NTNU – Trondheim
Norwegian University of
Science and Technology

Path Algebras of Coverings and Twisted Tensor Products

Tobias Dysthe Unneland

Master of Science in Physics and Mathematics
Submission date: June 2015
Supervisor: Øyvind Solberg, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Abstract

Given a quiver with relations (Γ_2, ρ_2) and its covering quiver (Γ_1, ρ_1) with respect to a subgroup H in a group G, we show that, for a field K, the path algebra $K[\Gamma_1]/\langle\rho_1\rangle$ is isomorphic to a twisted tensor product of the path algebra over the quiver with one vertex for each right coset in G/H and no arrows and the path algebra $K[\Gamma_2]/\langle\rho_2\rangle$.

1 Introduction

The purpose of this thesis is to elaborate further on the work done in [1] by explaining central concepts in greater detail and expanding proofs. The main theorem in [1] describes the path algebra of the covering Γ_1 with respect to a subgroup H in a group G of the quiver Γ_2 over a field K in terms of the twisted tensor product of Γ_2 and a quiver consisting of vertices corresponding to right cosets in G/H. Our main goal is to extend this theorem to include quivers with relations. Further, in [3] it is shown that taking Hochschild cohomology commutes with twisted tensor products of group graded algebras when only considering certain parts of the algebras. We show that one attempt to make an analogue better suited to our setting does not hold.

To summarize section by section: in section 2, we define concepts such as the tensor product and associative algebra and briefly discuss some basic properties. The tensor product of two algebras is then made into an algebra by defining multiplication using a twisting map. The rest of the section gives conditions for when this tensor product algebra is associative.

In Section 3, the twisted tensor product is defined for algebras with units and it is shown that all twisted tensor products have the form of a tensor product of two algebras. Viewing twisted tensor products in this way is better suited for generalizing to the definition of the twisted tensor product to include algebras without units.

In Section 4, we give a more general definition of the twisted tensor product where algebras are not required to be unital. This definition is motivated by considering algebras with adjoined units. We end the section by comparing our definition of the twisted tensor product to the one used in [3].

Section 5 introduces quivers, path algebras, coverings of quivers and quivers with relations. We show how covering quivers can be constructed. Necessary preliminary results are provided in order to prove the main theorem, which states, given a quiver with relations (Γ_2, ρ_2), it's covering quiver (Γ_1, ρ_1) with respect a subgroup H in a group G and the quiver $\Sigma_{G/H}$ which has one vertex for each right coset of G/H and no arrows, then, for a field K, the path algebra $K[(\Gamma_1, \rho_1)]$ is isomorphic to the twisted tensor product of the path algebras $K[\Sigma_{G/H}]$ and $K[(\Gamma_2, \rho_2)]$.

Finally, in section 6, the Hochschild cohomology is introduced and we show how it can be computed. The purpose the section is to discuss whether or not an analogue of Theorem (4.7) in [3] more suited to our twisted tensor product is possible. We suggest an analogue and provide a counterexample to show that this does not hold.

2 Tensor product algebra

In this section we introduce the tensor product of two algebras and make this into an algebra by defining a twisting map. The rest of the section aims to give conditions for when the tensor product algebra of two algebras with a twisting map is an associative algebra. We begin by defining the tensor product and give some basic facts which are needed for later.

Definition 2.1. Let M and N be a right and a left R-module, respectively, over a ring R with 1. The *tensor product* of M and N is an abelian group $M \otimes_R N$ together with a bilinear map $\phi\colon M \times N \to M \otimes_R N$ such that for any abelian group P with a bilinear map $f\colon M \times N \to P$, there exist a unique homomorphism $f'\colon M \otimes_R N \to P$ such that the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[d, "\phi"] \arrow[dr, "f"] & \\ M \otimes_R N \arrow[r, "f'"] & P \end{tikzcd}$$

It can be shown that the tensor product is exactly the quotient group $Z(M,N)/Y(M,N)$, where $Z(M,N)$ is the free abelian group generated by $M \times N$, and $Y(M,N)$ is the subgroup of $Z(M,N)$ generated by elements of the form

$$\begin{aligned} &(m_1 + m_2, n) - (m_1, n) - (m_2, n), \\ &(m, n_1 + n_2) - (m, n_1) - (m, n_2), \\ &(mr, n) - (m, rn), \end{aligned}$$

for all m, m_1, $m_2 \in M$, n, n_1, $n_2 \in N$, and $r \in R$. We will write $m \otimes n$ for $(m,n) + Y(M,N)$. The following are some basic properties of the tensor product

$$\begin{aligned} &(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \\ &m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \\ &mr \otimes n = m \otimes rn, \end{aligned}$$

for all $r \in R$, m, m_1, $m_2 \in M$, and n, n_1, $n_2 \in N$. We also have for families $(M_i)_{i \in I}$ and $(N_j)_{j \in J}$ of right and left R-modules, respectively, the isomorphism

$$\bigoplus_{i \in I} M_i \otimes_R \bigoplus_{j \in J} N_j \cong \bigoplus_{(i,j) \in I \times J} \sum (M_i \otimes_R N_j).$$

Given homomorphisms $f\colon M \to M'$ and $g\colon N \to N'$ between R-modules M, M' and N, N' the tensor product of f and g is defined to be a the map $f \otimes g\colon M \otimes_R N \to M' \otimes_R N'$ given by $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$. This map is itself a homomorphism. For a more thorough discussion of the tensor product see [2].

From here on, R will be a commutative ring with 1. An *algebra* over R is an R-module A together with a bilinear product $A \times A \to A$ given by $(a,b) \mapsto ab$ satisfying

$r(ab) = (ra)b = a(rb)$ for all $r \in R$ and $a, b \in A$. An algebra is associative if $(ab)c = a(bc)$ for all a, b, $c \in A$. By algebra we will mean an associative algebra throughout the text.

Let A and B be R-algebras. The tensor product $A \otimes_R B$ can naturally be made into a left A-module and a right B-module by defining the left action $a' \cdot (a \otimes b) = a'a \otimes b$ and the right action $(a \otimes b) \cdot b' = a \otimes bb'$. These actions are well defined, and since $a \cdot (x \cdot b) = (a \cdot x) \cdot b$, for all $a \in A$, $n \in B$, $x \in A \otimes B$, the actions commute, and $A \otimes_R B$ become a A-B bimodule.

Let $\tau \colon B \otimes_R A \to A \otimes_R B$ be an R-linear map. In order to make $A \otimes_R B$ into an algebra we define multiplication to be the composition

$$(A \otimes_R B) \times (A \otimes_R B) \to A \otimes_R B \otimes_R A \otimes_R B \xrightarrow{\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B} A \otimes_R A \otimes_R B \otimes_R B \to A \otimes_R B,$$

which maps $(a_1 \otimes b_1) \times (a_2 \otimes b_2) \mapsto a_1 \cdot \tau(b_1 \otimes a_2) \cdot b_2$ for all a_1, $a_2 \in A$ and b_1, $b_2 \in B$. The first map is defined by $(x, y) \mapsto x \otimes y$ for x, $y \in A \otimes_R B$, which is clearly bilinear. The second map is the tensor product of linear maps, and so is itself linear. The third map is defined by $a_1 \otimes a_2 \otimes b_1 \otimes b_2 \mapsto a_1 \cdot (a_2 \otimes b_1) \cdot b_2$, which is the composition of the two maps $a' \otimes a \otimes b \otimes b' \mapsto a' \cdot (a \otimes b \otimes b')$ and $a \otimes b \otimes b' \mapsto (a \otimes b) \cdot b'$. These are both the unique homomorphisms from the definition of the tensor product with natural maps and abelian groups, and so are both linear. The composition is thus a multiplication, which satisfies the distributive laws. Note that since the map is bilinear and τ is linear, all elements of $Y(A, B) \times Y(A, B)$ gets mapped to 0, which makes the multiplication well defined. With this multiplication $A \otimes_R B$ become a nonassociative algebra which we will denote by $A \otimes_\tau B$.

Next, we use τ to define a left and right multiplication maps

$$B \times (A \otimes_R B) \to A \otimes_R B \text{ and } (A \otimes_R B) \times A \to A \otimes_R B$$

where the multiplication is denoted by $\cdot_\tau$ and is given by $b' \cdot_\tau (a \otimes b) = \tau(b' \otimes a) \cdot b$ and $(a \otimes b) \cdot_\tau a' = a \cdot \tau(b \otimes a')$, respectively. The left multiplication $\cdot_\tau$ is defined by the composition

$$B \times (A \otimes_R B) \to B \otimes_R (A \otimes_R B) \xrightarrow{\tau \otimes \mathrm{id}_B} A \otimes_R B \otimes_R B \to A \otimes_R B,$$

where the first map is defined by $(b', a \otimes b) \mapsto b' \otimes a \otimes b$ and is easily seen to be bilinear. The third map is given by $(a \otimes b \otimes b') \mapsto (a \otimes b) \cdot b'$, which is the same linear map we encountered for τ, and so the left multiplication $\cdot_\tau$ is a bilinear map, which is also well defined.

Similarly, the right multiplication $\cdot_\tau$ is defined by the composition

$$(A \otimes_R B) \times A \to (A \otimes_R B) \otimes_R A \xrightarrow{\mathrm{id}_A \otimes \tau} A \otimes_R A \otimes_R B \to A \otimes_R B$$

where the first and third maps are given by $(a \otimes b, a') \mapsto a \otimes b \otimes a'$ and $a \otimes a' \otimes b \mapsto a \cdot (a' \otimes b)$. This map is bilinear and well defined by the same reasons as the left multiplication before.

We note that the left multiplication by $\cdot_\tau$ and the right scalar multiplication commute since

$$\begin{aligned}
[b_1 \cdot_\tau (a \otimes b)] \cdot b_2 &= [\tau(b_1 \otimes a) \cdot b] \cdot b_2 \\
&= \tau(b_1 \otimes a) \cdot (bb_2) \\
&= b_1 \cdot_\tau (a \otimes bb_2) \\
&= b_1 \cdot_\tau [(a \otimes b) \cdot b_2].
\end{aligned}$$

Lemma 2.2. *The tensor product $A \otimes_R B$ is a B-B bimodule with left and right actions $\cdot_\tau$ and scalar multiplication, respectively, if the condition*

$$b' \cdot_\tau \tau(b \otimes a) = \tau(b'b \otimes a) \tag{1a}$$

holds for all $a \in A$ and b, $b' \in B$. The converse is true if B has a unit.

Proof. We already know that $A \otimes_R B$ is a right B-module with scalar multiplication. To see that $A \otimes_R B$ is also a left B-module with multiplication $\cdot_\tau$, let $a \in A$ and b, b_1, $b_2 \in B$ and observe, using the condition $(1a)$, that

$$\begin{aligned}
(b_1 b_2) \cdot_\tau (a \otimes b) &= \tau(b_1 b_2 \otimes a) \cdot b \\
&= [b_1 \cdot_\tau \tau(b_2 \otimes a)] \cdot b \\
&= b_1 \cdot_\tau [\tau(b_2 \otimes a) \cdot b] \\
&= b_1 \cdot_\tau [b_2 \cdot_\tau (a \otimes b)].
\end{aligned}$$

Since $\cdot_\tau$ is bilinear, the other conditions for being a left B-module are satisfied.

Conversely, assume that B has a unit and that $A \otimes_R B$ is a B-B bimodule and observe that for all $a \in A$ and b, $b' \in B$

$$\begin{aligned}
\tau(b'b \otimes a) &= \tau(b'b \otimes a) \cdot 1_B \\
&= (b'b) \cdot_\tau (a \otimes 1_B) \\
&= b' \cdot_\tau [b \cdot_\tau (a \otimes 1_B)] \\
&= b' \cdot_\tau [\tau(b \otimes a) \cdot 1_B] \\
&= [b' \cdot_\tau \tau(b \otimes a)] \cdot 1_B \\
&= b' \cdot_\tau [\tau(b \otimes a)].
\end{aligned}$$

□

Naturally, we also have the following result.

