

Robust Transmit Power Control for Cognitive Radio

An algorithm is proposed, designed to dynamically control transmission power for maximum advantage in a wireless network where systems compete for limited resources.

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ABSTRACT | A cognitive radio network is a multiuser system, in which different users compete for limited resources in an opportunistic manner, interacting with each other for access to the available resources. The fact that both users and spectrum holes (i.e., unused spectrum subbands) can come and go makes a cognitive radio network a highly dynamic and challenging wireless environment. Therefore, finding robust resource-allocation algorithms, which are capable of achieving reasonably good solutions fast enough in order to guarantee an acceptable level of performance under worst case interference conditions, is crucial in such environment. The focus of this paper is the transmit-power control in cognitive radio networks, considering a noncooperative framework. Moreover, tools from control theory are used to study both the equilibrium and transient behaviors of the network under dynamically varying conditions.

KEYWORDS | Cognitive radio; functional differential equations (FDEs); hybrid dynamic systems (HDSs); iterative waterfilling algorithm (IWFA); piecewise affine (PWA) systems; projected dynamic systems (PDSs); robust optimization; transmit-power control; variational inequalities (VIs)

I. INTRODUCTION

In signal-processing terms, a feature that distinguishes cognitive radio from conventional wireless communication is the cognitive signal-processing cycle [1], [2]. This cycle applies to a secondary (unserved) user, where a transmitter at one location communicates with a receiver at some other location via a spectrum hole, that is, a licensed subband of the radio spectrum that is underutilized at a

particular point in time and at a particular location. The cognitive cycle encompasses two basic operations: radio-scene analysis of the surrounding wireless environment at the receiver and dynamic spectrum management/transmit-power control at the transmitter. Information on spectrum holes, extracted by the scene-analyzer at the receiver, is sent to the transmitter via a feedback channel. The combination of the radio-scene analyzer, the feedback channel, the dynamic spectrum manager/transmit-power controller, and the wireless link constitutes a closed-loop feedback system as depicted in Fig. 1 [1], [2]. In this paper, we focus attention on the theoretical and practical issues involved in the study of the transmit-power controller.

The operation of the transmit-power controller is complicated by a phenomenon that is peculiar to cognitive radio communication: the fact that spectrum holes come and go, depending on the availability of subbands as permitted by licensed users. To deal with this phenomenon and thereby provide the means for improved utilization of the radio spectrum, a cognitive radio system must have the ability to fill the spectrum holes rapidly and efficiently. In other words, cognitive radios have to be frequency-agile radios with flexible spectrum shaping abilities. The orthogonal frequency-division multiplexing (OFDM) modulation scheme can provide the required flexibility, and is therefore a good candidate for cognitive radio [1]–[5]. OFDM can be employed in a cognitive radio network by dividing the primary user's unused bandwidth into a number of subbands available for use by the cognitive radio systems. In order to achieve low mutual interference between primary and secondary users, an adaptive transmit filter can be used to prevent usage of a set of subcarriers, which are being used by the primary users. Moreover, the fast Fourier transform (FFT) block in the OFDM demodulator (Fig. 2) can be used for spectral analysis [3].

There are two primary resources in a cognitive radio network: channel bandwidth and transmit power. Using the OFDM-based modulation scheme, the bandwidth allocation

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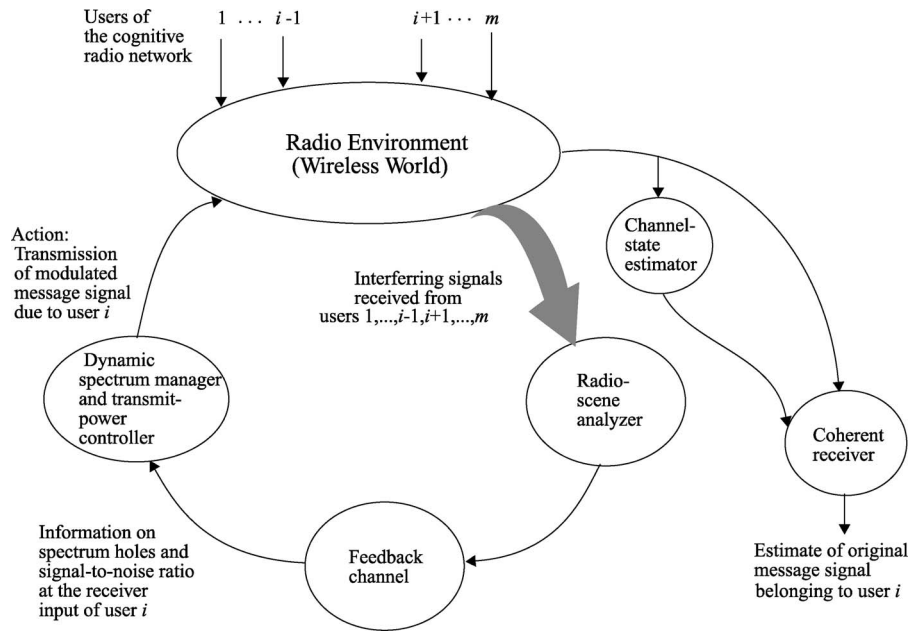


Fig. 1. Basic signal-processing cycle for user i in a cognitive radio network; the diagram also includes elements of the receiver of user i .

can be considered as a subcarrier assignment problem [6]. The resource management problem may then consist of subcarrier assignment and power control. While the availability of channel bandwidth depends on the communication patterns of primary users, a cognitive radio has complete control over its own transmit power. In other words, among the two primary resources, power is the only variable that can be manipulated by cognitive radio users. As mentioned previously, a subcarrier will not be assigned to a cognitive radio if its transmit power at that subcarrier is zero. Therefore, the resource-allocation problem can be reduced to the transmit power control and can be considered as a dis-

tributed control problem. Scalable decentralized algorithms with reasonable computational complexity are naturally preferred.

There are two ways to build a cognitive radio network: evolutionary and revolutionary. In the evolutionary viewpoint, the currently established communication infrastructures can be utilized, and cognitive radio networks are built around existing base stations. In this framework, the base stations or spectrum brokers [7] are responsible for assigning channels to cognitive radios; well-known algorithms proposed in the multicellular network literature for distributed and jointly optimal data rate and power control

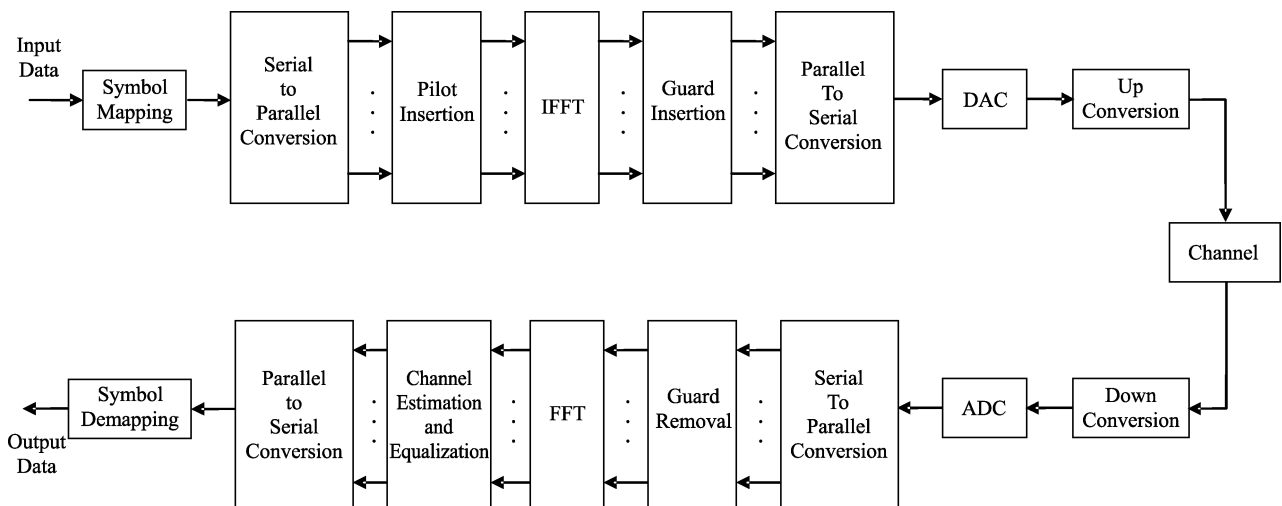


Fig. 2. Block diagram of an OFDM transceiver.

can be employed [8]–[14]. On the other hand, in the revolutionary viewpoint, which is the focus of this paper, there are no base stations or communication infrastructures; hence, channel assignment and power control would have to be performed jointly.

In a cognitive radio network, at any instant of time, new users may join the network or old users may leave the network. Also, primary users may start or stop communication, and therefore they may occupy or release some frequency bands in an uncertain manner. All of these occurrences can be considered as discrete events compared to the real-time evolution of each user's power vector, which can be considered as evolving in continuous time. It follows therefore that the cognitive-radio problem is a mixture of continuous dynamics and discrete events. In other words, a cognitive radio network is a hybrid dynamic system of the sort described in [15]. In this paper, the resource-allocation problem for worst case interference scenarios is considered in a dynamic framework, and both equilibrium and disequilibrium (transient) behaviors of the cognitive radio network are studied. Also, tools from control theory are utilized to investigate the stability and sensitivity of equilibrium points to external perturbations.

In the following sections, after a brief review of OFDM modulation scheme, a discussion on the constraints imposed by the cognitive-radio environment is followed by describing the attributes of the iterative waterfilling algorithm (IWFA) that shows why it is a potentially good choice for transmit-power control in cognitive radio networks. Different formulations of the IWFA are reviewed, and the problem is formulated as a robust game to guarantee an acceptable quality of service (QoS) requirement despite the continuously changing environment. A dynamic model for the evolution of the state of the network is derived to facilitate the study of transient behaviors as well as the stability and sensitivity analysis. Theoretical discussions are validated by simulations of the network, followed by concluding remarks and comments on future directions for the research. The Appendix identifies symbols used in this paper.

II. ORTHOGONAL FREQUENCY-DIVISION MULTIPLEXING (OFDM)

OFDM is a multicarrier modulation scheme, in which a wide-band signal is converted to a number of narrowband signals. Then closely spaced orthogonal subcarriers are used to transmit these narrowband data segments simultaneously. In effect, a frequency-selective fading channel is divided into a number of narrow-band flat fading subchannels. OFDM has many advantages over single-carrier transmission [6], [16]–[19].

- It improves the efficiency of spectrum utilization by the simultaneous use of multiple orthogonal subcarriers, which are densely packed.
- The OFDM waveform is first built in the frequency domain, and then it is transformed into the time

domain, thereby providing flexible bandwidth allocation.

- *Interleaving* the information over different OFDM symbols provides robustness against loss of information caused by flat-fading and noise effects.
- Although the spectrum tails of subcarriers overlap with each other, at the center frequency of each subcarrier all other subcarriers are zero. Theoretically this prevents *intercarrier interference* (ICI). However, time and frequency synchronization is critical for ICI prevention as well as correct demodulation, and is a major challenge in the physical layer design [20].
- Since a narrow-band signal has a longer symbol duration than a wide-band signal, OFDM takes care of *intersymbol interference* (ISI) caused by multipath delay of wireless channels. However, guard time intervals, which are longer than the channel impulse response, are introduced between OFDM symbols to eliminate the ISI by giving enough time for each transmitted OFDM symbol to dissipate considerably [6].
- Due to the low ISI, less complex equalization is required at the receiver, which leads to a simpler receiver structure.

In summary, frequency diversity enables OFDM to provide higher data rates, more flexibility in controlling the waveform characteristics, and greater robustness against channel noise and fading compared to single-carrier transmission schemes.

A. OFDM Modulator

The block diagram of an OFDM transceiver is shown in Fig. 2. Robustness is improved by adopting the *coded OFDM* (COFDM), which employs convolutional coding to encode the binary data used for creating the OFDM signal [21]. Since the focus of this paper is on the transmitter, in what follows, the function of each block in the transmitter side will be explained briefly [6].

1) *Symbol Mapping*: The subcarrier modulation is performed in the first step. The input binary data stream is divided into several groups of bits. The number of bits in each group is determined by the modulation scheme [i.e., binary phase-shift keying (BPSK), quadrature phase-shift keying (QPSK), 16-quadrature amplitude modulation (QAM), etc.]. For instance, this number is one, two, and four, respectively for BPSK, QPSK, and 16-QAM. Each group of bits is then mapped to a signal point on the constellation diagram, i.e., a complex number in the two-dimensional Cartesian coordinates. Accordingly, the first block generates a complex-valued symbol sequence from the input binary data stream.

2) *Serial-to-Parallel Conversion*: The complex-valued symbol sequence generated in the previous stage is broken down and rearranged into shorter sequences in the second

stage. Each of these new short sequences contains N symbols, where N is the number of available subcarriers for data transmission. The elements of these N -symbol sequences are transmitted in parallel by assigning each one of the symbols to a valid subcarrier. This way, symbols are *interleaved* in time and frequency, and therefore the nonsequential rearrangement of the symbols increases the robustness of OFDM signals against channel noise and fading.

3) *Pilot Subcarrier Insertion*: In addition to data-carrying subcarriers, there are some *pilot subcarriers*, which are not used for data transmission. To be specific, in an effort to avoid time and frequency errors between the transmitter and receiver, some predetermined *pilot symbols* are sent over the pilot subcarriers. Pilot symbols are selected from signal points on the constellation diagram that have maximum distance from the origin of the complex plane, and therefore have higher power compared to data symbols. Thus, they are distinguished from data symbols, and there is a higher probability that receiver receives them. Since the pilot symbols are known to the receiver beforehand, the receiver will be able to estimate the effect of the communication channel on the transmitted signal by comparing the received symbols with the pilot symbols.

4) *Inverse Fast Fourier Transform (IFFT)*: As mentioned before, the OFDM signal is created first in the frequency domain and then is converted to a time-domain signal. Since OFDM uses sinusoidal basis functions and the transmitted pulse is a rectangular-windowed sinusoid, the spectrum of a subcarrier is a *sinc* function. The main lobe of each subcarrier occupies a small proportion of the spectrum, and therefore characteristics of a frequency-selective channel can be considered uniform within a subcarrier bandwidth. Thus the wide-band frequency selective channel can be considered as a set of narrow-band flat fading subchannels.

The center frequency of each subcarrier is an integer multiple of a basis frequency, which is determined by the sampling rate of the transceiver. In other words, subcarriers are harmonics and therefore mutually orthogonal. The orthogonality property allows the subcarriers to be as close as possible so as to provide high spectral efficiency.

First, subcarriers are assigned to sequential FFT bins, and no FFT bin is left empty between subcarriers in order to achieve maximal spectral efficiency. This process is called *subcarrier insertion* [6]. Then, imaginary subcarriers, which are complex conjugates of real subcarriers, are inserted into the FFT array. An extra subcarrier is needed to represent the dc component. To reduce computational complexity, the FFT size is chosen to be an even number. This way, an OFDM signal is created in the frequency domain, which leads to a real-valued time-domain signal. It is obvious that less than half of the FFT bins will be available as subcarriers to represent the frequency range $[0, f_s/2]$, where f_s is the sampling frequency. If imaginary subcarriers are not used, then all of the FFT bins except the

one that corresponds to the dc component can be used as subcarriers. This results in a complex time-domain signal, which might be attractive from a maximum bandwidth efficiency viewpoint, although it is more computationally demanding.

The total number of subcarriers, including data-carrying subcarriers and pilot subcarriers as well as null subcarriers, are multiplexed, and the IFFT is then employed to transform the frequency-domain OFDM signal into the time domain. The resulting time-domain waveform representation is called an *OFDM symbol* [6].

