Generalized Evolutionary Spectral Analysis and the Weyl Spectrum of Nonstationary Random Processes

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***Abstract*— The *evolutionary spectrum* (ES) is a "time-varying power spectrum" of nonstationary random processes. Starting from an innovations system interpretation of the ES, we introduce the *generalized evolutionary spectrum* (GES) as a novel family of time-varying power spectra. The GES contains the ES and the recently introduced *transitory evolutionary spectrum* as special cases. We consider the problem of finding an innovations system for a process characterized by its correlation function, and we discuss the connection between GES analysis and the class of *underspread* processes.**

Furthermore, we show that another special case of the GES—a novel time-varying power spectrum that we call *Weyl spectrum*—has substantial advantages over all other members of the GES family. The properties of the Weyl spectrum are discussed, and its superior performance is verified experimentally for synthetic and real-data processes.

***Index Terms*—Nonstationary random processes, spectral analysis, time-frequency analysis, time-varying systems.**

I. Introduction

Spectral analysis of (wide-sense) stationary random processes by means of the *power spectral density* (PSD) is a useful concept. However, in many applications the signals must be modeled as nonstationary processes. Extensions of the PSD to nonstationary processes result in "time-varying power spectra" such as the Wigner–Ville spectrum [1]–[3], the physical spectrum [4], and the evolutionary spectrum (ES) [5]–[19].

This paper discusses and extends the ES. The original definition of the ES is based on an expansion of the nonstationary random process under analysis into complex exponentials with uncorrelated, random, *time-varying* amplitudes [5]. Alternatively, the ES can be expressed via the transfer function of a linear time-varying (LTV) innovations system [5], [9], [10]. Following the introduction of the ES by Priestley [5]–[7], numerous researchers have discussed the theoretical framework of the ES [2], [3], [8]–[12], applications of the ES to special types of nonstationary processes [13], [14], and the estimation of the ES [15]–[18]. Furthermore, extensions to parametric models have been established [19], and a concept dual to the ES—the transitory evolutionary spectrum (TES)—has been introduced recently [20].

In this paper, motivated by the innovations system interpretation of the ES, we define the *generalized evolutionary spectrum* (GES) as a family of time-varying power spectra extending both the ES and the TES. Subsequently, we concentrate on a specific member of the GES family—the novel *Weyl spectrum*—that has important advantages over all other GES members [21]. The paper is organized as follows. Section II reviews the ES. Section III reviews the TES and gives a novel innovations system interpretation of the TES. An important process classification (underspread/overspread) [22]–[24] is considered in Section IV. The GES is introduced in Section V using the generalized Weyl symbol [25], and the properties of the GES are discussed. The construction of an innovations system and, specifically, the advantages of the positive semidefinite innovations system are considered in Section VI. In Section VII, we introduce the *Weyl spectrum* [21] as a new member of the GES family with substantial advantages over all other GES members. In Section VIII, our theoretical results are verified experimentally for synthetic and real-data processes.

II. The Evolutionary Spectrum

A. The Stationary Case

The ES can be motivated by the stationary case. Therefore, we first consider a zero-mean, *wide-sense stationary* random process $x(t)$ with autocorrelation function $r_x(\tau) = \mathrm{E}\{x(t+\tau)x^*(t)\}$ and PSD[1]

$$S_x(f) = \int_\tau r_x(\tau) e^{-j2\pi f\tau}\, d\tau \geq 0.$$

Since $\int_f S_x(f)\, df = \mathrm{E}\{|x(t)|^2\}$, the PSD can be interpreted as a spectral distribution of the average power. It is related to an expansion[2] of the process $x(t)$ into complex sinusoids $e^{j2\pi ft}$ [26]

$$x(t) = \int_f X(f) e^{j2\pi ft}\, df \tag{1}$$

where the expansion coefficients can be shown to be *uncorrelated*, with the PSD as average intensity:

$$\mathrm{E}\{X(f_1)X^*(f_2)\} = S_x(f_1)\delta(f_1 - f_2). \tag{2}$$

[1] Integrals are from $-\infty$ to ∞ unless stated otherwise.

[2] This expansion and similar expansions used in the following are to be interpreted in the mean-square sense [26].