Lemma 2.3. *The tensor product $A \otimes_R B$ is an A-A bimodule with left and right actions scalar multiplication and $\cdot_\tau$, respectively, if the condition*

$$\tau(b \otimes a) \cdot_\tau a' = \tau(b \otimes aa') \tag{1b}$$

holds for all a, $a' \in A$ and $b \in B$. The converse is true if A has a unit.

Next we want to explore which conditions we can impose on τ in order to make $A \otimes_\tau B$ an associative algebra. We begin by giving some simple identities.

Lemma 2.4. *Let $x, y \in A \otimes_\tau B$, $a \in A$ and $b \in B$. Then,*

(i) $(a \otimes b)x = a \cdot (b \cdot_\tau x)$ *and* $x(a \otimes b) = (x \cdot_\tau a) \cdot b$,

(ii) $a \cdot (xy) = (a \cdot x)y$ *and* $(xy) \cdot b = x(y \cdot b)$.

Proof. (i) We only argue the first equality and note that the second is essentially the same. Since both multiplication in the algebra and the map $\cdot_\tau$ is bilinear, it is sufficient to prove this for $x = a_1 \otimes b_1$. We get

$$\begin{aligned}
(a \otimes b)x &= (a \otimes b)(a_1 \otimes b_1) \\
&= a \cdot \tau(b \otimes a_1) \cdot b_1 \\
&= a \cdot [b \cdot_\tau (a_1 \otimes b_1)] \\
&= a \cdot (b \cdot_\tau x).
\end{aligned}$$

(ii) Again, it is enough to consider $x = a_1 \otimes b_1$ and $y = a_2 \otimes b_2$. Then, using (i) and the fact that $A \otimes_\tau B$ is an A-B bimodule with scalar multiplication, we get

$$\begin{aligned}
a \cdot (xy) &= a \cdot [(a_1 \otimes b_1)(a_2 \otimes b_2)] \\
&= a \cdot [((a_1 \otimes b_1) \cdot_\tau a_2) \cdot b_2] \\
&= [a \cdot ((a_1 \otimes b_1) \cdot_\tau a_2)] \cdot b_2 \\
&= [a \cdot (a_1 \cdot \tau(b_1 \otimes a_2))] \cdot b_2 \\
&= [(aa_1) \cdot \tau(b_1 \otimes a_2)] \cdot b_2 \\
&= [((aa_1) \otimes b_1) \cdot_\tau a_2] \cdot b_2 \\
&= (aa_1 \otimes b_1)(a_2 \otimes b_2) \\
&= [a \cdot (a_1 \otimes b_1)](a_2 \otimes b_2) \\
&= (a \cdot x)y.
\end{aligned}$$

Again, the other equality is shown similarily. $\square$

When τ satisfies both conditions (1a) and (1b) we simply say that τ satisfies condition (1).

Lemma 2.5. *If τ satisfies condition* (1)*, then*

$$(x \cdot_\tau a)y = x(a \cdot y) \text{ and } (x \cdot b)y = x(b \cdot_\tau y)$$

for $a \in A$, $b \in B$ and x, $y \in A \otimes_\tau B$.

Proof. It is sufficient to look at $x = a_1 \otimes b_1$ and $y = a_2 \otimes b_2$. Using Lemma 2.4 and the commutativity of the A-A bimodule and B-B bimodule we get

$$\begin{aligned}
(x \cdot_\tau a)y &= [(a_1 \otimes b_1) \cdot_\tau a](a_2 \otimes b_2) \\
&= [a_1 \cdot \tau(b_1 \otimes a)](a_2 \otimes b_2) \\
&= ([a_1 \cdot \tau(b_1 \otimes a)] \cdot_\tau a_2) \cdot b_2 \\
&= (a_1 \cdot [\tau(b_1 \otimes a) \cdot_\tau a_2]) \cdot b_2 \\
&= a_1 \cdot \tau(b_1 \otimes aa_2) \cdot b_2 \\
&= (a_1 \otimes b_1)(aa_2 \otimes b_2) \\
&= (a_1 \otimes b_1)[a \cdot (a_2 \otimes b_2)] \\
&= x(a \cdot y).
\end{aligned}$$

The other equality has a similar proof. $\square$

We are now equipped to give our first condition on τ such that $A \otimes_\tau B$ is an associative algebra.

Theorem 2.6. *If τ satisfies* (1)*, then $A \otimes_\tau B$ is an associative algebra.*

Proof. It is enough to consider $x, y, z \in A \otimes_\tau B$ where $x = a_1 \otimes b_1$, $y = a_2 \otimes b_2$, $z = a_3 \otimes b_3$ for $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$. Using Lemmas 2.4 and 2.5 it now follows that

$$\begin{aligned}
(xy)z &= [(a_1 \otimes b_1)(a_2 \otimes b_2)](a_3 \otimes b_3) \\
&= [((a_1 \otimes b_1) \cdot_\tau a_2) \cdot b_2](a_3 \otimes b_3) \\
&= [(a_1 \otimes b_1) \cdot_\tau a_2][b_2 \cdot_\tau (a_3 \otimes b_3)] \\
&= (a_1 \otimes b_1)[a_2 \cdot (b_2 \cdot_\tau (a_3 \otimes b_3))] \\
&= (a_1 \otimes b_1)[(a_2 \otimes b_2)(a_3 \otimes b_3)] \\
&= x(yz),
\end{aligned}$$

so that $A \otimes_\tau B$ is an associative algebra. $\square$

While condition (1) consists of two parts the next theorem shows that we are able to achieve associativity in $A \otimes_\tau B$ with the single symmetric condition

$$[a' \cdot_\tau (b \otimes a)] \cdot_\tau b' = a' \cdot_\tau [(b \otimes a) \cdot_\tau b'], \tag{2}$$

which will be referred to as condition (2). This is the requirement that the right and left $\cdot_\tau$ multiplications commute.

Theorem 2.7. *If τ satisfies condition* (2)*, then $A \otimes_\tau B$ is an associative algebra. The converse is true when A and B both have units.*

Proof. It suffices to consider x, y, $z \in A \otimes_\tau B$ where $x = a_1 \otimes b_1$, $y = a_2 \otimes b_2$, $z = a_3 \otimes b_3$ for a_1, a_2, $a_3 \in A$ and b_1, b_2, $b_3 \in B$. Using Lemma 2.4 we get

$$\begin{aligned}
(xy)z &= [(a_1 \otimes b_1)(a_2 \otimes b_2)](a_3 \otimes b_3) \\
&= [a_1 \cdot (b_1 \cdot_\tau (a_2 \otimes b_2))](a_3 \otimes b_3) \\
&= ([a_1 \cdot (b_1 \cdot_\tau (a_2 \otimes b_2))] \cdot_\tau a_3) \cdot b_3 \\
&= (a_1 \cdot [(b_1 \cdot_\tau (a_2 \otimes b_2)) \cdot_\tau a_3]) \cdot b_3 \\
&= (a_1 \cdot [b_1 \cdot_\tau ((a_2 \otimes b_2) \cdot_\tau a_3)]) \cdot b_3 \\
&= a_1 \cdot ([b_1 \cdot_\tau ((a_2 \otimes b_2) \cdot_\tau a_3)] \cdot b_3) \\
&= a_1 \cdot (b_1 \cdot_\tau [((a_2 \otimes b_2) \cdot_\tau a_3) \cdot b_3]) \\
&= (a_1 \otimes b_1)[((a_2 \otimes b_2) \cdot_\tau a_3) \cdot b_3] \\
&= (a_1 \otimes b_1)[(a_2 \otimes b_2)(a_3 \otimes b_3)] \\
&= x(yz).
\end{aligned}$$

Thus, $A \otimes_\tau B$ is associative.

Conversely, assume that both A and B both have units and let a, $a' \in A$, b, $b' \in B$. Then, using Lemma 2.4 again, we get

$$\begin{aligned}
[b' \cdot_\tau (a \otimes b)] \cdot_\tau a' &= [\tau(b' \otimes a) \cdot b] \cdot_\tau a' \\
&= [1_A \cdot \tau(b' \otimes a) \cdot b)] \cdot_\tau a' \\
&= [(1_a \otimes b')(a \otimes b)] \cdot_\tau a' \\
&= ([(1_a \otimes b')(a \otimes b)] \cdot_\tau a') \cdot 1_B \\
&= [(1_a \otimes b')(a \otimes b)](a' \otimes 1_B) \\
&= (1_a \otimes b')[(a \otimes b)(a' \otimes 1_B)] \\
&= 1_a \cdot (b' \cdot_\tau [(a \otimes b)(a' \otimes 1_B)]) \\
&= b' \cdot_\tau [(a \otimes b)(a' \otimes 1_B)] \\
&= b' \cdot_\tau [(a \cdot \tau(b \otimes a') \cdot 1_B] \\
&= b' \cdot_\tau [(a \cdot \tau(b \otimes a')] \\
&= b' \cdot_\tau [(a \otimes b) \cdot_\tau a'],
\end{aligned}$$

and so (2) holds. $\square$

In the next sections we will see that when both A and B have units and τ satisfies a much simpler condition, then (1) and (2) are equivalent.

3 Twisted tensor product of algebras with units

We now introduce the twisted tensor product and discuss how it relates to the tensor product algebra $A \otimes_\tau B$ from the previous section. In this section all algebras have units and are associative. We begin with a definition.

Definition 3.1. If A and B are algebras, then their *twisted tensor product* is an algebra C along with algebra monomorphisms $\psi_1 \colon A \hookrightarrow C$ and $\psi_2 \colon B \hookrightarrow C$ such that the map $\psi \colon A \otimes_R B \to C$ given by $\psi(a \otimes b) = \psi_1(a)\psi_2(b)$ is a linear isomorphism.

We define the two maps $\zeta_A \colon A \to A \otimes_\tau B$ and $\zeta_B \colon B \to A \otimes_\tau B$ by $\zeta_A(a) = a \otimes 1_B$ and $\zeta_B(b) = 1_A \otimes b$ for all $a \in A$ and $b \in B$. Note that while ζ_A and ζ_B are both clearly linear, they are not necessarily injective. To see this, observe that if $R = \mathbb{Z}$, $A = \mathbb{Z}_6$ and $B = \mathbb{Z}_7$, then $\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_7 = 0$ since if we choose integers a, b such that $6a + 7b = 1$, then

$$\overline{x} \otimes \overline{y} = \overline{x}(6a + 7b) \otimes \overline{y} = \overline{x}6a \otimes \overline{y} + \overline{x}7b \otimes \overline{y} = \overline{0} + \overline{x}b \otimes 7\overline{y} = \overline{0},$$

and so ζ_A and ζ_B are not injective. We want to explore when $A \otimes_\tau B$ along with these two maps is a twisted tensor product of A and B. We begin with a few preliminary results.

Lemma 3.2. *The algebra $A \otimes_\tau B$ has a unit and ζ_A, ζ_B are both algebra homomorphisms if and only if τ satisfies the condition*

$$\tau(b \otimes 1_A) = 1_A \otimes b \text{ and } \tau(1_B \otimes a) = a \otimes 1_B \tag{3}$$

for all $a \in A$ and $b \in B$.

