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A categorical approach to Cuntz-Pimsner C*-algebras

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Abstract

Using a C^*-algebra A, a Hilbert A-module E and a C^*-correspondence (E, φ_E) we use the language of category theory to construct $\mathcal{O}_{(E,\varphi_E)}(J)$, the Cuntz-Pimsner representation relative to an ideal J. We provide a complete classification, up to isomorphism, of the bijective representations admitting a gauge action as relative Cuntz-Pimsner representations relative to some ideal. By doing this we obtain a simple proof of the gauge invariant uniqueness theorem for the Cuntz-Pimsner algebra $\mathcal{O}_{(E,\varphi_E)}$ over (E, φ_E).

Sammendrag

Ved å se på en C^*-algebra A, en Hilbert A-modul E og en C^*-korrespondanse (E, φ_E) konstruerer vi $\mathcal{O}_{(E,\varphi_E)}(J)$, kjent som Cuntz-Pimsner-representasjonen relativ til et ideal J, ved å hente terminologi fra kategoriteori. Vi gir en komplett klassifisering, opp til isomorfi, av de bijektive representasjonene med en gauge-virkning som relative Cuntz-Pimsner-representasjoner for visse ideal. Vi bruker så dette til å gi et enkelt bevis av det gauge-invariant unikhetsteoremet for Cuntz-Pimsner-algebraen $\mathcal{O}_{(E,\varphi_E)}$ over (E, φ_E).

Preface

This thesis is written over a period of one year, ranging from August 2014 to May 2015, to complete the degree of Master of Mathematics at the Norwegian University of Science and Technology.

Before the spring of 2013 I was split between wanting to write a thesis in algebra or analysis. Fortunately, I took a course in functional analysis that spring which suddenly made the choice apparent to me. The same spring I also met Professor Toke Meier Carlsen for the first time when writing on a project which he supervised. It then became natural to ask him to supervise my Master's thesis as well. He eventually suggested what became the topic for this thesis. At that time I only really knew the definition of a C^*-algebra, so needless to say the thesis would be an ambitious venture. It is then good that I thrive outside of my comfort zone and that I was very lucky to be advised by Toke Meier Carlsen. Even though the communication has mostly been over e-mail and Skype once a week he has been able to explain complex concepts relatively simple. For this very helpful guidance I owe him a debt of gratitude. It has also been very useful to get feedback and help from Eduardo Ortega at his office.

During the writing process I have seen some very intriguing mathematics and I have also discovered a deeper interest for the subject itself. As a result, I have decided that I will pursue a Ph.D. at some point in the near future.

I would also like to extend gratitude towards the closest people around me. Worthy of special mention is Kristine Lund for enduring me at both my best and worse throughout this period; Tobias Grøsfjeld for bringing clarity in desperate times and also for the immense effort of reading through the whole thesis and giving thorough and pedantic feedback; And at last my grandparents, especially my grandfather, for without them none of this would have been possible.

Contents

Introduction

In [Pim97], Pimsner introduced a class of C^*-algebras - now called Cuntz-Pimsner algebras - constructed from a C^*-correspondence X over a C^*-algebra A. This class of algebras have been associated with direct products by endomorphisms, graph C^*-algebras and also the generalization known as topological graph algebras as well as many other examples.bIn his article Pimsner assumed the left action of X to be injective remarking that this is required just for simplicity. To remove this assumption Katsura in [Kat03] introduced the ideal J_X and Condition $(*)$ which lead to the class of C^*-algebras $\mathcal{O}_X$ generalizing Pimsner's construction.

In [CaOr11], Carlsen and Ortega introduced an algebraic analogue of the C^*-algebra associated with a C^*-correspondence, and described them using terminology from noncommutative ring theory. They showed that if one is interested in the C^*-algebraic case, then there is some insight gained by considering the purely algebraic object. Moreover their arguments become more tangible than in the C^*-algebraic setting, this allows them, for example, to put everything into a frame of category theory, which make the whole construction more transparent. A ray of light in that belief come in [COP12], where they advance in the problem of finding conditions to have control of the ideal structure of the Cuntz-Pimsner algebras.

In this thesis we follow the approach of Carlsen and Ortega and give a similar description of the C^*-algebras associated with C^*-correspondences. We follow Pimsner's approach of constructing the Cuntz-Pimsner algebras using the Fock space ([Pim97]). We note that this C^*-algebra induces a universal representation - the Toeplitz representation - in the category of surjective representations $\mathcal{C}_{(E,\varphi_E)}$ (analogous to [CaOr11, Theorem 1.7]), that it is injective and that it admits a gauge action. As in [Kat04] we introduce the ideal J_E and use the universality of the Toeplitz representation to obtain a new universal representation in the subcategory of $\mathcal{C}_{(E,\varphi_E)}$ consisting of surjective representations that are Cuntz-Pimsner invariant relative to an ideal $J \subset J_E$. This differs slightly from [Kat04], as he only considers the ideal J_E. The other extreme case is $J = \{0\}$ which gives the Toeplitz representation.

The advantage of considering ideals J between $\{0\}$ and J_E is that we can provide a full classification of the bijective representations admitting a gauge action as Cuntz-Pimsner invariant representations relative to some ideal J (similar to [CaOr11, Theorem 3.18] in the algebraic case). This is done in Theorem 6.9.

In the case $J = J_E$ we obtain the Cuntz-Pimsner representation ("covariant" in [Kat04]) and the Cuntz-Pimsner algebra. The notion of Cuntz-Pimsner invariance is not very natural looking at first glance, but Lemma 6.8 shows where this condition comes from. The Cuntz-Pimsner representation is interesting since it contains a true copy of A and (E, φ_E), so by looking at the Cuntz-Pimsner algebra we do not lose any information. By using Theorem 6.9 this is seen to be a terminal object in the category of bijective representation admitting a gauge action. This allows us to provide a simple proof for the important gauge invariant uniqueness theorem for Cuntz-Pimnser algebras

(Theorem 6.14).

The thesis is almost self-contained and much of the contents is included for the sake of introducing the author to the content.

In section 1 an introduction to Hilbert A-modules and some basic results regarding those and the C^*-correspondences are given. Mainly, we define the Hilbert A-modules (Definition 1.2), the direct sum of these, and consider the space of adjointable maps $\mathscr{L}(E,F)$ from one Hilbert A-module E to another F. We prove that $\mathscr{L}(E,F)$ is a subset of the linear and bounded maps from E to F (Proposition 1.4). Furthermore, in the case where $E = F$ the set $\mathscr{L}(E,E)$ form a C^*-algebra (Theorem 1.5). We show that this C^*-algebra has a closed two-sided ideal, namely the generalized compact operators $\mathscr{K}(E)$ which is the closed linear span of the rank-1 operators $\theta_{x,y}$. The C^*-correspondences of a Hilbert A-module is defined (Definition 1.8) and we construct the higher-order tensor product of these (Definition 1.9).

The next section is concerned with representations of Hilbert A-modules on a C^*-algebra. The definition (Definition 2.1) is stated and some basic properties follow. We also shortly study the necessary operators to define the Fock representation (Definition 2.3 and Lemma 2.4), and the Fock space (Definition 2.5) as the direct sum of the higher order tensor products given in section one. This space and the operators are combined in Theorem 2.6 to define the Fock representation and to prove that it is injective.

Following in the tracks of representations of Hilbert A-modules we define in section 3 the category of surjective representations of a C^*-correspondence (Definition 3.1) with the aim of showing that this has an initial object. This is done by considering the universal $*$-algebra $G(A,E)$ generated by the C^*-correspondence (E,φ_E) over A (Definition 3.2) which is then equipped with a seminorm (Lemma 3.3). This is then made into a C^*-algebra $\mathcal{T}_{(E,\varphi_E)}$ by sending the norm zero elements to zero in the quotient, showing that the resulting quotient norm satisfies the C^*-property and then completing with respect to this norm. This construction gives a natural way to make a representation of (E,φ_E) on $\mathcal{T}_{(E,\varphi_E)}$, which is called the Toeplitz representation. Further, this representation is seen to be initial in the category of surjective representations (Theorem 3.4) and by using the Fock representation we also show that it is injective (Theorem 3.5).

After this, we again return to the rank-1 operators $\theta_{x,y}$ and explore some of their properties before we show the existence of a particular $*$-homomorphism $\psi_t : \mathscr{K}(E) \to B$ (Proposition 4.3). This is done by defining the $*$-homomorphism on each $\theta_{x,y}$ and then extending it to $\mathscr{K}(E)$. Finally we see how ψ_t corresponds with the already defined maps on (π, t, B).

Section 5 is concerned with modifying the Toeplitz representation from section 3 in such a way that we get a new smaller representation without losing any important information. We therefore introduce the ideal $J_E \subset A$ as in [Kat04] (Definition 5.2) and the notion of Cuntz-Pimsner invariance relative to an ideal $J \subset J_E$ (Definition 5.3). If (π, t, B) is a representation satisfying this invariance we can decompose π into $\psi_t \circ \varphi_E$ on J. We then define the closed two-sided ideal $\mathcal{T}(J)$ (Definition 5.4) which by definition makes the quotient $\mathcal{O}_{(E,\varphi_E)}(J) := \mathcal{T}_{(E,\varphi_E)}/\mathcal{T}(J)$ Cuntz-Pimsner invariant relative to J (Definition 5.5). Gauge actions are defined (Definition 5.6) and $\mathcal{O}_{(E,\varphi_E)}(J)$ is shown to be a universal representation admitting a gauge action in the category $\mathcal{C}^J_{(E,\varphi_E)}$ of surjective representations that are Cuntz-Pimsner invariant relative to J (Theorem 5.7). Finally, using an injective Cuntz-Pimsner invariant modification of the Fock representation, $\mathcal{O}_{(E,\varphi_E)}(J)$ is proven to be injective.

The final section starts by defining gauge invariance of a closed two-sided ideal I in $\mathcal{T}_{(E,\varphi_E)}$ (Definition 6.1) and relating this to the gauge action in a natural way (Proposition 6.2). Then for an ideal $I \in \mathcal{T}_{(E,\varphi_E)}$ the ideal $\mathcal{T}(J(I))$ of $\mathcal{T}_{(E,\varphi_E)}$ is defined (Definition 6.3) and we prove a bijective correspondence between $\mathcal{T}(J(I))$ for ideals in $\mathcal{T}_{(E,\varphi_E)}$ that are gauge invariant and satisfy $I \cap \iota_A(A) = \{0\}$ (Theorem 6.6). The ideal $J_{(\pi,t,B)} \subset A$ is introduced (Definition 6.7) and shown to coincide with J_E if the representation (π, t, B) is Cuntz-Pimsner invariant relative to J and injective (Lemma 6.8). This is then used to prove the main result of this section (Theorem 6.9) relating Cuntz-Pimsner invariance relative to an ideal $J \subset J_E$ with $*$-homomorphisms between $\mathcal{O}_{(E,\varphi_E)}(J)$ and B. The result also gives necessary conditions for this $*$-homomorphism to be a $*$-isomorphism. This allows us to classify all bijective representation admitting a gauge action as a Cuntz-Pimsner representation relative to some ideal (Corollary 6.10). Furthermore we see that in the case $J = J_E$ we get a terminal object in the category of bijective representations admitting a gauge action called the Cuntz-Pimsner representation (Definition 6.12). Finally we use this to give a simple proof of the gauge invariance uniqueness theorem for Cuntz-Pimsner algebras (Theorem 6.14).

In this text all C^* algebras are assumed to be complex. Furthermore we denote the unit circle as $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\overline{\text{span}}$ denotes the closure of the linear span of a set. In the diagrams we use the convention that $\hookrightarrow$ denotes injective maps and $\twoheadrightarrow$ for surjective maps.

1 Hilbert A-modules and C^*-correspondences

Hilbert A-modules can be thought of as a generalization of Hilbert spaces, in which the values of the inner product sits inside a C^*-algebra A instead of the more usual field $\mathbb{C}$. As is the case with the complex-valued inner product, the inner product over A also gives rise to a norm on the inner product module.

This chapter will serve as an introduction to the basic concepts needed in the later chapters and follow in many ways the approach of [Lan95, Chapter 1]. We refer to [BAA94, p. 246] for the definition of a module, and start by defining the structures analogous to inner product spaces and Hilbert spaces.

Definition 1.1. Let A be a C^*-algebra and let E be a complex linear space equipped with a compatible right A-module structure. Then E is called an *inner product A-module* if it is equipped with a map $E \times E \to A$ given by $(x, y) \mapsto \langle x, y\rangle_E$ (we omit the subscript when there is no chance of confusion) satisfying for all $x, y, z \in E$ and for all $a \in A$:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle, \quad \alpha, \beta \in \mathbb{C}$

(ii) $\langle x, ya\rangle = \langle x, y\rangle a$,

(iii) $\langle y, x\rangle = \langle x, y\rangle^*$,

(iv) $\langle x, x\rangle \geq 0$; if $\langle x, x\rangle = 0$ then $x = 0$,

(v) $\lambda(xa) = (\lambda x)a = x(\lambda a) \quad \lambda \in \mathbb{C}$.

We shall call the scalar multiplication on E by the elements of A for a *right action*. Let A be a C^*-algebra and E an inner product A-module. We define the map $\|\cdot\|_E : E \to \mathbb{R}$ given by $x \mapsto \|\langle x, x\rangle\|_A^{1/2}$, where the notation hints at this being a norm on E. But to see that this is true we need to check that it satisfies the norm properties. That the map separates points and is absolute homogeneous follows readily from Definition 1.1, so the only thing that remains to check is the triangle inequality.

Let now $x, y \in E$, then (by abuse of notation) the map $\|\cdot\|_E$ satisfies a version of the Cauchy-Schwarz inequality:

$$\|\langle x, y\rangle\|_A \leq \|x\|_E\|y\|_E. \tag{1.1}$$

This follows trivially by taking the norm of the result in [Lan95, Proposition 1.1]. Using Eq. (1.1) we can prove that $\|\cdot\|_E$ satisfies the triangle inequality. We are therefore justified in our abuse of notation and we see that $\|\cdot\|_E$ is a norm on E. The norm $\|\cdot\|_E$ is usually just denoted without the subscript.

So far we have developed an analogue to inner product spaces. As with Hilbert spaces, the Hilbert A-modules are just the complete counterpart to the inner product A-modules:

Definition 1.2. Let A be a C^*-algebra and let E be an inner product A-module. If E is complete with respect to the induced norm of the inner product ($\|\cdot\| = \|\langle\cdot,\cdot\rangle\|^{1/2}$), then it is called a *(right) Hilbert A-module*, or a *Hilbert C^*-module over the C^*-algebra A*.

For a C^*-algebra A an example of a Hilbert A-module is A itself with the inner product $\langle a, b\rangle = a^*b$. A useful result follows:

Proposition 1.3. *Let E be a Hilbert A-module over a C^*-algebra A. If $y, z \in E$ satisfy that $\langle y, x\rangle = \langle z, x\rangle$ for all $x \in E$, then $y = z$.*

Proof. Assume $\langle y, x\rangle = \langle z, x\rangle$ for all $x \in E$, then by Definition 1.1 we get $0 = \langle y, x\rangle - \langle z, x\rangle = \langle y - z, x\rangle$ and in particular, for $x = y - z$, we get $\|y - z\|^2 = 0$ and hence that $y = z$. □

Note that Eq. (1.1) yields a representation of the norm of any element x in a Hilbert A-module E: If $y \in E$ and $\|y\|_E \leq 1$, then $\|\langle x, y\rangle\|_A \leq \|x\|_E$, and it therefore follows that

$$\|x\|_E = \sup_{\|y\|_E \leq 1} \|\langle x, y\rangle\|_A. \tag{1.2}$$

Another example of a Hilbert A-module is motivated by the Hilbert space theory. We define the direct sum $\bigoplus_{i\in I} E_i$ of a family of Hilbert A-modules $\{E_i\}_{i\in I}$ as the set of the elements $x = (x_i)$ such that $\sum_{i\in I}\langle x_i, x_i\rangle_{E_i}$ converges in A. To make this a Hilbert A-module, define the inner product $\langle x, y\rangle = \sum_{i\in I}\langle x_i, y_i\rangle_{E_i}$. It can be shown that this is well-defined and that it makes $\bigoplus_{i\in I} E_i$ a Hilbert A-module ([Lan95, p. 6]).

Next, we continue developing analogues to the Hilbert theory by considering maps between Hilbert A-modules, and the first of these are the adjointable maps: Let E, F be two Hilbert A-modules. Denote by $\mathscr{L}(E, F)$ the set of all maps $t : E \to F$ for which there is an adjoint t^*, i.e. a map $t^* : F \to E$ such that $\langle tx, y\rangle = \langle x, t^*y\rangle$ for all $x \in E, y \in F$.

Proposition 1.4. *If $t \in \mathscr{L}(E, F)$, then t is a linear and bounded map (in the operator norm).*

Proof. Let $t \in \mathscr{L}(E, F)$. To show that t is a linear map we must show that it preserves the linearity of E, i.e. $t(\alpha x + y) = \alpha t(x) + t(y)$ for $x, y \in E, \alpha \in \mathbb{C}$ and furthermore that it is linear in the sense of A-modules, that is, $t(xa) = t(x)a$ for $x \in E, a \in A$.
By Proposition 1.3 it follows that it is sufficient to prove the linearity in terms of the inner product. So let $t \in \mathscr{L}(E, F)$; $x, y \in E$; $z \in F$; $\alpha \in \mathbb{C}$ and $a \in A$. Then:

$$\begin{aligned}\langle t(\alpha x + y), z\rangle &= \langle \alpha x + y, t^*z\rangle = \langle t^*z, x\rangle^*\overline{\alpha} + \langle t^*z, y\rangle^* = \langle z, tx\rangle^*\overline{\alpha} + \langle z, ty\rangle^* \\ &= \langle z, \alpha tx\rangle^* + \langle z, ty\rangle^* = \langle z, \alpha tx + ty\rangle^* = \langle \alpha tx + ty, z\rangle,\end{aligned} \tag{1.3}$$

and furthermore:

$$\langle t(xa), z\rangle = \langle t^*z, xa\rangle^* = (\langle t^*z, x\rangle a)^* = (\langle z, t(x)\rangle a)^* = \langle t(x)a, z\rangle. \tag{1.4}$$

This proves the linearity of t.

Next we prove that t is bounded in the operator norm, i.e. $\|t\|_{\mathrm{op}} := \sup_{x\in B_1} \|tx\|_F$. Denote the unit ball of E by $B_1 = \{x \in E : \|x\|_E \leq 1\}$ and define the function $f_x : F \to A$ by $f_x(y) = \langle tx, y\rangle$ for $y \in F$ and $x \in B_1$. The linearity of f_x follows from Definition 1.1 and by Eq. (1.1) it is bounded. Furthermore $\|f_x(y)\| = \|\langle tx, y\rangle\| = \|\langle x, t^*y\rangle\| \leq \|t^*y\|_E < \infty$, so f_x is pointwise bounded on B_1. So by the uniform boundedness principle [Con90, p. 95] we conclude that $\{\|f_x\|_{\mathrm{op}} : x \in B_1\}$ is bounded, where $\|f_x\|_{\mathrm{op}} = \sup_{y:\|y\|_F\leq 1} \|f_x(y)\|$.

