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Pursuing a Polynomial Invariant of 2-knots

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ABSTRACT

We construct an invariant of 2-knots akin to the Jones polynomial of a knot. To achieve this, we adopt a new point of view on the foundations of diagrammatic 2-knot theory, and introduce a series of ideas to address questions in 2-knot theory which we consider to be of fundamental significance.

SAMMENDRAG

Vi konstruerer en invariant av 2-knuter beslektet med Jones polynomet av en knute. For å oppnå dette, fremmer vi et nytt synspunkt på den diagramatiske 2-knuteteoriens grunnlag, og introduserer en rekke idéer for å besvare spørsmål innen 2-knuteteori som vi anser som fundamentale.

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my supervisor Richard Williamson for an intriguing and challenging topic. I also appreciated our highly interesting discussions and his patience through the process of writing this thesis.

I would also like to show my appreciation for my parents, who have always supported and encouraged me. A big thanks goes to my three flatmates, Heidi, Ingrid and Lisa, who reminded me on a daily basis that life outside of academia can also be fun and for picking me up when I was down.

Last, but not least, I would like to thank my friends at Matteland for keeping me company during the long working hours, and my friends, Fredrik and Kristine, for passing on their superior knowledge on how to write a master thesis.

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CHAPTER 1

INTRODUCTION

1.1. An invariant for 2-knots akin to the Jones polynomial of a knot

The discovery of the Jones polynomial in the 1980s revolutionised knot theory. It led swiftly to the resolution of long-standing conjectures [7], and forged deep links between knot theory and other fields.

In this thesis, we construct an invariant for 2-knots akin to the Jones polynomial for knots. As far as we are aware, no indication of this construction has been anticipated in the literature, even to the smallest degree. It may therefore come as quite a surprise.

We believe that our invariant can be a profound point of departure for fusing 2-knot theory and other fields, and moreover that it can lead to significant progress within 2-knot theory itself. This we leave as a challenge for future work.

In the course of constructing the invariant, we have been led to reflect deeply upon several points in 2-knot theory which we feel to be fundamental, and to introduce a series of new ideas thereafter. We feel these ideas to be of independent significance.

1.2. What is 2-knot theory?

In a nutshell, a 2-knot is a knotted sphere. Whilst knots live an $\mathbb{R}^3$, 2-knots live in $\mathbb{R}^4$. Just as in ordinary knot theory, the fundamental question in 2-knot theory is: when can a 2-knot be unknotted to a sphere? More generally, when are a pair of 2-knots isotopic to one another?

The model for the construction of our invariant is Kauffman's construction of the Jones polynomial [11]. This proceeds by first associating a monomial to each possible smoothing of a knot into a collection of disjoint circles, and summing them to obtain a polynomial. This polynomial, known as the *bracket polynomial* of a knot, is invariant under the $\mathsf{R2}$ and $\mathsf{R3}$ moves. Modifying the bracket polynomial by equipping the knot with an orientation and making use of the writhe, we arrive at a polynomial which is invariant under all three Reidemeister moves. This is the Jones polynomial.

This approach is fundamentally diagrammatic. It is not the knot itself which is significant, but rather its associated diagram, namely the combinatorial information of how its projection to a plane is built up of arcs and crossings.

Our approach to 2-knot theory is similarly diagrammatic. Rather than a 2-knot itself, it is the combinatorial information of how its projection to $\mathbb{R}^3$ is built up of planes and crossings. There are, however, important choices to be made in placing the diagrammatic theory of 2-knots on a firm foundation. The way we approach this is one of the technical innovation of the thesis.

1.3. The key ideas

We consider the following aspects of this thesis to be the heart of our work. As far as we are aware, none of these have previously been considered, and we consider all of them to be fundamental constructions in 2-knot theory.

1.3.1. A cubical approach

Throughout this thesis, we take a cubical approach to both knot theory and 2-knot theory. This appears to be novel even for ordinary knot theory, where either simplicial or smooth approach has been universally followed. The overwhelming majority of prior work in 2-knot theory has been carried out in a smooth setting: we refer the reader to the books [1], [2], [3] and [6] for an overview of much of the previous work that has been carried out in 2-knot theory.

A significant reason for our choice to work with cubes in 2-knot theory rather than a smooth setting, is that it allows us to define the notion of isotopy of 2-knots in an intuitive way, analogous to Reidemeister's original simplicial definition of an isotopy of knots [9]. We are then able to arrive at the cornerstone of a diagrammatic approach to 2-knot theory, that a pair of 2-knots are isotopic if and only if one can be obtained by the other by a sequence of Homma-Nagase-Roseman moves (see [4], [5] and [10] for the original papers), by means of a straightforward combinatorial argument. As a warm up, we demonstrate how to carry out a cubical version of Reidemeister's proof that a pair of knots are isotopic if and only if one can be obtained from the other by a sequence of Reidemeister moves.

The principal reason for our choice to work with cubes rather than simplices, is that we consider the gluing of cubes at their faces to be much easier to work with in practise than the gluing of tetrahedra. At first sight, this may not seem particularly conceptual, but in fact we regard this to be a fundamental advantage of cubes over simplices or other shape: if one looks at the difference between gluing cubes and gluing simplices from a sufficiently abstract perspective, the pleasant properties of gluing cubes along faces *do* have a conceptional explanation (endnote: this is not something I have looked into myself, but it comes from a universal property of the category of $\square$.)

1.3.2. Avoiding working too locally

In knot theory, one typically constructs diagrammatic invariants by focusing upon crossings. That is to say, it is a local structure of a knot at its crossings that is fundamental.

Whilst working on this thesis, we came to realise that a na"ive attempt to construct diagrammatic invariants of 2-knots in a similar way breaks down. It is a mistake to focus only on the possible ways in which planes can cross one another: It is not, for instance, possible to define a good notion of smoothing of a crossing in a 2-knot in this way.

Instead, it is vital to consider how planes can actually cross one another in a knotted cube. A naïve approach to smoothing crossings of a 2-knot, as in the previous paragraph, simply does not make any sense when considered in this light: it is impossible in general for the local smoothings to match up coherently.

This led us to realise that the way in which the Homma-Nagase-Roseman moves have been presented and worked with in all existing literature on 2-knot theory that we are aware of, is not appropriate for the construction of the kind of invariant which we are interested in in thesis. It is necessary to consider how these moves can arrive in a knotted cube. Especially with the saddle move, it took us a considerable time to arrive at the correct configuration.

We consider this shift away from working to locally in 2-knot theory, to be one of the most important innovations in this thesis.

1.3.3. Smoothing

Making crucial use of this realisation, we are able to define a natural notion of smoothing of a double plane crossing in a 2-knot. As far as we know, it has not previously been considered.

It is not, however, at all obvious how to go further to define a notion of smoothing of a triple point crossing. Such a notion must meet the requirement of being compatible with the corresponding double plane crossings. Nevertheless, we introduce in this thesis a notion of smoothing of a triple point crossing which we consider to be the correct one.

This is one of the most important aspects of this thesis. All proceeding work on diagrammatic 2-knot theory which we are aware of, does not truly take into account triple point crossings. It is vital to find invariants which see the triple point crossings, because it is this that is likely to lead to the deepest connections between 2-knot theory and other fields.

1.3.4. Making use of a little categorical algebra

With our notion of surgery at hand, we are able to smooth a 2-knot into a collection of disjoint unknotted pieces. Crucially, however, these may not be spheres. Any orientable surface can appear.

An innovation in our approach to an invariant of 2-knots which we consider to be highly significant, is that we do not merely count the number of disjoint surfaces that appear, as is the case with the bracket polynomial of a knot. Instead we make use of the classification of surfaces: we count the number of disjoint surfaces of each possible genus. To do this, we assign, to each possible smoothing of a 2-knot, not an ordinary menomial, but a gadget with an extra layer of structure: a categorification of a menomial. Summing these menomial, we arrive not at an ordinary polynomial, but a categorified version, which we refer to as a 2-polynomial.

1.3.5. Arriving at an invariant

In this way, we can associate a 2-polynomial to a 2-knot. In order for it to be an invariant of a 2-knot, it must be invariant under the Homma-Nagase-Roseman moves, which it is not at this point. To obtain an actual invariant involves three steps.

We consider the effect of the four Homma-Nagase-Roseman moves upon smoothing, in an analogous way to which we consider the effect of smoothing crossings upon the bracket polynomial in ordinary knot theory. Some of the resulting pieces match up, leading us to impose relations which our 2-polynomial must satisfy. These relations are considerably more complicated than those which arise in defining the bracket polynomial of a knot, and we consider it an interesting challenge for future work to understand the algebra of these relations better.

Unlike in ordinary knot theory, however, some of the pieces do knot match up. This leads us to a new idea, which we consider to be one of the most important in the this thesis. We introduce two supplementary kinds of surgery which we allow to be performed on a smooth 2-knot. One, coming from the saddle move, involves replacing some of the faces of a cylinder with the remaining ones. The other, coming from the triple point move, involves a more complicated configuration. To keep track of these surgeries, we introduce two new kinds of variables into our 2-polynomial, and adjust the relations we impose upon it accordingly.

Finally, we introduce a natural notion of writhe of an oriented 2-knot. This allows us to modify our 2-polynomial in such a way that it is an invariant under the bubble move. The resulting 2-polynomial is our 2-knot invariant. It is invariant under all four of the cubical Homma-Nagase-Roseman moves.

1.4. Analogies between the Reidemeister moves and the Homma-Nagase-Roseman moves

Intriguingly, we have come to realise that there does not seem to be a direct relationship between the Reidemeister moves in knot theory and the Homma-Nagase-Roseman moves in 2-knot theory.

The bubble move, for instance, may first be thought to be analogous to an R2 move, since taking an intersection with an orthogonal plane gives exactly an R2 move. However, it in many ways corresponds more closely to an R1 move: it introduces a double plane crossing, just as an R1 move introduces a crossing. This is reflected in the fact that we make use of a notion of writhe to modify our 2-polynomial in such a way that it is invariant under the bubble move, just as we make use of the writhe of a knot to modify the bracket polynomial in such a way that it is invariant under the R1 move.

It is the saddle move that behaves more like an R2 move. However, the analogy is not perfect: the writhe of a 2-knot is not the same before and after a saddle move, whereas the writhe of a knot is unchanged under the R2 move.

The triple point move may first be thought to be analogous to an R3 move, again since taking an intersection with an orthogonal plane gives exactly an R3 move. However, it is again more akin to an R1 move: we introduce a triple point crossing. Unlike an R1 move, however, the writhe of a 2-knot is unchanged by a triple point move.

It is the tetrahedral move that behaves more like an R3 move. We were in fact inspired by this analogy when carrying out the proof that our 2-polynomial invariant is invariant under the tetrahedral move.

1.5. How to read the thesis

An interesting practical aspect of our work on this thesis has been that we have worked throughout with the 3D graphics program Blender. We highly recommend that the thesis be read in conjunction with exploring the blender files corresponding to the figures: it is very difficult to acquire a good feeling for the more complicated cubical configurations from a 2D image.

To be able to work effectively in three dimensions is something that we feel will come to be increasingly important in several fields of mathematics. We feel that a 3D graphics tool such as Blender can be extremely useful in this regard. We feel it highly unlikely that we would have been able to carry out the parts of this work that concern triple points without the use of Blender.

CHAPTER 2

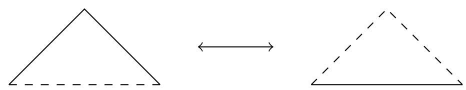
SQUARE REIDEMEISTER MOVES

As a prelude to our investigation of the Homma-Nagase-Roseman move in the next chapter, we introduce in this chapter the notion of a square move in ordinary knot theory, and define cubical versions of the Reidemeister moves. Defining an isotopy of knots by means of the two possible square moves, we explain how to prove, by a combinatorial argument, Reidemeister's theorem that a pair of knots are isotopic if and only if one can be obtained from the other by a sequence of Reidemeister moves.

2.1. Square moves and isotopy

2.1.1. Triangle move

2.1.1.1. In Reidemeister's original simplicial approach to knot theory (see [8] and [9]), isotopy of knots is defined by means of a triangle move. Given a solid triangle in $\mathbb{R}^3$, two of the edges of which are arcs of a knot, and which no other arcs of the knot intersect, a *triangle move* replaces these two arcs with the third edge of the triangle, or vice versa.

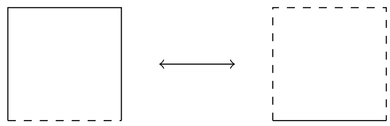


2.1.1.2. An isotopy of knots is then defined to be a sequence of triangle moves. This corresponds exactly to how we manipulate knots in practise, except that we restrict ourselves to working rigidly with triangles rather than arbitrary deformations.

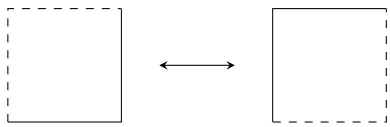
2.1.2. Square moves

2.1.2.1. We shall instead take a cubical approach to knot theory. The notion of a triangle move is replaced by a notion of square move. However, when working cubically, it turns out that we do not have only one square move to consider, but two, as we shall now explain.

2.1.2.2. Given a solid square in $\mathbb{R}^3$, three of the edges of which are arcs of a knot, and which no other arcs of the knot intersect, the first *square move* replaces these three arcs with the fourth edge of the square, or vice versa.



2.1.2.3. Given a solid square in $\mathbb{R}^3$, two of the edges of which are arcs of a knot, and which no other arcs of the knot intersect, the second *square move* replaces these two arcs with the two other edges of the square, or vice versa.



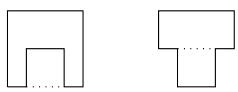
We will refer to this type of square move as the corner square move.

2.1.2.4. We define a pair of knots to be isotopic if we can obtain one from the other through a sequence of square moves.

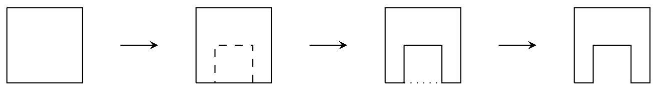
2.1.3. How do these moves arise?

2.1.3.1. There are two simple ways in which we can manipulate a square.

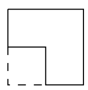
2.1.3.2. In the first manipulation, we *extrude* a small part of the side of the square in one of the two following ways.



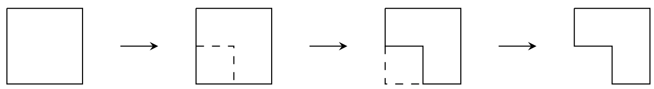
2.1.3.3. These manipulations give rise to the first square move. We can view them as replacing a small part of one side with three other sides, as demonstrated below.



2.1.3.4. In the second manipulation, we make an *indention* in one corner of the square in the following way.



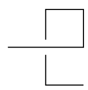
2.1.3.5. This manipulation gives rise to the second square move. We can view it as replacing a small corner with the opposite corner, as demonstrated below.



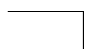
2.2. The cubical Reidemeister moves

2.2.1. What are the cubical Reidemeister moves?

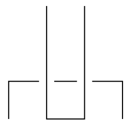
2.2.1.1. The cubical R1 *move* consists of replacing a configuration of arcs in the diagram of a knot as depicted below



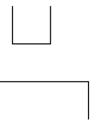
by the following arcs.



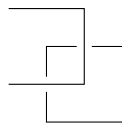
2.2.1.2. There are two variants of the cubical R2 *move.* The first consists of replacing a configuration of arcs in the diagram of a knot as depicted below



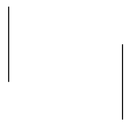
by the following arcs.



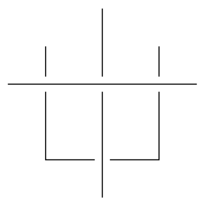
And the second consists of replacing a configuration of arcs in the diagram of a knot as depicted below



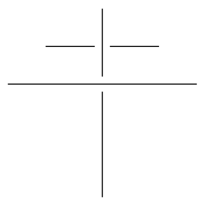
by the following arcs.



2.2.1.3. The cubical R3 *move* consists of replacing a configuration of arcs in the diagram of a knot as depicted below



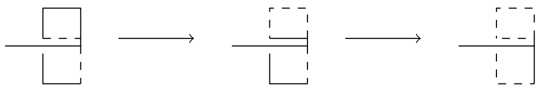
by the following arcs.



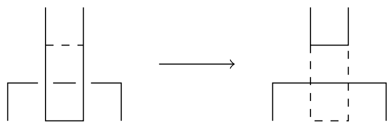
2.2.1.4. We have only illustrated the cubical Reidemeister moves above with a representative case. If we change the over and under crossings, we obtain variants of the moves.

2.2.2. How to obtain the cubical Reidemeister moves from square moves?

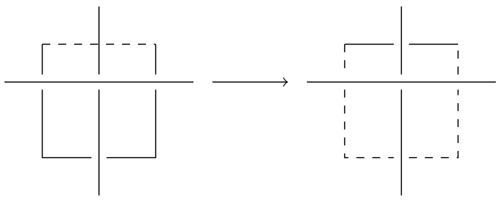
2.2.2.1. The R1 move can be obtained by two square moves as illustrated below.



2.2.2.2. The R2 move can be obtained by a square move as demonstrated below.

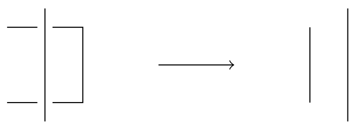


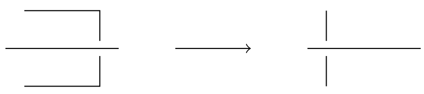
2.2.2.3. The R3 move can be obtained by a square move as illustrated below.

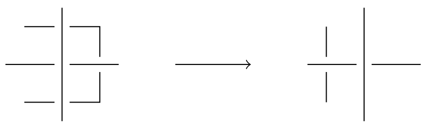


2.2.2.4. Thus the cubical Reidemeister moves arise as square moves. We now outline a proof that, conversely, every projection of a square move to a knot diagram can be obtained by a sequence of cubical Reidemeister moves.

The key point is that, after subdivision, every projection of a square move affecting at least two arcs is one of the following:



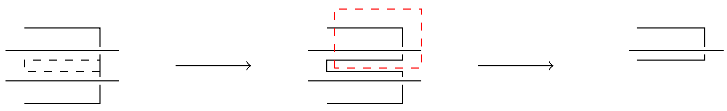


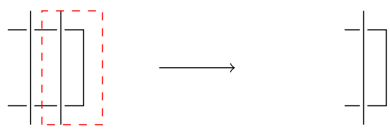


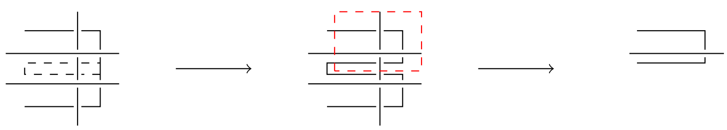
The first of these is exactly an R2 move, and the third is exactly an R3 move. The second is what is sometimes known as an R0 move, namely a manipulation of a diagram which does not affect the crossing information.

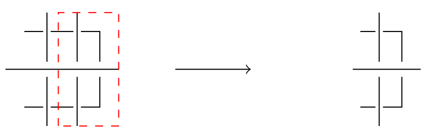
The R1 move arises in a similar way by considering square moves involving a single arc.

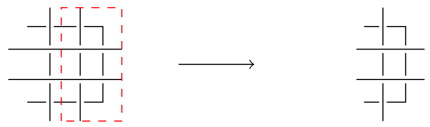
2.2.2.5. The pictures below indicate how we subdivide to obtain the configurations above. An inductive argument based upon this would treat configurations in which more arcs are involved.











CHAPTER 3

CUBICAL HOMMA-NAGASE-ROSEMAN MOVES

We introduce the notion of a cube move, and cubical versions of the Homma-Nagase-Roseman moves. As discussed in the introduction, a vital aspect of this thesis is that we do not work as locally as previous authors: we consider how the moves can arise in a knotted cube.

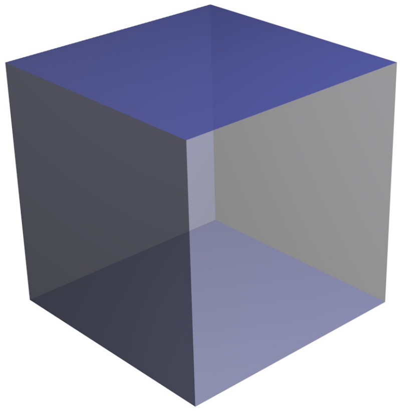
3.1. Cube moves and isotopy

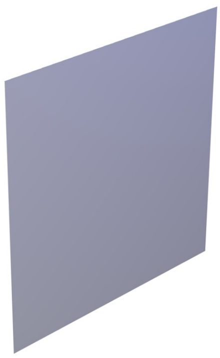
3.1.1. Cube moves

3.1.1.1. In the previous chapter, we saw that there are two square moves in ordinary knot theory, corresponding to the two possible ways of manipulating a square. In 2-knot theory, there are similarly several *cube moves*, corresponding to the different ways of manipulating a cube.

3.1.1.2. Given a solid cube in $\mathbb{R}^4$, some of the faces of which are planes of a 2-knot, and which no other planes of the 2-knot intersect, a cube move involves replacing these planes with the other faces of the cube. However, we can only do this if we can obtain the new faces by a manipulation of the initial ones.

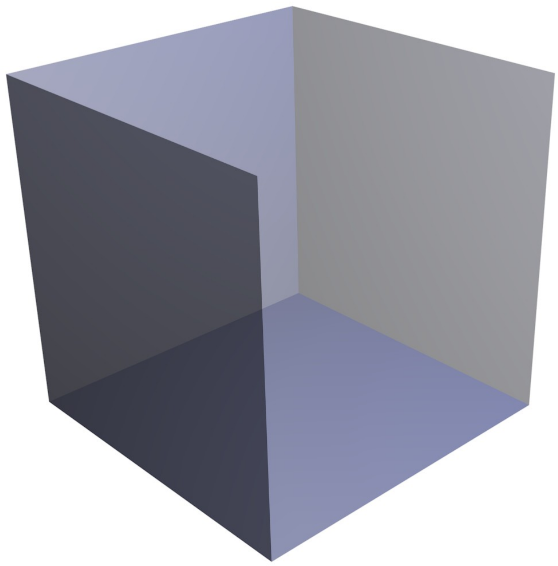
3.1.1.3. The first variant of the cube move arises by replacing one of the two configurations below with the other.

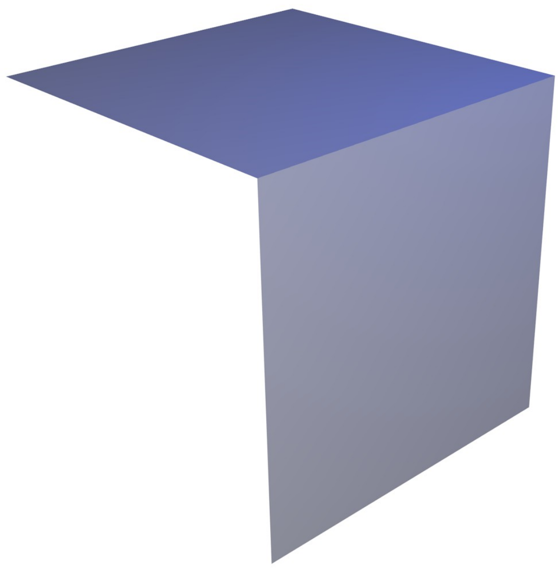




Here five faces are replaced by one, or vice versa.

3.1.1.4. The second variant of the cube move arises by replacing one of the two configurations below with the other.

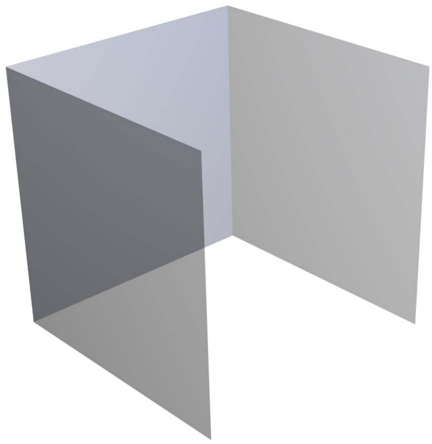


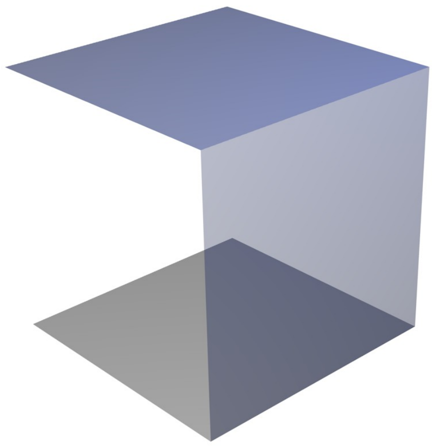


Here four faces are replaced by two, or vice versa.

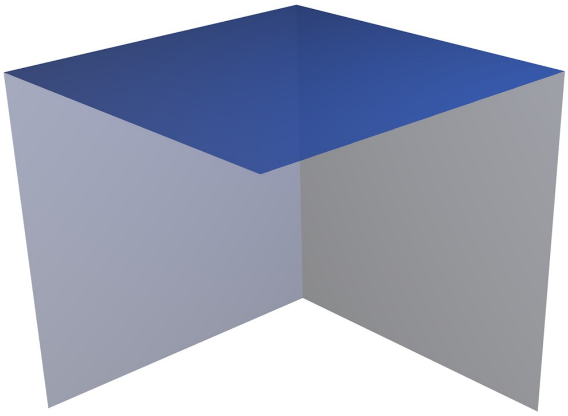
3.1.1.5. The third variant of the cube move arises by replacing three faces with the remaining three in one of the following two ways.

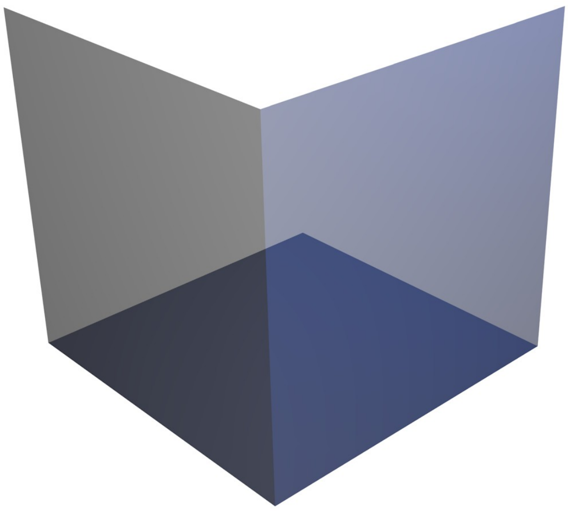
Firstly, we replace one of the two configurations below with the other.





Secondly, we replace one of the two configurations below with the other.





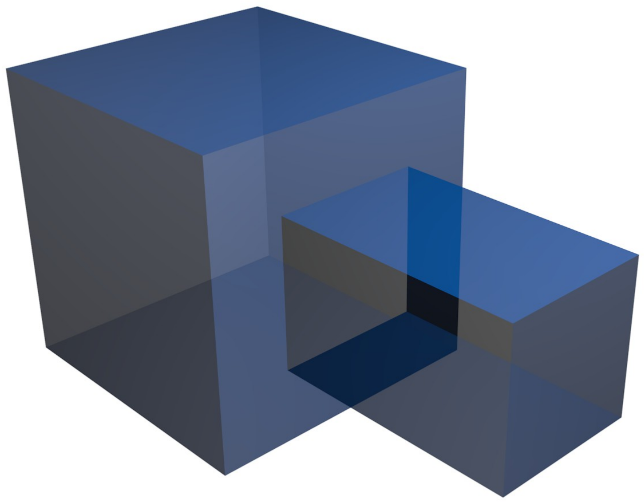
3.1.1.6. We define a pair of 2-knots to be *isotopic* if we can obtain one from the other through a sequence of cube moves.

3.1.1.7. Just as in ordinary knot theory, we feel that this notion of isotopy of 2-knots is intuitive, corresponding exactly, up to rigidifying, to how one would manipulate a 2-knot in practise.

3.1.2. How do these moves arise?

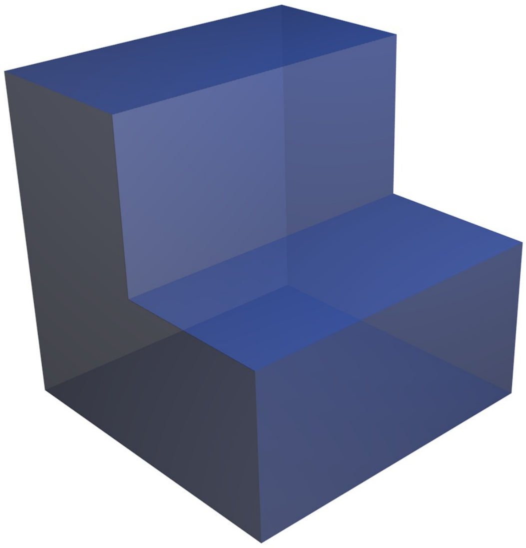
3.1.2.1. The cube moves as described above arise by considering the possible ways to manipulate a cube.

Firstly, we *extrude* regions in one face of a cube. In other words, we take a small part of the face and either stretch it away from the cube or into the cube. Both cases are demonstrated below.



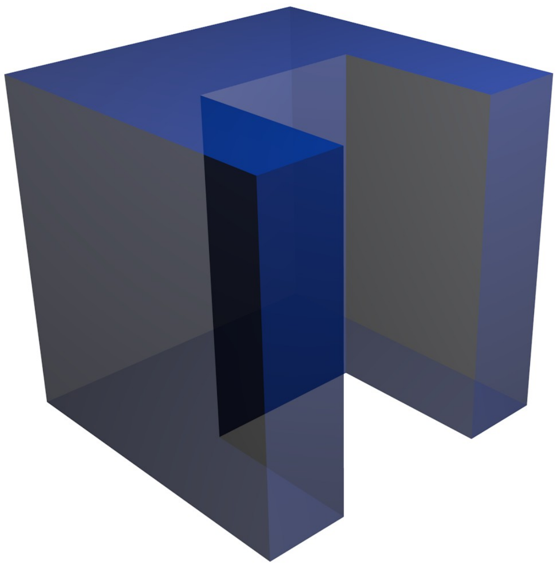
This manipulation gives rise to the first cube move described above.