5) *Guard-Interval Insertion*: Because of multipath fading phenomenon, the receiver receives several delayed copies of the transmitted signal. The delayed versions of the previous symbol affect the beginning of the current symbol and cause ISI, and possible loss of orthogonality. In order to mitigate this problem, the worst case delay caused by different propagation paths is considered, which is called *delay spread*. Guard intervals longer than the delay spread are inserted between OFDM symbols, obtained in the previous stage, to protect the beginning of each symbol from degradation by ISI. In other words, each OFDM symbol is extended, and the beginning of each symbol is shifted by a time interval larger than the delay spread.

Since, in reality, signals are continuous, instead of leaving the guard intervals empty, the end of the symbol is copied to these intervals, which is called *cyclic prefix*. However, while inclusion of cyclic prefixes results in an increase in OFDM transmission bandwidth and decreases throughput, it effectively reduces the effects of ISI and ICI. Therefore, spectral efficiency is traded for robustness. In the receiver, these guard intervals, and with them the cyclic prefixes, are removed to obtain the original symbols [6].

6) *Parallel-to-Serial Conversion*: The parallel-to-serial converter block uses the OFDM symbols prepended with cyclic prefixes, to create a serial sequence, which is called an *OFDM frame*. OFDM frames are separated by frame intervals that are larger than the guard intervals between OFDM symbols. These frame intervals are used for time and frequency synchronization between the transmitter and receiver as well as initial channel equalization. An OFDM frame is therefore a sequence of original OFDM symbols separated by guard intervals and its size is determined by the amount of data that should be transmitted [6].

7) *Digital-to-Analog Conversion*: The digital information is mapped into a waveform by the digital-to-analog converter block for transmission over the communication channel. Similarly, in the receiver, the received waveform is mapped into digital information by an analog-to-digital converter.

8) *Up-Conversion*: In the final stage, the baseband OFDM signal, created in the previous stages, is up-converted to

the passband of interest by mixing it with an appropriate radio-frequency carrier. A spectrum mask is used to limit the spectral content of the signal, and then it is linearly amplified and transmitted.

B. Constant Envelope OFDM (CE-OFDM)

Summation of the sinusoidal basis functions and therefore the resulting OFDM signal has a highly varying amplitude with probable large peaks, which results in a high peak-to-average power ratio (PAPR).

In order to avoid waveform distortions and therefore performance degradation, system components and especially transmitter power amplifier are needed to have a large linear range. The peak power of the input OFDM signal is maintained at a level less than the saturation point of the power amplifier so as to prevent distortion of the peaks. To this end, an input *power backoff* is needed, which is at least equal to the PAPR. Increasing the power backoff results in poor efficiency and reduces the battery lifetime of mobile devices [22].

In [22], it was proposed to use the OFDM waveform to phase modulate the carrier. This way the OFDM signal is transformed to a constant-amplitude signal with a PAPR of 0 dB, which is appropriate for efficient power amplification. In the transmitter, the phase modulator block is inserted between the IFFT block and the guard insertion block in Fig. 2. Constant envelope OFDM requires a real-valued OFDM signal, and therefore it should be implemented using conjugate symmetric subcarriers, as mentioned previously.

III. COGNITIVE RADIO ENVIRONMENT

In a cognitive radio, the receiver performs radio-scene analysis of the surrounding environment and sends the information extracted about the forward communication channel and availability of spectrum holes to the transmitter via a feedback channel. This information enables the transmitter to adaptively adjust the transmitted signal and update its transmit power. Using a predictive model, the cognitive radio is enabled to predict the availability duration of spectrum holes, which, in turn, determines the horizon of the transmit-power control [1], [2].

The feedback channel will naturally introduce some delay in the control loop, and some of the users may use inaccurate or outdated interference measurements to update their transmit powers. Also, they may update their transmit powers at different frequencies. Therefore, in a real-life situation, the resource-allocation algorithm would have to be implemented in a distributed asynchronous manner [9], [23]–[25].

In a competitive multiagent environment with limited resources, such as a cognitive radio network, where the actions of all agents (users) are coupled via available resources, finding a global optimum for the resource-allocation problem can be computationally intractable and time consuming. Moreover, such optimization would

require huge amounts of information exchange between different users that will consume precious resources. In a highly dynamic environment where both users and resources can freely come and go, finding a reasonably good solution (i.e., a suboptimal solution) that can be obtained fast enough is the only practical goal. Otherwise, spectrum holes may disappear before they can be utilized for communication. In such a situation, the concept of equilibrium is very important [26]. It is therefore not surprising that *game theory* has attracted the attention of many researchers in the field of communication networks. A nice survey on applications of game theory in wired communication systems is presented in [27]. For wireless communication systems and cognitive radio, the reader can refer to [28]–[34] and the references therein.

In game theory, the *Nash equilibrium* is considered to be a concept of fundamental importance. This equilibrium point is a solution such that none of the agents has an incentive to deviate from it unilaterally. In other words, in a Nash-equilibrium point, each user's chosen strategy is the best response to the other users' strategies [35]–[37]. Regarding the highly time-varying nature of communication networks in general and especially cognitive radio networks, a Nash-equilibrium solution is a reasonable candidate, even though it may not always be the best solution in terms of spectral efficiency [38].

The above discussion reveals that several key attributes such as distributed implementation, low complexity, and fast convergence to a reasonably good solution, provide an intuitively satisfying framework for choosing and designing resource-allocation algorithms for cognitive radio. It is with this kind of framework in mind that in [1] and [2], the IWFA has been proposed as a good candidate for finding a Nash-equilibrium solution for resource allocation in cognitive radio networks.

The IWFA was originally developed for digital subscriber lines (DSLs) [39]–[41]. In this algorithm, users sequentially update their transmit power vectors over the available frequency tones in a fixed updating order, considering the transmit power of other users as interference. The sequential nature of the algorithm requires some form of central scheduling to determine the order in which users update their transmit powers [24], [25]. In a cognitive radio network, which is an infrastructure-less network, such a central scheduling does not exist, and synchronization between the nodes is difficult. Therefore, users update their transmit powers in a totally asynchronous manner. The IWFA is well suited for cognitive radio networks. In particular, the practical virtues of the algorithm are the following.

- The transmit power control problem is formulated as a *game* or a distributed *convex optimization* problem.
- It is implemented in a *decentralized* manner.
- The algorithm has a *linear convergence* property under certain conditions [42].

- Each user acts greedily to optimize its own performance based on local information, and the users do not need to communicate with each other to establish coordination between themselves. This tends to *reduce the complexity* of the cognitive radio network.

Finding a Nash equilibrium for the DSL game was reformulated as a *nonlinear complementarity problem* (NCP) in [43]. In an NCP, the vector $\mathbf{x} \in \mathbb{R}^n$ should be found such that

$$\mathbf{x} \geq 0, \quad \mathbf{F}(\mathbf{x}) \geq 0, \quad \mathbf{x}^T \mathbf{F}(\mathbf{x}) \geq 0 \quad (1)$$

where \mathbf{F} is a nonlinear mapping from \mathbb{R}^n to \mathbb{R}^n . The problem will be a *linear complementarity problem* (LCP) if $\mathbf{F} = \mathbf{M}\mathbf{x} + \mathbf{q}$ for a matrix \mathbf{M} and a vector \mathbf{q} with appropriate dimensions [44]. In [42], the DSL game problem was reformulated as an LCP. Reformulation of the IWFA as an NCP and an LCP provides very interesting insights into this problem, such as establishing the linear convergence under certain conditions on interference gains. Also, conditions on interference gains are obtained to guarantee convergence of the algorithm to a unique Nash-equilibrium point [24], [25], [40], [42]. However, the algorithm has some drawbacks.

- It is suboptimal.
- It is defenseless against clever selfish users that try to exploit dynamic changes or limited resources.

Moreover, regarding the dynamic nature of the cognitive radio environment and the speed of changes, the current transmit power values may not provide a good initial point for the next iteration. In this case, it may be better to start the iterative procedure from a randomly picked initial point in the new feasible set.

In what follows, the resource-allocation problem in cognitive radio networks is presented in the IWFA framework. While the predictive model can help for dealing with the appearance and disappearance of spectrum holes, making the algorithm more robust is proposed to address the issues related to unavoidable changes in the number of users and their mobility.

IV. ITERATIVE WATERFILLING ALGORITHM (IWFA)

This section presents formulation of the transmit power control for cognitive radio in the IWFA framework. Assume that there are m active cognitive radio transmitter–receiver pairs in the region of interest and n subcarriers in an OFDM framework that potentially could be available for communication. Let PS denote the subset of subcarriers that are being used by primary users and cannot be assigned to cognitive radios. Since spectral efficiency is the main goal of cognitive radio, the utility function chosen by each user

to be maximized is the data rate. Thus, the IWFA lets user i solve the following optimization problem:

$$\begin{aligned} \max_{\mathbf{p}^i} \quad & f^i(\mathbf{p}^1, \dots, \mathbf{p}^m) = \sum_{k=1}^n \log \left(1 + \frac{p_k^i}{I_k^i} \right) \\ \text{subject to:} \quad & \sum_{k=1}^n p_k^i \leq p_{\max}^i \\ & p_k^i + I_k^i \leq CAP_k, \quad \forall k \notin PS \\ & p_k^i = 0, \quad \forall k \in PS \\ & p_k^i \geq 0 \end{aligned} \quad (2)$$

where p_k^i denotes user i 's transmit power over the subcarrier k . The noise plus interference experienced by user i at subcarrier k because of the transmission of other users is

$$I_k^i = \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} p_k^j. \quad (3)$$

Since cognitive radio is *receiver centric*, I_k^i is measured at receiver i .

The positive parameter σ_k^i is the normalized background noise power at the receiver input of user i on the k th subcarrier. The nonnegative parameter α_k^{ij} is the normalized interference gain from transmitter j to receiver i at subcarrier k , and we have $\alpha_k^{ii} = 1$. The term α_k^{ij} is the combined effect of two factors:

- propagation path-loss from transmitter j to receiver i at subcarrier k ;
- subcarrier amplitude reduction due to the frequency offset Δf .

Mathematically, α_k^{ij} is defined as

$$\alpha_k^{ij} = \frac{\Gamma |h_k^{ij}|^2}{|h_k^{ii}|^2} \quad (4)$$

where Γ is the signal-to-noise ratio (SNR) gap and h_k^{ij} is the channel gain from transmitter j to receiver i over the flat-fading subchannel associated with subcarrier k . Regarding the empirical formula for the path loss [45], we have

$$|h_k^{ij}|^2 = \frac{\beta_k}{(d_{ij})^r} \quad (5)$$

where d_{ij} is the distance from transmitter j to receiver i . The *path-loss exponent*, r varies from two to five, depending

on the environment, and the attenuation parameter β_k is frequency dependent. Therefore

$$\alpha_k^{ij} \propto \left(\frac{d_{ii}}{d_{ij}} \right)^r \quad (6)$$

and in general $\alpha_k^{ij} \neq \alpha_k^{ji}$. If user i 's receiver is closer to its transmitter compared to other active transmitters in the network, we will have $\alpha_k^{ij} \leq 1$.

Also, p_{\max}^i is user i 's maximum power and CAP_k is the maximum allowable interference at subcarrier k . CAP_k is determined in a way to make sure that the permissible interference power level limit will not be violated at the primary users' receivers [1], [2]. The formulation presented in (2) and (3) is built on [2] except for the fact that the subcarriers do not need to be confined to the same subband, thereby providing for broader applicability of the IWFA.

Regarding the first two constraint inequalities in (2), we follow an approach similar to [42] and assume that

$$\sum_{j=1}^m p_{\max}^j < \sum_{k \notin PS} (CAP_k - \sigma_k^i), \quad \forall i = 1, \dots, m \quad (7)$$

in order to avoid triviality. As will be shown later, this assumption leads to a nice reformulation of IWFA under the worst case interference condition similar to [42]. The previously mentioned properties of IWFA are further elaborated in the following based on the mathematical formulation of (2).

In IWFA, user i assumes that p_j^k is fixed for $j \neq i$. Therefore, the optimization problem in (2) is a concave maximization problem in $\mathbf{p}^i = [p_1^i, \dots, p_n^i]^T$, which can be converted to a convex minimization problem by considering $-f^i$ as the objective. The first constraint states that the total transmit power of the user i at all subcarriers should not exceed its maximum power (power budget). The second constraint set guarantees that the interference caused by all users at each subcarrier will be less than the maximum allowed interference at that subcarrier. If primary users do not let the secondary users use the nonidle subcarriers in their frequency bands, then cognitive radios should not use those subcarriers for transmission. The third constraint set guarantees this by forcing the related components of the secondary user i 's power vector to be zero. If primary users let the coexistence of secondary users at nonidle subcarriers in condition that they do not violate the permissible interference power level, then the third constraint set can be relaxed and the second constraint set suffices.

As mentioned previously, IWFA is implemented in a decentralized manner. In order to solve the optimization problem (2), it is not necessary for user i to know the value of p_j^k for $\forall j \neq i$. The I_k^i defined in (3) is measured by user i 's

receiver rather than calculated, and therefore users do not need to exchange information. It is not even necessary for user i to know the number of other users in the network. Therefore, changing the number of users in the network does not affect the complexity of the optimization problem that should be solved by each user. Hence, there is no scaling problem.

While the action of user i is denoted by its power vector \mathbf{p}^i , following the notation in the game theory literature, the joint actions of the other $m - 1$ users are denoted by \mathbf{p}^{-i} . Three major types of adjustment schemes \mathcal{S} can be used by the users to update their actions [46].

- i) Iterative waterfilling [39]–[41]: users update their actions in a predetermined order [46]

$$\mathbf{p}^{-i}(\mathcal{S}_t) = [\mathbf{p}^1(t+1), \dots, \mathbf{p}^{i-1}(t+1), \mathbf{p}^{i+1}(t), \dots, \mathbf{p}^m(t)]. \quad (8)$$

- ii) Simultaneous iterative-waterfilling [24]: users update their actions simultaneously regarding the most recent actions of the others [46]

$$\mathbf{p}^{-i}(\mathcal{S}_t) = \mathbf{p}^{-i}(t). \quad (9)$$

- iii) Asynchronous iterative-waterfilling [25] is an instance of an adjustment scheme that user i receives update information from user j at random times with delay [46]

$$\mathbf{p}^{-i}(\mathcal{S}_t) = [\mathbf{p}^1(\tau_t^{i,1}), \dots, \mathbf{p}^{i-1}(\tau_t^{i,i-1}), \mathbf{p}^{i+1}(\tau_t^{i,i+1}), \dots, \mathbf{p}^m(\tau_t^{i,m})] \quad (10)$$

where $\tau_t^{i,j}$ is an integer-valued random variable satisfying

$$\max(0, t-d) \leq \tau_t^{i,j} \leq t+1 \quad j \neq i \quad i, j \in \mathbb{N} \quad (11)$$

which means that the delay does not exceed d time units.

Due to the lack of central scheduling and difficulty of synchronization between different users in a cognitive radio network, the asynchronous adjustment scheme is more realistic than the others.

V. STOCHASTIC OPTIMIZATION VERSUS ROBUST OPTIMIZATION

The cognitive radio environment has a dynamic nature, in which users move all the time; they can leave the network,

and new users can join the network in a stochastic manner. Because of these activities, the interference-plus-noise term (3) in the objective function and the second constraint set of the optimization problem (2) are both time-varying; the IWFA therefore assumes the form of an optimization problem under uncertainty. Also, the appearance and disappearance of the spectrum holes, which depend on the activities of primary users, will change the third constraint set in (2). The behavior of the primary users, and therefore the availability as well as the duration of availability of spectrum holes, can be predicted to some extent using a predictive model. During the time intervals that the activity of primary users does not change and the available spectrum holes are fixed, two approaches can be taken to deal with the uncertainty caused by joining and leaving of other cognitive radios as well as their mobility: *stochastic optimization* and *robust optimization* [47]. The pros and cons of these two approaches are discussed here.

