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Moments of Random Multiplicative Functions and Truncated Characteristic Polynomials

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Abstract

An asymptotic formula for the $2k$th moment of a sum of multiplicative Steinhaus variables is given. This is obtained by expressing the moment as a $2k$-fold complex contour integral, from which one can extract the leading order term. The $2k$th moment of a truncated characteristic polynomial of a unitary matrix is also computed. This is done by expressing the moment as a combinatoric sum over a restricted region, and then invoking each restriction by introducing some complex integral. This gives a $2k$-fold integral that is very similar to the $2k$th moment of the sum of multiplicative Steinhaus variables, which in turn gives an asymptotic relation between the two.

Similarly, an asymptotic formula is given for the $2k$th moment of a sum of multiplicative Rademacher variables, and the $2k$th moment of the truncated characteristic polynomial of a special orthogonal matrix is found. This gives an asymptotic relation between these two.

Sammendrag

Det utledes en asymptotisk formel for det $2k$te momentet av en sum av multiplikative Steinhausvariabler. Dette gjøres ved å uttrykke momentet som et multipelt, komplekst konturintegral, for så å finne en asymptotisk formel for dette integralet. I tillegg beregnes det $2k$te momentet av det avkortede, karakteristiske polynomet til en unitær matrise. Dette blir gjort ved å uttrykke momentet som en kombinatorisk sum over et begrenset område, for så å uttrykke hver begrensning ved hjelp av et komplekst konturintegral. Dette gir et konturintegral, som ligner på uttrykket for det $2k$te momentet av summen av multiplikative Steinhausvariabler, som igjen gir en asymptotisk relasjon mellom de to.

På samme måte utledes en asymptotisk formel for det $2k$te momentet av en sum av multiplikative Rademachervariabler, og det gis en formel for det $2k$te momentet av det avkortede, karakteristiske polynomet til en spesiell-ortogonal matrise. Dette gir en asymptotisk relasjon mellom de to.

Preface

This master's thesis was written during six months in the spring of 2015, as the final part of the study program Industrial Mathematics, within Applied Physics and Mathematics at the Norwegian University of Science and Technology (NTNU).

The initial goal of this thesis was to find asymptotic expressions for the even moments of certain sums of random multiplicative functions using techniques from complex analysis. It soon became clear that Winston Heap was working on very similar things, and we therefore decided to combine our work into an article. In our article we compute the mentioned asymptotics in addition to the even moments of the truncated characteristic polynomial of a unitary and a special orthogonal matrix. After about a months work my thesis was therefore extended from analytic number theory to also deal with some random matrix theory.

As all the main results I have proved during the writing of this thesis are included in our article, this is the most essential part of my thesis, and it has been included in its entirety. The style of writing in a mathematical article is much more sparse than what one perhaps might expect from a masters thesis, and I have therefore also included an additional chapter containing a more detailed description of one of the derivations in our article. I have also included an introduction that is meant for someone who isn't already an expert on the topics discussed in the article.

I would like to thank Kristian Seip and Andry Bondarenko for being excellent supervisors. This thesis could not have been completed without their very useful input and help. During the past months I have been lucky enough to be able to write an article with Winston Heap, who I would like to thank for having the patience to work with a masters student. It has been a pleasure working with the three of you!

Contents

Chapter 1

Introduction

1.1 Outline of thesis

All the main results of this thesis are included in the article [9], which is included in its entirety as Chapter 2. In addition to the results in the article, a more general introduction to the central topics is given in the rest of this chapter, and in Chapter 3 one of the derivations in the article is presented in more detail. This means that there naturally will be a lot of overlap between Chapter 2 and the remaining chapters. In the rest of this text any mention of "the article" refers to [9].

1.2 Notation

The following asymptotic notation is used.

$$f(x) = O(g(x)) \text{ as } x \to a \qquad \Longleftrightarrow \qquad \exists C > 0 : |f(x)| \leq C|g(x)| \; \forall x \text{ near } a,$$

where $a \in \mathbb{R}$ or $a = \pm\infty$. If $a \in \mathbb{R}$,"x near a" means that this holds for all x in some neighborhood of a. If $a = +\infty$, "x near a" means that this holds for all $x \geq x_0$ for some $x_0 \in \mathbb{R}$, and similarly for $a = -\infty$. When it is clear what a is, the above will be stated as $f = O(g)$. In this text a is either $+\infty$ (when considering e.g. a truncated sum) or 0 (when considering functions of the type $\frac{1}{s}$ near $s = 0$).

Further,

$$f(x) = o(g(x)) \text{ as } x \to a \qquad \Longleftrightarrow \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

and

$$f(x) \sim g(x) \text{ as } x \to a \quad \iff \quad \lim_{x\to a} \frac{f(x)}{g(x)} = 1,$$

where $a \in \mathbb{R}$ or $a = \pm\infty$, as before. Also in these cases a will not be stated explicitly unless it is unclear which value is considered.

Expectation will be denoted by $\mathbb{E}[\cdot]$, possibly with a subscript to make it extra clear which probability measure the expectation is taken with respect to.

The expression $\prod_p f(p)$ always denotes a product over all primes p.

When using product notation several times in the same term, the convention is as follows. Instead of writing $(\prod_i f(i))(\prod_i g(i))$, the parentheses are occasionally dropped. When the same index is used in two consecutive products it is then understood that this is the intended meaning. Therefore $\prod_i f(i) \prod_i g(i) = (\prod_i f(i))(\prod_i g(i))$, but $\prod_i f(i) \prod_j g(i,j) = \prod_i \left(f(i) \prod_j g(i,j) \right)$. The same goes for more than two products. As soon as an index is reused, all previous products are "ended", i.e. $\prod_i \prod_j f(i,j) \prod_j g(j) = \left(\prod_i \prod_j f(i,j) \right) \left(\prod_j g(j) \right)$. Usually the intended meaning will be clear from the context, and if there is some ambiguity the necessary parentheses are included.

For a complex variable s the convention is to write $\Re s = \sigma$ and $\Im s = t$, i.e., $s = \sigma + it$.

1.3 Random multiplicative functions

1.3.1 Steinhaus variables

Let $\{X_p\}$ be independent identically distributed random variables indexed by the primes. Let each X_p be uniformly distributed on the complex unit circle, that is, X_p is uniformly distributed on $\{e^{i\theta} : 0 \leq \theta < 2\pi\}$. For $n \in \mathbb{N}$, let $n = \prod_p p^{\alpha(p)}$ be its prime factorization. The random variables $\{X_p\}$ are then extended to all natural numbers, by defining $X_n = \prod_p X_p^{\alpha(p)}$. This definition gives that X_n is multiplicative in n. The variables $\{X_n\}_n$ are referred to as multiplicative Steinhaus variables.

Denote the expectation with respect to this product measure by $\mathbb{E}[\cdot]$. Some basic properties of the Steinhaus variables are then that $\mathbb{E}X_n = \mathbb{E}\overline{X_n} =$

0 and $\mathbb{E}|X_n|^2 = 1$ for all $n \in \mathbb{N}$. This then gives that $\mathbb{E}X_n\overline{X_m} = 1$ if $n = m$ and zero otherwise.

The partial sums of X_n are of great interest. Specifically, consider the sum

$$F(x) = \sum_{n \le x} X_n. \tag{1.1}$$

If one associates p^{-it} with X_p for each prime p, this sum is expected to behave similarly to the sum

$$\sum_{n \le x} n^{-it}. \tag{1.2}$$

In fact, one has that these two sums have the same moments, as

$$\left(\mathbb{E}\left[\Big|\sum_{n \le x} X_n\Big|^q\right]\right)^{1/q} = \left(\lim_{T\to\infty} \frac{1}{T}\int_0^T \Big|\sum_{n \le x} n^{-it}\Big|^q dt\right)^{1/q} \tag{1.3}$$

for all $q > 0$. Note that this is a special case of a more general result with arbitrary coefficients. One way of showing this is by writing the sum in (1.2) as a power series in $(z_1 = 2^{-it}, z_2 = 3^{-it}, \dots, z_m = p_m^{-it}, \dots)$, and applying Birkhoff's ergodic theorem to the ergodic flow $(\tau_1, \tau_2, \dots) \mapsto (\tau_1 p_1^{-it}, \tau_2 p_2^{-it}, \dots)$. Alternatively, one can adopt the method mentioned in the article by showing the identity (1.3) first for even integer q and then extending it to all $q > 0$ by using the Weierstrass approximation theorem. This is done in [7].

The similarity of (1.2) and the Riemann zeta function is striking. As always, the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\sigma > 1) \tag{1.4}$$

and by the analytic continuation of this for $\sigma \le 1$. This gives a meromorphic function with a single simple pole at $s = 1$. In the rest of the complex plane ζ is holomorphic. The expression in (1.4) is a Dirichlet series, and (1.2) appears to be a partial sum of this series for $\sigma = 0$[1].

This comparison makes it natural to consider the sums (1.1)-(1.2) also for other values of σ than $\sigma = 0$. Therefore, consider

$$F_\sigma(x) = \sum_{n \le x} \frac{X_n}{n^\sigma} \tag{1.5}$$

[1]But of course the full sum doesn't converge for $\sigma = 0$

and

$$\sum_{n\leq x}\frac{1}{n^{\sigma+it}}.$$

It still holds that

$$\left(\mathbb{E}\left[\Big|\sum_{n\leq x}\frac{X_n}{n^\sigma}\Big|^q\right]\right)^{1/q}=\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big|\sum_{n\leq x}\frac{1}{n^{\sigma+it}}\Big|^q dt\right)^{1/q}, \tag{1.6}$$

so it is clear that there is some connection between $F_\sigma(x)$ and the partial sums of the Riemann zeta function.

In practice, computing the odd moments in (1.6) turns out to be a formidable task. On the other hand, for the even moments it is possible to compute asymptotic expressions for the moments, which is one of the main goals in this thesis.

It is also worthwhile mentioning the first moment in particular, as a conjecture of Helson [8] states that the first moment for $\sigma = 0$ is $o(\sqrt{x})$. In the article we present a conjecture stating that $\mathbb{E}\left[\big|\sum_{n\leq x} X_n\big|\right] \sim C\sqrt{x}$, where the constant C is given to a reasonable accuracy.

1.3.2 Rademacher variables

Similarly, one can instead consider i.i.d random variables Y_p that are uniformly distributed on $\{\pm 1\}$ and indexed by the primes. As before this is extended to all $n \in \mathbb{N}$ by requiring Y_n to be multiplicative, but now it is also required that Y_n is non-zero only if n is square-free. That is, $Y_n \neq 0$ iff $|\mu(n)| = 1$. If $n = \prod_p p^{\alpha(p)}$ it is thus defined that

$$Y_n = |\mu(n)|\prod_p Y_p^{\alpha(p)}.$$

The variables $\{Y_n\}_n$ are referred to as multiplicative Rademacher variables, and similarly as for the Steinhaus variables one has that $\mathbb{E}Y_n = 0$ and $\mathbb{E}Y_n^2 = 1$ for all square-free $n \in \mathbb{N}$. One also has that $\mathbb{E}Y_nY_m = 1$ if $n = m$ and n, m are square-free, and zero otherwise.

Note that $Y_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, and for square-free n one has $|Y_n| = 1$. In some sense one can therefore think of Rademacher variables as the real version of Steinhaus variables.

As in the Steinhaus case, consider the moments of a sum of multiplicative Rademacher variables, namely $\left(\mathbb{E}\left[\left|\sum_{n\leq x} Y_n\right|^q\right]\right)^{1/q}$. As before it is difficult to compute this for odd q, but for the even moments one can compute an asymptotic expression. This is done very similarly as for multiplicative Steinhaus variables, and is another main goal of this thesis.

1.4 Random matrix theory

1.4.1 Unitary group

One says that an $N \times N$ matrix M is unitary if $M^*M = MM^* = I_N$ where M^* denotes the conjugate transpose of M. The group of unitary $N \times N$ matrices is denoted by $U(N)$.

As $\det(M^*) = \overline{\det(M)}$, one has that $|\det(M)| = 1$ for all matrices $M \in U(N)$. It can easily be shown that all eigenvalues of a unitary matrix have absolute value 1. Indeed, let λ be an eigenvalue of $M \in U(N)$, and let x be a corresponding eigenvector. Then $Mx = \lambda x$ and $x^*M^* = \overline{\lambda}x^*$. Multiplying these two expressions together and using $M^*M = I$ gives $x^*x = |\lambda|^2 x^*x$, so one has $|\lambda| = 1$.

Now, consider the characteristic polynomial of a matrix $M \in U(N)$, given by

$$\Lambda(z) = \det(I - zM) = \prod_{j=1}^{N}(1 - \lambda_j z) = \prod_{j=1}^{N}(1 - e^{i\theta_j} z), \tag{1.7}$$

where the product is over all eigenvalues of M, and it is used that any eigenvalue can be expressed as $\lambda = e^{i\theta}$ for some θ. If one multiplies out the final expression in (1.7) one gets a polynomial in z of degree N, namely

$$\Lambda(z) = \sum_{n=0}^{N} c_M(n)(-z)^n,$$

where the coefficients $c_M(n)$ are known as the secular coefficients. Next, consider the truncated characteristic polynomial

$$\Lambda_{N'}(z) = \sum_{n=0}^{N'} c_M(n)(-z)^n, \tag{1.8}$$

where $N' \leq N$. Another main concern of this thesis is to find an expression for the moments of this truncated characteristic polynomial.

1.4.2 Haar measure and expectations

In order to be able to talk about moments of the truncated characteristic polynomial (1.8) one needs to introduce a probability measure. A natural choice is to use Haar measure, as this has many desirable properties. The existence and properties of Haar measure are given by Haar's theorem.

Theorem (Haar's theorem)**.** *Let G be a compact topological group. There exists a probability measure m defined on the σ-algebra $\mathcal{B}(G)$ of Borel subsets of G such that $m(xE) = m(E)$ for all $x \in G$ and all $E \in \mathcal{B}(G)$, and m is regular. There is only one such regular rotation invariant probability measure on $(G, \mathcal{B}(G))$.*

As $U(N)$ is a compact topological group one can apply Haar's theorem to get a probability measure μ. Denote expectation with respect to μ by $\mathbb{E}_{U(N)}[\cdot]$. One can then consider the moments of the truncated characteristic polynomial in (1.8), which are given by $\left(\mathbb{E}_{U(N)}\left[|\Lambda_{N'}(z)|^q\right]\right)^{1/q}$.

At this point it is worthwhile mentioning that there is a conjectured connection between the Riemann zeta function and random matrix theory. One of the most notable conjectures is the Hilbert-Pólya conjecture, which states that the nontrivial zeroes of the Riemann zeta function share the same distribution as the eigenvalues of some random matrix. Several interesting results concerning this connection are mentioned in the introduction to our article, see Section 1 of the article, and so they will not be repeated here.

Another goal of this thesis is to find an expression for the even moments of the truncated characteristic polynomial of a unitary matrix. Given the above connection it is not so surprising that this then leads to an asymptotic relation between these moments and the even moments of the sum in (1.5).