Secondly, we make *edge indentions* as follows.



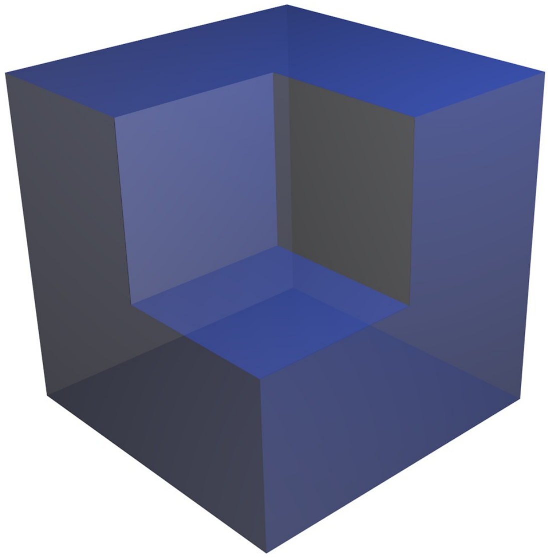
This manipulation gives rise to the second cube move described above.

Thirdly, we make *indentations* in the cube as follows.



This manipulation gives rise to the first of the two variants of the third cube move described above.

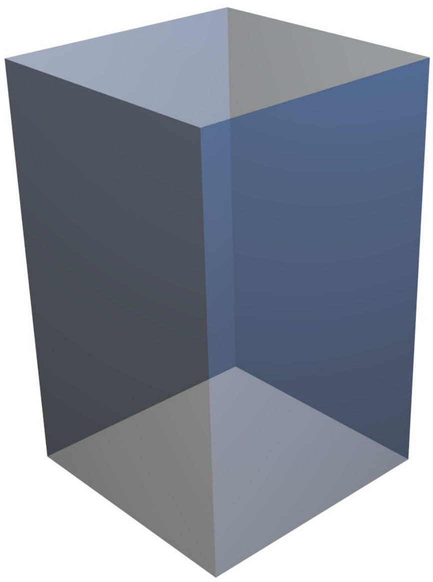
Lastly, we make *corner indentions* as depicted below.

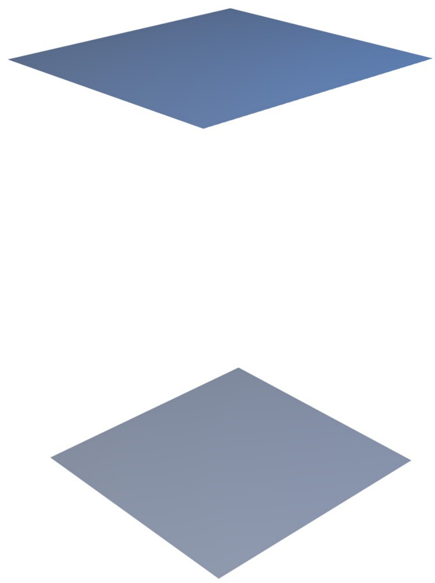


This manipulation gives rise to the second of the two variants of the third cube move described above.

3.1.3. A non-cube move

3.1.3.1. Replacing four faces of a cylinder with the missing two, or vice versa, is not a valid cube move. The four faces of the cylinder cannot be deformed to become the missing two. In other words, this replacement does not arise via a manipulation of a cube.





Nevertheless, this procedure of replacing four faces with the missing two or vice versa, will play a crucial role later in the thesis.

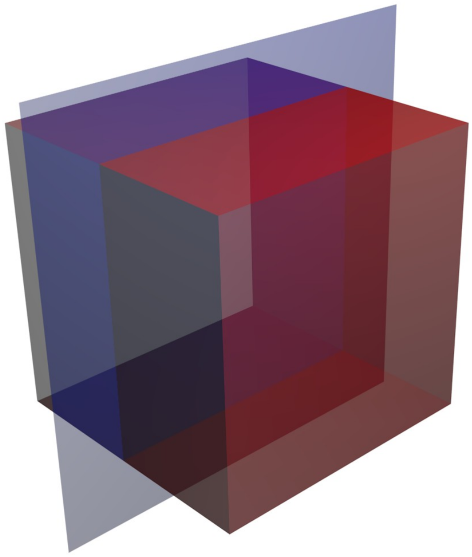
3.2. The cubical Homma-Nagase-Roseman moves

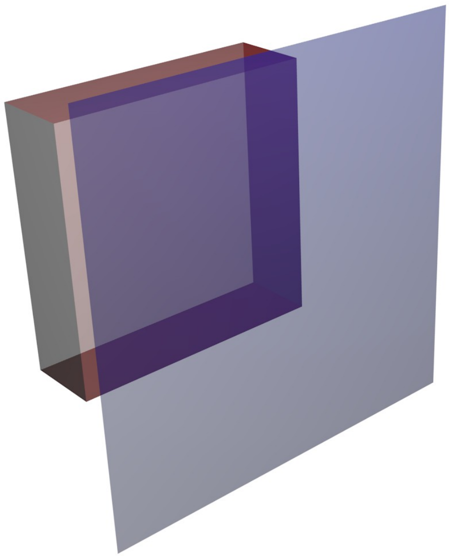
3.2.1. What are the cubical Homma-Nagase-Roseman moves?

3.2.1.1. The cubical *Homma-Nagase-Roseman moves* are the projections of cube moves to a 2-knot diagram which cannot be subdivided into configurations involving fewer planes.

3.2.1.2. We focus in this thesis only on those configurations which involve two or more planes. It is not yet clear to us how to understand the smooth Homma-Nagase-Roseman moves involving a branch point from a cubical perspective, and leave this as a challenge for future work. Given an understanding of branch points in a cubical setting, we are however convinced that the 2-polynomial we construct later in this thesis can be adapted to take them into account, and to be invariant under the Homma-Nagase-Roseman moves involving them.

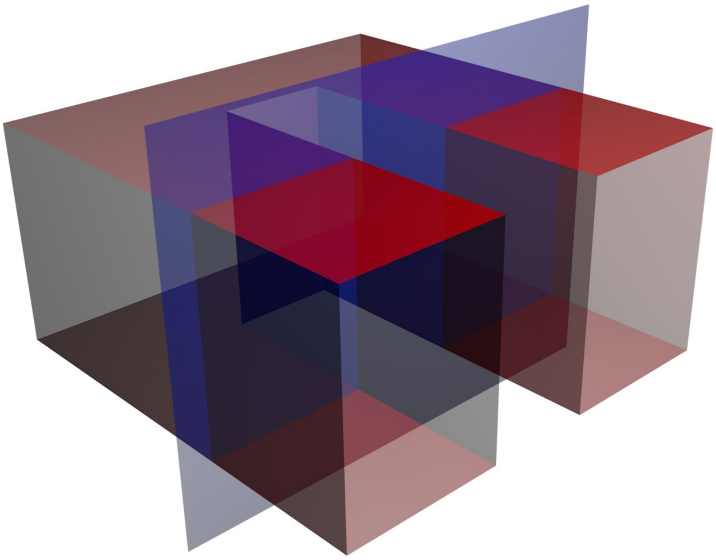
3.2.1.3. The cubical *bubble move* consists of replacing one of the two following configuration by the other.

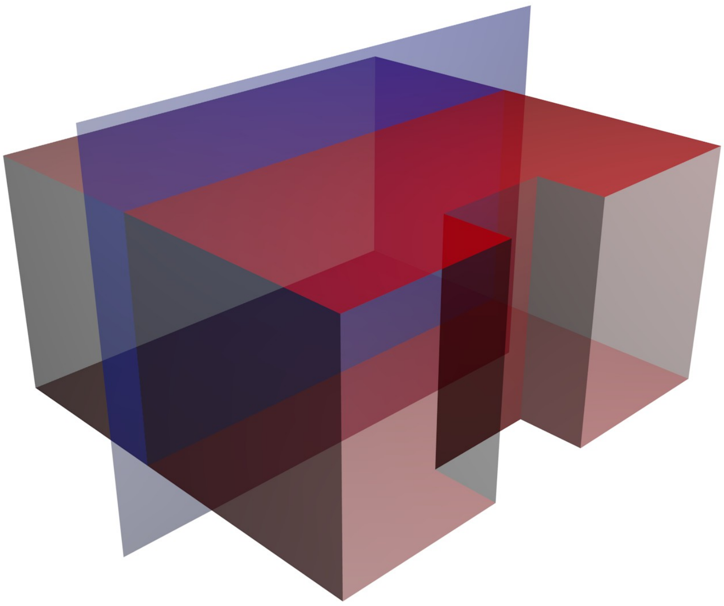




In words, the bubble move involves pushing a face of an open cube through a plane. By an open cube we mean a cube with only one missing face.

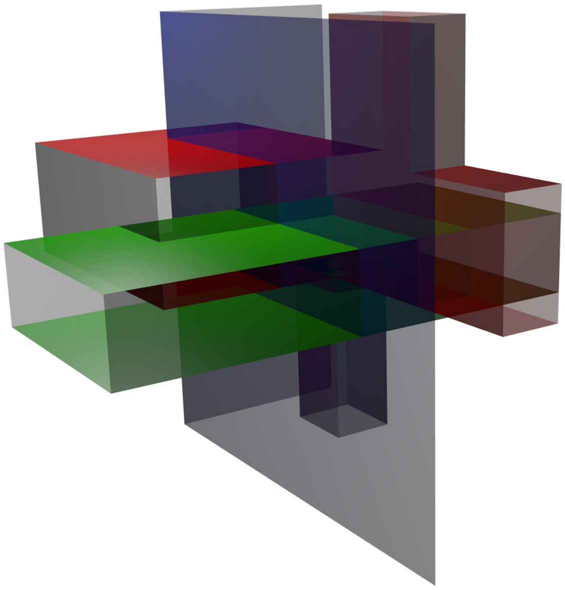
3.2.1.4. The cubical *saddle move* consists of replacing one of the two following configuration by the other.

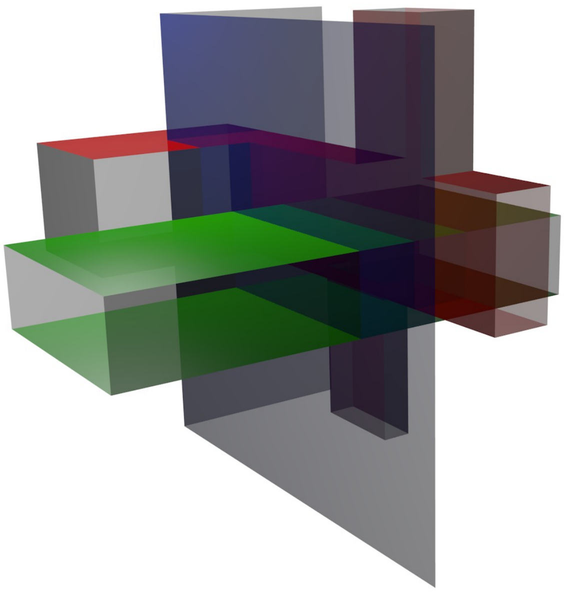




In words, the saddle move involves again a cube crossing a plane, but notice that the first configuration has a saddle, namely it is split into two cubes. Crucially, both ends of the cube are open. Therefore we cannot reduce this move to a sequence of bubble moves.

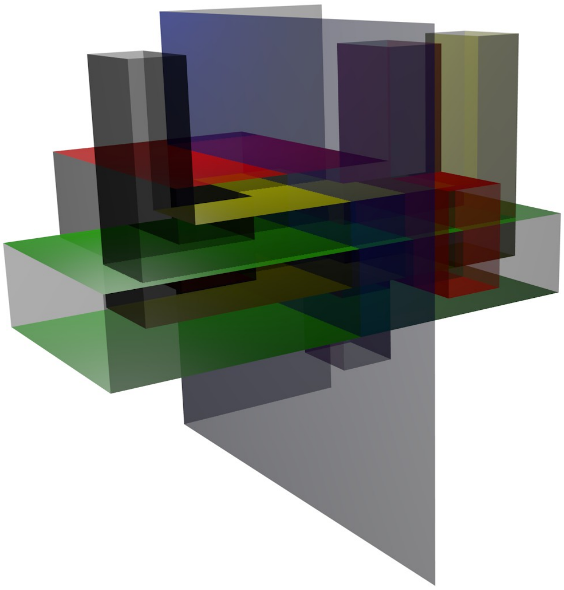
3.2.1.5. The cubical *triple point move* consists of replacing one of the two following configuration by the other.

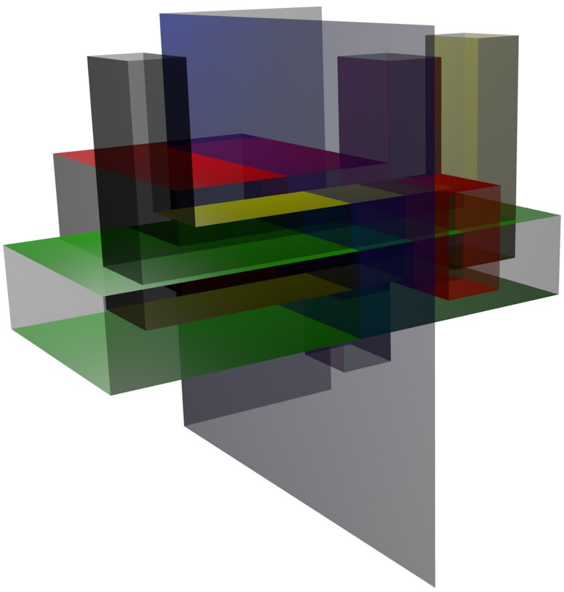




In words, the triple point move involves the crossing of two cubes and a plane. In the first configuration, all the elements cross at a point, whilst in the second configuration there is no such point. Again, since the green cube is open in both ends, the configuration cannot be simplified by a bubble move involving it. Similarly, the red cube has tunnels with missing faces, and is thereby locked in place.

3.2.1.6. The cubical *tetrahedral move* consists of replacing one of the two following configuration by the other.





In words, the tetrahedral move involves the first configuration of the triple point move together with an additional cube, depicted in yellow. In the first configuration, the yellow cube surrounds the triple point, whilst it does not in the second. Again, the open faces of the yellow cube lock it in place.

3.2.1.7. We have only illustrated the cubical Homma-Nagase-Roseman moves above with a representative case. If we take into account the different possibilities for breaking the crossing planes, we obtain variants of the moves.

3.2.2. How to obtain the cubical Homma-Nagase-Roseman moves from cube moves?

3.2.2.1. The bubble move can clearly be obtained by a cube move. We replace the five faces of the open cube with the missing one.

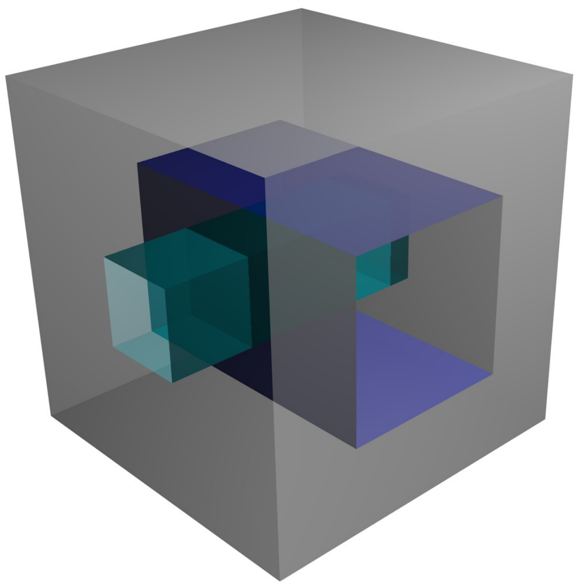
3.2.2.2. The saddle move can be obtained by a cube move involving replacing the three faces in the middle, where two are an opposite pair, with the missing three faces.

3.2.2.3. The triple point move can be obtained by a cube move in much the same manner as the bubble move, the only difference being that the cube move takes place around a double plane crossing.

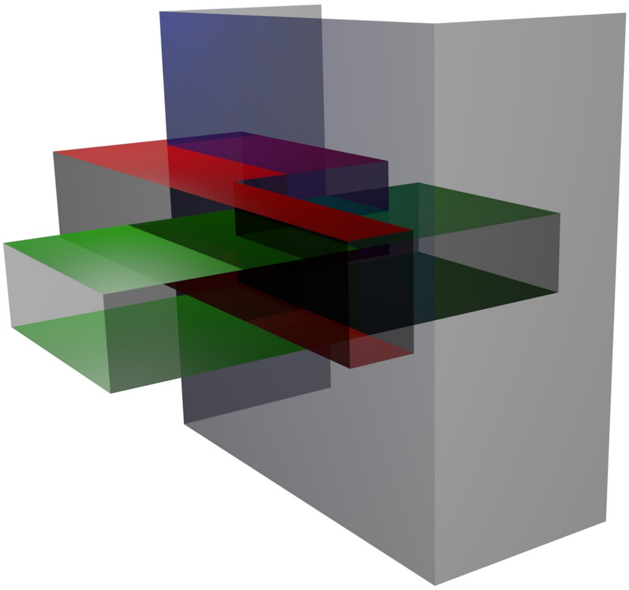
3.2.2.4. Finally, the tetrahedral move can be obtained by a cube move, again in much the same manner as the bubble move and the triple point move, the difference being that instead of taking place around a double plane crossing, it takes place around a triple point crossing.

3.2.3. Illustrations of how to use the cubical Homma-Nagase-Roseman moves

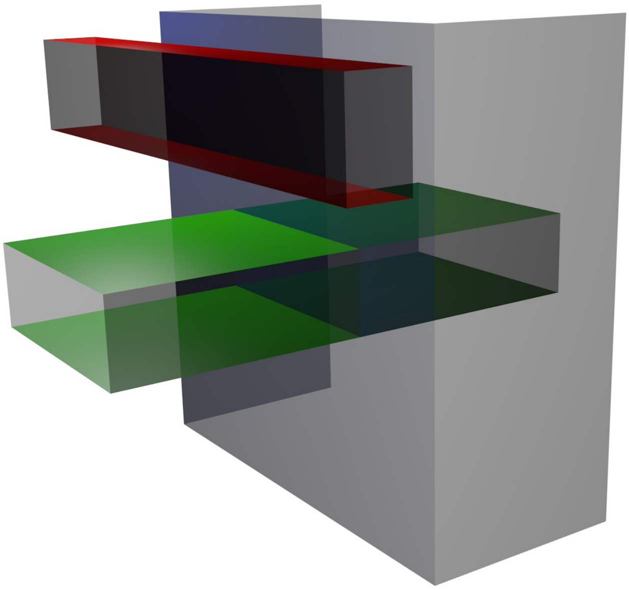
Consider the following configuration.



This is reducible by two bubble moves to an unknotted cube. For a more complicated example, consider the following configuration.



This can first be reduced by a triple point move, and then by a bubble move. We obtain with the following construction.



3.2.4. The Homma-Nagase-Roseman moves after subdivision

We conjecture that all cube moves can be obtained by sequence of Homma-Nagase-Roseman moves after subdivision. Even if this conjecture was incorrect, we consider the restriction of 2-knot theory to the moves we have described to be interesting and important in its own right.

CHAPTER 4

SMOOTHING A 2-KNOT

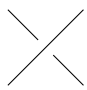
We first recall how to smooth crossings in ordinary knot theory. Then we explain that a naïve attempt to adapt this procedure to smooth double plane crossings in 2-knot theory runs into problems. Lastly we describe how to smooth double plane crossings and crossings which include triple points.

4.1. How to smooth a double crossing of a 2-knot?

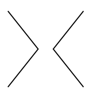
4.1.1. How to smooth a crossing of a knot?

4.1.1.1. The crucial ingredient in the construction of the Jones polynomial in ordinary knot theory via Kauffman's bracket polynomial is *smoothing* of crossings.

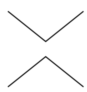
4.1.1.2. There are two possibilities when smoothing the following crossing.



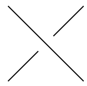
(1) Replace these three arcs with two arcs as follows.



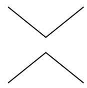
(2) Replace these three arcs with two arcs as follows.



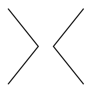
4.1.1.3. There are two possibilities when smoothing the following crossing.



(1) Replace these three arcs with two arcs as follows.



(2) Replace these three arcs with two arcs as follows.



4.1.2. States of a knot

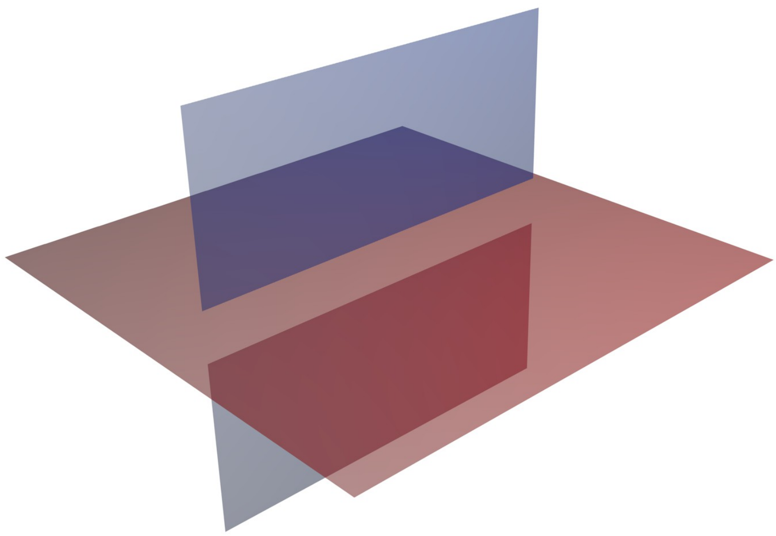
4.1.2.1. If we smooth all the crossings of a knot, we obtain a collection of disjoint circles.

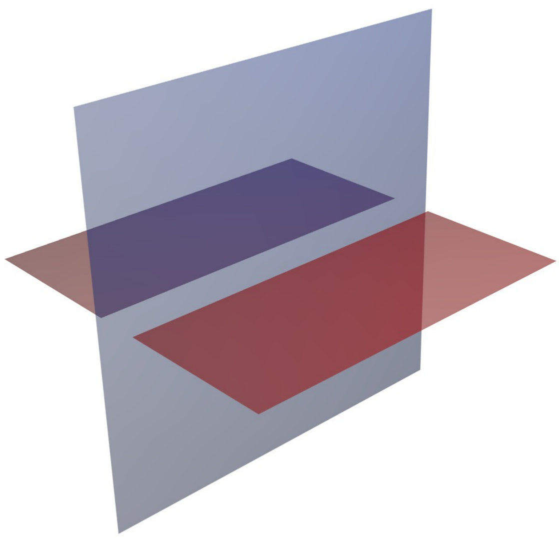
4.1.2.2. If a knot has n crossings, then there are 2^n possible ways to smooth all the crossings, since there are two choices for each crossing.

4.1.2.3. A *state* of a knot is one of the 2^n collections of disjoint circles that we obtain by smoothing its crossings.

4.1.3. A naïve approach to smoothing a double plane crossing

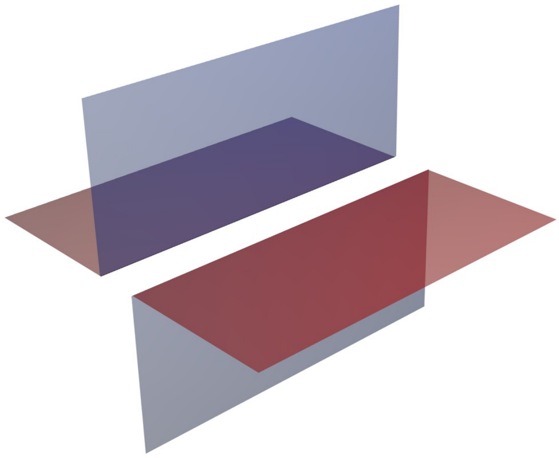
4.1.3.1. At first, it may seem that there is an obvious way to proceed for 2-knots without any triple points. There are two possible ways that two planes can cross each other, and they are depicted below.



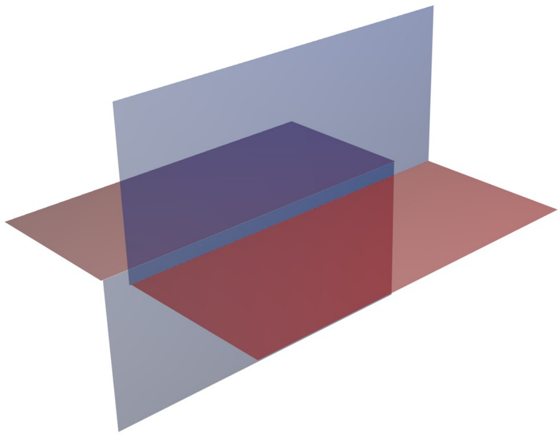


Given one of these two configurations it seems natural to smooth the crossing by doing one of the following.

Firstly, replace these three planes with two planes as follows.

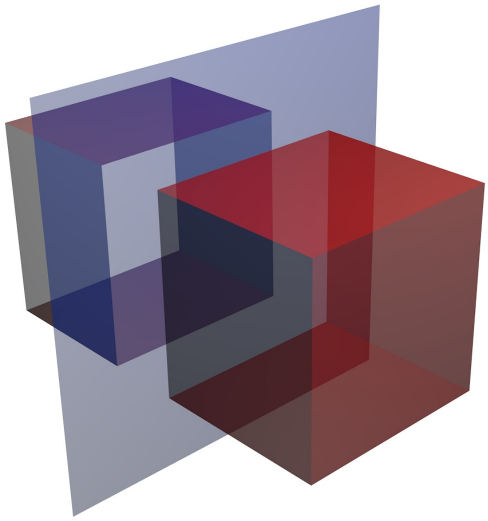


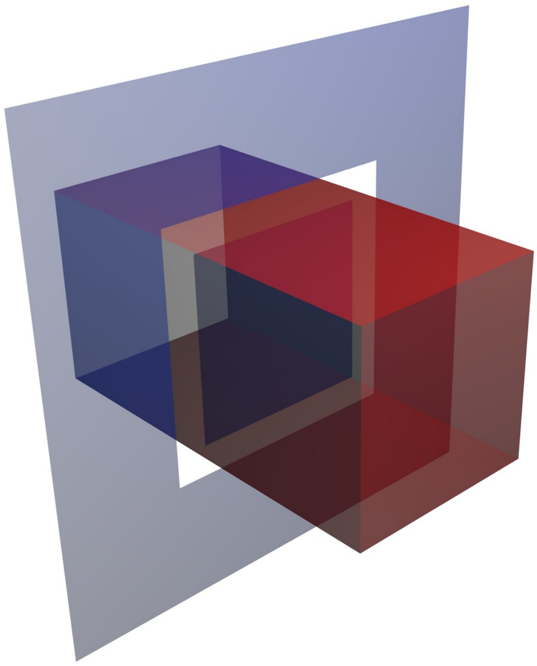
Alternatively, replace these three planes with two planes as follows.



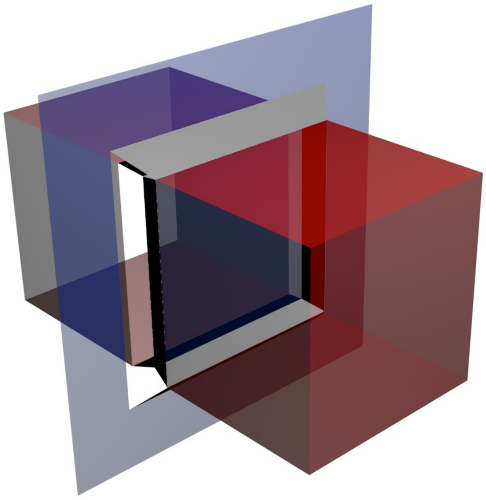
4.1.4. The naïve approach does not work

4.1.4.1. This is not a good approach. The problem comes from the fact that no matter how much we zoom in on a figure, we have to take into account that the planes are part of cubes. Therefore we have one of the two following configurations.





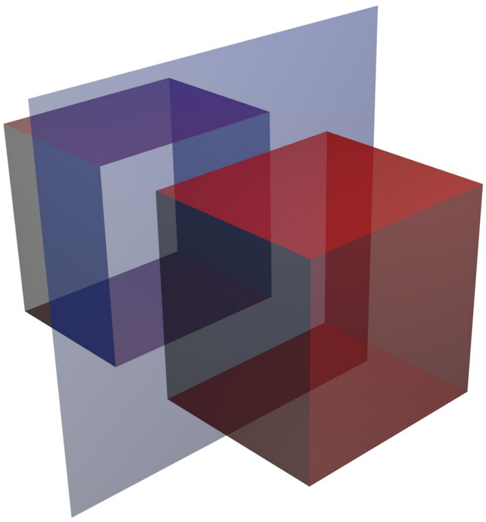
If we try to follow the naïve approach, we obtain configurations which are not valid in a cubical setting. For instance, the following one.

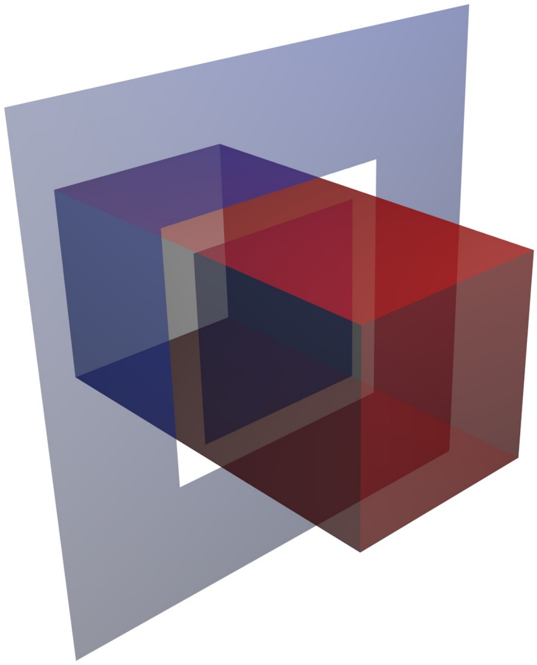


4.1.4.2. As discussed in the introduction, we take an alternative approach to smoothing a 2-knot.

4.1.5. Possible double crossing configurations

4.1.5.1. There are two possibilities depicted below.

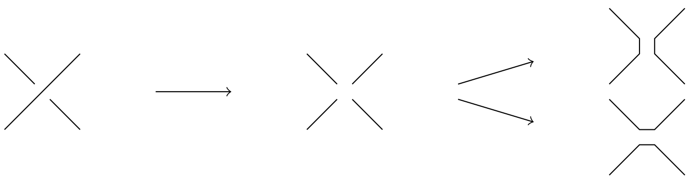




4.1.5.2. Crucially, we do not mean by these pictures that the 2-knot that we are working with looks exactly like this in this part of it. There may be other planes interacting with these, causing for example obstructions to applying a bubble move.