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Let us set $X(f) = N(f)A(f)$, where $N(f)$ denotes stationary white noise with normalized average intensity, i.e., $\mathrm{E}\{N(f_1)N^*(f_2)\} = \delta(f_1 - f_2)$, and $A(f)$ is a deterministic, complex-valued weighting function. The left-hand side of (2) then becomes $\mathrm{E}\{X(f_1)X^*(f_2)\} = |A(f_1)|^2\delta(f_1 - f_2)$ so that the PSD is seen to be

$$S_x(f) = |A(f)|^2. \tag{3}$$

A second important interpretation of the PSD is obtained by representing the process $x(t)$ as the output of a linear time-invariant (LTI) "innovations system" $\mathbf{H}$ whose input is stationary white noise $n(t)$ [26]:

$$x(t) = (\mathbf{H}n)(t) = \int_{t'} h(t-t')n(t')\,dt'$$

with $\mathrm{E}\{n(t_1)n^*(t_2)\} = \delta(t_1 - t_2)$. The PSD then equals the magnitude squared of the system's transfer function $H(f) = \int_t h(t)e^{-j2\pi ft}\,dt$

$$S_x(f) = |H(f)|^2. \tag{4}$$

It is easily seen that $H(f) = A(f)$ if we take $N(f)$ to be the Fourier transform of $n(t)$. Hence, the PSD expressions (3) and (4) are equivalent.

B. Definition and Interpretation of the Evolutionary Spectrum

We next consider a zero-mean, *nonstationary* random process $x(t)$ with correlation function $R_x(t_1, t_2) = \mathrm{E}\{x(t_1)x^*(t_2)\}$ [26]. Motivated by (1), we postulate an expansion of $x(t)$ into complex sinusoids $e^{j2\pi ft}$

$$x(t) = \int_f X_t(f)e^{j2\pi ft}\,df \tag{5}$$

where the expansion coefficients $X_t(f)$ are time-varying but again assumed to be uncorrelated:

$$\mathrm{E}\{X_t(f_1)X_t^*(f_2)\} = \mathrm{ES}_x(t, f_1)\delta(f_1 - f_2). \tag{6}$$

This constitutes an implicit definition of the ES. In order to make this definition more precise, we set $X_t(f) = N(f)A(t,f)$, where $N(f)$ denotes stationary white noise with normalized average intensity and $A(t,f)$ is a deterministic, complex-valued weighting function. The expansion (5) then becomes

$$x(t) = \int_f N(f)A(t,f)e^{j2\pi ft}\,df \tag{7}$$

and the left-hand side of (6) becomes $\mathrm{E}\{X_t(f_1)X_t^*(f_2)\} = |A(t,f_1)|^2\delta(f_1 - f_2)$ so that [5]

$$\mathrm{ES}_x(t,f) = |A(t,f)|^2. \tag{8}$$

With (7), it is easy to show that the process's average instantaneous power can be written as $\mathrm{E}\{|x(t)|^2\} = \int_f |A(t,f)|^2\,df$ so that the ES is a spectral distribution of the average instantaneous power, i.e., $\int_f \mathrm{ES}_x(t,f)\,df = \mathrm{E}\{|x(t)|^2\}$ ("marginal property").

We now ask if the expansion (7) underlying the ES exists and is unique and how the ES can be derived given the correlation function $R_x(t_1, t_2)$. Introducing $\phi(t,f) = A(t,f)e^{j2\pi ft}$, (7) becomes