1.4.3 Special Orthogonal group

If one instead considers real $N \times N$ matrices which satisfy $MM^T = M^TM = I_N$, where M^T is the transpose of M, one gets the orthogonal group $O(N)$. Consider the determinant of $M \in O(N)$. As $\det(M) = \det(M^T)$ one gets $(\det M)^2 = \det(I) = 1$. Now, $M \in O(N)$ is a real matrix, so one must have $\det M \in \mathbb{R}$. This in turn gives $\det M = \pm 1$. Note that an orthogonal matrix is just a real unitary matrix.

If one instead considers only those matrices in $O(N)$ that have determinant equal to $+1$ one gets the special orthogonal group $SO(N)$. In particular

$SO(2N)$ will be of interest.

As for the unitary group, Haar's theorem gives the existence of Haar measure on $SO(2N)$. Let $\mathbb{E}_{SO(2N)}[\cdot]$ denote expectation with respect to Haar measure on the special orthogonal group. The final goal of this thesis is to find an expression for the even moments of the truncated characteristic polynomial of a special orthogonal matrix of size $2N \times 2N$, i.e.,

$$\left(\mathbb{E}_{SO(2N)}\left[|\Lambda_{N'}(z)|^{2k}\right]\right)^{1/2k}, \tag{1.9}$$

where the truncated characteristic polynomial $\Lambda_{N'}(z)$ is defined as in (1.8). This will give an asymptotic relation between the moments in (1.9) and the corresponding even moments of sums of Rademacher variables.

In some sense it seems appropriate that there is a similar kind of relation between Steinhaus variables and $U(N)$ as there is between Rademacher variables and $SO(2N)$. Rademacher variables take values on the real subset of the complex unit circle, and orthogonal matrices are just real unitary matrices. It is on the other hand not clear why one has to consider $SO(2N)$ instead of just $O(N)$.

MOMENTS OF RANDOM MULTIPLICATIVE FUNCTIONS AND TRUNCATED CHARACTERISTIC POLYNOMIALS

WINSTON HEAP AND SOFIA LINDQVIST

ABSTRACT. We give an asymptotic formula for the $2k$th moment of a sum of multiplicative Steinhaus variables. This was recently computed independently by Harper, Nikeghbali and Radziwiłł. We also compute the $2k$th moment of a truncated characteristic polynomial of a unitary matrix. This provides an asymptotic equivalence with the moments of Steinhaus variables. Similar results for multiplicative Rademacher variables are given.

1. Introduction

In the study of the Riemann zeta function there are two probabilistic heuristics which have had significant recent attention. One of these is the use of random multiplicative functions in problems of an arithmetic nature and the other is the use of random matrix theory to model various statistics of the zeta function.

The study of random multiplicative functions was initiated by Wintner [23] when he modelled the Möbius function as the multiplicative extension to the squarefree integers of the random variables ϵ_p, each of which takes the values $\{\pm 1\}$ with equal probability. This provided a model for the reciprocal of the Riemann zeta function and hence an appropriate[1] probabilistic interpretation of the Riemann hypothesis. More recently, random models have been used in association with Dirichlet characters in the work of Granville and Soundararajan (e.g. in [9, 10]) and also for the quantities p^{it} when p ranges over the set of primes [11, 18, 19].

The connection between the Riemann zeta function and random matrix theory is well known and has been extensively studied. One of the more remarkable predictions of random matrix theory is the Keating–Snaith conjecture [16] regarding the moments of the zeta function. This states that

$$(1) \qquad \frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}dt \sim a(k)g(k)(\log T)^{k^2}$$

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[1]We say 'appropriate' here since previous models simply used a random ± 1 as the coefficients which, as objected to by Levy [21], did not take into account the multiplicative nature of the problem.

where $g(k)$ is a certain geometric factor involving the Barnes G-function and

$$a(k) = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^m} \tag{2}$$

with $d_k(n)$ being the k-fold divisor function. In essence, the reasoning behind the Keating–Snaith conjecture can be stated as follows. Since the zeros of the zeta function are conjectured to share the same distribution as eigenvalues of a random matrix in the CUE, it is reasonable to expect that the characteristic polynomial of a matrix provides a good model to the zeta function in the mean. Thus, for an appropriate choice of N one could expect

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \sim a(k)\mathbb{E}_{U(N)}\Big[|\Lambda(M,z)|^{2k}\Big] \tag{3}$$

where $\Lambda(M,z)$ denotes the characteristic polynomial of the matrix M and the expectation is over all unitary matrices of size N with respect to the Haar measure.

Recently, Conrey and Gamburd [7] showed that the asymptotic in (3) holds if one both truncates the characteristic polynomial and replaces the zeta function by a Dirichlet polynomial of length $x = o(T^{1/k})$. This allowed them to deduce that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Big|\sum_{n\le x} n^{-1/2-it}\Big|^{2k} dt \sim a(k)c(k)(\log x)^{k^2}, \qquad k \in \mathbb{N}$$

where $a(k)$ is given by (2) and $c(k)$ is the volume of a particular polytope in $\mathbb{R}^{k^2}$. This result may be stated in the more general framework of random multiplicative functions as follows.

Given the set of primes, associate a set of i.i.d. random variables $\{X_p\}$, equidistributed on the unit circle with variance 1. We extend these to the positive integers by requiring that X_n is multiplicative; that is, if $n = \prod_p p^{\alpha(p)}$ then $X_n = \prod_p X_p^{\alpha(p)}$. We let $\mathbb{E}[\cdot]$ denote the expectation. We refer to the X_n as multiplicative Steinhaus variables. The association $p^{it} \leftrightarrow X_p$ is then seen to be more than just formal in light of the identity

$$\mathbb{E}\Big[\Big|\sum_{n\le x} X_n/n^{\sigma}\Big|^{2q}\Big] = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Big|\sum_{n\le x} n^{-\sigma-it}\Big|^{2q} dt$$

which holds for all $\sigma \in \mathbb{R}$ and $q > 0$. This can be proved by first demonstrating it for $q \in \mathbb{N}$ and then applying the Weierstrass approximation theorem to the function $f : y \mapsto y^{q/2}$.

Our main aim is to extend the results of Conrey–Gamburd to more general σ, in particular to $\sigma = 0$, and to exhibit the connection between moments of random multiplicative functions and random matrix theory.

Theorem 1. *For fixed $k \in \mathbb{N}$ and $0 \le \sigma < 1/2$ we have*

$$\mathbb{E}\Big[\Big|\sum_{n\le x} X_n/n^{\sigma}\Big|^{2k}\Big] \sim \frac{a(k)\beta(k)}{(1-2\sigma)^{2k-1}}\frac{\Gamma(2k-1)}{\Gamma(k)^2}x^{k(1-2\sigma)}(\log x)^{(k-1)^2}. \tag{4}$$

where, $a(k)$ is given by (2), $\beta(1)=1$, and

$$\beta(k) =\frac{1}{(2\pi i)^{2k-1}}\int_{(b_{2k})}\cdots\int_{(b_2)}\Big[\prod_{i=2}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\Big]e^{s_2+\cdots+s_{2k}}\prod_{j=2}^{k}ds_j\prod_{j=k+1}^{2k}\frac{ds_j}{s_j} \tag{5}$$

for $k\ge 2$. Here, $\int_{(b_j)}=\int_{b_j-i\infty}^{b_j+i\infty}$ and $b_j>0$ for all j.

In the significant case $\sigma=0$, Theorem 1 has been proved independently by Harper, Nikeghbali and Radziwiłł [13]. It is of interest to note that the constant in their result involves the volume of the Birkhoff polytope $\mathcal{B}_k$. By comparing coefficients we get that $\mathrm{vol}(\mathcal{B}_k)=k^{k-1}\beta(k)$. It is an open problem to determine a closed form for the volume of the Birkhoff polytope [22] and a representation in terms of such contour integrals may be new. A direct proof of the equation $\mathrm{vol}(\mathcal{B}_k)=k^{k-1}\beta(k)$ can be found by applying the methods of section 4.2 to the formula for the Ehrhart polynomial given in [4]. Also, our methods work equally well in the case $\sigma=1/2$ and thus by comparing coefficients with Theorem 2 of Conrey–Gamburd [7], we get a contour integral representation for their constant (see equation (21)).

It should be noted that the expectation on the line $\sigma=0$ counts the number of solutions $(m_j)\in\mathbb{N}^{2k}$ to the equation $m_1m_2\cdots m_k=m_{k+1}\cdots m_{2k}$ with the restriction $1\leqslant m_j\leqslant x$. In the case $k=2$, Ayyad, Cochrane and Zheng [2] computed this quantity to a high accuracy. Theorem 1 therefore extends these results to $k\geqslant 3$, although we do not achieve their level of accuracy. By including the extra condition $(m_j,q)=1$ in the equation, a slight modification of our methods give the following asymptotic formula for Dirichlet character sums.

Theorem 2. *Let χ be a primitive Dirichlet character modulo q and suppose q has a bounded number of prime factors. Then for fixed $k\in\mathbb{N}$,*

$$\frac{1}{\varphi(q)}\sum_{\chi\ne\chi_0}\Big|\sum_{n\le x}\chi(n)\Big|^{2k}\sim a(k)\beta(k)\prod_{p|q}\Big(\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^m}\Big)^{-1}\frac{\Gamma(2k-1)}{\Gamma(k)^2}x^k(\log x)^{(k-1)^2} \tag{6}$$

as x, $q\to\infty$ with $x^k\le q$ where χ_0 denotes the principal character.

Our results on the random matrix theory side are as follows. Let $U(N)$ denote the group of unitary matrices of size N. For a matrix $M\in U(N)$ with eigenvalues

$(e^{i\theta_j})_{j=1}^N$ let

$$\Lambda(z) = \Lambda(M,z) = \det(I - zM) = \prod_{j=1}^{N}(1 - e^{i\theta_j} z) = \sum_{n=0}^{N} c_M(n)(-z)^n$$

where the $c_M(n)$ are the secular coefficients. For $N' \leqslant N$ we consider the truncated characteristic polynomial given by

$$\Lambda_{N'}(z) = \sum_{n \le N'} c_M(n)(-z)^n.$$

Let $\mathbb{E}_{U(N)}[\,\cdot\,]$ denote the expectation over $U(N)$ with respect to Haar measure.

Theorem 3. *Let $k \in \mathbb{N}$ be fixed and suppose $|z| > 1$. Then for all $N \geqslant k \log x$ we have*

$$\mathbb{E}_{U(N)}\left[|\Lambda_{\log x}(z)|^{2k}\right] \sim \frac{\beta(k)}{(1-|z|^{-2})^{2k-1}} \frac{\Gamma(2k-1)}{\Gamma(k)^2} F_k(z) x^{2k \log |z|} (\log x)^{(k-1)^2} \tag{7}$$

where $\beta(k)$ is given by (5) and

$$F_k(z) = {}_2F_1(1-k, 1-k; 2-2k; 1-|z|^{-2}) \tag{8}$$

with ${}_2F_1$ being Gauss' hypergeometric function.

One may notice a certain similarity between Theorems 1 and 3. Indeed, by including the work of [7] in the case $\sigma = 1/2$ we have the following.

Corollary 1. *Let $k \in \mathbb{N}$ be fixed and let z_σ be any comlpex number such that $|z_\sigma| = e^{1/2-\sigma}$. Then for $0 \le \sigma \le 1/2$ and $N \geqslant k \log x$ we have*

$$\mathbb{E}\left[\Big|\sum_{n \le x} X_n/n^\sigma\Big|^{2k}\right] \sim a(k) c_\sigma(k) \mathbb{E}_{U(N)}\left[|\Lambda_{\log x}(z_\sigma)|^{2k}\right] \tag{9}$$

where $a(k)$ is given by (2) and

$$c_\sigma(k) = \begin{cases} \left(\frac{1-e^{2\sigma-1}}{1-2\sigma}\right)^{2k-1} F_k(e^{1/2-\sigma})^{-1}, & 0 \le \sigma < 1/2 \\ 1, & \sigma = 1/2. \end{cases}$$

A problem which has garnered some attention recently is to determine the first moment of $\left|\sum_{n\leqslant x} X_n\right|$. A conjecture of Helson [14] states that this is $o(\sqrt{x})$, but this seems doubtful now given the evidence in [5, 13]. Another motivation for the present article was to provide a conjecture for the first moment via Corollary 1.

Let us then assume that Corollary 1 holds for $0 \leqslant k < 1$. Then (presumably) the average on the right side of (9) can be taken over matrices of size $N = \log x$ which leads to a computation of the full characteristic polynomial. By an application of Szegő's Theorem, Chris Hughes has shown ([15], formula (3.177)) that for $|z| < 1$

$$\mathbb{E}_{U(N)}\left[|\Lambda(z)|^{2s}\right] \sim \left(\frac{1}{1-|z|^2}\right)^{s^2} \tag{10}$$

as $N \to \infty$. On applying the functional equation

$$\Lambda(M, z) = \det M(-z)^N \Lambda(M^\dagger, 1/z)$$

with $|z| = e^{\sigma-1/2}$ we obtain the following conjecture.

Conjecture 1. *For* $0 \leqslant k < 1$ *and* $0 \leqslant \sigma < 1/2$ *we have*

$$\mathbb{E}\left[\Big|\sum_{n\le x} X_n/n^\sigma\Big|^{2k}\right] \sim \frac{a(k)F_k(e^{1/2-\sigma})^{-1}}{(1-e^{2\sigma-1})^{(k-1)^2}(1-2\sigma)^{2k-1}}\, x^{k(1-2\sigma)}. \tag{11}$$

For $k = 1/2$ and $\sigma = 0$ we can compute the constants to a reasonable accuracy. The arithmetic factor $a(k)$ admits a continuation to real values of k via the formula

$$d_k(p^m) = \binom{k+m-1}{m} = \frac{\Gamma(k+m)}{m!\Gamma(k)}.$$

We then find that $a(1/2) = 0.98849\ldots$. The other constants are given by

$$F_{1/2}(e^{1/2})^{-1} = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-e^{-1})^{-1} = \mathrm{agm}\Big(1-\sqrt{1-\tfrac{1}{e}}, 1+\sqrt{1-\tfrac{1}{e}}\Big) = 0.79099...$$

where $\mathrm{agm}(x, y)$ is Gauss' arithmetic-geometric mean and

$$\left(\frac{e}{e-1}\right)^{1/4} = 1.21250\ldots\,.$$

Thus, on combining the constants we acquire the conjecture

$$\mathbb{E}\left[\Big|\sum_{n\le x} X_n\Big|\right] \sim 0.8769\ldots\sqrt{x}. \tag{12}$$

One can instead consider multiplicative Rademacher variables. In this case, associate a set of i.i.d. random variables $\{Y_p\}$, which are ± 1 with uniform probability, to the set of primes. Extend this to all positive integers by requiring Y_n to be multiplicative and non-zero only on the square free integers; that is, $Y_n = |\mu(n)| \prod_{p|n} Y_p$.

Let

$$b(k) = \prod_p \left(1-\frac{1}{p}\right)^{k(2k-1)} \sum_{i=0}^{k} \binom{2k}{2i} \frac{1}{p^i} \tag{13}$$

and

$$\gamma(k)=\frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)}\prod_{1\le i<j\le 2k}\frac{1}{s_i+s_j}\prod_{j=1}^{2k}e^{2s_j}ds_j, \tag{14}$$

where $b_j>0$ for all j. We then have the following result.