By definition we have $\|t\|_{\mathrm{op}} := \sup_{x\in B_1} \|tx\|_F$, so we must show that this is bounded. First note that it follows from *Eq.* (1.2) that the norm satisfies:

$$\|tx\|_F = \sup_{\|y\|_E\leq 1} \|\langle tx, y\rangle\| = \sup_{\|y\|_E\leq 1} \|f_x(y)\| = \|f_x\|_{\mathrm{op}}. \tag{1.5}$$

But since we have proven that $\{\|f_x\|_{\mathrm{op}} : x \in B_1\}$ is bounded, it follows that $\|t\|_{\mathrm{op}} = \sup_{x\in B_1} \|tx\|_F = \sup_{x\in B_1} \|f_x\|_{\mathrm{op}} < \infty$ and we are done. $\square$

In the case that $E = F$ the space $\mathscr{L}(E, F)$ is just denoted by $\mathscr{L}(E)$. As the next theorem shows, this is in itself a C^*-algebra when equipped with suitable operations and norm.

Theorem 1.5. *The space $\mathscr{L}(E)$ equipped with pointwise addition and scalar multiplication, composition as multiplication, the operator norm, and the involution $t \mapsto t^*$, where t^* is the adjoint of t, is a C^*-algebra.*

Proof. It is not too difficult to verify that $\mathscr{L}(E)$ is an algebra with the operations defined above, so we first prove that $\mathscr{L}(E)$ is a $*$-algebra: Given $s, t \in \mathscr{L}(E)$ and $x, y \in E$ we see that:

$$\langle (t^*)^*(x), y\rangle = \langle y, (t^*)^*(x)\rangle^* = \langle t^*y, x\rangle^* = \langle x, t^*y\rangle = \langle tx, y\rangle, \tag{1.6}$$

hence $t^{**} := (t^*)^* = t$. Furthermore the map $s \cdot t = s \circ t$ is a bounded, linear map on E and the adjoint is computed as follows:

$$\langle (s \circ t)^*x, y\rangle = \langle x, (s \circ t)y\rangle = \langle s^*x, ty\rangle = \langle (t^* \circ s^*)x, y\rangle, \tag{1.7}$$

which gives $(st)^* = t^*s^*$. In the same manner $(s + t)^* = s^* + t^*$ and $(\alpha t)^* = \overline{\alpha} t^*$ for $\alpha \in \mathbb{C}$.

Since the operator norm is submultiplicative, we only need to show that that $\mathscr{L}(E)$ is complete relative to the operator norm and that this norm agrees on adjoints to show that $\mathscr{L}(E)$ is a Banach $*$-algebra.

We first prove that $\|t\|_{\mathrm{op}} = \|t^*\|_{\mathrm{op}}$ for $t \in \mathscr{L}(E)$. Let $t \in \mathscr{L}(E)$, then $\|t^*t\|_{\mathrm{op}} \leq \|t^*\|_{\mathrm{op}}\|t\|_{\mathrm{op}}$, so for $x \in E$:

$$\|t\|_{\mathrm{op}}^2 = \sup_{\|x\|\leq 1} \|tx\|^2 = \sup_{\|x\|\leq 1} \|\langle tx, tx\rangle\| = \sup_{\|x\|\leq 1} \|\langle t^*tx, x\rangle\| \leq \|t^*t\|_{\mathrm{op}}, \tag{1.8}$$

and therefore $\|t\|_{\mathrm{op}}^2 \leq \|t^*\|_{\mathrm{op}}\|t\|_{\mathrm{op}}$ which implies $\|t\|_{\mathrm{op}} \leq \|t^*\|_{\mathrm{op}}$. Using that $t^{**} = t$ the reverse inequality follows and we get that $\|t\|_{\mathrm{op}} = \|t^*\|_{\mathrm{op}}$.

Denote by $(\mathscr{B}(E), \|\cdot\|_{\text{op}})$ the space of all bounded operators on E with the natural operations and the operator norm. It can be verified that this is a Banach algebra ([Mur90, Example 1.1.7]), and from Proposition 1.4 it follows that $\mathscr{L}(E) \subset \mathscr{B}(E)$. Since a closed subalgebra of a Banach algebra is itself a Banach algebra, we need to show that $\mathscr{L}(E)$ is closed in $\mathscr{B}(E)$. To show this, let $(t_n) \in \mathscr{L}(E) \subset \mathscr{B}(E)$ be a Cauchy sequence converging to some $t \in \mathscr{B}(E)$. Since $\|t_n\| = \|t_n^*\|$ it follows that $(t_n^*) \in \mathscr{L}(E)$ is Cauchy too, and therefore converges to some $t' \in \mathscr{B}(E)$.

From continuity of the inner product in both arguments we have for $x, y \in E$ that:

$$\langle tx, y\rangle = \lim_{n\to\infty} \langle t_n x, y\rangle = \lim_{n\to\infty} \langle x, t_n^* y\rangle = \langle x, t'y\rangle. \tag{1.9}$$

This shows that t is adjointable with $t^* = t'$ and thus $t \in \mathscr{L}(E)$.

From Eq. (1.8) we have $\|t\|_{\text{op}}^2 \le \|t^*t\|_{\text{op}} \le \|t^*\|_{\text{op}}\|t\|_{\text{op}} = \|t\|_{\text{op}}^2$ and hence that $\|t\|_{\text{op}}^2 = \|t^*t\|_{\text{op}}$ which shows that the C^*-property is satisfied. □

As an important subclass of the adjointable operators we introduce the class $\mathscr{K}(E)$ which can be thought of as a generalization of the compact operators in the Hilbert space context. They will be central throughout this text.

Let E and F be Hilbert A-modules and fix $x \in E$, $y \in F$. We define the map $\theta_{y,x} : E \to F$ given by $z \mapsto y\langle x, z\rangle$ for $z \in E$. This map is well-defined since F is a right A-module. In the Hilbert space case these maps correspond to the rank-1 operators which linearly span the finite-rank operators and are dense in the set of compact operators. Furthermore $\theta_{y,x} \in \mathscr{L}(E, F)$ with the adjoint map $\theta_{y,x}^* = \theta_{x,y}$ as can be seen by:

$$\langle \theta_{y,x}(e), f\rangle = \langle y\langle x, e\rangle, f\rangle = (\langle f, y\rangle \cdot \langle x, e\rangle)^* = \langle e, x\langle y, f\rangle\rangle = \langle e, \theta_{x,y}(f)\rangle. \tag{1.10}$$

We denote by $\mathscr{K}(E, F)$ the subspace of $\mathscr{L}(E, F)$ given by the closed span of these functions, i.e. $\mathscr{K}(E, F) = \overline{\text{span}}\,\{\theta_{y,x} \in \mathscr{L}(E, F) : x \in E, y \in F\}$. If $E = F$ the resulting space is written $\mathscr{K}(E)$. In the same manner as above it can be shown that $\mathscr{K}(E)$ satisfies the algebraic properties to be a (closed, two-sided) ideal of $\mathscr{L}(E)$. It follows (for example from [Mur90, Theorem 3.1.3]), that $\mathscr{K}(E)$ is a C^*-subalgebra of $\mathscr{L}(E)$.

Motivated by the representation of a C^*-algebra on the space of bounded operators on a Hilbert space, we make the following definition:

Definition 1.6. Let E be a Hilbert A-module and le $\varphi_E : A \to \mathscr{L}(E)$ be a $*$-homomorphism. Then (E, φ_E) is said to be a C^*-correspondence over A.

The mapping φ_E is referred to as *the left action* of a C^*-correspondence. Note that A can be considered both as a C^*-algebra and as a C^*-correspondence over itself with left action given by multiplication and the inner product given by $\langle a, b\rangle = a^*b$ for $a, b \in A$. This C^*-correspondence is referred to as the *identity correspondence over* A, and by abuse of notation we just write A in both these cases.

Let A be the identity correspondence as above and recall that the linear span of $\theta_{x,y}$ for $x, y \in E$ is dense in $\mathscr{K}(E)$. Let further $a \in A$ and $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of A. Then for $b \in A$, we the limit $\lim_{\lambda \in \Lambda} \theta_{a,u_\lambda}(b) = \lim_{\lambda \in \Lambda} a(u_\lambda^* b) = ab = \varphi_A(a)b$. This shows that $\varphi_A(a) \subset \mathscr{K}(A)$, so we consider the map $\varphi_A : A \to \mathscr{K}(A)$ given by $a \mapsto \varphi_A(a)$. Since $\varphi_A(a)$ is a $*$-homomorphism it follows that φ_A is a $*$-homomorphism as well. Since $\|\varphi_A(a)\|_{\mathrm{op}} \leq \|a\|$ it is seen that $\|\varphi_A\|_{\mathrm{op}} = 1$, which implies that φ_A is isometric and therefore injective.

If $k \in \mathscr{K}(A)$ we can assume that $k = \theta_{a,b}$ for some $a, b \in A$ since the closure of these elements linearly span $\mathscr{K}(A)$. Since A is the identity correspondence we have $\theta_{a,b}(c) = a\langle b, c\rangle = ab^*c = \varphi_A(ab^*)(c)$ for any $c \in A$. This implies that $(ab^*) \mapsto \theta_{a,b}$ and φ_A is seen to be surjective and therefore a $*$-isomorphism. This proves that $\mathscr{K}(A) \simeq A$, a result that will be useful later.

As we did with the Hilbert A-modules, we again combine C^*-correspondences to form new ones. Two ways of doing this that we will be interested in, is by forming the direct sum over matrices and the n-fold tensor product. For the first of these we need the fact that for $n \in \mathbb{N}$ the set $M_n(A)$ of $n \times n$-matrices with entries in A is a C^*-algebra (see for example [Mur90, p. 94]).

Definition 1.7. Let A be a C^*-algebra and let (E, φ_E) be a C^*-correspondence over A. For $n \in \mathbb{N}$ the direct sum of n copies of E, denoted E^n is a C^*-correspondence over $M_n(A)$ with the operations:

$$\begin{aligned}
(x_1, \dots, x_n) + (y_1, \dots, y_n) &= (x_1 + y_1, \dots, x_n + y_n), \\
\lambda(x_1, \dots, x_n) &= (\lambda x_1, \dots, \lambda x_n), \\
(x_1, \dots, x_n)(a_{ij})_{i,j=1}^n &= \left(\sum_{k=1}^n x_k a_{k1}, \dots, \sum_{k=1}^n x_k a_{kn} \right), \\
\left(\Big\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \Big\rangle \right)_{i,j} &= \langle x_i, y_j \rangle, \\
\varphi_{E^n}\left((a_{ij})_{i,j=1}^n \right)(x_1, \dots, x_n) &= \left(\sum_{k=1}^n \varphi_E(a_{1k}) x_k, \dots, \sum_{k=1}^n \varphi_E(a_{nk}) x_k \right),
\end{aligned} \tag{1.11}$$

for $(x_1, x_2, \dots x_n), (y_1, y_2, \dots y_n) \in E^n$, $\lambda \in \mathbb{C}$ and $(a_{ij})_{i,j=1}^n \in M_n(A)$.

Note that the inner product is defined as entries in an $n \times n$-matrix. We can think of the elements in E^n as n-dimensional vectors $(x_1, x_2, \dots x_n)$ where each $x_i \in E$ and we use the short-hand notation (x_i), or simply x, to avoid too heavy notation.

The n-fold tensor product is defined by using the algebraic tensor product. That is, given two Hilbert A-modules E and F we can consider these as vector spaces over $\mathbb{C}$ and then construct the vector space tensor product $E \otimes_{\mathbb{C}} F$ which is another vector space over $\mathbb{C}$. We also have that $E \otimes_{\mathbb{C}} F = \operatorname{span}\{x \otimes_{\mathbb{C}} y : x \in E, y \in F\}$, where $x \otimes_{\mathbb{C}} y$ are the tensors in $E \otimes_{\mathbb{C}} F$ (see [Alu09, p. 502-504] for this construction). In this case we depend on the fact that $\mathbb{C}$ has commutative multiplication to ensure that the tensor product is a new vector space.

We want to extend this construction to handle right A-modules. The problem with the tensor product of modules over algebras is that it does not in general produce a new module over an algebra unless the algebra is abelian. We can get around this by considering bimodules and using [Pie82, 9.5 Lemma a] the resulting tensor product will be a right A-module. The natural way to do this is to use the added structure of the C^*-correspondence to introduce a left action to a Hilbert A-module F by setting $(a, y) \mapsto \varphi_F(a)y$, making it into an A-bimodule since $\varphi_F(a)(yb) = \varphi_F(a)(y)b$ for $a, b \in A$ and $y \in F$ (this follows by the A-module linearity of $\varphi_F(a)$).

From the vector space tensor product we already get some of the properties we want, except the one relating the left and right actions. This motivates the following definition:

Definition 1.8. Let $(E, \varphi_E), (F, \varphi_F)$ be C^*-correspondences over A and denote by $E \odot F$ the quotient space $(E \bigotimes_{\mathbb{C}} F)/S$ where S is the subspace of $E \bigotimes_{\mathbb{C}} F$ generated by $(xa) \otimes_{\mathbb{C}} y - x \otimes_{\mathbb{C}} (\varphi_F(a)y)$ for $x \in E, y \in F$ and $a \in A$. The A-valued inner product and the right and left actions of A on $E \odot F$ are given by:

1. $\langle x \otimes_{\mathbb{C}} y, x' \otimes_{\mathbb{C}} y' \rangle = \langle y, \varphi_F(\langle x, x' \rangle_E) y' \rangle_F$, for $x' \in E, y' \in F$,

2. $(x \otimes_{\mathbb{C}} y)a = x \otimes_{\mathbb{C}} (ya)$,

3. $\varphi_{E \otimes_{\mathbb{C}} F}(a)(x \otimes_{\mathbb{C}} y) = (\varphi_E(a)x) \otimes_{\mathbb{C}} y$.

The completion of $E \odot F$ with respect to the norm given by the inner product is a C^*-correspondence over A called the *tensor product* of E and F, denoted by $(E \bigotimes F, \varphi_{E \bigotimes F})$.

Since the Hilbert A-module $E \bigotimes F$ is given as the completion of $E \odot F$ it is seen that $E \bigotimes F = \overline{\text{span}}\{x \otimes y : x \in E, y \in F\}$, and that $(xa) \otimes_{\mathbb{C}} y = x \otimes_{\mathbb{C}} (\varphi_F(a)y)$ in $E \odot F$. This construction can be done repeatedly over the same C^*-correspondence:

Definition 1.9. Let (E, φ_E) be a C^*-correspondence A, for $n \in \mathbb{N}$ we define a new C^*-correspondence $(E^{\otimes n}, \varphi_{E^{\otimes n}})$ over A where $E^{\otimes n}$ is given by the iterative process: $E^{\otimes 0} = A$ (considered as a C^*-correspondence), $E^{\otimes 1} = E$ and $E^{\otimes (n+1)} = E \bigotimes E^{\otimes n}$ for $n \geq 1$. If $x \in E$ and $y \in E^{\otimes n}$, where again $n \geq 1$, the left action on $E^{\otimes (n+1)}$ is given by:

$$\varphi_{E^{\otimes (n+1)}}(a)(x \otimes y) = (\varphi_E(a)x) \otimes y, \tag{1.12}$$

and will simply be denoted $\varphi_n : A \to \mathscr{L}(E^{\otimes n})$.

By extending the case of two C^*-correspondences to the case of n we see that $E^{\otimes n} = \overline{\text{span}}\{x_1 \otimes x_2 \otimes \cdots \otimes x_n : x_1, x_2, \ldots, x_n \in E\}$. We also note that one can identify $E^{\otimes (n+m)}$ with $E^{\otimes n} \otimes E^{\otimes m}$ for non-zero $n, m \in \mathbb{N}$.

We introduce these higher order tensor products because they will be used in constructing the Fock space in the next section.

2 Representations and the Fock space

We can represent C^*-correspondences using C^*-algebras. That is, we can think of the C^*-correspondences somehow as a $*$-subalgebra of a C^*-algebra. More formally:

Definition 2.1. Let A and B be C^*-algebras and let (E, φ_E) be a C^*-correspondence over A. A *representation of the C^*-correspondence (E, φ_E) on B* is a triple (π, t, B) where $\pi : A \to B$ is a $*$-homomorphism and $t : E \to B$ is a linear map satisfying:

1. $t(x)^*t(y) = \pi(\langle x, y\rangle_E)$, for $x, y \in E$,
2. $\pi(a)t(x) = t(\varphi_E(a)x)$, for $a \in A, x \in E$.

Since B is a C^*-algebra it makes sense to talk about the C^*-subalgebras in B generated by $\pi(A) \cup t(E)$. We denote this generated C^*-subalgebra by $C^*(\pi, t)$ and if $C^*(\pi, t) = B$ the representation is said to be *surjective*. If π is injective the representation (π, t, B) is said to be *injective*. Note that π being injective immediately implies that t is injective as well, since for any $x \in E$:

$$\|t(x)\|^2 = \|t(x)^*t(x)\| = \|\pi(\langle x, x\rangle_E)\|, \tag{2.1}$$

so if $t(x) = 0$ we see that $\pi(\langle x, x\rangle_E) = 0$ and since π is injective $\langle x, x\rangle_E = 0$ but then $x = 0$.

In the language of category theory the representation of a C^*-correspondence (E, φ_E) over B is just a morphism from the (E, φ_E) to the identity correspondence of B. We can also illustrate the correspondence using this commutative diagram:

$$\begin{tikzcd}
 & & & A \times E \arrow[dl, "\pi(\cdot)t(\cdot)"'] \arrow[dr, "\varphi_E(\cdot)(\cdot)"] & \\
A \arrow[rr, "\pi"] & & B & & E \arrow[ll, "t"'] \\
 & E \times E \arrow[ul, "\langle \cdot, \cdot \rangle"] \arrow[ur, "t(\cdot)^*t(\cdot)"'] & & &
\end{tikzcd} \tag{2.2}$$

Given any representation (π, t, B) of (E, φ_E) we can extend this representation to a representation of the C^*-correspondence $(E^{\otimes n}, \varphi_n)$ for integer $n \geq 1$ as follows:

Using the convention that $E^{\otimes 0} = A$, we define a map $t^n : E^{\otimes n} \to C^*(\pi, t)$ given by:

$$t^n = \begin{cases} \pi & \text{if } n = 0 \\ t & \text{if } n = 1 \end{cases} \tag{2.3}$$

and

$$t^n(x \otimes y) = t(x)t^{n-1}(y) \quad \text{for } x \in E, y \in E^{\otimes(n-1)} \text{ if } n \geq 2 \tag{2.4}$$

Using the structure of $E^{\otimes n}$, arguing recursively that $\pi(a)t^n(x) = t^n(\varphi_n(x))$ for $x \in E^{\otimes n}$ and then using this to show, again recursively, that

$$t^n(x)^* t^n(y) = \pi(\langle x, y\rangle_{E^{\otimes n}}) \text{ for all } x, y \in E^{\otimes n}, \tag{2.5}$$

we see that (π, t^n, B) is the appropriate representation of $(E^{\otimes n}, \varphi_n)$.

In the definition of a representation (π, t, B) over (E, φ_E) we saw a relation between the left action and the maps π and t. There is also a relation between these maps and the right action:

Lemma 2.2. *If (π, t, B) is a representation of (E, φ_E), then $t(xa) = t(x)\pi(a)$ for all $a \in A$ and $x \in E$.*

Proof. The element $t(xa)$ is well-defined since E is a right A-module and so for any $a \in A$ and $x \in E$ the element xa is in E. By the C^*-property:

$$\|t(xa) - t(x)\pi(a)\|^2 = \|(t(xa) - t(x)\pi(a))^*(t(xa) - t(x)\pi(a))\|, \tag{2.6}$$

and by expanding and using that π is a $*$-homomorphism the right-hand side becomes:

$$\|t(xa)^* t(xa) - t(xa)^* t(x)\pi(a) - \pi(a^*)t(x)^* t(xa) + \pi(a^*)t(x)^* t(x)\pi(a)\|. \tag{2.7}$$

Since (π, t, B) is a representation of (E, φ_E), we have by definition that: $t(x)^* t(y) = \pi(\langle x, y\rangle)$ for $x, y \in E$. Using this, and the fact that $\pi(ab) = \pi(a)\pi(b)$, on the equation above yields:

$$\|\pi(\langle xa, xa\rangle) - \pi(\langle xa, x\rangle a) - \pi(a^*\langle x, xa\rangle) + \pi(a^*\langle x, x\rangle a)\|, \tag{2.8}$$

from which it follows that $\|t(xa) - t(x)\pi(a)\|^2 = 0$ due to property (ii) from Definition 1.1. $\square$

We shall be concerned with two basic, yet important ways of constructing operators between higher-order tensors. First, we can patch known operators to a higher order domain by using the identity operators id_m on $\mathscr{L}(E^{\otimes m})$ for any $m \in \mathbb{N}$, and secondly we can extend small tensors to larger ones. As we will see, these methods are naturally related.