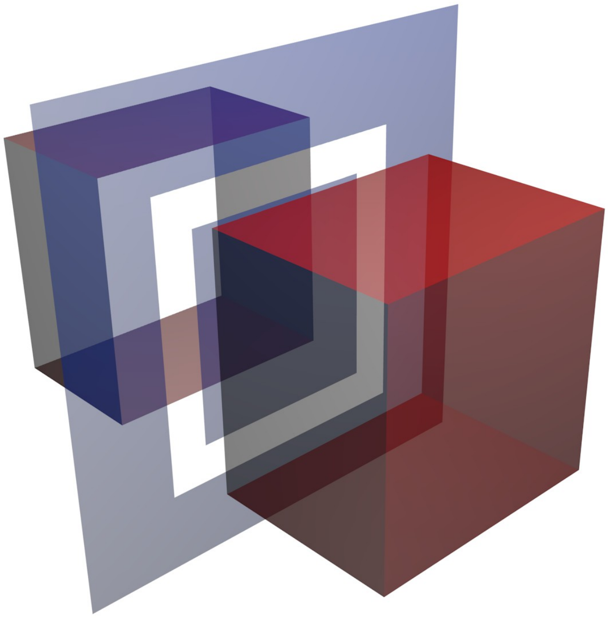
4.1.6. How to use surgery to smooth a double plane crossing of a 2-knot

4.1.6.1. Another way of looking at smoothing a crossing in ordinary knot theory, is that a type of surgery is used. First we cut the piece that is not broken, and then we glue the four pieces in one of two ways, as illustrated below.

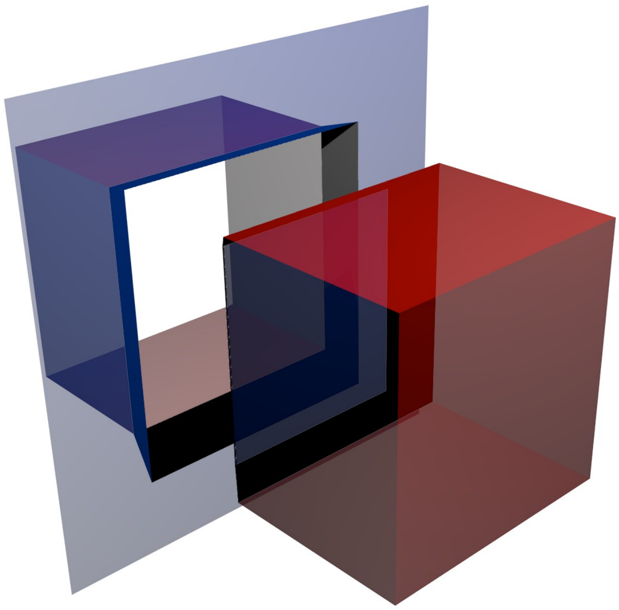


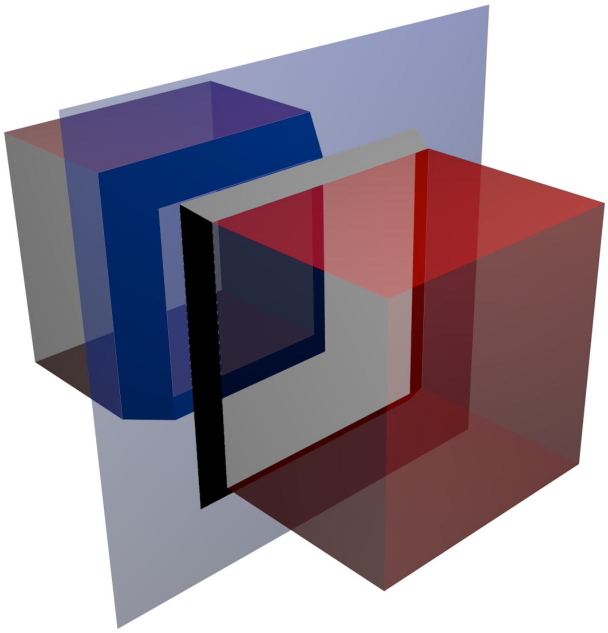
The same applies for the other type of crossing.

4.1.6.2. Our notion of smoothing of double plane crossings in 2-knot theory is inspired by this perspective. If a cube and a plane cross in one of the two ways described above, we smooth the crossing via a form of surgery as follows. First we cut out a small gap around where the pieces intersect. We obtain the following configuration.

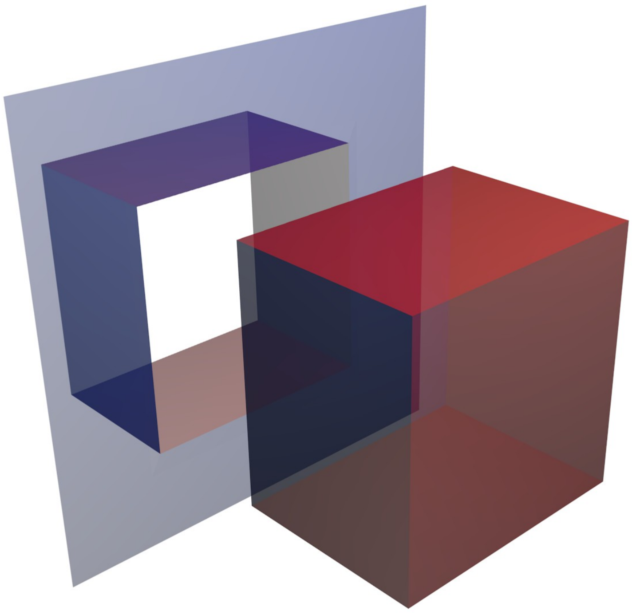


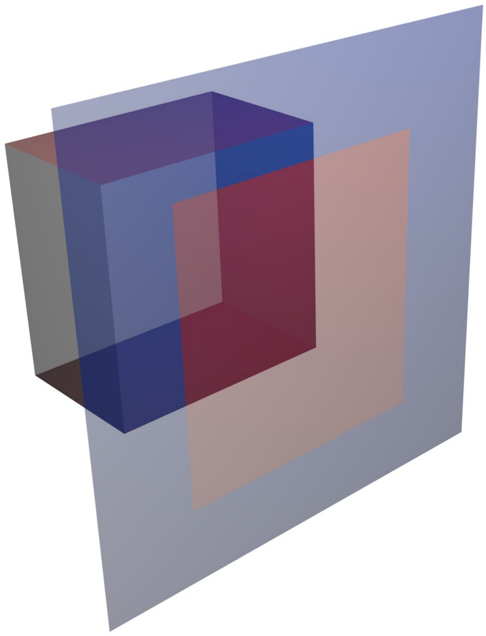
We then glue edges together in one of the two following ways.





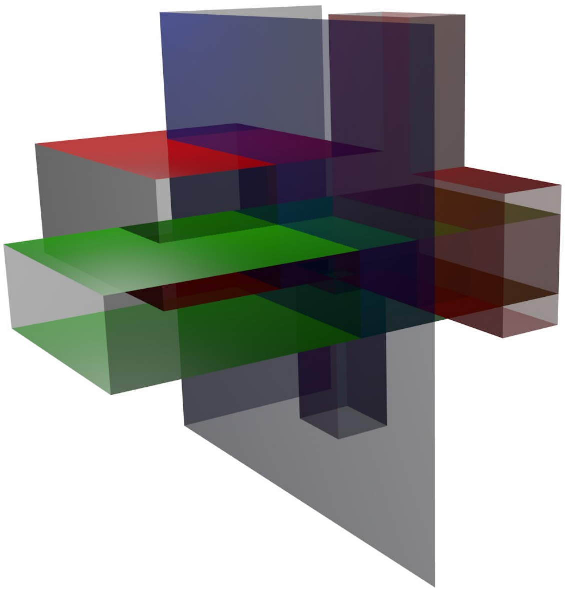
Finally, we tidy up the surgery to obtain a valid cubical configuration. We obtain the following two final configurations.





4.2. How to smooth a crossing with triple points?

When smoothing the crossings of the triple point configuration below, we first consider only two of the cubes, here for example the red and the green. Split them up and glue them in the aforementioned way, but respect where it intersects the remaining blue plane.



After we have done this we have to cut and glue in the last part, namely the blue plane in this picture. It is crucial to not only consider the new configurations we obtain from the first round of surgery: the last surgery has to respect the other double crossings as well. This ensures that no matter which double crossing we start with, namely if we start with the red and green, the red and blue, or the green and blue, the end result will be the same.

After a surgery on a configuration with triple points, there will be *holes* where the triple points used to be. We simply fill in these holes after having carried out the above procedure.

CHAPTER 5

ORIENTABILITY AND A NOTION OF WRITHE FOR 2-KNOTS

We first describe how to fix an orientation for 2-knots and explain how the approach used for ordinary knots will run into problems. Then we describe how to find the sign of a crossing and the notion of writhe for 2-knots.

5.1. Orientability of a 2-knot

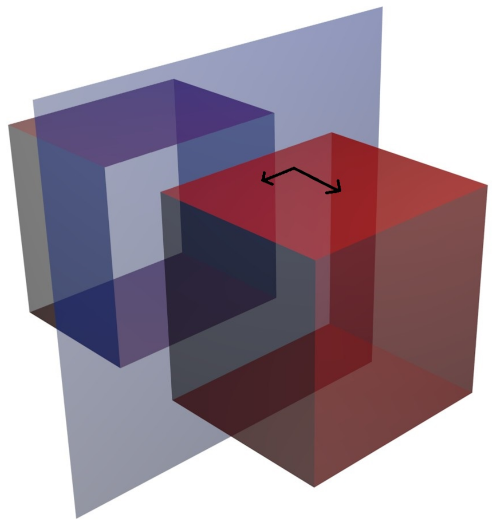
5.1.1. How to orient a 2-knot?

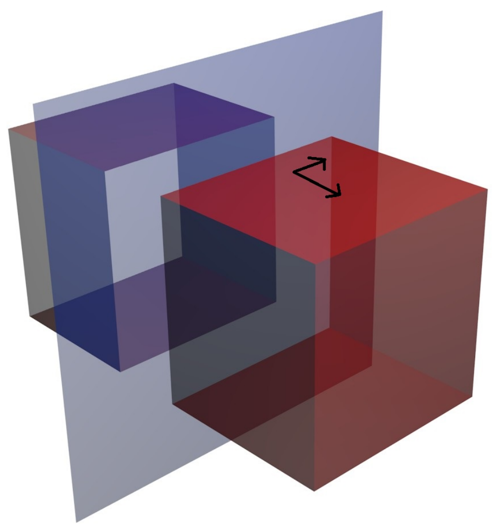
5.1.1.1. Orienting a knot in ordinary knot theory is easily done by choosing a direction and following it around the knot.

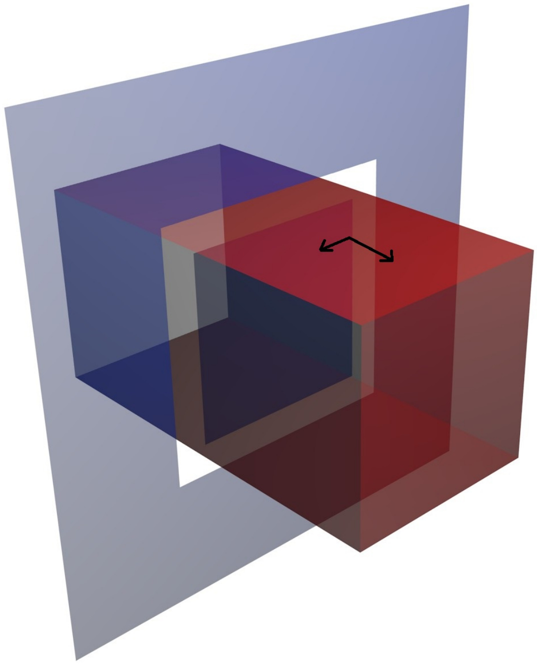
5.1.1.2. To naïvely fix an orientation by starting with an orientation on one face and following it arbitrarily around the cube, does not work for 2-knots. A face will not then have a unique orientation.

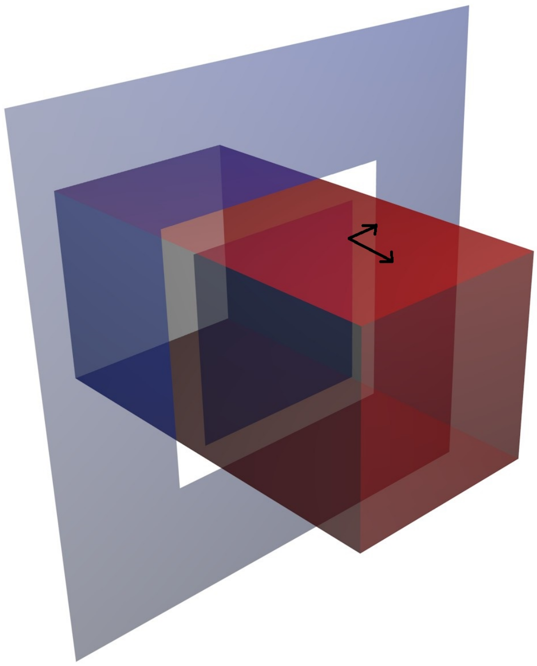
5.1.1.3. We therefore choose a path around the cube, passing once through each face, in addition to choosing a start orientation. Following our choice of orientation along this path equips each face with an orientation.

5.1.1.4. Let us consider the possible ways in which a cube and a plane which cross one another can be oriented. The orientation of the plane is of no consequence, since we can rotate the figure to obtain an arbitrary orientation of the plane without changing the orientation of the cube. The cube has four possible orientations when it is broken, and four when it is not. However, the possibilities are pairwise equal after rotation and we are therefore left with the following four possible orientations.



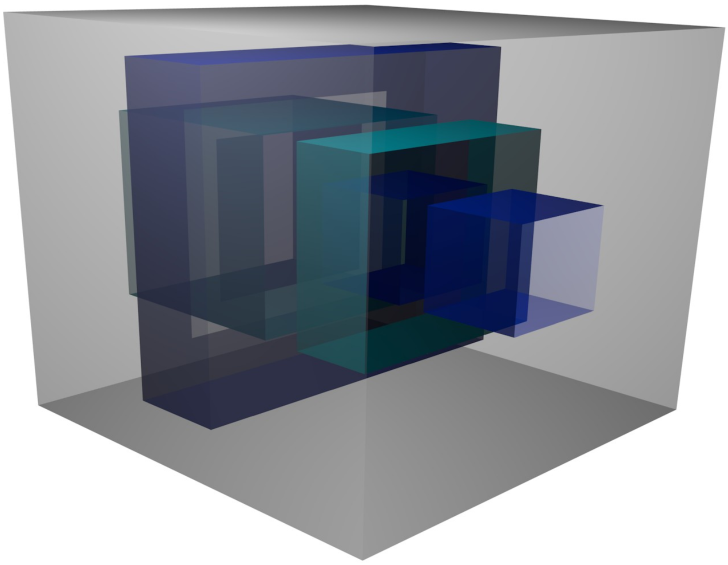




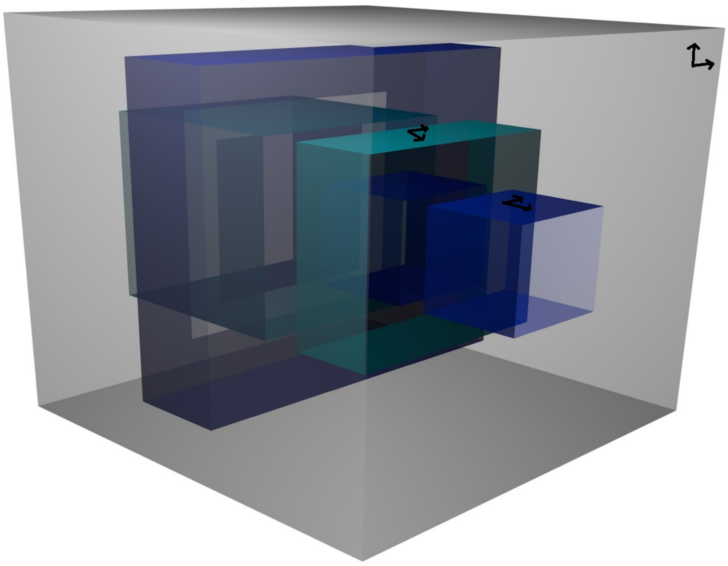


5.1.2. Orienting a 2-knot example

5.1.2.1. Let us consider how to equip the following 2-knot with an orientation.



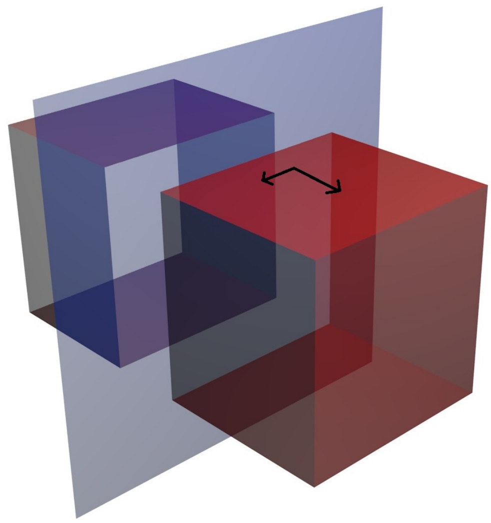
Let us call this example the *convoluted hoover*. We have to make a choice of orientation for the big cube and then choose to follow a path. Here we choose to travel along the top half of the cube by going over the top and into the hole on the back side and down and into the hole on the front side. Then one possible orientation we can obtain is the following.

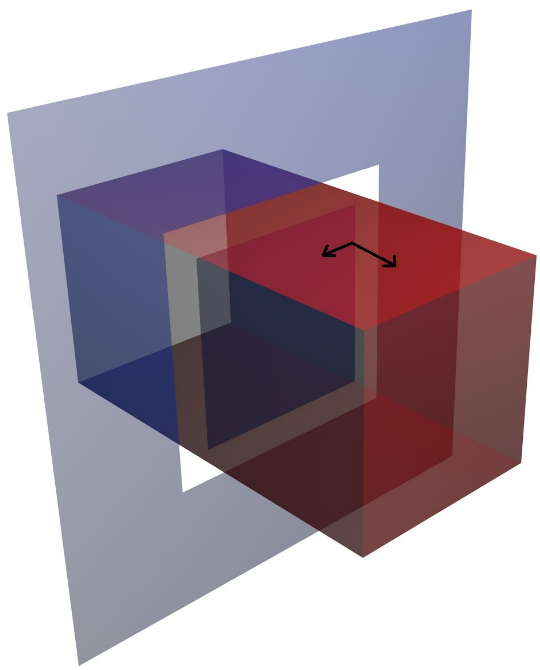


5.2. The writhe of a 2-knot

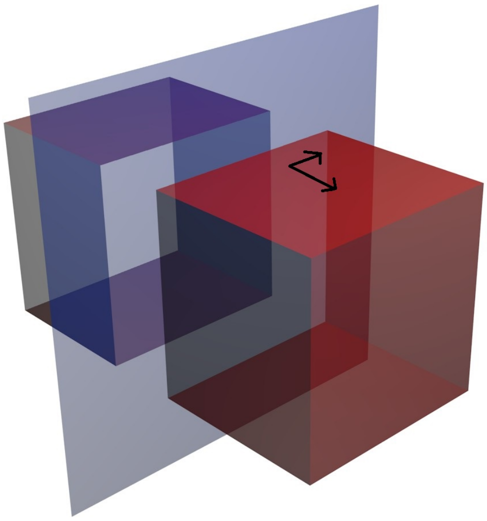
5.2.1. The sign of a crossing

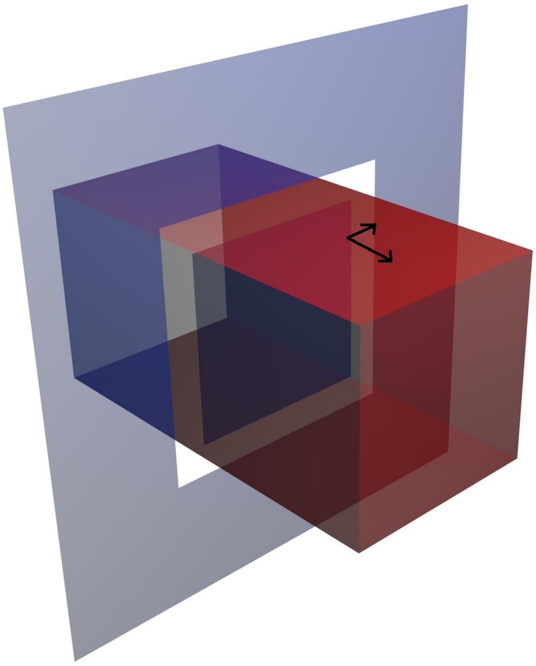
5.2.1.1. The *sign* of a crossing is $+1$ if we have one of the two following crossings.





5.2.1.2. The sign of a crossing is -1 if we have one of the two following crossings.





5.2.1.3. To find the sign of a triple point crossing, we ignore the triple point and find the sign for each pair of double crossing. In other words, fix an orientation, then look at all the combinations of two cubes that intersect independently.

5.2.2. Writhe of a 2-knot

5.2.2.1. We define the writhe of a 2-knot, which we denote by $\omega(K)$, as follows.

$$\omega(K) = \sum_{\text{crossings } C} sign(C)$$

5.2.2.2. The writhe of the unknot is 0, since it has no crossings.

5.2.2.3. When using the orientation from 5.1.2.1, the writhe of the convoluted hoover is -3: each crossing has sign -1.

CHAPTER 6

2-BRACKET POLYNOMIAL OF 2-KNOTS

We first define a gadget which we refer to as a 2-bracket polynomial of a 2-knot. We then show that this 2-bracket polynomial is invariant under the saddle move.

6.1. How to arrive at 2-bracket polynomial of 2-knots?

6.1.1. The 2-bracket polynomial

6.1.1.1. We shall denote the 2-*bracket polynomial* of a 2-knot K by $\langle K \rangle$.

6.1.1.2. When performing a surgery, we may obtain a cylindrical configuration which cannot be part of an orientable surface (without boundary). We allow ourselves to replace this cylinder with its two missing faces.

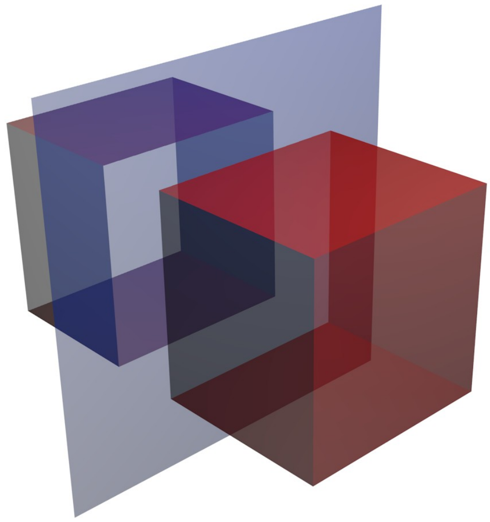
To keep track of these manipulations, we introduce a new variable γ. We then define the 2-bracket polynomial of the 2-knot after we have replaced the cylinder with its two missing faces to be the 2-bracket polynomial before we made the replacement multiplied by γ.

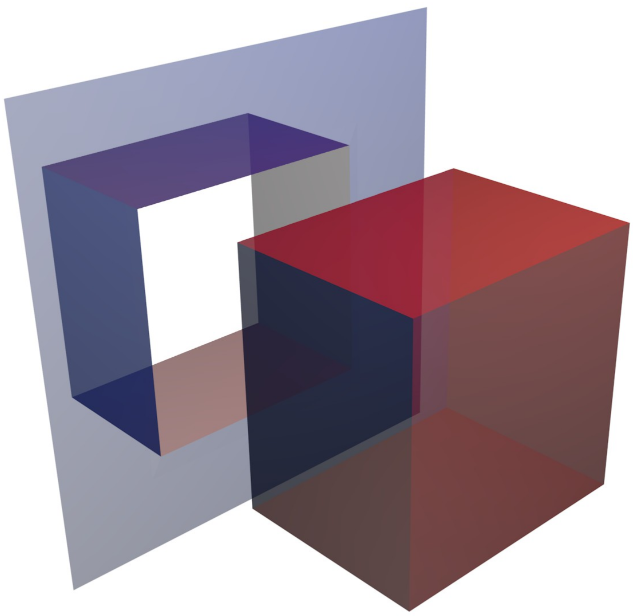
The other configuration that may occur consists of two parallel planes which are each connected in the same way to four planes which have a gap between two opposite corners. Then we can replace this configuration with a disconnected cylinder that corresponds to the four planes. This configuration occurs after a surgery on the triple point and this manipulation has been applied to simplify the configuration σ_{AB} which occurs in our discussion of a triple point later.

To keep track of this kind of manipulation, we introduce a new variable λ. Each time we perform it, we multiply the 2-bracket polynomial by λ.

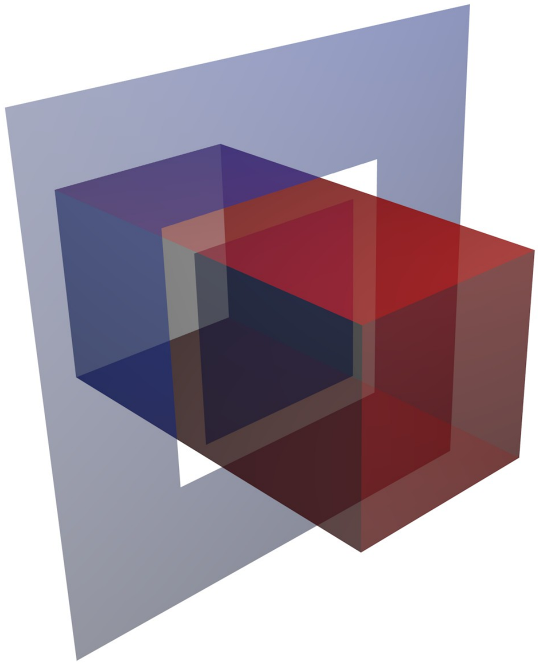
6.1.1.3. We define $\langle K \rangle$ inductively to be an expression in two variables A and B, beginning with defining $\langle K \rangle = 1$, and then proceeding as follows.

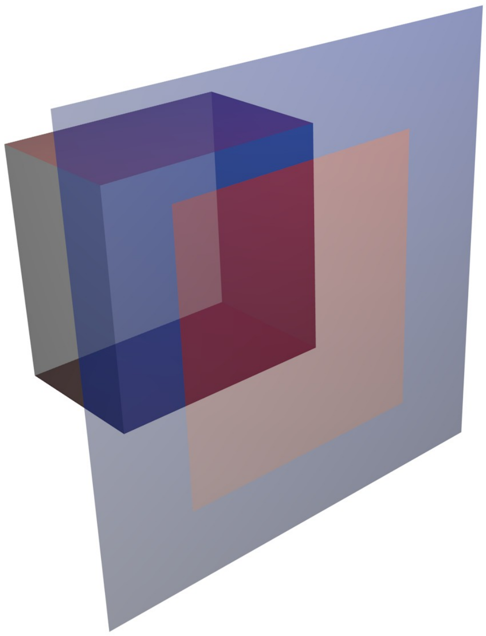
(1) If we replace a crossing as depicted below by the second configuration depicted below we multiply by A.



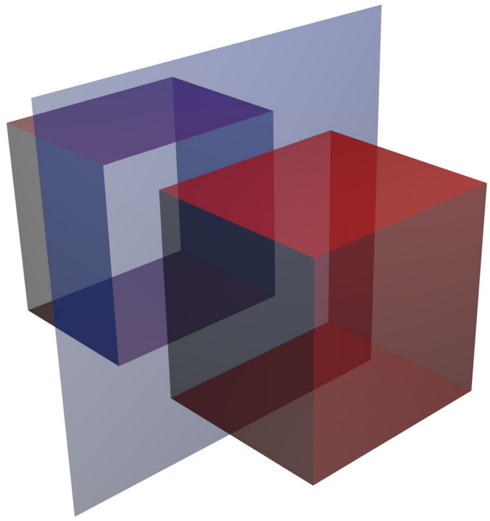


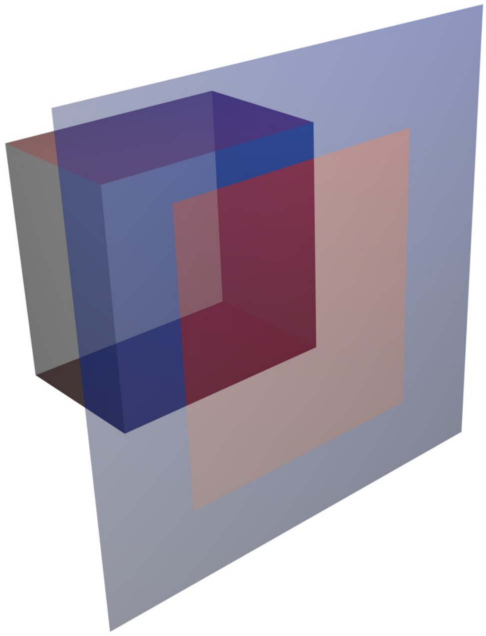
We also multiply by A if we replace a crossing as depicted below by the second configuration depicted below.