Theorem 4. *For fixed $k\in\mathbb{N}, k\geqslant 2$ we have*

$$\mathbb{E}\left[\left|\sum_{n\le x}Y_n\right|^{2k}\right]\sim\gamma(k)b(k)2^{2k}x^k(\log x)^{2k^2-3k}. \tag{15}$$

Let $SO(2N)$ denote the group of orthogonal $2N\times 2N$ matrices with determinant 1, and let $\mathbb{E}_{SO(2N)}[\cdot]$ denote the expectation over $SO(2N)$ with respect to Haar measure.

Theorem 5. *Let $k\in\mathbb{N}$ be fixed and suppose $z\in\mathbb{R}$, $|z|>1$. Then for all $N\geqslant k\log x$ we have*

$$\mathbb{E}_{SO(2N)}\left[|\Lambda_{\log x}(z)|^{2k}\right]\sim\frac{\gamma(k)}{(1-|z|^{-1})^{2k}}x^{2k\log|z|}(\log x)^{2k^2-3k} \tag{16}$$

where $\gamma(k)$ is given by (14).

Corollary 2. *For fixed $k\in\mathbb{N}$, $k\geqslant 2$ and all $N\geqslant k\log x$ we have*

$$\mathbb{E}\left[\Big|\sum_{n\le x}Y_n\Big|^{2k}\right]\sim b(k)2^{2k}(1-e^{-1/2})^{2k}\mathbb{E}_{SO(2N)}\left[|\Lambda_{\log x}(e^{1/2})|^{2k}\right]$$

where $b(k)$ is the arithmetic factor given by (13).

Similarly to the case of Steinhaus variables, we expect that the 1st moment is $\sim c\sqrt{x}$ for some constant c. Unfortunately we have not been able to find an analogue of (10) for the special orthogonal group and so cannot make a precise conjecture. For some recent results on the order of $\sum_{n\le x}Y_n$ see [12, 20].

2. Asymptotics for Steinhaus variables: Proof of Theorem 1

2.1. A contour integral representation for the expectation.

We have

$$\mathbb{E}\left[\Big|\sum_{n\le x}X_n/n^{\sigma}\Big|^{2k}\right]=\sum_{\substack{n_1\cdots n_k=\\ n_{k+1}\cdots n_{2k}\\ n_j\le x}}\frac{1}{(n_1\cdots n_{2k})^{\sigma}}.$$

We invoke the condition $n_j\le x$ in each j by using the contour integral

$$\frac{1}{2\pi i}\int_{(b)}y^s\frac{ds}{s}=\begin{cases}1, & y>1\\ 0, & y<1\end{cases}, \qquad (b>0) \tag{17}$$

with $y = x/n_j$. For each j we take a specific line of integration b_j. For reasons that will become clear we take $b_1 = \epsilon < 1 - 2\sigma$ if $\sigma < 1/2$ and $b_1 = 2$ if $\sigma = 1/2$. In both cases we may take the other lines to be sufficiently large so as to guarantee absolute convergence; $b_j = 2$ say ($j = 2, \ldots, 2k$). This gives

$$\begin{aligned}\mathbb{E}\Big[\Big|\sum_{n\le x} X_n/n^{\sigma}\Big|^{2k}\Big] &= \sum_{\substack{n_1\cdots n_k=\\ n_{k+1}\cdots n_{2k}}} \frac{1}{(n_1\cdots n_{2k})^{\sigma}} \frac{1}{(2\pi i)^{2k}} \int_{(b_{2k})} \cdots \int_{(b_1)} \prod_{j=1}^{2k} \Big(\frac{x}{n_j}\Big)^{s_j} \frac{ds_j}{s_j} \\ &= \frac{1}{(2\pi i)^{2k}} \int_{(b_{2k})} \cdots \int_{(b_1)} F_k(\sigma + s_1, \cdots, \sigma + s_{2k}) \prod_{j=1}^{2k} x^{s_j} \frac{ds_j}{s_j}.\end{aligned}$$

where

$$F_k(z_1, \cdots, z_{2k}) = \sum_{\substack{n_1\cdots n_k=\\ n_{k+1}\cdots n_{2k}}} \frac{1}{n_1^{z_1}\cdots n_{2k}^{z_{2k}}}.$$

Since the condition $n_1 \cdots n_k = n_{k+1} \cdots n_{2k}$ is multiplicative we may express $F_k(z)$ as an Euler product:

$$\begin{aligned}F_k(z_1, \ldots, z_{2k}) &= \prod_p \sum_{\substack{m_1+\cdots+m_k\\ =m_{k+1}+\cdots+m_{2k}}} \frac{1}{p^{m_1 z_1 + \cdots + m_{2k} z_{2k}}} \\ &= \prod_p \Big(1 + \sum_{i=1}^{k} \sum_{j=k+1}^{2k} \frac{1}{p^{z_i+z_j}} + O\Big(\sum \frac{1}{p^{z_{i_1}+z_{j_1}+z_{i_2}+z_{j_2}}}\Big)\Big) \qquad (18) \\ &= A_k(z_1, \ldots, z_{2k}) \prod_{i=1}^{k} \prod_{j=k+1}^{2k} \zeta(z_i + z_j)\end{aligned}$$

where

$$A_k(z_1, \ldots, z_{2k}) = \prod_p \Big[\prod_{i=1}^{k} \prod_{j=k+1}^{2k} \Big(1 - \frac{1}{p^{z_i+z_j}}\Big)\Big] \cdot \sum_{\substack{m_1+\cdots+m_k\\ =m_{k+1}+\cdots+m_{2k}}} \frac{1}{p^{m_1 z_1 + \cdots + m_{2k} z_{2k}}}. \qquad (19)$$

Upon expanding the inner products and sum whilst referring to the middle line of (18), we see that $A_k(z_1, \ldots, z_{2k})$ is an absolutely convergent product provided $\Re(z_i + z_j) > 1/2$ for $1 \le i \le k, k + 1 \le j \le 2k$.

We now have

$$\mathbb{E}\Big[\Big|\sum_{n\le x} X_n/n^\sigma\Big|^{2k}\Big] = \frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)} A_k(\sigma+s_1,\dots,\sigma+s_{2k}) \times \prod_{i=1}^{k}\prod_{j=k+1}^{2k}\zeta(2\sigma+s_i+s_j)\prod_{j=1}^{2k} x^{s_j}\frac{ds_j}{s_j}. \tag{20}$$

2.2. **The case $\sigma = 1/2$.** Although the case $\sigma = 1/2$ has already been investigated by Conrey-Gamburd [7], we will go over it with a proof that is instructive for the case $0 \le \sigma < 1/2$.

Set $\sigma = 1/2$ in (20). We write the resulting integral as

$$\frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)} B_k(s_1,\dots,s_{2k})\prod_{i=1}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\prod_{j=1}^{2k} e^{\mathcal{L}s_j}\frac{ds_j}{s_j}$$

where $\mathcal{L} = \log x$ and

$$B_k(s_1,\dots,s_{2k}) = A_k(\tfrac12+s_1,\dots,\tfrac12+s_{2k})\prod_{i=1}^{k}\prod_{j=k+1}^{2k}(s_i+s_j)\zeta(1+s_i+s_j).$$

This function is holomorphic in a neighbourhood of $(0,0,\dots,0)$ and the constant term in its Taylor expansion about this point is given by $A_k(\frac12,\dots,\frac12)$.

We now make the substitution $s_j \mapsto s_j/\mathcal{L}$ in each variable to give an integral of the form

$$\frac{\mathcal{L}^{k^2}}{(2\pi i)^{2k}}\int_{(c_{2k})}\cdots\int_{(c_1)} B_k(s_1/\mathcal{L},\dots,s_{2k}/\mathcal{L})\prod_{i=1}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\prod_{j=1}^{2k} e^{s_j}\frac{ds_j}{s_j}.$$

First, note that we may shift the contours so as to be independent of $\mathcal{L}$, to $\Re(s_j) = 2$ say. We now truncate the integrals at height $T = o(\mathcal{L})$ and take a Taylor approximation to $B_k(\underline{s})$ about the point $(0,0,\dots,0)$. Then upon letting $\mathcal{L}\to\infty$ we see that this integral is asymptotic to

$$A_k(\tfrac12,\dots,\tfrac12)\frac{\mathcal{L}^{k^2}}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)}\prod_{i=1}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\prod_{j=1}^{2k} e^{s_j}\frac{ds_j}{s_j}.$$

A short calculation gives $A_k(\frac12,\dots,\frac12) = a(k)$ where $a(k)$ is given by (2). The remaining constant is given by

$$\alpha(k) := \frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)}\prod_{i=1}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\prod_{j=1}^{2k} e^{s_j}\frac{ds_j}{s_j}. \tag{21}$$

We may express $\alpha(k)$ as a volume integral and hence recover the constant of Theorem 2 in [7]. This is achieved by first writing $(s_i+s_j)^{-1} = \int_0^\infty e^{-x_{ij}(s_i+s_j)}dx_{ij}$ for each term in the product over i,j so that the full product is then given by a k^2-fold integral. Upon exchanging the orders of integration and applying (17) the result follows.

2.3. **The case** $0 \le \sigma < 1/2$**.** Returning to our expression for the expectation given in (20), we first make the substitutions $s_j \mapsto s_j + 1 - 2\sigma$ for $k+1 \le j \le 2k$. This gives

$$\begin{aligned}(22)\quad \mathbb{E}\Big[\Big|\sum_{n\le x} X_n/n^\sigma\Big|^{2k}\Big] = x^{k(1-2\sigma)}\Big(\frac{1}{2\pi i}\Big)^{2k}\int_{(c_{2k})}\cdots\int_{(b_1)}\\ \times A_k(\sigma+s_1,\dots,\sigma+s_k,1-\sigma+s_{k+1},\cdots,1-\sigma+s_{2k})\\ \times \prod_{i=1}^{k}\prod_{j=k+1}^{2k}\zeta(1+s_i+s_j)\prod_{j=1}^{k}x^{s_j}\frac{ds_j}{s_j}\prod_{j=k+1}^{2k}x^{s_j}\frac{ds_j}{s_j+1-2\sigma}\end{aligned}$$

In the case $\sigma = 1/2$, the leading order term was essentially given by the poles at $s_j = 0$. In the present case we must first make the appropriate substitutions to bring the leading order contributions to $s_j = 0$. Only then can we make the substitution $s_j \mapsto s_j/\mathcal{L}$.

We first extract the polar behaviour of the integrand. Write

$$\begin{aligned}(23)\quad G_{k,\sigma}(s_1,\dots,s_{2k}) = A_k(\sigma+s_1,\dots,\sigma+s_k,1-\sigma+s_{k+1},\dots,1-\sigma+s_{2k})\\ \times\prod_{i=1}^{k}\prod_{j=k+1}^{2k}(s_i+s_j)\zeta(1+s_i+s_j)\end{aligned}$$

so that our integral becomes

$$\begin{aligned}x^{k(1-2\sigma)}\frac{1}{(2\pi i)^{2k}}\int_{(c_{2k})}\cdots\int_{(b_1)}G_{k,\sigma}(s_1,\dots,s_{2k})\prod_{i=1}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\times\\ \times e^{\mathcal{L}(s_1+\cdots+s_{2k})}\prod_{j=1}^{k}\frac{ds_j}{s_j}\prod_{j=k+1}^{2k}\frac{ds_j}{s_j+1-2\sigma}.\end{aligned}$$

The function $G_{k,\sigma}(s_1,\dots,s_{2k})$ is analytic in the region $\Re(s_i+s_j) > -1/2$ for $1 \le i \le k, k+1 \le j \le 2k$.

We now make the substitutions $s_j \mapsto s_j - s_1$ for $k+1 \le j \le 2k$ and $s_i \mapsto s_i + s_1$ for $2 \le i \le k$. This gives an integral of the form

$$x^{k(1-2\sigma)}\frac{1}{(2\pi i)^{2k}}\int_{(d_{2k})}\cdots\int_{(b_1)} G_{k,\sigma}(s_1,s_2+s_1\ldots,s_k+s_1,s_{k+1}-s_1,\ldots,s_{2k}-s_1)\times$$
$$\times\left[\prod_{i=2}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\right]e^{\mathcal{L}(s_2+\cdots+s_{2k})}\frac{ds_1}{s_1}\prod_{j=2}^{k}\frac{ds_j}{s_j+s_1}\prod_{j=k+1}^{2k}\frac{ds_j}{s_j(s_j-s_1+1-2\sigma)}.$$

Now, for $j=2,3,\ldots,2k$ we let $s_j\mapsto s_j/\mathcal{L}$. This gives the integral

$$x^{k(1-2\sigma)}\mathcal{L}^{(k-1)^2}\frac{1}{(2\pi i)^{2k}}\int_{(e_{2k})}\cdots\int_{(b_1)}$$
$$G_{k,\sigma}(s_1,s_1+s_2/\mathcal{L},\ldots,s_1+s_k/\mathcal{L},-s_1+s_{k+1}/\mathcal{L},\ldots,-s_1+s_{2k}/\mathcal{L})\times$$
$$\times\left[\prod_{i=2}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\right]e^{s_2+\cdots+s_{2k}}\frac{ds_1}{s_1}\prod_{j=2}^{k}\frac{ds_j}{\frac{s_j}{\mathcal{L}}+s_1}\prod_{j=k+1}^{2k}\frac{ds_j}{s_j(\frac{s_j}{\mathcal{L}}-s_1+1-2\sigma)}.$$

Once again, we may shift the lines of integration in the integrals over $s_2,s_3,\ldots,s_{2k}$ so as to be independent of $\mathcal{L}$; back to $\Re(s_j)=2$ say, and truncate the integrals at some height $T=o(\mathcal{L})$. From the definition of $G_{k,\sigma}$ given in (23), we see that

$$\begin{aligned}\lim_{\mathcal{L}\to\infty} & G_{k,\sigma}(s_1,s_1+s_2/\mathcal{L},\ldots,s_1+s_k/\mathcal{L},-s_1+s_{k+1}/\mathcal{L},\ldots,-s_1+s_{2k}/\mathcal{L})\\ &=A_k(\sigma+s_1,\ldots,\sigma+s_1,1-\sigma-s_1,\ldots,1-\sigma-s_1)\\ &=A_k(0,\ldots,0,1,\ldots,1)\\ &=A_k(\tfrac{1}{2},\ldots,\tfrac{1}{2})\end{aligned}$$

where in the last two lines we have used the symmetry of A_k. As previously claimed, this last quantity is given by (2). The other limits are easily evaluated.