Let the noise-plus-interference term be the summation of two components: a nominal term \bar{I} and a perturbation term ΔI , as

$$I_k^i = \bar{I}_k^i + \Delta I_k^i. \quad (12)$$

In the following, the objective functions for both stochastic and robust versions of the optimization problem (2) are presented.

If there is good knowledge about the probability distribution of the uncertainty term ΔI , then the uncertainty can be dealt with by means of probability and related concepts. In this case, calculation of the expected value will not be an obstacle, and therefore the IWFA problem (2) can be formulated as a stochastic optimization problem with the following objective function:

$$\max_{\mathbf{p}^i} \left[\mathbb{E}_{\Delta \mathbf{I}^i} \sum_{k=1}^n \log \left(1 + \frac{p_k^i}{\bar{I}_k^i + \Delta I_k^i} \right) \right] \quad (13)$$

where \mathbb{E} denotes the statistical expectation operator and

$$\Delta \mathbf{I}^i = [\Delta I_1^i, \dots, \Delta I_n^i]^T. \quad (14)$$

However, since in practice, little may be known about the probability distribution, the stochastic optimization approach that utilizes the expected value is not a suitable approach. In this case, robust optimization techniques that are based on the worst case analysis, without involving probability theory, are more appropriate, although such techniques may well be overly conservative in practice. Suboptimality in performance is traded in favor of robustness. The formulation of IWFA as a robust game

in the sense described in [48] is basically a max-min problem, in which each user tries to maximize its own utility while the environment and the other users are trying to minimize that user's utility [49], [50]. Worst case interference scenarios have been studied for DSL in [51]. Considering an ellipsoidal uncertainty set, the IWFA problem (2) can be formulated as the following robust optimization problem:

$$\begin{aligned} \max_{\mathbf{p}^i} \quad & \left[\min_{\|\Delta \mathbf{I}^i\| \leq \varepsilon} \sum_{k=1}^n \log \left(1 + \frac{p_k^i}{\bar{I}_k^i + \Delta I_k^i} \right) \right] \\ \text{subject to:} \quad & \sum_{k=1}^n p_k^i \leq p_{\max}^i \\ & \max_{\|\Delta \mathbf{I}^i\| \leq \varepsilon} (p_k^i + \bar{I}_k^i + \Delta I_k^i) \leq \text{CAP}_k, \quad \forall k \notin \text{PS} \\ & p_k^i = 0, \quad \forall k \in \text{PS} \\ & p_k^i \geq 0. \end{aligned} \quad (15)$$

A larger ε accounts for larger perturbations, and the second set of constraints guarantee that the permissible interference power level will not be violated for any perturbation from the considered uncertainty set.

Stochastic optimization guarantees some level of performance on average, and sometimes the desired quality of service may not be achieved, which means a lack of reliable communication. On the other hand, robust optimization guarantees an acceptable level of performance under the worst case conditions. It is a conservative approach because real-life systems are not always in their worst behavior, but it can provide seamless communication even in the worst situations. Regarding the dynamic nature of the cognitive radio network and the delay introduced by the feedback channel, the statistics of interference that is used by the transmitter to adjust its power may not represent the current situation of the network. In these cases, robust optimization is equipped to prevent permissible interference power level violation by taking into account the worst case uncertainty in the interference and noise. Therefore, sacrificing optimality for robustness seems to be a reasonable proposition. However, the use of a predictive model may make it possible for the user to choose the uncertainty set adaptively according to environmental conditions and, therefore, may lead to less conservative designs.

A. The Cost of Robustness

In addition to conservatism, there is yet another price to be paid for achieving robustness. Although the IWFA problem (2) is a convex optimization problem, appearance of the perturbation term ΔI in the denominator of signal-to-(interference plus noise) ratio (SINR) in the objective function of the robust IWFA problem (15) makes it a nonconvex optimization problem. A robust optimization

technique is proposed in [52] for solving nonconvex and simulation-based problems. The proposed method is based on the assumption that the cost and constraints as well as their gradient values are available. The required values can even be provided by numerical simulation subroutines. It operates directly on the surface of the objective function, and therefore does not assume any specific structure for the problem. In this method, the robust optimization problem is solved in two steps, which are applied repeatedly in order to achieve better robust designs.

- *Neighborhood search:* The algorithm evaluates the worst outcomes of a decision by obtaining knowledge of the cost surface in a neighborhood of that specific design.
- *Robust local move:* The algorithm excludes neighbors with high costs and picks an updated design with lower estimated worst case cost. Therefore, the decision is adjusted in order to counterbalance the undesirable outcomes.

Linearity of constraints of the robust optimization problem (15), especially the second set of constraints that involve the perturbation terms, improves the efficiency of the algorithm.

VI. IWFA AS A VARIATIONAL INEQUALITY (VI) PROBLEM

A Nash-equilibrium game can be reformulated as a variational inequality (VI) problem [53], [54]. To be specific, denoting the feasible set of (2) by K^i , we may rewrite the optimization problem (2) as

$$\begin{aligned} \min_{\mathbf{p}^i} \quad & -f^i(\mathbf{p}^1, \dots, \mathbf{p}^m) \\ \text{subject to: } & \mathbf{p}^i \in K^i \end{aligned} \quad (16)$$

where K^i is a closed convex subset of \mathbb{R}^n and $-f^i$ is a convex and continuously differentiable function for $i = 1, \dots, m$. Then, $\mathbf{p}^* = [\mathbf{p}^{*1^T}, \dots, \mathbf{p}^{*m^T}]^T$ is a Nash equilibrium of the game if, and only if, it is a solution of the following VI problem $\text{VI}(K, \mathbf{F})$:

$$(\mathbf{p} - \mathbf{p}^*)^T \mathbf{F}(\mathbf{p}^*) \geq 0 \quad (17)$$

where

$$K = \left\{ \mathbf{p} \in \mathbb{R}^{m \times n} \mid \begin{aligned} & p_k^i = 0 \quad \forall k \in PS, \forall i = 1, \dots, m; \\ & 0 \leq p_k^i + I_k^i \leq CAP_k \quad \forall k \notin PS, \forall i = 1, \dots, m; \\ & \sum_{k=1}^n p_k^i \leq p_{\max}^i, \forall i = 1, \dots, m \end{aligned} \right\} \quad (18)$$

and

$$\mathbf{F}(\mathbf{p}) = [\nabla_{\mathbf{p}} f^i]_{i=1}^m. \quad (19)$$

Calculating the gradients in (19) leads to fractional terms with the sum of the power and interference plus noise in the denominators

$$\begin{aligned} \nabla_{\mathbf{p}} f^i &= \left[\frac{1}{p_1^i + I_1^i}, \dots, \frac{1}{p_n^i + I_n^i} \right]^T \\ &= \left[\frac{1}{\sigma_1^i + \sum_{j=1}^m \alpha_{1j}^{ij} p_1^j}, \dots, \frac{1}{\sigma_n^i + \sum_{j=1}^m \alpha_{nj}^{ij} p_n^j} \right]^T. \end{aligned} \quad (20)$$

Alternatively, following the approach of [42], a nice formulation of the IWFA as a VI problem is obtained for the worst case interference that facilitates study of the network in a dynamic framework. The discussion presented in this section is built on [42] and extends the proposed reformulation of IWFA to the cognitive-radio problem. In particular, we are allowed to utilize some existing mathematical tools and also benefit from ongoing research in other fields. The *Lagrangian* of the optimization problem in (2) for the user i is now written as

$$\begin{aligned} L^i(\mathbf{p}^1, \dots, \mathbf{p}^m) &= -f^i + u^i \left(\sum_{k=1}^n p_k^i - p_{\max}^i \right) \\ &+ \sum_{k \notin PS} \gamma_k^i \left(\sigma_k^i + \sum_{j=1}^m \alpha_{kj}^{ij} p_k^j - CAP_k \right) + \sum_{k \in PS} \lambda_k^i p_k^i. \end{aligned} \quad (21)$$

Therefore, we have

$$\begin{cases} \gamma_k^i = 0 & k \in PS \\ \lambda_k^i = 0 & k \notin PS. \end{cases} \quad (22)$$

The Karush–Kuhn–Tucker (KKT) conditions [55], [56] for the user i and $\forall k = 1, \dots, n$ are as follows:

$$\begin{aligned} 0 &\leq p_k^i \perp -\frac{1}{\sigma_k^i + \sum_{j=1}^m \alpha_{kj}^{ij} p_k^j} + u^i + \gamma_k^i + \lambda_k^i \geq 0 \\ 0 &\leq u^i \perp p_{\max}^i - \sum_{k=1}^n p_k^i \geq 0 \\ 0 &\leq \gamma_k^i \perp CAP_k - \sigma_k^i - \sum_{j=1}^m \alpha_{kj}^{ij} p_k^j \geq 0, \quad \forall k \notin PS \\ p_k^i &= 0, \quad \forall k \in PS \end{aligned} \quad (23)$$

where \perp signifies orthogonality of the corresponding variables. Similar to [42, Proposition 1], it can be shown that the system described in (23) is equivalent to a *mixed linear complementarity system* (mixture of a linear complementarity problem with a system of linear equations) [57].

Proposition 1: Suppose that (7) holds; then (23) is equivalent to the following mixed linear complementarity system:

$$\begin{aligned} 0 &\leq p_k^i \perp \sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j + \nu^i + \varphi_k^i + \varsigma_k^i \geq 0 \\ 0 &\leq \varphi_k^i \perp CAP_k - \sigma_k^i - \sum_{j=1}^m \alpha_k^{ij} p_k^j \geq 0, \quad \forall k \notin PS \\ p_{\max}^i - \sum_{k=1}^n p_k^i &= 0 \\ p_k^i &= 0, \quad \forall k \in PS. \end{aligned} \quad (24)$$

Proof: Let $(p_k^i, u^i, \gamma_k^i, \lambda_k^i)$ satisfy (23) and assume that the complement set of PS is nonempty. Since power is nonnegative and $\sigma_k^i > 0$, $0 \leq \alpha_k^{ij} \leq 1$, it is known that

$$\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j > 0 \quad \forall k = 1, \dots, n. \quad (25)$$

It can be proved by contradiction that $u^i > 0$. To show this, we first note that $u^i = 0$; then

$$\gamma_k^i + \lambda_k^i \geq \frac{1}{\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j} > 0 \quad \forall k = 1, \dots, n. \quad (26)$$

If $k \notin PS$, then $\lambda_k^i = 0$, and from (26) we must have $\gamma_k^i > 0$. Regarding the third complementarity condition in (23), $\gamma_k^i > 0$ leads to

$$CAP_k - \sigma_k^i - \sum_{j=1}^m \alpha_k^{ij} p_k^j = 0. \quad (27)$$

Therefore, we have

$$CAP_k - \sigma_k^i = \sum_{j=1}^m \alpha_k^{ij} p_k^j. \quad (28)$$

Taking the summation over $k \notin PS$ from both sides of this equation leads to

$$\sum_{k \notin PS} (CAP_k - \sigma_k^i) = \sum_{k \notin PS} \sum_{j=1}^m \alpha_k^{ij} p_k^j \quad (29)$$

$p_k^i = 0$, $\forall k \in PS$, and $\forall i = 1, \dots, m$, so we have

$$\sum_{k \in PS} \sum_{j=1}^m \alpha_k^{ij} p_k^j = 0. \quad (30)$$

Therefore, we can rewrite (29) as

$$\begin{aligned} \sum_{k \notin PS} (CAP_k - \sigma_k^i) &= \sum_{k \notin PS} \sum_{j=1}^m \alpha_k^{ij} p_k^j + \sum_{k \in PS} \sum_{j=1}^m \alpha_k^{ij} p_k^j \\ &= \sum_{k=1}^n \sum_{j=1}^m \alpha_k^{ij} p_k^j. \end{aligned} \quad (31)$$

Since $0 \leq \alpha_k^{ij} \leq 1$, we have

$$\sum_{j=1}^m \alpha_k^{ij} p_k^j \leq \sum_{j=1}^m p_k^j \quad (32)$$

and therefore

$$\sum_{k \notin PS} (CAP_k - \sigma_k^i) \leq \sum_{k=1}^n \sum_{j=1}^m p_k^j. \quad (33)$$

Changing the order of the two summations on the right-hand side of (33), we get

$$\sum_{k \notin PS} (CAP_k - \sigma_k^i) \leq \sum_{j=1}^m \sum_{k=1}^n p_k^j. \quad (34)$$

From the first inequality constraint of (2), we know that

$$\sum_{k=1}^n p_k^j \leq p_{\max}^j \quad (35)$$

Thus

$$\sum_{k \notin PS} (CAP_k - \sigma_k^i) \leq \sum_{j=1}^m p_{\max}^j \quad (36)$$

which contradicts (7). Thus $\forall k \notin PS$ and $\forall i = 1, \dots, m$, in addition to λ_k^i , γ_k^i , must be zero too, and we must therefore have $u^i > 0$ in order to satisfy the first complementary condition in (23). Defining the following variables:

$$\begin{aligned} \nu^i &= -\frac{1}{u^i} \\ \varphi_k^i &= \frac{\gamma_k^i (\sigma_k^i + \sum_{j=1}^m \alpha_{kj}^{ij} p_k^j)}{u^i} \\ \varsigma_k^i &= \frac{\lambda_k^i (\sigma_k^i + \sum_{j=1}^m \alpha_{kj}^{ij} p_k^j)}{u^i} \end{aligned} \quad (37)$$

we do get a solution to (24).

Conversely, assume that $(p_k^i, \nu^i, \varphi_k^i, \varsigma_k^i)$ satisfies (24). This time, we must have $\nu^i < 0$. Otherwise

$$\sigma_k^i + \sum_{j=1}^m \alpha_{kj}^{ij} p_k^j + \nu^i + \varphi_k^i + \varsigma_k^i > 0 \quad (38)$$

and then the first complementarity condition in (24) yields

$$p_k^i = 0, \quad \forall k = 1, \dots, n \quad (39)$$

which contradicts the equality constraint in (24). Therefore, (23) holds by having

$$\begin{aligned} u^i &= -\frac{1}{\nu^i} \\ \gamma_k^i &= -\frac{\varphi_k^i}{\nu^i (\sigma_k^i + \sum_{j=1}^m \alpha_{kj}^{ij} p_k^j)} \\ \lambda_k^i &= -\frac{\varsigma_k^i}{\nu^i (\sigma_k^i + \sum_{j=1}^m \alpha_{kj}^{ij} p_k^j)}. \end{aligned} \quad (40)$$

This completes the proof. \blacksquare

While each user solves the above *mixed linear complementarity problem* (MLCP) with time-varying constraints, they should finally reach an equilibrium. The linear equation in (24) dictates that each user transmits with its maximum power, which leads to the worst case interference condition. Intuitively, it makes sense that each user

transmits with its maximum power in order to achieve maximum data rate.