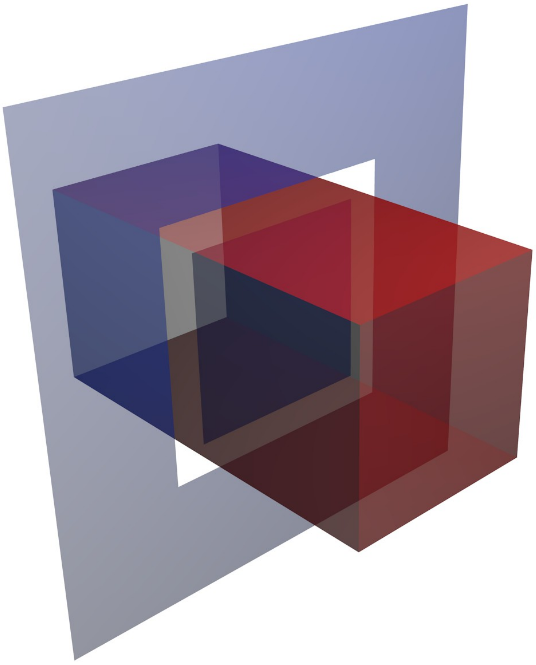


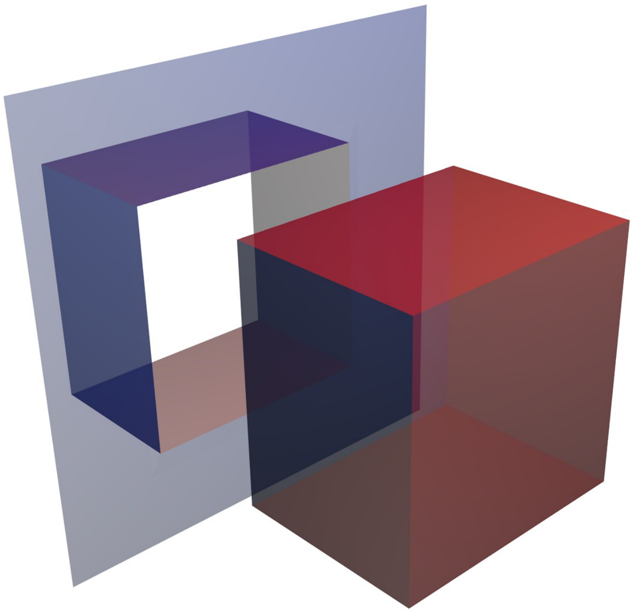
(2) If we replace a crossing as depicted below by the second configuration depicted below we multiply by B.





We also multiply by B if we replace a crossing as depicted below by the second configuration depicted below.





6.1.1.4. We impose the following relations.

(1) $A^2\gamma = A\gamma$

(2) $AB\langle K\rangle = -BA\langle K'\rangle$ where K and K' are symmetric configurations

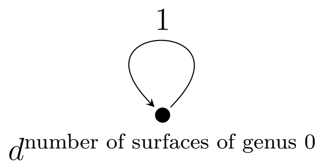
(3) $B^2 = B$

The necessity of these relations will become apparent when we discuss the invariance of the 2-bracket polynomial under the saddle move.

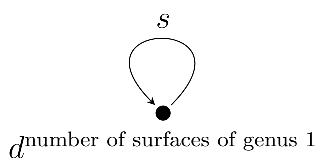
6.1.2. How to denote the surfaces obtained when evaluating a 2-bracket

6.1.2.1. A 2-bracket of complete smoothing of a 2-knot consists of a disjoint collection of orientable surfaces of arbitrary genus, not only spheres. To take this into account, we evaluate this 2-bracket to be the formal sum of the following.

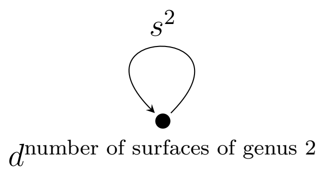
(1) For the surfaces of genus 0, the following loop.



(2) For the surfaces of genus 1, the following loop.



(3) For the surfaces of genus 2, the following loop.

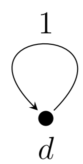


(4) And so on.

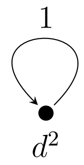
We allow ourselves only sum two loops if their dots have the same label.

6.1.2.2. To simplify, we adopt the following notation (we will not come across surfaces with genus more than 26).

(1) We assign a instead of the following loop,

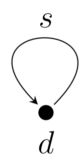


a_2 instead of the following loop,

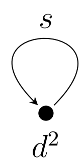


and so on.

(2) We assign b instead of the following loop,

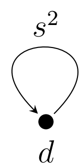


b_2 instead of the following loop,

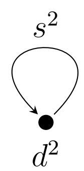


and so on.

(3) We assign c instead of the following loop,



c_2 instead of the following loop,



and so on.

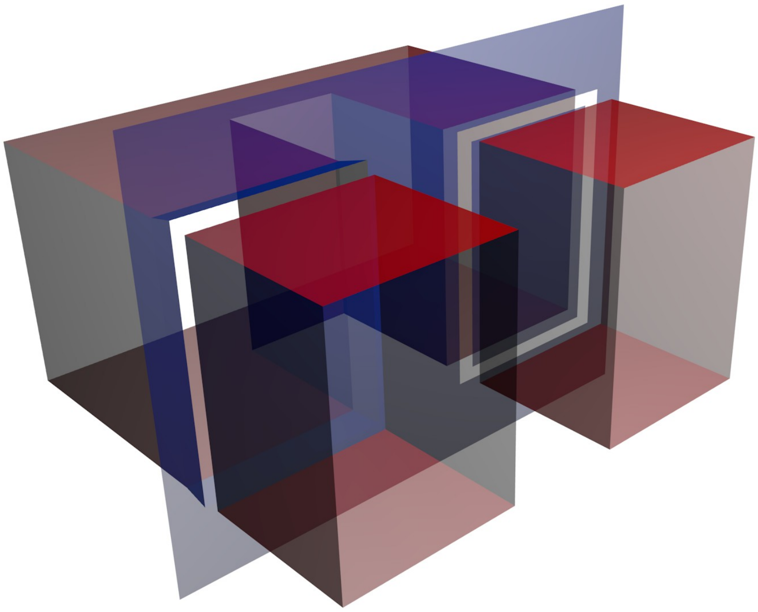
(4) Et cetera.

6.1.3. Invariance under the saddle move

6.1.3.1. We now prove that the 2-bracket polynomial $\langle K \rangle$ is unchanged under the cubical saddle move.

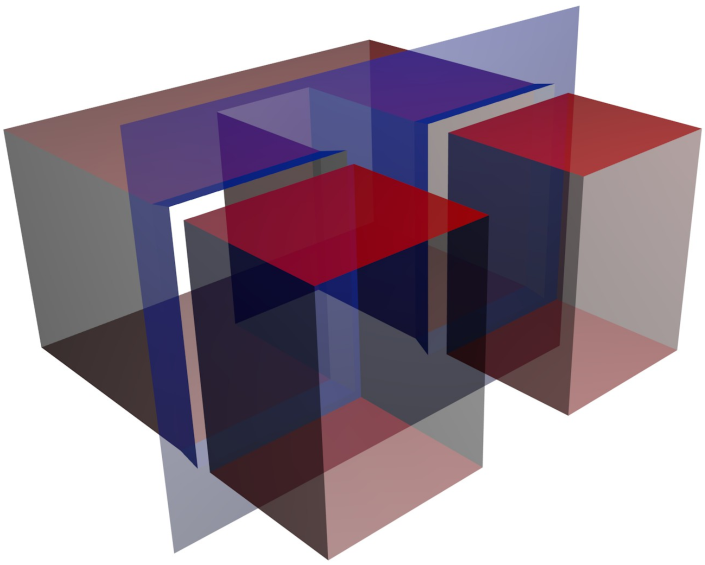
6.1.3.2. We first make the calculations for the first part of the saddle move. We concentrate on one crossing at a time and get the following configurations.

(1) The following configuration we denote by α_A and multiply by A,

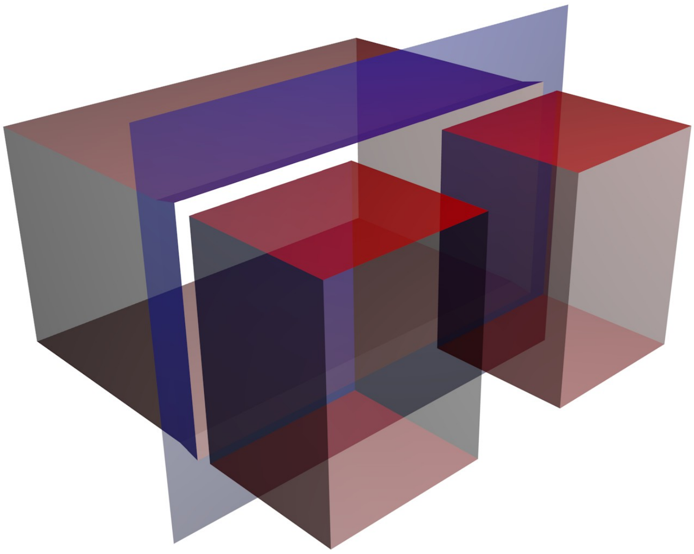


From this configuration, when we do the surgery on the second crossing, we get one of the two following cases.

(1.1) First we multiply the following configuration by A.

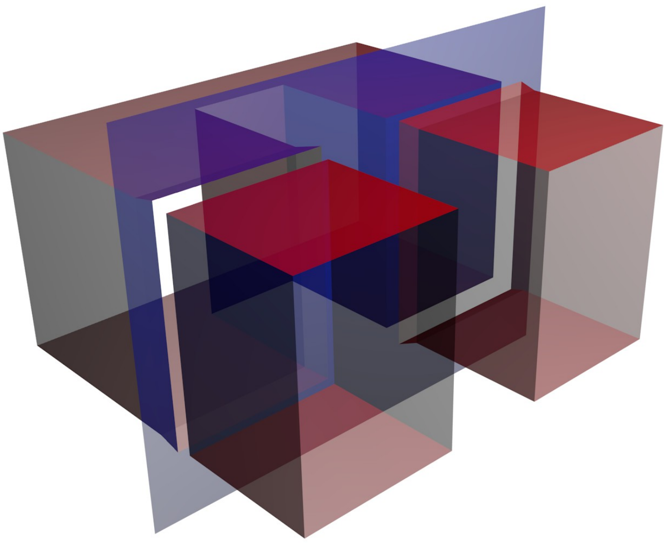


Notice that we have a cylinder which we can replace with the two missing faces as follows.

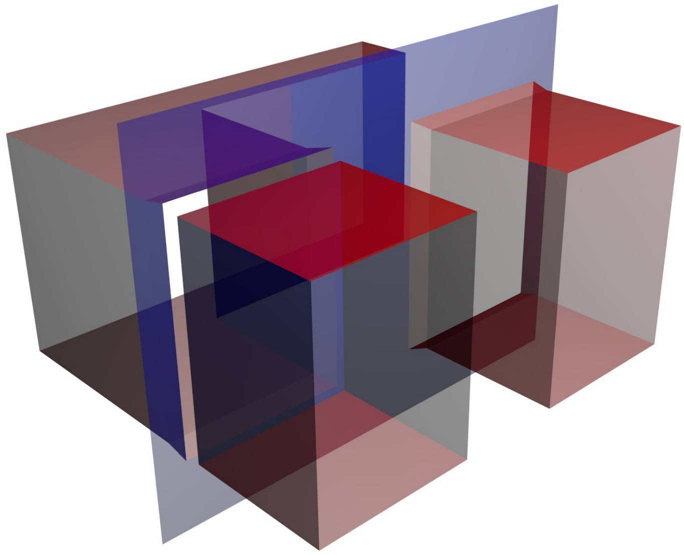


We call this configuration α_{AA}, but we also have to multiply by the unknown γ to account for the cylinder replacement.

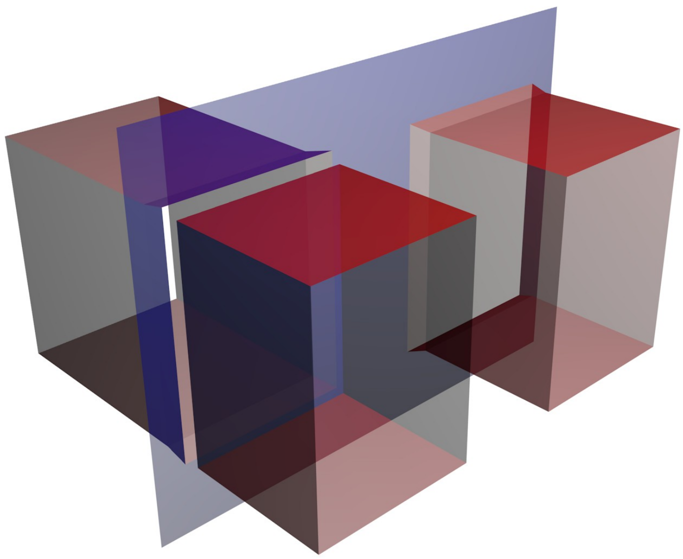
(1.2) Then we multiply the following configuration by B.



Notice that we can simplify it by doing a cube move.

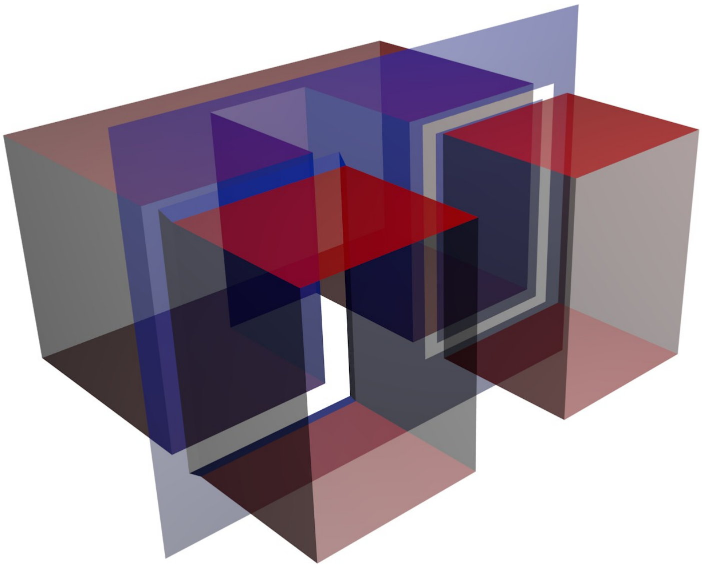


After this we can reduce the piece we got from the last manipulation as follows.



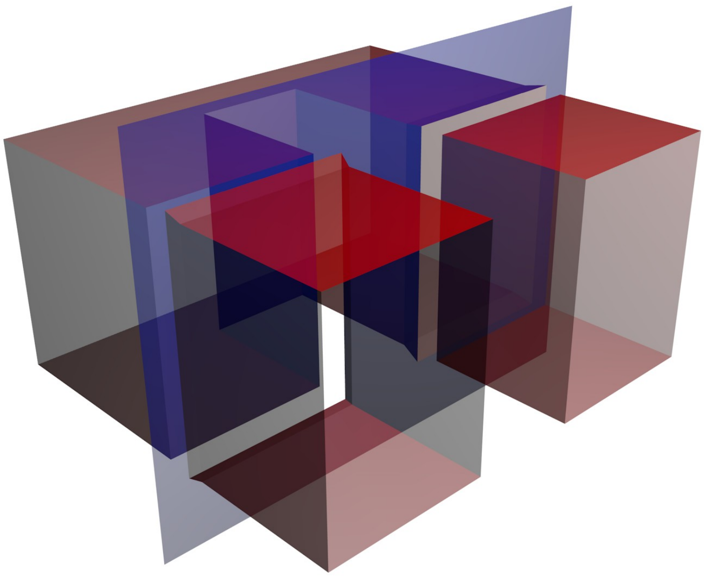
We call this configuration α_{AB}.

(2) We denote the following configuration by α_B and multiply it by B.

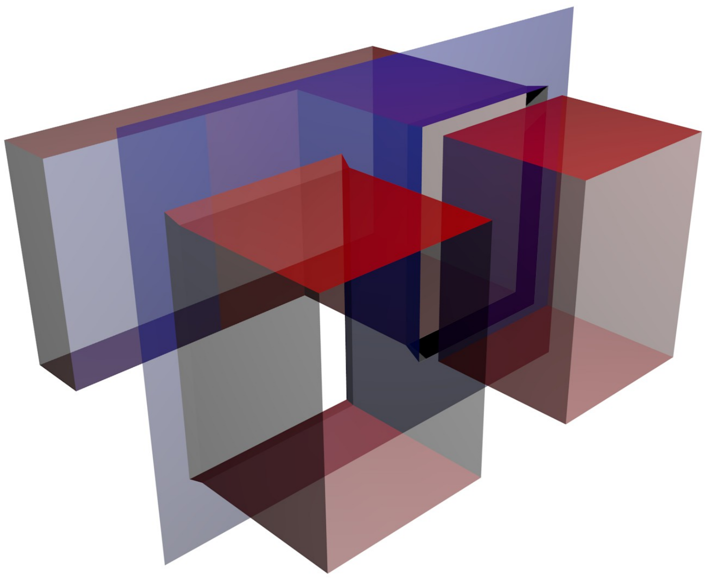


From this configuration, when we do the surgery on the second crossing, we get one of the two following cases.

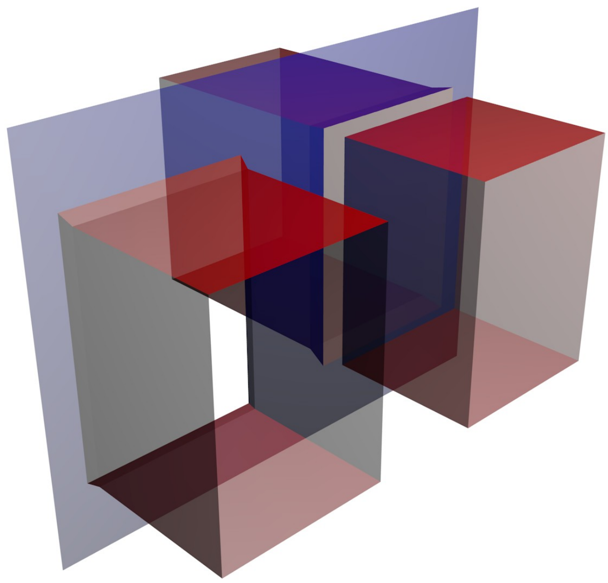
(2.1) First we multiply the following configuration by A.



Notice that we can simplify it by doing a cube move.

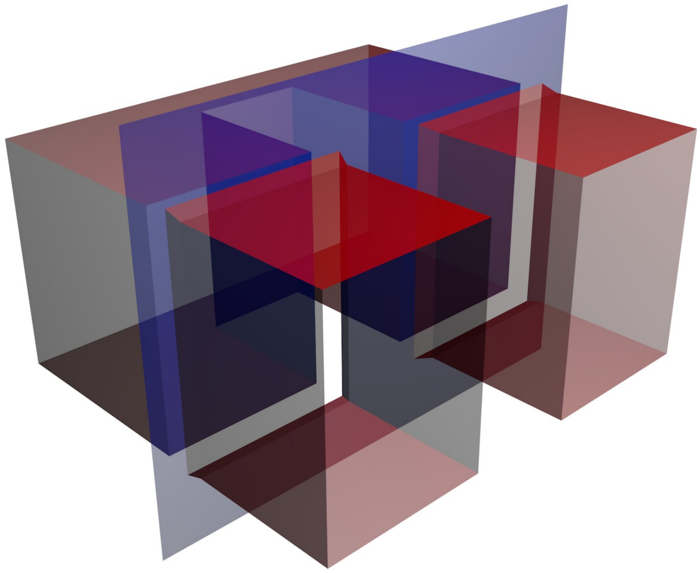


After this we reduce the piece we got from the last manipulation as follows.

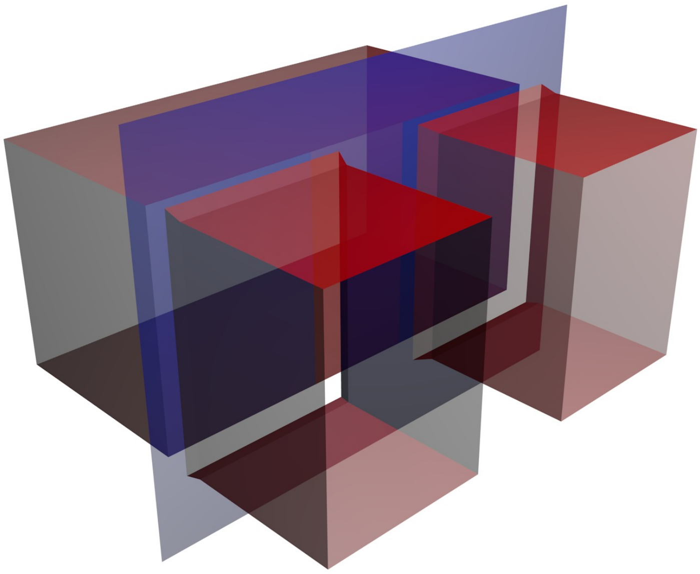


We call this configuration α_{BA}.

(2.2) Then we multiply the following configuration by B.



Notice that we can do the cube move which replaces three faces where two are an opposite pair, with the other three missing faces as follows.



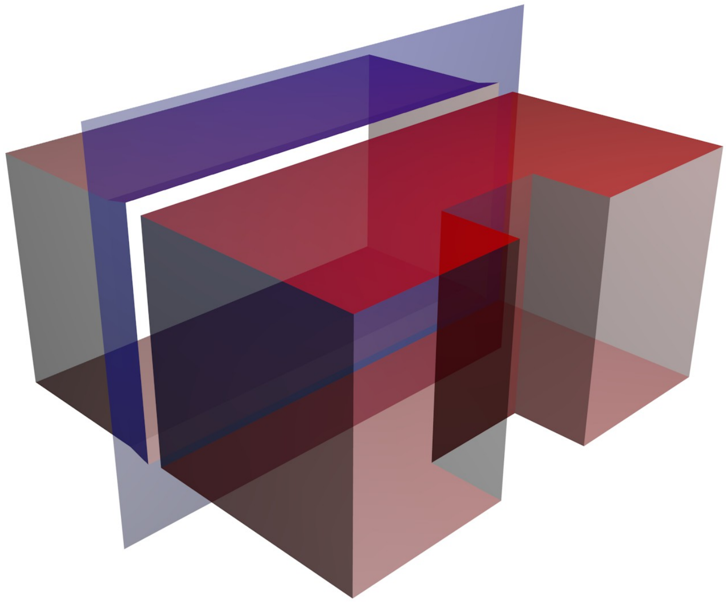
We call it α_{BB}.

We are thus left with the 2-bracket polynomial below.

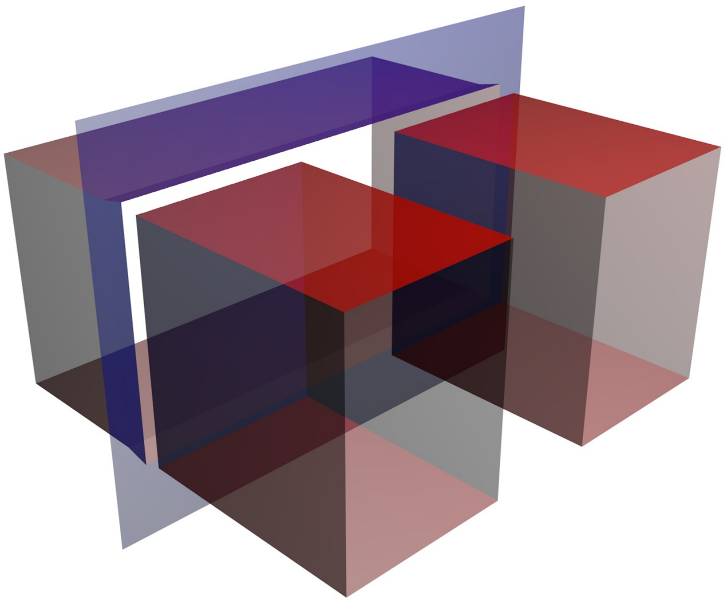
$$\begin{aligned}
\langle saddle_1 \rangle &= A\alpha_A + B\alpha_B \\
&= A(A\gamma\langle \alpha_{AA} \rangle + B\langle \alpha_{AB} \rangle) + B(A\langle \alpha_{BA} \rangle + B\langle \alpha_{BB} \rangle) \\
&= A^2\gamma\langle \alpha_{AA} \rangle + AB\langle \alpha_{AB} \rangle + BA\langle \alpha_{BA} \rangle + B^2\langle \alpha_{BB} \rangle
\end{aligned}$$

6.1.3.3. We then make the calculations for the second part of the saddle move. We only have one crossing which results in the following calculations.

(1) We multiply the following configuration by A.

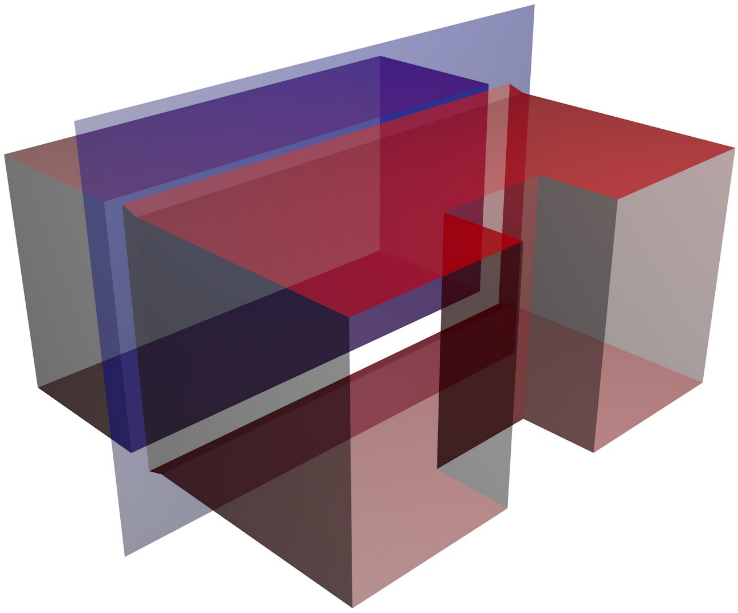


Notice that we can simplify by replacing the cylinder with its two missing faces, as follows.

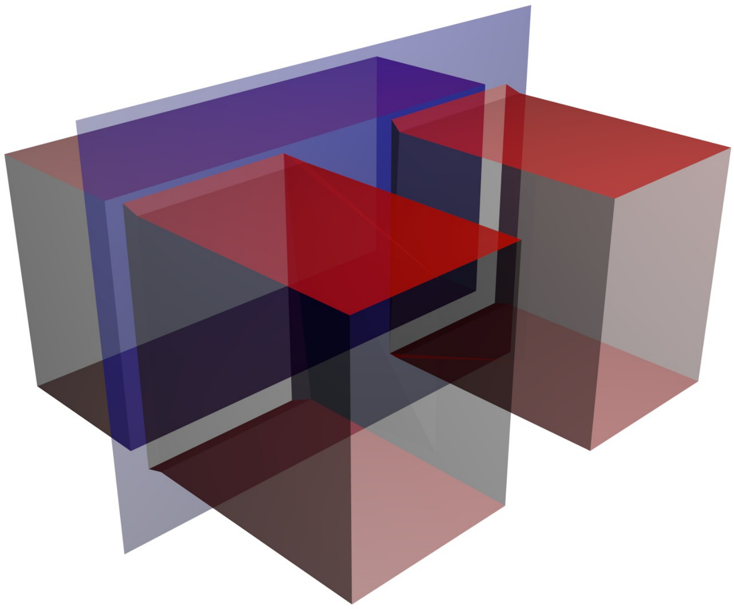


We have to multiply this with γ and we call the configuration β_A.

(2) We multiply the following configuration by B.



Notice that we can simplify by replacing the three faces, where two are an opposite pair, with the other three faces as follows.



We call this configuration β_B.

Then we are left with the 2-bracket polynomial as follows.

$$\langle saddle_2 \rangle = A\gamma\langle \beta_A \rangle + B\langle \beta_B \rangle$$

6.1.3.4. Observe that $\alpha_{AA} = \beta_A$ and $\alpha_{BB} = \beta_B$. Also notice that α_{AB} is symmetric to α_{BA}. This gives us the following three relations.

(1) $A^2\gamma = A\gamma$

(2) $AB = -BA$ if we have symmetric configurations

(3) $B^2 = B$

We know these hold from 6.1.1.4.

6.1.4. The 2-bracket polynomial of some 2-knots

6.1.4.1. The 2-bracket polynomial $\langle K \rangle$ of the unknot is 1.

6.1.4.2. The 2-bracket polynomial of the convoluted hoover from 5.1.2.1 is as follows.

$$\langle \text{Convoluted hoover} \rangle = A^3ab + B^3a + 2A^2Ba + A^2Bab_2 + 3AB^2ab$$

See Appendix A for the calculations and corresponding configurations.

CHAPTER 7

2-POLYNOMIAL INVARIANT OF 2-KNOTS

We modify the 2-bracket polynomial of a 2-knot to obtain a gadget which we refer to as the 2-polynomial of a 2-knot. We then prove that it is invariant under the four cubical Homma-Nagase-Roseman moves.

7.1. How to arrive at a 2-polynomial of 2-knots?

7.1.1. The 2-polynomial

7.1.1.1. We define the 2-*polynomial* of a 2-knot to be

$$J(K) = v^{-\omega(K)}\langle K\rangle,$$

where $v = Aa\gamma + B$ and the following relations hold.

(1) $v^{-2}A^2\gamma = v^{-1}A\gamma$

(2) $v^{-2}AB\langle K\rangle = -v^{-2}BA\langle K'\rangle$ where K and K' are symmetric configurations

(3) $v^{-2}B^2 = v^{-1}B$

(4) $2A^2\gamma = A^2$

(5) $AB\gamma\lambda = AB$

The reason for these relations will become apparent when we discuss the invariance of the 2-polynomial of a 2-knot under the Homma-Nagase-Roseman moves.