$$x(t) = \int_f N(f)\phi(t,f)\,df \tag{9}$$

and it is easily shown that the correlation function can be expressed as $R_x(t_1,t_2) = \int_f \phi(t_1,f)\phi^*(t_2,f)\,df$. In operator notation, this reads $\mathbf{R}_x = \mathbf{\Phi}\mathbf{\Phi}^+$, where $\mathbf{R}_x$ is the correlation operator (i.e., the positive definite or semidefinite, self-adjoint linear operator whose kernel is $R_x(t_1,t_2)$), $\mathbf{\Phi}$ is the linear operator whose kernel is $\phi(t,f)$, and $\mathbf{\Phi}^+$ is the adjoint of the operator $\mathbf{\Phi}$ [27]. Hence, for given correlation operator $\mathbf{R}_x$, $\mathbf{\Phi}$ is a solution to the *factorization problem* $\mathbf{\Phi}\mathbf{\Phi}^+ = \mathbf{R}_x$. Such a solution always exists since $\mathbf{R}_x$ is positive semidefinite, but it is not unique: If $\mathbf{\Phi}$ is a solution, then so is $\tilde{\mathbf{\Phi}} = \mathbf{\Phi}\mathbf{U}$, where $\mathbf{U}$ is an arbitrary unitary operator (i.e., $\mathbf{U}\mathbf{U}^+ = \mathbf{I}$ with $\mathbf{I}$ the identity operator). Hence, the ES as defined in (8) is not unique [5]; the specific ES obtained depends on the particular solution to the factorization problem $\mathbf{\Phi}\mathbf{\Phi}^+ = \mathbf{R}_x$.

It is important to note that the interpretation of the ES as a time-varying power spectrum is restricted to the case where the "amplitude function" $A(t,f)$ weighting the complex sinusoids $e^{j2\pi ft}$ in (7) is *slowly* time-varying. Indeed, only in this case can the function $\phi(t,f) = A(t,f)e^{j2\pi ft}$ in (9) be interpreted (as a function of t) as a narrowband, amplitude-modulated signal (an "oscillatory function" [5]) spectrally localized around f so that the parameter f in $\mathrm{ES}_x(t,f) = |A(t,f)|^2$ can be interpreted as "frequency" in a meaningful sense. This restriction amounts to a kind of quasistationarity assumption.

C. Innovations System Interpretation

We can express $x(t)$ as the response of an LTV innovations system $\mathbf{H}$ to stationary white noise $n(t)$ [28]:

$$x(t) = (\mathbf{H}n)(t) = \int_{t'} H(t,t')n(t')\,dt' \quad \text{with} \quad \mathrm{E}\{n(t_1)n^*(t_2)\} = \delta(t_1 - t_2). \tag{10}$$

Calculating the correlation function of $x(t)$ from (10) yields

$$R_x(t_1,t_2) = \int_{t'} H(t_1,t')H^*(t_2,t')\,dt' \tag{11}$$

so that the innovations system $\mathbf{H}$ is obtained as (nonunique) solution to the factorization problem $\mathbf{H}\mathbf{H}^+ = \mathbf{R}_x$, i.e., $\mathbf{H}$ is a "square root" of the correlation operator $\mathbf{R}_x$ (cf. Section VI). It is easily shown that (10) is equivalent to (9), i.e., $x(t) = \int_f N(f)\phi(t,f)\,df$ with $N(f)$ the Fourier transform of $n(t)$ and $\phi(t,f) = \int_{t'} H(t,t')e^{j2\pi ft'}\,dt'$. We then obtain

$$\begin{aligned} A(t,f) &= \phi(t,f)e^{-j2\pi ft} \\ &= \int_\tau H(t,t-\tau)e^{-j2\pi f\tau}\,d\tau = Z_{\mathbf{H}}(t,f) \end{aligned}$$

where $Z_{\mathbf{H}}(t,f)$ is the time-varying transfer function of $\mathbf{H}$ as defined by Zadeh [29]. Hence, the ES can be reformulated

TABLE I
DUALITY OF EVOLUTIONARY SPECTRUM AND TRANSITORY EVOLUTIONARY SPECTRUM ($\mathcal{F}$ DENOTES THE FOURIER TRANSFORM OPERATOR)