Thus, as $\mathcal{L}\to\infty$ we have

$$\mathbb{E}\left[\Big|\sum_{n\le x}X_n/n^{\sigma}\Big|^{2k}\right]\sim a(k)x^{k(1-2\sigma)}\mathcal{L}^{(k-1)^2}\frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)}$$
$$\left[\prod_{i=2}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}\right]e^{s_2+\cdots+s_{2k}}\frac{ds_1}{s_1^k(1-2\sigma-s_1)^k}\prod_{j=2}^{k}ds_j\prod_{j=k+1}^{2k}\frac{ds_j}{s_j}. \tag{24}$$

For the integral over s_1 we push the line of integration to the far left encountering a pole at $s_1 = 0$. The integral over the new line vanishes in the limit and so

$$\begin{aligned}\frac{1}{2\pi i}\int_{(b_1)}\frac{ds_1}{s_1^k(1-2\sigma-s_1)^k} =&\frac{1}{(k-1)!}\frac{d^{k-1}}{ds_1^{k-1}}\Big((1-2\sigma-s_1)^{-k}\Big)\Big|_{s_1=0}\\ =&\frac{1}{(k-1)!}\frac{\Gamma\big(k+(k-1)\big)/\Gamma(k)}{(1-2\sigma)^{2k-1}}\\ =&\frac{\Gamma(2k-1)}{\Gamma(k)^2}\frac{1}{(1-2\sigma)^{2k-1}}.\end{aligned}$$

The remaining integrals are given by $\beta(k)$ of equation (5).

Note that although it appears as if one should be able to make the substitution $s_j \mapsto s_j/\mathcal{L}$ directly in (22) without first shifting the variables by s_1, this is not the case. Upon truncating the integrals at height $T = o(\mathcal{L})$, the largest error terms arise from the ζ-factors in $G_{k,\sigma}$ when they are evaluated close to $t = 0$. For this to occur in all terms of the form $\zeta(1 + (s_i + s_j)/\mathcal{L})$ and $\zeta(1 + s_j/\mathcal{L})$ for $i = 2, \ldots, k$, $j = k + 1, \ldots, 2k$, one must have $t_i \approx -t_j \approx 0$ for $i = 2, \ldots, 2k$ and $j = k + 1, \ldots, 2k$. When looking at the error arising from cutting some s_i at height T, this large contribution clearly is excluded, as one has $|t_i| \geq T \gg 0$ for this i. On the other hand, if one makes the substitution $s_j \mapsto s_j/\mathcal{L}$ directly in (22) and attempts to cut all integrals at height $T = o(\mathcal{L})$, a large error arises from $t_i \approx -t_j$ for $i = 1, \ldots, k$ and $j = k + 1, \ldots, 2k$.

3. Character sums: sketch proof of Theorem 2

We shall only sketch the proof of Theorem 2 since it is very similar to the proof of Theorem 1. Recall the orthogonality property of Dirichlet characters: for m, n coprime to q

$$\frac{1}{\varphi(q)}\sum_{\chi}\chi(m)\overline{\chi(n)} = \begin{cases}1 & \text{if } m \equiv n \mod q,\\ 0 & \text{otherwise.}\end{cases}$$

This implies that for all $x^k \leq q$

$$\begin{aligned}\frac{1}{\varphi(q)}\sum_{\chi}\Big|\sum_{m\leq x}\chi(m)\Big|^{2k} =&\frac{1}{\varphi(q)}\sum_{\chi}\sum_{m_i\leqslant x}\chi(m_1\cdots m_k)\overline{\chi}(m_{k+1}\cdots m_{2k})\\ =&\sum_{\substack{m_1\cdots m_k=m_{k+1}\cdots m_{2k}\\ m_i\leqslant x\\ (m_i,q)=1}} 1.\end{aligned}$$

On applying the line integral (17) we acquire

$$\sum_{\substack{m_1\cdots m_k=m_{k+1}\cdots m_{2k}\\ m_i\leqslant x\\ (m_i,q)=1}} 1=\frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)} H_{k,q}(s_1,\ldots,s_{2k})\prod_{j=1}^{2k} x^{s_j}\frac{ds_j}{s_j}$$

where

$$H_{k,q}(s_1,\ldots,s_{2k})=\sum_{\substack{m_1\cdots m_k=m_{k+1}\cdots m_{2k}\\ (m_j,q)=1}}\frac{1}{m_1^{s_1}\cdots m_{2k}^{s_{2k}}}.$$

Expressing this as an Euler product gives

$$\begin{aligned}H_{k,q}(s_1,\ldots,s_{2k})=&\prod_{p|q}\Bigg(\sum_{\substack{m_1+\cdots+m_k=\\ m_{k+1}+\cdots+m_{2k}}}\frac{1}{p^{m_1s_1+\cdots+m_{2k}s_{2k}}}\Bigg)^{-1}\prod_p\sum_{\substack{m_1+\cdots+m_k=\\ m_{k+1}+\cdots+m_{2k}}}\frac{1}{p^{m_1s_1+\cdots+m_{2k}s_{2k}}}\\ =&\prod_{p|q}\Bigg(\sum_{\substack{m_1+\cdots+m_k=\\ m_{k+1}+\cdots+m_{2k}}}\frac{1}{p^{m_1s_1+\cdots+m_{2k}s_{2k}}}\Bigg)^{-1}A_k(s_1\ldots,s_{2k})\prod_{i,j}\zeta(s_i+s_j)\\ =&C_{k,q}(s_1,\ldots,s_{2k})\prod_{i,j}\zeta(s_i+s_j),\end{aligned}$$

say. Here, the function $A_k(s_1,\ldots s_{2k})$ is that of equation (19). Since the number of prime factors of q remains fixed, $C_{k,q}(s_1,\ldots,s_{2k})$ is holomorphic in the same regions as $A_k(s_1,\ldots,s_{2k})$. The arguments of the previous section now follow, with the arithmetic constant being given by

$$\begin{aligned}C_{k,q}(\tfrac{1}{2},\ldots,\tfrac{1}{2})=&A_k(\tfrac{1}{2},\ldots,\tfrac{1}{2})\prod_{p|q}\Bigg(\sum_{\substack{m_1+\cdots+m_k=\\ m_{k+1}+\cdots+m_{2k}}}\frac{1}{p^{m_1+\cdots+m_k}}\Bigg)^{-1}\\ =&a(k)\prod_{p|q}\Bigg(\sum_{n=0}^{\infty}\frac{d_k(p^m)^2}{p^n}\Bigg)^{-1}.\end{aligned}$$

Now

$$\frac{1}{\varphi(q)}\Big|\sum_{n\le x}\chi_0(n)\Big|^{2k}=\frac{1}{\varphi(q)}\Big|\sum_{\substack{n\le x\\ (n,q)=1}}1\Big|^{2k}=\frac{1}{\varphi(q)}\Big(\frac{\varphi(q)}{q}x+O(2^{\omega(q)})\Big)^{2k}$$

where $\omega(q)$ represents the number of distinct prime factors of q. Since we're assuming $\omega(q)$ is bounded this last error term is $O(1)$ as $q \to \infty$. Hence,

$$\frac{1}{\varphi(q)}\Big|\sum_{n\leq x}\chi_0(n)\Big|^{2k} \sim \left(\frac{\varphi(q)}{q}\right)^{2k-1}\frac{x^{2k}}{q} \leq \left(\frac{\varphi(q)}{q}\right)^{2k-1} x^k.$$

Since this is of a lower order than the main term when $\omega(q)$ is bounded the result follows.

4. Moments of the truncated characteristic polynomial in the unitary case: Proof of Theorem 3

4.1. **A formula for the expectation.** We begin by recalling the definitions. Let $U(N)$ denote the group of unitary matrices of size N. For a matrix $M \in U(N)$ with eigenvalues $(e^{i\theta_j})_{j=1}^N$ let

$$\Lambda(z) = \det(I - zM) = \prod_{j=1}^{N}(1 - e^{i\theta_j}z) = \sum_{n=0}^{N} c_M(n)(-z)^n.$$

The coefficients $c_M(n)$ are known as the secular coefficients. We have $c_M(0) = 1$, $c_M(1) = \mathrm{Tr}(M)$ and $c_M(N) = \det(M)$. In general, note that these coefficients are symmetric functions of the eigenvalues. For $N' \leq N$, consider the truncated characteristic polynomial given by

$$\Lambda_{N'}(z) = \sum_{n\leq N'} c_M(n)(-z)^n.$$

We will compute the expectation of this object as the following multiple contour integral.

Proposition 3. *Let $k \in \mathbb{N}$. Then for all $z \in \mathbb{C}$ and $N \geq k\mathcal{L}$ we have*

$$\mathbb{E}_{U(N)}\left[|\Lambda_{\mathcal{L}}(z)|^{2k}\right] = \frac{1}{(2\pi i)^{2k}}\int\cdots\int \frac{(u_1\cdots u_{2k})^{-\mathcal{L}}}{\prod_{i=1}^{k}\prod_{j=k+1}^{2k}(1-|z|^2u_iu_j)}\prod_{j=1}^{2k}\frac{du_j}{u_j(1-u_j)}$$

where the integration is around small circles of radii less than $\min(|z|^{-1}, 1)$.

Our plan is to expand $|\Lambda(z)|^{2k}$, push the expectation through, and then use the results of Diaconis-Gamburd [8] regarding the expectation of products of the coefficients $c_M(j)$. To state their result we must first detail some notation.

For an $m \times n$ matrix A denote the row and column sums by r_i and c_j respectively and define the vectors

$$\mathrm{row}(A) = (r_1, \ldots, r_m),$$
$$\mathrm{col}(A) = (c_1, \ldots, c_n).$$

Given two partitions $\mu = (\mu_1, \ldots, \mu_m)$ and $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_n)$ we let $N_{\mu\tilde{\mu}}$ denote the number of matrices A with $\mathrm{row}(A) = \mu$ and $\mathrm{col}(A) = \tilde{\mu}$. The notation $\langle 1^{a_1} 2^{a_2} \cdots \rangle$ is used to represent a partition with a_1 parts equal to 1, a_2 parts equal to 2 etc. For example, $(5, 3, 3, 2, 1) = \langle 1^1 2^1 3^2 4^0 5^1 \rangle$.

Theorem ([8]). *Let $(a_j)_{j=1}^l$, $(b_j)_{j=1}^l$ be sequences of nonnegative integers. Then for $N \geqslant \max\big(\sum_{j=1}^l j a_j, \sum_{j=1}^l j b_j\big)$ we have*

$$\mathbb{E}_{U(N)}\Bigg[\prod_{j=1}^{l} c_M(j)^{a_j} \overline{c_M}(j)^{b_j}\Bigg] = N_{\mu\tilde{\mu}}$$

where $\mu = \langle 1^{a_1} 2^{a_2} \cdots \rangle$ and $\tilde{\mu} = \langle 1^{b_1} 2^{b_2} \cdots \rangle$.

On expanding the polynomial and pushing the expectation through we get

$$\begin{aligned}
&\mathbb{E}_{U(N)}\big[|\Lambda_{\mathcal{L}}(z)|^{2k}\big] \\
=&\sum_{n_1,\ldots,n_{2k} \leq \mathcal{L}} \mathbb{E}_{U(N)}\big[c_M(n_1)\cdots c_M(n_k)\overline{c_M}(n_{k+1})\cdots\overline{c_M}(n_{2k})\big] z^{n_1+\cdots+n_k}\overline{z}^{n_{k+1}+\cdots+n_{2k}} \\
=&\mathbb{E}_{U(N)}\big[|c_M(\mathcal{L})|^{2k}\big]|z|^{2k\mathcal{L}} \\
&+ |z|^{(2k-2)\mathcal{L}} \sum_{\substack{m \leqslant \mathcal{L} \\ n \leqslant \mathcal{L}-1}} \mathbb{E}_{U(N)}\big[c_M(\mathcal{L})^{k-1}c_M(m)\overline{c_M}(\mathcal{L})^{k-1}\overline{c_M}(n)\big] z^m \overline{z}^n + \cdots .
\end{aligned}$$

On taking $N \geqslant k\mathcal{L}$ the condition of the Theorem is satisfied for all terms in the sum, and can thus be applied.

Consider the first term. Note that we may write

$$\begin{aligned}
\mathbb{E}_{U(N)}\big[|c_M(\mathcal{L})|^{2k}\big]|z|^{2k\mathcal{L}} &= \sum_{(m_{ij})_{i,j=1}^k \in B_k(\mathcal{L})} z^{(\sum_{i=1}^k \sum_{j=1}^k m_{ij})} \overline{z}^{(\sum_{j=1}^k \sum_{i=1}^k m_{ij})} \\
&= \sum_{(m_{ij})_{i,j=1}^k \in B_k(\mathcal{L})} |z|^{2\sum_{i=1}^k \sum_{j=1}^k m_{ij}}
\end{aligned}$$

where

$$B_k(\mathcal{L}) = \Bigg\{(m_{ij}) \in \mathbb{Z}_{\geqslant 0}^{k^2} : \sum_{j=1}^{k} m_{ij} = \mathcal{L};\ \sum_{i=1}^{k} m_{ij} = \mathcal{L}\Bigg\}.$$

Similarly, for the second term we may write

$$|z|^{(2k-2)\mathcal{L}} \sum_{\substack{m\leqslant \mathcal{L}\\ n\leqslant \mathcal{L}-1}} \mathbb{E}_{U(N)}\left[c_M(\mathcal{L})^{k-1}c_M(m)\overline{c_M}(\mathcal{L})^{k-1}\overline{c_M}(n)\right]z^m\overline{z}^n$$

$$= \sum_{\substack{m\leqslant \mathcal{L}\\ n\leqslant \mathcal{L}-1}} \sum_{(m_{ij})_{i,j=1}^k \in C_k(\mathcal{L},m,n)} z^{\sum_{i=1}^k\sum_{j=1}^k m_{ij}}\overline{z}^{\sum_{i=1}^k\sum_{j=1}^k m_{ij}}$$

where

$$C_k(\mathcal{L},m,n) = \Big\{(m_{ij})\in\mathbb{Z}_{\geqslant 0}^{k^2} : \sum_{j=1}^k m_{ij} = \mathcal{L} \text{ for } 0\leqslant i\leqslant k-1, \sum_{j=1}^k m_{kj} = m;$$
$$\sum_{i=1}^k m_{ij} = \mathcal{L} \text{ for } 0\leqslant j\leqslant k-1, \sum_{i=1}^k m_{ik} = n\Big\}.$$

This set is empty unless $m=n$ in which case we can write the second term as a sum of $|z|^2$. Continuing in this fashion we see that

$$\mathbb{E}_{U(N)}\left[|\Lambda_{\mathcal{L}}(z)|^{2k}\right] = \sum_{(m_{ij})_{i,j=1}^k\in D_k(\mathcal{L})} |z|^{2\sum_{i=1}^k\sum_{j=1}^k m_{ij}} \tag{25}$$

where

$$D_k(\mathcal{L}) = \Big\{(m_{ij})\in\mathbb{Z}_{\geqslant 0}^{k^2} : \sum_{j=1}^k m_{ij}\leqslant \mathcal{L};\ \sum_{i=1}^k m_{ij}\leqslant\mathcal{L}\Big\}.$$

We now invoke the conditions $\sum m_{ij}\leqslant\mathcal{L}$ with the formula

$$\frac{1}{2\pi i}\int_{|u|=\varepsilon} u^{m-\mathcal{L}}\frac{du}{u(1-u)} = \begin{cases}1, & m\leqslant\mathcal{L}\\ 0, & m>\mathcal{L}\end{cases} \tag{26}$$

which follows on expanding $(1-u)^{-1}$ as a geometric series. This gives

$$\mathbb{E}_{U(N)}\left[|\Lambda_{\mathcal{L}}(z)|^{2k}\right] = \sum_{m_{ij}\geqslant 0}|z|^{2\sum_{i=1}^k\sum_{j=1}^k m_{ij}}\frac{1}{(2\pi i)^{2k}}\int_{|u_{2k}|=\varepsilon_{2k}}\cdots\int_{|u_1|=\varepsilon_1}\times$$
$$\times\, u_1^{m_{11}+m_{12}+\cdots+m_{1k}-\mathcal{L}}u_2^{m_{21}+m_{22}+\cdots+m_{2k}-\mathcal{L}}\cdots u_k^{m_{k1}+m_{k2}+\cdots+m_{kk}-\mathcal{L}}$$
$$\times\, u_{k+1}^{m_{11}+m_{21}+\cdots+m_{k1}-\mathcal{L}}u_{k+2}^{m_{12}+m_{22}+\cdots+m_{k2}-\mathcal{L}}\cdots u_{2k}^{m_{1k}+m_{2k}+\cdots+m_{kk}-\mathcal{L}}\prod_{j=1}^{2k}\frac{du_j}{u_j(1-u_j)}.$$

On collecting like powers and computing the geometric series we acquire Proposition 3.