Proof. Suppose first that (3) holds. We observe that $1_A \otimes 1_B$ is the unit for $A \otimes_\tau B$ since

$$\begin{aligned} (1_A \otimes 1_B)(a \otimes b) &= 1_A \cdot \tau(1_B \otimes a) \cdot b \\ &= (1_A \otimes 1_B)(a \otimes b) \\ &= 1_A \cdot \tau(1_B \otimes a) \cdot b \\ &= 1_A \cdot (a \otimes 1_B) \cdot b \\ &= 1_A a \otimes 1_B b \\ &= a \otimes b \end{aligned}$$

and $(a \otimes b)(1_A \otimes 1_B) = a \otimes b$ similarly. Further, we observe that for all $a, a' \in A$ we have

$$\begin{aligned} \zeta_A(a)\zeta_A(a') &= (a \otimes 1_B)(a' \otimes 1_B) \\ &= a \cdot \tau(1_B \otimes a') \cdot 1_B \\ &= a \cdot (a' \otimes 1_B) \\ &= aa' \otimes 1_B \\ &= \zeta_A(aa') \end{aligned}$$

and similarly $\zeta_B(b)\zeta_B(b') = \zeta_B(bb')$ for all $b, b' \in B$ and thus, since both ζ_A and ζ_B are linear, they are also algebra homomorphisms.

Next, suppose that $A \otimes_\tau B$ has a unit, then $\tau(b \otimes 1_A) = 1_A \cdot \tau(b \otimes 1_A) \cdot 1_B = (1_A \otimes b)(1_A \otimes 1_B) = 1_A \otimes b$. The second equality is shown similarly. $\square$

We will refer to the first and second parts of condition (3) as (3a) and (3b), respectively. It turns out that when τ satisfies (3), then conditions (1) and (2) from the previous section are equivalent. We prove this next.

Lemma 3.3. *If τ satisfies* (3)*, then τ satisfies* (1) *if and only if it satisfies* (2)*.*

Proof. Assume first that τ satisfies (1). Using Lemma 3.2 and the fact that $A \otimes_\tau B$ is associative from Theorem 2.6, we get for all a, $a' \in A$ and all b, $b' \in B$

$$\begin{aligned}
[b' \cdot_\tau (a \otimes b)] \cdot_\tau a' &= [1_A \cdot (b' \cdot_\tau (a \otimes b))] \cdot_\tau a') \cdot 1_B \\
&= [(1_A \otimes b')(a \otimes b)] \cdot_\tau a') \cdot 1_B \\
&= [(1_A \otimes b')(a \otimes b)](a' \otimes 1_B) \\
&= (1_A \otimes b')[(a \otimes b)(a' \otimes 1_B)] \\
&= (1_A \otimes b')[(a \otimes b) \cdot_\tau a') \cdot 1_B] \\
&= 1_A \cdot (b' \cdot_\tau [(a \otimes b) \cdot_\tau a')] \\
&= b' \cdot_\tau [(a \otimes b) \cdot_\tau a'].
\end{aligned}$$

Next we assume that τ satisfies (2), then by Theorem 2.7, $A \otimes_\tau B$ is associative and so

$$\begin{aligned}
b' \cdot_\tau \tau(b \otimes a) &= b' \cdot_\tau [1_A \cdot \tau(b \otimes a)] \\
&= b' \cdot_\tau [(1_A \otimes b) \cdot_\tau a] \\
&= [b' \cdot_\tau (1_A \otimes b)] \cdot_\tau a \\
&= [\tau(b' \otimes 1_A) \cdot b] \cdot_\tau a \\
&= [(1_A \otimes b') \cdot b] \cdot_\tau a \\
&= (1_A \otimes b'b) \cdot_\tau a \\
&= 1_A \cdot \tau(b'b \otimes a) \\
&= \tau(b'b \otimes a),
\end{aligned}$$

and so τ satisfies (1). $\square$

We are now ready to show that as long as τ satisfies conditions (2) and (3) and both ζ_A, ζ_B are both injective, then $A \otimes_\tau B$ is a twisted tensor product of A and B.

Theorem 3.4. *Suppose that ζ_A and ζ_B are both injective. Then, $A \otimes_\tau B$ is a twisted tensor product if and only if τ satisfies condition* (2) *and* (3)*.*

Proof. Suppose first that $A \otimes_\tau B$ along with ζ_A and ζ_B is a twisted tensor product. Then, since all algebras in this section are unital and associative, we have from Theorem 2.7 that τ satisfies (2). Further, since the definition of the twisted tensor product make ζ_A and ζ_B algebra homomorphisms we get that τ satisfies (3) from Lemma 3.2.

Conversely, assume that τ satisfies (2) and (3), then $A \otimes_\tau B$ has a unit by Lemma 3.2 and is associative by Theorem 2.7. Further, ζ_A and ζ_B are assumed to be injective and so are both algebra monomorphisms by Lemma 3.2. It only remains to show that

$\zeta\colon A \otimes_R B \to A \otimes_\tau B$ given by $\zeta(a \otimes b) = \zeta_A(a)\zeta_B(b)$ is a linear isomorphism. ζ can be seen to be linear from the definition of the tensor product with natural bilinear maps and abelian groups, where ζ become the unique homomorphism. To see that ζ is one-to-one and onto, observe that $\zeta(a \otimes b) = \zeta_A(a)\zeta_B(b) = (a \otimes 1_B)(1_A \otimes b) = a \cdot \tau(1_B \otimes 1_A) \cdot b = a \cdot (1_A \otimes 1_B) \cdot b = a \otimes b$. It follows that $A \otimes_\tau B$ is the twisted tensor product from the definition. $\square$

In fact, all twisted tensor products of A and B are of the form $A \otimes_\tau B$, where τ satisfies (2) and (3), as we now show.

Theorem 3.5. *If C is a twisted tensor product of A and B, then $C \cong A \otimes_\tau B$ for some τ satisfying* (2) *and* (3).

Proof. We begin by determining τ. Let $\psi_A\colon A \hookrightarrow C$, $\psi_B\colon B \hookrightarrow C$ be the inclusion algebra monomorphisms from A and B, respectively, into C, and let $\psi\colon A \otimes_R B \to C$ be the linear isomorphism from $A \otimes_R B$ into C given by $\psi(a \otimes b) = \psi_A(a)\psi_B(b)$. Define $t\colon B \times A \to A \otimes B$ by $t(b, a) = \psi^{-1}[\psi_A(a)\psi_B(b)]$ which is a linear map since ψ, ψ_A, ψ_B are all linear. From the definition of the tensor product t determines a linear map $\tau\colon B \otimes_R A \to A \otimes_R B$ which is given by $\tau(b \otimes a) = t(b, a)$.

Next, we show that $\psi\colon A \otimes_\tau B \to C$ is an algebra isomorphism. We already know that ψ is a linear isomorphism from the definition of the twisted tensor product. Further, observe that ψ is an A-B bimodule homomorphism since

$$\begin{aligned}
\psi(a' \cdot (a \otimes b) \cdot b') &= \psi(a'a \otimes bb') \\
&= \psi_A(a'a)\psi_B(bb') \\
&= \psi_A(a')\psi_A(a)\psi_B(b)\psi_B(b') \\
&= \psi_A(a')\psi(a \otimes b)\psi_B(b') \\
&= a'\psi(a \otimes b)b'.
\end{aligned}$$

To see that ψ is an algebra homomorphism we observe that

$$\begin{aligned}
\psi((a \otimes b)(a' \otimes b')) &= \psi(a \cdot \tau(b \otimes a') \cdot b') \\
&= \psi_A(a)\psi(\tau(b \otimes a'))\psi_B(b') \\
&= \psi_A(a)\psi(\psi^{-1}(\psi_B(b)\psi_A(a')))\psi_B(b') \\
&= \psi_A(a)\psi_B(b)\psi_A(a')\psi_B(b') \\
&= \psi(a \otimes b)\psi(a' \otimes b').
\end{aligned}$$

Since $A \otimes_\tau B \cong C$, it follows that $A \otimes_\tau B$ is associative, since all algebras are assumed to be associative, and so by Theorem 2.7 τ satisfies (2). To see that τ also satisfies (3), we observe that $\psi(\tau(b \otimes 1_A)) = \psi(\psi^{-1}(\psi_B(b)\psi_A(1_A))) = \psi_B(b)\psi_A(1_A) = \psi(1_A \otimes b)$, which gives $\tau(b \otimes 1_A) = 1_A \otimes b$, and similarly $\tau(1_B \otimes a) = a \otimes 1_B$. $\square$

4 General twisted tensor product of algebras

In the previous section we required all algebras to be unital. In this section we want to expand the definition of the twisted tensor product to a more general setting, where algebras do not necessarily have units. We begin by studying what can be learnt by adjoining units. If A is an algebra over a ring R, then the algebra with unit adjoined, denoted by $\hat{A}$, is $\hat{A} = R \oplus A$. Note that $\hat{A}$ is an R-module, since it is the direct sum of R-modules and that $\hat{A}$ contains A as a submodule. In order to make $\hat{A}$ into an algebra we define multiplication to be $(r_1 + a_1)(r_2 + a_2) = r_1 r_2 + r_1 a_2 + a_1 r_2 + a_1 a_2$. The unit in $\hat{A}$ coincides with the unit in R, and so, if A is unitary, then the units of A and $\hat{A}$ are not the same.

If A and B are R-algebras and $\tau \colon B \otimes A \to A \otimes B$ is a R-linear map, then we want to make an extension $\widehat{\tau} \colon \hat{B} \otimes_R \hat{A} \to \hat{A} \otimes_R \hat{B}$ of τ that satisfies condition (3) from the previous section. Since $\hat{A} \otimes_R \hat{B} = (R \oplus A) \otimes_R (R \oplus B) = (R \otimes_R R) \oplus (R \otimes_R B) \oplus (A \otimes_R R) \oplus (A \otimes_R B)$ we observe that is there is a natural way to do this. It is clear that $\widehat{\tau}$ satisfies (3) if and only if

$$\begin{aligned}
\widehat{\tau}(1_R \otimes 1_R) &= 1_R \otimes 1_R, \\
\widehat{\tau}(1_R \otimes a) &= a \otimes 1_R, \\
\widehat{\tau}(b \otimes 1_R) &= 1_R \otimes b,
\end{aligned}$$

and because $\widehat{\tau}$ is to be an extension of τ, we want it to be R-linear as well as satisfy $\widehat{\tau}(a \otimes b) = \tau(a \otimes b)$ for $a \in A$ and $b \in B$. We get

$$\begin{aligned}
\widehat{\tau}((r_2 + b) \otimes (r_1 + a)) &= \widehat{\tau}((r_2 \otimes r_1) + (r_2 \otimes a) + (b \otimes r_1) + (b \otimes a)) \\
&= \widehat{\tau}(r_2 \otimes r_1) + \widehat{\tau}(r_2 \otimes a) + \widehat{\tau}(b \otimes r_1) + \widehat{\tau}(b \otimes a) \\
&= r_2 \cdot \widehat{\tau}(1_R \otimes 1_R) \cdot r_1 + r_2 \cdot \widehat{\tau}(1_R \otimes a) + \widehat{\tau}(b \otimes 1_R) \cdot r_1 + \tau(b \otimes a) \\
&= r_2 \cdot (1_R \otimes 1_R) \cdot r_1 + r_2 \cdot (a \otimes 1_R) + (1_R \otimes b) \cdot r_1 + \tau(b \otimes a) \\
&= r_2 1_R \otimes 1_R r_1 + r_2 a \otimes 1_R + 1_R \otimes b r_1 + \tau(b \otimes a) \\
&= r_1 \otimes r_2 + a \otimes r_2 + r_1 \otimes b + \tau(b \otimes a).
\end{aligned}$$

We observe that $\widehat{\tau}$ satisfies (3a) since

$$\begin{aligned}
\widehat{\tau}((r + b \otimes 1_{\hat{A}}) &= \widehat{\tau}(r \otimes 1_R) + \widehat{\tau}(b \otimes 1_R) \\
&= 1_R \otimes r + 1_R \otimes b \\
&= 1_R \otimes (r + b) \\
&= 1_{\hat{A}} \otimes (r + b).
\end{aligned}$$

Similarly (3b) is satisfied.

