}\rho\int_0^T|\phi'(t)|\mathrm{d}t,$$

where we have used that

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi}|v_\delta(x,s)-v_\delta(y,s)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)dy\,dx &= \int_0^{2\pi}\int_0^{2\pi}|v_\delta(x,s)-v_\delta(x+y)|\hat{\omega}_\rho\left(\frac{y}{2}\right)dy\,dx\\ &\le |v_\delta|_{BV}\int_{-\pi}^{\pi}|y|\hat{\omega}_\rho\left(\frac{y}{2}\right)\mathrm{d}y\,\mathrm{d}x\\ &\le C|u_0|_{BV}\rho.\end{aligned}$$

Furthermore, when we integrate over y and s, we get

$$\begin{aligned}&\iint_{Q_T}\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(x,t)|\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\gamma\left(\frac{t-s}{2}\right)\phi'(t)\mathrm{d}w\\ &=\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(x,t)|\phi'(t)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

where we use that $\omega(\cdot)$ integrates to 1. We put all this back into (B.10), and take it together with (B.9) into (B.8) to arrive at

$$-\iint_{Q_T}|v_\epsilon(x,t)-v_\delta(x,t)|\phi'(t)dx\,dt \le C|u_0|_{BV}\rho\int_0^T|\phi'(t)|\mathrm{d}t + C\int_0^T\phi(t)\mathrm{d}t|u_0|_{BV}\frac{\epsilon+\delta}{\rho}. \tag{B.11}$$

Determining ϕ and finishing up: Take $0<t_1<t_2<T$, and define

$$\phi(t)=\int_{-\infty}^{t}\omega_{\tilde{\gamma}}(\tau-t_1)-\omega_{\tilde{\gamma}}(\tau-t_2)\mathrm{d}\tau,$$

where $\tilde{\gamma}$ is sufficiently small to make the support ϕ contained in $[0,T]$. We would also like the mollification around t_1 and t_2 not to interfere. Since the support of $\omega_{\tilde{\gamma}}$ is contained in $[-\tilde{\gamma},\tilde{\gamma}]$, we see that $\tilde{\gamma}<\min\left\{\frac{t_2-t_1}{4},\frac{t_1}{2},\frac{T-t_2}{2}\right\}$ will do. To help understanding, ϕ is now a smooth approximation of $\mathbf{1}_{(t_1,t_2)}$. By direct calculation, it is seen that

$$\phi'(t)=\omega_{\tilde{\gamma}}(t-t_1)-\omega_{\tilde{\gamma}}(t-t_2),$$

and

$$\int_0^T|\phi'(t)|\mathrm{d}t=2$$

$$\int_0^T\phi(t)\mathrm{d}t=t_2-t_1\le T.$$

Taking the limit $\tilde{\gamma}\to 0$ in (B.11) yields

$$\int_0^{2\pi}|v_\epsilon(x,t_2)-v_\delta(x,t_2)|\mathrm{d}x \le \int_0^{2\pi}|v_\epsilon(x,t_1)-v_\delta(x,t_1)|\mathrm{d}x + C|u_0|_{BV}\left(\rho+\frac{\epsilon+\delta}{\rho}\right).$$

Letting t_1 approach 0 and using that v_ϵ and v_δ share initial data results in

$$||v_\epsilon(\cdot,t_2)-v_\delta(\cdot,t_2)||_1 \le C|u_0|_{BV}\left(\rho+\frac{\epsilon+\delta}{\rho}\right),$$

and taking $\rho=\sqrt{\epsilon+\delta}$, we at long last end up at the desired result.

□

Remark 1. The technique used in this proof may seem overly complicated for our purposes, and additionally it does not coincide well with the intuition that $v_\delta \to v_\epsilon$ as $\delta \to \epsilon$ in some sense. However, more straightforward approaches will require an estimate of $\frac{\partial^2 v_\epsilon}{\partial^2 x^2}$ in L^∞ that is independent of ϵ.