4.2. **Asymptotics for the multiple contour integral.** Denote the integral in Proposition 3 by I. Thus,

$$I = \frac{1}{(2\pi i)^{2k}} \int \cdots \int \frac{(u_1 \cdots u_{2k})^{-\mathcal{L}}}{\prod_{i=1}^{k} \prod_{j=k+1}^{2k} (1 - |z|^2 u_i u_j)} \prod_{j=1}^{2k} \frac{du_j}{u_j(1-u_j)}$$

where the contours of integration are positively oriented circles of radii $|u_j| = \varepsilon_j$. We choose $1/|z|^2 < \varepsilon_1 < 1/|z|$, $\varepsilon_j = \varepsilon_1^{-1}|z|^{-2}\delta_j^{1/\mathcal{L}}$ $(j = k+1, \ldots, 2k)$ and $\varepsilon_j = \varepsilon_1 \delta_j^{1/\mathcal{L}}$ $(j = 2, \ldots, k)$ with $\delta_j < 1/|z|^{\mathcal{L}}$. With these choices of ε_j the conditions of Proposition 3 are satisfied, provided $|z| > 1$. We will now perform similar manipulations to those in section 2.3

First, let $u_j \mapsto u_1^{-1}|z|^{-2}u_j$ for $j = k+1, \ldots, 2k$. Then

$$I = \frac{x^{2k \log|z|}}{(2\pi i)^{2k}} \int \cdots \int \frac{(u_1^{1-k} u_2 \cdots u_{2k})^{-\mathcal{L}}}{\prod_{i=1}^{k} \prod_{j=k+1}^{2k} (1 - u_1^{-1} u_i u_j)} \prod_{j=1}^{k} \frac{du_j}{u_j(1-u_j)} \prod_{j=k+1}^{2k} \frac{du_j}{u_j(1 - u_1^{-1}|z|^{-2}u_j)}.$$

Now let $u_j \mapsto u_1 u_j$ for $j = 2, \ldots, k$. After a bit of rearranging we have

$$I = \frac{x^{2k \log|z|}}{(2\pi i)^{2k}} \int \cdots \int \frac{(u_2 \cdots u_{2k})^{-\mathcal{L}}}{\prod_{i=2}^{k} \prod_{j=k+1}^{2k} (1 - u_i u_j)} \frac{du_1}{u_1(1-u_1)} \prod_{j=2}^{k} \frac{du_j}{u_j(1 - u_1 u_j)} \prod_{j=k+1}^{2k} \frac{du_j}{u_j(1-u_j)(1 - u_1^{-1}|z|^{-2}u_j)}.$$

Consider the integral with respect to $u_2, \ldots, u_{2k}$:

$$J := \frac{1}{(2\pi i)^{2k-1}} \int \cdots \int \frac{(u_2 \cdots u_{2k})^{-\mathcal{L}}}{\prod_{i=2}^{k} \prod_{j=k+1}^{2k} (1 - u_i u_j)} \prod_{j=2}^{k} \frac{du_j}{u_j(1 - u_1 u_j)} \prod_{j=k+1}^{2k} \frac{du_j}{u_j(1-u_j)(1 - u_1^{-1}|z|^{-2}u_j)}.$$

The contours of integration are given by circles of radii $\varepsilon_j = \delta_j^{1/\mathcal{L}}$, $j = 2, \ldots, 2k$. We may now choose the δ_j to be small but independent of $\mathcal{L}$ since this does not alter the value of the integral. Then, writing J in its parametrised form gives

$$J = \frac{1}{(2\pi)^{2k-1}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{(\delta_2 \cdots \delta_{2k})^{-1} e^{-i\mathcal{L}(\theta_2+\cdots+\theta_{2k})}}{\prod_{i=2}^{k} \prod_{j=k+1}^{2k} (1-(\delta_i\delta_j)^{1/\mathcal{L}} e^{i\theta_i+i\theta_j})} \prod_{j=2}^{k} \frac{d\theta_j}{(1-u_1\delta_j^{1/\mathcal{L}} e^{i\theta_j})} \prod_{j=k+1}^{2k} \frac{d\theta_j}{(1-\delta_j^{1/\mathcal{L}} e^{i\theta_j})(1-u_1^{-1}|z|^{-2}\delta_j^{1/\mathcal{L}} e^{i\theta_j})}.$$

Performing the substitutions $\theta_j \mapsto \theta_j/\mathcal{L}$ and writing $c_j = \log \delta_j$ we have

$$J = \frac{\mathcal{L}^{(k-1)^2}}{(2\pi)^{2k-1}} \int_{-\pi\mathcal{L}}^{\pi\mathcal{L}} \cdots \int_{-\pi\mathcal{L}}^{\pi\mathcal{L}} \frac{e^{-(c_2+i\theta_2+\cdots+c_{2k}+i\theta_{2k})}}{\prod_{i=2}^{k} \prod_{j=k+1}^{2k} \mathcal{L}(1-e^{(c_i+i\theta_i+c_j+i\theta_j)/\mathcal{L}})} \prod_{j=2}^{k} \frac{d\theta_j}{(1-u_1 e^{(c_j+i\theta_j)/\mathcal{L}})} \prod_{j=k+1}^{2k} \frac{d\theta_j}{\mathcal{L}(1-e^{(c_j+i\theta_j)/\mathcal{L}})(1-u_1^{-1}|z|^{-2}e^{(c_j+i\theta_j)/\mathcal{L}})}.$$

We now divide by $\mathcal{L}^{(k-1)^2}$ and take the limit as $\mathcal{L} \to \infty$. By an argument involving dominated convergence we may pass the limit through the integral. Then, since $\mathcal{L}(1-e^{-z/\mathcal{L}}) \sim z$, we acquire

$$J \sim \frac{\mathcal{L}^{(k-1)^2}}{(1-u_1)^{k-1}(1-u_1^{-1}|z|^{-2})^k} \frac{1}{(2\pi)^{2k-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e^{-(c_2+i\theta_2+\cdots+c_{2k}+i\theta_{2k})}}{\prod_{i=2}^{k} \prod_{j=k+1}^{2k} (-c_i-i\theta_i-c_j-i\theta_j)} \prod_{j=2}^{k} d\theta_j \prod_{j=k+1}^{2k} \frac{d\theta_j}{(-c_j-i\theta_j)}.$$

Finally, we let $\theta_j \mapsto -\theta_j$. Upon noting that $-c_j$ is postive, we see that this last integral is the parametrised form of $\beta(k)$ of equation (5). Therefore,

$$I \sim \beta(k) x^{2k\log|z|} \mathcal{L}^{(k-1)^2} \frac{1}{2\pi i} \int \frac{1}{(1-u_1)^k (1-u_1^{-1}|z|^{-2})^k} \frac{du_1}{u_1}.$$

It remains to show that

$$\begin{aligned} I_1 :=& \frac{1}{2\pi i} \int \frac{1}{(1-u_1)^k(1-u_1^{-1}|z|^{-2})^k} \frac{du_1}{u_1} = \frac{1}{2\pi i} \int \frac{u_1^{k-1}}{(1-u_1)^k(u_1-|z|^{-2})^k} du_1 \\ =& \frac{1}{(1-|z|^{-2})^{2k-1}} \frac{\Gamma(2k-1)}{\Gamma(k)^2} F_k(z) \end{aligned} \tag{27}$$

where

$$F_k(z) = {}_2F_1(1-k, 1-k; 2-2k; 1-|z|^{-2}) \tag{28}$$

and ${}_2F_1$ is the usual hypergeometric function.

Note that in I_1 we are still integrating over a circle of radius $\varepsilon < 1/|z|$ since we made no substitutions in the variable u_1. Thus, the only contribution is from the pole at $u_1 = |z|^{-2}$. Therefore,

$$\begin{aligned} I_1 =& \frac{1}{(k-1)!}\frac{d^{k-1}}{du^{k-1}}\left(\frac{u^{k-1}}{(1-u)^k}\right)\Bigg|_{u=|z|^{-2}} \\ =& \frac{1}{(k-1)!}\sum_{m=0}^{k-1}\binom{k-1}{m}\frac{d^m}{du^m}\left[(1-u)^{-k}\right]\frac{d^{k-1-m}}{du^{k-1-m}}\left[u^{k-1}\right]\Bigg|_{u=|z|^{-2}} \\ =& \frac{1}{\Gamma(k)}\sum_{m=0}^{k-1}\binom{k-1}{m}\cdot\frac{\Gamma(k+m)/\Gamma(k)}{(1-|z|^{-2})^{k+m}}\cdot\frac{\Gamma(k)}{\Gamma(m+1)}|z|^{-2m} \\ =& \frac{1}{(1-|z|^{-2})^k}\cdot\frac{1}{\Gamma(k)}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}\frac{\Gamma(k+m)}{\Gamma(m+1)}\left(\frac{1}{1-|z|^2}\right)^m. \end{aligned} \tag{29}$$

As a quick aside, we note that

$$\frac{1}{\Gamma(k)}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}\frac{\Gamma(k+m)}{\Gamma(m+1)}\left(\frac{1}{1-|z|^2}\right)^m = {}_2F_1\big(1-k,k;1;1/(1-|z|^2)\big) \tag{30}$$

and that this can be written in terms of the Legendre polynomials $P_n(x)$ via the formula (see [3], section 3.2)

$${}_2F_1(-\lambda,\lambda+1;1;z) = P_\lambda(1-2z).$$

Continuing with our manipulations, the last line of equation (29) may be rewritten as

$$\begin{aligned} &\frac{|z|^{2k}}{(|z|^2-1)^{2k-1}}\frac{1}{\Gamma(k)}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}\frac{\Gamma(2k-1-m)}{\Gamma(k-m)}(1-|z|^2)^m \\ &\qquad = \frac{|z|^{2k}}{(|z|^2-1)^{2k-1}}\frac{\Gamma(2k-1)}{\Gamma(k)^2}\sum_{m=0}^{k-1}\frac{(1-k)_m}{m!}\frac{(1-k)_m}{(2-2k)_m}(1-|z|^2)^m \end{aligned} \tag{31}$$

where $(x)_m$ is rising factorial or Pochammer symbol defined by

$$(x)_m = \begin{cases} 1, & m=0, \\ x(x+1)\cdots(x+m-1), & m\geqslant 1. \end{cases}$$

This last sum is the hypergeometric function ${}_2F_1(1-k,1-k;2-2k;1-|z|^2)$. By formula (18) in section 2.9 of [3] we have

$${}_2F_1(1-a,1-b;2-c;z) = (1-z)^{b-1}{}_2F_1(a+1-c,1-b;2-c;z/(z-1)).$$

On applying this the result follows.

5. Moments of Rademacher variables: Proof of Theorem 4

5.1. **A contour integral representation for the norm.** Note that $\mathbb{E}\left[Y_{n_1}\cdots Y_{n_{2k}}\right] = 1$ if $n_1\cdots n_{2k}$ is a square number and n_i is square-free for $i = 1,\ldots,2k$, and otherwise it equals zero. Therefore we have

$$\mathbb{E}\left[\left|\sum_{n\le x} Y_n\right|^{2k}\right] = \sum_{\substack{n_1\cdots n_{2k}\ \text{square}\\ n_j\le x}} |\mu(n_1)|\cdots|\mu(n_{2k})|.$$

As earlier we invoke the condition $n_j \le x$ in each j by using (17) with $y = x/n_j$. For each j we integrate along the lines $b_j = 2$. This gives

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{n\le x} Y_n\right|^{2k}\right] &= \sum_{\substack{n_1\cdots n_{2k}\ \text{square}\\ |\mu(n_j)|=1}} \frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)}\prod_{j=1}^{2k}\left(\frac{x}{n_j}\right)^{s_j}\frac{ds_j}{s_j} \\ &= \frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_1)} F_k(s_1,\ldots,s_{2k})\prod_{j=1}^{2k} x^{s_j}\frac{ds_j}{s_j}\end{aligned} \tag{32}$$

where

$$F_k(z_1,\ldots,z_{2k}) = \sum_{n_1\cdots n_{2k}\ \text{square}} \frac{|\mu(n_1)|\cdots|\mu(n_{2k})|}{n_1^{s_1}\cdots n_{2k}^{s_{2k}}}.$$

Since the condition $n_1\cdots n_{2k}$ being square is multiplicative we may express $F_k(z)$ as an Euler product:

$$\begin{aligned}F_k(z_1,\ldots,z_{2k}) &= \prod_p \sum_{\substack{m_1+\cdots+m_{2k}\ \text{even}\\ 0\leqslant m_j\leqslant 1}} \frac{1}{p^{m_1z_1+\cdots m_{2k}z_{2k}}} \\ &= \prod_p\left(1+\sum_{1\le i<j\le 2k}\frac{1}{p^{z_i+z_j}} + O\left(\sum\frac{1}{p^{z_{i_1}+\cdots+z_{i_4}}}\right)\right) \\ &= B_k(z_1,\ldots,z_{2k})\prod_{1\le i<j\le 2k}\zeta(z_i+z_j)\end{aligned}$$

where

$$B_k(z_1,\ldots,z_{2k}) = \prod_p\left[\prod_{1\le i<j\le 2k}\left(1-\frac{1}{p^{z_i+z_j}}\right)\right]\cdot\sum_{\substack{m_1+\cdots+m_{2k}\ \text{even}\\ 0\leqslant m_j\leqslant 1}}\frac{1}{p^{m_1z_1+\cdots+m_{2k}z_{2k}}}.$$

Similarly as in the proof of Theorem 1 we get that $B_k(z)$ is an absolutely convergent product provided $\Re(z_i + z_j) > 1/2$ for $1 \le i < j \le 2k$.