From $\hat{A} \otimes_R \hat{B} = (R \otimes_R R) \oplus (R \otimes_R B) \oplus (A \otimes_R R) \oplus (A \otimes_R B)$ and the fact that $A \otimes_R B$ is a direct summand, it follows that it is a subalgebra of $\hat{A} \otimes_R \hat{B}$.

Next, we want to explore if and when $\widehat{\tau}$ satisfies conditions (1) and (2). We begin by observing that $\widehat{\tau}$ fits well with (1). For this we need the following simple identity.

Lemma 4.1. *$x \cdot_{\widehat{\tau}} r = rx$ for $r \in R$ and $x \in \hat{A} \otimes_R \hat{B}$.*

Proof. It is enough to consider $x = (r_1 \otimes r_2) + (r_1 \otimes b) + (a \otimes r_2) + (a \otimes b)$ for r_1, $r_2 \in R$, $a \in A$ and $b \in B$. We get

$$\begin{aligned} x \cdot_{\widehat{\tau}} r &= ((r_1 \otimes r_2) + (r_1 \otimes b) + (a \otimes r_2) + (a \otimes b)) \cdot_{\widehat{\tau}} r \\ &= r_1 \cdot \widehat{\tau}(r_2 \otimes r) + r_1 \cdot \widehat{\tau}(b \otimes r) + a \cdot \widehat{\tau}(r_2 \otimes r) + a \cdot \widehat{\tau}(b \otimes r) \\ &= r_1 \cdot (r \otimes r_2) + r_1 \cdot (r \otimes b) + a \cdot (r \otimes r_2) + a \cdot (r \otimes b) \\ &= r \cdot (r_1 \otimes r_2) + r \cdot (r_1 \otimes b) + r \cdot (a \otimes r_2) + r \cdot (a \otimes b) \\ &= rx \end{aligned}$$

It follows that the identity holds. □

Using this we are now able to prove the following theorem.

Theorem 4.2. *The linear map $\widehat{\tau}$ satisfies* (1) *if and only if τ satisfies* (1).

Proof. Firstly, assume $\widehat{\tau}$ satisfies (1), then as $\widehat{\tau}$ is just an extension of τ we get $\tau(b \otimes a) \cdot_\tau a' = \widehat{\tau}(b \otimes a) \cdot_{\widehat{\tau}} a' = \widehat{\tau}(b \otimes aa') = \tau(b \otimes aa')$, so that τ satisfies (1a). Similarly, τ satisfies (1b).

Conversely, if τ satisfies (1), then using Lemma 4.1 we can see that

$$\begin{aligned} \widehat{\tau}[(s+b) \otimes (r+a)] \cdot_{\widehat{\tau}} (r'+a') &= \widehat{\tau}[(s+b) \otimes (r+a)] \cdot_\tau r' + \widehat{\tau}[(s+b) \otimes (r+a)] \cdot_{\widehat{\tau}} a' \\ &= r'\widehat{\tau}[(s+b) \otimes (r+a)] + \widehat{\tau}(s \otimes r + s \otimes a + b \otimes r + b \otimes a) \cdot_{\widehat{\tau}} a' \\ &= r'\widehat{\tau}[(s+b) \otimes (r+a)] + (r \otimes s + a \otimes s + r \otimes b + \tau(b \otimes a)) \cdot_{\widehat{\tau}} a' \\ &= r'\widehat{\tau}[(s+b) \otimes (r+a)] + r\widehat{\tau}(s \otimes a') + a\widehat{\tau}(s \otimes a') + r\tau(b \otimes a') \\ &\quad + \tau(b \otimes a) \cdot_{\widehat{\tau}} a' \\ &= r'\widehat{\tau}[(s+b) \otimes (r+a)] + ra' \otimes s + aa' \otimes s + \tau(b \otimes ra') + \tau(b \otimes aa') \\ &= r'\widehat{\tau}[(s+b) \otimes (r+a)] + (ra' + aa') \otimes s + \tau(b \otimes ra' + aa') \\ &= r'\widehat{\tau}[(s+b) \otimes (r+a)] + \widehat{\tau}(s \otimes (ra' + aa')) + \widehat{\tau}(b \otimes ra' + aa') \\ &= \widehat{\tau}[(s+b) \otimes (r'r + r'a)] + \widehat{\tau}((s+b) \otimes (ra' + aa')) \\ &= \widehat{\tau}[(s+b) \otimes (r'r + r'a + ra' + aa')] \\ &= \widehat{\tau}[(s+b) \otimes (r'+a)(r+a')], \end{aligned}$$

so $\widehat{\tau}$ satisfies (1a). In a similar manner $\widehat{\tau}$ also satisfies (1b). □

From this theorem we get the following facts about τ.

Corollary 4.3. *If τ satisfies* (1)*, then it also satisfies* (2).

Proof. Since τ satisfies (1), then by the previous theorem $\widehat{\tau}$ satisfies (1), and since $\widehat{\tau}$ also satisfies (3), we get from Lemma 3.3 that $\widehat{\tau}$ satisfies (2). Since $\widehat{\tau}$ is an extension of τ, then if we restrict to τ, we get that τ satisfies (2). □

Corollary 4.4. *τ satisfies* (1) *if and only if $\widehat{\tau}$ satisfies* (2)*.*

Proof. By Lemma 3.3 conditions (1) and (2) are equivalent for $\widehat{\tau}$, and so, if τ satisfies (1), then the result follows from Theorem 4.2. □

If either A or B already has a unit, then we may restrict $\widehat{\tau}$ to the maps $\widehat{\tau}_1\colon \hat{B}\otimes A \to A\otimes \hat{B}$ and $\widehat{\tau}_2\colon B\otimes \hat{A} \to \hat{A}\otimes B$, respectively. We observe that $\widehat{\tau}_1$ satisfies (3) if and only if τ satisfies (3a), since $\widehat{\tau}_1((r+b)\otimes 1_A) = 1_A\otimes r + \tau(b\otimes 1_A)$ and likewise $\widehat{\tau}_2$ satisfies (3) if and only if τ satisfies (3b). Further, since $\widehat{\tau}_1$ and $\widehat{\tau}_2$ are extensions of τ and restrictions of $\widehat{\tau}$, we immediately get the following result from Theorem 4.2.

Corollary 4.5. *Either all of τ, $\widehat{\tau}_1$, $\widehat{\tau}_2$, $\widehat{\tau}$ satisfy* (1) *or none of them do.*

Also, the same argument used in Corollary 4.4 applies to $\widehat{\tau}_1$ and $\widehat{\tau}_2$ as well.

Corollary 4.6. *τ satisfies* (1) *if and only if $\widehat{\tau}_1$ satisfies* (2)*.*

Corollary 4.7. *τ satisfies* (1) *if and only if $\widehat{\tau}_2$ satisfies* (2)*.*

If A and B are algebras, let A_1 be $\hat{A}$ if A does not have a unit and A otherwise, and define B_1 similarly. Next, let $\sigma\colon B_1\otimes_R A_1 \to A_1\otimes_R B_1$ be the appropriate linear map among τ, $\widehat{\tau}_1$, $\widehat{\tau}_2$, $\widehat{\tau}$. From Theorem 3.5 in the previous section we know that all twisted tensor products of A_1 and B_1 are of the form $A_1\otimes_\sigma B_1$ where σ satisfies (1) and (3). Using Corollaries 4.4, 4.5, 4.6 and 4.7 we are able to express this condition in terms of τ.

Corollary 4.8. *σ satisfies conditions* (2) *and* (3) *if and only if τ satisfies the conditions:*

(1) *τ satisfies* (1)*;*

(3a) *If A has a unit, then τ satisfies* (3a)*;*

(3b) *If B has a unit, then τ satisfies* (3b)*.*

Since $A\otimes_\tau B$ is a subalgebra of $A_1\otimes_\sigma B_1$, this corollary is our justification for making the following definition of the twisted tensor product to include algebras without units.

Definition 4.9. Let A and B be R-algebras. A linear map $\tau\colon B\otimes_R A \to A\otimes_R B$ is a *twisting map* if it satisfies conditions (1), (3a) and (3b). A *twisted tensor product* of A and B is an algebra of the form $A\otimes_\tau B$ where τ is a twisting map.

In [3] Bergh and Oppermann define a twisted tensor product for graded algebras. We will briefly discuss how their definition compares to the one we gave in Definition 4.9, but first we define what is means for an algebra to be graded (by an abelian group).

Definition 4.10. Let A be an abelian group and Λ an associative k-algebra. We say that Λ is an *A-graded algebra* if $\Lambda = \bigoplus\limits_{a\in A} \Lambda_a$ as k-modules and $\Lambda_a\cdot\Lambda_{a'} \subseteq \Lambda_{a+a'}$. An element λ in the factor Λ_a is called a homogeneous elements of degree a, and we denote the degree of λ by $|\lambda|$.

Note that if the A-graded algebra Λ has a unit, then $|1_\Lambda| = 1_A = 0$ since for all $\lambda \in \Lambda_a$ we have $1_\Lambda \cdot \lambda \in \Lambda_a$.

Definition 4.11. Let A and B be abelian groups, Λ an A-graded algebra and Γ a B-graded algebra. If $k^\times$ denotes the multiplicative group of nonzero elements in k, let $t\colon A \otimes_{\mathbb{Z}} B \to k^\times$ be a homomorphism of abelian groups. We write $t(a \otimes b) = t^{\langle a|b\rangle}$. The twisted tensor product of Λ and Γ, denoted by $\Lambda \otimes_t \Gamma$, is the tensor product algebra with multiplication

$$(\lambda \otimes \gamma)(\lambda' \otimes \gamma') = t^{\langle|\lambda'||\gamma|\rangle} \lambda\lambda' \otimes \gamma\gamma'$$

for all λ, $\lambda' \in \Lambda$, γ, $\gamma' \in \Gamma$.

Clearly, as this twisted tensor product is defined on graded algebras only, the twisted tensor product from Definition 4.9 does not in general satisfy the conditions of being a twisted tensor product in Definition 4.11. However, the opposite is true, as we now show.

For abelian groups A and B, let Λ be an A-graded algebra and Γ a B-graded algebra. It is clear that $\Lambda \otimes_\tau \Gamma$ and $\Lambda \otimes_t \Gamma$ are isomorphic as k-vector spaces with linear isomorphism $\phi\colon \Lambda \otimes_\tau \Gamma \to \Lambda \otimes_t \Gamma$ defined by $\phi(\lambda \otimes \gamma) = \lambda \otimes \gamma$. To avoid confusion we will denote multiplication in $\Lambda \otimes_\tau \Gamma$ by $\cdot^\tau$ (not to be confused with $\cdot_\tau$) and multiplication in $\Lambda \otimes_t \Gamma$ by $\cdot^t$. If the twisted tensor products are to be isomorphic as algebras, then we must have

$$\begin{aligned} \phi((\lambda \otimes \gamma) \cdot^\tau (\lambda' \otimes \gamma')) &= \phi(\lambda \otimes \gamma) \cdot^t \phi(\lambda' \otimes \gamma') \\ &= (\lambda \otimes \gamma) \cdot^t (\lambda' \otimes \gamma') \\ &= t^{\langle|\lambda'||\gamma|\rangle} \lambda\lambda' \otimes \gamma\gamma' \\ &= \lambda(t^{\langle|\lambda'||\gamma|\rangle} \lambda' \otimes \gamma)\gamma', \end{aligned}$$

and since

$$\begin{aligned} \phi((\lambda \otimes \gamma) \cdot^\tau (\lambda' \otimes \gamma')) &= \phi(\lambda\tau(\gamma \otimes \lambda')\gamma') \\ &= \lambda\tau(\gamma \otimes \lambda')\gamma' \end{aligned}$$

for all λ, $\lambda' \in \Lambda$ and γ, $\gamma' \in \Gamma$, we have that

$$\tau(\gamma \otimes \lambda) = t^{\langle|\lambda||\gamma|\rangle} \lambda \otimes \gamma.$$

We need to show that τ satisfies conditions (1), (3a) and (3b) from Definition 4.9.