For simplicity, assume that $\epsilon>\delta$ and take the difference of (2.11) for v_ϵ and v_δ. Then

$$\begin{aligned}\frac{\partial}{\partial t}(v_\epsilon-v_\delta) = \mathcal{L}[\Phi(v_\epsilon)-\Phi(v_\delta)]+\delta\frac{\partial^2}{\partial x^2}(v_\epsilon-v_\delta)\\ +(\epsilon-\delta)\frac{\partial^2 v_\epsilon}{\partial x^2}.\end{aligned}$$

If we now multiply by $\mathrm{sgn}_\rho(v_\epsilon-v_\delta)$, where sgn_ρ is the standard mollification of sgn, and integrate over $(0,2\pi)$ we get

$$\begin{aligned}\left\langle\frac{\partial}{\partial t}(v_\epsilon-v_\delta),\mathrm{sgn}_\rho(v_\epsilon-v_\delta)\right\rangle &= \left\langle\mathcal{L}[\Phi(v_\epsilon)-\Phi(v_\delta)],\mathrm{sgn}_\rho(v_\epsilon-v_\delta)\right\rangle\\ &+\delta\left\langle\frac{\partial^2}{\partial x^2}(v_\epsilon-v_\delta),\mathrm{sgn}_\rho(v_\epsilon-v_\delta)\right\rangle\\ &+(\epsilon-\delta)\left\langle\frac{\partial^2 v_\epsilon}{\partial x^2},\mathrm{sgn}_\rho(v_\epsilon-v_\delta)\right\rangle.\end{aligned}$$

Using a similar argument as that in the proof Lemma 2.4 we get in the limit $\rho\to 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}||v_\epsilon-v_\delta||_1 \le (\epsilon-\delta)\left\langle\frac{\partial^2 v_\epsilon}{\partial x^2},\mathrm{sgn}(v_\epsilon-v_\delta)\right\rangle.$$

The trouble comes in estimating the right hand side of this, which we need some sort of estimate of $\frac{\partial^2 v_\epsilon}{\partial x^2}$ that is independent of ϵ for. Or at least an estimate where the dependence on ϵ is made explicit. Regrettably we do not have this sort of estimate at our disposal.

C Software documentation

The code for this project was written mainly in Python, and this appendix is devoted to documenting what constituents were needed in making a working code. An example of how to build a script for running simulations is also provided. But first we need to establish how we can numerically compute the coefficient for the Fourier weighting of the fractional laplacian, C_α.

C.1 Computing Fresnel integral

As already stated we have analytic expressions for C_α when $\alpha \in (0,1]$, but for $\alpha \in (1,2)$, numerical quadrature is needed to approximate the generalized Fresnel integral

$$\int_0^\infty x^{-\alpha} \sin(x)\,\mathrm{d}x. \tag{C.1}$$

In [21], Takuya Ooura and Masatake Mori provide an algorithm for evaluating such integrals, and an implementation in C can be found on Ooura's home page, `http://www.kurims.kyoto-u.ac.jp/~ooura/`. However, the implementation requires an analytic integrand, and ours is singular at $x = 0$. The solution is to split up the integral as

$$\int_0^\infty x^{-\alpha} \sin(x)\,\mathrm{d}x = \underbrace{\int_0^b x^{-\alpha} \sin(x)\,\mathrm{d}x}_{=:I_1} + \underbrace{\int_b^\infty x^{-\alpha} \sin(x)\,\mathrm{d}x}_{=:I_2},$$

for some $b > 0$. In the implementation in this project $b = 1$ was used.