Now we make the substitution $s_j \mapsto s_j + 1/2$ for $j = 1, \dots, 2k$ in the second line of (32) to get

$$\mathbb{E}\left[\left|\sum_{n\le x} Y_n\right|^{2k}\right] = x^k \frac{1}{(2\pi i)^{2k}} \int_{(b'_1)} \cdots \int_{(b'_{2k})} G_k(s_1, \dots, s_{2k}) \prod_{1\le i<j\le 2k} \frac{1}{s_i + s_j} \prod_{j=1}^{2k} x^{s_j} \frac{ds_j}{s_j + \frac{1}{2}}$$

where we define

$$G_k(z_1, \dots, z_{2k}) = B_k(s_1 + 1/2, \dots, s_{2k} + 1/2) \prod_{1\le i<j\le 2k} \zeta(1 + s_i + s_j)(s_i + s_j).$$

The function $G_k(z)$ is analytic in the region $\Re(z_i + z_j) > -1/2$ for $1 \le i < j \le 2k$, and $G_k(0, \dots, 0) = B_k(1/2, \dots, 1/2)$. Finally we make the substitution $s_j \mapsto s_j/\mathcal{L}$ for $j = 1, \dots, 2k$ to get

$$x^k \mathcal{L}^{2k^2-3k} \frac{1}{(2\pi i)^{2k}} \int \cdots \int G_k(s_1/\mathcal{L}, \dots, S_{2k}/\mathcal{L}) \prod_{1\le i<j\le 2k} \frac{1}{s_i + s_j} \prod_{j=1}^{2k} e^{s_j} \frac{ds_j}{s_j/\mathcal{L} + \frac{1}{2}}.$$

Shift the lines of integration to be independent of $\mathcal{L}$, say back to $\Re(s_j) = 2$ for $j = 1, \dots, 2k$ and truncate each line at height $T = o(\mathcal{L})$. Computing the Taylor expansions and then taking the limit as $\mathcal{L} \to \infty$ gives

$$\mathbb{E}\left[\left|\sum_{n\le x} Y_n\right|^{2k}\right] \sim b(k) 2^{2k} x^k \mathcal{L}^{2k^2-3k} \gamma(k)$$

where $b(k) = B_k(1/2, \dots, 1/2)$ is the arithmetic factor given in (13) and $\gamma(k)$ is the integral given by (14).

6. Moments of the truncated characteristic polynomial in the special orthogonal case: Proof of Theorem 5

6.1. **A formula for the expectation.** As for the $U(N)$ case we begin by expressing the expectation as a multiple contour integral.

Proposition 4. *Let $k \in \mathbb{N}$, $\mathcal{L} > 1$. Then for all $z \in \mathbb{R}$ and $N \ge k\mathcal{L}$ we have*

$$\mathbb{E}_{SO(2N)}\left[\Lambda_{\mathcal{L}}(z)^{2k}\right] = \frac{1}{(2\pi i)^{2k}} \int \cdots \int \frac{(u_1 \cdots u_{2k})^{-\mathcal{L}}}{\prod_{1\le i<j\le 2k}(1 - z^2 u_i u_j)} \prod_{j=1}^{2k} \frac{du_j}{u_j(1 - u_j)}$$

where the integration is around small circles of radii less than $\min(|z|^{-1}, 1)$.

To prove this we must use an alternative method to before since there is no analogous result of Diaconis–Gamburd [8] for the special orthogonal group. Instead, we use the following result of Conrey-Farmer-Keating-Rubinstein-Snaith.

Theorem ([6]). *Let $d\mu$ denote the Haar measure on $SO(2N)$. Then for $m \geqslant 1$ we have*

$$\int_{SO(2N)} \Lambda(w_1)\cdots\Lambda(w_m)d\mu = w_1^N\cdots w_m^N\left[\sum_{\epsilon_j\in\{1,-1\}}\left(\prod_{j=1}^m w_j^{N\epsilon_j}\right)\prod_{1\le i<j\le m}\frac{1}{1-w_i^{-\epsilon_i}w_j^{-\epsilon_j}}\right].$$

To begin with, we use the integral

$$\frac{1}{2\pi i}\int_{(c)} e^{Ys}\frac{ds}{s} = \begin{cases}1, & Y>0\\ 0, & Y<0\end{cases} \qquad (c>0) \tag{33}$$

to write

$$\Lambda_{\mathcal{L}}(z) = \frac{1}{2\pi i}\int_{(c)}\Lambda(ze^{-s})e^{\mathcal{L}s}\frac{ds}{s}$$

for each of the factors $\Lambda_{\mathcal{L}}(z)$. Pushing through the expectation then gives

$$\mathbb{E}_{SO(2N)}\left[\Lambda_{\mathcal{L}}(z)^{2k}\right] = \frac{1}{(2\pi i)^{2k}}\int_{(c_{2k})}\cdots\int_{(c_1)}\mathbb{E}\left[\prod_{i=1}^{2k}\Lambda(ze^{-s_i})\right]\prod_{i=1}^{2k}e^{\mathcal{L}s_i}\frac{ds_i}{s_i}$$

where, for reasons that will become apparent, we take $c_1 > c_2 > \cdots > c_{2k} > \max\{0, \log|z|\}$ and $c_i - c_{i-1} > 2\log|z|$ for $i = 2, \ldots, 2k$. Using the theorem with $w_i = ze^{-s_i}$ for $i = 1, \ldots, 2k$ then gives

$$\mathbb{E}_{SO(2N)}\left[\Lambda_{\mathcal{L}}(z)^{2k}\right] = \sum_{\epsilon_j\in\{1,-1\}}\frac{1}{(2\pi i)^{2k}}\int_{(c_{2k})}\cdots\int_{(c_1)}|z|^{2Nk}\left(\prod_{i=1}^{2k}z^{N\epsilon_i}e^{-N\epsilon_i s_i}\right)\times$$
$$\times\prod_{1\le i<j\le 2k}\frac{1}{1-z^2e^{\epsilon_i s_i+\epsilon_j s_j}}\prod_{j=1}^{2k}e^{(\mathcal{L}-N)s_j}\frac{ds_j}{s_j}.$$

We would like to show that any term with $\epsilon_j = 1$ for some $j \in \{1, \ldots, 2k\}$ gives zero contribution. To this end, let $n = \min\{j \in \{1, \ldots, 2k\} : \epsilon_j = 1\}$. When integrating over $s_1, \ldots, s_{n-1}$ we only need to keep track of the highest power of e^{s_n}. For the s_1 integral, write

$$\frac{1}{1-z^2e^{-s_1+\epsilon_j s_j}} = \sum_{m\ge 0} z^{2m}e^{m(-s_1+\epsilon_j s_j)}$$

for $j = 2, \dots, 2k$. Using (33) we see that $m \le \mathcal{L}$ in each sum, so the highest possible contribution of powers of e^{s_n} is $e^{\mathcal{L}s_n}$. Further, as $\epsilon_2 = \cdots = \epsilon_{n-1}$, this integral contributes a nonpositive power of $e^{s_2}, \dots, e^{s_{n-1}}$.

Continuing in this fashion and integrating $s_2, \dots, s_{n-1}$, we deduce that one is left with

$$\frac{1}{(2\pi i)^{2k-n+1}} \int_{(c_{2k})} \cdots \int_{(c_{n+1})} \left(\int_{(c_n)} \prod_{j=n+1}^{2k} \frac{1}{1 - z^2 e^{s_n + \epsilon_j s_j}} e^{(\mathcal{L}-2N)s_n} \left(e^{(n-1)\mathcal{L}s_n} + \cdots \right) \frac{ds_n}{s_n} \right) \times$$
$$\times \left(\prod_{j=n+1}^{2k} z^{N\epsilon_j} e^{(\mathcal{L}-N-N\epsilon_j)s_j} \right) \prod_{n+1 \le i < j \le 2k} \frac{1}{1 - z^2 e^{\epsilon_i s_i + \epsilon_j s_j}} \prod_{j=n+1}^{2k} \frac{ds_j}{s_j}$$

multiplied by some power of z, where the additional terms in $\left(e^{(n-1)\mathcal{L}s_n} + \cdots\right)$ are all lower powers of e^{s_n}. For the innermost integral, expanding the factors in power series gives

$$\frac{1}{2\pi i} \int_{(c_n)} \prod_{j=n+1}^{2k} \frac{1}{1 - z^2 e^{s_n + \epsilon_j s_j}} e^{(\mathcal{L}-2N)s_n} \left(e^{(n-1)\mathcal{L}s_n} + \dots \right) \frac{ds_n}{s_n}$$
$$= \frac{1}{2\pi i} \int_{(c_n)} \left(\prod_{j=n+1}^{2k} z^{-2} e^{-s_n - \epsilon_j s_j} \sum_{m \ge 0} x^{-2m} e^{(-s_n - \epsilon_j s_j)m} \right) e^{(\mathcal{L}-2N)s_n} \left(e^{(n-1)\mathcal{L}s_n} + \dots \right) \frac{ds_n}{s_n}.$$

The highest possible power of e^{s_n} amongst all terms is $e^{(2k-n+(n-1)\mathcal{L}+\mathcal{L}-2N)s_n}$ which is negative for $N \ge \mathcal{L}k$ and $\mathcal{L} > 1$. By (33) all terms are thus zero.

This leaves

$$\mathbb{E}_{SO(2N)} \left[\Lambda_{\mathcal{L}}(z)^{2k} \right] = \frac{1}{(2\pi i)^{2k}} \int_{(c_{2k})} \cdots \int_{(c_1)} \prod_{1 \le i < j \le 2k} \frac{1}{1 - z^2 e^{-s_i - s_j}} \prod_{j=1}^{2k} e^{\mathcal{L}s_j} \frac{ds_j}{s_j}.$$

In order to arrive at the contour integral, we expand each factor of the product as

$$\frac{1}{1 - z^2 e^{-s_i - s_j}} = \sum_{m_{ij} \ge 0} z^{2m_{ij}} e^{(-s_i - s_j)m_{ij}}.$$

Separating the integrals and using (33) then gives

$$\mathbb{E}_{SO(2N)} \left[\Lambda_{\mathcal{L}}(z)^{2k} \right] = \sum_{(m_{ij})_{i,j=1}^{2k} \in A_k(\mathcal{L})} z^{2 \sum_{1 \le i < j \le 2k} m_{ij}}$$

where

$$A_k(\mathcal{L}) = \left\{ (m_{ij}) \in \mathbb{Z}_{\ge 0}^{k(2k-1)} : \sum_{j=1}^{i-1} m_{ji} + \sum_{j=i+1}^{2k} m_{ij} \le \mathcal{L}, i = 1, \dots, 2k \right\}.$$

As for $U(N)$ we invoke the conditions $\sum m_{ij} \leq \mathcal{L}$ with the formula (26), giving

$$\mathbb{E}_{SO(2N)}\left[\Lambda_{\mathcal{L}}(z)^{2k}\right] = \sum_{m_{ij}\geq 0} z^{2\sum_{1\leq i<j\leq 2k} m_{ij}} \frac{1}{(2\pi i)^{2k}} \int_{|u_{2k}|=\varepsilon_{2k}} \cdots \int_{|u_1|=\varepsilon_1} \times$$

$$u_1^{m_{12}+m_{13}+\cdots+m_{1,2k}-\mathcal{L}} u_2^{m_{12}+m_{23}+\cdots+m_{2,2k}-\mathcal{L}} \cdots u_{2k}^{m_{1,2k}+m_{2,2k}+\cdots+m_{2k-1,2k}-\mathcal{L}} \prod_{j=1}^{2k} \frac{du_j}{u_j(1-u_j)}.$$

On collecting like powers and computing the geometric series we aquire Proposition 4.

6.2. **Asymptotics for the multiple contour integral.** Denote the integral in Proposition 4 by I. Again we perform manipulations similar to those in section 4.2.

Let $u_j \mapsto |z|^{-1}u_j$ for $j = 1, \ldots, 2k$. Then

$$I = \frac{x^{2k\log|z|}}{(2\pi i)^{2k}} \int \cdots \int \frac{(u_1\cdots u_{2k})^{-\mathcal{L}}}{\prod_{1\leq i<j\leq 2k}(1-u_iu_j)} \prod_{j=1}^{2k} \frac{du_j}{u_j(1-|z|^{-1}u_j)}.$$

Next, let $u_j \mapsto u_j^{1/\mathcal{L}}$ for $j = 1, \ldots, 2k$. This gives

$$I = \frac{x^{2k\log|z|}}{(2\pi i)^{2k}} \int \cdots \int \frac{(u_1\cdots u_{2k})^{-1}}{\prod_{1\leq i<j\leq 2k}(1-(u_iu_j)^{1/\mathcal{L}})} \prod_{j=1}^{2k} \frac{du_j}{\mathcal{L}u_j(1-|z|^{-1}u_j^{1/\mathcal{L}})}$$

which can be expressed as

$$\frac{I}{x^{2k\log|z|}\mathcal{L}^{2k^2-3k}} = \frac{1}{(2\pi i)^{2k}} \int \cdots \int \frac{(u_1\cdots u_{2k})^{-1}}{\prod_{1\leq i<j\leq 2k}\mathcal{L}(1-(u_iu_j)^{1/\mathcal{L}})} \prod_{j=1}^{2k} \frac{du_j}{u_j(1-|z|^{-1}u_j^{1/\mathcal{L}})}.$$

Here, the contours wind around the origin $\mathcal{L}$ times. We now choose the radii of the contours to be independent of $\mathcal{L}$ and then write the integral in parametrised form. Upon taking the limit as $\mathcal{L} \to \infty$ and pushing the limit through the integrals we acquire

$$I \sim \frac{x^{2k\log|z|}\mathcal{L}^{2k^2-3k}}{(1-|z|^{-1})^{2k}}\gamma(k)$$

where $\gamma(k)$ is given by (14).

7. Concluding remarks

Admittedly, our evidence for Conjecture 1 is rather weak and there is a certain level of ambiguity in choosing the size N of the matrices. However, it is interesting that for our choice of $N = \log x$ the random matrix expectation seems to capture the phase change that we expect to see from the expectation of the Steinhaus variables. Indeed, if the conjecture holds for $k = 1/2$, then we can obtain the order of magnitude predicted by the conjecture for $0 \leqslant k \leqslant 1$ using Hölders inequality since we know the value at $k = 1$. Also, it seems a little strange, but not impossible, that one could obtain more than square-root cancellation in the case $k = 1/2$ as conjectured by Helson.

Finally, we note the following argument taken from [5] which gives an upper bound on the Steinhaus expectation for $k = 1/2$. Let $0 \leqslant u, v < 1$ and let $S_x = \sum_{n \leqslant x} X_n$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned} \mathbb{E}\big[|S_x|\big]^2 \leqslant & \mathbb{E}\big[|(1-uX_2)(1-vX_3)S_x|^2\big] \cdot \mathbb{E}\big[|(1-uX_2)(1-vX_3)|^{-2}\big] \\ = & \frac{1}{(1-u^2)(1-v^2)} \mathbb{E}\big[|(1-uX_2)(1-vX_3)S_x|^2\big] \end{aligned} \tag{34}$$

Now,

$$\begin{aligned} & \mathbb{E}\big[|(1-uX_2)(1-vX_3)S_x|^2\big] \\ & = \mathbb{E}\bigg[\Big(1 - 2u\Re(X_2) + u^2 - 2v\Re(X_3) + 4uv\Re(X_2)\Re(X_3) - 2u^2v\Re(X_3) \\ & \qquad + v^2 - 2uv^2\Re(X_2) + u^2v^2\Big)|S_x|^2\bigg] \\ & \sim x\Big(1 - u + u^2 - \tfrac{2}{3}v + \tfrac{2}{3}uv - \tfrac{2}{3}u^2v + v^2 - uv^2 + u^2v^2\Big). \end{aligned} \tag{35}$$

In this last line we have expanded the square of S_x and used

$$\mathbb{E}\Big[\sum_{m,n \leq x} X_{am}\overline{X}_{bn}\Big] = \sum_{\substack{am=bn \\ m,n \leqslant x}} 1 \sim \frac{1}{ab}x.$$

For $0 \leqslant u, v < 1$, the minimum of the function

$$f(x,y) = \frac{1 - u + u^2 - \frac{2}{3}v + \frac{2}{3}uv - \frac{2}{3}u^2v + v^2 - uv^2 + u^2v^2}{(1-u^2)(1-v^2)}$$

is found to be $\approx 0.8164965809...$. Taking square roots gives

$$\mathbb{E}\big[|S_x|\big] \leqslant (1 + o(1)) \cdot 0.903...\sqrt{x}.$$

Of course, further optimisations may prove to disprove conjecture (12).