Definition 2.3. Let $n, m \in \mathbb{N}$. For any $S \in \mathscr{L}(E^{\otimes n})$ and fixed $n > 0$ we define for each m, the adjointable operator:

$$\begin{aligned} S \otimes \mathrm{id}_m : E^{\otimes(n+m)} &\to E^{\otimes(n+m)}, \\ x \otimes y &\mapsto S(x) \otimes y, \end{aligned} \tag{2.9}$$

where $x \in E^{\otimes n}, y \in E^{\otimes m}$. For any n, m we fix an element $z \in E^{\otimes n}$ and define the adjointable operator:

$$\begin{aligned} \tau^n_m(z) : E^{\otimes m} &\to E^{\otimes(n+m)}, \\ y &\mapsto z \otimes y. \end{aligned} \tag{2.10}$$

We can think of the first operator as an inclusion from $\mathscr{L}(E^{\otimes n})$ to $\mathscr{L}(E^{\otimes n+m})$ and the second as extension of tensors.

It is claimed in the definition that these operators are adjointable, and it is readily seen that $(S \otimes \mathrm{id}_m)^* = S^* \otimes \mathrm{id}_m$ and $\tau_m^n(z)^*(x \otimes y) = \varphi_m(\langle z, x\rangle_{E^{\otimes n}})y$ for $x \in E^{\otimes m}$, $y \in E^{\otimes n}$. The operator $\tau_m^n(z)$ plays along nicely with the structure already defined on $E^{\otimes m}$ and as we will see later, this result ensures that a variation of $\tau_m^n(z)$ forms a linear map on the Fock space satisfying the properties needed in the definition of a representation.

Lemma 2.4 ([Kat04, Lemma 1.9])**.** *Let $n, m \in \mathbb{N}$, $x, y \in E^{\otimes n}$ and $a \in A$, then*

1. $\tau_m^n(x)\tau_m^n(y)^* = \theta_{x,y} \otimes \mathrm{id}_m \in \mathscr{L}(E^{\otimes(n+m)})$,
2. $\tau_m^n(x)^*\tau_m^n(y) = \varphi_m(\langle x, y\rangle_{E^{\otimes n}}) \in \mathscr{L}(E^{\otimes m})$,
3. $\tau_m^n(x)\varphi_m(a) = \tau_m^n(xa) \in \mathscr{L}(E^{\otimes m}, E^{\otimes(n+m)})$,
4. $\varphi_{n+m}(a)\tau_m^n(x) = \tau_m^n(\varphi_n(a)x) \in \mathscr{L}(E^{\otimes m}, E^{\otimes(n+m)})$.

Proof. For the first property, let $z = x' \otimes y' \in E^{\otimes(n+m)}$ where $x' \in E^{\otimes n}$ and $y' \in E^{\otimes m}$, then:

$$\begin{aligned} \tau_m^n(x)(\tau_m^n(y)^*(x' \otimes y')) &= x \otimes (\varphi_m(\langle y, x'\rangle_{E^{\otimes n}})y') \\ &= (x\langle y, x'\rangle_{E^{\otimes n}}) \otimes y' = \theta_{x,y}(x') \otimes \mathrm{id}_m(y'), \end{aligned} \tag{2.11}$$

where the equalities follows from the structure on the tensor product space and the definition of the operators.

For the second, let $x' \in E^{\otimes m}$, then:

$$\tau_m^n(x)^*(\tau_m^n(y)(x')) = \tau_m^n(x)^*(y \otimes x') = \varphi_m(\langle x, y\rangle_{E^{\otimes n}})x'. \tag{2.12}$$

For the third and fourth, if $x' \in E^{\otimes m}$, then:

$$\tau_m^n(x)\varphi_m(a)(x') = x \otimes (\varphi_m(a)(x')) = (xa) \otimes x' = \tau_m^n(xa)(x'), \tag{2.13}$$

and

$$\varphi_{n+m}(a)\tau_m^n(x)(x') = \varphi_{n+m}(a)(x \otimes x') = (\varphi_n(a)x) \otimes x' = \tau_m^n(\varphi_n(a)x)(x'), \tag{2.14}$$

again by using the tensor product space properties. □

We are now ready to define the Fock space, a central concept in the theory of representations. Using this space we will show that every C^*-correspondence has an injective (non-trivial) representation. To define the space we employ a very specific construction using the C^*-correspondences $(E^{\otimes n}, \varphi_n)$. To define the space we employ both the tensor products of C*-correspondences and direct sums of Hilbert A-modules.

Definition 2.5. Let (E, φ_E) be a C^*-correspondence. Denote by $\mathscr{F}(E)$ the Hilbert A-module that is the direct sum of the Hilbert A-modules $E^{\otimes n}$ $(n \geq 0)$, that is:

$$\mathscr{F}(E) = \bigoplus_{n=0}^{\infty} E^{\otimes n}. \tag{2.15}$$

This space is called the *Fock space.*

We can think of the elements in $\mathscr{F}(E)$ as infinite-dimensional vectors where the i-th entry lies inside $E^{\otimes i}$ and τ_m^n as a shift operator.

Using the notion of the Fock space every $E^{\otimes n}$ can be thought of as a submodule of $\mathscr{F}(E)$ in the natural way, and the adjointable operators from $E^{\otimes n}$ to $E^{\otimes m}$ as a subspace of $\mathscr{L}(\mathscr{F}(E))$. The notion of convergence used on the space $\mathscr{L}(\mathscr{F}(E))$ is the strong operator topology, that is, we say that $T_n \in \mathscr{L}(\mathscr{F}(E))$ converges to T if $T_n x \to Tx$ for all $x \in \mathscr{F}(E)$.

Theorem 2.6. *The triple $(\varphi_\infty, \tau_\infty, \mathscr{L}(\mathscr{F}(E)))$, where $\varphi_\infty : A \to \mathscr{L}(\mathscr{F}(E))$ is a $*$-homomorphism and $\tau_\infty : E \to \mathscr{L}(\mathscr{F}(E))$ a linear map given by:*

$$\varphi_\infty(a) = \sum_{m=0}^{\infty} \varphi_m(a), \quad \tau_\infty(x) = \sum_{m=0}^{\infty} \tau_m^1(x), \quad \textit{for } a \in A, x \in E, \tag{2.16}$$

is an injective representation of (E, φ_E) on $\mathscr{L}(\mathscr{F}(E))$.

Proof. To see that $(\varphi_\infty, \tau_\infty, \mathscr{L}(\mathscr{F}(E)))$ is a representation we must show that:

$$\begin{aligned} \tau_\infty(x)^* \tau_\infty(y) &= \varphi_\infty(\langle x, y\rangle_E), \quad \text{for } x, y \in E, \\ \varphi_\infty(a)\tau_\infty(x) &= \tau_\infty(\varphi_E(a)x), \quad \text{for } a \in A, x \in E. \end{aligned} \tag{2.17}$$

To prove the first of these, let $x, y \in E$ and note that:

$$\tau_\infty(x)^* \tau_\infty(y) = \sum_{m=0}^{\infty} \tau_m^1(x)^* \tau_m^1(y) = \sum_{m=0}^{\infty} \varphi_m\left(\langle x, y\rangle_E\right) = \varphi_\infty(\langle x, y\rangle_E), \tag{2.18}$$

where the second equality follows from Lemma 2.4 (2) with $n = 1$. The second follows in the same way by Lemma 2.4 (4), and hence $(\varphi_\infty, \tau_\infty, A)$ is a representation of (E, φ_E) on $\mathscr{L}(\mathscr{F}(E))$.

To prove injectivity we assume that $\varphi_\infty(a) = 0$ for $a \in A$. Then since $\varphi_\infty(a) \in \mathscr{F}(E)$ it follows that $\varphi_m(a) = 0$ for all $m \geq 0$, and specifically that $\varphi_0(a) = 0$. This in turn implies that $aa^* = \varphi(a)(a^*) = 0$ and by the C^*-property it follows that $a = 0$. This shows that the kernel is trivial and therefore the map is injective. $\square$

Since $(\varphi_\infty, \tau_\infty, \mathscr{L}(\mathscr{F}(E)))$ is an injective representation the linear mapping τ_∞ is also injective.

3 The Toeplitz representation

Of central interest in the study of category theory is existence of universal objects. That is, an object that in a way gives us information about all the other objects in the category. In this section we introduce the most general category we shall consider in this paper, namely the category of surjective representations. We proceed to show that this category has a universal element. Recall that an object in a category is called universal if it is either an initial or a terminal object, i.e. an object I, (terminal: T) such that for every other object X there exists exactly one morphism $I \to X$ (terminal: $X \to T$).

Definition 3.1. Let A be a C^*-algebra and let (E, φ_E) be a C^*-correspondence over A. Denote by $\mathcal{C}_{(E,\varphi_E)}$ the category where the objects are surjective representations (π, t, B) of (E, φ_E). The morphisms between (π_1, t_1, B_1) and (π_2, t_2, B_2) are $*$-homomorphisms $f : B_1 \to B_2$ such that $f \circ t_1 = t_2$ and $f \circ \pi_1 = \pi_2$ and are denoted by $\hom_{\mathcal{C}_{(E,\varphi_E)}}(B_1, B_2)$.

We can illustrate how the morphisms act in this category by considering this commuting diagram:

$$\begin{tikzcd}
A \arrow[dr, "\pi_1"'] \arrow[drr, bend left, "\pi_2"] & & \\
& B_1 \arrow[r, "f"] & B_2 \\
E \arrow[ur, "t_1"] \arrow[urr, bend right, "t_2"'] & &
\end{tikzcd} \tag{3.1}$$

To construct an initial object for $\mathcal{C}_{(E,\varphi_E)}$ we will start with a C^*-algebra A and a C^*-correspondence (E, φ_E) over A. We then construct the universal $*$-algebra $G(A, E)$ generated by A and E subject to relations reflecting the structure of A and (E, φ_E). These relations are given in the following definition.

Definition 3.2. Let A and B be C^*-algebras and let (E, φ_E) be a C^*-correspondence over A. We write by $G(A, E)$ the universal $*$-algebra generated by A and E subject to the following set of relations for $a, b \in A$, and $x, y \in E$:

1. $a + b = c$ if $c = a + b \in A$,
2. $\lambda a = b$ if $b = \lambda a \in A$,
3. $ab = c$ if $c = ab \in A$,
4. $a^* = b$ if $b = a^* \in A$,
5. $x + y = z$ if $z = x + y \in E$,
6. $\lambda x = y$ if $y = \lambda x \in E$,
7. $xa = y$ if $y = xa \in E$,
8. $ax = y$ if $y = \varphi_E(a)x \in E$,
9. $x^*y = a$ if $\langle x, y\rangle = a \in A$.

The relations ensures that $G(A,E)$ preserves the wanted algebraic structure of A and (E,φ_E). Specifically we turn the left and right action on E into left and right multiplication in $G(A,E)$ and the inner product into the "standard" inner product on an involutive algebra. The universal construction also gives rise to two injective inclusion maps $\iota'_A : A \to G(A,E)$ and $\iota'_E : E \to G(A,E)$, where ι'_A is a $*$-homomorphism and ι'_E a linear map. That ι'_A is a $*$-homomorphism and t is linear follows from the relations given in Definition 3.2.

To extend $G(A,E)$ into a C^*-algebra we define a seminorm on $G(A,E)$ using $*$-homomorphisms from $G(A,E)$ to B:

$$\|x\|_0 = \sup\{\|\psi(x)\| : \psi \text{ is a } *\text{-homomorphism from } G(A,E) \text{ to } B\}. \tag{3.2}$$

Using the properties of the norm in B it is seen that $\|\cdot\|_0$ is submultiplicative and that $\|x^*\|_0 = \|x\|$ for $x \in G(A,E)$. The only difficult part in showing that $\|\cdot\|_0$ is a seminorm is to show that it is pointwise bounded. This is done in the next lemma.

Lemma 3.3. *The map $\|\cdot\|_0$ is pointwise bounded.*

Proof. We will prove that for each $x \in G(A,E)$ there exists a $C_x \in \mathbb{R}^+$ such that $\|\psi(x)\| \leq C_x$ for any $*$-homomorphism $\psi : G(A,E) \to B$ as this will imply that $\sup_\psi \|\psi(x)\| < \infty$. To do this we will exploit the fact that elements in $G(A,E)$ by construction can be written as formal sums over finite words of A and E. That is, for every $x \in G(A,E)$ we can write:

$$x = \sum_{i\in I} x_i, \quad |I| < \infty. \tag{3.3}$$

where the x_i are finite products of the elements $a \in A, e \in E, f^* \in E$. By the relations given in Definition 3.2 we can be even more precise in our expression. If somewhere in the product x_i we have the expression f^*e for $e, f \in E$ we can by relation 9 reduce this to an element a in A. Furthermore ae and ea for $a \in A, e \in E$ is reduced to elements in E. Trivially we also have that $ab \in A$ where $a, b \in A$. In other words the only elements that cannot be reduced are products of the form ef^*, ef and e^*f^* for $e, f \in E$. Altogether this implies that every x_i can be written in the form:

$$x_i = e_1 e_2 \cdots e_n f_1^* f_2^* \cdots f_m^*, \quad n, m \in \mathbb{N}, e_1, \cdots e_n \in E, f_1, \cdots, f_m \in E. \tag{3.4}$$

Of course an element in $G(A,E)$ may also be purely in A. Therefore we can write every $x \in G(A,E)$ on the form:

$$x = a_0 + \sum_{i\in I} x_i, \quad \text{where each } x_i \text{ is as in 3.4}. \tag{3.5}$$

Note that both $a_0 \in A$ and x_i can be zero. Let B be a C^*-algebra and $\psi : G(A,E) \to B$ be a $*$-homomorphism, then we have:

$$\|\psi(x)\| = \left\|\psi\left(a_0 + \sum_{i\in I} x_i\right)\right\| \leq \|\psi(a_0)\| + \sum_{i\in I} \|\psi(x_i)\| \leq \|a_0\|_A + \sum_{i\in I} \|\psi(x_i)\|. \tag{3.6}$$

Now by submultiplicativity and the fact that for $b \in B$ we have $\|b^*\| = \|b\|$ we get that:

$$\|\psi(x_i)\| = \|\psi(e_1 e_2 \cdots e_n f_1^* f_2^* \cdots f_m^*)\| \leq \|\psi(e_1)\| \cdots \|\psi(e_n)\| \cdot \|\psi(f_1)\| \cdots \|\psi(f_m)\|, \tag{3.7}$$

and therefore it is sufficient to prove that for each $e \in E$ there is a $c_e \in \mathbb{R}^+$ such that $\|\psi(e)\| \leq c_e$ for all $*$-homomorphisms ψ. This is not difficult, for by the C^*-property of B we have:

$$\|\psi(e)\|^2 = \|\psi(e^* e)\| = \|\psi(\langle e, e\rangle)\|. \tag{3.8}$$

By the relations imposed on $G(A,E)$ the inner product $\langle e, e\rangle$ is an element $a_e \in A$ and thus:

$$\|\psi(e)\|^2 = \|\psi(a_e)\| \leq \|a_e\|_A < \infty, \tag{3.9}$$

which gives the desired bound. This proves that $G(A,E)$ is equipped with a well-defined seminorm $\|\cdot\|_0$. $\square$

The reason $\|\cdot\|_0$ is only a seminorm, and not a norm, is that it might happen that an element x is nonzero in $G(A,E)$, but $\psi(x) = 0$ for any $*$-homomorphism $\psi : G(A,E) \to B$.

By forming the norm completion of this algebra we get a new C^*-algebra which forms the representation space of the promised initial object. Before doing this we remove some of the troublesome elements; I.e. let $I = \{x \in G(A,E) : \|x\|_0 = 0\}$, which is seen to be a closed two-sided ideal. Then the seminorm $\|\cdot\|_0$ on $G(A,E)$ induces a norm $\|\cdot\|$ on $G' := G(A,E)/I$. By straight computation and using the C^*-property of B (i.e. that $\|x\|^2 = \|x^* x\|$ for all $x \in B$) it is seen that this property also holds on $(G', \|\cdot\|)$. We denote the norm completion of $(G', \|\cdot\|)$ by $\mathcal{T}_{(E,\varphi_E)}$, which by construction is a C^*-algebra.

To complete the construction of a representation we define a $*$-homomorphism and a linear map mapping to $\mathcal{T}_{(E,\varphi_E)}$ acting on A and E respectively. To do this let q be the quotient map from $G(A,E)$ to G', let ι be the inclusion of G' into $\mathcal{T}_{(E,\varphi_E)}$, and let $\iota_A := \iota \circ q \circ \iota'_A$ and $\iota_E := \iota \circ q \circ \iota'_E$. This is illustrated by the following diagram.

$$\begin{tikzcd}
A \arrow[dr, hook, "\iota'_A"'] \arrow[drrr, bend left, "\iota_A"] & & & \\
& G(A,E) \arrow[r, two heads, "q"] & G' \arrow[r, hook, "\iota"] & \mathcal{T}_{(E,\varphi_E)} \\
E \arrow[ur, hook, "\iota'_E"] \arrow[urrr, bend right, "\iota_E"'] & & &
\end{tikzcd} \tag{3.10}$$

By relation 8 and 9 we see that $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ is a representation of (E, φ_E). By construction, $\mathcal{T}_{(E,\varphi_E)}$ is generated by $\iota_A(A) \cup \iota_E(E)$, so the representation $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ is surjective, and therefore an object in $\mathcal{C}_{(E,\varphi_E)}$.

We prove that there for any representation (π, t, B) of (E, φ_E) exists a unique morphism $f \in \hom_{\mathcal{C}_{(E,\varphi_E)}}(\mathcal{T}_{(E,\varphi_E}, B)$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"'] \arrow[drr, bend left, "\pi"] & & \\
& \mathcal{T}_{(E,\varphi(E))} \arrow[r, dashed, "f"] & B \\
E \arrow[ur, "\iota_E"] \arrow[urr, bend right, "t"'] & &
\end{tikzcd} \tag{3.11}$$

This shows that $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ is an initial object in $\mathcal{C}_{(E,\varphi_E)}$.