Suppose Λ has a unit, then, since t is a homomorphism of abelian groups, we get

$$\begin{aligned} \tau(\gamma \otimes 1_\Lambda) &= t^{\langle|1_\Lambda||\gamma|\rangle} 1_\Lambda \otimes \gamma \\ &= t(|1_\Lambda| \otimes |\gamma|) 1_\Lambda \otimes \gamma \\ &= t(0 \otimes |\gamma|) 1_\Lambda \otimes \gamma \\ &= 1_\Lambda \otimes \gamma \end{aligned}$$

and so τ satisfies (3a). Similarly, τ satisfies (3b) when Γ has a unit.

Before we show that τ also satisfies (1), observe that for λ, λ' such that $|\lambda| = a$ and $|\lambda'| = a'$ we have, since t is a homomorphism of abelian groups, that

$$\begin{aligned}
t^{\langle|\lambda|||\gamma|\rangle}t^{\langle|\lambda'|||\gamma|\rangle} &= t(|\lambda| \otimes |\gamma|)t(|\lambda'| \otimes |\gamma|) \\
&= t(|\lambda| \otimes |\gamma| + |\lambda'| \otimes |\gamma|) \\
&= t((|\lambda| + |\lambda'|) \otimes |\gamma|) \\
&= t^{\langle a+a'||\gamma|\rangle} \\
&= t^{\langle|\lambda\lambda'|||\gamma|\rangle}.
\end{aligned}$$

Using this identity we get

$$\begin{aligned}
\tau(\gamma \otimes \lambda) \cdot_\tau \lambda' &= t^{\langle|\lambda|||\gamma|\rangle}(\lambda \otimes \gamma) \cdot_\tau \lambda' \\
&= t^{\langle|\lambda|||\gamma|\rangle}\lambda\tau(\gamma \otimes \lambda') \\
&= t^{\langle|\lambda|||\gamma|\rangle}t^{\langle|\lambda'|||\gamma|\rangle}\lambda\lambda' \otimes \gamma \\
&= t^{\langle|\lambda\lambda'|||\gamma|\rangle}\lambda\lambda' \otimes \gamma \\
&= \tau(\gamma \otimes \lambda\lambda'),
\end{aligned}$$

and so τ satisfies (1).

In what follows we will use the twisted tensor product of Definition 4.9.

5 Quiver algebras and coverings

In this section we introduce quivers, path algebras and covering quivers. Our goal is to relate the structure of the path algebra of a quiver to the path algebra of it's covering quiver. We begin this section by giving some basic definitions. In what follows the ground ring will be a field, which we denote by K.

Definition 5.1. A *quiver* $\Gamma = (V(\Gamma), E(\Gamma))$ is a directed graph, where $V(\Gamma)$ is a set of *vertices* and $E(\Gamma)$ is a set of *arrows* between the vertices. Associated to the quiver are two functions s, $t\colon E(\Gamma) \to V(\Gamma)$, called the *source* and *target* function, respectively, which for an arrow $e\colon i \to j$ going from the vertex i to the vertex j, give $s(e) = i$ and $t(e) = j$.

A *path* γ in Γ is a sequence of arrows $\gamma = e_1e_2\cdots e_n$ where $s(e_{i+1}) = t(e_i)$ for $1 \le i < n$. The length of a path is defined to be the number of arrows in the sequence. The source and target functions are naturally extended to paths by $s(\gamma) = s(e_1)$ and $t(\gamma) = t(e_n)$. We associate with each vertex $v \in V(\Gamma)$ a *trivial path* of length 0, denoted by ϵ_v, for which $s(\epsilon_v) = t(\epsilon_v) = v$. A *map of quivers* $f\colon \Gamma_1 \to \Gamma_2$ is two maps, both denoted by f, $f\colon V(\Gamma_1) \to V(\Gamma_2)$ and $f\colon E(\Gamma_1) \to E(\Gamma_2)$ which preserve sources and targets, that is, $s(f(e)) = f(s(e))$ and $t(f(e)) = f(t(e))$ for all $e \in E(\Gamma_1)$. We say that a quiver is *connected* if for any two vertices $u, v \in V(\Gamma)$ there is a path of undirected arrows going from u to v of finite length. In what follows, given a quiver that is not connected it would suffice to look at each component separately and so we will assume that all quivers are connected.

Definition 5.2. The *path algebra* of Γ over the field K, denoted by $K[\Gamma]$, is the K-algebra which has a basis consisting of all paths in Γ. For two paths $p, q \in \Gamma$ multiplication pq is defined to be composition of paths if $s(q) = t(s)$ and 0 otherwise.

$K[\Gamma]$ has identity if and only if $V(\Gamma)$ is finite, in which case $1_{K[\Gamma]} = \sum_{v \in V(\Gamma)} \epsilon_v$. Note also that the trivial paths are defined to be idempotents, that is $\epsilon_v^2 = \epsilon_v$ for all $v \in V(\Gamma)$.

For a quiver Γ_1 and a vertex $v \in V(\Gamma_1)$, the *link* of v, denoted by L_v, is the disjoint union of all arrows with source v and all arrows with target v, that is,

$$L_v = \{e \in E(\Gamma) \colon s(e) = v\} \sqcup \{e \in E(\Gamma) \colon t(e) = v\}.$$

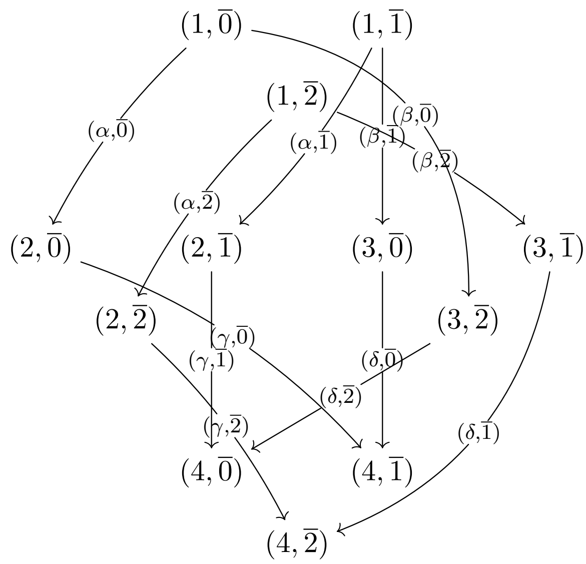
This means that if Γ has an edge that is a loop, that is $s(e) = t(e)$, then this edge contributes two elements to L_v. A *covering map* $f \colon \Gamma_1 \to \Gamma_2$ is a map of quivers that maps the elements of L_v bijectively onto $L_{f(v)}$ for all $v \in V(\Gamma_1)$. We say that Γ_1 is a covering of Γ_2.

We now introduce a simple way of constructing covering quivers. Given a quiver Γ_2 and a group G, let $\lambda \colon E(\Gamma) \to G$ be an arrow labelling function which associates with each arrow in $E(\Gamma)$ an element $g \in G$. The map λ extends naturally to paths $\gamma = e_1 e_2 \cdots e_n$ by letting $\lambda(\gamma) = \lambda(e_1)\lambda(e_2)\cdots\lambda(e_n)$. Further, we define λ in such a way that $\lambda(\epsilon_v) = 1_G$ for all $v \in V(\Gamma)$. For a subgroup H of G let G/H be the right cosets of H in G, and let Γ_1 be the quiver with vertex set $V(\Gamma_1) = V(\Gamma_2) \times G/H$, and arrow set $E(\Gamma_1) = E(\Gamma_2) \times G/H$ with $s(e, Hg) = (s(e), Hg)$ and $t(e, Hg) = (t(e), Hg\lambda(e))$ for all $e \in \Gamma_2$. The covering map $f \colon \Gamma_1 \to \Gamma_2$ is defined to be the functions $f(e, Hg) = e$ for all $e \in E(\Gamma_2)$ and $f(v, Hg) = v$ for all $v \in V(\Gamma_2)$. To see that this is in fact a covering map, let $v \in V(\Gamma_2)$ and $g \in G$ and look at the two sets $A = \{(e, Hg) \mid e \in E(\Gamma_2), s(e) = v\}$ and $B = \{e \mid e \in E(\Gamma_2), s(e) = v\}$. For any $e \in B$, $(e, Hg) \in A$ is such that $f(e, Hg) = e$, and given $(e_1, Hg), (e_2, Hg) \in A$ where $e_1 \neq e_2$, then clearly $f(e_1, Hg) = e_1 \neq e_2 = f(e_2, Hg)$. So f maps A bijectively onto B. The same is easily seen to be true for the sets $\{(e, Hg) \mid e \in E(\Gamma_2), t(e) = v\}$ and $\{e \mid e \in E(\Gamma_2), t(e) = v\}$, and so f is a covering map.

Example 5.3. Let Γ_1 be the quiver

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$

, $G = \mathbb{Z}$ with subgroup $H = 3\mathbb{Z}$ where $\lambda(\alpha) = 3, \lambda(\beta) = 17, \lambda(\gamma) = 103$ and $\lambda(\delta) = 58$. The constructed covering quiver is



A fundamental property of covering quivers is unique path lifting, which is stated and proved in the following lemma.

Lemma 5.4. *Let $f\colon \Gamma_1 \to \Gamma_2$ be a covering map of quivers. Given a path p in Γ_2 and $g \in G$, there exists a unique path $\tilde{p}_{Hg} \in \Gamma_1$ with $s(\tilde{p}_{Hg}) = (s(p), Hg)$ such that $f(\tilde{p}_{Hg}) = p$. Furthermore, $t(\tilde{p}_{Hg}) = (t(p), Hg\lambda(p))$.*

Proof. The proof is by induction on the length of the path p. If p is a trivial path, the result is clear. Assume the property holds for paths up to length k and let p be a path of length $k+1$. Then $p = p'e$ for some path $p' \in \Gamma_1$ of length k and $e \in E(\Gamma_1)$. If $g \in G$, then by the induction hypothesis p' has a unique lift $\tilde{p'}_{Hg}$ with target $t(\tilde{p}'_{Hg}) = (t(p'), Hg\lambda(p'))$. Further, since the length of e is 1, the element $\tilde{e}_{Hg\lambda(p')}$ is the unique lift of e starting at the target of $\tilde{p'}_{Hg}$ with $t(\tilde{e}_{Hg\lambda(p')}) = (t(e), Hg\lambda(p')\lambda(e)) = (t(e), Hg\lambda(p))$. Then $\tilde{p'}\tilde{e}$ is a path in Γ_1 with the desired source and target. Furthermore, since $f(\tilde{p'}\tilde{e}) = f(\tilde{p'})f(\tilde{e}) = p'e = p$ it is a lift, and since f is a covering map, the choice of e is unique making the lift unique also. $\square$

We define $L\colon \Gamma_2 \times G \to \Gamma_1$ to be the lift function which for a path $p \in \Gamma_2$ and $g \in G$ gives $L(p, g) = \tilde{p}_{Hg}$. We naturally extend L to k-linear sums of paths $t = \sum_{i=1}^n k_i p_i$ for $k_i \in K$ and $p_i \in \Gamma_2$ by $L(t, g) = \sum_{i=1}^n k_i L(p_i, g)$. Note further that for any path $p = q_2 q_1$ the lift function has the property $L(p, g) = L(q_2, g\lambda(q_1))L(q_1, g)$, as can easily be seen by looking at p as a sequence of arrows.