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Chapter 3

Derivation of Theorem 1: taking the limit as $\mathcal{L} \to \infty$

3.1 Outline

In the proof of Theorem 1 in the article, we say that we truncate the integrals at height $T = o(\mathcal{L})$, which then makes it safe to take the limit as $\mathcal{L} \to \infty$. This gives equation (24) in the article. In this chapter a similar truncation and the resulting limit are performed in full detail by using Taylor series, which gives an explicit error term. In the rest of this chapter, set $\mathcal{L} = \log x$.

Define

$$I_k(x) = \frac{x^{(1-2\sigma)k}}{(2\pi i)^{2k}} \int_{(b_{2k})} \cdots \int_{(b_1)} A_k(\sigma+s_1, \dots, \sigma+s_k, 1-\sigma+s_{k+1}, \dots, 1-\sigma+s_{2k}) \prod_{i=1}^{k} \prod_{j=k+1}^{2k} \zeta(1+s_i+s_j) \prod_{j=1}^{k} x^{s_j} \frac{ds_j}{s_j} \prod_{j=k+1}^{2k} x^{s_j} \frac{ds_j}{s_j+1-2\sigma} \quad (3.1)$$

with $b_1 = \epsilon < 1-2\sigma$ and $b_j = 2$ for $j = 2, \cdots, 2k$ and

$$A_k(z_1, \cdots, z_{2k}) = \prod_p \left[\prod_{i=1}^{k} \prod_{j=k+1}^{2k} \left(1 - \frac{1}{p^{z_i+z_j}}\right) \right] \cdot \sum_{\substack{m_1+\ldots+m_k \\ =m_{k+1}+\cdots+m_{2k}}} \frac{1}{p^{m_1 z_1 + \cdots + m_{2k} z_{2k}}}. \quad (3.2)$$

Referring to equation (22) in the article this gives

$$\mathbb{E}\left[\Big|\sum_{n\le x} X_n/n^\sigma\Big|^{2k}\right] = I_k(x). \tag{3.3}$$

The goal is to show that this is asymptotically equal to $a(k)g(k)x^{(1-2\sigma)k}\mathcal{L}^{(k-1)^2}$, where $a(k)$ is the usual arithmetic factor

$$a(k) = \prod_p \left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^m}, \tag{3.4}$$

$d_k(n) = \sum_{n_1\cdots n_k=n} 1$ is the k-fold divisor function, and

$$g(k) = \frac{1}{(2\pi i)^{2k}}\int_{(b_{2k})}\cdots\int_{(b_2)}\left(\int_{(b_1)}\frac{ds_1}{s_1^k(1-2\sigma-s_1)^k}\right) \times\prod_{i=2}^{k}\prod_{j=k+1}^{2k}\frac{1}{s_i+s_j}e^{s_2+\cdots s_{2k}}\prod_{j=2}^{k}ds_j\prod_{j=k+1}^{2k}\frac{ds_j}{s_j}. \tag{3.5}$$

Substitute $s_j \mapsto s_j + s_1$ for $j = 2,\ldots,k$ and $s_j \mapsto s_j - s_1$ for $j = k+1,\ldots,2k$. The necessity of this substitution will become clear when computing the error term from the truncation. This gives

$$I_k(x) = \frac{x^{(1-2\sigma)k}}{(2\pi i)^{2k}}\int_{(c_{2k})}\cdots\int_{(c_2)}\int_{(b_1)} A_k(\sigma+s_1,\sigma+s_2+s_1,\ldots,\sigma+s_k+s_1,1-\sigma+s_{k+1}-s_1,\ldots,1-\sigma+s_{2k}-s_1) \prod_{i=2}^{k}\prod_{j=k+1}^{2k}\zeta(1+s_i+s_j)\prod_{j=k+1}^{2k}\zeta(1+s_j)\frac{ds_1}{s_1}\prod_{j=2}^{k}\frac{x^{s_j}ds_j}{s_j+s_1}\prod_{j=k+1}^{2k}\frac{x^{s_j}ds_j}{s_j+1-2\sigma-s_1} \tag{3.6}$$

where $c_j = b_j - b_1 = 2-\epsilon$ for $j = 2,\ldots,k$ and $c_j = b_j + b_1 = 2+\epsilon$ for $j = k+1,\ldots,2k$.

From here the first step is to cut the integrals over $s_2,\ldots,s_{2k}$ at some height $T = C$. Then the correct power of $\mathcal{L}$ is extracted by substituting $s_j \mapsto s_j/\mathcal{L}$ for $j = 2,\ldots,2k$. Because the integrals are cut at an appropriate height one can express the integrand as a Taylor series to get the leading order term.

3.2 Truncation error

The integrals in (3.1) are not absolutely convergent, so one can therefore not bound the truncation error directly. Instead the same smoothing technique as in [4, Chapter II.5] and [1] is used. Namely, we first integrate the expression for $I_k(x)$ with respect to x, then cut the integrals at the desired height and finally take the derivative with respect to x.

Integrating with respect to x gives

$$\int_0^x I_k(t)dt = \frac{1}{(2\pi i)^{2k}} \int_{(c_{2k})} \cdots \int_{(c_2)} \int_{(b_1)} A_k(\dots) \prod_{i=2}^{k} \prod_{j=k+1}^{2k} \zeta(1+s_i+s_j)$$
$$\prod_{j=k+1}^{2k} \zeta(1+s_j) \frac{x^{1+(1-2\sigma)k+s_2+\cdots+s_{2k}}}{1+(1+2\sigma)k+s_2+\cdots+s_{2k}} \frac{ds_1}{s_1}$$
$$\prod_{j=2}^{k} \frac{ds_j}{s_j+s_1} \prod_{j=k+1}^{2k} \frac{ds_j}{s_j+1-2\sigma-s_1}. \quad (3.7)$$

At this point, one would like to cut the integrals over $s_2, \dots, s_{2k}$ at some height $T = C$. In order to avoid a too high error term from the factor $x^{1+(1-2\sigma)k+s_2+\cdots+s_{2k}}$ one first has to shift the lines of integration over $s_2, \cdots, s_{2k}$ to $d_j = 2\mathcal{L}^{-1}$, which can be done, as the integrals are absolutely convergent. The error terms after cutting the integrals at height C will be of the form

$$\int_{(d_{2k})} \cdots \int_{d_j+iC}^{d_j+i\infty} \cdots \int_{(d_2)} \int_{(b_1)} (\cdots) \quad (3.8)$$

and

$$\int_{(d_{2k})} \cdots \int_{d_j-i\infty}^{d_j-iC} \cdots \int_{(d_2)} \int_{(b_1)} (\cdots) \quad (3.9)$$

for $j = 2, \cdots, 2k$, with the integrand the same as in (3.7).

In order to bound these error terms the following estimates will be used to bound the zeta function terms. The Laurent series of $\zeta(1+s)$ is given by [2, Formula (2.1.16)]

$$\zeta(1+s) = \frac{1}{s} + \gamma + O(|s|) \quad (3.10)$$

where γ is Euler's Gamma constant. This will be used for $|s| \leq C$, which gives $\zeta(1+s) = \frac{1}{s} + O(1)$. For $|s|$ near zero it will be used that

$$\zeta(1+s) = \sum_{n=1}^{\infty} \frac{1}{n^{1+s}} \ll \sum_{n=1}^{\infty} \frac{1}{n^{1+\sigma}} \ll \frac{1}{\sigma}. \tag{3.11}$$

The last relation in the above can be shown using e.g. Euler summation [3, p. 54].

For $|s|$ large, the following theorem is used.

Theorem 3.1 (Theorem 3.5 in [2]). *We have*

$$\zeta(s) = O(\log|t|) \tag{3.12}$$

uniformly in the region

$$1 - \frac{A}{\log t} \leq \sigma \leq 2 \quad (t > t_0)$$

where A is any positive constant and t_0 is a fixed constant.

Note that there exist stronger results for the region $\sigma > 1$, to which this will be applied, in e.g. [6]. As such an improvement will not have any effect on the order of the error term, the bound in (3.12) is sufficient.

It is instructive to consider three different cases before computing the full error. First, consider the integral (3.8) integrated only over those regions where $|t_i + t_j| > t_0$ and $|t_j| > t_0$ for $i = 2, \ldots, k$ and $j = k+1, \ldots, 2k$. Bounding $|x^{1+(1-2\sigma)k+s_2+\cdots+s_{2k}}| = x^{1+(1-2\sigma)k+d_2+\cdots+d_{2k}} \ll x^{1+(1-2\sigma)k}$ and $A_k(\ldots) \ll 1$[1], and using Theorem 3.1 for all zeta-terms gives that this is

$$\ll x^{(1-2\sigma)k+1} \int \cdots \int \prod_{i=2}^{k} \prod_{j=k+1}^{2k} \log|t_i + t_j| \prod_{j=k+1}^{2k} \log|t_j|$$
$$\frac{1}{|1 + s_2 + \cdots + s_{2k}|} \frac{|ds_1|}{|s_1|} \prod_{j=2}^{k} \frac{|ds_j|}{|s_j + s_1|} \prod_{j=k+1}^{2k} \frac{|ds_j|}{|s_j + 1 - 2\sigma - s_1|}. \tag{3.13}$$

The remaining integral can be bounded independently of x, as $b_1 = \epsilon$ is independent of x. The integral converges in the mentioned region, giving an error $O(x^{1+(1-2\sigma)k})$.

[1]See the discussion after Equation (19) in the article

Next, consider the region where $\mathcal{L}^{-a} \le |t_2 + t_{2k}| \le t_0$, and $|t_i + t_j| > t_0$, $|t_j| > t_0$ for all other values of i, j. Further, assume $|t_j - t_{2k}| > t_0$ for $j = k+1, \ldots, 2k-1$. The integral over s_2 is then of length less than $t_0 = O(1)$. For the factor $\zeta(1 + s_2 + s_{2k})$, (3.10) is used to give

$$\zeta(1 + s_2 + s_{2k}) \ll \frac{1}{|s_2 + s_{2k}|} \ll \frac{1}{|t_2 + t_{2k}|} \ll \mathcal{L}^a,$$

and for the other zeta terms the bound in Theorem 3.1 is used as before. When integrating over s_2 in this region one can essentially use the M-L inequality to replace all instances of s_2 by $-s_{2k}$ and multiply by the length of the integral. This leaves an error

$$\ll x^{1+(1-2\sigma)k}\mathcal{L}^a \int \cdots \int \prod_{i=3}^{k} \prod_{j=k+1}^{2k} \log |t_i + t_j| \prod_{j=k+1}^{2k} \log |t_j| \prod_{j=k+1}^{2k-1} \log |t_j - t_{2k}|$$
$$\frac{1}{|1 + (1-2\sigma)k + s_3 + \cdots + s_{2k-1}|} \frac{1}{|s_1 - s_{2k}|} \frac{|ds_1|}{|s_1|}$$
$$\prod_{j=3}^{k} \frac{|ds_j|}{|s_j + s_1|} \prod_{j=k+1}^{2k} \frac{|ds_j|}{|s_j + 1 - 2\sigma - s_1|} \qquad (3.14)$$

where the integral converges in the mentioned region, and is independent of x once more, to give an error $O(x^{1+(1-2\sigma)k}\mathcal{L}^a)$.

Thirdly, consider the region where $|t_2 + t_{2k}| \le \mathcal{L}^{-a}$ and $|t_i + t_j| > t_0$, $|t_j| > t_0$, $|t_j - t_{2k}| > t_0$ as before. Using (3.11) gives $\zeta(1 + s_2 + s_{2k}) \ll \mathcal{L}$ in this region. When integrating over s_2 the M-L inequality is used again, to give something very similar to (3.14), except that the length over the s_2-integral is $O(\mathcal{L}^{-a})$. This gives an error $O(x^{1+(1-2\sigma)k}\mathcal{L}\mathcal{L}^{-a}) = O(x^{1+(1-2\sigma)k}\mathcal{L}^{1-a})$.

One could continue like this to bound each region separately, but it is clear that the above three tricks should cover everything. Depending on our choice of a, the largest error term will occur in one of the following cases.

(i) $|t_i + t_j| \le \mathcal{L}^{-a}$ and $|t_j| \le \mathcal{L}^{-a}$ for as many as possible terms.

(ii) $\mathcal{L}^{-a} \le |t_i + t_j| \le t_0$ and $\mathcal{L}^{-a} \le |t_j| \le t_0$ for as many as possible terms.

So far it hasn't been specified which integral is being truncated, so assume that the s_2-integral is cut. This gives that $|t_2| > C$, and choosing $C = 2t_0$ gives $|t_2| > 2t_0$. It is clearly not possible to have $|t_2 + t_j| \le t_0$ and $|t_j| < t_0$ at the same time as $|t_2| > 2t_0$. This gives that there are at most $k^2 - k$ ζ-terms that contribute a factor $\mathcal{L}$ in the first case, and a factor $\mathcal{L}^a$ in the second case. The remaining k ζ-terms can be bounded by Theorem 3.1.

In the first case, the integrals over $s_3, \cdots, s_{2k}$ are each of length $O(\mathcal{L}^{-a})$, and the M-L inequality will give $2k-2$ factors $\mathcal{L}^{-a}$. In the second case each of these $2k-2$ integrals are of length $O(1)$. In both cases the final integrals over s_1, s_2 are convergent and can be bounded independently of x. Combining this gives an error

$$O\left(x^{1+(1-2\sigma)k}\left(\mathcal{L}^{a(k^2-k)} + \mathcal{L}^{k^2-k-2a(k-1)}\right)\right).$$

The leading order term for the full integral will be of order $O\left(x^{1+(1-2\sigma)k}\mathcal{L}^{(k-1)^2}\right)$, so one needs to choose a in such a way that

$$\begin{aligned} a(k^2-k) &< (k-1)^2, \\ k^2-k-2a(k-1) &< (k-1)^2. \end{aligned} \tag{3.15}$$

The first requirement reduces to $a \le 1 - \frac{1}{k}$ and the second requirement reduces to $a > \frac{1}{2}$. Choosing e.g $a = \frac{3}{5}$ then gives a lower order error for $k \ge 3$.