For the existence of a such morphism f, note that since $G(A, E)$ is a universal $*$-algebra there exists a $*$-homomorphism $f' : G(A, E) \to B$ such that $\pi = f' \circ \iota'_A$ and $t = f' \circ \iota'_E$ (see 3.12).

$$\begin{tikzcd}
A \arrow[dr, hook, "\iota'_A"'] \arrow[drr, bend left, "\pi"] & & \\
& G(A,E) \arrow[r, dashed, "f'"] & B \\
E \arrow[ur, hook, "\iota'_E"] \arrow[urr, bend right, "t"'] & &
\end{tikzcd} \tag{3.12}$$

We again consider the ideal $I = \{x \in G(A, E) : \|x\| = 0\}$. Let $q : G(A, E) \to G'$ be defined as $q(x) = x + I$, i.e. the quotient map. If $x \in I$, then $\|x\| = 0$ and therefore $\|f'(x)\| = 0$ since $\|x\| \geq \|f'(x)\|$ for any $*$-homomorphism f'. This in turn implies that $f'(x) = 0$ since B is a C^*-algebra. This shows that $I \subset \ker f'$ and therefore there exists a unique $*$-homomorphism $\tilde{f} : G' \to B$ such that $\tilde{f}(q(x)) = f'(x)$ (see 3.13).

$$\begin{tikzcd}
G(A,E) \arrow[r, two heads, "q"] \arrow[d, "f'"'] & G' \arrow[dl, dashed, "\tilde{f}"] \\
B &
\end{tikzcd} \tag{3.13}$$

By the density of G' in $\mathcal{T}_{(E,\varphi_E)}$ and the fact that B is complete the map $\tilde{f}$ can be extended to a $*$-homomorphism $f : \mathcal{T}_{(E,\varphi_E)} \to B$ preserving the properties of $\tilde{f}$; That is, $\pi = f \circ \iota_A$ and $t = f \circ \iota_E$.

Now let $g : T_{(E,\varphi_E)} \to B$ be another $*$-homomorphism such that $\pi = g \circ \iota_A$ and $t = g \circ \iota_E$. Then by the surjectivity of $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ it follows that f and g agrees on

$G(A, E)$ and therefore also on G' which is dense in $T_{(E,\varphi_E)}$ implying that $f = g$. Hence f is unique.

All of this is summarized in the next theorem:

Theorem 3.4. *The category $\mathcal{C}_{(E,\varphi_E)}$ has an initial object, namely $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$. This representation is called the* Toeplitz representation *of (E, φ_E) on $\mathcal{T}_{(E,\varphi_E)}$.*

In Theorem 2.6 we constructed a particular representation of (E, φ_E), namely the Fock representation $(\varphi_\infty, \tau_\infty, \mathscr{L}(\mathscr{F}(E)))$ and saw that this representation is injective. The reason we introduced the Fock representation is because we will use the injectivity to show that the Toeplitz representation is injective too.

Theorem 3.5. *The Toeplitz representation $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ of (E, φ_E) is injective.*

Proof. Recall that $C^*(\varphi_\infty, \tau_\infty)$ is the C^*-algebra generated by $\varphi_\infty(A) \cup \tau_\infty(E)$. Then the representation $(\varphi_\infty, \tau_\infty, C^*(\varphi_\infty, \tau_\infty))$ is an injective and surjective representation of (E, φ_E). By the universal property of $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ there exists a unique $*$-homomorphism $f : \mathcal{T}_{(E,\varphi_E)} \to C^*(\varphi_\infty, \tau_\infty)$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"'] \arrow[drr, hook, bend left, "\varphi_\infty"] & & \\
& \mathcal{T}_{(E,\varphi(E))} \arrow[r, dashed, "f"] & C^*(\varphi_\infty, \tau_\infty) \\
E \arrow[ur, "\iota_E"] \arrow[urr, hook, bend right, "\tau_\infty"'] & &
\end{tikzcd} \tag{3.14}$$

Since $\varphi_\infty = f \circ \iota_A$ is injective it also follows that ι_A is injective. $\square$

Generally speaking, since the Toeplitz representation $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ is initial it has to be the most general object in the category of surjective representations we have defined. This level of generality also means that the representation is too large to be an interesting representation of (E, φ_E). Therefore we will introduce the Cuntz-Pimsner representations relative to an ideal in the next section.

4 A $*$-homomorphism on $\mathscr{K}(E)$

Let A be a C^*-algebra and let (E, φ_E) be a C^*-correspondence over A. Given a representation (π, t, B) of (E, φ_E) we now construct a special $*$-homomorphism from $\mathscr{K}(E)$ to B. This $*$-homomorphism will give rise to a notion of invariance of representations relative to the ideals we will be considering.

Let E be a Hilbert A-module and recall that for $x, y \in E$ the map $\theta_{x,y} : E \to E$ is given by $z \mapsto x\langle y, z\rangle$ for $z \in E$. Some basic results, yet very useful throughout this paper, regarding this map is given in the next lemma.

Lemma 4.1. *Let E be a Hilbert A-module, let $x, y, x', y' \in E$ and let $t \in \mathscr{L}(E)$. Then:*

$$\begin{aligned} \theta_{x,y}\theta_{x',y'} &= \theta_{x\langle y,x'\rangle,y'}, \\ t\theta_{x,y} &= \theta_{tx,y}, \\ \theta_{x,y}t &= \theta_{x,t^*y}. \end{aligned} \tag{4.1}$$

Proof. Note that for any $z \in E$:

$$\theta_{x,y}\theta_{x',y'}(z) = x\langle y, \theta_{x',y'}(z)\rangle = x\langle y, x'\rangle\langle y', z\rangle = \theta_{x\langle y,x'\rangle,y'}(z), \tag{4.2}$$

and

$$(t\theta_{x,y})(z) = t(x\langle y, z\rangle) = (tx)\langle y, z\rangle = \theta_{tx,y}, \tag{4.3}$$

and that

$$(\theta_{x,y}t)(z) = x\langle y, tz\rangle = x\langle t^*y, z\rangle = \theta_{x,t^*y}(z). \tag{4.4}$$

□

Using this result it follows that $\mathscr{K}(E)$ is a closed two-sided ideal of $\mathscr{L}(E)$. Before we continue to the next theorem we need a lemma:

Lemma 4.2. *If $\theta_{x,y} \in \mathscr{K}(E)$ and $x, y \in E$, then:*

$$\|\theta_{x,y}\|_{op} = \|\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}\|. \tag{4.5}$$

Proof. We first prove that $\|\theta_{x,x}\|_{\mathrm{op}} = \|x\|^2$. It can be shown that $\|xa\| \leq \|x\|\|a\|$ for $x \in E, a \in A$ (see [Lan95, p. 4]). This implies that if $x, z \in E$, then:

$$\|\theta_{x,x}(z)\| = \|x\langle x, z\rangle\| \leq \|x\|\|\langle x, z\rangle\| \leq \|x\|^2\|z\|, \tag{4.6}$$

since $\|\langle x, z\rangle\| \leq \|x\|\|z\|$. Thus $\|\theta_{x,x}\|_{\mathrm{op}} \leq \|x\|^2$. For the reverse inequality note that $\|\theta_{x,x}(x)\| = \|x\langle x, x\rangle\| = \|x\|^3$, so by taking the supremum we have that $\|\theta_{x,x}\|_{\mathrm{op}} \geq \|x\|^2$.

Using the C^*-property on the space of linear operators with the operator norm we get:

$$\|\theta_{x,y}\|_{\mathrm{op}}^2 = \|\theta_{x,y}^*\theta_{x,y}\|_{\mathrm{op}} = \|\theta_{y,x}\theta_{x,y}\|_{\mathrm{op}} = \|\theta_{y\langle x,x\rangle,y}\|_{\mathrm{op}}, \tag{4.7}$$

where the last equality follows from Lemma 4.1.

Since $\langle x,x\rangle$ is positive there exists a positive element $\langle x,x\rangle^{1/2}$ such that $\langle x,x\rangle = \left(\langle x,x\rangle^{1/2}\right)^2$. Hence $(y\langle x,x\rangle)\langle y,z\rangle = y\langle x,x\rangle^{1/2}\langle y\langle x,x\rangle^{1/2}, z\rangle$ for all $z \in E$ and therefore $\theta_{y\langle x,x\rangle,y} = \theta_{y\langle x,x\rangle^{1/2},y\langle x,x\rangle^{1/2}}$. Combining this with the above we get that:

$$\begin{aligned}\|\theta_{x,y}\|^2_{\mathrm{op}} &= \|\theta_{y\langle x,x\rangle^{1/2},y\langle x,x\rangle^{1/2}}\|_{\mathrm{op}} = \|y\langle x,x\rangle^{1/2}\|^2 \\ &= \|\langle y\langle x,x\rangle^{1/2}, y\langle x,x\rangle^{1/2}\rangle\| = \|\langle x,x\rangle^{1/2}\langle y,y\rangle\langle x,x\rangle^{1/2}\| \\ &= \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\left(\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right)^*\right\| \\ &= \|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\|^2.\end{aligned} \tag{4.8}$$

From which it follows that:

$$\|\theta_{x,y}\|_{\mathrm{op}} = \|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\|. \tag{4.9}$$

□

We are now ready to prove the existence of the $*$-homomorphism mentioned at the beginning of this section.

Proposition 4.3. *Let C^*-correspondence (E, φ_E) of A and (π, t, B) be a representation of (E, φ_E). Then there exists a unique $*$-homomorphism $\psi_t : \mathscr{K}(E) \to B$ given by $\psi_t(\theta_{x,y}) = t(x)t(y)^*$ for $x, y \in E$.*

Proof. Since B is a C^*-algebra and $\mathscr{K}(E)$ is the closure of $\mathscr{D} := \mathrm{span}\{\theta_{x,y} \in \mathscr{L}(E) : x, y \in E\}$, it is sufficient to construct a bounded $*$-homomorphism with the desired properties on $\mathscr{D}$ and then extending it onto $\mathscr{K}(E)$.

So consider first the elements $\theta_{x,y}$ which span $\mathscr{D}$. Since we require that $\psi_t(\theta_{x,y}) = t(x)t(y)^*$ we are already deciding the action of ψ_t on the generators of $\mathscr{D}$ and therefore there cannot be more than one such linear map. Given any $k \in \mathscr{D}$ there may however be more than one way of writing k as a sum of elements of the form $\theta_{x,y}$. Hence we must show that the linear map is well-defined.

To do this let $k := \sum_{i=1}^n \theta_{x_i,y_i} = 0$ for some $x_i, y_i \in E$. We must then show that $\psi_t(k) = \sum_{i=1}^n t(x_i)t(y_i)^* = 0$. By the C^*-property of B it is sufficient to prove that:

$$\left(\sum_{i=1}^n t(x_i)t(y_i)^*\right)\left(\sum_{j=1}^n t(x_j)t(y_j)^*\right)^* = 0. \tag{4.10}$$

Since (π, t, B) is a representation we have for each j that

$$\begin{aligned}\sum_{i=1}^n t(x_i)t(y_i)^*t(y_j)t(x_j)^* = \sum_{i=1}^n t(x_i)\pi(\langle y_i, y_j\rangle)t(x_j)^* = \sum_{i=1}^n t(x_i\langle y_i, y_j\rangle)t(x_j)^* \\ = t\left(\sum_{i=1}^n \theta_{x_i,y_i}(y_j)\right)t(x_j)^* = 0,\end{aligned} \tag{4.11}$$

since $k = \sum_i \theta_{x_i,y_i}(z) = 0$.

Hence $(\sum_{i=1}^n t(x_i)t(y_i)^*)(\sum_{j=1}^n t(x_j)t(y_j)^*)^* = 0$. It follows that there is a linear map $\psi_t : \mathscr{D} \to B$ satisfying $\psi_t(\theta_{x,y}) = t(x)t(y)^*$. To see that ψ_t is an algebra homomorphism note that by Lemma 4.1, if $x, y, x', y' \in E$, then $\theta_{x,y}\theta_{x',y'} = \theta_{x\langle y,x'\rangle,y'}$, so:

$$\begin{aligned} \psi_t(\theta_{x,y}\theta_{x',y'}) &= t(x\langle y,x'\rangle)t(y')^* = t(x)\pi(\langle y,x'\rangle)t(y')^* \\ &= t(x)t(y)^*t(x')t(y')^* = \psi_t(\theta_{x,y})\psi_t(\theta_{x',y'}), \end{aligned} \tag{4.12}$$

again since (π, t, B) is a representation. Furthermore ψ_t is a $*$-homomorphism, since for every $\theta_{x,y} \in \mathscr{L}(E)$ we have:

$$\psi_t(\theta_{x,y}^*) = \psi_t(\theta_{y,x}) = t(y)t(x)^* = (t(x)t(y)^*)^* = \psi_t(\theta_{x,y})^*. \tag{4.13}$$

It remains to show that ψ_t is bounded so that we can extend it to $\mathscr{K}(E)$.

Recall that we in Definition 1.7 defined for $n \in \mathbb{N}$ the C^*-correspondence E^n over $M_n(A)$. If $x, y, z \in E^n$, we have:

$$\begin{aligned} \theta_{x,y}(z) = x\langle y,z\rangle = x\Big(\langle y_i,z_j\rangle\Big)_{ij} &= \left(\sum_{k=1}^n x_k\langle y_k,z_1\rangle, \dots, \sum_{k=1}^n x_k\langle y_k,z_n\rangle\right) \\ &= \left(\sum_{k=1}^n \theta_{x_k,y_k}(z_1), \dots, \sum_{k=1}^n \theta_{x_k,y_k}(z_n)\right). \end{aligned} \tag{4.14}$$

Using the matrix structure we can go from a sum of θ_{x_k,y_k} to a single $\theta_{x,y}$. For any $v \in E^n$ we can define the norm $\|v\|_\infty := \max_{1\le i\le n}\|v_i\|$, and from this it directly follows that:

$$\|\theta_{x,y}(z)\|_\infty = \max_{1\le i\le n}\left\|\sum_{k=1}^n \theta_{x_k,y_k}(z_i)\right\|. \tag{4.15}$$

Using the definition of the operator norm and Lemma 4.2 we see the following equalities:

$$\begin{aligned} \left\|\sum_{k=1}^n \theta_{x_k,y_k}\right\|_{\mathrm{op}} &= \sup\left\{\max_{1\le i\le n}\left\|\sum_{k=1}^n \theta_{x_k,y_k}(z_i)\right\| : \max_{1\le i\le n}\|z_i\| = 1\right\} \\ &= \sup\{\|\theta_{x,y}(z)\|_\infty : \|z\|_\infty = 1\} \\ &= \|\theta_{x,y}\|_{\mathrm{op}} \\ &= \|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\| \\ &= \left\|\Big(\big(\langle x_i,x_j\rangle\big)_{i,j=1}^n\Big)^{1/2}\Big(\big(\langle y_i,y_j\rangle\big)_{i,j=1}^n\Big)^{1/2}\right\|, \end{aligned} \tag{4.16}$$

where $\big(\langle x_i,x_j\rangle\big)_{i,j=1}^n$ denotes the $n \times n$-matrix with ij-th entry $\langle x_i,x_j\rangle$.

Next, consider the identity correspondence (B, φ_B). Then:

$$\begin{aligned}
\|\psi_t(\theta_{x,y})\| &= \left\| \sum_{k=1}^{n} t(x_k)t(y_k)^* \right\| = \left\| \sum_{k=1}^{n} \theta_{t(x_k),t(y_k)} \right\| \\
&= \left\| \Big(\big(\langle t(x_i), t(x_j) \rangle \big)_{i,j=1}^{n} \Big)^{1/2} \Big(\big(\langle t(y_i), t(y_j) \rangle \big)_{i,j=1}^{n} \Big)^{1/2} \right\| \\
&= \left\| \Big(\big(\pi\big(\langle x_i, x_j \rangle \big) \big)_{i,j=1}^{n} \Big)^{1/2} \Big(\big(\pi\big(\langle y_i, y_j \rangle \big) \big)_{i,j=1}^{n} \Big)^{1/2} \right\|.
\end{aligned} \tag{4.17}$$

Let π' be the natural extension of π to $M_n(A)$. Then $\pi' : M_n(A) \to M_n(B)$ is given by $(a_{ij})_{i,j=1}^{n} \mapsto (\pi(a_{ij}))_{i,j=1}^{n}$ which is a $*$-homomorphism and therefore norm-decreasing. We can now conclude that:

$$\begin{aligned}
\|\psi_t(\theta_{x,y})\| &= \left\| \Big(\Big(\pi\big(\langle x_i, x_j \rangle \big) \Big)_{i,j=1}^{n} \Big)^{1/2} \Big(\big(\pi\big(\langle y_i, y_j \rangle \big) \big)_{i,j=1}^{n} \Big)^{1/2} \right\| \\
&\leq \left\| \Big(\big(\langle x_i, x_j \rangle \big)_{i,j=1}^{n} \Big)^{1/2} \Big(\big(\langle y_i, y_j \rangle \big)_{i,j=1}^{n} \Big)^{1/2} \right\| \\
&= \left\| \sum_{k=1}^{n} \theta_{x_k,y_k} \right\|_{\mathrm{op}}.
\end{aligned} \tag{4.18}$$

This shows that $\psi_t : \mathscr{D} \to B$ is bounded. We can now extend ψ_t uniquely to $\overline{\mathscr{D}} = \mathscr{K}(E)$ to conclude the proof. □

The following proposition assures that ψ_t is injective if the representation (π, t, B) is injective.

Proposition 4.4 ([Kat04, Lemma 2.4])**.** *Let (π, t, B) be a representation of (E, φ_E) on B. Then:*

$$\begin{aligned}
\pi(a)\psi_t(k) &= \psi_t(\varphi_E(a)k), \\
\psi_t(k)\pi(a) &= \psi_t(k\varphi_E(a)), \\
\psi_t(k)t(x) &= t(kx)
\end{aligned} \tag{4.19}$$

for $a \in A$, $x \in E$ and $k \in \mathscr{K}(E)$.

Proof. By density of $\operatorname{span}\{\theta_{\xi,\eta} : \xi, \eta \in E\}$ in $\mathscr{K}(E)$ and the linearity of ψ_t it is enough to consider the case $k = \theta_{\xi,\eta}$ for some $\xi, \eta \in E$. Using that (π, t, B) is a representation and Lemma 4.1 we see that:

$$\pi(a)\psi_t(k) = \pi(a)(t(\xi)t(\eta)^*) = t(\varphi_E(a)\xi)t(\eta)^* = \psi_t(\theta_{\varphi_E(a)\xi,\eta}) = \psi_t(\varphi_E(a)k), \tag{4.20}$$

and:

$$\psi_t(k)\pi(a) = t(x)(\pi(a^*)t(y))^* = t(x)t(\varphi_E(a^*)y)^* = \psi_t(\theta_{x,\varphi_E(a^*)y}) = \psi_t(k\varphi_E(a)). \tag{4.21}$$

In the same way we have:

$$\psi_t(k)t(x) = t(\xi)t(\eta)^*t(x) = t(\xi)\pi(\langle \eta, x \rangle) = t(\xi\langle \eta, x \rangle) = t(kx). \tag{4.22}$$

□

For an injective representation both π and t is injective, so if $\psi_t(k) = 0$, then $t(kx) = 0$ by the proposition above. By injectivity of t we then have $kx = 0$ for all $x \in E$ which implies that $k = 0$ and we see that ψ_t is injective.

It follows from Proposition 4.3 that there for integer $n \geq 0$ exists a unique $*$-homomorphism $\psi_{t^n} : \mathscr{K}(E^{\otimes n}) \to C^*(\pi, t)$ given by $\psi_{t^n}(\theta_{x,y}) = t^n(x)t^n(y)$ where $x, y \in E^{\otimes n}$. Furthermore, if (π, t, B) is injective then both t_n and ψ_{t_n}.

5 The relative Cuntz-Pimsner representation

Now that we have constructed the initial object in the category of surjective representations we turn to the somewhat dual problem of constructing a terminal object of a particular subcategory of $\mathcal{C}_{(E,\varphi_E)}$. The idea on how to obtain this is to consider $\mathcal{T}_{(E,\varphi_E)}$ and then take the quotient of this with respect to specific ideals thereby removing enough freeness to make it terminal. Before constructing the terminal element we need to discuss the mentioned quotient structure which associates to a specific class of ideals a class of representations. That is the contents of this section.