7.2. Invariance under the bubble move

7.2.1. Writhe in the setting of the bubble move

7.2.1.1. The writhe of the first part of the bubble move is as follows.

$$\omega(bubble_1) = 1$$

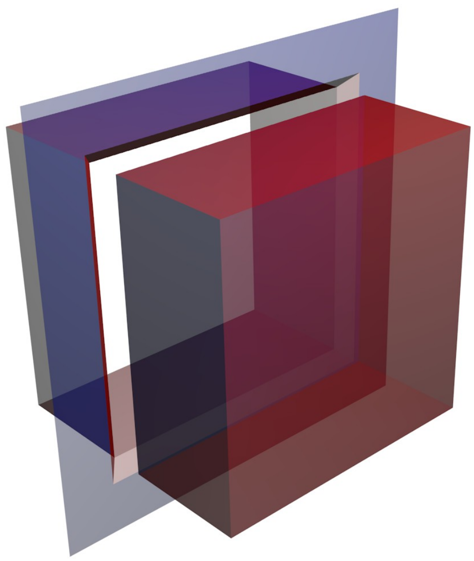
7.2.1.2. The writhe of the second part of the bubble move is as follows.

$$\omega(bubble_2) = 0$$

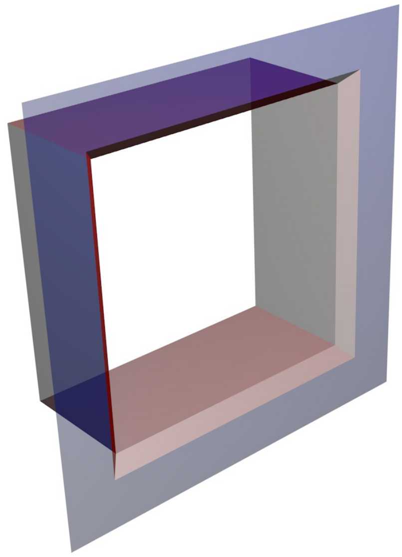
7.2.2. The 2-bracket polynomial of the bubble move

7.2.2.1. Since we only have one crossing in the first part of the bubble move, we have two states, which are as follows.

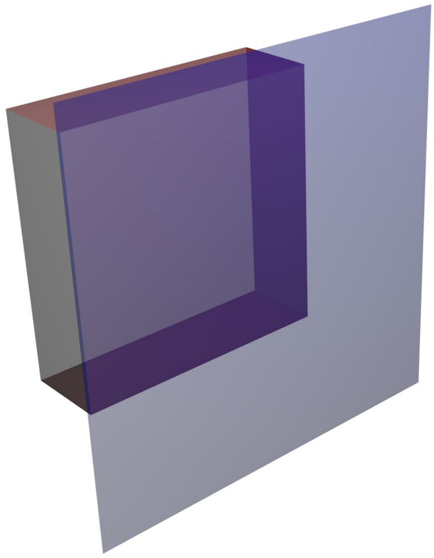
(1) The first state we multiply by A and is depicted below.



Notice that this configuration contains a closed cube which we can remove while multiplying the polynomial by a. Then we are left with the following configuration.

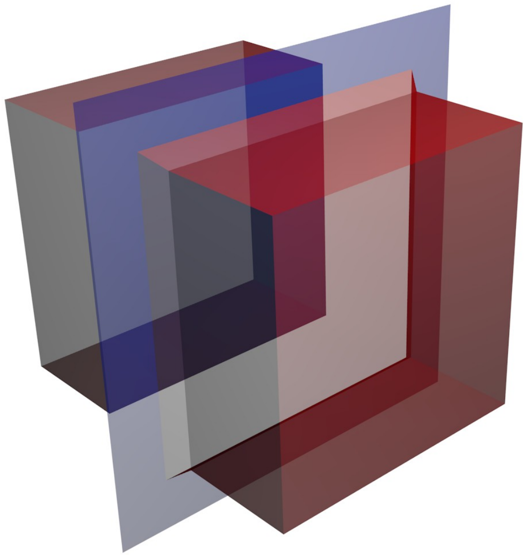


Also notice that we can replace a cylinder with its two missing faces and multiply by γ. Then we obtain the following configuration.

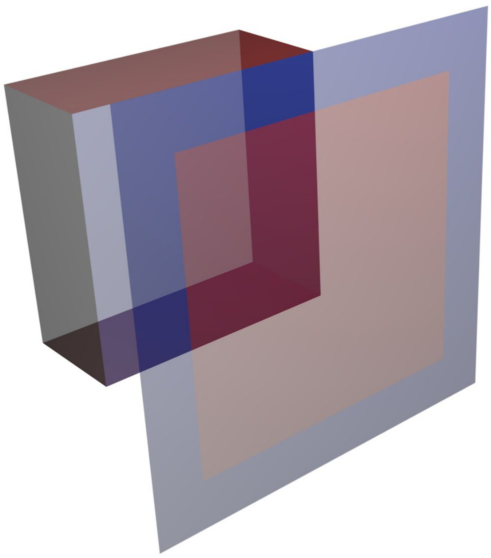


Let us call this configuration δ_1.

(2) The second state we multiply by B and is depicted below.



Notice that this configuration can be simplified by a cube move as follows.

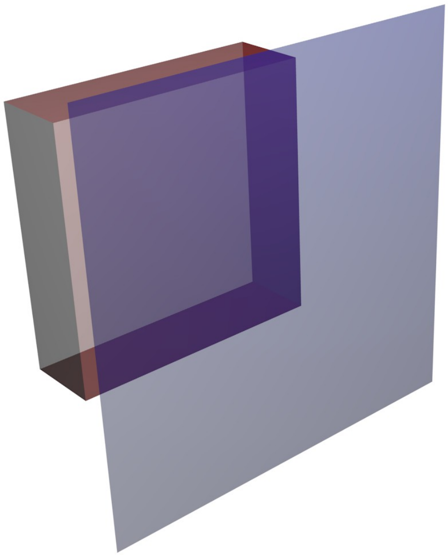


We call this configuration δ_2.

This leaves us with the calculation as below.

$$\langle bubble_1 \rangle = Aa\gamma\langle \delta_1 \rangle + B\langle \delta_2 \rangle$$

7.2.2.2. Since the second part of the bubble move has no crossings and we only have the following configuration.



We call this ϵ.

7.2.2.3. Observe that $\delta_1 = \delta_2 = \epsilon$, so let us call them collectively for δ. As the second part of the bubble move has no crossings, we have that $\langle bubble_2 \rangle = \langle \delta \rangle = 1$.

7.2.3. The 2-polynomial of the bubble move

7.2.3.1. We deduce that we have the following 2-polynomial for the first part of the bubble move.

$$\begin{aligned} J(bubble_1) &= v^{-\omega(bubble_1)} \langle bubble_1 \rangle \\ &= v^{-\omega(bubble_1)} (Aa\gamma\langle \delta_1 \rangle + B\langle \delta_2 \rangle) \\ &= v^{-1}(Aa\gamma\langle \delta \rangle + B\langle \delta \rangle) \\ &= v^{-1}(Aa\gamma + B)\langle \delta \rangle \\ &= (Aa\gamma + B)^{-1}(Aa\gamma + B)\langle \delta \rangle \\ &= \langle \delta \rangle \end{aligned}$$

7.2.3.2. We deduce that we have the following 2-polynomial for the second part of the bubble move.

$$\begin{aligned} J(bubble_2) &= v^{-\omega(bubble_2)} \langle bubble_2 \rangle \\ &= v^{-\omega(bubble_2)} \langle \delta \rangle \\ &= v^0 \langle \delta \rangle \\ &= \langle \delta \rangle \end{aligned}$$

7.2.3.3. Thus $J(bubble_1) = J(bubble_2)$ and the 2-polynomial is invariant under the bubble move.

7.3. Invariance under the saddle move

7.3.1. Writhe in the setting of the saddle move

7.3.1.1. The writhe of the first part of the saddle move is as follows.

$$\omega(saddle_1) = 2$$

7.3.1.2. The writhe of the second part of the saddle move is as follows.

$$\omega(saddle_2) = 1$$

7.3.2. The 2-bracket polynomial of the saddle move

7.3.2.1. We have already shown that $\langle K \rangle$ is not changed by the saddle move in 6.1.3.4.

7.3.3. The 2-polynomial of the saddle move

7.3.3.1. We deduce that we have the following 2-polynomial for the first part of the saddle move.

$$\begin{aligned} J(saddle_1) &= v^{-\omega(saddle_1)}\langle saddle_1 \rangle \\ &= v^{-2}(A^2\gamma\langle \alpha_{AA} \rangle + AB\langle \alpha_{AB} \rangle + BA\langle \alpha_{BA} \rangle + B^2\langle \alpha_{BB} \rangle) \\ &= v^{-2}A^2\gamma\langle \alpha_{AA} \rangle + v^{-2}AB\langle \alpha_{AB} \rangle + v^{-2}BA\langle \alpha_{BA} \rangle + v^{-2}B^2\langle \alpha_{BB} \rangle \end{aligned}$$

7.3.3.2. We deduce that we have the following 2-polynomial for the second part of the saddle move.

$$\begin{aligned} J(saddle_2) &= v^{-\omega(saddle_2)}\langle saddle_2 \rangle \\ &= v^{-1}(A\gamma\langle \beta_A \rangle + B\langle \beta_B \rangle) \\ &= v^{-1}A\gamma\langle \beta_A \rangle + v^{-1}B\langle \beta_B \rangle \end{aligned}$$

7.3.3.3. Since we have that $\alpha_{AA} = \beta_A$, $\alpha_{BB} = \beta_B$, α_{AB} and α_{BA} are mirror images, and the relations from 7.1.1.1, we get the following.

$$\begin{aligned} J(saddle_1) &= v^{-2}A^2\gamma\langle \alpha_{AA} \rangle + v^{-2}AB\langle \alpha_{AB} \rangle + v^{-2}BA\langle \alpha_{BA} \rangle + v^{-2}B^2\langle \alpha_{BB} \rangle \\ &= v^{-1}A\gamma\langle \beta_A \rangle + v^{-2}AB\langle \alpha_{AB} \rangle - v^{-2}AB\langle \alpha_{AB} \rangle + v^{-1}B\langle \beta_B \rangle \\ &= v^{-1}A\gamma\langle \beta_A \rangle + v^{-1}B\langle \beta_B \rangle \\ &= v^{-\omega(saddle_2)}\langle saddle_2 \rangle \\ &= J(saddle_2) \end{aligned}$$

7.3.3.4. Thus $J(saddle_1) = J(saddle_2)$ and the 2-polynomial is invariant under the saddle move.

7.4. Invariance under the triple point move

7.4.1. Writhe in the setting the triple point move

7.4.1.1. To find the writhe of the first part of the triple point move, notice that the red and green cube only intersect once, and therefore the sign is either $+1$ or -1 depending on the orientation. Also notice that once the orientation is fixed, the green cube will intersect the blue plane in the same manner as it intersect the red cube. It will therefore have the same sign for these two crossings. This means that the writhe of the green cube crossings is either $+2$ or -2. The sign obtained from the red cube crossing the blue plane is either $+1$ or -1 depending on the orientation. Thus the writhe of this triple point is either $+3$, $+1$, -1 or -3.

7.4.1.2. To find the writhe of the second part of the triple point move, notice that it does not have any triple points, but that the green cube still crosses the red once and in the same manner as it crosses the blue plane. Thus the writhe is still either $+3$, $+1$, -1 or -3.

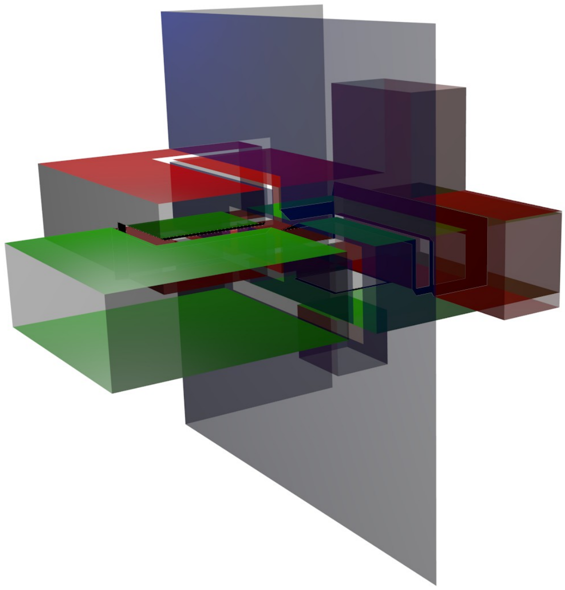
7.4.1.3. Thus we conclude that $\omega(triple_1) = \omega(triple_2)$.

7.4.2. The 2-bracket polynomial of the triple point move

7.4.2.1. We start with gluing the green and red cubes together, and then glue in the blue plane, being careful to respect the double plane crossings of the plane with each cube.

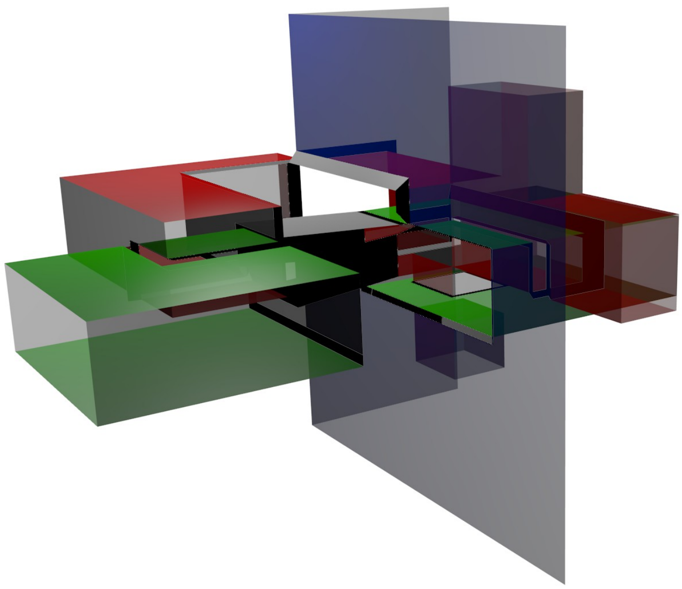
7.4.2.2. The first part of the triple point move has the following calculations.

(1) We denote the following configuration by σ_A and multiply it by A.

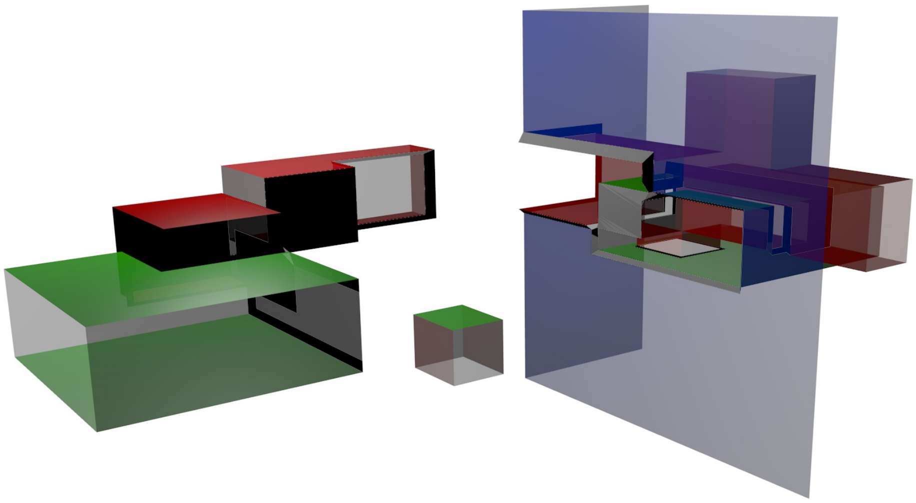


From this configuration, when we glue in the plane, we get one of the two following configuration.

(1.1) First we multiply the following configuration by A.

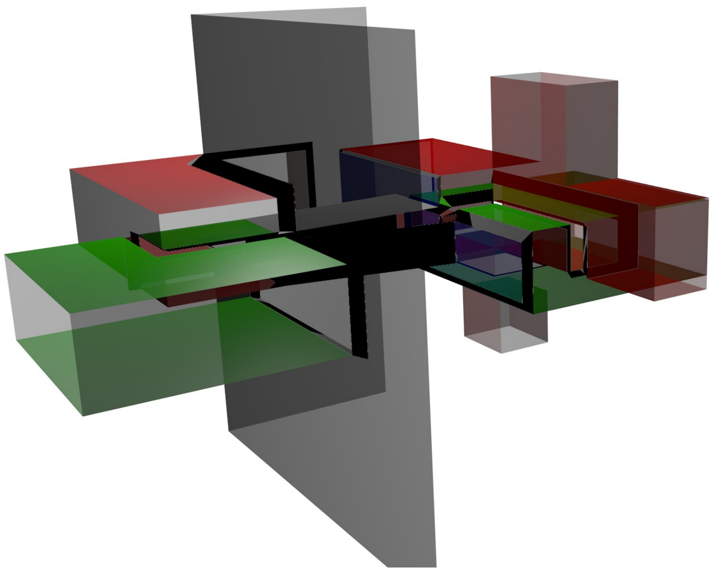


After doing some manipulations and replacing two cylinders with their corresponding two missing faces, we get the following.

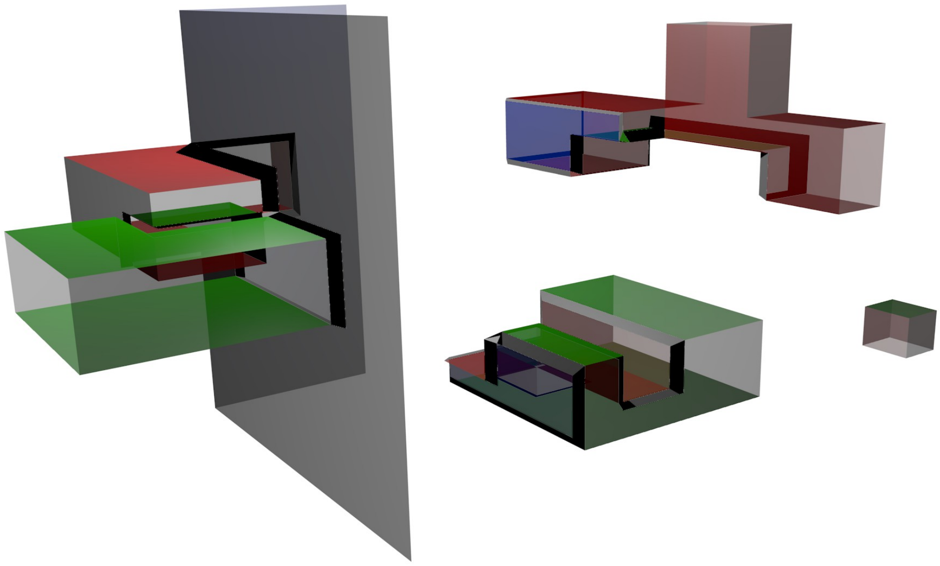


Let us call this configuration σ_{AA}. We also have to multiply by 2γ since we replaced two cylinders. Notice that σ_{AA} consists of three open cubes and a plane connected to a torus with one missing face and a cube with tunnels and two missing faces.

(1.2) Then we multiply the following configuration by B.

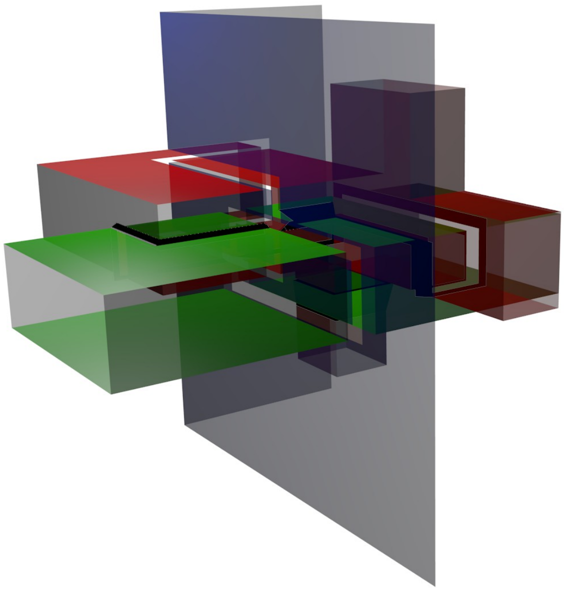


After replacing a cylinder with its two missing faces and also replacing two parallel faces which has two disconnected corners with the corresponding disconnected cylinder, we obtain the following configuration.



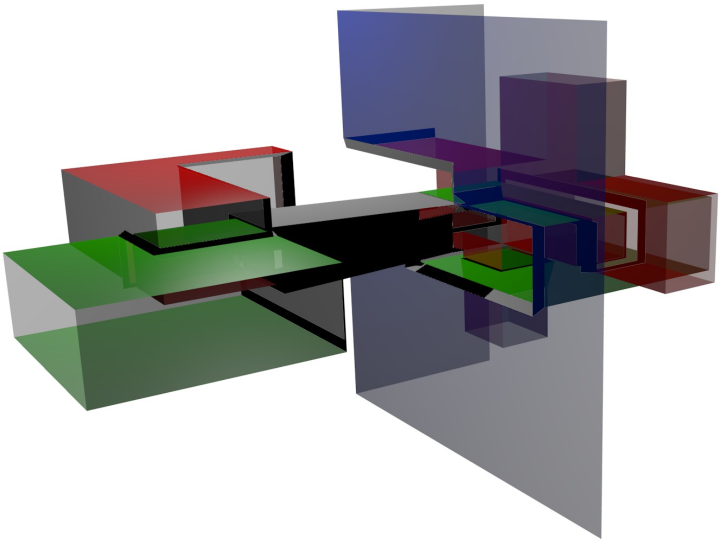
Let us call this σ_{AB}. We also have two multiply by γ and λ to account for the cylinder operations we performed. Notice that σ_{AB} consists of a open cube, a cube with two missing faces, a torus with a missing face and a plane connected to two open cubes.

(2) We denote the following configuration by σ_B and multiply it by B.

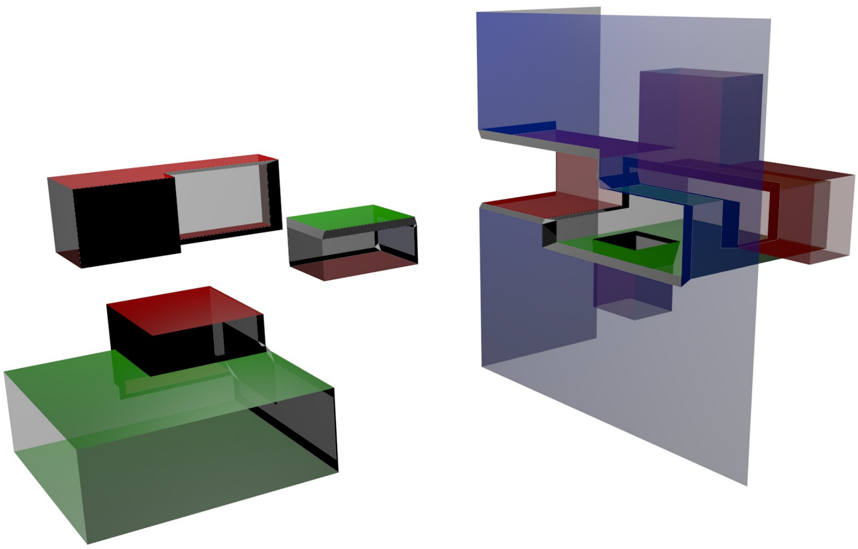


From this configuration, when we glue in the plane, we get one of the two following configurations.

(2.1) First we multiply the following by A.

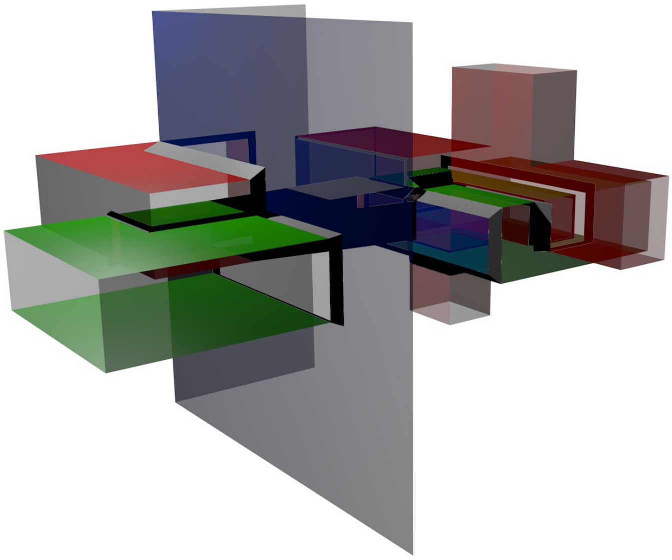


After replacing a cylinder with its two missing faces, we can split the cube with two missing faces such that it becomes two open cubes. Then we obtain the following configuration.

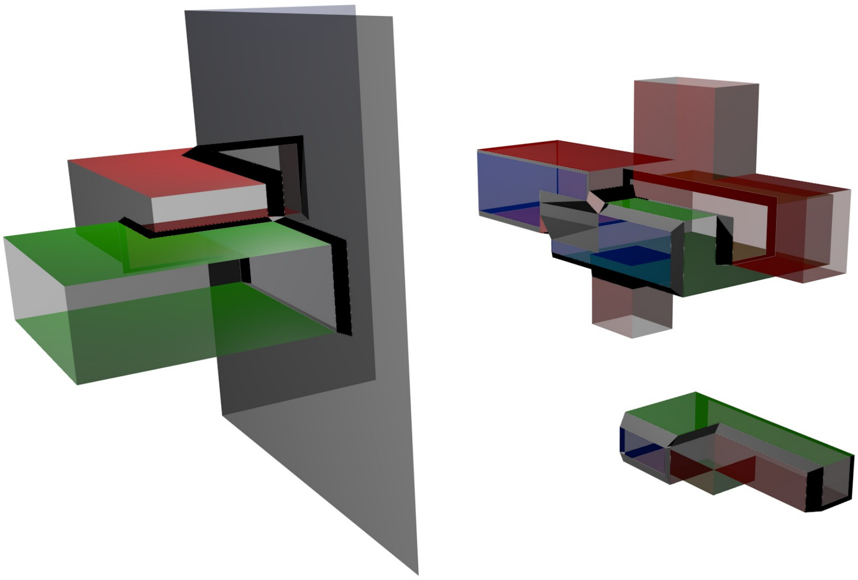


Let us call this σ_{BA}. We also have to multiply by γ because of the cylinder replacement. Notice that σ_{BA} consists of two open cubes, a complete cube and a plane connected to a cube with tunnels and 4 open faces. Since it contains a complete cube, we can remove it and multiply by a.

(2.2) Then we multiply the following configuration by B.



After performing a cube move manipulation, we have the following.



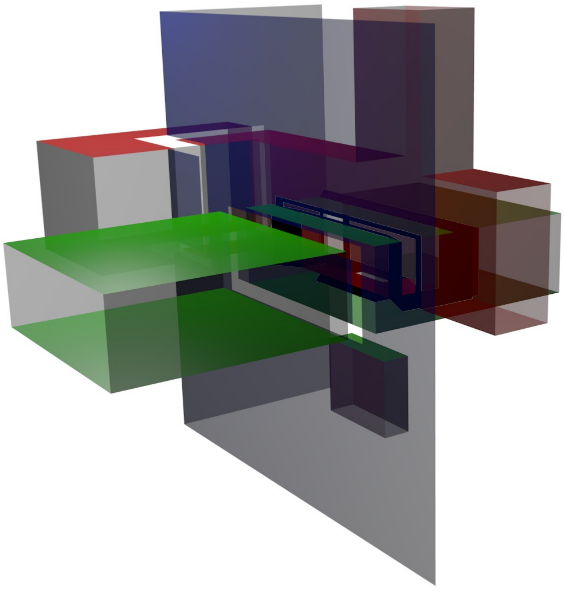
Let us call this σ_{BB}. Notice that this configuration consist of a cube, a cube with tunnels and 4 open faces, and a plane connected to two open cubes. Again we can remove the cube and multiply by a instead.

Thus we are left with the following 2-bracket polynomial for the first part of the triple point move.

$$\begin{aligned}\langle triple_1 \rangle &= A\langle \sigma_A \rangle + B\langle \sigma_B \rangle \\ &= A(A2\gamma\langle \sigma_{AA} \rangle + B\gamma\lambda\langle \sigma_{AB} \rangle) + B(Aa\gamma\langle \sigma_{BA} \rangle + Ba\langle \sigma_{BB} \rangle) \\ &= 2A^2\gamma\langle \sigma_{AA} \rangle + AB\gamma\lambda\langle \sigma_{AB} \rangle + BAa\gamma\langle \sigma_{BA} \rangle + B^2a\langle \sigma_{BB} \rangle\end{aligned}$$

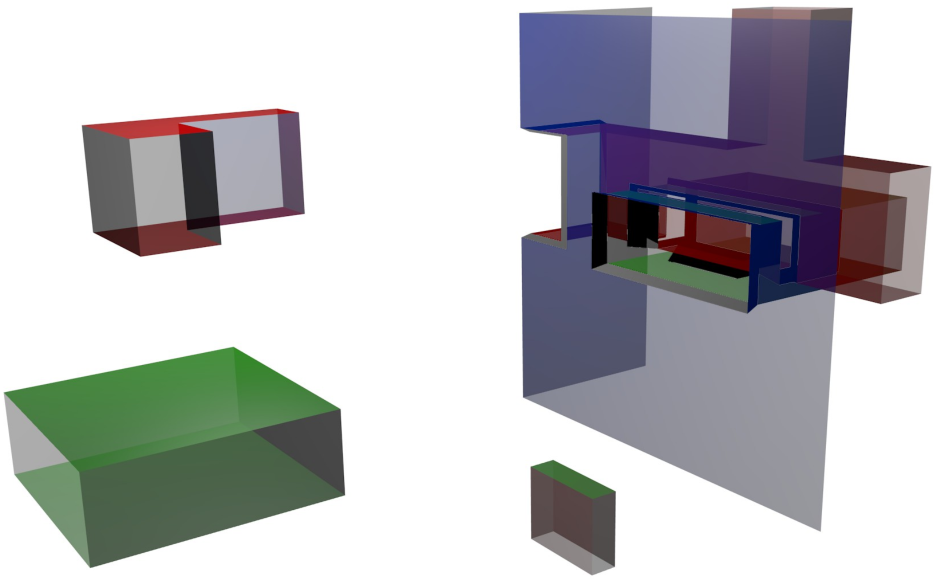
7.4.2.3. The second part of the triple point move has the following calculations.