To handle I_1 we do the same as proposed in [5], and use a Taylor series with remainder for the integrand as

$$x^{-\alpha} \sin(x) = \sum_{k=0}^{N-1} (-1)^k \frac{x^{2k+1-\alpha}}{(2k+1)!} + r_N,$$

which then gives that

$$\int_0^b x^{-\alpha} \sin(x)\mathrm{d}x = \sum_{k=0}^{N-1} (-1)^k \frac{b^{2k+2-\alpha}}{(2k+2-\alpha)(2k+1)!} + R_N.$$

This being an alternating series, the remainder term, R_N, is smaller in magnitude then the next term of the series, so

$$|R_N| \leq \frac{b^{2N+2-\alpha}}{(2N+2-\alpha)(2N+1)!},$$

which is for sufficiently large N, depending on b, decreasing. Thus, given an error tolerance level of ϵ, one needs to find N so that

$$b^{2N+2-\alpha}(2N+2-\alpha)(2N+1)! \leq \epsilon. \tag{C.2}$$

With an N satisfying this inequality we may approximate

$$I_1 \approx \sum_{k=0}^{N-1} (-1)^k \frac{b^{2k+2-\alpha}}{(2k+2-\alpha)(2k+1)!}. \tag{C.3}$$

To handle the integral I_2, we use the C code package by Ooura, and more specifically the function

```
void intdeo(double (*f)(double), double a, double omega,
            double eps, double *i, double *err)
```

which is suited for integrals on (a, ∞) with oscillating integrand. Since the code for this project is written in Python, and the code package for evaluating the oscillatory integrals is written in C, we need to create a C extension in Python, to wrap the function with a Python interface. Luckily this is quite manageable, and we will now give a short step-by-step description on how to do it. The reader should also be aware that the amount of tutorials and examples on this subject are ample, and just a web search away.

Step 1: Create .c-file: First we create "oscintmodule.c", the file which will contain the module.

```c
/*
 *                oscintmodule.c
 */

#include <Python.h>
/* This pulls in the Python API and
 *must come before any other include statements*/
#include <math.h>

/* The oscillating integrand: */
double f(double x, double alpha)
{
   return pow(x,-alpha)*sin(x);
}

/*
 *          Provided C-function:
 */
void intdeo(double a, double alpha, double omega,
double eps, double *i, double *err)
{
/* See web page for implementation. Note that
 * we have also changed the interface to take
 * in the double alpha, instead of the
 * function pointer f
 */
}

/*
```

```
 * Function to be called from Python
 */
static PyObject* py_oscillatoryQuadrature(PyObject* self,
                              PyObject* args)
{
    const double b, alpha, omega, eps;
    double i=0, err=256;

        /* Parses the input given in Python and
         * converts to corresponding C types.
         */
    if(!PyArg_ParseTuple(args, "dddd", &b, &alpha,
     &omega, &eps))
        return NULL;

        /* Call the intdeo function */
    intdeo(b, alpha, omega, eps, &i, &err);

    /* Convert back to Python object and return */
    return Py_BuildValue("d", i);
}


/*
 *  Set up methods table:
 */
static PyMethodDef oscintmodule_methods[] = {
    {''oscillatoryQuadrature'', py_oscillatoryQuadrature,
     METH_VARARGS,
      ''Function for evaluating oscillating integrals''},
    {NULL, NULL, 0, NULL}
};

/*
 * Python calls this to let us initialize our module
 */
PyMODINIT_FUNC initoscintmodule(void)
{
    (void) Py_InitModule("oscintmodule",
      oscintmodule_methods);
}
/*
 * NOTE: The function name must be
```

```
    * init[exact name of module].
    */
```

The key part of "oscintmodule.c" is the function "py_oscillatoryQuadrature" that takes in arguments from Python, and then converts them to C before doing what we want on them there and then return a double for Python. See that we also need to create a methods table that links each function in the module to the name we want in the final Python module. Finally, we have a function that initializes the module, which will be called when we compile the file.

Step 2:Compiling and creating necessary links: The next step is to compile oscintmodule.c and create the necessary links so that the oscillatoryQuadrature can be used in a Python script. This can readily be done with Python script as follows:

```
# Setup.py
from distutils.core import setup, Extension

setup(name='oscintmodule', version='1.0', \
   ext_modules=[Extension('oscintmodule',
                                              ['oscintmodule.c'])])
```

All that remains to do is then to run "setup.py" as

```
# python setup.py install
```

in the terminal.