Evolutionary Spectrum	**Transitory Evolutionary Spectrum**
$x(t) = \int_f N(f)\,\phi(t,f)\,df$	$X(f) = \int_t n(t)\,\psi(t,f)\,dt$
$R_x(t_1,t_2) = \int_f \phi(t_1,f)\,\phi^*(t_2,f)\,df$	$R_X(f_1,f_2) = \int_t \psi(t,f_1)\,\psi^*(t,f_2)\,dt$
$\phi(t,f) = A(t,f)\,e^{j2\pi ft}$	$\psi(t,f) = a(t,f)\,e^{-j2\pi ft}$
$x(t) = (\mathbf{H}n)(t) = \int_{t'} H(t,t')\,n(t')\,dt'$	
$\phi(t,f) = \mathcal{F}^{-1}_{t'\to f}\{H(t,t')\}$	$\psi(t',f) = \mathcal{F}_{t\to f}\{H(t,t')\}$
$A(t,f) = Z_{\mathbf{H}}(t,f) = L^{(1/2)}_{\mathbf{H}}(t,f)$	$a(t,f) = \tilde{Z}_{\mathbf{H}}(t,f) = L^{(-1/2)}_{\mathbf{H}}(t,f)$

as the squared magnitude of Zadeh's transfer function of the innovations system $\mathbf{H}$ [5], [9], [10]:

$$\mathrm{ES}_x(t,f) = |Z_{\mathbf{H}}(t,f)|^2. \tag{12}$$

This result is very intuitive and extends the PSD expression (4) obtained in the stationary case. However, recall that the interpretation of the ES as a time-varying power spectrum requires $A(t,f)$ to be *slowly* time-varying. Since $A(t,f) = Z_{\mathbf{H}}(t,f)$, this means that $Z_{\mathbf{H}}(t,f)$ should be slowly time-varying as well. Furthermore, (12) is intuitively meaningful only if the LTV system $\mathbf{H}$ acts as a weighting in the time–frequency (TF) domain, i.e., if the TF shifts caused by the innovations system $\mathbf{H}$ are small. These considerations will lead us to the class of *underspread* systems and processes discussed in Section IV.

III. THE TRANSITORY EVOLUTIONARY SPECTRUM

The *transitory evolutionary spectrum* (TES) has recently been introduced [20] as a time-varying spectrum that is dual to the ES (see Table I). The TES is matched to "quasiwhite" processes.

A. The Nonstationary White Case

We first consider a zero-mean, *nonstationary white* random process $x(t)$ with correlation function $R_x(t_1,t_2) = q_x(t_1)\delta(t_1-t_2)$, where $q_x(t) \geq 0$ is the process's *average instantaneous intensity* [26]. The process allows a trivial expansion into Dirac impulses

$$x(t') = \int_t x(t)\delta(t'-t)\,dt \tag{13}$$

or, taking the Fourier transform

$$X(f) = \int_t x(t)e^{-j2\pi tf}\,dt. \tag{14}$$

Here, the expansion coefficients $x(t)$ are uncorrelated

$$\mathrm{E}\{x(t_1)x^*(t_2)\} = q_x(t_1)\delta(t_1-t_2). \tag{15}$$

Let us set $x(t) = n(t)a(t)$, where $n(t)$ is stationary white noise with normalized average intensity and $a(t)$ is a deterministic function. The left-hand side of (15) is then $\mathrm{E}\{x(t_1)x^*(t_2)\} = |a(t_1)|^2\delta(t_1-t_2)$ so that

$$q_x(t) = |a(t)|^2.$$

The relation $x(t) = n(t)a(t)$ can be interpreted in the sense that stationary white noise $n(t)$ is passed through a linear "frequency invariant" (LFI) system, i.e., an LTV system acting as a multiplier. This is dual to the interpretation of stationary processes in terms of an LTI innovations system (cf. Section II-A).

B. Definition and Interpretation of the Transitory Evolutionary Spectrum

We next consider a zero-mean, *nonstationary* random process $x(t)$ with correlation function $R_x(t_1,t_2)$. Motivated by the expansion (14) in the nonstationary white case, let us postulate an expansion

$$X(f) = \int_t x_f(t)e^{-j2\pi tf}\,dt = \int_t n(t)a(t,f)e^{-j2\pi ft}\,dt \tag{16}$$

$$= \int_t n(t)\psi(t,f)\,dt. \tag{17}$$

Here, $n(t)$ is stationary white noise with average intensity one, $a(t,f)$ is a deterministic, complex-valued weighting function defined by $x_f(t) = n(t)a(t,f)$, and $\psi(t,f) = a(t,f)\,e^{-j2\pi ft}$. The expansion coefficients $x_f(t)$ are frequency-varying but still assumed uncorrelated

$$\mathrm{E}\{x_f(t_1)x_f^*(t_2)\} = \mathrm{TES}_x(t_1,f)\delta(t_1-t_2). \tag{18}$$

The frequency variation in $x_f(t)$ will be seen to imply a broadening of the Dirac impulses in (13).