(1) We denote the following configuration by ρ_A and multiply it by A.



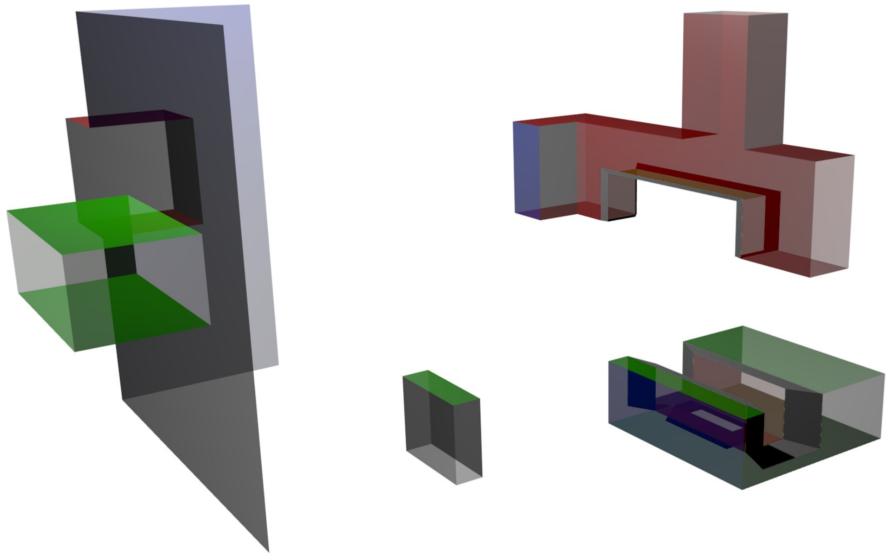
From this configuration, when we glue in the plane, we get one of the two following configuration.

(1.1) First we multiply the following by A.



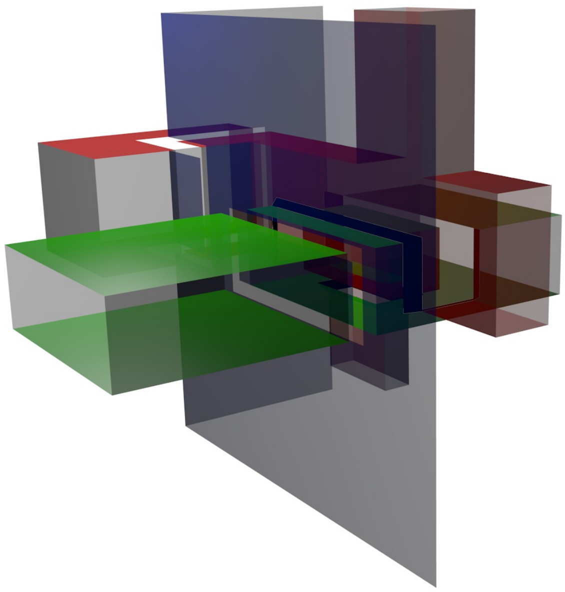
Let us call it ρ_{AA}. Notice that ρ_{AA} consists of three open cubes and a plane connected to a torus with one missing face and a cube with tunnels and two missing faces. Also notice that this means $\sigma_{AA} = \rho_{AA}$.

(1.2) Then multiply the following configuration by B.



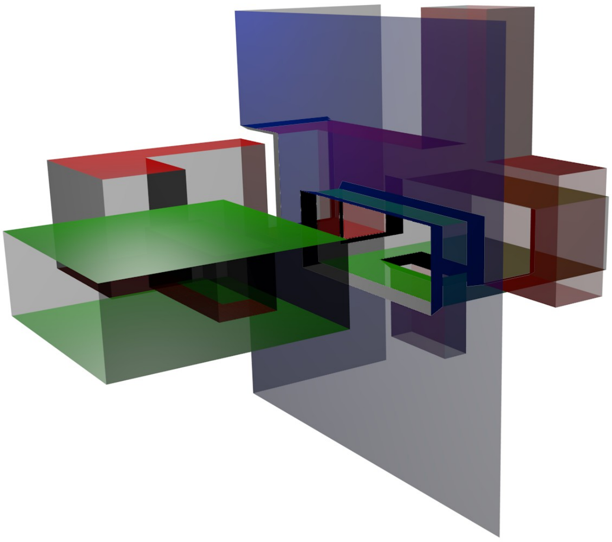
Let us call it ρ_{AB}. Notice that ρ_{AB} consists of a open cube, a cube with two missing faces, a torus with a missing face and a plane connected to two open cubes. Also notice that this means $\sigma_{AB} = \rho_{AB}$.

(2) We denote the following configuration by ρ_B and multiply it by B.

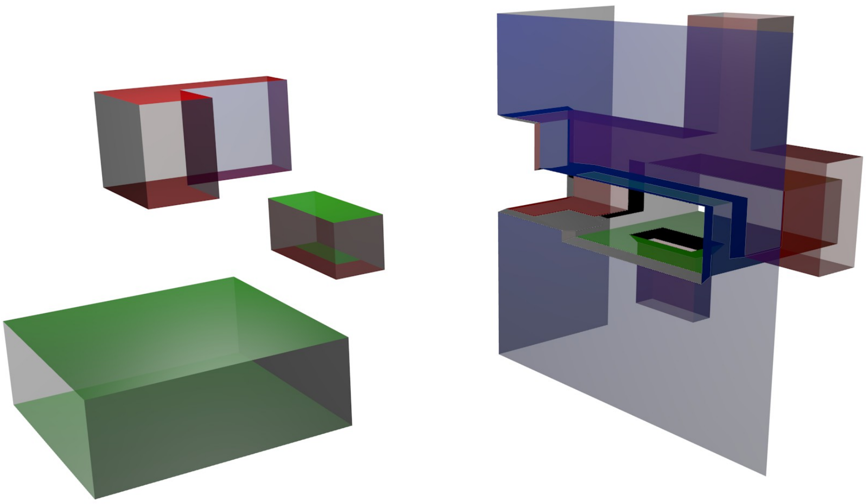


From this configuration, when we glue in the plane, we get one of the two following configurations.

(2.1) First we multiply the following configuration by A.

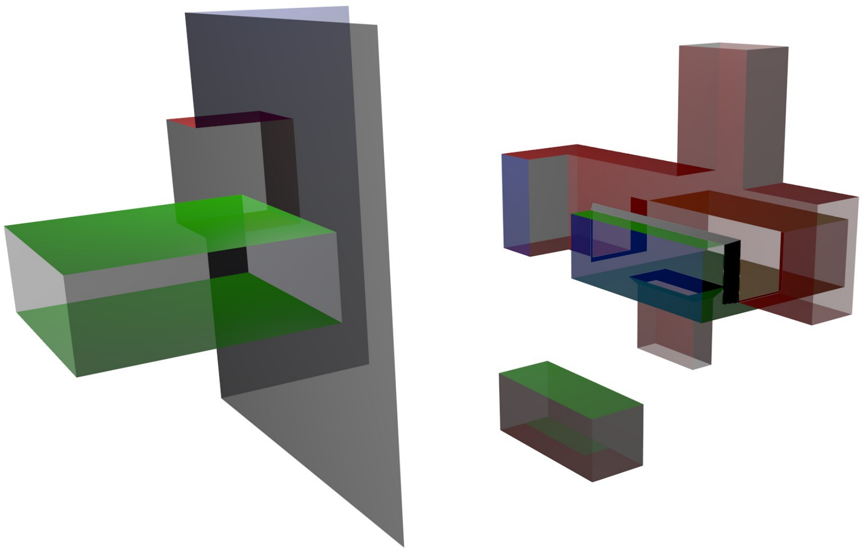


After replacing a cylinder with the two missing faces, we have eliminated one hole in the plane. Then we obtain the following configuration.



Let us call it ρ_{BA}. We also have to multiply by γ because of the cylinder replacement. Notice that ρ_{BA} consists of two open cubes, one complete cube and a plane connected to a cube with tunnels and 4 open faces. The complete cube can be removed by multiplying by a instead. Also notice that this means $\sigma_{BA} = \rho_{BA}$.

(2.2) Then multiply the following configuration by B.



Let us call it ρ_{BB}. Notice that this configuration consist of a cube, a cube with tunnels and 4 open faces, and a plane connected to two open cubes. The cube can again be removed by multiplying by a instead. Also notice that this means $\sigma_{BB} = \rho_{BB}$.

This leaves us with the following bracket polynomial for the second part of the triple point move.

$$\begin{aligned}\langle triple_2\rangle &= A\langle\rho_A\rangle + B\langle\rho_B\rangle\\ &= A(A\langle\rho_{AA}\rangle + B\langle\rho_{AB}\rangle) + B(Aa\gamma\langle\rho_{BA}\rangle + Ba\langle\rho_{BB}\rangle)\\ &= A^2\langle\rho_{AA}\rangle + AB\langle\rho_{AB}\rangle + BAa\gamma\langle\rho_{BA}\rangle + B^2a\langle\rho_{BB}\rangle\\ &= A^2\langle\sigma_{AA}\rangle + AB\langle\sigma_{AB}\rangle + BAa\gamma\langle\sigma_{BA}\rangle + B^2a\langle\sigma_{BB}\rangle\end{aligned}$$

7.4.3. The 2-polynomial of the triple point move

7.4.3.1. We deduce that we have the following 2-polynomial for the first part of the triple point move.

$$\begin{aligned}J(triple_1) &= v^{-\omega(triple_1)}\langle triple_1\rangle\\ &= v^{-\omega(triple_1)}(2A^2\gamma\langle\sigma_{AA}\rangle + AB\gamma\lambda\langle\sigma_{AB}\rangle + BAa\gamma\langle\sigma_{BA}\rangle + B^2a\langle\sigma_{BB}\rangle)\end{aligned}$$

7.4.3.2. We deduce that we have the following 2-polynomial for the second part of the triple point move.

$$\begin{aligned}J(triple_2) &= v^{-\omega(triple_2)}\langle triple_1\rangle\\ &= v^{-\omega(triple_2)}(A^2\langle\sigma_{AA}\rangle + AB\langle\sigma_{AB}\rangle + BAa\gamma\langle\sigma_{BA}\rangle + B^2a\langle\sigma_{BB}\rangle)\end{aligned}$$

7.4.3.3. Since we have that $\omega(triple_1) = \omega(triple_2)$ and the relations in 7.1.1.1, we get the following.

$$\begin{aligned} J(triple_1) &= v^{-\omega(triple_1)}(2A^2\gamma\langle\sigma_{AA}\rangle + AB\gamma\lambda\langle\sigma_{AB}\rangle + BAa\gamma\langle\sigma_{BA}\rangle + B^2a\langle\sigma_{BB}\rangle) \\ &= v^{-\omega(triple_2)}(A^2\langle\sigma_{AA}\rangle + AB\langle\sigma_{AB}\rangle + BAa\gamma\langle\sigma_{BA}\rangle + B^2a\langle\sigma_{BB}\rangle) \\ &= J(triple_2) \end{aligned}$$

7.4.3.4. Thus $J(triple_1) = J(triple_2)$ and the 2-polynomial is invariant under the triple point move.

7.5. Invariance under the tetrahedral move

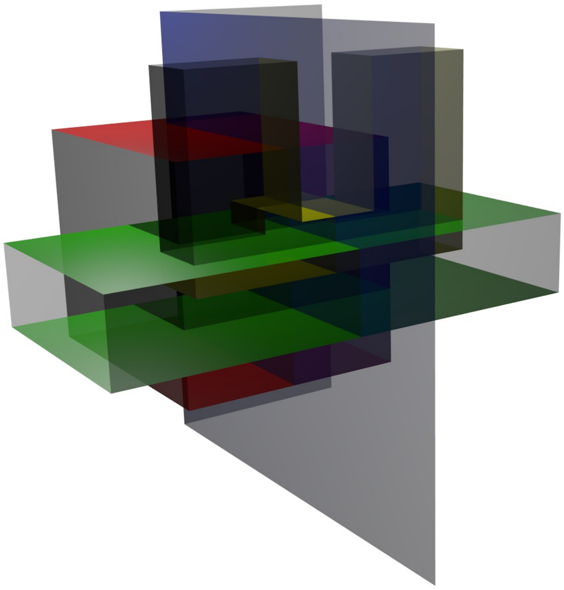
7.5.1. Writhe in the setting of the tetrahedral move

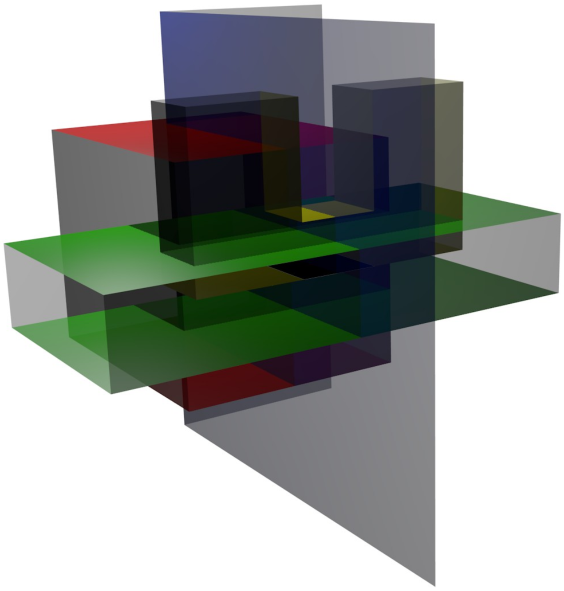
7.5.1.1. To find the writhe of the first part of the tetrahedral move notice that the green cube only intersects the other components once and in the same manner once the orientation is set. Thus the writhe of the green cube crossings is either $+3$ or -3 depending on the orientation. We also notice that the yellow cube cross the red cube three times and will have the same sign for two of the crossings. The writhe when only considering the red and yellow cube is then either $+1$ or -1. The sign where the yellow cube crosses the blue plane will be the same as for the two crossings between the yellow and red cube. Therefore we have that the writhe when only considering the blue, yellow and red is either $+2$ or -2. That leaves the sign we obtain when the red cube crosses the blue plane, and this is either $+1$ or -1. Thus $\omega(tetrahedral_1)$ is either ± 6, ± 4, ± 2 or 0.

7.5.1.2. To find the writhe of the second part of the tetrahedral move notice that the green cube still only intersects the other components once and in the same manner once the orientation is set. So the writhe is either $+3$ or -3. We also notice that the yellow cube only cross the red cube the two times where the sign is the same now. Thus the writhe when only considering the red and yellow cube is then either $+2$ or -2. The sign where the yellow cube crosses the blue plane will still be the same as for when it crosses the red cube. Therefore we have that the writhe, when only considering the blue, yellow and red, is either $+3$ or -3. That leaves the sign when the red cube crosses the blue plane, and that still is either $+1$ or -1. Thus $\omega(tetrahedral_2)$ is either ± 7, ± 5 or ± 1.

7.5.2. The 2-bracket polynomial of the tetrahedral move

7.5.2.1. Observe that the only difference between the two configuration in the tetrahedral move is where the yellow cube crosses around the triple point. We can therefore simplify the configurations without losing any information. The simplified configurations we will use to find the 2-bracket polynomial, are depicted below.

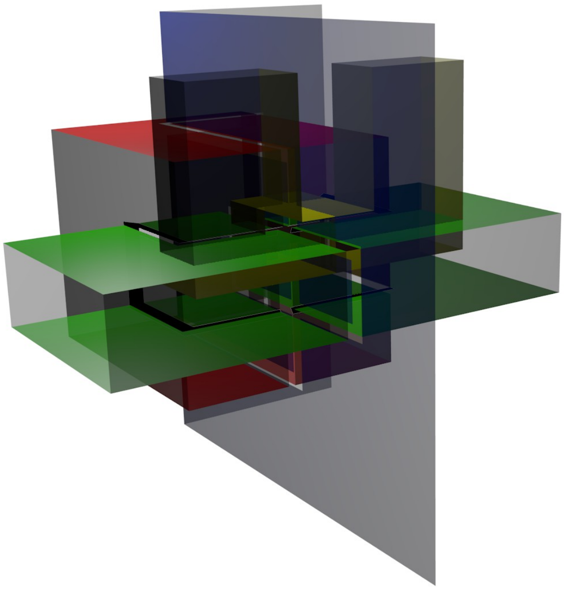




7.5.2.2. We start by performing the surgery to the triple point which the yellow cube crosses in the first part of the tetrahedral move. Then we see that after a bubble move they are in fact the same configuration.

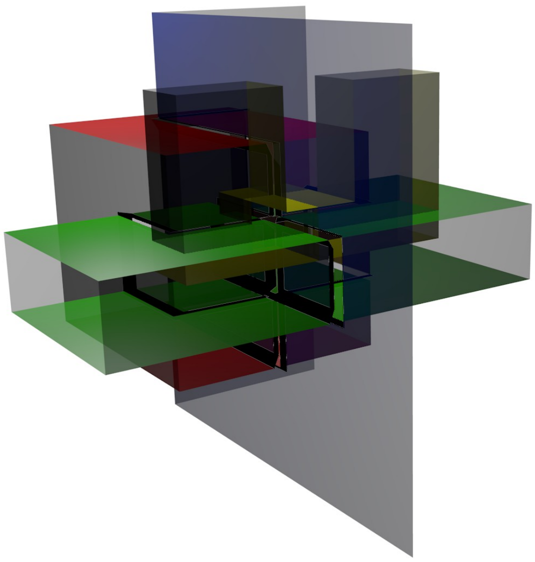
7.5.2.3. The first part of the tetrahedral move has the following calculations.

(1) We denote the following configuration by τ_A and multiply it by A.

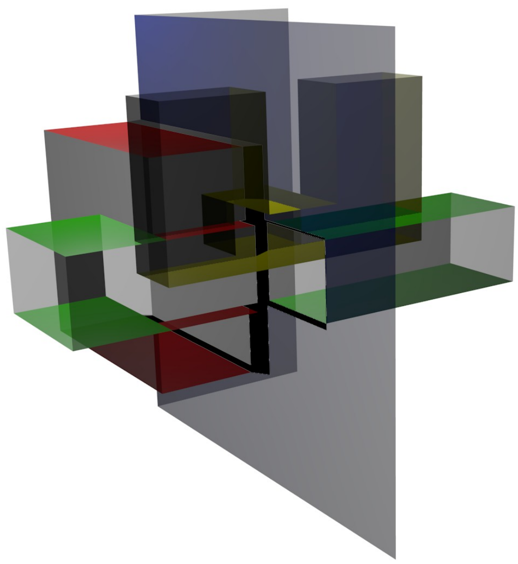


From this configuration we get one of the two following configurations.

(1.1) First we multiply the following configuration by A.

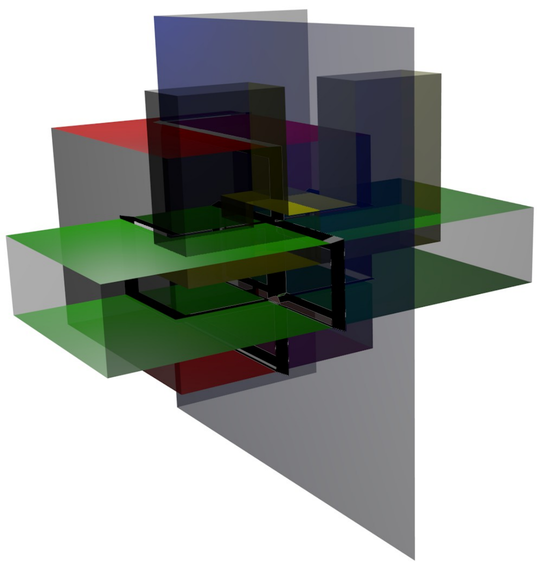


Let us call it τ_{AA}. After simplifying this configuration, we get the following.

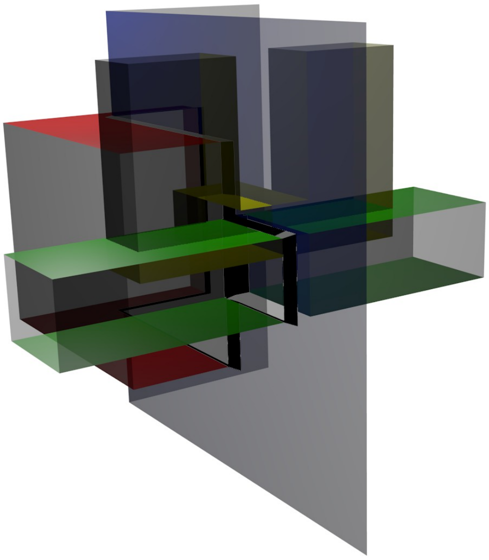


Notice that we can reduce the yellow cube with a bubble move.

(1.2) Then we multiply the following configuration by B.

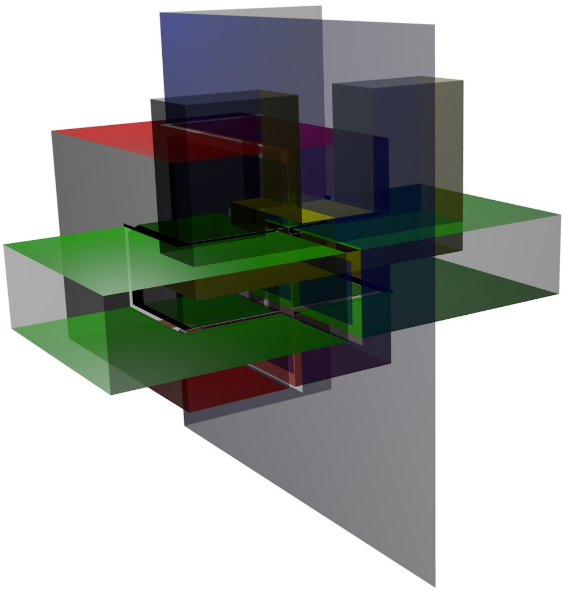


Let us call it τ_{AB}. After simplifying this configuration, we get the following.



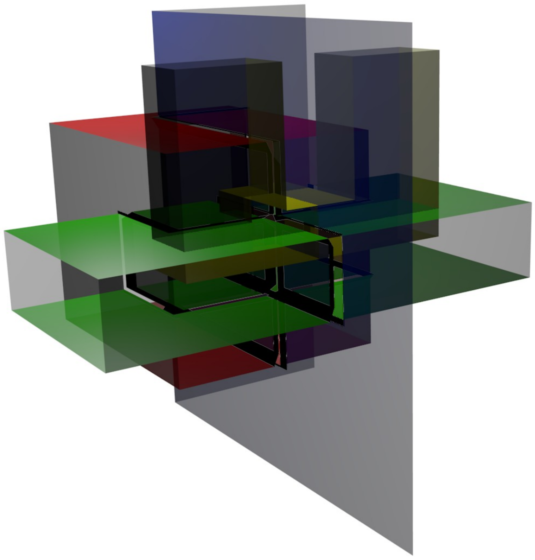
Again notice that we can reduce the yellow cube with a bubble move.

(2) We denote the following configuration by τ_B and multiply it by B.

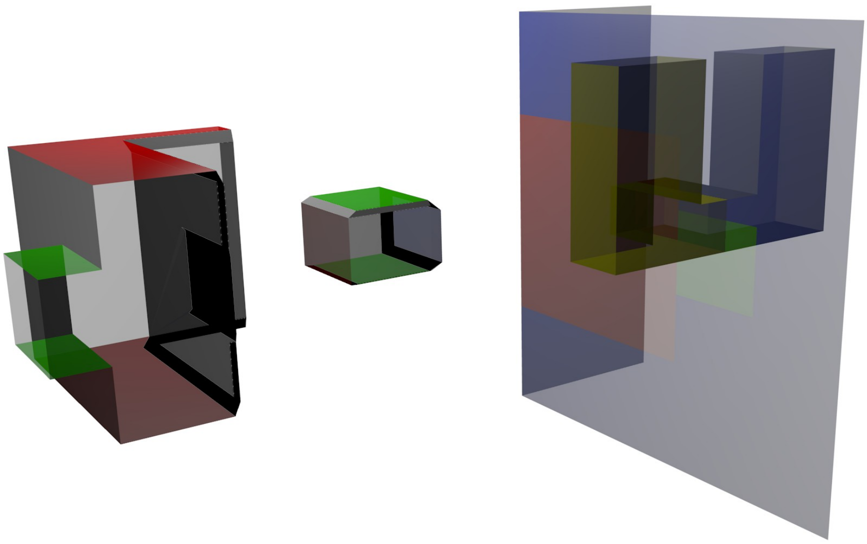


From this configuration, we get one of the two following configurations.

(2.1) First we multiply the following configuration by A.

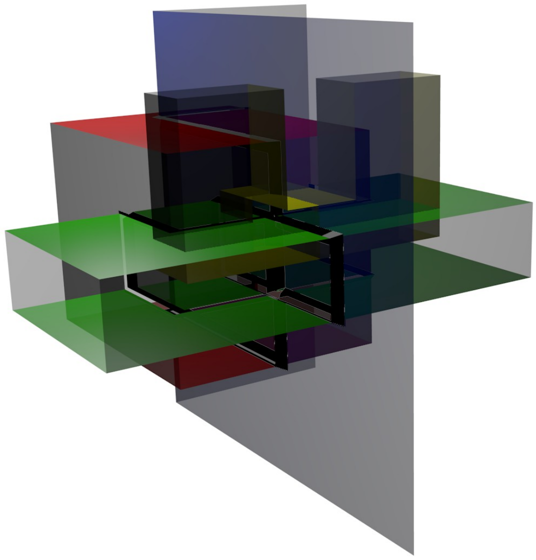


Let us call it τ_{BA}. After simplifying this configuration, we get the following.

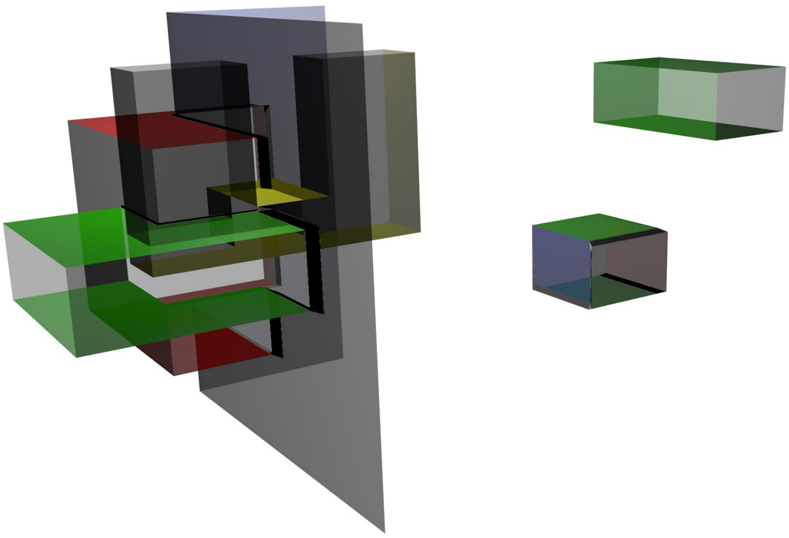


Notice that again there is nothing stopping us from reducing the yellow cube by a bubble move.

(2.2) Then multiply the following configuration by B.



Let us call it τ_{BB}. After simplifying this configuration, we get the following configuration.



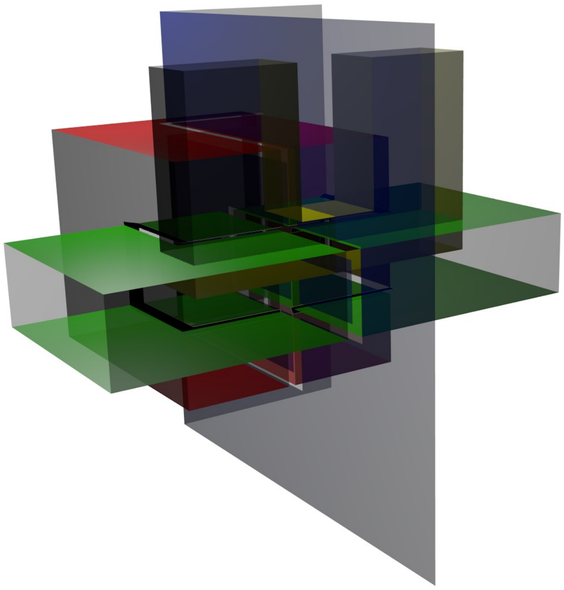
Again notice that the yellow cube can be reduced by a bubble move.

This gives us the following 2-bracket polynomial for the first part of the tetrahedral move.

$$\begin{aligned}\langle tetrahedral_1\rangle &= A\langle\tau_A\rangle + B\langle\tau_B\rangle\\ &= A(A\langle\tau_{AA}\rangle + B\langle\tau_{AB}\rangle) + B(A\langle\tau_{BA}\rangle + B\langle\tau_{BB}\rangle)\\ &= A^2\langle\tau_{AA}\rangle + AB\langle\tau_{AB}\rangle + BA\langle\tau_{BA}\rangle + B^2\langle\tau_{BB}\rangle\end{aligned}$$

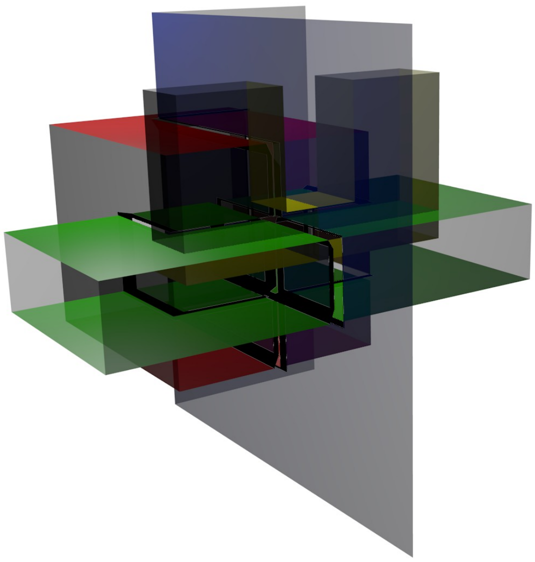
7.5.2.4. The second part of the tetrahedral move has the following calculations.