Definition 5.1. Let A be a C^*-algebra and let I be an ideal in A. We then define:

$$I^{\perp} = \{a \in A : ab = 0 \text{ for } b \in I\} \tag{5.1}$$

This is easily seen to be an ideal of A such that $I \cap I^{\perp} = \{0\}$. This implies that if J is another ideal of A such that $J \subset I^{\perp}$, then $J \cap I = \{0\}$. Suppose conversely that $J \cap I = \{0\}$, and let $a \in J$. For any $b \in I$ we then have $ab \in J \cap I = \{0\}$ which implies that $a \in I^{\perp}$. From this it follows that if $J \cap I^{\perp} = \{0\}$, then $J \subset I^{\perp}$.

The left action φ_E of a representation (E, φ_E) assigns to each $a \in A$ an operator in $\mathscr{L}(E)$. Since $\mathscr{K}(E) \subset \mathscr{L}(E)$ it may be the case that $\varphi_E(a) \in \mathscr{K}(E)$ for some $a \in A$. We therefore introduce the following ideal J_E.

Definition 5.2. Let (E, φ_E) be a C^*-correspondence over A and define the two-sided closed ideal J_E of A as:

$$J_E = \varphi_E^{-1}(\mathscr{K}(E)) \cap \ker(\varphi_E)^{\perp}, \tag{5.2}$$

where $\ker(\varphi_E)^{\perp} := \{a \in A : ab = 0 \text{ for } b \in \ker(\varphi_E)\}$.

The ideal J_E is well-defined: The set J_E is non-empty since $0 \in J_E$. Furthermore, if $x, y \in J_E$ then $\varphi_E(x - y) = \varphi_E(x) - \varphi_E(y) \in \mathscr{K}(E)$, since $\mathscr{K}(E)$ is an ideal of $\mathscr{L}(E)$. We also have that $(x - y)b = xb - yb = 0$ for any $b \in \ker(\varphi_E)$. Hence $x - y \in J_E$ and J_E is an additive subgroup of A. Now let $x \in J_E$ and $a \in A$. Then $\varphi_E(ax) = \varphi_E(a)\varphi_E(x) \in \mathscr{K}(E)$ since $\mathscr{K}(E)$ is an ideal in $\mathscr{L}(E)$. In addition, $(ax)b = a(xb) = 0$ for every $b \in \ker(\varphi_E)$ and hence $ax \in J_E$. Similarly $xa \in J_E$ which shows that J_E is a (two-sided) ideal of A. Furthermore, since $\mathscr{K}(E)$ is a closed set, the ideal is closed by continuity of φ_E.

Suppose that $J \subset A$ is another ideal such that the restriction of φ_E to J is injective on $\mathscr{K}(E)$. Then if $a \in J \cap \ker(\varphi_E)$, then $a = 0$ since φ_E is injective restricted to J, and since $\ker(\varphi_E)^{\perp}$ is the largest ideal satisfying that $\ker(\varphi_E)^{\perp} \cap \ker(\varphi_E) = \{0\}$ this implies that $J \subset \ker(\varphi_E)^{\perp}$. It follows that J_E is the largest ideal such that the restriction of φ_E on it is an injection onto $\mathscr{K}(E)$.

By considering ideals J of J_E we will see that this gives rise to an entirely new class of representations known as the Cuntz-Pimsner representations relative to J.

Definition 5.3. Let (E, φ_E) be a C^*-correspondence over A and let J be an ideal in A such that $J \subset J_E$. We say that a representation (π, t, B) of (E, φ_E) is *Cuntz-Pimsner invariant relative to* J if:

$$\psi_t(\varphi_E(a)) = \pi(a), \quad \text{for all } a \in J. \tag{5.3}$$

This property can be visualised by the commutative diagram:

$$\begin{tikzcd} J \arrow[r, "\pi"] \arrow[d, hook, "\varphi_E"'] & B \\ \mathscr{K}(E) \arrow[ur, "\psi_t"'] & \end{tikzcd} \tag{5.4}$$

Motivated by this definition we will now construct the Cuntz-Pimsner representation relative to an ideal $J \subset J_E$. These representations are constructed from the Toeplitz representation $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ by taking the quotient of an ideal in $\mathcal{T}_{(E,\varphi_E)}$ such that the resulting structure is Cuntz-Pimsner invariant relative to J. To ensure this we first define the ideal that we will take the quotient by.

Definition 5.4. Let (π, t, B) be a representation of the C^*-correspondence (E, φ_E) over A and let J be an ideal in A such that $J \subset J_E$. Denote by $\mathcal{T}(J)$ the closed, two-sided ideal in $\mathcal{T}_{(E,\varphi_E)}$ generated by the set:

$$\{\psi_{\iota_E}(\varphi_E(a)) - \iota_A(a) : a \in J\}. \tag{5.5}$$

By definition this is the smallest closed, two-sided ideal containing the $\{\psi_{\iota_E}(\varphi_E(a)) - \iota_A(a) : a \in J\}$. Due to $\mathcal{T}(J) \subset \mathcal{T}_{(E,\varphi_E)}$ being closed the quotient $\mathcal{T}_{(E,\varphi_E)}/\mathcal{T}(J)$ is a C^*-algebra. It is also easily seen to be Cuntz-Pimsner invariant relative to J by the definition above.

Definition 5.5. Let A be a C^*-algebra, let (E, φ_E) be a C^*-correspondence over A and let J be an ideal in A such that $J \subset J_E$. We define a C^*-algebra $\mathcal{O}_{(E,\varphi_E)}(J)$ by setting $\mathcal{O}_{(E,\varphi_E)}(J) := \mathcal{T}_{(E,\varphi_E)}/\mathcal{T}(J)$. Let $\rho_J : \mathcal{T}_{(E,\varphi_E)} \to \mathcal{O}_{(E,\varphi_E)}(J)$ denote the canonical surjective $*$-homomorphism. The C^*-algebra $\mathcal{O}_{(E,\varphi_E)}(J)$ is called the *Cuntz-Pimsner C^*-algebra relative to the ideal* J.

The following diagram accompanies this definition:

$$\begin{tikzcd}
A \arrow[dr, hook, "\iota_A"'] & & \\
& \mathcal{T}_{(E,\varphi_E)} \arrow[r, two heads, "\rho_J"] & \mathcal{T}_{(E,\varphi_E)}/\mathcal{T}(J) := \mathcal{O}_{(E,\varphi_E)}(J) \\
E \arrow[ur, hook, "\iota_E"] & &
\end{tikzcd} \tag{5.6}$$

Note that in the case where $J = \{0\}$ this simply reduces to the Toeplitz algebra $\mathcal{T}_{(E,\varphi_E)}$. In the other extreme, we have $J = J_E$ which gives the *Cuntz-Pimsner algebra*. We return to this algebra in the next section.

Before proving that $\mathcal{O}_{(E,\varphi_E)}(J)$ can be made into a universal representation in a suitable category we introduce the concept of gauge action. If (π, t, B) is a representation of (E, φ_E) the gauge action is a $*$-homomorphism acting on π and t. In a way we can think of the gauge action as a form of rotation on the images of π and t.

Definition 5.6. Let A be a C^*-algebra, let (E, φ_E) a C^*-correspondence over A and let (π, t, B) be a representation of (E, φ_E) on B. The representation (π, t, B) is said to *admit a gauge action* if for every $z \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ there exists a $*$-homomorphism $\beta_z : C^*(\pi, t) \to C^*(\pi, t)$ such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(t(x)) = zt(x)$ for all $a \in A$ and $x \in E$.

Since β_z is defined on the C^*-algebra generated by the set $\pi(A) \cup t(E)$ and the action of β_z is independent of z on $\pi(a)$ and linear in z on $t(x)$ for $a \in A$, and $x \in E$ it follows that β_z is a continuous map from $\mathbb{T}$ to $\mathrm{Aut}(C^*(\pi, t))$ (set the inverse to be β_{z^*}).

From the diagram Eq. (5.6) it is easy to see how to define the $*$-homomorphism and linear map to make $\mathcal{O}_{(E,\varphi_E)}(J)$ into a representation.

Theorem 5.7 ([CaOr11, Theorem 3.18])**.** *Let A be a C^*-algebra, let (E, φ_E) a C^*-correspondence over A and let $J \subset A$ be an ideal such that $J \subset J_E$. Set $\iota_A^J := \rho_J \circ \iota_A$ and $\iota_E^J := \rho_J \circ \iota_E$. Then $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ is a surjective representation of (E, φ_E) on $\mathcal{O}_{(E,\varphi_E)}(J)$, called the* Cuntz-Pimsner representation relative to J, *which is Cuntz-Pimsner invariant relative to J and admits a gauge action. Furthermore $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ satisfies the universal property:*

> *If (π, t, B) is a representation of (E, φ_E) which is Cuntz-Pimsner invariant relative to J then there exists a unique $*$-homomorphism $\eta : \mathcal{O}_{(E,\varphi_E)} \to B$ such that $\eta \circ \iota_A^J = \pi$ and $\eta \circ \iota_E^J = t$. Visually, this amounts to the following*

diagram being commutative:

$$
\begin{tikzcd}
A \arrow[dr, "\iota_A^J"'] \arrow[drr, bend left, "\pi"] & & \\
& \mathcal{O}_{(E,\varphi_E)}(J) \arrow[r, dashed, "\eta"] & B \\
E \arrow[ur, "\iota_E^J"] \arrow[urr, bend right, "t"'] & &
\end{tikzcd}
\tag{5.7}
$$

In addition, if (π, t, B) is a surjective representation of (E, φ_E) which is Cuntz-Pimsner invariant relative to J and $\xi : B \to \mathcal{O}_{(E,\varphi_E)}(J)$ is $*$-homomorphism such that $\xi \circ \pi = \iota_A^J$ and $\xi \circ t = \iota_E^J$, then ξ is an isomorphism.

Proof. We first prove that $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ is a surjective and injective representation. Since ρ_J is a $*$-homomorphism the composition with a linear map or a $*$-homomorphisms is again, respectively, linear and a $*$-homomorphism. Therefore $\iota_A^J : A \to \mathcal{O}_{(E,\varphi_E)}(J)$ is a $*$-homomorphism and $\iota_E^J : E \to \mathcal{O}_{(E,\varphi_E)}(J)$ is a linear map.

Now for any $x, y \in E$:

$$
\begin{aligned}
\iota_E^J(x)^* \iota_E^J(y) &= (\rho_J \circ \iota_E)(x)^* (\rho_J \circ \iota_E)(y) = \rho_J(\iota_E(x)^* \iota_E(y)) \\
&= \rho_J\big(\iota_A(\langle x, y\rangle_E)\big) = \iota_A^J\big(\langle x, y\rangle_E\big),
\end{aligned}
\tag{5.8}
$$

and for any $a \in A$ and $x \in E$:

$$
\iota_A^J(a)\iota_E^J(x) = \rho_J(\iota_A(a)\iota_E(x)) = \rho_J(\iota_E(\varphi_E(a)x)) = \iota_E^J(\varphi_E(a)x),
\tag{5.9}
$$

which shows that $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ is a representation. Since ρ_J is surjective, and $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ is a surjective representation it follows that $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ is also a surjective representation.

If $a \in J$, then $\psi_{\iota_E}(\varphi_E(a)) - \iota_A(a) = 0$ in $\mathcal{O}_{(E,\varphi_E)}(J)$. This implies that:

$$
\psi_{\iota_E^J}(\varphi_E(a)) - \iota_A^J(a) = \rho_J(\psi_{\iota_E}(\varphi_E(a)) - \iota_A(a)) = 0,
\tag{5.10}
$$

which establishes that $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ is Cuntz-Pimsner invariant relative to J.

Suppose now that (π, t, B) is another representation of (E, φ_E) which is Cuntz-Pimsner invariant relative to J. By Theorem 3.4 $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ is a universal representation of (E, φ_E), so there exists a unique $*$-homomorphism $f : \mathcal{T}_{(E,\varphi_E)} \to B$ such that $f \circ \iota_A = \pi$ and $f \circ \iota_E = t$. Since (π, t, B) is Cuntz-Pimsner invariant relative to J, for any $a \in J$ we have that:

$$
0 = \psi_t(\varphi_E(a)) - \pi(a) = \psi_{f\circ\iota_E}(\varphi_E(a)) - (f \circ \iota_A)(a) = f(\psi_{\iota_E}(\varphi_E(a)) - \iota_A(a)),
\tag{5.11}
$$

which implies that $\mathcal{T}(J) \subset \ker f$. This allows us, by the quotient construction, to conclude that there exists a unique $*$-homomorphism $\eta : \mathcal{O}_{(E,\varphi(E))}(J) \to B$ such that $f = \eta \circ \rho_J$. Furthermore we have $\pi = f \circ \iota_A = (\eta \circ \rho_J) \circ \iota_A = \eta \circ \iota_A^J$ and similarly $t = \eta \circ \iota_E^J$.

Assume now that (π, t, B) is a surjective representation which is Cuntz-Pimsner invariant relative to J and let $\xi : B \to \mathcal{O}_{(E,\varphi_E)}(J)$ be a $*$-homomorphism such that $\xi \circ \pi = \iota_A^J$ and $\xi \circ t = \iota_E^J$. Visually we have:

$$\begin{tikzcd}
A \arrow[dr, "\iota_A^J"'] \arrow[drr, bend left, "\pi"] & & \\
& \mathcal{O}_{(E,\varphi_E)}(J) & B \arrow[l, "\xi"'] \\
E \arrow[ur, "\iota_E^J"] \arrow[urr, bend right, "t"'] & &
\end{tikzcd} \tag{5.12}$$

Let $a \in A$, then by the above $\eta \circ \xi(\pi(a)) = \eta \circ \iota_A^J(r) = \pi(a)$ and similarly for $x \in E$ we get $\eta \circ \xi(t(x)) = t(x)$. Since (π, t, B) is surjective B is generated by $\pi(A) \cup t(E)$ which implies that $\eta \circ \xi$ acts as the identity on B. Since $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ also is a surjective representation of (E, φ_E) the same argument applies to show that $\xi \circ \eta$ acts as the identity on $\mathcal{O}_{(E,\varphi_E)}(J)$. Together this shows that ξ and η are inverse of each other and therefore ξ is an isomorphism.

Finally, to show that $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ admits a gauge action let $z \in \mathbb{T}$ and define:

$$t_z : E \to \mathcal{O}_{(E,\varphi_E)}(J)) \text{ by } t_z(x) = z\iota_E^J(x). \tag{5.13}$$

Note that:

$$t_z(x)^* t_z(y) = \iota_E^J(x)^*(z^* z)\iota_E^J(y) = \iota_E^J(x)^* \iota_E^J(y) = \iota_A^J\big(\langle x, y\rangle_E\big), \tag{5.14}$$

for $x, y \in E$, and:

$$\iota_A^J(a) t_z(x) = \iota_A^J(a) z \iota_E^J(x) = z \iota_A^J(a) \iota_E^J(x) = t_z(\varphi(a)x), \tag{5.15}$$

for $a \in A, x \in E$.

This implies that $(\iota_A^J, t_z, \mathcal{O}_{(E,\varphi_E)}(J))$ is a representation of (E, φ_E). Furthermore it is Cuntz-Pimsner invariant relative to J since this is a property depending on ι_A^J in the representation. Then, by the universal property of $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$, there exists a $*$-homomorphism $\beta_z : \mathcal{O}_{(E,\varphi_E)}(J) \to \mathcal{O}_{(E,\varphi_E)}(J)$ such that $\beta_z \circ \iota_A^J = \iota_A^J$ and $\beta_z \circ \iota_E^J = t_z$. $\square$

Because this results provides us with a universality claim for Cuntz-Pimsner invariant representations relative to an ideal $J \subset J_E$ we define this subcategory in which $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ is initial.

Definition 5.8. Let A be a C^*-algebra, let (E, φ_E) a C^*-correspondence over A and let J be an ideal in A such that $J \subset J_E$. Denote by $\mathcal{C}^J_{(E,\varphi_E)}$ the subcategory of $\mathcal{C}_{(E,\varphi_E)}$ consisting of all surjective representations that are Cuntz-Pimsner invariant relative to J.

There is more to say about this representation; As in the case of the Toeplitz representation the Cuntz-Pimsner representation relative to an ideal $J \subset J_E$ is also injective. The idea of how to prove this is similar: Using the universality of the relative Cuntz-Pimsner representation, we can construct a quotient of the Fock representation that is injective in $\mathcal{C}^J_{(E,\varphi_E)}$.

Let $J \subset J_E$ and consider the set $\mathscr{F}(E)J := \overline{\operatorname{span}}\{xj : x \in \mathscr{F}(E), j \in J\}$. It can be shown that this is a Hilbert J-module with the sum, inner product and right action given by $\mathscr{F}(E)$, and also that $\mathscr{F}(E)J \subset \mathscr{F}(E)$ as a closed linear subspace ([Kat07, Corollary 1.4]).

By definition we have that $\mathscr{K}(\mathscr{F}(E)J) = \overline{\operatorname{span}}\{\theta_{x,y} : x, y \in \mathscr{F}(E)J\}$. Say now that $x, y \in \mathscr{F}(E)J$, then we can write $x = \lim_{n\to\infty} x_n$ where x_n is a sum of elements on the form xj for $x \in \mathscr{F}(E)$ and $j \in J$. Similarly we write $y = \lim_{n\to\infty} y_n$ for y_n of the same form as x_n. We see that $\theta_{x,y} = \lim_{n\to\infty} \theta_{x_n,y_n}$ (since both the inner product and multiplication is continuous operators), and that θ_{x_n,y_n} is just a sum of elements on the form $\theta_{xj,yk} = \theta_{xjk^*,y}$ for $x, y \in \mathscr{F}(E)$ and $j, k \in J$. Hence:

$$\mathscr{K}(\mathscr{F}(E)J) = \overline{\operatorname{span}}\{\theta_{xj,y} : x, y \in \mathscr{F}(E),\ j \in J\}. \tag{5.16}$$

This is actually an ideal in $\mathscr{L}(\mathscr{F}(E))$:

Proposition 5.9 ([Kat04, Proposition 4.6])**.** *$\mathscr{K}(\mathscr{F}(E)J)$ is an ideal in $\mathscr{L}(\mathscr{F}(E))$ contained in $C^*(\varphi_\infty, \tau_\infty)$.*

Proof. Let $k \in \mathscr{K}(\mathscr{F}(E)J)$ and $t \in \mathscr{L}(\mathscr{F}(E))$. By Eq. (5.16) it suffices to show that $kt \in \mathscr{K}(\mathscr{F}(E)J)$ for $k = \theta_{xj,y}$ for some $x, y \in \mathscr{F}(E)$ and $j \in J$. By Lemma 4.1 it follows that $kt = \theta_{xj,y}t = \theta_{xj,t^*y} = \theta_{xj,z}$ where $z = t^*y \in \mathscr{F}(E)$, it is therefore seen that $kt \in \mathscr{K}(\mathscr{F}(E)J)$. Similarly $tk = t\theta_{xj,y} = \theta_{txj,y} = \theta_{z'j,y}$ where $z' = tx \in \mathscr{F}(E)$ and again $kt \in \mathscr{K}(\mathscr{F}(E)J)$.

We now show that $\mathscr{K}(\mathscr{F}(E)J) \subset C^*(\varphi_\infty, \tau_\infty)$. Since $\operatorname{span}\{\theta_{ax,y} : x \in E^{\otimes n}, y \in E^{\otimes m}, a \in J\}$ is dense in $\mathscr{K}(\mathscr{F}(E)J)$ (by the way $\mathscr{F}(E)$ is defined) it suffies to show that $\theta_{ax,y} \in C^*(\varphi_\infty, \tau_\infty)$.