Next we introduce quivers with relations and extend the notion of coverings to these. A relation ρ on a quiver Γ over K is a K-linear combination of paths in Γ, $\rho = \sum_{i=1}^n k_i p_i$ for $k_i \in K$ and $p_i \in \Gamma$, with $s(p_1) = s(p_2) = \cdots = s(p_n)$ and $t(p_1) = t(p_2) = \cdots = t(p_n)$. We denote the quiver Γ with relation ρ as (Γ, ρ) and the path algebra of (Γ, ρ) is defined to be $K[(\Gamma, \rho)] = K[\Gamma]/\langle\rho\rangle$ where $\langle\rho\rangle$ is the ideal in $K[\Gamma]$ generated by ρ. If $t = \sum_{i=1}^n k_i p_i$

is a k-linear combination of paths in Γ we define the (u, v)-component, denoted by $c_{u,v}$, for vertices $u, v \in \Gamma$ to be the sum $c_{u,v} = \sum_{i=1}^{m} k_i p_i$ where $\{p_i\}_{i=1}^{m}$ is the set of paths p_i in the sum with source u and target v.

If (Γ_1, ρ_1) and (Γ_2, ρ_2) are two quivers with relations we define $f\colon (\Gamma_1, \rho_1) \to (\Gamma_2, \rho_2)$ to be a morphism of quivers with relations if $f\colon \Gamma_1 \to \Gamma_2$ is a regular covering map and $\rho_1 = \{L(t, g) \mid g \in G \text{ and } t \in \rho_2\}$ and for all $s \in \rho_1$ and $u, v \in V(\Gamma_2)$ there exist $u', v' \in V(\Gamma_1)$ such that $f(c_{u',v'}(s)) = c_{u,v}(f(s))$. We note that for ρ_1 to be a relation, we require that given any $s \in \rho_1$ all its paths must have the same source and the same target. For this to be true for arbitrary quivers we need to impose a condition on the weights of arrows in Γ_2 as we show next.

Lemma 5.5. *Given a set of relations ρ_2 on Γ_2, there exists a unique set of relations ρ_1 on Γ_1 such that f is a morphism of quivers with relations if and only if for any $t = \sum_{i=1}^{n} k_i p_i \in \rho_2$ we have $\lambda(p_1) = \lambda(p_2) = \cdots = \lambda(p_n)$.*

Proof. First observe that if f is to be a morphism of quivers with relations, then ρ_1 is completely determined by ρ_2, since by the definition, ρ_1 is just all possible lifts of $t \in \rho_2$. Further, let $\sigma \in \rho_1$ and observe that the source of any paths in σ are equal, since $\sigma = L(t, Hg) = \sum_{i=1}^{n} k_i L(p_i, Hg)$ for some $t = \sum_{i=1}^{n} k_i p_i \in \rho_2$, and $s(L(p_i, Hg)) = (s(p_i), Hg)$ for all $1 \le i \le n$. Next, if $p_i = e_1 e_2 \cdots e_m$ is any path in t, then

$$\begin{aligned} L(p_i, Hg) &= L(e_1 e_2 \cdots e_m, Hg) \\ &= L(e_1, Hg) L(e_2, Hg\lambda(e_1)) \cdots L(e_m, Hg \prod_{k=1}^{m-1} \lambda(e_k)), \end{aligned}$$

and so $t(L(p_i, Hg)) = (t(p_i), Hg \prod_{k=1}^{m} \lambda(e_k))$. It is now clear that if for any $t = \sum_{i=1}^{n} k_i p_i$ we have $\lambda(p_1) = \lambda(p_2) = \cdots = \lambda(p_n)$, then ρ_1 is a relation. Conversely, if ρ_1 is a relation and p_i, p_j are any two paths in t, then $\lambda(p_i) = \lambda(p_j)$ so that their targets coincide. For any $\sigma \in \rho_1$ with $t = f(\sigma) = \sum_{i=1}^{n} k_i p_i$ the second condition is trivially satisfied with $u' = (u, Hg)$ and $v' = (v, Hg\lambda(p_i))$ for any summand p_i in t. $\square$

Before we continue it is useful to note that all elements in $\langle \rho_1 \rangle$ are sums of lifts of elements in $\langle \rho_2 \rangle$ since $L(p, g)L(p', g') = L(pp', g)$ when $Hg\lambda(p) = Hg'$ and 0 otherwise, we have

$$\sum_{i=1}^{n} k_i q_i' s_i q_i'' = \sum_{i=1}^{n} k_i L(p_i', g_i') L(t_i, g_i) L(q_i'', g_i'') = \sum_{i=1}^{n} k_j L(p_j' t_j p_j'', g_j')$$

for paths $q_i', q_i'' \in \Gamma_1$, $p_i', p_i'' \in \Gamma_2$, $s_i \in \rho_1$, $t_i \in \rho_2$, $k_i \in K$, $1 \le i \le n$ and $1 \le j \le m$.

Next we show that lifts in $K[\Gamma_2]/\langle \rho_2 \rangle$ are still unique.

Lemma 5.6. *For paths $\overline{p}$, $\overline{p}' \in K[\Gamma_2]/\langle \rho_2 \rangle$ and g, $g' \in G$ we have*

(a) $\overline{\overline{p}}_{Hg} = \overline{\overline{p}}'_{Hg'} = 0$ *in $K[\Gamma_1]/\langle \rho_1 \rangle$ if and only if $\overline{p} = \overline{p'} = 0$;*

(b) $\overline{\overline{p}}_{Hg} = \overline{\overline{p}}'_{Hg'} \neq 0$ *in $K[\Gamma_1]/\langle \rho_1 \rangle$ if and only if $Hg = Hg'$ and $\overline{p} = \overline{p}'$.*

Proof. Since elements of $\langle \rho_1 \rangle$ are just sums of lifts of elements in $\langle \rho_2 \rangle$ it is clear that (a) holds. To see that (b) holds, first note that if $Hg \neq Hg'$, then $s(\tilde{p}_{Hg}) \neq s(\tilde{p}'_{Hg'})$ and so clearly $\overline{\tilde{p}}_{Hg} \neq \overline{\tilde{p}}'_{Hg'}$. Next, observe that $\tilde{p}_{Hg} - \tilde{p}'_{Hg} = L(p, g) - L(p', g) = L(p - p', g)$, and so it is clear that if $\overline{p} = \overline{p}'$ then $\overline{\tilde{p}}_{Hg} = \overline{\tilde{p}}'_{Hg}$ and conversely if $\overline{p} \neq \overline{p}'$ then $\overline{\tilde{p}}_{Hg} \neq \overline{\tilde{p}}'_{Hg}$. $\square$

For the main theorem we introduce the quiver $\Sigma_{G/H}$ with vertex set $V(\Sigma_{G/H}) = \{Hg \mid g \in G\}$ and no arrows. The algebra $K[\Sigma_{G/H}]$ consists only of trivial paths ϵ_{Hg} which are all orthogonal idempotents. We define a right G-action on $K[\Sigma_{G/H}]$ by $\epsilon_{Hg}g' = \epsilon_{Hgg'}$ and note the following simple fact.

Lemma 5.7. *For all ϵ_{Hg_1}, $\epsilon_{Hg_2} \in K[\Sigma_{G/H}]$ with g_1, $g_2 \in G$ and all $g \in G$, we have $(\epsilon_{Hg_1}\epsilon_{Hg_2})g = (\epsilon_{Hg_1}g)(\epsilon_{Hg_2}g)$.*

Proof. If $Hg_1 \neq Hg_2$, then $(\epsilon_{Hg_1}\epsilon_{Hg_2})g = 0 = \epsilon_{Hg_1g}\epsilon_{Hg_2g}$. Assume $Hg_1 = Hg_2$, then $Hg_1g = Hg_2g$ and so $(\epsilon_{Hg_1}\epsilon_{Hg_2})g = (\epsilon_{Hg_1})g = \epsilon_{Hg_1g} = (\epsilon_{Hg_1g}\epsilon_{Hg_2g}) = (\epsilon_{Hg_1}g)(\epsilon_{Hg_2}g)$. $\square$

For the main theorem we need to apply the arrow labelling function to basis elements of the path algebra $K[\Gamma_2]/\langle \rho_2 \rangle$ and so we define $\lambda^* \colon K[\Gamma_2]/\langle \rho_2 \rangle \to G$ by $\lambda^*(\overline{p}) = \begin{cases} \lambda(p) & \text{if } \overline{p} \neq \overline{0} \\ 1_G & \text{if } \overline{p} = \overline{0} \end{cases}$. To see that this is well defined, let p, $q \in \Gamma_2$ be such that $\overline{p} = \overline{q}$, then $p - q \in \langle \rho_2 \rangle$ and so $p - q = \sum_{i=1}^{n} k_i\gamma_i t_i \gamma'_i$ for $t_i \in \rho_2$, γ_i, $\gamma'_i \in \Gamma_2$, $k_i \in K$ and $1 \leq i \leq n$. For $\overline{p} = \overline{q} \neq \overline{0}$, we know that no subpaths p' of p or q' of q are in ρ_2. If any one of $\gamma_i t_i \gamma'_i$ contains both p and q as summands, then $\lambda(p) = \lambda(q)$, and so consider only the case where this does not occur. Define the three sets $A = \{k_i\gamma_i t_i\gamma'_i \mid k_i\gamma_i t_i\gamma'_i$ contains p as a summand$\}$, $B = \{k_i\gamma_i t_i\gamma'_i \mid k_i\gamma_i t_i\gamma'_i$ contains q as a summand$\}$ and $C = \{k_i\gamma_i t_i\gamma'_i \mid k_i\gamma_i t_i\gamma'_i$ does not contain p or q among it's summands$\}$. If any $k_i\gamma_i t_i\gamma'_i \in C$ shares a summand with both an element of A and B, then $\lambda(p) = \lambda(q)$ so suppose this is not the case. Go through all elements of C and if they share a summand with an element in A or B, then remove it from C and add it to A or B, respectively. Repeat this process until either one element of C shares a summand with both an element of A and B, in which case $\lambda(p) = \lambda(q)$, or no element of C shares a summand with any element of A and B. The latter case cannot occur, since then all elements of A and B would sum to p and q, respectively, so that $\overline{p} = \overline{q} = \overline{0}$.

Finally, our main theorem.

Theorem 5.8. *There is an isomorphism of algebras $K[\Gamma_1]/\langle \rho_1 \rangle \cong K[\Sigma_{G/H}] \otimes_\tau K[\Gamma_2]/\langle \rho_2 \rangle$ where $\tau \colon K[\Gamma_2]/\langle \rho_2 \rangle \otimes K[\Sigma_{G/H}] \to K[\Sigma_{G/H}] \otimes K[\Gamma_2]/\langle \rho_2 \rangle$ is the twisting map defined on the basis elements by $\tau(\overline{p} \otimes \epsilon_{Hg}) = \epsilon_{Hg\lambda(\overline{p})^{-1}} \otimes \overline{p}$ for all $p \in \Gamma_2$ and $\epsilon_{Hg} \in \Sigma_{Hg}$.*

Proof. Let us begin by checking that τ is in fact a twisting map by verifying the defining conditions from Definition 4.9. As K is a field and τ is defined on the basis elements, we know that it is linear and so it is sufficient to verify for the basis elements only.