(1) We multiply the following configuratoin by A.



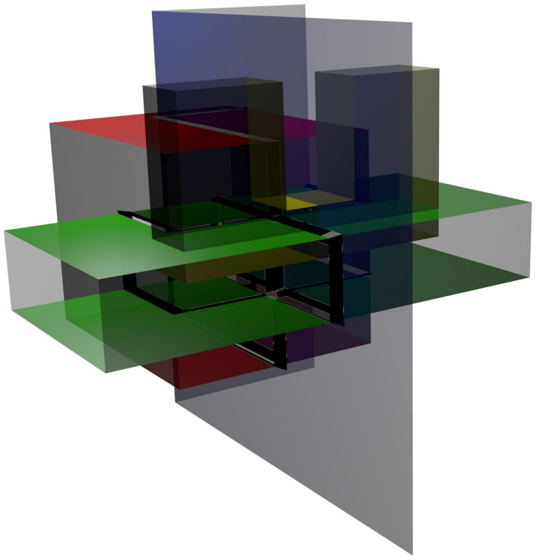
From this configuration, we get one of the two following.

(1.1) First we multiply the following by A.



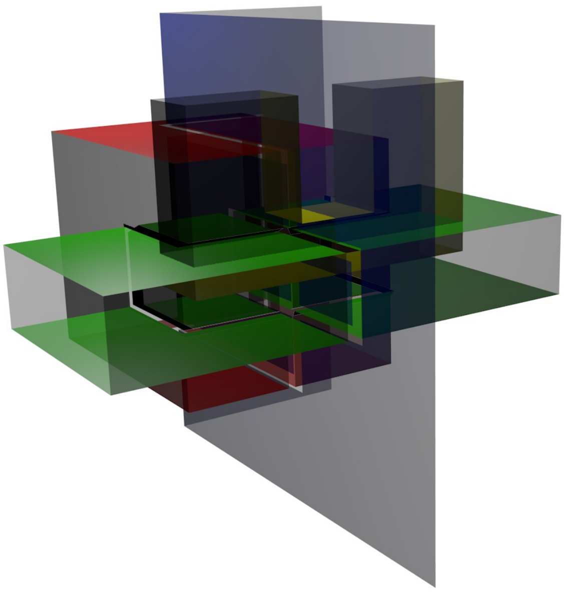
Notice that this configuration is equal to τ_{AA}, which means we can simplify in the same way.

(1.2) Then multiply the following configuration by B.



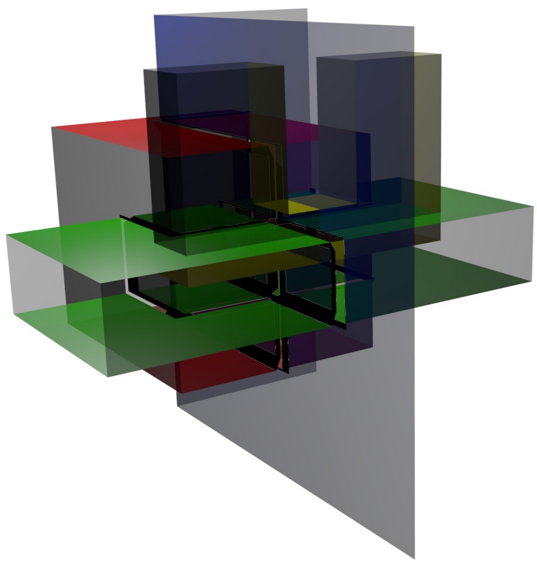
Notice that this configuration is equal to τ_{AB}, we can therefore simplify in the same way.

(2) We multiply the following configuration by B.



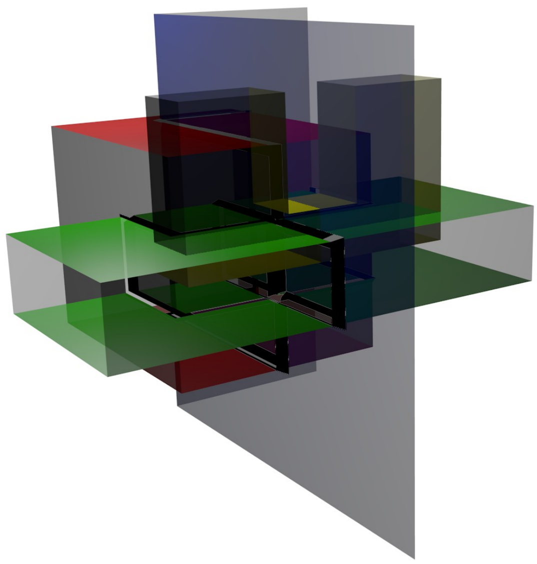
From this configuration, we get one of the two following.

(2.1) First we multiply the following configuration by A.



Notice that this configuration is equal to τ_{BA}, which means we can simplify in the same way.

(2.2) Then multiply the following configuration by B.



Notice that this configuration is equal to τ_{BB}, we can therefore simplify in the same way.

This gives us the following 2-bracket polynomial for the second part of the tetrahedral move.

$$\begin{aligned}\langle tetrahedral_2 \rangle &= A(A\langle \tau_{AA} \rangle + B\langle \tau_{AB} \rangle) + B(A\langle \tau_{BA} \rangle + B\langle \tau_{BB} \rangle) \\ &= A^2\langle \tau_{AA} \rangle + AB\langle \tau_{AB} \rangle + BA\langle \tau_{BA} \rangle + B^2\langle \tau_{BB} \rangle\end{aligned}$$

7.5.2.5. We observe that $\langle tetrahedral_1 \rangle = \langle tetrahedral_2 \rangle$.

7.5.3. The 2-polynomial of the tetrahedral move

7.5.3.1. The 2-polynomial of the tetrahedral move is as follows.

$$\begin{aligned} J(tetrahedral_1) &= J(tetrahedral_1 \textit{ after surgery}) \\ &= J(tetrahedral_1 \textit{ after surgery and bubble move}) \end{aligned}$$

7.5.3.2. Since the 2-polynomial is invariant under the bubble move, we have the following calculation.

$$\begin{aligned} J(tetrahedral_2) &= J(tetrahedral_2 \textit{ after surgery}) \\ &= J(tetrahedral_1 \textit{ after surgery and bubble move}) \end{aligned}$$

7.5.3.3. Thus $J(tetrahedral_1) = J(tetrahedral_2)$ and the 2-polynomial is invariant under the tetrahedral move.

7.6. Conclusion and an example

7.6.1. In conclusion

We have proven that if a pair of oriented 2-knots K and K' are isotopic, then $J(K) = J(K')$. In other words, the 2-polynomial of a 2-knot is an invariant.

7.6.2. Examples

7.6.2.1. We have that the 2-polynomial of the unknot is 1, since $\omega(K) = 0$ and $\langle K \rangle = 1$.

7.6.2.2. We calculated in 5.2.2.3 that the writhe of the convoluted hoover is -3, and in 6.1.4.2 that its bracket polynomial is the following.

$$\langle \text{Convoluted hoover} \rangle = A^3ab + B^3a + 2A^2Ba + A^2Bab_2 + 3AB^2ab$$

We conclude that the 2-polynomial of the convoluted hoover is given by the following.

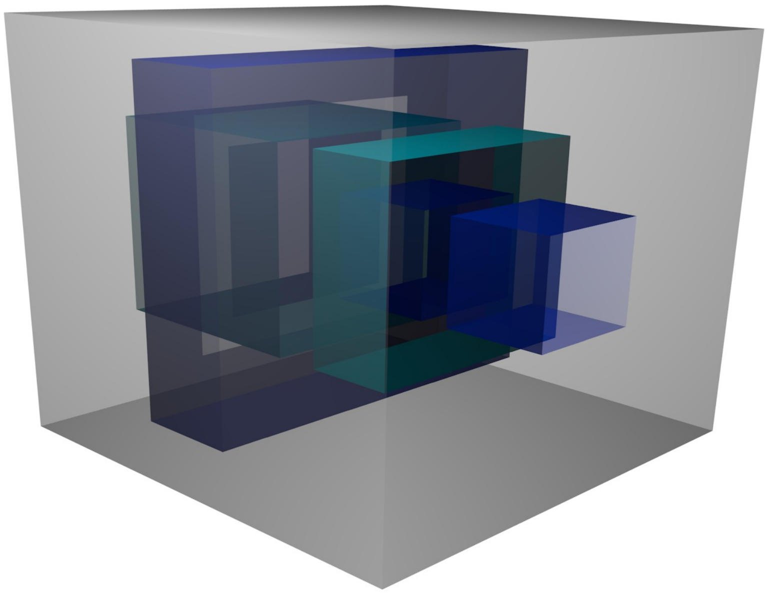
$$\begin{aligned} J(\text{Convoluted hoover}) &= v^{-\omega(\text{Convoluted hoover})} \langle \text{Convoluted hoover} \rangle \\ &= v^3(A^3ab + B^3a + 2A^2Ba + A^2Bab_2 + 3AB^2ab) \end{aligned}$$

Appendices

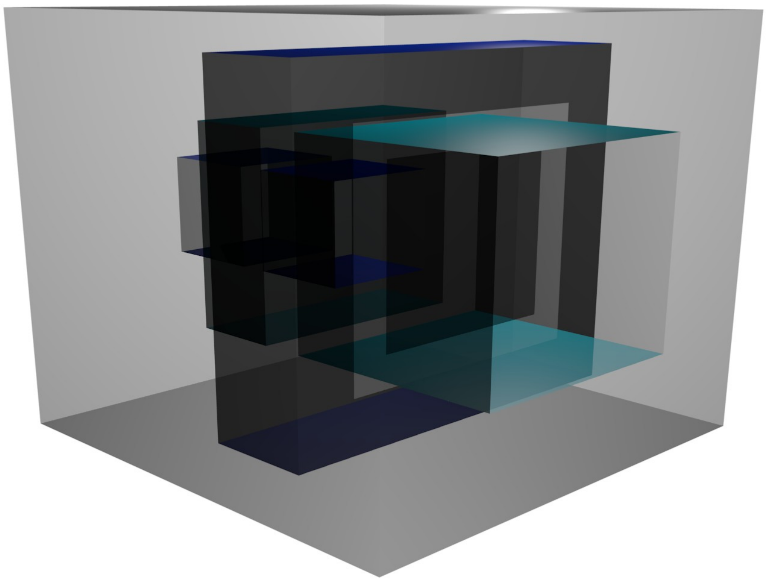
APPENDIX A

CALCULATING THE 2-BRACKET POLYNOMIAL OF THE CONVOLUTED HOOVER EXAMPLE

The convoluted hoover has three crossings, and we shall perform the surgery on all of them. All the configurations obtained in the calculation of the 2-bracket polynomial for the convoluted hoover example are illustrated from two angles in this appendix. Below you can see the alternating convoluted hoover depicted from the first angle.

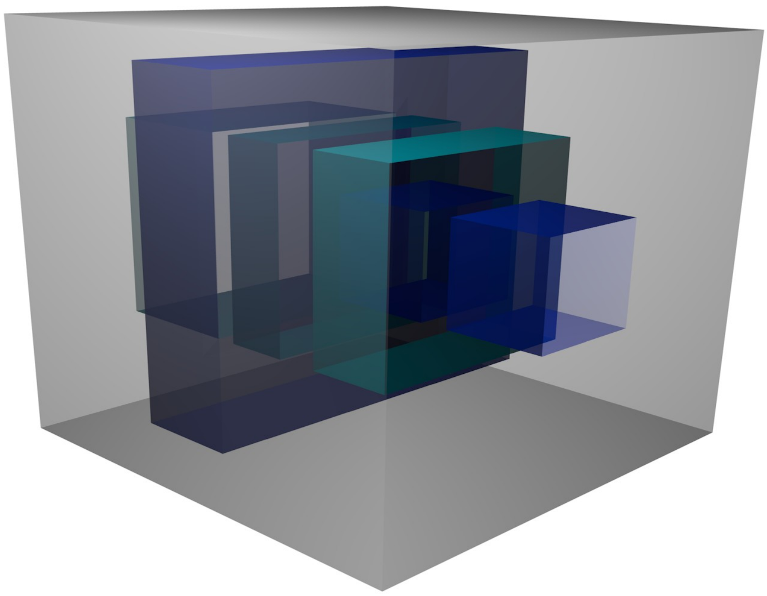


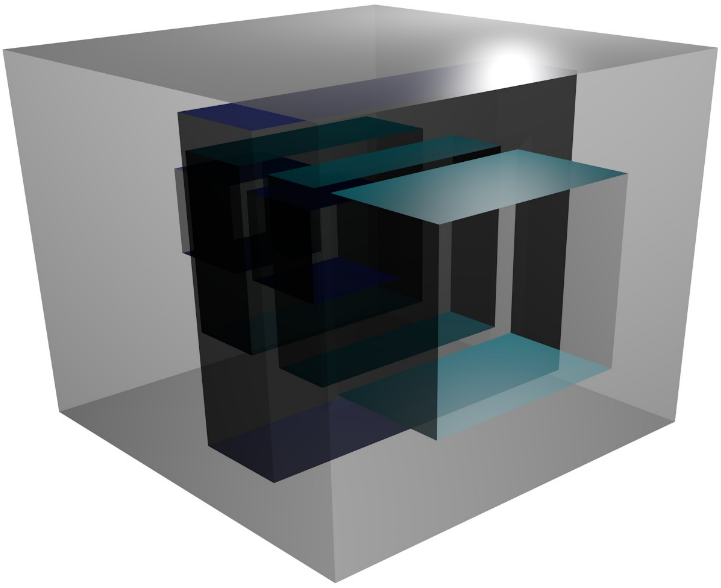
The alternating convoluted hoover from the opposite angle is as follows.



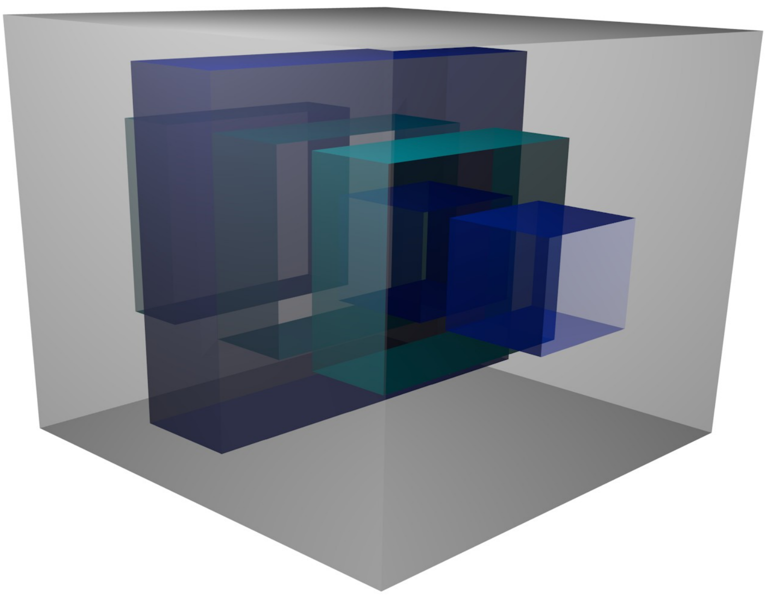
A.1. Pictures after surgery on the first crossing

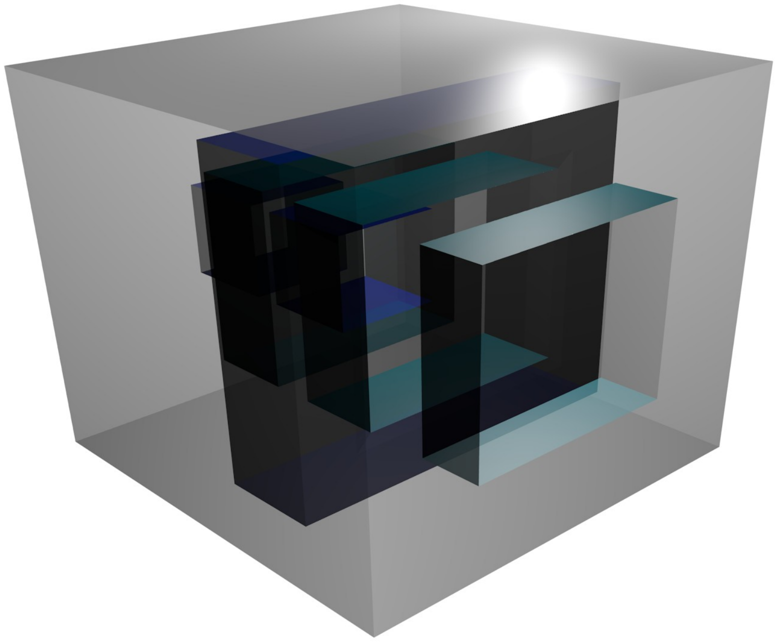
First we perform the surgery on the crossing between the large dark blue tunnel and the lighter blue one. We denote by s_A the following configuration, which is illustrated from the two angles.





The other configuration we get for this surgery, we denote by s_B and is depicted from the two angles below.





From this first surgery, we obtain the following calculation.

$$\langle \text{Convoluted hoover} \rangle = A\langle s_A \rangle + B\langle s_B \rangle$$

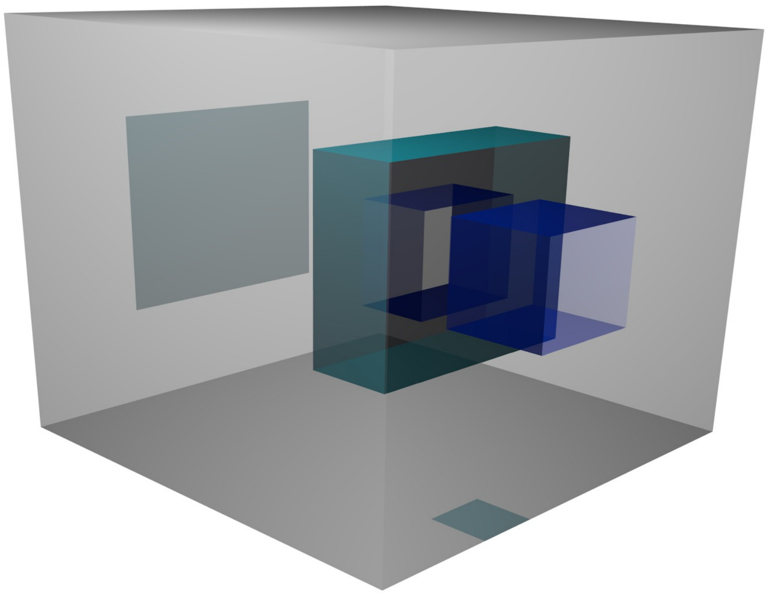
Both s_A and s_B are still knotted, thus it will be necessary to do a second round of surgery.

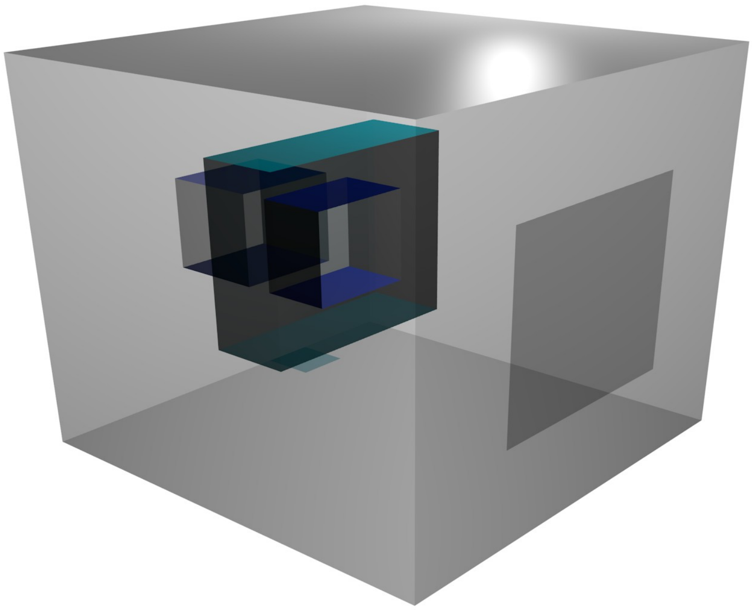
A.2. Pictures after surgery on the second crossing

There are two possible results of the second surgery for each of the configurations in the last section. We therefore end up with four possible configurations.

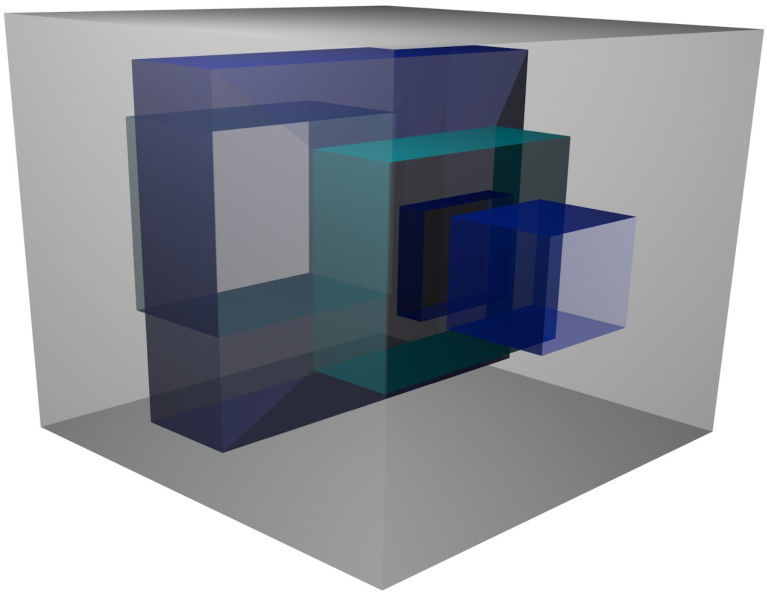
Pictures after surgery on the second crossing of s_A

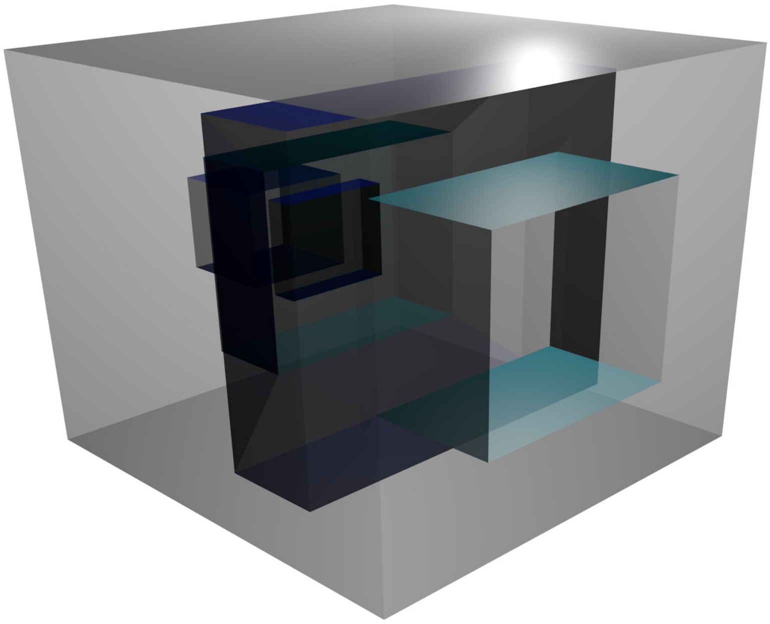
First we continue with the surgery on s_A. Then the first possible configuration, after being simplified by a couple of cube moves, is depicted below.





Let us call it s_{AA}. The other possible configuration obtained from the surgery on s_A, after being simplified by a cube move, is the configuration depicted below.

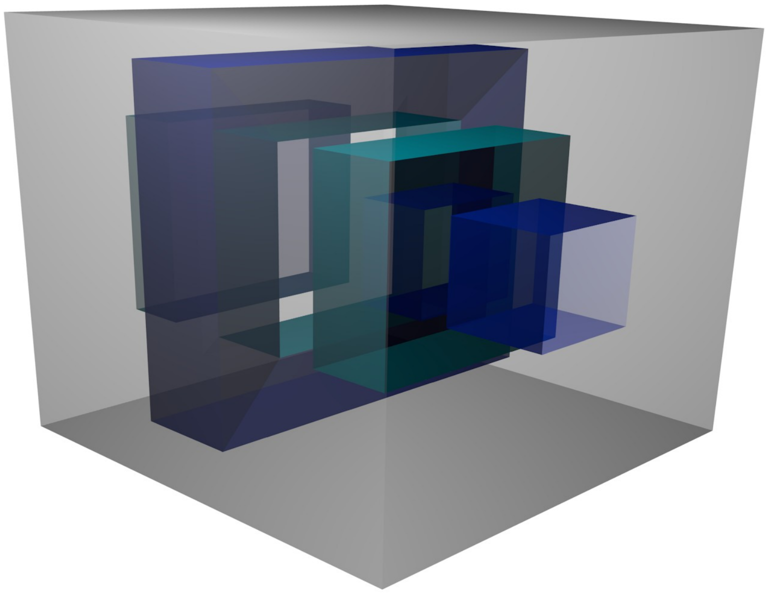


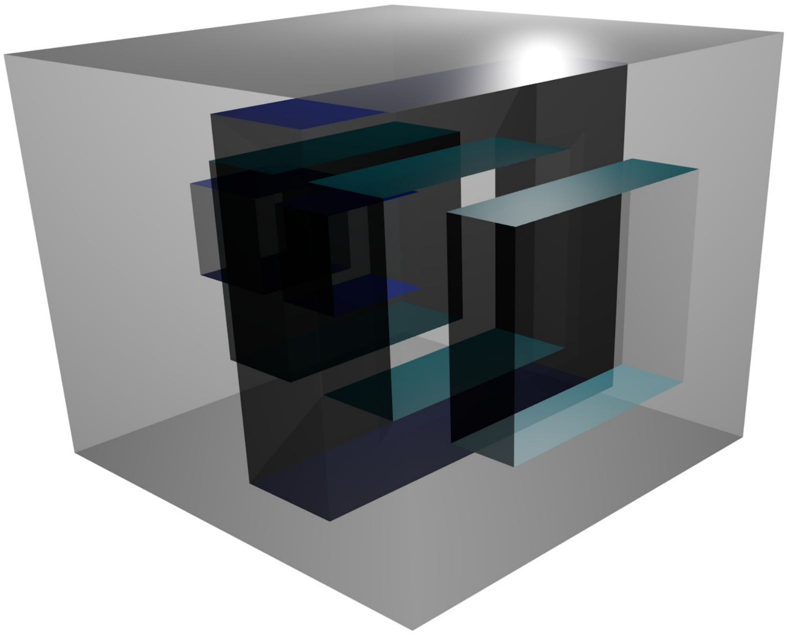


Let us call it s_{AB}.

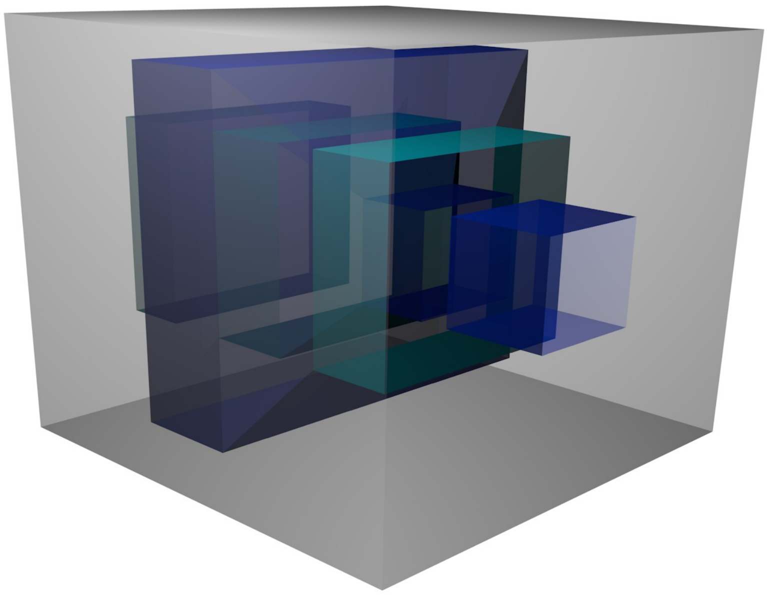
Pictures after surgery on the second crossing of s_B

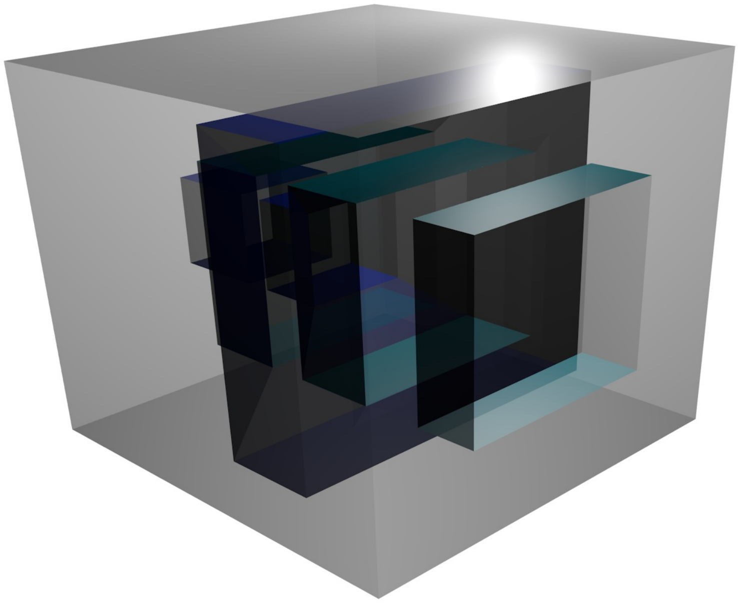
Now we perform the surgery on the second crossing of s_B. Then the first possible configuration is depicted below.





Let us call it s_{BA}. Observe that inside there is a cubical torus. We can remove this torus and multiply the remaining configuration by b. Then we call the remaining configuration s_{BA2}. The other possible configuration obtained from the surgery on the second crossing of s_B, is depicted below.