Let us concatenate the variables in (24) as follows:

$$\mathbf{p} = [\mathbf{p}^i]_{i=1}^m = \begin{bmatrix} \begin{bmatrix} p_1^1 \\ \vdots \\ p_n^1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} p_1^m \\ \vdots \\ p_n^m \end{bmatrix} \end{bmatrix}$$

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}^i]_{i=1}^m = \begin{bmatrix} \begin{bmatrix} \sigma_1^1 \\ \vdots \\ \sigma_n^1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \sigma_1^m \\ \vdots \\ \sigma_n^m \end{bmatrix} \end{bmatrix} \quad (41)$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^{11} & \dots & \mathbf{M}^{1m} \\ \vdots & \dots & \vdots \\ \mathbf{M}^{m1} & \dots & \mathbf{M}^{mm} \end{bmatrix} \quad (42)$$

where \mathbf{M}^{ij} s are diagonal matrices

$$\mathbf{M}^{ij} = \begin{bmatrix} \alpha_1^{ij} & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \alpha_n^{ij} \end{bmatrix}. \quad (43)$$

The MLCP (24) is the KKT condition for an *affine variational inequality* (AVI) problem [54], defined by the affine mapping

$$\mathbf{F}(\mathbf{p}) = \boldsymbol{\sigma} + \mathbf{M}\mathbf{p} \quad (44)$$

and the polyhedron [42]

$$X = \left\{ \mathbf{p} \in \mathbb{R}^{m \times n} \mid \begin{aligned} &p_k^i = 0 \quad \forall k \in PS, \quad \forall i = 1, \dots, m; \\ &0 \leq p_k^i + I_k^i \leq CAP_k \quad \forall k \notin PS, \quad \forall i = 1, \dots, m; \\ &\sum_{k=1}^n p_k^i = p_{\max}^i, \quad \forall i = 1, \dots, m \end{aligned} \right\}. \quad (45)$$

Hence, the IWFA can be reformulated as an AVI problem $VI(X, \sigma + M\mathbf{p})$ or $VI(X, \sigma, \mathbf{M})$. The vector \mathbf{p}^* is a Nash equilibrium point of the IWFA if, and only if, $\mathbf{p}^* \in X$ and $\forall \mathbf{p} \in X$ [42], [54]

$$(\mathbf{p} - \mathbf{p}^*)^T (\sigma + M\mathbf{p}^*) \geq 0. \quad (46)$$

In the next sections, it is shown that the AVI reformulation of IWFA facilitates the study of the disequilibrium behavior and stability analysis of the cognitive radio network.

VII. TRANSIENT BEHAVIOR ANALYSIS OF COGNITIVE RADIO NETWORKS

Although the components of the network may remain unchanged in complex and large-scale networks, the general behavior of the network can change drastically over time. If the SINR of a communication link drops below a specified threshold for a relatively long time, the connection between the transmitter and receiver will be lost. For this reason, in addition to the equilibrium resource allocation, which was discussed in the previous sections, the transient behavior of the network deserves attention too [58]. Therefore, studying the equilibrium states in a dynamic framework by methods that provide information about the disequilibrium behavior of the system is critical, which is the focus of this section.

In the previous sections, IWFA was proposed as an approach to find an equilibrium solution for the resource-allocation problem in cognitive radio networks. Following the approach of [42], the IWFA was reformulated as a VI problem. The *projected dynamic systems* (PDS) theory [53] can be utilized to associate an ordinary differential equation (ODE) to the obtained VI. A projection operator, which is discontinuous, appears on the right-hand side of the ODE to incorporate the feasibility constraints of the VI problem into the dynamics. This ODE provides a dynamic model for the competitive system whose equilibrium behavior is described by the VI. Also, the stationary points of the ODE coincide with the set of solutions of the VI, which are the equilibrium points. Thus, the equilibrium problem can be studied in a dynamic framework. This dynamic model enables us not only to study the transient behavior of the network but also to predict it.

Before we proceed, we need to recall some mathematical definitions from [53]. The set of inward normals at $\mathbf{p} \in X$ is defined as

$$S(\mathbf{p}) = \{\gamma : \|\gamma\| = 1, \langle \gamma, \mathbf{p} - \mathbf{y} \rangle \leq 0, \forall \mathbf{y} \in X\}. \quad (47)$$

Then the projection of $\mathbf{b} \in \mathbb{R}^n$ onto X at $\mathbf{p} \in X$ can be written as

$$\Pi_X(\mathbf{p}, \mathbf{b}) = \mathbf{b} + \max(0, \langle \mathbf{b}, -\mathbf{s}^* \rangle) \cdot \mathbf{s}^* \quad (48)$$

where \mathbf{s} is a vector in $S(\mathbf{p})$ that satisfies the condition

$$\langle \mathbf{b}, -\mathbf{s}^* \rangle = \max_{\mathbf{s} \in S(\mathbf{p})} \langle \mathbf{b}, -\mathbf{s} \rangle. \quad (49)$$

By this projection operator, a point in the interior of X is projected onto itself, and a point outside of X is projected onto the closest point on the boundary of X . The following ODE:

$$\dot{\mathbf{p}} = \Pi_X(\mathbf{p}, \mathbf{b}(\mathbf{p})) \quad (50)$$

with the initial condition

$$\mathbf{p}(t_0) = \mathbf{p}_0 \in X \quad (51)$$

is called a projected dynamic system.

Let us replace $\mathbf{b}(\mathbf{p})$ with $-\mathbf{F}(\mathbf{p}) = -(\sigma + M\mathbf{p})$ in (50). Then, the *stationary points* of the following PDS:

$$\dot{\mathbf{p}} = \Pi_X(\mathbf{p}, -\mathbf{F}(\mathbf{p})) \quad (52)$$

coincide with the solutions of the VI problem of (46) [53]. The associated dynamic model to the equilibrium problem will be realistic only if there is a unique solution path from a given initial point. The existence and uniqueness of the solution path for the above ODE are established in [53]. When $\mathbf{p}(t)$ is in the interior of the feasible set $\mathbf{p}(t) \in \text{int}X$, the projection operator in (52) is

$$\Pi_X(\mathbf{p}, -\mathbf{F}(\mathbf{p})) = -\mathbf{F}(\mathbf{p}). \quad (53)$$

If $\mathbf{p}(t)$ reaches the boundary of the feasible set $\mathbf{p}(t) \in \partial X$, we have

$$\Pi_X(\mathbf{p}, -\mathbf{F}(\mathbf{p})) = -\mathbf{F}(\mathbf{p}) + z(\mathbf{p})\mathbf{s}^*(\mathbf{p}) \quad (54)$$

where

$$\mathbf{s}^*(\mathbf{p}) = \operatorname{argmax}_{\mathbf{s} \in S(\mathbf{p})} \langle -\mathbf{F}(\mathbf{p}), -\mathbf{s} \rangle \quad (55)$$

and

$$z(\mathbf{p}) = \max(0, \langle -\mathbf{F}(\mathbf{p}), -\mathbf{s}^*(\mathbf{p}) \rangle). \quad (56)$$

From (53) and (54), it is obvious that

$$\|\Pi_X(\mathbf{p}, -\mathbf{F}(\mathbf{p}))\| \leq \|-\mathbf{F}(\mathbf{p})\|. \quad (57)$$

Therefore, because of the projection operator, the right-hand side of the differential equation (52) is discontinuous on the boundary of X . If, at some t , $\mathbf{p}(t)$ reaches the boundary of X and $-\mathbf{F}(\mathbf{p}(t))$ points out of the boundary, then the right-hand side becomes the projection of $-\mathbf{F}$ onto the boundary. The state trajectory then evolves on the boundary. In summary, the projection operator keeps the state trajectory in the feasible set. At some later time, the state trajectory may enter a lower dimensional part of the boundary or even go to the interior of X [53], where the evolution of the state trajectory is governed by the differential equation

$$\dot{\mathbf{p}} = -\mathbf{F}(\mathbf{p}) = -(\boldsymbol{\sigma} + \mathbf{M}\mathbf{p}), \quad \forall \mathbf{p}(t) \in X. \quad (58)$$

Since $-\mathbf{F}(\mathbf{p})$ is an affine mapping, the differential equation in (58) represents an *affine system*. Moreover, the state trajectory of this affine system must remain in the feasible set X . Therefore, the system described by (58) is a constrained affine system [59].

As mentioned before, a cognitive radio network is a hybrid dynamic system with both continuous and discrete dynamics. Changes occur in the network due to discrete events such as the appearance and disappearance of users and spectrum holes, as well as continuous dynamics described by differential equations that govern the evolution of transmit power vectors of users over time. When conditions change due to these kinds of discrete events, each user will have to solve a new optimization problem similar to the one described in (2). The network deviates from the achieved equilibrium point, and it is desirable to converge to a new one reasonably fast. Also, the occurrence of an event such as a change in the number of users or available subcarriers will change the parameters $\boldsymbol{\sigma}$ and \mathbf{M} in (58). Accordingly, the problem is formulated in terms of an ensemble of subsystems, and the global state space is:

- partitioned into polyhedral regions described in (45) that follow the varying realizations of the network at different time intervals;
- an affine state equation, similar to that described in (58), associated with each polyhedral region that governs the evolution of state trajectory in that region.

It follows, therefore, that the whole network can be modeled as a constrained *piecewise affine* (PWA) system [59]:

$$\dot{\mathbf{p}} = -\mathbf{M}(\mathbf{v})\mathbf{p} - \boldsymbol{\sigma}(\mathbf{v}), \quad \forall \mathbf{p}(t) \in X(\mathbf{v}) \quad (59)$$

where \mathbf{v} is a key vector, which is a function of time and discrete events, and describes which affine subsystem is currently a valid representation of the network [60].

The stationary points of each one of these dynamic subsystems coincide with the equilibrium points of the corresponding game resulting from solving the related optimization problems. In summary, the occurrence of discrete events changes the equilibrium point and causes the state trajectory to deviate from an equilibrium point and therefore converge to another equilibrium point. Each one of these equilibrium points may have a *region of attraction* around it such that if the system is perturbed, the solution remains in that region close to the solution of the unperturbed system. This issue will be studied in the next section.

Iterative algorithms based on time discretization of the PDS (52) are proposed in [53] for computation of the system state trajectory. At each time-step t , the proposed algorithms solve the minimum-norm problem

$$\min_{\mathbf{p}(t+1) \in X} \|\mathbf{p}(t+1) - [\mathbf{p}(t) - a(t)\mathbf{F}(\mathbf{p}(t))]\| \quad (60)$$

or, equivalently, solve the following quadratic programming problem:

$$\min_{\mathbf{p}(t+1) \in X} \frac{1}{2} \mathbf{p}^T(t+1) \mathbf{p}(t+1) - [\mathbf{p}(t) - a(t)\mathbf{F}(\mathbf{p}(t))] \cdot \mathbf{p}(t+1) \quad (61)$$

where “ \cdot ” signifies the dot product. A good approximation of the state trajectory may be achieved by choosing a small value for the step-size $a(t)$. It should be noted that although quadratic programming is indeed computationally demanding, it does not feature in operation of the robust IWFA; rather it is a burden incurred in carrying out simulation experiments to study the behavior of the whole network.

VIII. SENSITIVITY ANALYSIS OF EQUILIBRIUM SOLUTIONS

The stability of a system is an important issue and deserves special attention. It can be interpreted as the ability of the system to maintain or restore its equilibrium state against external perturbations. In other words, system stability is linked to system sensitivity to perturbations.

The formulation of the IWFA as AVI($X, \boldsymbol{\sigma}, \mathbf{M}$) is helpful to study the sensitivity of a solution \mathbf{p}^* as the pair ($X, \boldsymbol{\sigma} + \mathbf{M}\mathbf{p}$) is perturbed. It would be interesting to know if the perturbed system has a solution close to \mathbf{p}^* ; and in the case such a solution exists, if it will converge to \mathbf{p}^* as the perturbed AVI approaches the original one. This way of thinking leads to the concept of *solution stability* [54]. The *monotonicity* conditions play a key role in the analysis of both local and global stability [53], [54]. The following definition and theorem are recalled from [54].

Definition: A mapping $F : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be:

a) *monotone* on X if

$$(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq 0, \quad \forall \mathbf{x}, \mathbf{y} \in X \quad (62)$$

b) *strictly monotone* on X if

$$(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) > 0, \quad \forall \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y} \quad (63)$$

c) ξ -*monotone* on X for some $\xi > 1$ if there exists a constant $c > 0$ such that

$$(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq c\|\mathbf{x} - \mathbf{y}\|^\xi, \quad \forall \mathbf{x}, \mathbf{y} \in X \quad (64)$$

d) *strongly monotone* on X if there exists a constant $c > 0$ such that

$$(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq c\|\mathbf{x} - \mathbf{y}\|^2, \quad \forall \mathbf{x}, \mathbf{y} \in X \quad (65)$$

i.e., if F is 2-monotone on X .

Strong monotonicity implies ξ -monotonicity, ξ -monotonicity implies strict monotonicity, and strict monotonicity implies monotonicity, but the reverse is not true.

Theorem 1: Let $X \subseteq \mathbb{R}^n$ be closed convex and $F : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous.

- If F is strictly monotone on X , then $\text{VI}(X, F)$ has at most one solution.
- If F is ξ -monotone on K for some $\xi > 1$, then $\text{VI}(X, F)$ has a unique solution.

Therefore, $\text{VI}(X, \boldsymbol{\sigma} + \mathbf{M}\mathbf{p})$ has at most one solution if $\boldsymbol{\sigma} + \mathbf{M}\mathbf{p}$ is strictly monotone, and it has a unique solution, \mathbf{p}^* if $\boldsymbol{\sigma} + \mathbf{M}\mathbf{p}$ is ξ -monotone for some $\xi > 1$. The local uniqueness of \mathbf{p}^* is not sufficient to guarantee the solvability of the perturbed AVI, but it is important for sensitivity analysis because every unique solution of a VI problem is an *attractor* of all solutions of nearby VIs [54]. Alternatively, the following theorems can be recalled from [53] about stability of the corresponding PDS.

Theorem 2: Suppose that \mathbf{p}^* solves $\text{VI}(X, \boldsymbol{\sigma} + \mathbf{M}\mathbf{p})$. If the mapping $\boldsymbol{\sigma} + \mathbf{M}\mathbf{p}$ is strictly monotone at \mathbf{p}^* , then \mathbf{p}^* is a strictly monotone attractor for the PDS($X, \boldsymbol{\sigma} + \mathbf{M}\mathbf{p}$).

Theorem 3: Suppose that \mathbf{p}^* solves $\text{VI}(X, \boldsymbol{\sigma} + \mathbf{M}\mathbf{p})$. If the mapping $\boldsymbol{\sigma} + \mathbf{M}\mathbf{p}$ is ξ -monotone at \mathbf{p}^* with $\xi < 2$, then \mathbf{p}^* is a finite-time attractor.

Theorem 4: Suppose that \mathbf{p}^* solves $\text{VI}(X, \boldsymbol{\sigma} + \mathbf{M}\mathbf{p})$. If the mapping $\boldsymbol{\sigma} + \mathbf{M}\mathbf{p}$ is strongly monotone at \mathbf{p}^* , then \mathbf{p}^* is exponentially stable.

Monotonicity of the affine map $\mathbf{M}\mathbf{p} + \boldsymbol{\sigma}$, where \mathbf{M} is not necessarily symmetric, is equivalent to the condition that all of the eigenvalues of \mathbf{M} have nonnegative real parts. Also, strict monotonicity, ξ -monotonicity, and strong monotonicity of $\mathbf{M}\mathbf{p} + \boldsymbol{\sigma}$, as well as the condition that all of the eigenvalues of \mathbf{M} have positive real parts, are all equivalent [54]. The latter condition is equivalent to saying that $-\mathbf{M}$ is a Hurwitz matrix because in a Hurwitz matrix, all of its eigenvalues have negative real parts [61]. Since \mathbf{M} is a nonnegative real matrix, in this case the symmetric part of \mathbf{M} , $(1/2)(\mathbf{M} + \mathbf{M}^T)$, will be positive definite. Therefore, if matrix $-\mathbf{M}$ is Hurwitz, the existence of a unique equilibrium solution for the IWFA game, which is exponentially stable, will be guaranteed. As will be clear later, the Hurwitz property of matrix $-\mathbf{M}$ is also needed to guarantee the robust exponential stability of the system in the presence of multiple time-varying delays.