If $K[\Gamma_2]/\langle\rho_2\rangle$ has a unit, then $V(\Gamma_2)$ is finite and $\overline{1}_{K[\Gamma_2]/\langle\rho_2\rangle} = \sum_{v\in V(K[\Gamma_2]/\langle\rho_2\rangle)} \overline{\epsilon}_v$ so that

$$\begin{aligned}
\tau(\sum \overline{\epsilon}_v \otimes \epsilon_{Hg}) &= \sum \tau(\overline{\epsilon}_v \otimes \epsilon_{Hg}) \\
&= \sum(\epsilon_{Hg\lambda^*(\overline{\epsilon}_v)^{-1}} \otimes \overline{\epsilon}_v) \\
&= \sum(\epsilon_{Hg1_G} \otimes \overline{\epsilon}_v) \\
&= \epsilon_{Hg} \otimes \sum \overline{\epsilon}_v \\
&= \epsilon_{Hg} \otimes \overline{1}_{K[\Gamma_2]/\langle\rho_2\rangle}.
\end{aligned}$$

This verifies (3a).

If $K[\Sigma_{G/H}]$ has a unit, then $V(\Sigma_{G/H})$ is finite and $1_{K[\Sigma_{G/H}]} = \sum_{Hg\in G/H} \epsilon_{Hg}$ and so

$$\begin{aligned}
\tau(\overline{p} \otimes \sum \epsilon_{Hg}) &= \sum \tau(\overline{p} \otimes \epsilon_{Hg}) \\
&= \sum(\epsilon_{Hg\lambda^*(\overline{p})^{-1}} \otimes \overline{p}) \\
&= \sum \epsilon_{Hg\lambda^*(\overline{p})^{-1}} \otimes \overline{p} \\
&= 1_{K[\Sigma_{G/H}]} \otimes \overline{p}.
\end{aligned}$$

This verifies (3b).

For condition (1a), using Lemma 5.7, we get

$$\begin{aligned}
\tau(\overline{p} \otimes \epsilon_{Hg}) \cdot_\tau \epsilon_{Hg'} &= (\epsilon_{Hg\lambda^*(\overline{p})^{-1}} \otimes \overline{p}) \cdot_\tau \epsilon_{Hg'} \\
&= \epsilon_{Hg\lambda^*(\overline{p})^{-1}} \cdot \tau(\overline{p} \otimes \epsilon_{Hg'}) \\
&= \epsilon_{Hg\lambda^*(\overline{p})^{-1}} \epsilon_{Hg'\lambda^*(\overline{p})^{-1}} \otimes \overline{p} \\
&= (\epsilon_{Hg}\epsilon_{Hg'}) \cdot \lambda^*(\overline{p})^{-1} \otimes \overline{p} \\
&= \tau(\overline{p} \otimes \epsilon_{Hg}\epsilon_{Hg'}).
\end{aligned}$$

And finally, we verify condition (1b). In the following we will use the identity $\lambda^*(\overline{p})\lambda^*(\overline{p}') = \lambda^*(\overline{pp'})$ to show that $\epsilon_{Hg\lambda^*(\overline{p})^{-1}\lambda^*(\overline{p}')^{-1}} \otimes \overline{p}'\overline{p} = \epsilon_{Hg(\lambda^*(\overline{p'p}))^{-1}} \otimes \overline{p'p}$. This identity has not been proved. Since the identity holds for λ, the only potential problem occurs when $\overline{p} \neq 0$ and $\overline{p}' \neq 0$ while $\overline{pp'} = 0$, but we note that in this case, since $\overline{pp'} = 0$, the equality holds even if $\lambda^*(\overline{p})\lambda^*(\overline{p}') \neq \lambda^*(\overline{pp}')$. We get

$$\begin{aligned}
\overline{p}' \cdot_\tau \tau(\overline{p} \otimes \epsilon_{Hg}) &= \overline{p}' \cdot_\tau (\epsilon_{Hg\lambda^*(\overline{p})^{-1}} \otimes \overline{p}) \\
&= \tau(\overline{p}' \otimes \epsilon_{Hg\lambda^*(\overline{p})^{-1}})\overline{p} \\
&= (\epsilon_{Hg\lambda^*(\overline{p})^{-1}\lambda^*(\overline{p}')^{-1}} \otimes \overline{p}')\overline{p} \\
&= \epsilon_{Hg(\lambda^*(\overline{p}'\overline{p}))^{-1}} \otimes \overline{p}'\overline{p} \\
&= \tau(\overline{p}'\overline{p} \otimes \epsilon_{Hg}).
\end{aligned}$$

This confirms that τ is a twisting map.

Next, let $\phi\colon K[\Sigma_{G/H}] \otimes_\tau K[\Gamma_2]/\langle\rho_2\rangle \to K[\Gamma_1]/\langle\rho_1\rangle$ be the map defined on the basis elements by $\phi(\epsilon_{Hg} \otimes \overline{p}) = \overline{\tilde{p}}_{Hg}$. We want to show that this is our isomorphism and begin by showing it is an algebra homomorphism. Since K is a field and the map is defined on the basis elements it is linear, and from Lemma 5.6 we immediately get that it is well defined. Further, we have

$$\begin{aligned}
\phi[(\epsilon_{Hg} \otimes \overline{p})(\epsilon_{Hg'} \otimes \overline{p'})] &= \phi[\epsilon_{Hg} \cdot \tau(\overline{p} \otimes \epsilon_{Hg'}) \cdot \overline{p'}] \\
&= \phi[\epsilon_{Hg} \cdot (\epsilon_{Hg'\lambda^*(\overline{p})^{-1}} \otimes \overline{p}) \cdot \overline{p}'] \\
&= \phi[\epsilon_{Hg}\epsilon_{Hg'\lambda^*(\overline{p})^{-1}} \otimes \overline{pp'}].
\end{aligned}$$

This is 0 unless $Hg = Hg'\lambda^*(\bar{p})^{-1}$, $pp' \notin \langle\rho_2\rangle$ and $t(p) = s(p')$, otherwise it is the unique lift $\overline{\widetilde{pp'}_{Hg}}$. On the other hand, $\phi(\epsilon_{Hg} \otimes \overline{p})\phi(\epsilon_{Hg'} \otimes \overline{p}') = \overline{\tilde{p}_{Hg}\tilde{p}'_{Hg'}}$ which again is 0 unless $Hg\lambda^*(\bar{p}) = Hg'$, $pp' \notin \langle\rho_2\rangle$ and $t(p) = s(p')$, otherwise it is $\overline{\widetilde{pp'}_{Hg}}$ by the unique path lift property. It follows that ϕ is an algebra homomorphism.

To show that ϕ is bijective, we verify that the linear map $\psi\colon K[\Gamma_1]/\langle\rho_1\rangle \to K[\Sigma_{G/H}]\otimes_\tau K[\Gamma_2]/\langle\rho_2\rangle$ defined on paths by $\psi(\overline{q}) = \epsilon_{Hg} \otimes \overline{p}$ where $\overline{q} = \overline{L(p,g)}$, is an inverse. First we show that it is an algebra homomorphism. The map ψ is well defined by Lemma 5.6. Further, we have for paths $\overline{q_1}, \overline{q_2} \in K[\Gamma_1]/\langle\rho_1\rangle$ where $\overline{q_1} = \overline{L(p_1, g)}$ and $\overline{q_2} = \overline{L(p_2, g')}$ that $\psi(\overline{q_1 q_2}) = \epsilon_{Hg} \otimes \overline{p_1 p_2}$ and

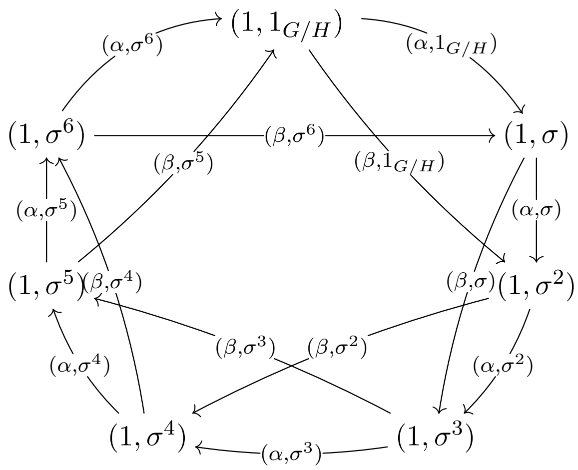
$$\begin{aligned}
\psi(\overline{q_1})\psi(\overline{q_2}) &= (\epsilon_{Hg} \otimes \overline{p_1})(\epsilon_{Hg'} \otimes \overline{p_2}) \\
&= \epsilon_{Hg} \cdot \tau(\overline{p_1} \otimes \epsilon_{Hg'}) \cdot \overline{p_2} \\
&= \epsilon_{Hg}\epsilon_{Hg'\lambda^*(p_1)^{-1}} \otimes \overline{p_1 p_2} \\
&= \epsilon_{Hg} \otimes \overline{p_1 p_2}.
\end{aligned}$$

The last equality follows from the fact that unless $\overline{q_1 q_2} = 0$ (in which case equality holds trivially) we have $s(\overline{q_2}) = (s(\overline{p_2}), Hg') = t(\overline{q_1}) = (t(\overline{p_1}), Hg\lambda^*(\overline{p_1}))$ so that $Hg = Hg'\lambda^*(\overline{p_1})^{-1}$.

Finally, to see that ψ is an inverse of ϕ, observe that an arbitrary basis element $\overline{q} \in K[\Gamma_1]/\langle\rho_1\rangle$ where $\overline{q} = \overline{L(p,g)}$ we have $\phi\psi(\overline{q}) = \phi(\epsilon_{Hg} \otimes \overline{p}) = \overline{\tilde{p}_{Hg}} = \overline{q}$ and for any basis element of $K[\Sigma_{G/H}] \otimes_\tau K[\Gamma_2]/\langle\rho_2\rangle$ we have $\psi\phi(\epsilon_{Hg} \otimes \overline{p}) = \psi(\overline{L(p,g)}) = \epsilon_{Hg} \otimes \overline{p}$. This shows that ϕ is a bijection. □

We illustrate this theorem with an example.

Example 5.9. Let Γ_2 be the quiver $\alpha \circlearrowright 1 \circlearrowleft \beta$ with relations $\rho_2 = \{\alpha^2, \alpha\beta + \beta\alpha, \beta^2\}$. The basis of the path algebra $K[\Gamma_2]/\langle\rho_2\rangle$ consists of the paths $\{\overline{e_1}, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}\}$. Let $G/H = \langle\sigma\rangle$ where σ is such that $\sigma^7 = 1_{G/H}$ so that $G/H \cong \mathbb{Z}_7$. Further, let $\lambda(\alpha) = \sigma$ and $\lambda(\beta) = \sigma^2$, and note that these weights satisfy the condition in Lemma 5.5. By construction, we then get the following covering quiver Γ_1



with relations $\rho_1 = \{(\alpha,\sigma^i)(\alpha,\sigma^{i+1}, (\alpha,\sigma^i)(\beta,\sigma^{i+1}) + (\beta,\sigma^i)(\alpha,\sigma^{i+2}, (\beta,\sigma^i)(\beta,\sigma^{i+2}) \mid$ for $i = 1, 2, ..., 7\ \}$. Theorem 5.8 says that $K[\Gamma_1]/\langle\rho_1\rangle \cong K[\Sigma_{\mathbb{Z}_7}] \otimes_\tau K[\Gamma_2]/\langle\rho_2\rangle$ by the isomorphism $\phi\colon K[\Sigma_{\mathbb{Z}_7}] \otimes_\tau K[\Gamma_2]/\langle\rho_2\rangle \to K[\Gamma_1]/\langle\rho_1\rangle$ which maps the basis elements by $\phi(\epsilon_{\sigma^i} \otimes p) = \bar{\tilde{p}}_{\sigma^i}$.