To use the function in Python, all that is required is to import the oscintmodule, like so:

```
import oscintmodule as oscint

# Using oscillatoryQuadrature from C extension:
integral = oscint.oscillatoryQuadrature(1., 1.5, 1., 1E-8)
# Arguments are: start of interval,
#                alpha,
#                frequency,
#                error tolerance, respectively.
```

C.2 Software documentation

What follows is a list of methods, together with a short despriction, that were used in the implementation. For sake of completeness, and short example of a script for running a simulation on the fractional porous medium equation is also added.

physical_to_fourier(u_evals, N): Given the values of a function u at points x_j, $j = 0, \dots, N-1$, returns the discrete Fourier coefficients using FFT. Assumes real input and returns a complex array of length $N/2+1$.

fourier_to_physical(u_coeffs,N): Given the Fourier coefficients of a real function u, returns the evaluations of u at x_j.

nonlinear_coeffs(u_points, N, phi): Given the values of u at x_j, with N nodes, and the nonlinearity Φ, returns the discrete Fourier coefficients of $\Phi(u)$.

nonlinear_coeffs_from_fourier(u_coeffs, N, phi): Given the discrete Fourier coefficients of u, returns the discrete Fourier coefficients of $\Phi(u)$.

get_svv_eps(theta,N): Given SVV parameter θ and number of degrees of freedom, returns ϵ_N according to assumption 4.1.

get_svv_m(theta, N): Given the SVV parameter θ and number of degrees of freedom, returns m_N according to assumption 4.1.

get_svv_qhat(m,N): Given the SVV spectrum limit m_N and degrees of freedom N returns an array of length $N/2+1$ containing the SVV components $\hat{Q}_\xi$ for $0 \leq \xi \leq N/2$.

get_time_derivative(u_coeff, N, phi, qhat, xisq, xial, epsN, c_alpha): Given the discrete Fourier coefficients of u, degrees of freedom N, nonlinearity Φ, SVV components $\hat{Q}_\xi$, the derivative weightings $|\xi|^2$ and $|\xi|^\alpha$, together with the parameters ϵ_N and C_α returns the time derivative as

$$-C_\alpha|\xi|^\alpha\hat{\Phi}_\xi - \epsilon_N|\xi|^2\hat{Q}_\xi\hat{u}_\xi$$

for $0 \leq \xi \leq N/2$.

fourier_svv_RK4(u_points, N, phi, alpha, c_alpha, qhat, dt, epsN): Uses the values of u at x_j and "get_time_derivative(...)" to evolve u one time step of magnitude dt with the explicit fourth order Runge-Kutta method.

get_little_c_alpha(alpha): Given α, returns c_α according to (2.4).

get_theta_alpha(alpha, eps): Given α and an error tolerance "eps", returns an approximation of the Fresnel integral

$$\int_0^\infty x^{-\alpha}\sin(x)\,dx.$$

get_boundary_layer_integral(alpha,b, eps): Given α, end point b and error tolerance ϵ, returns an approximation of I_1 according to (C.3).

get_big_c_alpha(alpha, eps): Given α and error tolerance ϵ, returns $C_\alpha = 2c_\alpha\alpha^{-1}\int_0^\infty x^{-\alpha}\sin(x)$.

oscillatoryQuadrature(b, alpha, omega, eps): Given start of interval b, α, frequency ω and error tolerance ϵ uses the C-extension as previously outlined and approximates I_2.

fejer_approximation(u,N): Given the number of points N and the values of u at the points, returns the value of the discrete Fejer approximation at the same points.

vallee_poussin_approximation(u,N): Given the number of points N and the values of u at the points, returns the value of the discrete de la Vallée Poussin approximation at the same points.