Let $x \in E^{\otimes n}$, $y \in E^{\otimes m}$, $a \in J$ and $z = (z_i)_{i\in\mathbb{N}} \in \mathscr{F}(E)$. Since $E^{\otimes m} \subset \mathscr{F}(E)$ we can identify y with $(y_i)_{i\in\mathbb{N}}$ where $y_i = 0$ for every $i \neq m$. Because of how the adjoint $\tau^m_\infty(y)$ acts on $\mathscr{F}(E)$ we get:

$$z' := \tau^m_\infty(y)^*(z_i) = \sum_{k\geq 0} \tau^m_k(y_m)^*(z_{k+m}). \tag{5.17}$$

The map $\varphi_0(a)$ is zero outside of $E^{\otimes 0}$, so when we apply it to z' we get:

$$z'' := \varphi_0(a)(z') = (a\tau_0^m(y_m)^*(z_m), 0, \ldots) \in \mathscr{F}(E), \tag{5.18}$$

and because:

$$\tau_0^m(y_m)^*(z_m) = \varphi_0(\langle y_m, z_m\rangle) = \langle y_m, z_m\rangle, \tag{5.19}$$

we have:

$$\begin{aligned} \tau_\infty^n(x)(z'') &= \sum_{k=0}^{\infty} \tau_k^n(x_n)(z_k'') \\ &= (0, \ldots, x_n a\langle y_m, z_m\rangle, 0, 0, \ldots) \\ &= (0, \ldots, \theta_{x_n a, y_m}(z_m), 0, \ldots), \end{aligned} \tag{5.20}$$

which shows that:

$$\theta_{xa,y} = \tau_\infty^n(x)\varphi_0(a)\tau_\infty^m(y)^*, \tag{5.21}$$

and by [Kat04, Proposition 4.4] it follows that the closed ideal $\mathscr{K}(\mathscr{F}(E)J)$ of $\mathscr{L}(\mathscr{F}(E))$ is contained in $C^*(\varphi_\infty, \tau_\infty) \subset \mathscr{L}(\mathscr{F}(E))$. □

Using this result we see that $(\varphi_\infty^J, \tau_\infty^J, \mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J))$ is a representation where $\varphi_\infty^J = q_J \circ \varphi_\infty$, $\tau_\infty^J = q_J \circ \tau_\infty$ and $q_J : \mathscr{L}(\mathscr{F}(E)) \to \mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J)$ is the quotient map. This is illustrated by the diagram:

$$\begin{tikzcd}
A \arrow[dr, hook, "\varphi_\infty"'] \arrow[drr, bend left, "\varphi_\infty^J"] & & \\
& \mathscr{L}(\mathscr{F}(E)) \arrow[r, two heads, "q_J"] & \mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J) \\
E \arrow[ur, hook, "\tau_\infty"] \arrow[urr, bend right, "\tau_\infty^J"'] & &
\end{tikzcd} \tag{5.22}$$

To prove the injectivity claim of the Cuntz-Pimsner representation relative to an ideal $J \subset J_E$ we first need to prove that $(\varphi_\infty^J, \tau_\infty^J, \mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J))$ is Cuntz-Pimsner invariant relative to J and that it is injective.

Theorem 5.10. *Let A be a C^*-algebra, let (E, φ_E) a C^*-correspondence over A and let $J \subset A$ be an ideal such that $J \subset J_E$. Then the representation:*

$$(\varphi_\infty^J, \tau_\infty^J, \mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J)) \in \mathcal{C}_{(E,\varphi_E)}^J, \tag{5.23}$$

and it is injective.

Proof. [Kat04, Proposition 4.4] states that for $a \in J_E$, we have $\varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_E(a)) = \varphi_0(a) \in \mathscr{L}(E^{\otimes 0})$, where $(E^{\otimes 0}, \varphi_0)$ represents the identity correspondence over A. In particular this holds for all $a \in J$ as well, since $J \subset J_E$. By applying the quotient map we get that $\varphi_\infty^J(a) - \psi_{\tau_\infty^J}(\varphi_E(a)) = q_J(\varphi_0(a))$, but since $\mathscr{K}(E^{\otimes 0}) = \mathscr{L}(E^{\otimes 0})$ and $\varphi_0(a)$ is the left multiplication of elements in $E^{\otimes 0} = A$ by elements in J it follows that $q_J(\varphi_0(a)) = 0$ and consequently that $\varphi_\infty^J(a) = \psi_{\tau_\infty^J}(\varphi_E(a))$. This shows that $(\varphi_\infty^J, \tau_\infty^J, \mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J))$ is a Cuntz-Pimsner invariant representation relative to J, and the representation is therefore in $\mathcal{C}^J_{(E,\varphi_E)}$.

Take now $a \in A$ with $\varphi_\infty^J(a) = 0$, then $\varphi_\infty(a) \in \mathscr{K}(\mathscr{F}(E)J)$. We can write:

$$\varphi_n(a) = P_n \varphi_\infty(a) P_n, \tag{5.24}$$

where P_n is the projection from $\mathscr{F}(E)$ onto the nth direct summand $E^{\otimes n} \subset \mathscr{F}(E)$. Since $\varphi_\infty(a) \in \mathscr{K}(\mathscr{F}(E)J)$ it follows that:

$$\varphi_n(a) \in P_n \mathscr{K}(\mathscr{F}(E)J) P_n. \tag{5.25}$$

We first show that $P_n \mathscr{K}(\mathscr{F}(E)J) P_n = \mathscr{K}(E^{\otimes n}J)$, for then $\varphi_n(a) \in \mathscr{K}(E^{\otimes n}J)$ which implies that for $n = 0$ we get that $a \in J$.

Take $x, y \in \mathscr{F}(E)$ and $a \in J$ and consider $\theta_{xa,y} \in \mathscr{K}(\mathscr{F}(E)J)$, the linear span of these elements are dense in $\mathscr{K}(\mathscr{F}(E)J)$. Since P_n is a projection it is self-adjoint and by Lemma 4.1 it follows that $P_n \theta_{ax,y} P_n = \theta_{P_n xa, P_n y}$. This shows that $P_n \mathscr{K}(\mathscr{F}(E)J) P_n \subset \mathscr{K}(E^{\otimes n})$. The converse inclusion, follows from the fact that if $x, y \in E^{\otimes n}$ and $a \in J$, then $\theta_{xa,y} = P_n \theta_{xa,y} P_n$.

For $a \in J \subset J_E$ the map $\varphi_1 = \varphi_E$ is injective into $\mathscr{K}(E)$ by definition of J_E, and since φ_E is a $*$-homomorphism it is isometric and therefore $\|\varphi_1(a)\| = \|a\|$. For $n \geq 1$ [Kat04, Proposition 4.7] implies that the map sending $\varphi_n(a) \mapsto \varphi_n(a) \otimes \mathrm{id}_1$ is injective by Eq. (5.25). From definition of $\varphi_{n+1}(a)$ we also see that $\varphi_{n+1}(a) = \varphi_n(a) \otimes \mathrm{id}_1$. Putting this together we have that $\|\varphi_n(a)\| = \|\varphi_{n+1}(a)\|$ for all $n \geq 1$. Combining this with $\|\varphi_1(a)\| = \|a\|$ we see that $\|\varphi_n(a)\| = \|a\|$ for all $n \geq 1$.

We now show that $\lim_{n\to\infty} \|\varphi_n(a)\| = 0$ which implies that $\|a\| = 0$ and therefore that $a = 0$, ensuring injectivity. Recall from Eq. (5.24) that $\varphi_n(a) = P_n \varphi_\infty(a) P_n$, so if we can prove that $\lim_{n\to\infty} \|P_n \varphi_\infty(a) P_n\| = 0$ we are done.

Since $\varphi_\infty(a) \in \mathscr{K}(\mathscr{F}(E)J)$ and $\mathscr{K}(\mathscr{F}(E)J)$ is linearly spanned by the elements $\theta'_{xa,y}$ for $x, y \in \mathscr{F}(E)$ which again is linearly spanned (with closure) by $\theta_{xa,y}$ for $x \in E^{\otimes k}$, $y \in E^{\otimes l}$ it is enough to prove that $\lim_{n\to\infty} \|P_n \theta_{xa,y} P_n\| = 0$. But since $\theta_{xa,y} P_n = 0$ for all $l \neq n$ it follows that the limit goes to zero as $n \to \infty$. $\square$

Using the injective representation from Theorem 5.10 it is not difficult to see that the Cuntz-Pimsner representation relative to an ideal $J \subset J_E$ is injective.

Theorem 5.11. *Let A be a C^*-algebra, let (E, φ_E) a C^*-correspondence over A and let $J \subset A$ be an ideal such that $J \subset J_E$. Then the Cuntz-Pimsner representation relative to J is injective.*

Proof. From Theorem 5.7 $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$, the Cuntz-Pimsner representation relative to J, is universal in $\mathcal{C}^J_{(E,\varphi_E)}$ so therefore there exists a unique $*$-homomorphism η such that the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[dr, "\iota_A^J"'] \arrow[drr, hook, bend left, "\varphi_\infty^J"] & & \\
& \mathcal{O}_{(E,\varphi_E)}(J) \arrow[r, dashed, "\eta"] & \mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J) \\
E \arrow[ur, "\iota_E^J"] \arrow[urr, hook, bend right, "\tau_\infty^J"'] & &
\end{tikzcd}
\tag{5.26}
$$

This implies that $\varphi_\infty^J = \eta \circ \iota_A^J$ and since φ_∞^J is injective it follows that ι_A^J is injective. $\square$

6 A correspondence between ideals and representations

Let (E, φ_E) be a C^*-correspondence over the C^*-algebra A. In this section we prove that we can completely classify the bijective representations admitting a gauge action, as a relative Cuntz-Pimsner representations using certain ideals of A. From this classification we will get a final object in the category of bijective representation admitting a gauge action. By Theorem 5.7 the surjective representation $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ admits a gauge action. That is, for each $z \in \mathbb{T}$ there exists a $*$-homomorphism $\beta_z : \mathcal{T}_{(E,\varphi_E)} \to \mathcal{T}_{(E,\varphi_E)}$ such that $\beta_z(\iota_A(a)) = \iota_A(a)$ and $\beta_z(\iota_E(x)) = z\iota_E(x)$ for all $a \in A, x \in E$.

First a definition:

Definition 6.1. Let I be a closed, two-sided ideal in $\mathcal{T}_{(E,\varphi_E)}$. We say that I is *gauge invariant* if $\beta_z(I) \subset I$ for all $z \in \mathbb{T}$, where β_z is the gauge action on $(\iota_A, \iota_E, \mathcal{T}_{(e,\varphi_E)})$.

Gauge invariance and the property of admitting a gauge action is related in a natural way.

Proposition 6.2. *Let I be a closed ideal of $\mathcal{T}_{(E,\varphi_E)}$, let β_z be the gauge action of $\mathcal{T}_{(E,\varphi_E)}$, and let $q_I : \mathcal{T}_{(E,\varphi_E)} \to \mathcal{T}_{(E,\varphi_E)}/I$ be the quotient map. Then I is gauge invariant if and only if the representation $(q_I \circ \iota_A, q_I \circ \iota_E, \mathcal{T}_{(E,\varphi_E)}/I)$ admits a gauge action.*

Proof. Assume first that I is gauge invariant, that is $\beta_z(I) \subset I$ for all $z \in \mathbb{T}$. Fix any $z \in \mathbb{T}$ and consider the mapping

$$q_I \circ \beta_z : \mathcal{T}_{(E,\varphi_E)} \to \mathcal{T}_{(E,\varphi_E)}/I. \tag{6.1}$$

Since I is gauge invariant it follows that $q_I \circ \beta_z(I) = 0$, which is to say that $I \subset \ker(q_I \circ \beta_z)$. This induces by the quotient structure a $*$-homomorphism:

$$\beta'_z : \mathcal{T}_{(E,\varphi_E)}/I \to \mathcal{T}_{(E,\varphi_E)}/I, \tag{6.2}$$

such that $q_I \circ \beta_z = \beta'_z \circ q_I$. See this diagram:

$$\begin{tikzcd} & \mathcal{T}_{(E,\varphi_E)}/I \\ \mathcal{T}_{(E,\varphi_E)} \arrow[ur, "q_I \circ \beta_z"] \arrow[r, two heads, "q_I"] & \mathcal{T}_{(E,\varphi_E)}/I \arrow[u, dashed, "\beta'_z"'] \end{tikzcd} \tag{6.3}$$

Now if $a \in A$, then we have $\beta_z(\iota_A(a)) = \iota_A(a)$, so:

$$q_I(\iota_A(a)) = q_I(\beta_z(\iota_A(a))) = \beta'_z(q_I(\iota_A(a))). \tag{6.4}$$

Similarly for $x \in E$ we have $\beta_z(\iota_E(x)) = z\iota_E(x)$ and therefore:

$$zq_I(\iota_E(x)) = q_I(z\iota_E(x)) = q_I(\beta_z(\iota_E(x))) = \beta'_z(q_I(\iota_E(x))), \tag{6.5}$$

which shows that β'_z is a gauge action on $(q_I \circ \iota_A, q_I \circ \iota_E, \mathcal{T}_{(E,\varphi_E)}/I)$.

Assume now that the representation $(q_I \circ \iota_A, q_I \circ \iota_E, \mathcal{T}_{(E,\varphi_E)}/I)$ admits a gauge action. Then there exists for every $z \in \mathbb{T}$ a $*$-homomorphism:

$$\beta'_z : \mathcal{T}_{(E,\varphi_E)}/I \to \mathcal{T}_{(E,\varphi_E)}/I, \tag{6.6}$$

such that $\beta'_z(q_I \circ \iota_A) = q_I \circ \iota_A$ and $\beta'_z(q_I \circ \iota_E) = z(q_I \circ \iota_E)$. It can easily be verified that the identity $q_I \circ \beta_z = \beta'_z \circ q_I$ holds. But then:

$$q_I(\beta_z(I)) = \beta'_z(q_I(I)) = \beta'_z(0) = 0, \tag{6.7}$$

and therefore $\beta_z(I) \in \ker(q_I)$, which implies that $\beta_z(I) \subset I$. □

Definition 6.3. Let I be an ideal in $\mathcal{T}_{(E,\varphi_E)}$ such that $I \cap \iota_A(A) = \{0\}$ and define

$$J(I) = \{a \in J_E : \iota_A(a) - \psi_{\iota_E}(\varphi_E(a)) \in I\}. \tag{6.8}$$

Denote by $\mathcal{T}(J(I))$ the minimal closed, two-sided ideal of $\mathcal{T}_{(E,\varphi_E)}$ generated by $J(I)$.

Before proving Lemma 6.5 we need to define the core of $\mathcal{T}_{(E,\varphi_E)}$:

Definition 6.4 ([Kat04, Definition 5.5])**.** Let A be a C^*-algebra and let (E, φ_E) be a C^*-correspondence over A. Define the *core* of $\mathcal{T}_{(E,\varphi_E)}$ as the C^*-subalgebra $B_{[0,\infty]}$ as follows:

$$B_{[0,\infty]} = \overline{\bigcup_{n=0}^{\infty} B_{[0,n]}}, \tag{6.9}$$

where $B_{[0,n]} = B_0 + \cdots + B_n$ and $B_n = \psi_{\iota_E^n}(\mathscr{K}(E^{\otimes n})) \subset \mathcal{T}_{(E,\varphi_E)}$ for $n \in \mathbb{N}$. Here $\iota_E^n : E^{\otimes n} \to \mathcal{T}_{(E,\varphi_E)}$ is defined as $e_1 \otimes \cdots \otimes e_n = \iota_E(e_1) \cdots \iota_E(e_n)$ in accordance with Eq. (2.4).

As an immediate consequence, since the representation $(\iota_A, \iota_E, \mathcal{T}_{(E,\varphi_E)})$ is injective, it follows that $\psi_{\iota_E^n}$ is injective for every $n \in \mathbb{N}$ and hence that $\mathscr{K}(E^{\otimes n}) \simeq B_n$, i.e. $\psi_{\iota_E^n}$ is a $*$-isomorphism.

Lemma 6.5. *Let A be a C^*-algebra, let (E, φ_E) be a C^*-correspondence over A and let I be a closed ideal in $\mathcal{T}_{(E,\varphi_E)}$. Then $I \cap B_{[0,\infty]} \subset \mathcal{T}(J(I))$.*

Proof. Note first that since I is a closed ideal of $\mathcal{T}_{(E,\varphi_E)}$ and since $B_{[0,\infty]}$ is a C^*-subalgebra of $\mathcal{T}_{(E,\varphi_E)}$, the intersection $I \cap B_{[0,\infty]}$ is a closed ideal of $B_{[0,\infty]}$. Then since $B_{[0,\infty]} = \overline{\bigcup_{n=0}^{\infty} B_{[0,n]}}$ it follows from [ALNR94, Lemma 1.3] that:

$$I \cap B_{[0,\infty]} = \overline{\bigcup_{n=0}^{\infty} I \cap B_{[0,n]}}, \tag{6.10}$$

since $B_{[0,n]} \subset B_{[0,\infty]} \Rightarrow B_{[0,n]} \cap B_{[0,\infty]} = B_{[0,n]}$. It is therefore sufficient to prove that $I \cap B_{[0,n]} \subset \mathcal{T}(J(I))$ for every $n \in \mathbb{N}$. We continue by induction over n.

For $n = 0$ we claim that $B_{[0,0]} = B_0 = \iota_A(A)$. To see that $B_0 \subset \iota_A(A)$ consider $A = E^{\otimes 0}$ as the identity correspondence over A. Then $\iota_{E^{\otimes 0}} = \iota_A$, and this combined with the fact that ψ_t is a $*$-homomorphism and that the linear span of $\theta_{a,b}$ is dense in $\mathscr{K}(E)$ results in:

$$\psi_{\iota_{E^{\otimes 0}}}(\theta_{a,b}) = \iota_A(a)\iota_A(b)^* = \iota_A(ab^*) \in A, \tag{6.11}$$

for $a, b \in A$. For the reverse inclusion let $a \in A$ and $(x_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A, then $\iota_A(a) = \iota_A(\lim_{\lambda\in\Lambda} ax_\lambda) = \lim_{\lambda\in\Lambda} \iota_A(ax_\lambda^*) = \lim \psi_{\iota_{E^{\otimes 0}}}(\theta_{a,x_\lambda})$. This shows that $B_0 = \iota_A(A)$ and hence $I \cap B_{[0,0]} = I \cap \iota_A(A) = \{0\}$ by assumption. Clearly $\{0\} \subset \mathcal{T}(J(I))$ and the base case is covered.