6 Hochschild cohomology of the twisted tensor product

In this section we will introduce the Hochschild cohomology groups and show how they can be computed. We then propose an analogue to Theorem (4.7) in [3] better suited to our setting and show through a counterexample that it does not hold. We begin with some basic definitions and then introduce the Hocschild cohomology.

Let R be a ring. An R-module P is *projective* if and only if for every epimorphism $f\colon N \to M$ between R-modules N and M and every homomorphism $g\colon P \to M$, there exists a homomorphism $h\colon P \to N$ such that $fh = g$.

A sequence

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} M_n$$

of R-modules is called an *exact sequence* if $\mathrm{Im}(f_k) = \mathrm{Ker}(f_{k+1})$ for $k = 1, 2, \ldots, n-1$. A left *projective resolution* of M over R is a possibly infinite exact sequence of projective left R-modules P_i of the form

$$\cdots \xrightarrow{f_2} P_2 \xrightarrow{f_1} P_1 \xrightarrow{f_0} M \to 0.$$

A *complex* is a sequence of R-modules $\cdots B_{-1}, B_0, B_1 \cdots$ with maps $f^n\colon B^n \to B^{n+1}$ such that $f^{n+1} \circ f^n = 0$ for all n.

Given a k-algebra A, we define the enveloping algebra, denoted by A^e, as $A \otimes_k A^{op}$ where A^{op} is the same algebra as A except that multiplication is reversed. Multiplication in A^e is given by $(a \otimes b)(a' \otimes b') = aa' \otimes b'b$.

It can be shown that the tensor product $A^{\otimes n} = A \otimes_k \cdots \otimes_k A$, which takes the tensor product of A with itself n times, is a projective A^e-module and that the map

$b_{n-1}\colon A^{\otimes n+1} \to A^{\otimes n}$ given by $b_{n-1}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1}(-1)^i a_o \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$ is a homomorphism of A^e-modules. Further, it is possible to show that the sequence

$$\cdots \to A^{\otimes n+1} \xrightarrow{b_{n-1}} A^{\otimes n} \to \cdots \to A^{\otimes 3} \xrightarrow{b_1} A^{\otimes 2} \xrightarrow{b_0} A \to 0,$$

is a projective resolution of A over A^e, which is known as the *Hochschild resolution* of A.

Let M be an A^e-module. Consider the sequence

$$0 \to \mathrm{Hom}_{A^e}(A^{\otimes 2}, M) \xrightarrow{-\circ b_1} \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \xrightarrow{-\circ b_2} \mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \to \cdots .$$

Note that $\mathrm{Hom}_{A^e}(A, M)$ is not included. This is no longer an exact sequence, but is still a complex. Using the isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes n}, M) \cong \mathrm{Hom}_k(A^{\otimes n-2}, M)$ given by $f \mapsto \tilde{f}$, with $\tilde{f} = f(1 \otimes x \otimes 1)$, we get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{A^e}(A^{\otimes 2}, M) \arrow[r, "-\circ b_1"] \arrow[d, "\cong"'] & \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \arrow[r, "-\circ b_2"] \arrow[d, "\cong"'] & \mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \arrow[r] \arrow[d, "\cong"'] & \cdots . \\
0 \arrow[r] & M \arrow[r, "d_1"] & \mathrm{Hom}_k(A, M) \arrow[r, "d_2"] & \mathrm{Hom}_k(A^{\otimes 2}, M) \arrow[r] & \cdots
\end{tikzcd}$$

The two complexes are equivalent if all the squares commute and it can be verified that the maps $d^n\colon\ \mathrm{Hom}_k(A^{\otimes n}, M) \to \mathrm{Hom}_k(A^{\otimes n+1}, M)$ given by

$$\begin{aligned}
(d^n f)(a_0 \otimes \cdots \otimes a_n) &= a_0 f(a_1 \otimes \cdots \otimes a_n) \\
&+ \sum_{i=0}^{n-1}(-1)^{i+1} f(a_0 \otimes \cdots \otimes a_1 a_{i+1} \otimes \cdots \otimes a_n) \\
&+ (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1}) a_n
\end{aligned}$$

accomplishes this.

The *Hochschild cohomology* of A with coefficients in M is defined as $H^i(A, M) \cong \mathrm{Ker} d^i / \ \mathrm{Im} d^{i-1}$. We write $H^i(A) = H^i(A, A)$ for the Hochschild cohomology of A with coefficients in A. For the 0-Hochschild cohomology group of A over A we have

$$H^0(A) = \mathrm{Ker}(d^0) = \{a \in A \colon a'a - aa' = 0, \forall a' \in A\} = Z(A),$$

where $Z(A)$ is called the center of A. The next Hochschild cohomology group is given by $H^1(A) = \mathrm{Ker}(d^1)/\mathrm{Im}(d^0)$ where

$$\mathrm{Ker}(d^1) = \{f \in \mathrm{Hom}_k(A, A) \colon af(a') - f(aa') + f(a)a' = 0, \forall a, a' \in A\}$$

and

$$\mathrm{Im}(d^0) = \{f_a \in \mathrm{Hom}_k(A, A), a \in A \colon f_a(a') = a'a - aa', \forall a \in A\}.$$

Higher Hochschild cohomology groups become increasingly more difficult to compute.

Theorem (4.7) in [3] show that taking Hochschild cohomology commutes with twisted tensor products of graded algebras when only considering the graded parts corresponding to subgroups $\cap_{b\in B}$ Ker$t^{\langle -|b\rangle} \leq A$ and $\cap_{a\in A}$ Ker$t^{\langle a|-\rangle} \leq B$. We will illustrate that the more general claim, that is, for algebras A and B and twisted tensor product in the sense of Definition 4.9, the isomorphism

$$\bigoplus_{i=0}^{n} H^i(A) \otimes_k H^{n-i}(B) \cong H^n(A \otimes_\tau B), \tag{1}$$

does not hold in general. We do this with an example.

Example 6.1. Let Q be the quiver $\alpha \circlearrowleft 1 \circlearrowright \beta$ with relations $\rho = \{\alpha^2, \alpha\beta + \beta\alpha, \beta^2\}$. Further, for σ such that $\sigma^7 = 1_{G/H}$, let $G/H = \langle\sigma\rangle \cong \mathbb{Z}_7$. We denote $k[Q]/\langle\rho\rangle$ by Λ, and similarly, $k[\Sigma_{G/H}]$ by $\Sigma_{G/H}$. We want to calculate the first two Hochschild cohomology groups for the two path algebras Λ and $\Sigma_{G/H}$.

The algebra Λ has basis $\{\overline{e_1}, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}\}$ where we note that only $\overline{e_1}$ and $\overline{\alpha\beta}$ commute with all the basis elements, and so $H^0(\Lambda) = Z(\Lambda) = \{k_1\overline{e_1} + k_2\overline{\alpha\beta} \mid k_1, k_2 \in k\}$. To calculate $H^1(\Lambda) =$Kerd^1/Imd^0 we note that Λ as a k-module is a 4-dimensional vector space so that $\mathrm{Hom}_k(\Lambda, \Lambda) = M_{4x4}(k)$. We will use vector notation for elements in Λ and write $k_1\overline{e_1} + k_2\overline{\alpha} + k_3\overline{\beta} + k_4\overline{\alpha\beta} = [k_1, k_2, k_3, k_4]^T$. In order to find Ker$d^1$ we need to determine which $A \in M_{4x4}(k)$ satisfy $(A\lambda)\cdot\lambda' = (A\lambda)\cdot\lambda' + \lambda\cdot(A\lambda')$ for all $\lambda, \lambda' \in \Lambda$. Solving the equation for each of the basis elements give

$$\mathrm{Ker}(d^1) = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & e & f & a+d \end{pmatrix} \mid a, b, c, d, e, f \in k\}.$$

Similarly, to calculate $\mathrm{Im}(d^0)$ we need to determine which $A \in M_{4x4}(k)$ satisfy $A\lambda = \lambda\cdot\lambda' - \lambda'\cdot\lambda$ for all $\lambda \in \Lambda$. Solving this using the basis elements yield

$$\mathrm{Im}(d^0) = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \end{pmatrix} \mid a, b \in k\}.$$

We now have $\dim_k H^0(\Lambda) = 2$ and $\dim_k H^1(\Lambda) = 4$.

Next, consider $\Sigma_{G/H}$ which has basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. This is a commutative algebra, and so $H^0(\Sigma_{G/H}) = Z(\Sigma_{G/H}) = \Sigma_{G/H}$. Performing the same kind of calculation on $A(\sigma\cdot\sigma') = (A\sigma)\cdot\sigma' + \sigma\cdot(A\sigma')$ for all σ, $\sigma' \in \Sigma_{G/H}$ give that $\mathrm{Ker}(d^1) = 0$ and so $H^1(\Sigma_{G/H}) = 0$.

With this we have

$$\dim_k(H^1(\Lambda) \otimes H^0(\Sigma_{G/H}) \oplus H^0(\Lambda) \otimes H^1(\Sigma_{G/H})) = 28.$$

Lastly, we want to calculate $H^1(\Lambda \otimes_\tau \Sigma_{G/H})$. As this algebra has 28 basis elements, solving the linear equation $A((\lambda\otimes\sigma)\cdot(\lambda'\otimes\sigma')) = (A(\lambda\otimes\sigma))\cdot(\lambda'\otimes\sigma') + (\lambda\otimes\sigma)\cdot(A(\lambda'\otimes\sigma'))$

by hand is not an option. Instead, we use the GAP package QPA [4] to calculate and get $\dim_k H^1(\Lambda \otimes_\tau \Sigma_{G/H}) = 2$. The code is included in Appendix A. It is now clear that (1) does not hold in general.

References

[1] Jon M. Corson, Thomas J. Ratkovich *Quiver Algebras of Coverings and Twisted Tensor Products*

[2] P.B.Bhattacharya, S.K.Jain, S.R.Nagpaul *Basic Abstract Algebra*, Cambridge University Press 2nd edition, 1994.

[3] Petter Andreas Bergh, Steffen Oppermann *Cohomology of Twisted Tensor Products*

[4] QPA: http://www.math.ntnu.no/∼oyvinso/QPA/

A Code for calculation used in Example 6.1

```
Q := Quiver(7, [[1,2,"a1"],[2,3,"a2"],[3,4,"a3"],[4,5,"a4"],
[5,6,"a5"],[6,7,"a6"],[7,1,"a7"],[1,3,"b1"],[3,5,"b3"],
[5,7,"b5"],[7,2,"b7"],[2,4,"b2"],[4,6,"b4"],[6,1,"b6"]]);
KQ := PathAlgebra(Rationals, Q);
AssignGeneratorVariables(KQ);
relations := [a1*a2, a2*a3, a3*a4, a4*a5, a5*a6, a6*a7, a7*a1,
              b1*b3, b3*b5, b5*b7, b7*b2, b2*b4, b4*b6, b6*b1,
              a1*b2 + b1*a3, a2*b3 + b2*a4, a3*b4 + b3*a5,
              a4*b5 + b4*a6, a5*b6 + b5*a7, a6*b7 + b6*a1,
              a7*b1 + b7*a2];
A := KQ/relations;
B := AlgebraAsModuleOverEnvelopingAlgebra(A);
ExtOverAlgebra(B);
```

ExtOverAlgebra(B); returns three elements where the second is the basis vectors for the Hochschild cohomology.