In order to get an idea about the conditions under which matrix $-\mathbf{M}$ is Hurwitz in a practical cognitive radio network, let us regroup the elements of the power vector in (41) based on subcarriers instead of users

$$\mathbf{q} = [\mathbf{p}_k]_{k=1}^n = [[p_1^1 \cdots p_1^m] \cdots [p_n^1 \cdots p_n^m]]^T. \quad (66)$$

Accordingly, by rearranging rows and columns of matrix \mathbf{M} , the following block diagonal matrix is obtained:

$$\mathbf{N} = \begin{bmatrix} \mathbf{M}_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{M}_n \end{bmatrix} \quad (67)$$

where \mathbf{M}_k s are *tone matrices* [42]

$$\mathbf{M}_k = \begin{bmatrix} 1 & \cdots & \alpha_k^{1m} \\ \vdots & \ddots & \vdots \\ \alpha_k^{m1} & \cdots & 1 \end{bmatrix}. \quad (68)$$

Matrices \mathbf{M} and \mathbf{N} have the same set of eigenvalues. Regarding the block diagonal structure of matrix \mathbf{N} , if all of the eigenvalues of every tone matrix \mathbf{M}_k have positive real values or if the symmetric part of every tone matrix $(1/2)(\mathbf{M}_k + \mathbf{M}_k^T)$ is positive definite, then $-\mathbf{M}$ will be Hurwitz. If tone matrices are strictly diagonally dominant, then their symmetric parts will be positive definite.

Therefore, the following condition guarantees that $-\mathbf{M}$ will be Hurwitz:

$$\sum_{j=1, j \neq i}^m \alpha_k^{ij} < 1, \quad \forall i = 1, \dots, m, \quad \forall k = 1, \dots, n. \quad (69)$$

For instance, if

$$\alpha_k^{ij} < \frac{1}{m-1}, \quad \forall i, j = 1, \dots, m, \quad \forall k = 1, \dots, n \quad (70)$$

the Hurwitz condition will be guaranteed [39]–[42].

As shown in (6), the interference gains α_k^{ij} depend on the distance between a receiver and its own transmitter relative to its distance from other active transmitters in the network. Therefore, the Hurwitz condition of matrix $-\mathbf{M}$ depends on the topology of the network. Roughly speaking, if each user's receiver has the proper distance from its own transmitter, which is short compared to its distance from other active transmitters in the network, then it can be guaranteed that the network will reach a stable unique equilibrium.

The existence and uniqueness results for IWFA are extended in [25] and [42], and broader conditions are obtained compared to those presented in [39]–[41]. However, the condition (69) provides insight on the stability of real-life cognitive radio networks.

IX. RETARDED EQUILIBRIA

The feedback channel plays a fundamental role in the design and operation of cognitive radio. Indeed, feedback is the *facilitator of intelligence*, without which the radio loses its cognitive capability. The discovery of spectrum holes prompts the need to establish the feedback channel from the receiver to the transmitter of a cognitive radio. In other words, a fraction of the available spectrum holes is used for the feedback channel to send relevant information from a user's receiver to its transmitter to take the appropriate action. In effect, therefore, feedback channels are not fixed and instead of having permanent feedback, we have sporadic feedback. Also, in order to be conservative in consuming the precious bandwidth that can be used for data transmission, the feedback should be low-rate and quantized. Therefore, rather than sending the actual values of the required parameters identified by the radio scene-analyzer, the practical approach is to feed their respective quantized values back to the transmitter [2].

Feedback may naturally introduce delay in the control loop and different transmitters may receive statistics of noise and interference with different time delays. Moreover, the sporadic feedback causes users to use outdated statistics to update their power vectors. The time-varying

delay in the control loop degrades the performance and may cause stability problems. Analysis of stability and control of time-delay systems is a topic of practical interest and has attracted the attention of many researchers [62]–[65]. Robust stability of the system under time-varying delays is the focus of this section. The dynamic model of the previous sections can be used to find out if the network is able to achieve a *retarded equilibrium*, which is stable. If an equilibrium point is not stable, the system may not be able to maintain that state long enough because of perturbations, and there is the potential possibility that an equilibrium cannot even be established.

The dynamic model of (52) will be a PDS with delay (PDSD) [66] in the form of the following *functional differential equation* (FDE) [67], [68]:

$$\dot{\mathbf{p}}(t) = \Pi_X(\mathbf{p}(t), -\mathbf{F}_d(\mathbf{p})). \quad (71)$$

The dot in $\dot{\mathbf{p}}(t)$ denotes differentiation. \mathbf{F}_d can be written as

$$\mathbf{F}_d(\mathbf{p}) = \begin{bmatrix} \mathbf{F}^1(\mathbf{p}^1(t), \mathbf{p}^{-1}(\mathcal{S}_t)) \\ \vdots \\ \mathbf{F}^i(\mathbf{p}^i(t), \mathbf{p}^{-i}(\mathcal{S}_t)) \\ \vdots \\ \mathbf{F}^m(\mathbf{p}^m(t), \mathbf{p}^{-m}(\mathcal{S}_t)) \end{bmatrix} \quad (72)$$

where $\mathbf{p}^{-i}(\mathcal{S}_t)$ denotes a continuous-time asynchronous adjustment scheme similar to (10).

Let the given initial point be t_0 . In order to determine the continuous solution $\mathbf{p}(t)$ of (71) for $t > t_0$, we need to know a continuous *initial function* $\phi(t)$, where $\mathbf{p}(t) = \phi(t)$ for $t_0 - \tau^{ij} \leq t \leq t_0$, $\forall i, j = 1, \dots, m$. The initial function may be obtained from measurements. Since the system described in (71) and (72) is a multiple-delay system, each deviation defines an initial set $\Psi_{t_0}^{ij}$ consisting of point t_0 and those values $t - \tau^{ij}(t)$ for which $t - \tau^{ij}(t) < t_0$ when $t \geq t_0$ [69].

Therefore, the initial condition for system (71) is

$$\mathbf{p}(\theta) = \phi(\theta), \quad \forall \theta \in \Psi_{t_0} \quad (73)$$

where $\phi: \Psi_{t_0} \rightarrow \mathbb{R}^{m \times n}$ is a continuous norm-bounded initial function [69], [70] and

$$\begin{aligned} \Psi_{t_0} &= \bigcup_{i,j=1}^m \Psi_{t_0}^{ij} \\ &= \bigcup_{i,j=1}^m \{t \in \mathbb{R} : t = \kappa - \tau^{ij}(\kappa) \leq 0, \kappa \geq t_0\}. \end{aligned} \quad (74)$$

$\mathbf{F}_d(\mathbf{p})$ in (72) can be written as the following summation:

$$\begin{aligned} \mathbf{F}_d(\mathbf{p}) = & \mathbf{p}(t) + \sum_{i=1}^m \sum_{j=1, \neq i}^m \mathbf{M}_d^{ij} \mathbf{p}(t - \tau^{ij}(t)) \\ & + \sum_{i=1}^m \sum_{j=1, \neq i}^m \Delta \mathbf{M}_d^{ij} \mathbf{p}(t - \tau^{ij}(t)) + \mathbf{q}(t) \end{aligned} \quad (75)$$

where \mathbf{M}_d^{ij} is obtained by replacing all the blocks in \mathbf{M} except \mathbf{M}^{ij} by $n \times n$ zero matrices; and $\Delta \mathbf{M}_d^{ij}$ is a perturbation in \mathbf{M}_d^{ij} . The term \mathbf{q} is the combined effect of the background noise in both forward and feedback channels.

Therefore, the associated constrained affine system that governs the network's dynamics is described by the differential equation

$$\begin{aligned} \dot{\mathbf{p}}(t) = & -\mathbf{p}(t) - \sum_{i=1}^m \sum_{j=1, \neq i}^m \mathbf{M}_d^{ij} \mathbf{p}(t - \tau^{ij}(t)) \\ & - \sum_{i=1}^m \sum_{j=1, \neq i}^m \Delta \mathbf{M}_d^{ij} \mathbf{p}(t - \tau^{ij}(t)) - \mathbf{q}(t) \end{aligned} \quad (76)$$

$\forall \mathbf{p}(t) \in X$, which is a multiple time-varying delay system with uncertainty. It can be written as

$$\begin{aligned} \dot{\mathbf{p}}(t) = & -\mathbf{p}(t) - \sum_{\ell=1}^{m(m-1)} \mathbf{M}_d^\ell \mathbf{p}(t - \tau^\ell(t)) \\ & - \sum_{\ell=1}^{m(m-1)} \Delta \mathbf{M}_d^\ell \mathbf{p}(t - \tau^\ell(t)) - \mathbf{q}(t). \end{aligned} \quad (77)$$

This reformulation is an instance of the general systems that were studied in [70]. Following the approach of [70], we assume that $\forall t \geq t_0$, the time-varying delays $\tau^\ell(t)$ satisfy

$$\tau^\ell(t) \leq \tau(t) \leq \bar{\tau} \quad (78)$$

$$\dot{\tau} \leq \delta < 1 \quad (79)$$

where $\bar{\tau} > 0$, $\delta \geq 0$, and $\tau(t)$ is a strictly positive continuous differentiable function. Also, the uncertainties are assumed to be bounded for all \mathbf{p} and at all times, such that

$$\|\mathbf{q}(t)\| \leq b_d \|\mathbf{p}\| \quad (80)$$

$$\|\Delta \mathbf{M}_d^\ell \mathbf{p}(t)\| \leq b_d^\ell \|\mathbf{p}\| \quad (81)$$

where $b_d \geq 0$ and $b_d^\ell \geq 0$. If there exist $\zeta \geq 1$ and $\lambda > 0$ such that

$$\|\mathbf{p}(t)\| \leq \zeta \sup_{\theta \in \Psi_{t_0}} \{\|\mathbf{p}(\theta)\|\} e^{-\lambda(t-t_0)} \quad (82)$$

then the uncertain time-delay system of (77) is said to be robustly exponentially stable with a decay rate of λ . In other words, the trivial solution $\mathbf{p} = 0$ of the system is exponentially stable with a decay rate of λ for all admissible uncertainties [70].

Regarding the fact that

$$\mathbf{I} + \sum_{i=1}^{m(m-1)} \mathbf{M}_d^\ell = \mathbf{M} \quad (83)$$

we conclude the robust exponential stability of the network from [70, Th. 4], which is repeated here with some modification to conform to our problem.

Theorem 5: Consider the system (77) with initial condition (73) and assume that $-\mathbf{M}$ is a Hurwitz stable matrix satisfying

$$\|e^{\mathbf{M}t}\| \leq ce^{-\eta t} \quad (84)$$

for some real numbers $c \geq 1$ and $\eta > 0$. On the left-hand side of the above equation, e denotes a matrix exponential operator. If the inequality

$$\frac{c}{\eta} \left[\bar{\tau} \sum_{i=1}^{m(m-1)} (\mu_1^i + \mu_2^i) + b_d + \sum_{i=1}^{m(m-1)} b_d^i \right] < 1 \quad (85)$$

holds, then the transient response of $\mathbf{p}(t)$ satisfies

$$\|\mathbf{p}(t)\| \leq \zeta \sup_{\theta \in \Psi_{t_0}} \{\|\mathbf{p}(\theta)\|\} e^{-\rho \int_{t_0}^t \frac{d\theta}{\tau(\theta)}}, \quad \forall t \geq 0, \zeta \geq 1 \quad (86)$$

where

$$\mu_1^i = \|\mathbf{M}_d^i\| + \|\mathbf{M}_d^i\| b_d \quad (87)$$

$$\mu_2^i = \sum_{j=1}^{m(m-1)} \|\mathbf{M}_d^i \mathbf{M}_d^j\| + \|\mathbf{M}_d^i\| \sum_{j=1}^{m(m-1)} b_d^j \quad (88)$$

and $\rho > 0$ is the unique positive solution of the transcendental equation

$$1 - \frac{c}{\eta} b_d - \frac{\rho}{\eta \tau(0)} = \mu_3 \frac{c}{\eta} e^{\frac{\rho}{\tau}} \quad (89)$$

where

$$\mu_3 = \bar{\tau} \sum_{i=1}^{m(m-1)} \mu_1^i + \bar{\tau} e^{\frac{\rho}{\tau}} \sum_{i=1}^{m(m-1)} \mu_2^i + \sum_{i=1}^{m(m-1)} b_d^i. \quad (90)$$

Furthermore, the system described by (77) and (73) is robustly exponentially stable with a decay rate $\rho/\bar{\tau}$.

The left-hand side of the transcendental equation in (89) is a continuous decreasing function of ρ , and its right-hand side is a continuous increasing function. By virtue of (85) at $\rho = 0$, the right-hand side is less than the left-hand side. Therefore, (89) has a unique positive solution, as desired.

X. COMPUTER EXPERIMENTS

Simulation results are now presented to support the theoretical discussions of the previous sections. In simulations, similar to [42] and [43], the background noise levels σ_k^i , the normalized interference gains α_k^{ij} , and the power budgets p_{\max}^i are chosen randomly from the intervals $(0, 0.1/(m-1))$, $(0, 1/(m-1))$, and $(n/2, n)$, respectively, with uniform distributions. The randomly chosen values for interference gains α_k^{ij} , which are less than $1/(m-1)$, guarantee that the tone matrices will be strictly diagonally dominant [39]–[41]. Thus, the corresponding matrix $-\mathbf{M}$ will be Hurwitz. Also, the chosen values for noise levels and power budgets guarantee that the worst case p_{\max}^i/σ_k^i per subcarrier will be close to 7 dB. For scenarios that consider the time-varying delay in the control loops, delays are chosen randomly in a broader range than what is allowed by the constraints given in Section IX on the delays, but constraints on perturbation terms are met.

A. Robust IWFA Versus Classic IWFA

The transmit power control problem in a cognitive radio network using the classic IWFA and its robust version were presented in Sections IV and V, respectively. In a cognitive radio network, when a spectrum hole disappears, users may have to increase their transmit powers at other spectrum holes, and this increases the interference. Also, when new users join the network, current users in the network experience more interference. Therefore, the joining of new users or the disappearance of spectrum holes makes the interference condition worse. Also, the cross-interference between users is time-varying because of the mobility of the users. Results related to two typical but extreme scenarios are

presented here to show superiority of the robust IWFA (15) over the classic IWFA (2) in dealing with the above issues.