Let us call this s_{BB}.

Where are we after the second surgery?

After the surgery is performed on the second crossing, we obtain the following calculations.

$$\begin{aligned}\langle \text{Convoluted hoover} \rangle &= A\langle s_A \rangle + B\langle s_B \rangle \\ &= A(A\langle s_{AA}\rangle + B\langle s_{AB}\rangle) + B(A\langle s_{BA}\rangle + B\langle s_{BB}\rangle) \\ &= A^2\langle s_{AA}\rangle + AB\langle s_{AB}\rangle + BAb\langle s_{BA2}\rangle + B^2\langle s_{BB}\rangle\end{aligned}$$

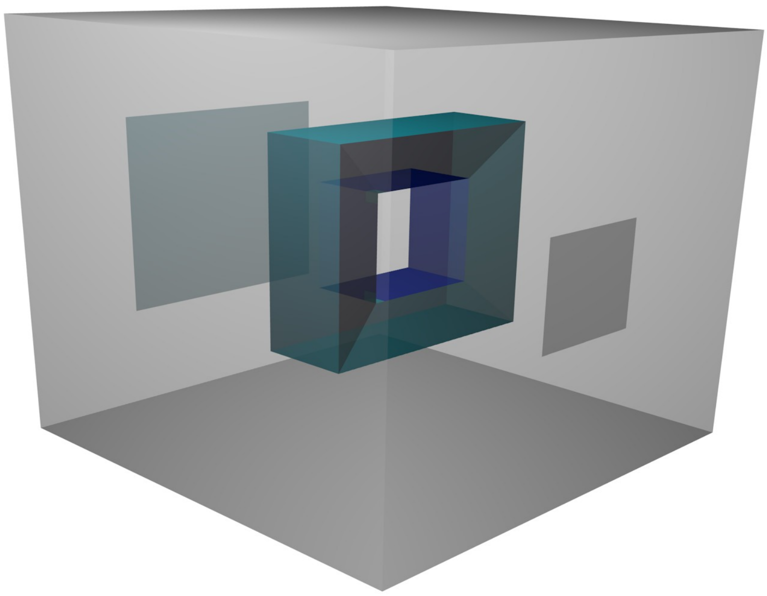
The configurations we obtain after the surgery on the two crossings, are still knotted. We must therefore apply surgery to the third and last crossing in those cases.

A.3. Pictures after surgery on the third crossing

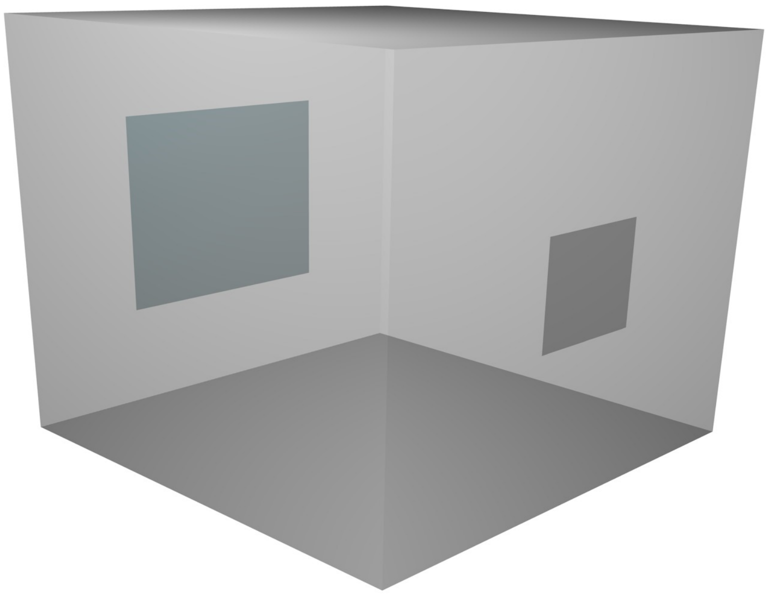
There are again two possible configurations for each of the four in the last section. We end up with eight possible states. These configurations will only be shown from one angle.

Pictures after surgery on the third crossing of s_{AA}

The first possible state we obtain by performing the last surgery on s_{AA} and it is depicted below.



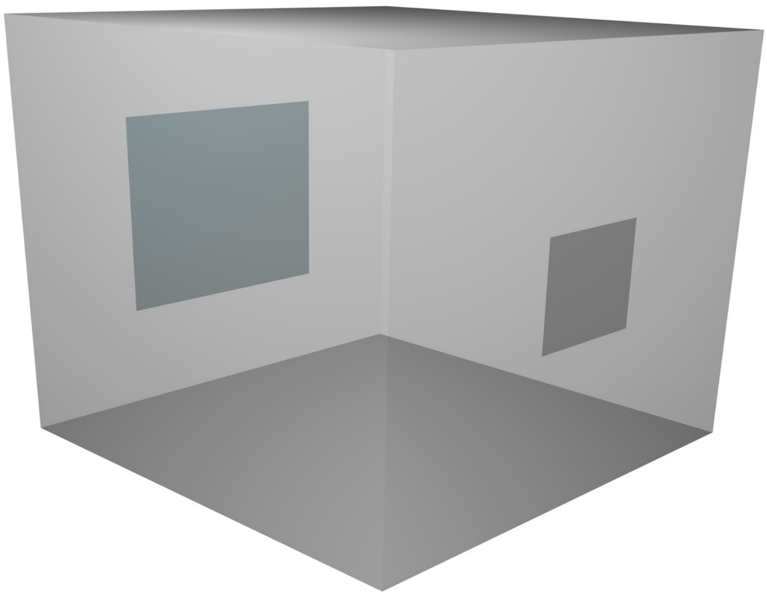
Let us call this state s_{AAA}. Observe that this configuration consists of one torus inside a cube. The second possible state we obtain from the surgery on s_{AA}, after applying a couple of cube moves, is depicted below.



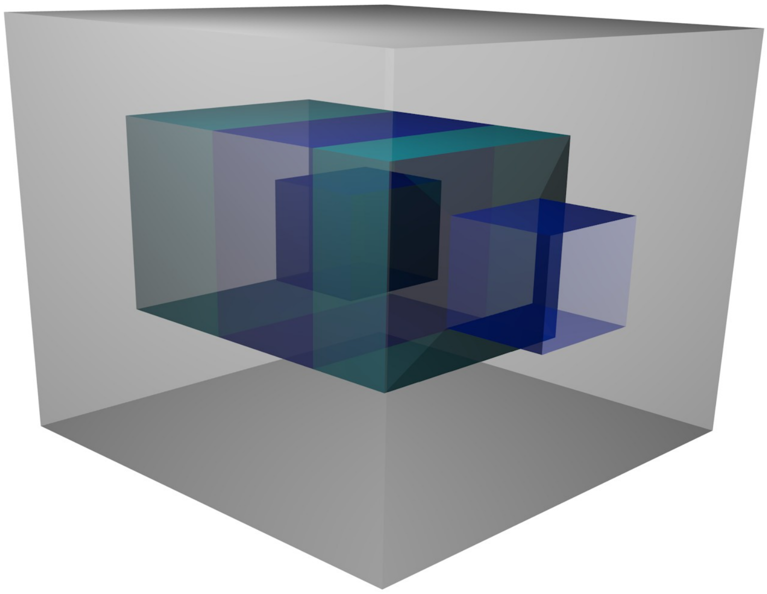
Let us call this state s_{AAB}. Observe that this configuration consist of just a cube.

Pictures after surgery on the third crossing of s_{AB}

The first possible state we obtain from the last surgery on s_{AB}, after being simplified by some cube moves, is depicted below.



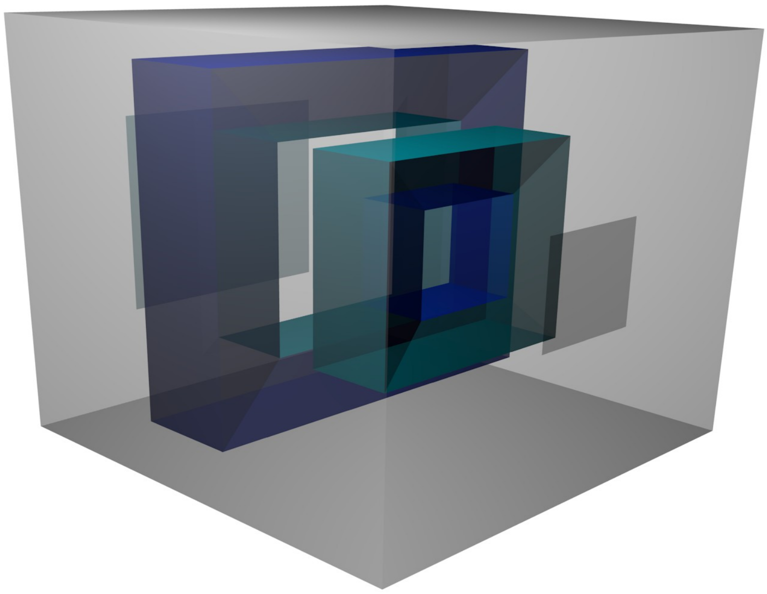
Let us call this state s_{ABA}. Observe that this configuration consists of just one cube. The second possible state we obtain from the surgery on s_{AB}, is depicted below.



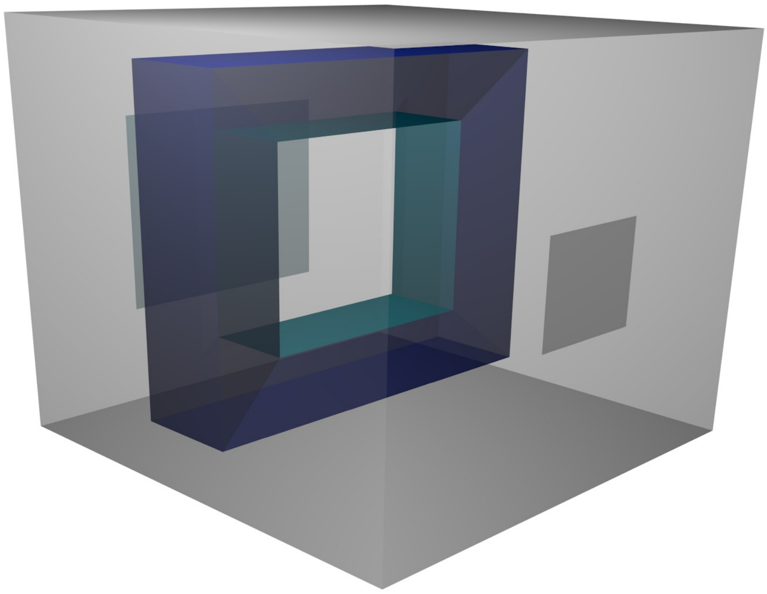
Let us call this state s_{ABB}. Observe that this configuration consists of a cube inside the hole of a torus.

Pictures after surgery on the third crossing of s_{BA}

The first possible state we obtain from the last surgery on s_{BA}, after a cube move, is depicted below.



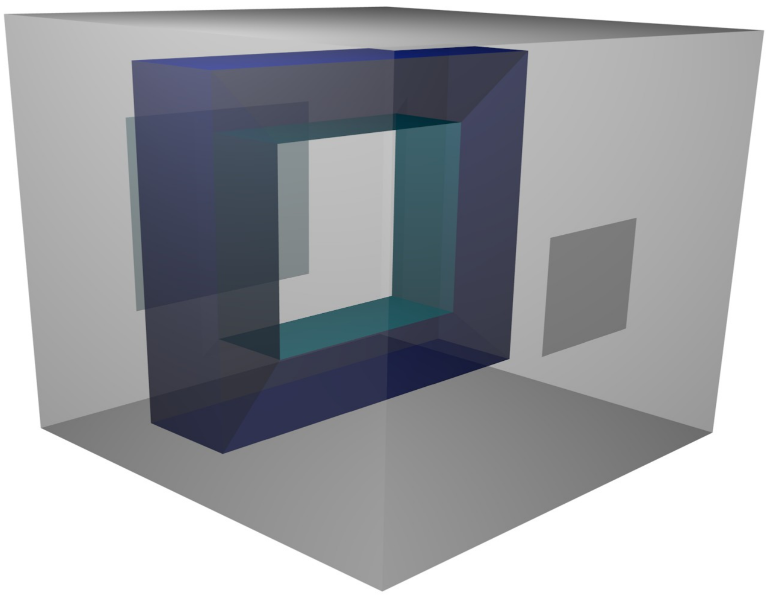
Let us call this state s_{BAA}. Observe that this configuration consists of two tori inside a cube. The second possible state we obtain from the surgery on s_{BA}, after two cube moves, is depicted below.



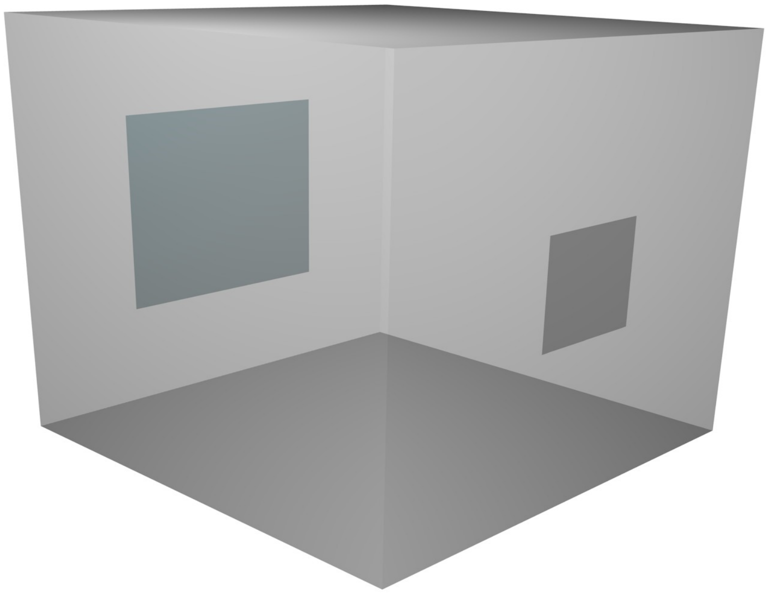
Let us call this state s_{BAB}. Observe that this configuration consists of a torus inside a cube.

Pictures after surgery on the third crossing of s_{BB}

The first possible state we obtain from the last surgery on s_{BB}, after a cube move, is depicted below.



Let us call this state s_{BBA}. Observe that this configuration consists of a torus inside a cube. The second possible state we obtain from the surgery on s_{BB}, after a couple of cube moves, is depicted below.



Let us call this state s_{BBB}. Observe that this configuration consists of just one cube.

Where are we after the third surgery?

After this last surgery, we continue where we left the calculations in the last section. Then we obtain the following 2-bracket polynomial.

$$\begin{aligned}\langle \text{Convoluted hoover} \rangle &= A^2\langle s_{AA}\rangle + AB\langle s_{AB}\rangle + BA\langle s_{BA}\rangle + B^2\langle s_{BB}\rangle \\ &= A^2(A\langle s_{AAA}\rangle + B\langle s_{AAB}\rangle) + AB(A\langle s_{ABA}\rangle + B\langle s_{ABB}\rangle) \\ &\quad + BA(A\langle s_{BAA}\rangle + B\langle s_{BAB}\rangle) + B^2(A\langle s_{BBA}\rangle + B\langle s_{BBB}\rangle \\ &= A^3ab + A^2Ba + ABAa + AB^2ab + BA^2ab_2 + BABab \\ &\quad + B^2Aab + B^3a \\ &= A^3ab + B^3a + 2A^2Ba + A^2Bab_2 + 3AB^2ab\end{aligned}$$

This completes the calculation of the 2-bracket polynomial of the convoluted hoover example.

APPENDIX B

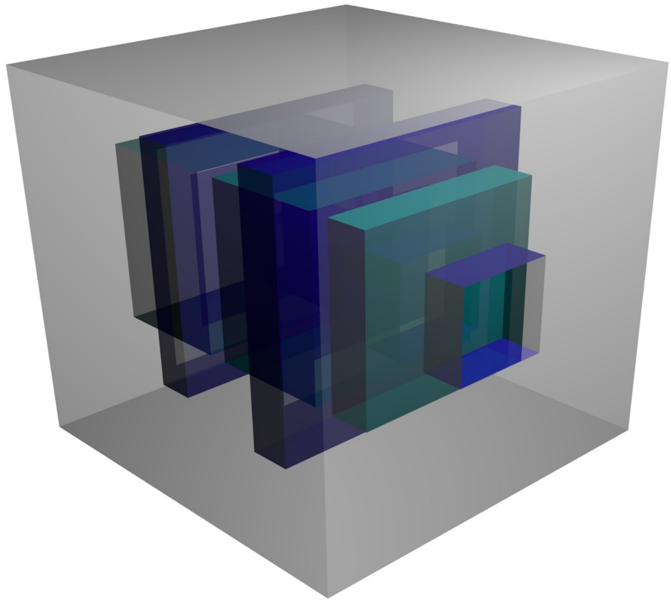
EXAMPLES OF 2-KNOTS

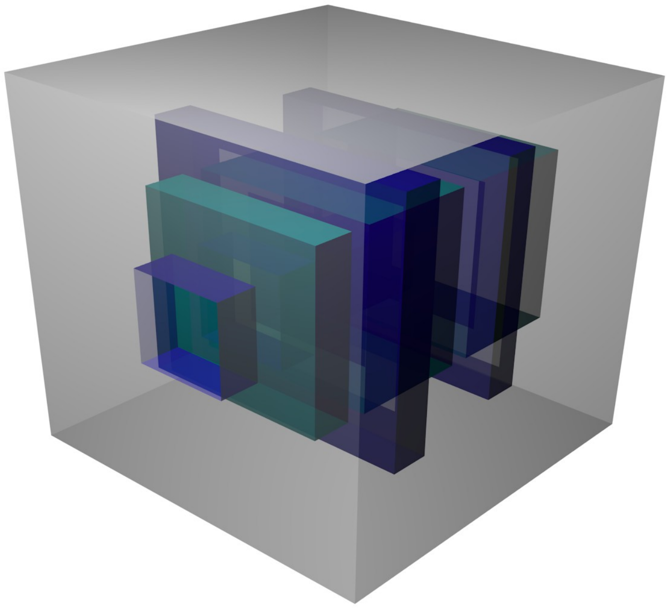
B.1. Generating 2-knots

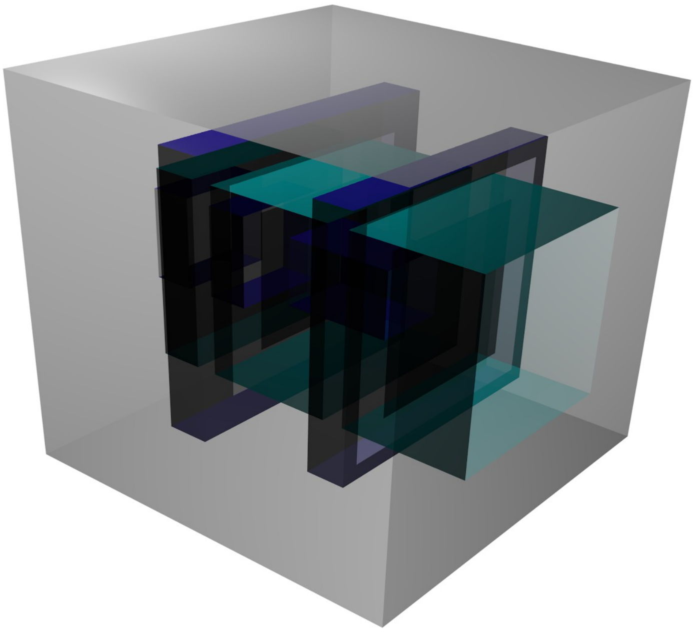
We feel that the convoluted hoover is a prototypical example of a 2-knot with only double plane crossings, akin to the trefoil in ordinary knot theory. The way in which the tunnels interact, seem to be fundamental in obtaining 2-knottedness. In this appendix we explain how to modify the convoluted hoover to obtain two infinite families of 2-knots.

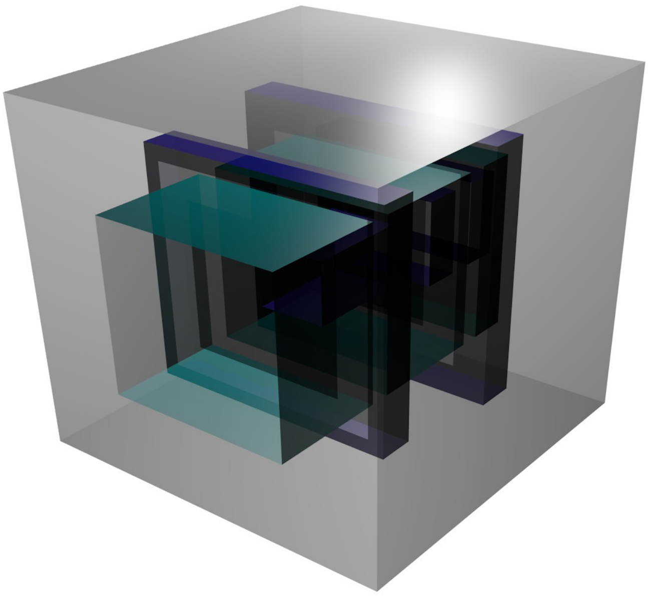
B.2. The convoluted double hoover

The *convoluted double hoover* is an extension of the convoluted hoover example in Appendix A. After the first enlarged dark blue part in the convoluted hoover, it reduces back down to become smaller than the light blue tube before it again enlarges in a second hoover part. Below you find the convoluted double hoover from four angles.







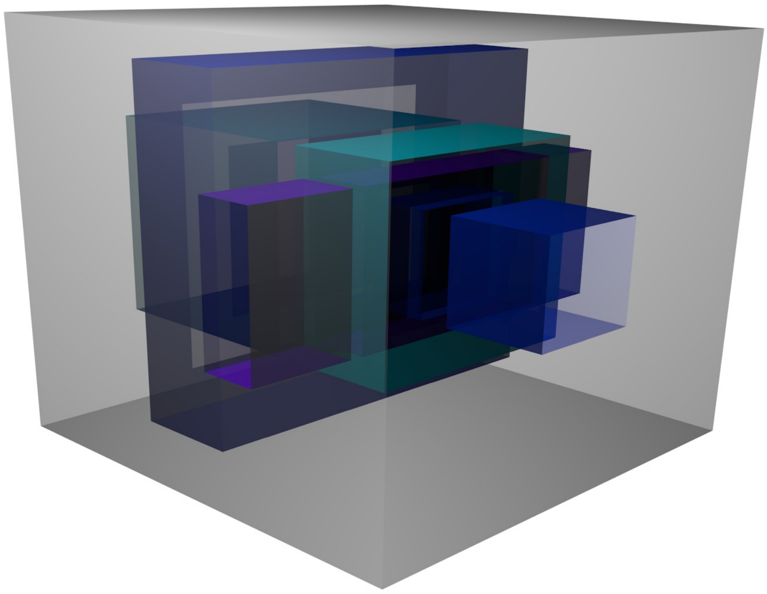


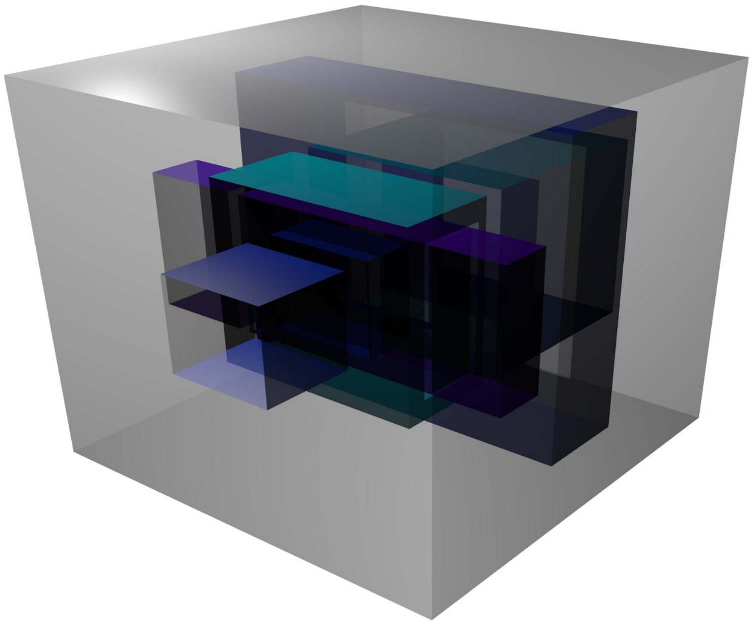
This example has 5 crossing, which means it has $2^5 = 32$ possible end states, if it cannot be reduced by some combination of Homma-Nagase-Roseman moves along the way.

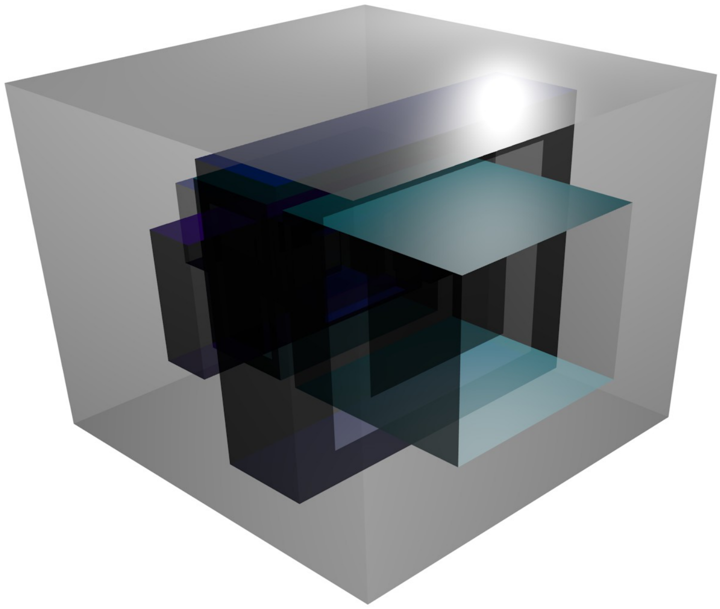
This example can be generalised to include an arbitrary number of hoovers.

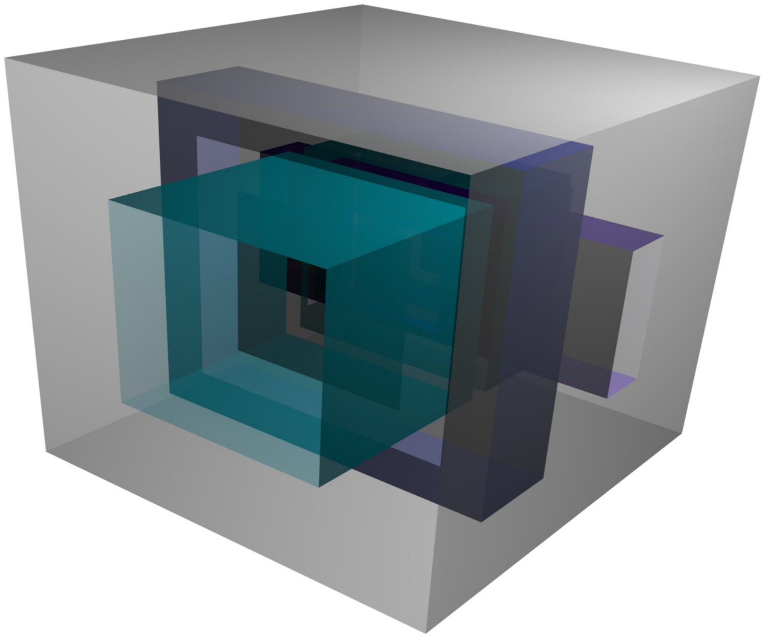
B.3. The convoluted hoover with an extra tunnel

This example is simply put, the convoluted hoover example from Appendix A with an extra tunnel running through it. The important thing is that the tunnel is larger than the small dark blue tunnel, but smaller than the light blue, such that the new purple tunnel gets locked in. The new tunnel is in fact locked since we can not apply any sequence of Homma-Nagase-Roseman moves to reduce it. Below you find the configuration from four different angles.









This example has 5 crossing as well, which means it too has $2^5 = 32$ possible end states, if it cannot be reduced by some combination of Homma-Nagase-Roseman moves along the way.

This example can be generalised to include an arbitrary number of tunnels.

BIBLIOGRAPHY

[1] J. S. Carter, *How surfaces intersect in space. An introduction to topology*, vol. 2 of Series on Knots and Everything, World Scientific, Singapore, 1995.

[2] J. S. Carter, S. Kamada, and M. Saito, *Surfaces in 4-spaces*, vol. 142 of Encyclopedia of Mathematical Sciences, Springer-Verlag, Berlin, 2004.

[3] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, vol. 55 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1998.

[4] T. Homma and T. Nagase, *On elementary deformations of maps of surfaces into 3-manifolds. I*, Yokohama Math. J., 33 (1985), pp. 103–119.

[5] ——, *On elementary deformations of maps of surfaces into 3-manifolds. II*, in Topology and computer science (Atami, 1986), Tokyo, 1987, Kinokuniya, pp. 1–20.

[6] S. Kamada, *Braid and knot theory in dimension four*, vol. 95 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2002.

[7] L. H. Kauffman, *New invariants in the theory of knots*, Amer. Math. Monthly, 95 (1988), pp. 195–242.

[8] ——, *Knot diagrammatics*, in Handbook of knot theory, W. Menasco et al., eds., Elsevier, Amsterdam, 2005, pp. 233–318. arXiv:math/0410329.

[9] K. Reidemeister, *Knotenteorie*, Springer-Verlag, Berlin-New York, 1974. Reprint of the 1932 original.

[10] D. Roseman, *Reidemeister-type moves for surfaces in four-dimensional space*, in Knot theory. Proceedings of the mini-semester, Warsaw, Poland, July 13 – August 17, 1995, V. F. R. Jones et al., eds., vol. 42, Warszawa: Polish Academy of Sciences, Institute of Mathematics, 1998, Banach Cent. Publ., pp. 347–380.

[11] R. Wiliamson, *General topology lecture notes*, 2013. Lecture 20 and Lecture 21.