Lastly, we give the promised code example for running a simple simulation of the fractional porous medium equation, which will yield an animation of the solution evolving through time. This is a working example of generating an animation, so there it is unavoidable to have a lot of code that draws attention away for the numerical method itself. For the reader who is foremost interested in that, the pertinent parts lies in the parameter initializations together with the animation function "animate".

```
# Script for numerical simulation of the

# Importing modules:
import numpy as np
import numpy.fft as fft
import fourier
# All functions except
# oscillatoryQuadrature is in the "fourier" module.

import matplotlib.pyplot as plt
import matplotlib.animation as animation

##################
# Parameters:
##################
N = 256 #Dofs

# Initial data:
def initial_data(x):
        return 1.*(np.abs(x-np.pi)<0.5*np.pi)

# Make the function so it works componentwise:
```

```
initial_data = np.vectorize(initial_data)

# Nonlinearity:
m = 2.
def phi(u):
        return np.sign(u)*np.abs(u)**(m)

phi = np.vectorize(phi)

# Fractional laplacian:
alpha = 1.5
eps = 1E-8
c_alpha = fourier.get_big_c_alpha(alpha,eps)

# End time:
T = 0.5

# Delta time:
dt = 1E-3

# SVV data:
theta = 0.5
epsN = fourier.get_svv_eps(theta, N)
mN = fourier.get_svv_m(theta, N)
qhat = fourier.get_svv_qhat(theta, N)

# Get spatial domain
x = 2.*np.pi/N*np.arange(N)

# Initialize data:
u = initial_data(x)

########################
#          ANIMATION:
########################

# Wanted length of animation (in secs):
ani_T = 20.

# Frame rate:
fps = 30

# Number of frames:
```

```
num_frames = int(fps*ani_T)

# Number of time steps per frame:
time_steps = int(T/(dt*num_frames))

# Start figure:
fig = plt.figure()
ax = fig.add_subplot(111, autoscale_on=False,
        xlim=(0.,2*np.pi), ylim=(min(u)-0.1, max(u)+0.5))

# Initialize line to be drawn:
line, = ax.plot([], [], 'b-', lw=1.5)

# Initialize text saying elapsed time in animation:
time_text = ax.text(0.02, 0.95, '', transform=ax.transAxes)

# Initialize time:
t = 0.

# And now to create functions to be passed into Matplotlib's
# animation interface:

def init():
        ''''''Initialize animation''''''
        line.set_data([],[])
        time_text.set_text('')
        return line, time_text

def animate(i):
        ''''''Perform animation step''''''
        # Variables changed in this step needs
        # to be declared global:
        global u, t

        # Do some time steps:
        for i in range(time_step):
                t += dt
                u = fourier.fourier_svv_RK4(u, N, phi, alpha,
                                c_alpha, qhat, dt, epsN)

        # Update figure data:
        line.set_data(x,u)
        time_text.set_text('time=%.4f' %t)
```

```
        return line, time_text


# Do the animation:
ani = animation.FuncAnimation(fig, animate, frames=num_frames,
                              blit=True, init_func=init)

# Prepare for writing animation to file
# (Should check which writer your system is
 using as default):
mywriter = animation.AVConWriter(fps=fps)

# Save animation:
ani.save('fourier_fraclap_pme1d.mp4', writer=mywriter,
                extra_args = ['-vcodec', 'libx264'])
```

References

[1] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang. *Spectral Methods: Fundamentals in Single Domains*. Springer Verlag, 2010.

[2] Gui Qiang Chen, Qiang Du, and Eitan Tadmor. Spectral viscosity approximations to multidimensional scalar conservation laws. *Mathematics of Computation*, 61(204): 629–643, 1993.

[3] Simone Cifani. *On nonlinear fractional convection-diffusion equations*. PhD thesis, PhD Thesis 2011: 282, NTNU, 2011.

[4] Simone Cifani and Espen Jakobsen. On the spectral vanishing viscosity method for periodic fractional conservation laws. *Mathematics of Computation*, 82(283):1489–1514, 2013.