Equation (18) constitutes an implicit definition of the TES. With $x_f(t) = n(t)a(t,f)$, (18) can be rewritten as $\mathrm{E}\{x_f(t_1)x_f^*(t_2)\} = |a(t_1,f)|^2\delta(t_1-t_2)$ so that

$$\mathrm{TES}_x(t,f) = |a(t,f)|^2. \tag{19}$$

With (16), one can show that the TES is a temporal distribution of the average spectral energy density, i.e., the TES satisfies the "marginal property" $\int_t \mathrm{TES}_x(t,f)\,dt = \mathrm{E}\{|X(f)|^2\}$.

Taking the inverse Fourier transform of (16) or (17), the process $x(t')$ is represented as

$$x(t') = \int_t n(t)\xi(t',t)\,dt$$

where $\xi(t',t) = \int_f \psi(t,f)e^{j2\pi t'f}\,df = \int_f [a(t,f)e^{-j2\pi tf}]\,e^{j2\pi t'f}\,df$. If $a(t,f)$ is *slowly* varying with respect to f, then $\xi(t',t)$, as a function of t', is localized about $t' = t$

(in the limiting case where $a(t,f) = a(t)$, we get $\xi(t',t) = a(t)\delta(t'-t)$). The TES is then based on an expansion of $x(t')$ into narrow pulses with uncorrelated coefficients, which justifies the interpretation of the TES parameter t as "time." Note that the assumption that $a(t,f)$ is slowly varying with respect to f amounts to a "quasiwhiteness" property of $x(t')$.

With (17), the spectral correlation function $R_X(f_1,f_2) = \mathrm{E}\{X(f_1)X^*(f_2)\}$ (which is related to the temporal correlation function $R_x(t_1,t_2)$ by a 2-D Fourier transform) can be expressed as $R_X(f_1,f_2) = \int_t \psi(t,f_1)\psi^*(t,f_2)\,dt$. Hence, calculation of $\psi(t,f)$ given $R_x(t_1,t_2)$ amounts to solving the factorization problem $\mathbf{\Psi}\mathbf{\Psi}^+ = \mathbf{R}_X$, where $\mathbf{\Psi}$ is the linear operator whose kernel is $\Psi(f,t) = \psi(t,f)$. Again, $\mathbf{\Psi}$ always exists, but it is not unique; therefore, the TES as given by (19) is not unique either.

C. Innovations System Interpretation

We now establish a novel reformulation of the TES in terms of LTV innovations systems. In (10), we have modeled $x(t)$ as the response of an LTV system $\mathbf{H}$ to stationary white noise $n(t)$. Taking the Fourier transform of (10) with respect to t, one reobtains (17) with $\psi(t',f) = \int_t H(t,t')e^{-j2\pi ft}\,dt$ so that

$$a(t,f) = \psi(t,f)e^{j2\pi ft} = \int_\tau H(t+\tau,t)e^{-j2\pi f\tau}\,d\tau = \tilde{Z}_{\mathbf{H}}(t,f).$$

Hence, the amplitude function $a(t,f)$ can be interpreted as a time-varying transfer function $\tilde{Z}_{\mathbf{H}}(t,f)$ of $\mathbf{H}$, which, however, is different from Zadeh's function $Z_{\mathbf{H}}(t,f)$ arising in the case of the ES. With (19), the TES can be expressed as the squared magnitude of the new transfer function of the innovations system $\mathbf{H}$:

$$\mathrm{TES}_x(t,f) = |\tilde{Z}_{\mathbf{H}}(t,f)|^2.$$

Compared with (12), we see that the only difference between the ES and the TES is in the definition of the time-varying transfer function. This viewpoint will motivate the definition of the GES in Section V.