The above gives a bound for the error introduced from cutting the s_2-integral. Similarly, the truncation error from the integrals over $s_3, \ldots, s_{2k}$ can be bounded, to give

$$\int_0^x I_k(t)dt = \frac{1}{(2\pi i)^{2k}} \int_{d_{2k}-iC}^{d_{2k}+iC} \cdots \int_{d_2-iC}^{d_2+iC} \int_{(b_1)} A_k(\ldots) \prod_{i=2}^{k} \prod_{j=k+1}^{2k} \zeta(1+s_i+s_j)$$

$$\prod_{j=k+1}^{2k} \zeta(1+s_j) \frac{x^{1+(1-2\sigma)k+s_2+\cdots+s_{2k}}}{1+(1-2\sigma)k+s_2+\cdots+s_{2k}} \frac{ds_1}{s_1} \prod_{j=2}^{k} \frac{ds_j}{s_j+s_1} \prod_{j=k+1}^{2k} \frac{ds_j}{s_j+1-2\sigma-s_1}$$

$$+ O\left(x^{1+(1-2\sigma)k}\mathcal{L}^{a(k^2-k)} + x^{1+(1-2\sigma)k}\mathcal{L}^{k^2-k-2a(k-1)}\right) \tag{3.16}$$

3.3 Taylor series

The next step is to expand the integrand in (3.16) as a Taylor series in $\mathcal{L}$. In order to do this, substitute $s_j \mapsto s_j/\mathcal{L}$ for $j = 2, \dots, 2k$. This gives

$$\int_0^x I_k(t)dt = \frac{x^{1+(1-2\sigma)k}\mathcal{L}^{(k-1)^2}}{(2\pi i)^{2k}} \int_{b_{2k}-iT}^{b_{2k}+iT} \cdots \int_{b_2-iT}^{b_2+iT} \int_{(b_1)}$$

$$A_k(\sigma+s_1, \sigma-s_1+s_2/\mathcal{L}, \dots, \sigma-s_1+s_k/\mathcal{L}, 1-\sigma-s_1+s_{k+1}/\mathcal{L}, \dots, 1-\sigma-s_1+s_{2k}/\mathcal{L})$$

$$\prod_{i=2}^{k} \prod_{j=k+1}^{2k} \frac{1}{\mathcal{L}}\zeta\left(1 + \frac{s_i+s_j}{\mathcal{L}}\right) \prod_{j=k+1}^{2k} \frac{1}{\mathcal{L}}\zeta\left(1+\frac{s_j}{\mathcal{L}}\right)$$

$$\frac{e^{s_2+\cdots+s_{2k}}}{1+(1-2\sigma)k+\frac{s_2+\cdots+s_{2k}}{\mathcal{L}}} \frac{ds_1}{s_1} \prod_{j=2}^{k} \frac{ds_j}{\frac{s_j}{\mathcal{L}}+s_1} \prod_{j=k+1}^{2k} \frac{ds_j}{\frac{s_j}{\mathcal{L}}+1-2\sigma-s_1}$$

$$+ O\left(x^{1+(1-2\sigma)k}\mathcal{L}^{a(k^2-k)} + x^{1+(1-2\sigma)k}\mathcal{L}^{k^2-k-2a(k-1)}\right)$$

where $b_j = 2$ for $j = 2, \dots, 2k$ as before, and $T = C\mathcal{L}$.

As $T = O(\mathcal{L})$ one can use (3.10) to get $\frac{1}{\mathcal{L}}\zeta(1+(s_i+s_j)/\mathcal{L}) = \frac{1}{s_i+s_j} + O\left(\frac{1}{\mathcal{L}}\right)$ and $\frac{1}{\mathcal{L}}\zeta(1+s_j/\mathcal{L}) = \frac{1}{s_j} + O\left(\frac{1}{\mathcal{L}}\right)$ for $i = 2, \dots, k$ and $j = k+1, \dots, 2k$. The remaining terms in the integrand are all nicely bounded as $\mathcal{L} \to \infty$, so replacing $\frac{1}{\mathcal{L}}\zeta(1+(s_i+s_j)/\mathcal{L})$ by $\frac{1}{s_i+s_j}$ everywhere, and $\frac{1}{\mathcal{L}}\zeta(1+s_j/\mathcal{L})$ by $\frac{1}{s_j}$ everywhere introduces an error $O(x^{1+(1-2\sigma)k}\mathcal{L}^{(k-1)^2-1})$.

For $A_k(\cdots)$ one has that $\lim_{\mathcal{L}\to\infty} A_k(\sigma+s_1, \sigma-s_1+s_2/\mathcal{L}, \dots, \sigma+s_1+s_k/\mathcal{L}, 1-\sigma-s_1+s_{k+1}/\mathcal{L}, \dots, 1-\sigma-s_1+s_{2k}/\mathcal{L}) = A_k(\sigma+s_1, \cdots, \sigma+s_1, 1-\sigma-s_1, \cdots, 1-\sigma-s_1) = a(k)$ by (3.2) and (3.4). Expanding the remaining terms that depend on $\mathcal{L}$ as a Taylor series in $\mathcal{L}^{-1}$ then gives

$$\int_0^x I_k(t)dt = \frac{a(k)x^{1+(1-2\sigma)k}\mathcal{L}^{(k-1)^2}}{(1+(1-2\sigma)k)(2\pi i)^{2k}} \int_{b_{2k}-iT}^{b_{2k}+iT} \cdots \int_{b_2-iT}^{b_2+iT} \int_{(b_1)} \prod_{i=2}^{k}\prod_{j=k+1}^{2k} \frac{1}{s_i+s_j}$$

$$e^{s_2+\cdots+s_{2k}} \frac{ds_1}{s_1^k(1-2\sigma-s_1)^k} \prod_{j=2}^{k} ds_j \prod_{j=k+1}^{2k} \frac{ds_j}{s_j}$$

$$+ O\left(x^{1+(1-2\sigma)k}\left(\mathcal{L}^{a(k^2-k)} + \mathcal{L}^{k^2-k-2a(k-1)} + \mathcal{L}^{(k-1)^2-1}\right)\right). \quad (3.17)$$

It now only remains to extend the integrals back to the full vertical lines, and then take the derivative with respect to x. Extending the integrals back to their full height gives error terms of the form (3.8)-(3.9), where the integrand is equal to the integrand in (3.17), the truncation is at height T instead of C, and one has $\Re s_j = b_j = 2$ instead of $\Re s_j = \frac{2}{\mathcal{L}}$ for $j = 2, \dots, 2k$.

In order to bound these terms, one essentially uses that $\frac{1}{|s_i+s_|} \ll \frac{1}{|\sigma_i+\sigma_j|+|t_i+t_j|}$ and then treats the cases $|t_i+t_j| < |\sigma_i+\sigma_j| = O(1)$ and $|t_i+t_j| > |\sigma_i+\sigma_j|$ separately. In all cases one gets an error of lower order than $x^{1+(1-2\sigma)k}\mathcal{L}^{(k-1)-1}$, so the integrals can be extended back to their full height.

3.4 Derivation with respect to x

Combining the definition of $g(k)$ in (3.5) and the manipulations in the previous section gives

$$\int_0^x I_k(t)dt = \frac{x^{1+(1-2\sigma)k}}{1+(1-2\sigma)k}\mathcal{L}^{(k-1)^2}a(k)g(k) + O\left(x^{1+(1-2\sigma)k}\left(\mathcal{L}^{(k-1)^2-1} + \mathcal{L}^{a(k^2-k)} + \mathcal{L}^{k^2-k-2a(k-1)}\right)\right). \quad (3.18)$$

Using (3.18) with $x+h$ and x, and subtracting, one arrives at

$$\int_x^{x+h} I_k(t)dt = \frac{(x+h)^{1+(1-2\sigma)k}}{1+(1-2\sigma)k}(\log(x+h))^{(k-1)^2}a(k)g(k) - \frac{x^{1+(1-2\sigma)k}}{1+(1-2\sigma)k}(\log x)^{(k-1)^2}a(k)g(k) + O\left(x^{1+(1-2\sigma)k}\left(\mathcal{L}^{(k-1)^2-1} + \mathcal{L}^{a(k^2-k)} + \mathcal{L}^{k^2-k-2a(k-1)}\right)\right), \quad (3.19)$$

where h is chosen in such a way that $h = O(x)$. Then, using

$$f(x+h) - f(x) = hf'(x) + h^2\int_0^1 (1-t)f''(x+th)dt \quad (3.20)$$

with $f(x) = \frac{x^{1+(1-2\sigma)k}}{1+(1-2\sigma)k}\mathcal{L}^{(k-1)^2}a(k)g(k)$ gives

$$\int_x^{x+h} I_k(t)dt = a(k)g(k)hx^{(1-2\sigma)k}\mathcal{L}^{(k-1)^2} + O\left(x^{1+(1-2\sigma)k}\left(\mathcal{L}^{(k-1)^2-1} + \mathcal{L}^{a(k^2-k)} + \mathcal{L}^{k^2-k-2a(k-1)}\right)\right) + O\left(h^2x^{(1-2\sigma)k-1}\mathcal{L}^{(k-1)^2}\right)$$

and then

$$I_k(x) = a(k)g(k)x^{(1-2\sigma)k}\mathcal{L}^{(k-1)^2} + O\left(hx^{(1-2\sigma)k-1}\mathcal{L}^{(k-1)^2} + h^{-1}L\right) + O\left(h^{-1}x^{1+(1-2\sigma)k}\left(\mathcal{L}^{(k-1)^2-1} + \mathcal{L}^{a(k^2-k)} + \mathcal{L}^{k^2-k-2a(k-1)}\right)\right), \quad (3.21)$$

with $L = \int_x^{x+h} |I_k(t) - I_k(x)|dt$.

At this point it is needed that I_k is increasing in x. To see this, expand the powers on the left in (3.3) and push the expectation through. This is then a multiple sum of non-negative terms. The terms are non-negative because $\mathbb{E}\prod_i X_i \prod_i \overline{X}_i$ is either 0 or 1 for finite products (see Section 1.3.1). Increasing x then adds more terms, so $I_k(x) \leq I_k(y)$ if $x \leq y$. Therefore,

$$\int_x^{x+h} |I_k(t) - I_k(x)|dt \leq \int_x^{x+h} I_k(t)dt - \int_{x-h}^{x} I_k(t)dt,$$

and if one uses (3.20) and similarly

$$f(x) - f(x-h) = hf'(x) - h^2\int_0^1 (1-t)f''(x-th)dt$$

for $f(x) = \frac{x^{1+(1-2\sigma)k}}{1+(1-2\sigma)k}(\log x)^{(k-1)^2}a(k)g(k)$ together with (3.19), one then gets

$$L \ll x^{1+(1-2\sigma)k}\left(\mathcal{L}^{(k-1)^2-1} + \mathcal{L}^{a(k^2-k)} + \mathcal{L}^{k^2-k-2a(k-1)}\right) + h^2x^{(1-2\sigma)k-1}\mathcal{L}^{(k-1)^2}.$$

The term $h^{-1}L$ in (3.21) is then included in the other error terms. Finally, choosing $h = x\mathcal{L}^{-1/2}$ gives

$$I_k(x) = a(k)g(k)x^{(1-2\sigma)k}(\log x)^{(k-1)^2} + O\left(x^{(1-2\sigma)k}\left((\log x)^{(k-1)^2-1/2} + (\log x)^{a(k^2-k)-1/2} + (\log x)^{k^2-k-1/2-2a(k-1)}\right)\right)$$

for $\frac{1}{2} < a < 1$ and $k \geq \frac{1}{1-a}$. As mentioned previously, choosing e.g. $a = \frac{3}{5}$ gives

$$I_k(x) \sim a(k)g(k)x^{(1-2\sigma)k}(\log x)^{(k-1)^2}$$

for $k \geq 3$. The case $k = 1$ can be easily computed and the case $k = 2$ has been computed to a greater accuracy in [5], so it doesn't matter that these cases are lost here.

3.5 Shifting the variables by s_1

To get from (3.1) to (3.6) the substitutions $s_j \mapsto s_j + s_1$ for $j = 2, \ldots, k$ and $s_j \mapsto s_j - s_1$ for $j = k+1, \ldots, 2k$ were made. These substitutions might have seemed somewhat unmotivated at the time, but the necessity of the substitutions becomes clear when considering the error introduced in Section 3.2. Here it was essential that one couldn't have $|t_j + t_i| < t_0$ and $|t_j| < t_0$ for $i = 2, \ldots, k$ and $j = k+1, \ldots, 2k$ all at the same time. This was ensured by always cutting some integral at a height greater than t_0.

If one tries to do the same things directly to the integral in (3.1), the equivalent region contributing a large error will be $|t_i + t_j| < t_0$ for $i = 1, \ldots, k$ and $j = k+1, \ldots, 2k$. But even though some integral is cut, all of these inequalities can be satisfied at once. This in turn results in an error term larger than the main order term, so one doesn't get any results from this.

3.6 Rademacher case

The derivation in the Rademacher case is very similar to the derivation in the Steinhaus case, so it will not be repeated here in any greater detail than presented in Section 5 of the article.

It is still worthwhile noting why the error term is small enough also in the Rademacher case. As before, the main error term will come from all ζ-factors being evaluated near zero. For the Rademacher expression this translates to $|t_i + t_j| < t_0$ for $1 \leq i < j \leq 2k$. If one has $|t_j| > C = 2t_0$ for some j these conditions can't all be fulfilled at once, which makes the truncation possible.

Chapter 4

Final comments

4.1 Higher accuracy

The results in theorems 1 and 4 in the article are both asymptotic results. For Theorem 1 in the article an explicit error term is given in Chapter 3. In [5], Ayyad–Cochrane–Zheng show that $\mathbb{E}\left[\left|\sum_{n\leq x} X_n\right|^4\right] = Ax^2\log x + Bx^2 + o(x^2)$ with explicit constants A and B. It is interesting to see if one could obtain a similar polynomial for general k, i.e. something like

$$\mathbb{E}\left[\left|\sum_{n\leq x} X_n\right|^{2k}\right] = A_{(k-1)^2}x^{2k}(\log x)^{(k-1)^2} + A_{(k-1)^2-1}x^{2k}(\log x)^{(k-1)^2-1} + \cdots \\ \cdots + A_1x^{2k}\log x + A_0x^{2k} + o\left(x^{2k}\right) \quad (4.1)$$

for explicit constants A_i, $i = 0, \ldots, (k-1)^2$, and a similar expression for $\sigma \neq 0$.

If one follows the exact same procedure as in Chapter 3, with $\sigma = 0$, it is not immediately clear how to be more precise for the truncation error in Section 3.2, but for the Taylor series in Section 3.3 one could in principle include more terms in the Taylor expansion to get a more precise result. If one were to do this, the limit for the number of terms in the polynomial (4.1) will come from the error $O\left(x^{2k}(\log x)^{a(k^2-k)-1/2}) + x^{2k}(\log x)^{k^2-k-1/2-2a(k-1)}\right)$. Choosing a very close to 1, one sees that the best obtainable error term is $o\left(x^{2k}(\log x)^{k^2-3k+2}\right)$, which holds for $k > \frac{1}{1-a}$. Nevertheless, this means that one in principle could obtain the first k coefficients in (4.1) for large k, by including more terms in the Taylor expansion.

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