Assume now that $I \cap B_{[0,n]} \subset \mathcal{T}(J(I))$ and let $x \in I \cap B_{[0,n+1]}$, by definition of $B_{[0,n+1]}$ there is then $x_1 \in B_{[0,n]}$ and $x_2 \in B_{n+1}$ such that $x = x_1 + x_2$ (note that both $x_1, x_2 \in I$ as well). Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for B_n. Since $B_n \simeq \mathscr{K}(E^{\otimes n})$ under $\psi_{\iota_E^n}$ the approximate unit in B_n corresponds to an approximate unit $(k_\lambda)_{\lambda\in\Lambda}$ such that $\psi_{\iota_E^n}(k_\lambda) = u_\lambda$. We also have that $x_2 = \psi_{\iota_E^{n+1}}(k)$ for some $k \in \mathscr{K}(E^{\otimes n+1})$. Now by [Kat04, Lemma 5.10] we get the limit:

$$k = \lim_{\lambda\in\Lambda}(k_\lambda \otimes \mathrm{id}_1)k, \tag{6.12}$$

By applying $\psi_{\iota_E^{n+1}}$ and [Kat04, Lemma 5.4] we see that:

$$x_2 = \lim_{\lambda\in\Lambda} \psi_{\iota_E^{n+1}}((k_\lambda \otimes \mathrm{id}_1)k) = \lim_{\lambda\in\Lambda} \psi_{\iota_E^n}(k_\lambda)\psi_{\iota_E^{n+1}}(k) = \lim_{\lambda\in\Lambda} u_\lambda x_2. \tag{6.13}$$

Since every u_λ is self adjoint we can write $x_2 = \lim_{\lambda\in\Lambda} u_\lambda x_2 u_\lambda$. Now set $x_1' := \lim_{\lambda\in\Lambda} u_\lambda x_1 u_\lambda$, then:

$$x_1' := \lim_{\lambda\in\Lambda} u_\lambda x_1 u_\lambda = \lim_{\lambda\in\Lambda} u_\lambda(x - x_2)u_\lambda = \lim_{\lambda\in\Lambda} u_\lambda x u_\lambda - x_2 = x' - x_2, \tag{6.14}$$

where $x' = \lim_{\lambda\in\Lambda} u_\lambda x u_\lambda \in I$ (hence implying that $x_1' \in I$ too). Again by using [Kat04, Lemma 5.4] we see that $\psi_{\iota_E^m}(k)\psi_{\iota_E^n}(k') = \psi_{\iota_E^n}((k \otimes \mathrm{id}_{n-m})k')$ for integer $n \geq m$ and $k \in \mathscr{K}(E^{\otimes m}), k' \in \mathscr{K}(E^{\otimes n})$. This implies, in particular, that B_n is an ideal of $B_{[0,n]}$. From this we conclude that $x_1' \in B_n$ since $u_\lambda x_1 u_\lambda \in B_n$ and B_n is complete. Therefore $x - x' = x_1 - x_1' \in I \cap B_{[0,n]} \subset \mathcal{T}(J(I))$ by the inductive hypothesis. If we can show that $x' \in \mathcal{T}(J(I))$ we are done.

We can further reduce to proving that $u_\lambda x' u_\lambda \in \mathcal{T}(J(I))$ for all $\lambda \in \Lambda$ for then it will follow that $\lim_{\lambda\in\Lambda} u_\lambda x' u_\lambda = x' \in \mathcal{T}(J(I))$ since $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit in B_n. The C^*-algebra B_n is given by $\psi_{\iota_E^n}(\mathscr{K}(E^{\otimes n}))$, where $\mathscr{K}(E^{\otimes n}) = \overline{\mathrm{span}}\{\theta_{x,y} : x, y \in E^{\otimes n}\}$, and since $\psi_{\iota_E^n}(\theta_{x,y}) = \iota_E^n(x)\iota_E^n(y)^*$ we get:

$$\begin{aligned} B_n &= \overline{\mathrm{span}}\{\iota_E^n(x)\iota_E^n(y)^* : x, y \in E^{\otimes n}\} \\ &= \overline{\mathrm{span}}\{\iota_E(e_1)\cdots\iota_E(e_n)\iota_E(f_1)^*\cdots\iota_E(f_n)^* : e_1, \ldots, e_n, f_1, \ldots, f_n \in E\}. \end{aligned} \tag{6.15}$$

This implies that for each $\lambda \in \Lambda$ we can write u_λ as a limit of a linear combination of:

$$\iota_E(e_1)\cdots\iota_E(e_n)\iota_E(f_1)^*\cdots\iota_E(f_n)^* \text{ where } e_1,\ldots,e_n,f_1,\ldots,f_n \in E. \tag{6.16}$$

So to show that $u_\lambda x' u_\lambda \in \mathcal{T}(J(I))$ it is sufficient to show that:

$$\iota_E(f_1)^*\cdots\iota_E(f_n)^* x' \iota_E(e_1)\cdots\iota_E(e_n) \in \mathcal{T}(J(I)), \tag{6.17}$$

since then the result follows by the closedness of the ideal structure of $\mathcal{T}(J(I))$.

Now for $e_1,\ldots,e_n,f_1,\ldots,f_n \in E$ form the elements:

$$\begin{aligned} z_1 &:= \iota_E^n\left(\bigotimes_{i=1}^n f_i\right)^* x_1' \iota_E^n\left(\bigotimes_{i=1}^n e_i\right) = \iota_E(f_1)^*\cdots\iota_E(f_n)^* x_1' \iota_E(e_1)\cdots\iota_E(e_n), \\ z_2 &:= -\iota_E^n\left(\bigotimes_{i=1}^n f_i\right)^* x_2 \iota_E^n\left(\bigotimes_{i=1}^n e_i\right) = -\iota_E(f_1)^*\cdots\iota_E(f_n)^* x_2 \iota_E(e_1)\cdots\iota_E(e_n), \end{aligned} \tag{6.18}$$

and note that $\iota_E(f_1)^*\cdots\iota_E(f_n)^* x' \iota_E(e_1)\cdots\iota_E(e_n) = z_1 - z_2 \in I$ since both $x_1', x_2 \in I$. Furthermore, [Kat04, Lemma 2.6] implies that $t^m(E^{\otimes m})^* t^n(E^{\otimes n}) \in t^{n-m}(E^{\otimes(n-m)})$, and since $x_1' \in B_n$ and $x_2 \in B_{n+1}$ it is seen by Eq. (6.15) that $z_1 \in B_0 = \iota_A(A)$ and $z_2 \in B_1$.

Now choose $a \in A$ such that $\iota_A(a) = z_1$ and $\theta \in \mathscr{K}(E)$ such that $\psi_{\iota_E}(\theta) = z_2$ and consider the representation (π, t, B) where $\pi := q_I \circ \iota_A, t := q_I \circ \iota_E$, and $B = \mathcal{T}_{(E,\varphi_E)}/I$. Then we have $\pi(a) = \psi_t(\theta)$ since $z_1 - z_2 \in I$, so for all $x \in E$ we have:

$$t(\varphi_E(a)x) = \pi(a)t(x) = \psi_t(\theta)t(x) = t(\theta x). \tag{6.19}$$

Since π is injective t is injective and therefore by linearity $\varphi_E(a)x = \theta x$ for every $x \in E$ which implies that $\varphi_E(a) = \theta$ or that $a \in \varphi_E^{-1}(\mathscr{K}(E))$. Suppose now that $b \in \ker(\varphi_E)$, then:

$$\pi(ab) = \pi(a)\pi(b) = \psi_t(\varphi_E(a))\pi(b) = \psi_t(\varphi_E(a)\varphi_E(b)), \tag{6.20}$$

where the last equality essentially follows from Lemma 4.1. Since $b \in \ker(\varphi_E)$ we therefore get $\pi(ab) = 0$ which implies that $ab = 0$ since π is injective. Hence $a \in \ker(\varphi_E)^\perp$ and therefore by definition $a \in J_E$. Since $z_1 - z_2 \in I$ and $\theta = \varphi_E(a)$, by definition of $J(I)$ we have that $a \in J(I)$. We can therefore finally conclude that:

$$\iota_E(f_1)^*\cdots\iota_E(f_n)^* x' \iota_E(e_1)\cdots\iota_E(e_n) = z_1 - z_2 = \iota_A(a) - \psi_{\iota_E}(\varphi_E(a)) \in \mathcal{T}(J(I)). \tag{6.21}$$

□

The next theorem states that the ideal $\mathcal{T}(J(I))$ is the smallest ideal that contains the elements $\{\iota_A(a) - \psi_{\iota_E}(\varphi_E(a)) : a \in J_E\}$ and furthermore that I is generated by this set if and only if $I \cap \iota_A(A) = \{0\}$ and I is gauge invariant.

Theorem 6.6. *Let (E, φ_E) be a C^*-correspondence over A and let I be a closed ideal in $\mathcal{T}_{(E,\varphi_E)}$. Then $\mathcal{T}(J(I)) \subset I$, and $\mathcal{T}(J(I)) = I$ if and only if $I \cap \iota_A(A) = \{0\}$ and I is gauge invariant.*

Proof. If $a \in J(I)$, then $\iota_A(a) - \psi_{\iota_E}(\varphi_E(a)) \in I$ and therefore the ideal generated by $\{\iota_A(a) - \psi_{\iota_E}(\varphi_E(a)) : a \in J(I)\}$ is contained in I, i.e. $\mathcal{T}(J(I)) \subset I$. Now suppose that $\mathcal{T}(J(I)) = I$. We prove that $\mathcal{T}(J(I)) \cap \iota_A(A) = \{0\}$ and that $\mathcal{T}(J(I))$ is gauge invariant. Since $\mathcal{T}(J(I)) \subset I$ and $0 \in \mathcal{T}(J(I))$ it follows that $\mathcal{T}(J(I)) \cap \iota_A(A) = \{0\}$. By Theorem 5.7 and Proposition 6.2 it follows that $\mathcal{T}(J(I))$ is gauge invariant.

Now assume that $I \cap \iota_A(A) = \{0\}$ and I is gauge invariant. We need to show that $I \subset \mathcal{T}(J(I))$, so assume that $x \in I$.

Since $z \mapsto \beta_z(x^*x)$ is a continuous map with compact support the integral:

$$\int_{\mathbb{T}} \beta_z(x^*x)\,\mathrm{d}z, \tag{6.22}$$

is well-defined ([RaWi98, Lemma C.3, p. 274-275]) and belongs to $\mathcal{T}_{(E,\varphi_E)}$. Set:

$$y = \int_{\mathbb{T}} \beta_z(x^*x)\,\mathrm{d}z, \tag{6.23}$$

then $y \in B_{[0,\infty]}$ by [Kat04, Proposition 5.7]. Since I is gauge invariant we have that $\beta_z(x^*x) \in I$ for all $z \in \mathbb{T}$ and therefore $y \in I$. Hence $y \in I \cap B_{[0,\infty]} \subset \mathcal{T}(J(I))$, by Lemma 6.5.

Now let:

$$q_{\mathcal{T}(J(I))} : \mathcal{T}_{(E,\varphi_E)} \to \mathcal{O}_{(E,\varphi_E)}(J(I)), \tag{6.24}$$

be the quotient map and $\beta_z^{J(I)}$ the gauge action on $\mathcal{O}_{(E,\varphi_E)}(J(I))$. By the construction of $\mathcal{O}_{(E,\varphi_E)}(J(I))$ it is easy to see that

$$q_{\mathcal{T}(J(I))} \circ \beta_z = \beta_z^{J(I)} \circ q_{\mathcal{T}(J(I))}, \tag{6.25}$$

for all $z \in \mathbb{T}$. It then follows that:

$$\int_{\mathbb{T}} \beta_z^{J(I)}(q_{\mathcal{T}(J(I))}(x^*x))\,\mathrm{d}z = \int_{\mathbb{T}} q_{\mathcal{T}(J(I))}(\beta_z(x^*x))\,\mathrm{d}z = q_{\mathcal{T}(J(I))}(y) = 0. \tag{6.26}$$

Where the next to last equality follows by the properties given in [RaWi98, Lemma C.3, p. 274-275]. Since x^*x is positive and $*$-homomorphisms preserve positivity the element $\beta_z^{J(I)}(q_{\mathcal{T}(J(I))}(x^*x))$ is also positive for every $z \in \mathbb{T}$. Since the integral is zero over $\mathbb{T}$, this forces $\beta_z^{J(I)}(q_{\mathcal{T}(J(I))}(x^*x)) = 0$, and in particular we have $q_{\mathcal{T}(J(I))}(x^*x) = 0$. Since $\mathcal{T}(J(I))$ is an ideal, this implies that $x \in \mathcal{T}(J(I))$. □

This theorem provides us with the technicality needed to prove the classification result about bijective representations admitting a gauge action. Before doing this, however, we define a specific ideal that is central in the promised classification.

Definition 6.7. Let A be a C^*-algebra, let (E, φ_E) a C^*-correspondence over A and let (π, t, B) be a representation of A over B. We define:

$$J_{(\pi,t,B)} = \{a \in A : \pi(a) \in \psi_t(\mathscr{K}(E))\}. \tag{6.27}$$

We see that this is an ideal of A: If $j \in J_{(\pi,t,B)}$, then $\pi(j) \in \psi_t(\mathscr{K}(E))$ so we must have that that $\pi(j) = \psi_t(k)$ for some $k \in \mathscr{K}(E)$, but since the linear span of $\theta_{x,y}$ for some $x, y \in E$ are dense in $\mathscr{K}(E)$ we can assume that $k = \theta_{x,y}$.

For $j_1, j_2 \in J_{(\pi,t,B)}$ we therefore have

$$\pi(j_1 - j_2) = \psi_t(\theta_{x_1,y_1}) - \psi_t(\theta_{x_2,y_2}) = \psi_t(\theta_{x_1,y_1} - \theta_{x_2,y_2}), \tag{6.28}$$

since ψ_t is a $*$-homomorphism. Since $\mathscr{K}(E)$ is an ideal we therefore see that $\pi(j_1 - j_2) \in \psi_t(\mathscr{K}(E))$, and $J_{(\pi,t,B)}$ is an additive subgroup of A.

If $a \in A, j \in J_{(\pi,t,B)}$, then we have $\pi(j) = \psi_t(\theta_{x,y})$ for some $x, y \in E$ and $\pi(aj) = \pi(a)\psi_t(\theta_{x,y}) = \psi_t(\varphi_E(a)\theta_{x,y})$ by Proposition 4.4. Now since $\mathscr{K}(E) \subset \mathscr{L}(E)$ is an ideal $\varphi_E(a)\theta_{x,y} \in \mathscr{K}(E)$. That $\pi(ja) \in \psi_t(\mathscr{K}(E))$ follows in the same way from Proposition 4.4. This proves that $J_{(\pi,t,B)}$ is a two-sided ideal of A.

If the representation (π, t, B) is injective there is a relation between J_E and $J_{(\pi,t,B)}$.

Lemma 6.8 ([CaOr11, Lemma 3.24])**.** *Let A be a C^*-algebra, let (E, φ_E) be a C^*-correspondence over A and let (π, t, B) an injective representation of (E, φ_E) on B. For $a \in A$ we have $a \in J_{(\pi,t,B)}$ if and only if $a \in J_E$ and $\pi(a) = \psi_t(\varphi_E(a))$.*

Proof. If $a \in J_E$ then $\varphi_E(a) \in \mathscr{K}(E)$ so $\pi(a) = \psi_t(\varphi_E(a)) \in \psi_t(\mathscr{K}(E))$. For the other direction see [Kat04, Proposition 3.3]. □

As an immediate result of this it follows that if $J \subset J_E$ is an ideal of A, and (π, t, B) is injective, then (π, t, B) is Cuntz-Pimsner invariant relative to J if and only if $J \subset J_{(\pi,t,B)} = J_E$.

The next theorem is perhaps the main result of this section. The two first parts roughly states that (π, t, B) is Cuntz-Pimsner invariant if and only if there is a $*$-homomorphism from $\mathcal{O}_{(E,\varphi_E)}(J)$ to B satisfying some extra conditions. The last parts provides us with the promised classification result.

Theorem 6.9 ([CaOr11, Theorem 3.29])**.** *Let A and B be C^*-algebras, let (E, φ_E) a C^*-correspondence over A, let J be an ideal of A such that $J \subset J_E$ and let (π, t, B) be a representation of (E, φ_E) over B. Then:*

- *i) If there exists a $*$-homomorphism $\eta : \mathcal{O}_{(E,\varphi_E)}(J) \to B$ such that $\eta \circ \iota_A^J = \pi$ and $\eta \circ \iota_E^J = t$, then (π, t, B) is Cuntz-Pimsner invariant relative to J.*
- *ii) If (π, t, B) is Cuntz-Pimsner invariant relative to J, then there exists a unique $*$-homomorphism $\eta^J_{(\pi,t,B)} : \mathcal{O}_{(E,\varphi_E)}(J) \to B$ such that $\eta^J_{(\pi,t,B)} \circ \iota_A^J = \pi$ and $\eta^J_{(\pi,t,B)} \circ \iota_E^J = t$.*
- *iii) If (π, t, B) is Cuntz-Pimsner invariant relative to J, then $\eta^J_{(\pi,t,B)}$ is a $*$-isomorphism if and only if (π, t, B) is surjective, injective, admits a gauge action and $J = J_{(\pi,t,B)}$.*

Proof. Assume that there exists a $*$-homomorphism $\eta : \mathcal{O}_{(E,\varphi_E)}(J) \to B$ such that $\eta \circ \iota_A^J = \pi$ and $\eta \circ \iota_E^J = t$. We need to show that $\psi_t(\varphi_E(a)) = \pi(a)$ for all $a \in J$, which by assumption is equivalent to $\psi_{\eta \circ \iota_E^J}(\varphi_E(a)) - \eta \circ \iota_A^J(a) = 0$. Note that we then have

$\psi_{\eta\circ\iota_E^J}(\varphi_E(a)) - \eta \circ \iota_A^J(a) = \eta(\psi_{\iota_E^J}(\varphi_E(a)) - \iota_A^J(a)) = 0$ since η is a $*$-homomorphism and $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ is Cuntz-Pimsner invariant relative to J by Theorem 5.7. This proves the first assertion.

For the second assertion assume that (π, t, B) is Cuntz-Pimsner invariant relative to J. Then by the universality of $\mathcal{O}_{(E,\varphi_E)}(J)$ proven in Theorem 5.7 there exists a unique $*$-homomorphism $\eta^J_{(\pi,t,B)} : \mathcal{O}_{(E,\varphi_E)}(J) \to B$ such that $\eta^J_{(\pi,t,B)} \circ \iota_A^J = \pi$ and $\eta^J_{(\pi,t,B)} \circ \iota_E^J = t$.

For the third assertion we provide a nice diagram illustrating the setting:

$$\begin{tikzcd}
A \arrow[d, hook, "\iota_A"'] \arrow[dr, hook, "\iota_A^J"] \arrow[drr, bend left, "\pi"] & & \\
\mathcal{T}_{(E,\varphi(E))} \arrow[r, two heads, "\rho_J"] & \mathcal{O}_{(E,\varphi(E))}(J) \arrow[r, "\eta^J_{(\pi,t,B)}"] & B \\
E \arrow[u, hook, "\iota_E"] \arrow[ur, hook, "\iota_E^J"'] \arrow[urr, bend right, "t"'] & &
\end{tikzcd} \tag{6.29}$$

Assume first that (π, t, B) is Cuntz-Pimsner invariant relative to J and that $\eta^J_{(\pi,t,B)}$ is a $*$-isomorphism. By Theorem 5.7 ι_A^J is injective and therefore $\eta^J_{(\pi,t,B)} \circ \iota_A^J = \pi$ is injective as well. The surjectivity of (π, t, B) follows from the surjectivity of $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ and the assumption that $\eta^J_{(\pi,t,B)}$ is a $*$-isomorphism. Let $z \in \mathbb{T}$, from Theorem 5.7 $(\iota_A^J, \iota_E^J, \mathcal{O}_{(E,\varphi_E)}(J))$ admits a gauge action β'_z and by setting $\beta_z = \eta^J_{(\pi,t,B)} \circ \beta'_z \circ \left(\eta^J_{(\pi,t,B)}\right)^{-1}$ we obtain a gauge action on (π, t, B).

Finally since (π, t, B) is Cuntz-Pimsner invariant relative to J and injective it follows by Lemma 6.8 that $J \subset J_{(\pi,t,B)}$.