The identity $a(t,f) = \tilde{Z}_{\mathbf{H}}(t,f)$ suggests that $\tilde{Z}_{\mathbf{H}}(t,f)$ should be a smooth function of f as it was required for $a(t,f)$. In addition, the interpretation of $\mathrm{TES}_x(t,f)$ via the transfer function of $\mathbf{H}$ is meaningful only if $\mathbf{H}$ acts as a weighting in the TF domain (i.e., the TF shifts caused by $\mathbf{H}$ are negligible). This restriction was already encountered in the context of the ES and will be further considered in the next section.

IV. Underspread Systems and Processes

We have argued above that a meaningful interpretation of the ES in terms of a process expansion into uncorrelated narrowband signals is restricted to quasistationary processes, while the interpretation of the TES in terms of an expansion into uncorrelated short pulses is restricted to quasiwhite processes. These restrictions correspond to the requirement that the transfer functions $Z_{\mathbf{H}}(t,f)$ and $\tilde{Z}_{\mathbf{H}}(t,f)$ be smooth with respect to t and f, respectively, and that the innovations system $\mathbf{H}$ act as a pure TF weighting in the sense that it introduces negligible TF displacements. For later use, we will now discuss characterizations of the TF displacements of the innovations system and of the correlation structure of the resulting process.

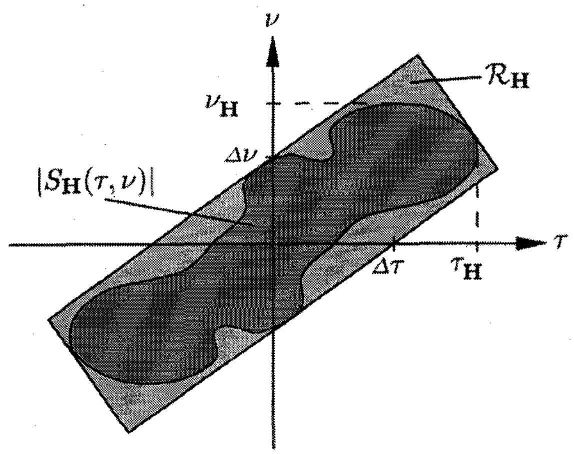


Fig. 1. Effective support of the spreading function of a self-adjoint LTV system $\mathbf{H}$.

A. Underspread Systems

The TF shifts caused by an LTV system $\mathbf{H}$ are characterized by the *generalized spreading function* [23], [25]

$$S_{\mathbf{H}}^{(\alpha)}(\tau,\nu) = \int_t H^{(\alpha)}(t,\tau)e^{-j2\pi\nu t}\,dt \tag{20}$$

with

$$H^{(\alpha)}(t,\tau) = H\left(t+\left(\tfrac{1}{2}-\alpha\right)\tau, t-\left(\tfrac{1}{2}+\alpha\right)\tau\right).$$

Here, τ and ν denote time lag and frequency lag, respectively, and α is a real parameter. Since the magnitude of $S_{\mathbf{H}}^{(\alpha)}(\tau,\nu)$ is independent of α, we will use the simplified notation $|S_{\mathbf{H}}(\tau,\nu)| = |S_{\mathbf{H}}^{(\alpha)}(\tau,\nu)|$. For given (τ,ν), $|S_{\mathbf{H}}(\tau,\nu)|$ indicates how much the TF-shifted input signal $(\mathbf{S}_{\tau,\nu}x)(t) = x(t-\tau)e^{j2\pi\nu t}$ contributes to the output signal [25]. It follows that the TF shifts caused by an LTV system are crudely characterized by the effective support of $|S_{\mathbf{H}}(\tau,\nu)|$. Let us define the "displacement spread" $\sigma_{\mathbf{H}}$ [23], [24] as the area of the smallest rectangle $\mathcal{R}_{\mathbf{H}}$ (centered about the origin of the (τ,ν) plane) containing the effective support of $|S_{\mathbf{H}}(\tau,\nu)|$, as shown in Fig. 1. A system $\mathbf{H}$ is called *underspread* if $\sigma_{\mathbf{H}} \ll 1$, meaning that $|S_{\mathbf{H}}(\tau,\nu)|$ is concentrated about the origin of the (τ,ν) plane so that $\mathbf{H}$ causes only small TF shifts. Furthermore, an underspread system $\mathbf{H}$ will be called *strictly underspread* if $\mathcal{R}_{\mathbf{H}}$ is oriented parallel to the τ and ν axes, such that $\Delta\tau = \tau_{\mathbf{H}}$ and $\Delta\nu = \nu_{\mathbf{H}}$, where $\Delta\tau$, $\Delta\nu$, $\tau_{\mathbf{H}}$, and $\nu_{\mathbf{H}}$ are defined in Fig. 1. Here, $\sigma_{\mathbf{H}} = 4\tau_{\mathbf{H}}\nu_{\mathbf{H}}$ so that a system is strictly underspread if $4\tau_{\mathbf{H}}\nu_{\mathbf{H}} \ll 1$. Quasi-LTI systems with small frequency shifts (small $\nu_{\mathbf{H}}$) and quasi-LFI systems with small time shifts (small $\tau_{\mathbf{H}}$) are potentially strictly underspread systems.