The first scenario addresses a network with $m = 5$ nodes and $n = 2$ available subcarriers, and all of the users simultaneously update their transmit powers using the interference measurements from the previous time-step. At the fourth time-step, two new users join the network, which increases the interference. The interference gains are also changed randomly at different time instants to consider mobility of the users. Figs. 3 and 4 show the transmit power of three users (users one, four, and seven) at two different subcarriers for the classic IWFA and robust IWFA, respectively. At the second subcarrier, the classic IWFA is not able to reach an equilibrium. Data rates achieved by the chosen users are shown too. Also, the total

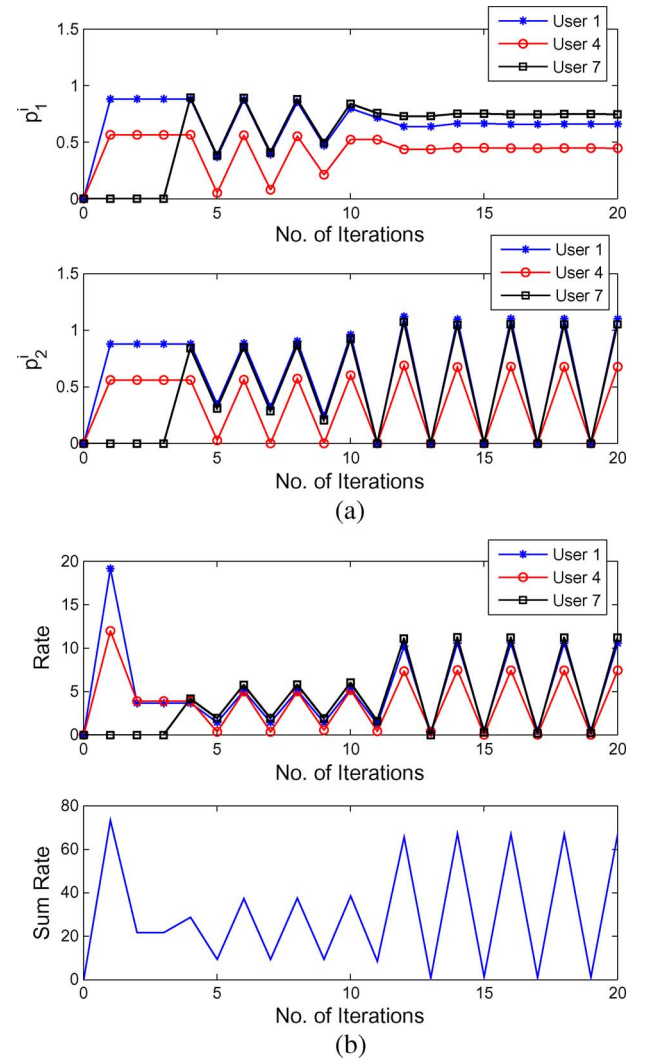


Fig. 3. Resource allocation results of simultaneous IWFA when two new users join a network of five users and interference gains are changed randomly to address the mobility of the users. (a) Transmit powers of three users at two subcarriers. (b) Data rates of three users and the total data rate in the network.

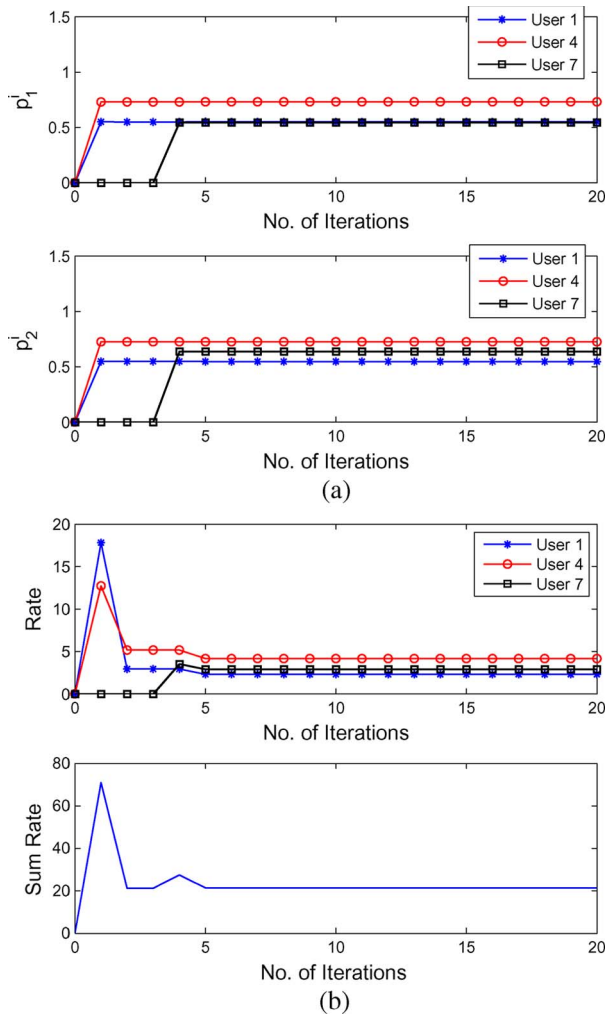


Fig. 4. Resource allocation results of simultaneous robust IWFA when two new users join a network of five users and interference gains are changed randomly to address the mobility of the users. (a) Transmit powers of three users at two subcarriers. (b) Data rates of three users and the total data rate in the network.

data rate in the network is plotted against time, which is a measure of spectral efficiency. Although the average sum rate achieved by the classic IWFA is close to the average sum rate of the robust IWFA, it fluctuates, and in some time instants the data rate is very low, which indicates lack of spectrally efficient communication. Although the oscillation occurs mainly because of using simultaneous update scheme, it also highlights practical effectiveness of the robust IWFA.

In the second scenario, a network with $m = 5$ nodes and $n = 4$ available subcarriers is considered. Again, at the fourth time-step, two new users join the network, but at the eighth time-step, the third subcarrier is no longer available (i.e., a spectrum hole disappears). Results are shown in Figs. 5 and 6, which show superiority of the robust IWFA. For classic IWFA, immediately after the

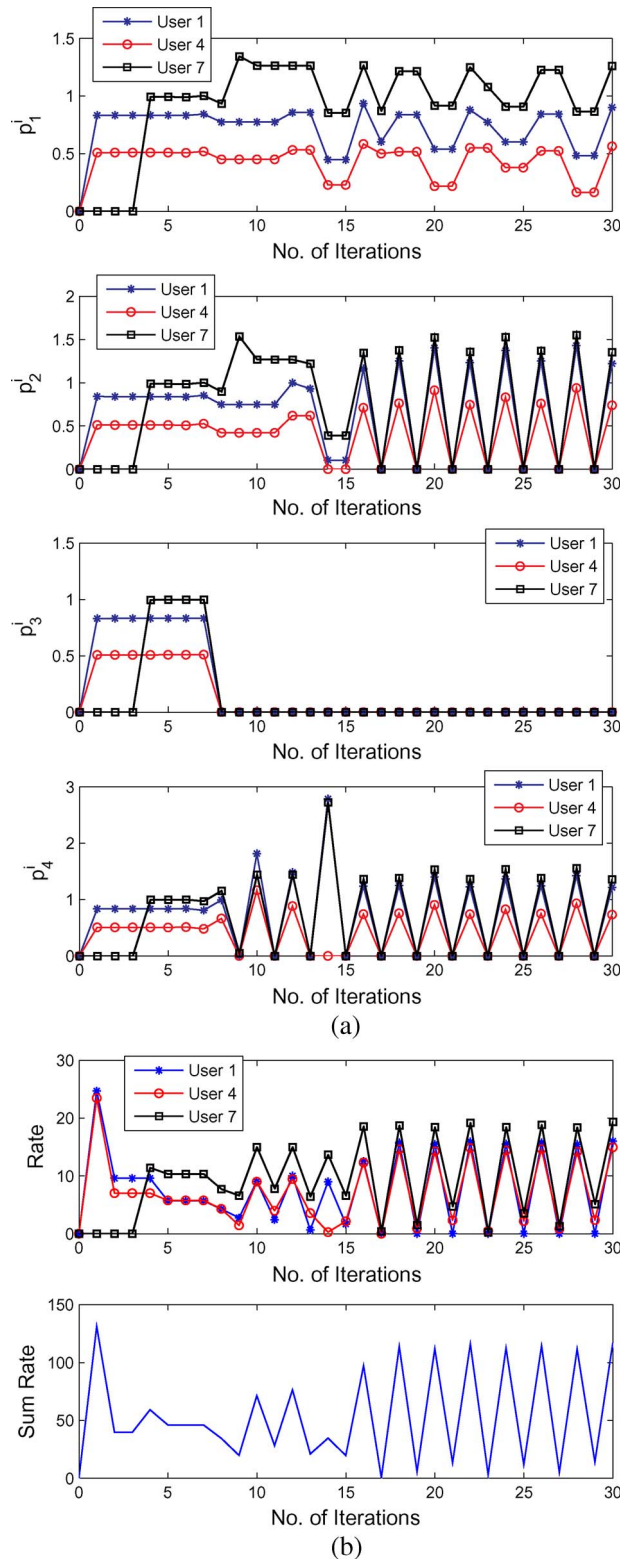


Fig. 5. Resource allocation results of simultaneous IWFA when two new users join a network of five users, a subcarrier disappears, and interference gains are changed randomly to address the mobility of the users. (a) Transmit powers of three users at four subcarriers. (b) Data rates of three users and the total data rate in the network.

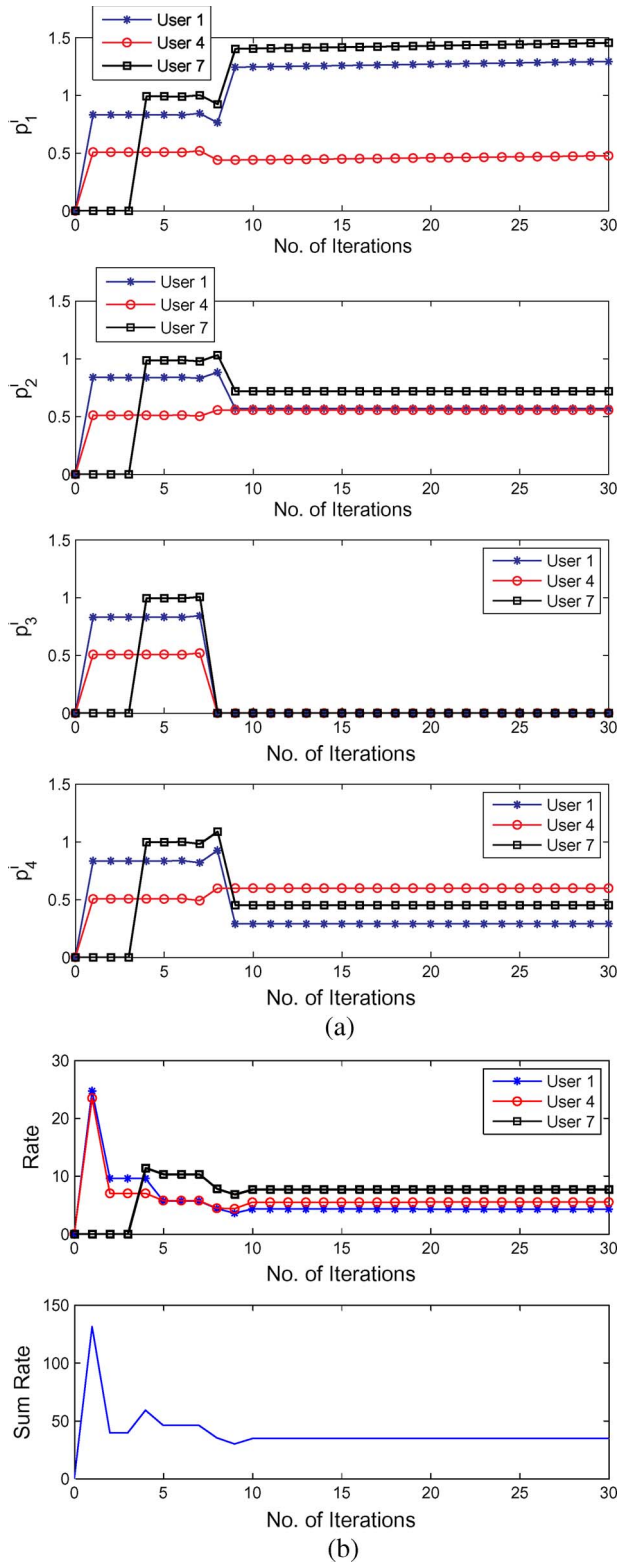


Fig. 6. Resource allocation results of simultaneous robust IWFA when two new users join a network of five users, a subcarrier disappears, and interference gains are changed randomly to address the mobility of the users. (a) Transmit powers of three users at four subcarriers. (b) Data rates of three users and the total data rate in the network.

disappearance of the third subcarrier, power in the fourth subcarrier starts to oscillate. After changing the interference gains randomly, we observe the same behavior in other subcarriers. In contrast to the robust IWFA, the classic IWFA fails again to achieve an equilibrium.

As mentioned previously, sporadic feedback introduces a time-varying delay in the transmit power control loop, which causes different users to update their transmit powers based on outdated statistics. For instance, when the network configuration and therefore interference pattern changes, some users receive the related information after a delay. If the interference at a subcarrier increases and the transmitter is not informed immediately, it will not reduce its transmit power and may violate the permissible interference power level for awhile until it receives updated statistics of the interference in the forward channel. Similarly, this may happen to some users that update their transmit powers at lower rates compared to others. In the third scenario, a new user joins a network of three users, who are competing for utilizing two subcarriers. Each user's transmitter receives statistics of the interference plus noise with a time-varying delay. Fig. 7(a) shows the randomly chosen time-varying delays introduced by each user's feedback channel. The sum of transmit power and interference plus noise at the second subcarrier at the receiver of each user is plotted in Fig. 7(b) and (c) for classic IWFA and robust IWFA, respectively. Dashed lines show the limit imposed by the permissible interference power level. Although the classic IWFA is less conservative, it is not as successful as the robust IWFA at preventing violations of the permissible interference power level. Similar results are obtained when users update their transmit powers with different frequencies.

B. Projected Dynamic System

To study the transient behavior of a cognitive radio network, a scenario is considered for three users and three subcarriers. "Three" is chosen merely for the sake of visualization. Numerical values for parameters are chosen in the same way that was described at the beginning of this section. It is assumed that all the users update their power vectors simultaneously, considering a worst case interference condition.