[5] Stein-Olav Hagen Davidsen. Nonlinear integro-differential equations: Numerical solutions by using spectral methods. Master's thesis, NTNU, 2013.

[6] Arturo de Pablo, Fernando Quirós, Ana Rodriguez, and Juan Luis Vázquez. A fractional porous medium equation. *Advances in Mathematics*, 226(2):1378–1409, 2011.

[7] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez. A general fractional porous medium equation. *Communications on Pure and Applied Mathematics*, 65(9):1242–1284, 2012.

[8] Hongjie Dong, Dapeng Du, and Dong Li. Finite time singularities and global well-posedness for fractal burgers equations. *Indiana University mathematics journal*, 58 (2):807–821, 2009.

[9] Albert Einstein. *Investigations on the Theory of the Brownian Movement*. Courier Corporation, 1956.

[10] Jørgen Endal, Espen R. Jakobsen, and Felix del Toso. Uniqueness and existence for very general nonlocal equations of porous medium type. Preprint, 2015.

[11] Lawrence C Evans. *Partial differential equations*. Providence, Rhode Land: American Mathematical Society, 1998.

[12] Claude Gasquet and Patrick Witomski. *Fourier analysis and applications: filtering, numerical computation, wavelets*, volume 30. Springer Science & Business Media, 1999.

[13] Dorothee Haroske and Hans Triebel. *Distributions, Sobolev spaces, elliptic equations*. European Mathematical Society, 2008.

[14] Helge Holden and Nils Henrik Risebro. *Front tracking for hyperbolic conservation laws*, volume 152. Springer Science & Business Media, 2011.

[15] Alexander Kiselev, Fedor Nazarov, and Roman Shterenberg. Blow up and regularity for fractal burgers equation. *arXiv preprint arXiv:0804.3549*, 2008.

[16] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva. *Linear and quasi-linear equations of parabolic type*, volume 23. American Mathematical Soc., 1968.

[17] Yvon Maday and Alfio Quarteroni. Legendre and chebyshev spectral approximations of burgers' equation. *Numerische Mathematik*, 37(3):321–332, 1981.

[18] Yvon Maday and Eitan Tadmor. Analysis of the spectral vanishing viscosity method for periodic conservation laws. *SIAM Journal on Numerical Analysis*, 26(4):854–870, 1989.

[19] Yvon Maday, Sidi M Ould Kaber, and Eitan Tadmor. Legendre pseudospectral viscosity method for nonlinear conservation laws. *SIAM Journal on Numerical Analysis*, 30(2):321–342, 1993.

[20] Harsh Mehta. The l1 norms of de la vallée poussin kernels. *Journal of Mathematical Analysis and Applications*, 422(2):825–837, 2015.

[21] Takuya Ooura and Masatake Mori. The double exponential formula for oscillatory functions over the half infinite interval. *Journal of Computational and Applied Mathematics*, 38(1):353–360, 1991.

[22] Alfio Quarteroni. *Numerical models for differential problems*, volume 2. Springer Science & Business Media, 2010.

[23] S. Schochet. The rate of convergence of spectral-viscosity methods for periodic scalar conservation laws. *SIAM Journal on Numerical Analysis*, 27(5):1142–1159, 1990. doi: 10.1137/0727066. URL `http://dx.doi.org/10.1137/0727066`.

[24] Eitan Tadmor. Convergence of spectral methods for nonlinear conservation laws. *SIAM Journal on Numerical Analysis*, 26(1):30–44, 1989.

[25] Eitan Tadmor. Total variation and error estimates for spectral viscosity approximations. *Mathematics of Computation*, 60(201):245–256, 1993.

[26] Luc Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics, Heriot-Watt symposium*, volume 4, pages 136–211. Pitman, 1979.

[27] Juan Luis Vázquez. *The porous medium equation: mathematical theory*. Oxford University Press, 2007.