For the other inclusion take the C^*-algebra $B' := \mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J)$, the quotient map $q_J : \mathscr{L}(\mathscr{F}(E)) \to B'$ and the representation $(\varphi^J_\infty, \tau^J_\infty, B')$ as in the proof of Theorem 5.10. In the same proof it is shown that this is an object in $\mathcal{C}^J_{(E,\varphi_E)}$ and therefore there exists a unique $*$-homomorphism $\eta : \mathcal{O}_{(E,\varphi_E)}(J) \to B'$ such that $\eta \circ \iota_A^J = \varphi^J_\infty$ and $\eta \circ \iota_E^J = \tau^J_\infty$. This is illustrated below:

$$
\begin{tikzcd}
 & A \arrow[dl, hook, "\iota_A^J"'] \arrow[dr, hook, "\varphi_\infty^J"] \arrow[drr, hook, bend left, "\varphi_\infty"] & & \\
\mathcal{O}_{(E,\varphi(E))}(J) \arrow[rr, dashed, "\eta"] & & B' & \mathscr{L}(\mathscr{F}(E)) \arrow[l, two heads, "q_J"'] \\
 & E \arrow[ul, hook, "\iota_E^J"] \arrow[ur, hook, "\tau_\infty^J"'] \arrow[urr, hook, bend right, "\tau_\infty"'] & &
\end{tikzcd}
\tag{6.30}
$$

Now assume that $a \in J_{(\pi,t,B)}$. Since (π, t, B) is an injective representation it follows from Lemma 6.8 that $a \in J_E$ and that:

$$0 = \pi(a) - \psi_t(\varphi_E(a)) = (\eta^J_{(\pi,t,B)})^{-1}(\iota_A^J(a) - \psi_{\iota_E^J}(\varphi_E(a))), \tag{6.31}$$

from which it follows that $\iota_A^J(a) - \psi_{\iota_E^J}(\varphi_E(a)) = 0$ since $\eta^J_{(\pi,t,B)}$ is a $*$-isomorphism. We therefore have:

$$\varphi_\infty^J(a) - \psi_{\tau_\infty^J}(\varphi_E(a)) = \eta(\iota_A^J(a) - \psi_{\iota_E^J}(\varphi_E(a))) = 0, \tag{6.32}$$

and since

$$\varphi_\infty^J(a) - \psi_{\tau_\infty^J}(\varphi_E(a)) = q_J(\varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_E(a))), \tag{6.33}$$

it follows that $\varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_E(a)) \in \mathscr{K}(\mathscr{F}(E)J)$ so by [Kat04, Proposition 4.4] we see that $\varphi_0(a) \in \mathscr{K}(\mathscr{F}(E)J)$ and since $\varphi_0(a) \in \mathscr{L}(E^{\otimes 0})$ is the multiplication on the left by elements in A we therefore have that $a \in J$. Hence $J_{(\pi,t,B)} = J$, which proves the first implication.

For the other direction, assume now that (π, t, B) is surjective, injective, admits a gauge action and that $J = J_{(\pi,t,B)}$. Since (π, t, B) is surjective, $\eta^J_{(\pi,t,B)} \circ \iota_A^J = \pi$ and $\eta^J_{(\pi,t,B)} \circ \iota_E^J = t$ it follows that $\eta^J_{(\pi,t,B)}$ is surjective.

Let $\eta_{(\pi,t,B)} : \mathcal{T}_{(E,\varphi_E)} \to B$ be the unique $*$-homomorphism such that $\eta_{(\pi,t,B)} \circ \iota_A = \pi$ and $\eta_{(\pi,t,B)} \circ \iota_E = t$ which exists by Theorem 3.4. It is then seen that $\eta_{(\pi,t,B)} = \eta^J_{(\pi,t,B)} \circ \rho_J$, so $\eta^J_{(\pi,t,B)}$ is injective if $\ker(\rho_J) = \mathcal{T}(J)$ and because $\ker(\rho_J) = \ker(\eta_{(\pi,t,B)})$ it suffices to show that $H := \ker(\eta_{(\pi,t,B)}) = \mathscr{T}(J)$. The kernel H is a closed, two-sided ideal of $\mathcal{T}_{(E,\varphi_E)}$. We first show that H is gauge invariant and that $H \cap \iota_A(A) = \{0\}$, since Theorem 6.6 then implies that $H = \mathcal{T}(J(H))$.

First, note that since $\pi = \eta_{(\pi,t,B)} \circ \iota_A$ is assumed to be injective so if $a \in H \cap \iota_A(A)$, then $\pi(a) = 0$ and therefore $a = 0$. Hence $H \cap \iota_A(A) = \{0\}$.

For the gauge invariance of H, let $z \in \mathscr{T}$ and let β_z denote the gauge action admitted by (π, t, B). In addition to this gauge action it follows from Theorem 5.7 that $\mathcal{T}_{(E,\varphi(E))}$

also admits a gauge action β'_z. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{T}_{(E,\varphi(E))} \arrow[r, "\eta_{(\pi,t,B)}"] & B \\
\mathcal{T}_{(E,\varphi(E))} \arrow[r, "\eta_{(\pi,t,B)}"] \arrow[u, "\beta'_z"] & B \arrow[u, "\beta_z"']
\end{tikzcd} \tag{6.34}$$

Suppose for the sake of argument that we can show that the diagram commutes. To show that H is gauge invariant in $\mathcal{T}_{(E,\varphi(E))}$ we need to show that $\beta'_z(H) \subset H$. We obviously have $\beta_z(\{0\}) = \{0\}$, but since H is the kernel of $\eta_{(\pi,t,B)}$ we have $\eta_{(\pi,t,B)}(H) = \{0\}$, so $\beta_z(\eta_{(\pi,t,B)}(H)) = \beta_z(\{0\}) = \{0\}$. Since the diagram is assumed to commute we then get $\eta_{(\pi,t,B)}(\beta'_z(H)) = \{0\}$, but this implies that $\beta_z(H)$ is a subset of the kernel of $\eta_{(\pi,t,B)}$, or $\beta'_z(H) \subset H$.

We now prove that the diagram commutes. To do this we will show that $\eta_{(\pi,t,B)} \circ \beta'_z = \beta_z \circ \eta_{(\pi,t,B)}$ agrees on A and E. This, combined with the fact that the composition is continuous since both functions are $*$-homormophisms implies that the functions are equal on $\mathcal{T}_{(E,\varphi(E))}$ since A and E generates $\mathcal{T}_{(E,\varphi(E))}$.

Take $a \in A$ and $x \in E$. We abuse the notation slightly by just writing a for the element $\iota_A(a) \in \mathcal{T}_{(E,\varphi(E))}$ and similarly for $x \in E$. Then using the fact that both β_z and β'_z are gauge actions and that $\eta_{(\pi,t,B)} \circ \iota_A = \pi$ and $\eta_{(\pi,t,B)} \circ \iota_E = t$ we see the following:

$$\begin{aligned}
\eta_{(\pi,t,B)} \circ \beta'_z(a) &= \eta_{(\pi,t,B)}(a) = \pi(a) = \beta_z \circ \pi(a) = \beta_z \circ \eta_{(\pi,t,B)}(a), \\
\eta_{(\pi,t,B)} \circ \beta'_z(x) &= \eta_{(\pi,t,B)}(zx) = zt(x) = \beta_z \circ t(x) = \beta_z \circ \eta_{(\pi,t,B)}(x).
\end{aligned} \tag{6.35}$$

Hence $\eta_{(\pi,t,B)} \circ \beta'_z = \beta_z \circ \eta_{(\pi,t,B)}$ are equal on A and E, and the diagram commutes.

We have now shown that $H = \mathcal{T}(J(H))$, so to show that $H = \mathcal{T}(J)$ we need to show that $J(H) = J$. To do this, we use Lemma 6.8 and that $J = J_{(\pi,t,B)}$ by assumption, to see that $a \in J = J_{(\pi,t,B)}$ if and only if $a \in J_E$ and $\pi(a) = \psi_t(\varphi_E(a))$.

First, if $a \in J = J_{(\pi,t,B)}$ then $a \in J_E$ and $\pi(a) = \psi_t(\varphi_E(a))$. Because $\eta_{(\pi,t,B)} \circ \iota_A = \pi$ and $\eta_{(\pi,t,B)} \circ \iota_E = t$ we get:

$$\eta_{(\pi,t,B)}(\iota_A(a) - \psi_{\iota_E}(\varphi_E(a))) = 0, \tag{6.36}$$

or

$$\iota_A(a) - \psi_{\iota_E}(\varphi_E(a)) \in H, \tag{6.37}$$

which implies that $a \in J(H)$.

For the other inclusion, let $a \in J(H)$. Then $a \in J_E$ and $\iota_A(a) - \psi_{\iota_E}(\varphi_E(a)) \in H$ so by applying $\eta_{(\pi,t,B)}$ we see that:

$$\eta_{(\pi,t,B)}(\iota_A(a) - \psi_{\iota_E}(\varphi_E(a))) = \{0\}, \tag{6.38}$$

or

$$\pi(a) = \psi_t(\varphi_E(a)), \tag{6.39}$$

and therefore $a \in J_{(\pi,t,B)} = J$.

To conclude we have shown that $\ker(\eta_{(\pi,t,B)}) = H = \mathcal{T}(J(H)) = \mathcal{T}(J)$. Therefore $\eta^J_{(\pi,t,B)}$ satisfying $\eta_{(\pi,t,B)} = \eta^J_{(\pi,t,B)} \circ \rho_J$ must be injective, and therefore an isomorphism. $\square$

This theorem enables us to characterize every bijective representation admitting a gauge action as a Cuntz-Pimsner representation relative to some ideal $J \subset J_E$. The idea is to use the theorem above and the ideal $J_{(\pi,t,B)}$.

Corollary 6.10. *Let (π, t, B) be a bijective representation admitting a gauge action. Then for $J = J_{(\pi,t,B)}$ there exists a $*$-isomorphism $\eta : \mathcal{O}_{(E,\varphi(E))}(J) \to B$ such that $\eta \circ \iota^J_A = \pi$ and $\eta \circ \iota^J_E = t$.*

Proof. By the remark after Lemma 6.8 the representation (π, t, B) is Cuntz-Pimsner invariant relative to J and from the third property of Theorem 6.9 it follows that there exists an isomorphism $\eta : \mathcal{O}_{(E,\varphi_E)}(J) \to B$ producing the commuting diagram:

$$
\begin{tikzcd}
A \arrow[dr, hook, "\iota^J_A"] \arrow[drr, bend left, "\pi"] & & \\
& \mathcal{O}_{(E,\varphi(E))}(J) \arrow[r, dashed, "\eta"] & B \\
E \arrow[ur, hook, "\iota^J_E"'] \arrow[urr, bend right, "t"'] & &
\end{tikzcd}
\tag{6.40}
$$

$\square$

Theorem 6.9 also enables us to observe an ordering structure on the relative Cuntz-Pimsner representation. This will allow us to consider the largest and smallest relative Cuntz-Pimsner representation.

Proposition 6.11. *Let $J_1, J_2 \subset J_E$, then $J_1 \subset J_2$ if and only if there exists a $*$-homomorphism $\sigma : \mathcal{O}_{(E,\varphi_E)}(J_1) \to \mathcal{O}_{(E,\varphi_E)}(J_2)$ such that $\sigma \circ \iota^{J_1}_A = \iota^{J_2}_A$ and $\sigma \circ \iota^{J_1}_E = \iota^{J_2}_E$.*

Proof. Assume first that $J_1 \subset J_2$, then the representation $(\iota^{J_2}_A, \iota^{J_2}_E \mathcal{O}_{(E,\varphi_E)}(J_2))$ is Cuntz-Pimsner invariant relative to J_1. Then by Theorem 6.9 (ii) there exists a unique $*$-homomorphism $\sigma : \mathcal{O}_{(E,\varphi_E)}(J_1) \to \mathcal{O}_{(E,\varphi_E)}(J_2)$ satisfying $\sigma \circ \iota^{J_1}_A = \iota^{J_2}_A$ and $\sigma \circ \iota^{J_1}_E = \iota^{J_2}_E$ (see Eq. (6.41)).

$$\begin{tikzcd}
A \arrow[dr, hook, "\iota_A^{J_1}"] \arrow[drr, hook, bend left, "\iota_A^{J_2}"] & & \\
& \mathcal{O}_{(E,\varphi_E)}(J_1) \arrow[r, dashed, "\sigma"] & \mathcal{O}_{(E,\varphi_E)}(J_2) \\
E \arrow[ur, hook, "\iota_E^{J_1}"'] \arrow[urr, hook, bend right, "\iota_E^{J_2}"'] & &
\end{tikzcd} \tag{6.41}$$

Assume now that there exists a $*$-homomorphism σ as described in the statement of this proposition. By Theorem 6.9 (i) the representation $(\iota_A^{J_2}, \iota_E^{J_2}\mathcal{O}_{(E,\varphi_E)}(J_2))$ is Cuntz-Pimsner invariant relative to J_1. It is also injective, so by Lemma 6.8 $J_1 \subset J_E = J_{(\iota_A^{J_2}, \iota_E^{J_2}\mathcal{O}_{(E,\varphi_E)}(J_2))}$. If we can show that $J_2 = J_{(\iota_A^{J_2}, \iota_E^{J_2}\mathcal{O}_{(E,\varphi_E)}(J_2))}$ we are done. For simplicity of notation and readability denote $J_{(\iota_A^{J_2}, \iota_E^{J_2}\mathcal{O}_{(E,\varphi_E)}(J_2))}$ by J'.

The inclusion $J_2 \subset J'$ follows from Lemma 6.8 since $(\iota_A^{J_2}, \iota_E^{J_2}\mathcal{O}_{(E,\varphi_E)}(J_2))$ is Cuntz-Pimsner invariant relative to J_2.

For the reverse inclusion, let $a \in J'$. To simplify notation we denote the C^*-algebra $\mathscr{L}(\mathscr{F}(E))/\mathscr{K}(\mathscr{F}(E)J_2)$ by B', let further $q_{J_2} : \mathscr{L}(\mathscr{F}(E)) \to B'$ be the quotient map. Recall that by Theorem 5.10 the representation $(\varphi_\infty^{J_2}, \tau_\infty^{J_2}, B')$ is injective and Cuntz-Pimsner invariant relative to J_2. By Theorem 6.9 (ii) there exists a unique $*$-homomorphism $\eta : \mathcal{O}_{(E,\varphi_E)}(J_2) \to B'$ such that $\eta \circ \iota_A^{J_2} = \varphi_\infty^{J_2}$ and $\eta \circ \iota_E^{J_2} = \tau_\infty^{J_2}$ for all $a \in A, x \in E$. All this is summarized in the following commuting diagram:

$$\begin{tikzcd}
& A \arrow[dl, hook, "\iota_A^{J_2}"'] \arrow[dr, hook, "\varphi_\infty^{J_2}"] \arrow[drr, hook, bend left, "\varphi_\infty"] & & \\
\mathcal{O}_{(E,\varphi(E))}(J_2) \arrow[rr, dashed, "\eta"] & & B' & \mathscr{L}(\mathscr{F}(E)) \arrow[l, "q_{J_2}"'] \\
& E \arrow[ul, hook, "\iota_E^{J_2}"] \arrow[ur, hook, "\tau_\infty^{J_2}"'] \arrow[urr, hook, bend right, "\tau_\infty"'] & &
\end{tikzcd} \tag{6.42}$$

Note then that:

$$\varphi_\infty^{J_2}(a) - \psi_{\tau_\infty^{J_2}}(\varphi_E(a)) = \eta\left(\iota_A^{J_2}(a) - \psi_{\iota_E^{J_2}}(\varphi_E(a))\right) = 0, \tag{6.43}$$

since $(\iota_A^{J_2}, \iota_E^{J_2}\mathcal{O}_{(E,\varphi_E)}(J_2))$ is Cuntz-Pimsner invariant relative to J_2. This also implies that:

$$0 = \varphi_\infty^{J_2}(a) - \psi_{\tau_\infty^{J_2}}(\varphi_E(a)) = q_{J_2}\left(\varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_E(a))\right), \tag{6.44}$$

so $\varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_E(a)) \in \mathscr{K}(\mathscr{F}(E)J_2$. By [Kat04, Proposition 4.4], then $\varphi_0(a) \in \mathscr{K}(\mathscr{F}(E)J_2$ and further:

$$\varphi_0(a) = P_0\varphi_0(a)P_0 \in P_0\mathscr{K}(\mathscr{F}(E)J_2P_0 = \mathscr{K}(E^{\otimes 0}J_2), \tag{6.45}$$

which implies that $a \in J_2$. □

This result tells us that the relative Cuntz-Pimsner representations depend, up to isomorphism, on the size of the ideals in J_E. As mentioned it is natural to ask what happens in the extreme cases. Recall that when the ideal is the zero ideal, we get the Toeplitz representation, so the interesting question is which representation corresponds to the largest possible ideal? The largest possible ideal we can consider is simply the ideal J_E and corresponding to this we get a representation called the Cuntz-Pimsner representation.

Definition 6.12. Let A be a C^*-algebra and let (E, φ_E) be a C^*-correspondence over A. The *Cuntz-Pimsner algebra* of (E, φ_E) is defined as:

$$\mathcal{O}_{(E,\varphi_E)} := \mathcal{O}_{(E,\varphi_E)}(J_E), \tag{6.46}$$

and the *Cuntz-Pimsner representation* on B as:

$$(\iota_A^{J_E}, \iota_E^{J_E}, \mathcal{O}_{(E,\varphi_E)}) \tag{6.47}$$

By Proposition 6.11 the representation $(\iota_A^{J_E}, \iota_E^{J_E}, \mathcal{O}_{(E,\varphi_E)})$ is final in the category of bijective representations admitting a gauge action:

Theorem 6.13. *Let A and B be C^*-algebras, let (E, φ_E) be a C^*-correspondence over A and let (π, t, B) be a representation of (E, φ_E) on B. The Cuntz-Pimsner representation $(\iota_A^{J_E}, \iota_E^{J_E}, \mathcal{O}_{(E,\varphi_E)})$ is final in the category of bijective representations on B admitting a gauge action.*

The next result is called the gauge invariant uniqueness theorem for $\mathcal{O}_{(E,\varphi_E)}$. When considering the Cuntz-Pimsner algebra $\mathcal{O}_{(E,\varphi_E)}$ the maps $\iota_A^{J_E}$, and $\iota_E^{J_E}$ from the Cuntz-Pimsner representation are always defined.

Theorem 6.14. *Let A be a C^*-algebra and let (E, φ_E) be a C^*-correspondence over A. If B is a C^*-algebra and $\eta : \mathcal{O}_{(E,\varphi_E)} \to B$ a surjective $*$-homomorphism, then η is injective if and only if $\eta \circ \iota_A^{J_E}$ is injective and for each $z \in \mathbb{T}$ there exists a $*$-homomorphism $\gamma_z : B \to B$ such that $\gamma_z(\eta(\iota_A^{J_E}(a))) = \eta(\iota_A^J(a))$ and $\gamma_z(\eta(\iota_E^{J_E}(x))) = z\eta(\iota_E^J(x))$ for $a \in A$, $x \in E$.*

Proof. First off we construct a representation of (E, φ_E) over B by setting $\pi = \eta \circ \iota_A^J$ and $t = \eta \circ \iota_E^J$. Then by Theorem 6.9 (i) (π, t, B) is Cuntz-Pimsner invariant relative to J.

Assume that $\eta : \mathcal{O}_E \to B$ is injective, then Theorem 6.9 (iii) implies that (π, t, B) is injective and admits a gauge action.

Now assume the converse statement. Then (π, t, B) is a surjective, injective, Cuntz-Pimsner invariant representation relative to J that admits a gauge action and therefore Theorem 6.9 (iii) implies that η is a $*$-isomorphism and in particular injective. $\square$

Using this theorem it is in some cases possible to construct a Cuntz-Pimsner algebra from a C^*-algebra B: Suppose we can construct a C^*-correspondence over B and a surjective and injective Cuntz-Pimsner invariant representation (relative to J_E) of this C^*-correspondence on B such that this representation admits a gauge action.

Then, since the representation is Cuntz-Pimsner invariant (relative to J_E) it follows from Theorem 6.9 that there exists a unique (surjective) $*$-homomorphism from $\mathcal{O}_{(E,\varphi_E)}$ to B. The injectivity and gauge action of the representation now implies by Theorem 6.14 that the map is injective as well and therefore that $\mathcal{O}_{(E,\varphi_E)} \simeq B$.

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