Fig. 8 depicts state trajectories for three users obtained from a discrete-time approximation of the PDS by solving the quadratic programming described in (61), when the following sequence of events happens. First, all three subcarriers are idle and can be used by the secondary users. Therefore, the state trajectories evolve in three-dimensional space (i.e., $p_1^i p_2^i p_3^i$ space). Then, the second subcarrier is no longer available and state trajectories enter the two-dimensional space and evolve in $p_1^i p_3^i$ plane. After that, the same thing happens to the third subcarrier and state trajectories evolve in one-dimensional space (i.e., p_1^i line). After a while, subcarrier three becomes idle and therefore available again. Thus, the state trajectories enter from p_1^i line to $p_1^i p_3^i$ plane. When subcarrier two becomes available again,

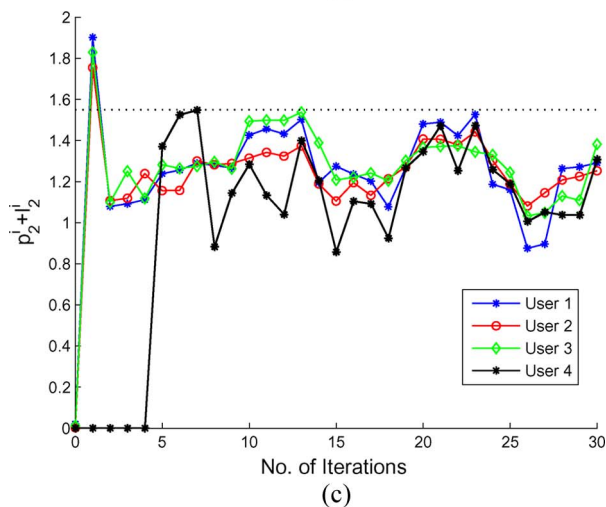
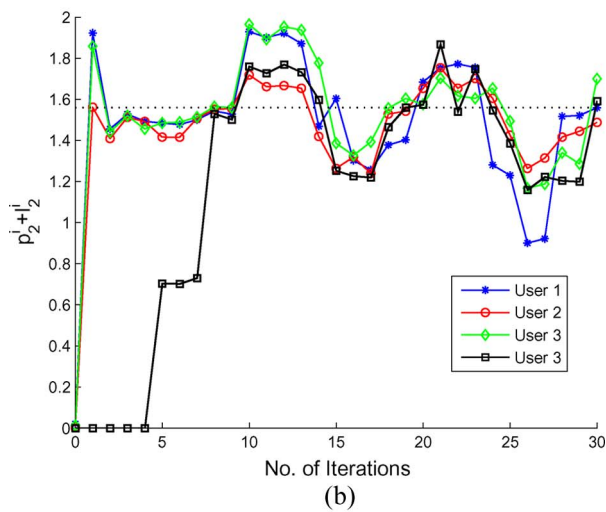
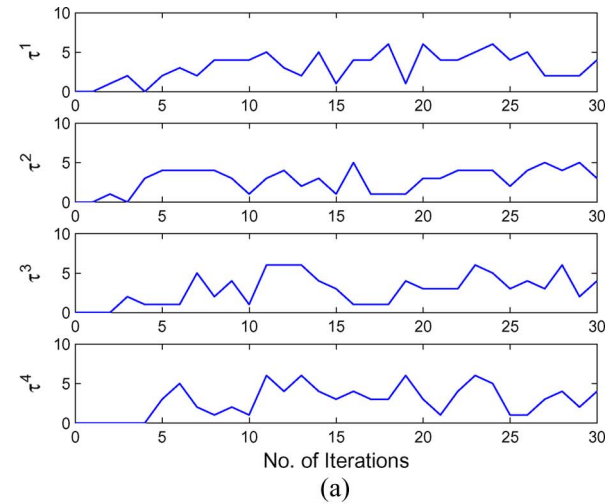


Fig. 7. Resource allocation results of IWFA when interference gains change randomly with time and users use outdated information to update their transmit powers. (a) Time-varying delays introduced by each user's feedback channel. Sum of transmit power and interference plus noise for four users achieved by (b) classic IWFA. (c) Robust IWFA. Dashed lines show the limit imposed by the permissible interference power level.

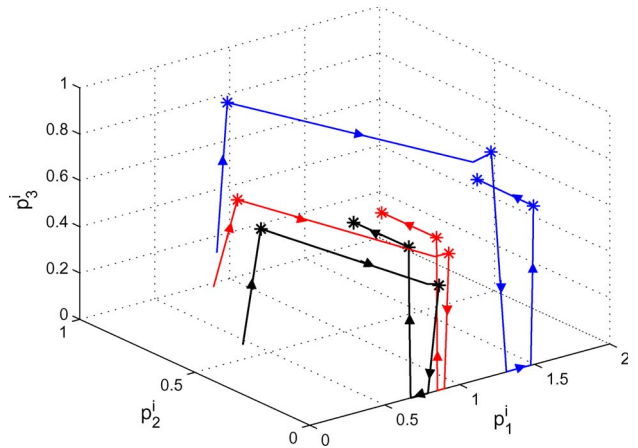


Fig. 8. Power trajectories for a network of three users with three available subcarriers obtained from the associated PDS when both the interference gains and the number of subcarriers change by time. Direction of evolution of states and the achieved equilibrium points are shown by arrows and stars, respectively. Trajectories enter lower dimensional spaces when spectrum holes disappear and then again go back to higher dimensional spaces when new spectrum holes are available. When the second subcarrier is not idle, trajectories enter $p_1^i p_3^i$ plane, and when the third subcarrier is not also idle anymore, trajectories enter p_1^i line. After a while, when third and then second subcarriers become available again, state trajectories go back to $p_1^i p_3^i$ plane and then $p_1^i p_2^i p_3^i$ space.

state trajectories enter from $p_1^i p_3^i$ plane to $p_1^i p_2^i p_3^i$ space. It is obvious that the power trajectories enter from higher dimensional spaces to lower dimensional spaces according to the number of available subcarriers, and again they go back to higher dimensional spaces when users have access to more subcarriers, which is what should happen during a successful operation. The achieved equilibrium points for different users as they exist between occurrences of the mentioned events are shown by stars on their state trajectories. Also, arrows in Fig. 8 show the direction of evolution of states for different users.

C. Sensitivity Analysis

To study the solution stability via simulation, the system of the previous subsection is perturbed and the equilibrium point of the perturbed system is calculated. Results at three different subcarriers are shown separately in Fig. 9. As the perturbation terms decay and the perturbed system approaches the original one, the behavior of the perturbed system converges to the solution of the original system, which is shown by stars in Fig. 9. The arrows show the direction, in which the solution of the perturbed system converges to the solution of the original system. This experiment validates the notion of solution stability that was discussed previously.

When delays, introduced by the feedback channels, are considered, it may take longer for both the original system and the perturbed systems to achieve an equilibrium.

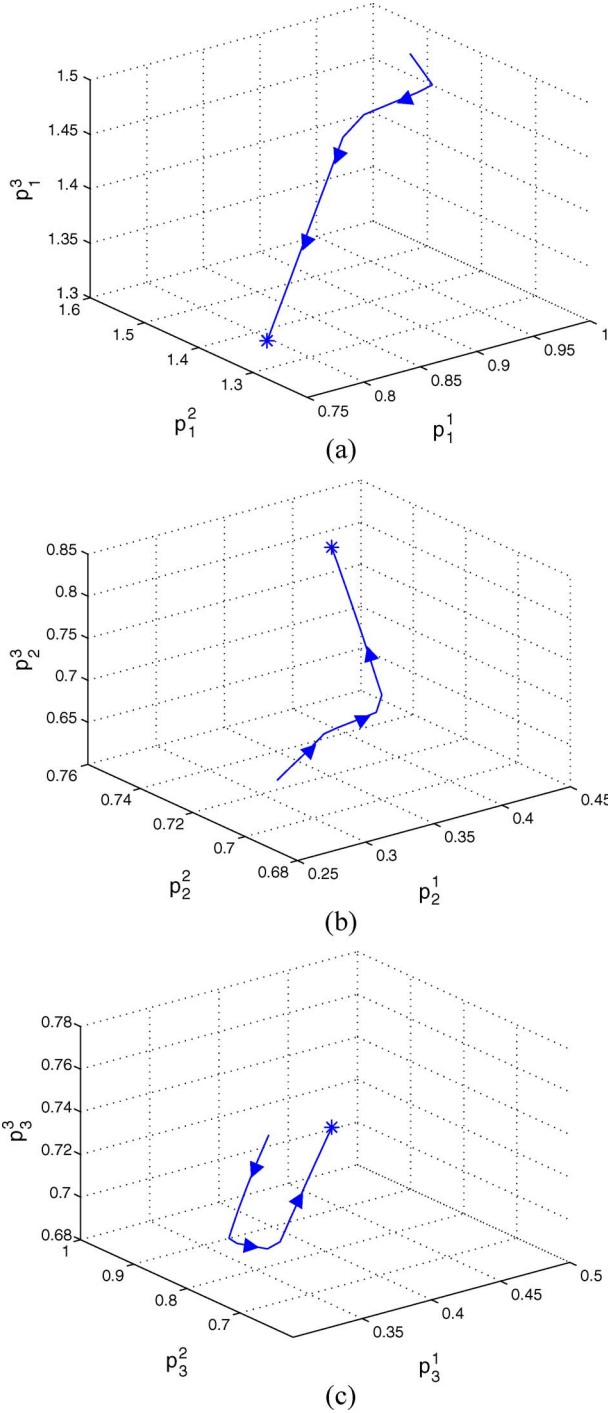


Fig. 9. Solution stability analysis. Solution of the perturbed system converges to the solution of the original system (shown by stars) as the perturbed system approaches the original system. Results are depicted for different subcarriers separately. Arrows show the direction of convergence.

Under the conditions mentioned in Section IX, the robust exponential stability of the system is guaranteed, and similar results are obtained in simulations for the time-delay cases with constraints on delays.

D. Multiple Time-Varying Delay System With Uncertainty

Simulation results for the above network of three users and three potentially available subcarriers, with a similar sequence of events, mentioned in Section X-B, are repeated with asynchronous adjustment scheme. In the beginning, all three subcarriers are idle and can be used by secondary users. Then, the second subcarrier is no longer available, and after that the same thing happens to the third subcarrier. After a while, subcarriers two and then three become idle and therefore available again. Power trajectories and achieved equilibrium points are shown in Fig. 10. Fig. 11 depicts the random delays in adjustment schemes $\tau^i(t)$ used by different users, which shows that most of the time users have used outdated information to update their power vectors. Results confirm the ability of the system to achieve retarded equilibria under the conditions given in Theorem 5. By increasing the delay, the performance of the system will degrade, and eventually the system becomes unstable.

XI. SUMMARY AND DISCUSSION

A cognitive radio network is a dynamic environment, in which both users and resources can freely come and go in a stochastic manner. Therefore, the employed resource-allocation algorithms must converge fast to be able to utilize

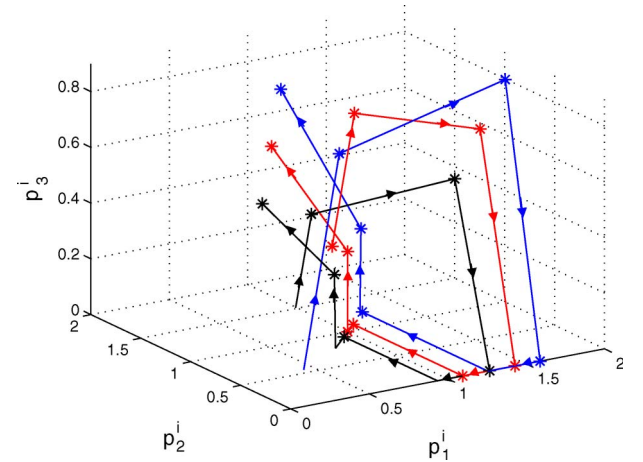


Fig. 10. Power trajectories for a network of three users with three available subcarriers obtained from the associated multiple time-varying delay PDS with uncertainty, when both the interference gains and the number of subcarriers change by time. Direction of evolution of states and the achieved equilibrium points are shown by arrows and stars, respectively. Trajectories enter lower dimensional spaces when spectrum holes disappear and then again go back to higher dimensional spaces when new spectrum holes are available. When the second subcarrier is not idle, trajectories enter $p_1^i p_3^i$ plane; and when the third subcarrier is not also idle anymore, trajectories enter p_1^i line. After a while, when second and then third subcarriers become available again, state trajectories go back to $p_1^i p_2^i$ plane and then $p_1^i p_2^i p_3^i$ space.

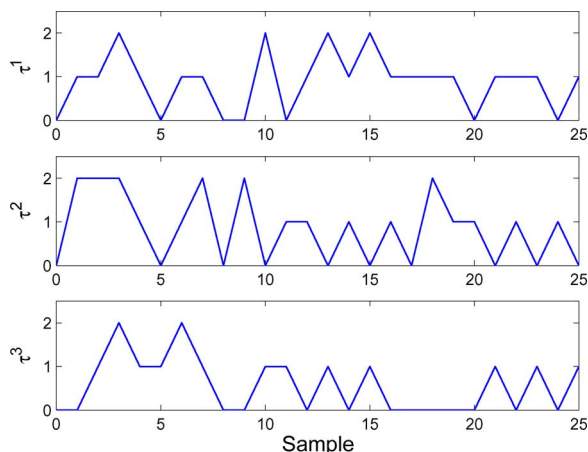


Fig. 11. Time-varying delay introduced by feedback channels in transmit power control loops for a network of three users.

the available resources before they disappear. Also, these algorithms need to have the ability to be implemented in a decentralized manner. Moreover, accounting for uncertainties of the environment is crucial. Having these points in mind, the best available algorithm in the literature was picked and modified to suit the cognitive radio environment.

A. The Robust IWFA

The IWFA is a potentially good candidate for resource allocation in cognitive radio networks because of its low complexity, fast convergence, distributed nature, and convexity. Propagation effects are considered in the formulation of IWFA through the interference gains α_k^{ij} . Dimension of the global state space (i.e., dimension of the problem) is equal to the product of two quantities; the number of users m and the number of subcarriers n . The classic IWFA is able to achieve a unique Nash-equilibrium solution under certain conditions.

A Nash-equilibrium game such as the IWFA can be reformulated as a VI problem. A nice reformulation of the IWFA under worst interference conditions, in which all users transmit with their maximum powers, was presented in [42] as an AVI problem. Using the AVI reformulation of IWFA, conditions on interference gains were presented in [42] that guarantee the existence of a unique Nash-equilibrium solution. The AVI reformulation also allows us to associate a dynamic model to the network.

Using the permissible interference power level criterion, the resource-allocation problem in a cognitive radio network was formulated in the IWFA framework. By trading optimality for robustness, a robust version of the IWFA, based on the max-min theory, was formulated to address issues such as appearance and disappearance of spectrum holes as well as coming and going of users in light of their mobility. This robust game formulation guarantees an acceptable level of performance even under worst case conditions.

B. Control-Theoretic Underpinning of the Robust IWFA

Building on [42], the transmit power control problem in a cognitive radio network under worst interference conditions was presented as an AVI problem. Employing the theory of PDS, an affine dynamic model was obtained for the evolution of the network's state. This dynamic model allows us to study both equilibrium and disequilibrium behavior of the network under worst interference conditions. The proposed dynamic framework also facilitates sensitivity and stability analysis of the system. The fact that changes happen in a cognitive radio network because of continuous dynamics as well as discrete events makes it a hybrid dynamic system. Modeling the system using the theory of PDS lends itself to describing the cognitive radio network as a constrained PWA system and therefore benefiting from various mathematical tools, which have been well demonstrated in control theory.

The AVI formulation of the transmit power control problem allows us to study the solution stability. Alternatively, the same results are obtained from a stability analysis of the associated PDS. Under some conditions on interference gains, if the system is perturbed, the behavior of the perturbed system converges to the behavior of the original system as the perturbed system approaches the original one.

Usually users use asynchronous update schemes, and they update their transmit powers at different rates. The feedback channel introduces a time-varying delay in the control loop of a cognitive radio system, which means sometimes users update their transmit powers using outdated information. Therefore, the network is, practically speaking, a multiple time-varying delay system with uncertainty. Also, the robust exponential stability of the network was studied in this framework.

C. What We Learned From the Computer Experiments

Extensive computer experiments were conducted, and typical results were presented to support the theoretical discussions. The performance of classic IWFA and robust IWFA was compared. In some extreme cases because of occurrence of discrete events such as appearance and disappearance of spectrum holes and users, the IWFA cannot achieve an equilibrium solution, and calculated results oscillate in subsequent time-steps especially if we use the simultaneous update scheme. This confirms the point that in a PWA system, even if all subsystems are stable, the whole system may become unstable because of switching between the subsystems. In the presented cases, the robust IWFA was able to achieve an equilibrium solution. Also, when some users update their transmit powers with lower frequencies or use outdated information, the robust IWFA can prevent violating the permissible interference power level. Classic IWFA lacks this ability, although it achieves less conservative data rates. These results show the superiority of the robust

IWFA over the classic IWFA in dealing with different practical issues in a cognitive radio environment.

Simulations were conducted to show the concept of solution stability. The system was perturbed and its equilibrium solution calculated. By decaying the perturbation terms, the equilibrium solution of the perturbed system converged to the equilibrium solution of the original system.

The ability of the dynamic model, obtained using PDS theory, was validated by simulations for both delay-free and multiple time-varying delay cases. The results presented here show that by appearance and disappearance of spectrum holes, the state trajectory of the network enters higher and lower dimensional subspaces in the global state space, respectively.

D. Future Work

This paper studied the resource allocation problem in a cognitive radio network considering a totally competitive framework. For future work, we plan to consider cooperative scenarios and study what can be gained by consuming a portion of the resources for coordination between users. ■

APPENDIX I GLOSSARY OF SYMBOLS

α_k^{ij}	Normalized interference gain from transmitter j to receiver i at subcarrier k .
β_k	Frequency-dependent attenuation parameter associated with subcarrier k .
γ_k^i	Lagrange multiplier associated with the constraint imposed by the permissible interference power level.
λ_k^i	Lagrange multiplier associated with the constraint that prevents cognitive radios to transmit over non-idle subcarriers.
λ	Decay rate.
Γ	Signal-to-noise ratio gap.
Π_X	Projection operator onto the feasible set X .
\mathbf{Q}	Combined effect of the background noise in both forward and feedback channels in the network.
σ_k^i	Normalized background noise power at the receiver input of user i on subcarrier k .
$\boldsymbol{\sigma}^i$	Normalized background noise power vector at the receiver input of user i .
$\boldsymbol{\sigma}$	Normalized background noise power vector of the network.
$\tau^i(t)$	Time-varying delay introduced by user i 's feedback channel.
$\tau^{ij}(t)$	Time-varying delay with which user i receives update information from user j .
ϕ	Initial function.
Ψ_{i0}^i	Initial set associated with user i .
Ψ_{i0}	Initial set for the network.
$a(t)$	Step-size.
CAP_k	Permissible interference power level at subcarrier k .
d_{ij}	Distance from transmitter j to receiver i .
f^i	Objective function of user i .
Δf	Frequency offset.

h_k^{ij}	Channel gain from transmitter j to receiver i over the flat-fading subchannel associated with subcarrier k .
I_k^i	Noise plus interference experienced by user i at subcarrier k .
\bar{I}_k^i	Nominal noise plus interference experienced by user i at subcarrier k .
ΔI_k^i	Perturbation term in noise plus interference experienced by user i at subcarrier k .
\mathbf{I}^i	Noise plus interference vector experienced by user i .
$\Delta \mathbf{I}^i$	Perturbation term in noise plus interference vector experienced by user i .
K^i	Feasible set of user i .
K	Feasible set of the network.
L^i	Lagrangian of the optimization problem for user i .
\mathbf{M}	Matrix of normalized interference gains.
\mathbf{M}_k	Tone matrix associated with subcarrier k .
\mathbf{M}^{ij}	Diagonal matrix, which its diagonal elements are normalized interference gains from transmitter j to receiver i at different subcarriers.
m	Number of active cognitive radio transceivers in the region of interest.
n	Total number of subcarriers in an OFDM framework that can be potentially available for communication.
p_k^i	User i 's transmit power over subcarrier k .
\mathbf{p}^i	User i 's power vector.
\mathbf{p}^{-i}	Joint power vectors of users other than user i .
\mathbf{p}	Network power vector.
\mathbf{p}^*	Nash equilibrium point.
p_{\max}^i	User i 's maximum power.
PS	Subset of subcarriers that cannot be assigned to cognitive radios.
r	Path-loss exponent.
$S(\mathbf{p})$	Set of inward normals at $\mathbf{p} \in X$.
\mathcal{S}_t	Adjustment scheme at time t .
t	Time.
u^i	Lagrange multiplier associated with maximum power constraint.
X	Feasible set of the network under worst-case interference condition that each transmitter transmits with its maximum power.
$intX$	Interior of the feasible set X .
∂X	Boundary of the feasible set X .

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REFERENCES

- [1] S. Haykin, "Cognitive radio: Brain-empowered wireless communications," *IEEE J. Sel. Areas Commun.*, vol. 23, no. 2, pp. 201–220, 2005.
- [2] S. Haykin, "Fundamental issues in cognitive radio," in *Cognitive Wireless Communication Networks*, E. Hossain and V. K. Bhargava, Eds. New York: Springer, 2007.
- [3] T. A. Weiss and F. K. Jondral, "Spectrum pooling: An innovative strategy for the enhancement of spectrum efficiency," *IEEE Commun. Mag.*, vol. 42, no. 3, pp. S8–S14, 2004.
- [4] U. Berthold, F. K. Jondral, S. Brandes, and M. Schnell, "OFDM-based overlay systems: A promising approach for enhancing spectral efficiency," *IEEE Commun. Mag.*, vol. 45, no. 12, pp. 52–58, 2007.
- [5] H. Arslan, H. A. Mahmoud, and T. Yucek, "OFDM for cognitive radio: Merits and challenges," in *Cognitive Radio, Software Defined Radio, and Adaptive Wireless Systems*, H. Arslan, Ed. New York: Springer, 2007.
- [6] K. E. Nolan, "Reconfigurable OFDM Systems," Ph.D. dissertation, Univ. of Dublin, Trinity College, Dublin, Ireland, 2005.
- [7] I. F. Akyildiz, W. Y. Lee, M. C. Vuran, and S. Mohanty, "NeXt generation/dynamic spectrum access/cognitive radio wireless networks: A survey," *Comput. Netw.*, vol. 50, pp. 2127–2159, 2006.
- [8] G. J. Foschini and Z. Miljanic, "A simple distributed autonomous power control algorithm and its convergence," *IEEE Trans. Veh. Technol.*, vol. 42, no. 4, pp. 641–646, 1993.
- [9] R. D. Yates, "A framework for uplink power control in cellular radio systems," *IEEE J. Sel. Areas Commun.*, vol. 13, no. 7, pp. 1341–1347, 1995.
- [10] N. Bambos, S. C. Chen, and G. J. Pottie, "Channel access algorithms with active link protection for wireless communication networks with power control," *IEEE/ACM Trans. Netw.*, vol. 8, no. 5, pp. 583–597, 2000.
- [11] C. U. Saraydar, N. B. Mandayam, and D. J. Goodman, "Efficient power control via pricing in wireless data networks," *IEEE Trans. Commun.*, vol. 50, no. 2, pp. 291–303, 2002.
- [12] D. O'Neill, D. Julian, and S. Boyd, "Seeking Foschinis genie: Optimal rates and powers in wireless networks,"
- [13] S. Stanczak, M. Wiczanowski, and H. Boche, *Resource Allocation in Wireless Networks: Theory and Algorithms*. New York: Springer, 2006.
- [14] C. W. Tan, D. Palomar, and M. Chiang, "Robustness-energy tradeoff in cellular network power control," *IEEE Trans. Wireless Commun.*, 2007.
- [15] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*. New York: Springer, 2000.
- [16] T. Keller and L. Hanzo, "Adaptive multicarrier modulation: A convenient framework for time-frequency processing in wireless communications," *Proc. IEEE*, vol. 88, no. 5, pp. 611–640, 2000.
- [17] A. R. S. Bahai, B. R. Saltzberg, and M. Ergen, *Multi-Carrier Digital Communications: Theory and Applications of OFDM*, 2nd ed. New York: Springer, 2004.
- [18] Y. Li and G. Stuber, *Orthogonal Frequency Division Multiplexing for Wireless Communications*. New York: Springer, 2006.
- [19] M. Sterand, T. Svensson, T. Ottosson, A. Ahlen, A. Svensson, and A. Brunstrom, "Towards systems beyond 3G based on adaptive OFDMA transmission," *Proc. IEEE*, vol. 95, no. 12, pp. 2432–2455, 2007.
- [20] M. Morelli, C. C. J. Kuo, and M. O. Pun, "Synchronization techniques for orthogonal frequency division multiple access (OFDMA): A tutorial review," *Proc. IEEE*, vol. 95, no. 7, pp. 1394–1427, 2007.
- [21] W. Y. Zou and Y. Wu, "COFDM: An overview," *IEEE Trans. Broadcast.*, vol. 41, no. 1, pp. 1–8, 1995.
- [22] S. C. Thompson, "Constant envelope OFDM phase modulation," Ph.D. dissertation, Univ. of California, San Diego, 2005.
- [23] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Englewood Cliffs: Prentice-Hall, 1989.
- [24] G. Scutari, D. P. Palomar, and S. Barbarossa, "Simultaneous iterative water-filling for Gaussian frequency-selective interference channels," in *Proc. IEEE Int. Symp. Inf. Theory*, 2006, pp. 600–604.
- [25] G. Scutari, D. P. Palomar, and S. Barbarossa, "Asynchronous iterative waterfilling for Gaussian frequency-selective interference channels: A unified framework," in *Proc. Inf. Theory Applicat. Workshop*, 2007, pp. 349–358.
- [26] I. Konnov, *Equilibrium Models and Variational Inequalities*. New York: Springer, 2007.
- [27] E. Altman, T. Boulogne, R. El-Azouzi, T. Jimenez, and L. Wynter, "A survey on networking games in telecommunication," *Comput. Oper. Res.*, vol. 24, no. 3, pp. 58–68, 2007.
- [28] A. MacKenzie and L. DaSilva, *Game Theory for Wireless Engineers*. San Rafael, CA: Morgan and Claypool, 2006.
- [29] M. Felegyhazi and J. P. Hubaux, "Game theory in wireless networks: A tutorial," *ACM Comput. Surv.*, pp. 1–14, 2006.
- [30] M. Biagi, "Cross-layer optimization of multi-antenna 4G-WLANs," Ph.D. dissertation, Univ. of Rome La Sapienza, Rome, Italy, 2005.
- [31] N. Nei and C. Comaniciu, "Adaptive channel allocation spectrum etiquette for cognitive radio networks," in *Proc. 1st IEEE Int. Symp. Dyn. Spectrum Access Netw. (DySPAN)*, 2005, pp. 269–278.
- [32] J. O. Neel, "Analysis and design of cognitive radio networks and distributed radio resource management algorithms," Ph.D. dissertation, Virginia Polytechnic Inst. and State Univ., Blacksburg, 2006.
- [33] C. Liang, "A game-theoretic approach to power management in MIMO-OFDM ad hoc networks," Ph.D. dissertation, Drexel Univ., Philadelphia, PA, 2006.
- [34] M. Felegyhazi, "Non-cooperative behavior in wireless networks," Ph.D. dissertation, Ecole Polytechnique Federale De Lausanne (EPFL), Lausanne, Switzerland, 2007.
- [35] J. F. Nash, "Equilibrium points in n-person games," *Proc. Nat. Acad. Sci.*, vol. 36, pp. 48–49, 1950.
- [36] J. F. Nash, "Non-cooperative games," *Ann. Math.*, vol. 54, pp. 286–295, 1951.
- [37] M. J. Osborne and A. Rubenstein, *A Course in Game Theory*. Cambridge, MA: MIT Press, 1994.
- [38] M. Felegyhazi, M. Cagalj, S. S. Bidokhti, and J. P. Hubaux, "Non-cooperative multi-radio channel allocation in wireless networks," in *Proc. IEEE Int. Conf. Comput. Commun. (INFOCOM)*, 2007, pp. 1442–1450.
- [39] K. B. Song, S. T. Chung, G. Ginis, and J. M. Cioffi, "Dynamic spectrum management for next-generation DSL systems," *IEEE Commun. Mag.*, vol. 40, no. 10, pp. 101–109, 2002.
- [40] W. Yu, G. Ginis, and J. M. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE J. Sel. Areas Commun.*, vol. 20, no. 5, pp. 1105–1115, 2002.
- [41] S. T. Chung, S. J. Kim, J. Lee, and J. M. Cioffi, "A game theoretic approach to power allocation in frequency-selective Gaussian interference channels," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2003, pp. 316–316.
- [42] Z. Q. Luo and J. S. Pang, "Analysis of iterative waterfilling algorithm for multiuser power control in digital subscriber lines," *EURASIP J. Appl. Signal Process.*, vol. 2006, pp. 1–10, 2006.
- [43] N. Yamashita and Z. Q. Luo, "A nonlinear complementarity approach to multiuser power control for digital subscriber lines," *Optim. Methods Software*, vol. 19, no. 5, pp. 633–652, 2004.
- [44] Z. Q. Luo, J. S. Pang, and D. Ralph, *Mathematical Programs With Equilibrium Constraints*. Cambridge, U.K.: Cambridge Univ. Press, 1996.
- [45] S. Haykin and M. Moher, *Modern Wireless Communications*. Englewood Cliffs: Prentice-Hall, 2005.
- [46] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*. Philadelphia, PA: SIAM, 1999.
- [47] M. Fukushima, "Stochastic and robust approaches to optimization problems under uncertainty," in *Proc. Int. Conf. Inf. Res. Develop. Knowl. Soc. Infrastruct. (ICKS)*, 2007, pp. 87–94.
- [48] M. Aghassi and D. Bertsimas, "Robust game theory," *Math. Program. B*, vol. 107, pp. 231–273, 2006.
- [49] J. M. Danskin, *The Theory of Max-Min*. Springer-Verlag, 1967.
- [50] T. Basar and P. Bernhard, *H_∞ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd ed. Berlin, Germany: Birkhauser, 1995.
- [51] M. K. Brady, "Techniques for interference analysis and spectrum management of digital subscriber lines," Ph.D. dissertation, Univ. of California, Berkeley, 2006.
- [52] K. M. Teo, "Nonconvex Robust Optimization," Ph.D. dissertation, Massachusetts Inst. of Technology, Cambridge, 2007.
- [53] A. Nagurney and D. Zhang, *Projected Dynamical Systems and Variational Inequalities With Applications*. New York: Springer, 1996.
- [54] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*. New York: Springer, 2003.
- [55] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.
- [56] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*, 3rd ed. New York: Wiley-Interscience, 2006.
- [57] R. W. Cottle, J. S. Pang, and R. E. Stone, *The Linear Complementarity Problem*. New York: Academic, 1992.
- [58] M. Fazel, D. M. Gayme, and M. Chiang, "Transient analysis for wireless power control," in *Proc. IEEE Global Telecommun. Conf. (GLOBECOM)*, 2006, pp. 1–6.

- [59] F. J. Christophersen, *Optimal Control of Constrained Piecewise Affine Systems*. New York: Springer, 2007.
- [60] J. Roll, "Local and piecewise affine approaches to system identification," Ph.D. dissertation, Linköping Univ., Linköping, Sweden, 2003.
- [61] H. K. Khalil, *Nonlinear Systems*, 3rd ed. New York: Prentice-Hall, 2001.
- [62] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*. New York: Academic, 1986.
- [63] V. B. Kolmanovskii and L. E. Shauikhet, *Control of Systems With Aftereffect*. Oxford, U.K.: Oxford Univ. Press, 1997.
- [64] K. Gu and S. Niculescu, "Survey on recent results in the stability and control of time-delay systems," *J. Dyn. Syst., Meas., Contr.*, vol. 125, pp. 158–165, 2003.
- [65] J. P. Richard, "Time-delay systems: An overview of some recent advances and open problems," *Automatica*, vol. 39, pp. 1667–1694, 2003.
- [66] L. Engelson, "On dynamics of traffic queues in a road network with route choice based on real time traffic information," *Transp. Res. C*, vol. 11, pp. 161–183, 2003.
- [67] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*. Berlin, Germany: Springer-Verlag, 1993.
- [68] V. B. Kolmanovskii and A. D. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Norwood, MA: Kluwer Academic, 1999.
- [69] L. E. El'sgol'ts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations With Deviating Arguments*. New York: Academic, 1973.
- [70] S. I. Niculescu, C. E. de Souza, L. Dugard, and J. M. Dion, "Robust exponential stability of uncertain systems with time-varying delays," *IEEE Trans. Autom. Control*, vol. 43, no. 5, pp. 743–748, 1998.

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