B. Underspread Processes

Quasistationary processes have small spectral correlation, whereas quasiwhite processes have small temporal correlation. These two situations are generalized by the concept of *underspread processes*. We first define the *expected ambiguity function* (EAF) [22], [23] of a nonstationary process $x(t)$ as

$$\begin{aligned}\overline{A}_x(\tau,\nu) &= \mathrm{E}\{\langle \mathbf{S}_{-\tau/2,-\nu/2}\,x, \mathbf{S}_{\tau/2,\nu/2}\,x\rangle\} \\ &= \int_t R_x\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big) e^{-j2\pi\nu t}\, dt\end{aligned}$$

where again $(\mathbf{S}_{\tau,\nu}x)(t) = x(t-\tau)e^{j2\pi\nu t}$. The EAF describes the average correlation of all TF locations separated by τ in time and by ν in frequency. Noting that the EAF is the spreading function (with $\alpha = 0$) of the correlation operator $\mathbf{R}_x$, that is, $\overline{A}_x(\tau,\nu) = S^{(0)}_{\mathbf{R}_x}(\tau,\nu)$, we are led to define the *TF correlation spread* σ_x of $x(t)$ as the displacement spread of the correlation operator, i.e., $\sigma_x \triangleq \sigma_{\mathbf{R}_x}$. Then, a process $x(t)$ is underspread if $\sigma_x \ll 1$ [22], [23], which means that the EAF is concentrated about the origin of the (τ,ν)-plane, and hence, that components of $x(t)$ that are sufficiently separated in the TF plane will be nearly uncorrelated.

The effective support of the EAF can also be described by the quantities $\tau_x \triangleq \tau_{\mathbf{R}_x}$ and $\nu_x \triangleq \nu_{\mathbf{R}_x}$ (see Fig. 1), and $x(t)$ will be called *strictly underspread* if the correlation operator $\mathbf{R}_x$ is strictly underspread, i.e., $4\tau_x\nu_x \ll 1$. Two potential special cases of strictly underspread processes are quasistationary processes (with small ν_x) and quasiwhite processes (with small τ_x).

The TF shifts caused by the innovations system $\mathbf{H}$ are related to the TF correlation structure of the associated process $x(t)$. It can be shown [23] that the correlation spread of $x(t)$ is bounded in terms of the displacement spread of $\mathbf{H}$ as $\sigma_x \le 4\sigma_{\mathbf{H}}$. Hence, an underspread innovations system implies an underspread process. Conversely, if $x(t)$ is (strictly) underspread, then it is not true that every innovations system $\mathbf{H}$ is (strictly) underspread, even though a (strictly) underspread $\mathbf{H}$ can always be found (see Section VI).

V. The Generalized Evolutionary Spectrum

A. Definition and Interpretation

The expressions $\mathrm{ES}_x(t,f) = |Z_{\mathbf{H}}(t,f)|^2$ and $\mathrm{TES}_x(t,f) = |\tilde{Z}_{\mathbf{H}}(t,f)|^2$ suggest an extension of the ES and TES. Indeed, the time-varying transfer functions $Z_{\mathbf{H}}(t,f)$ and $\tilde{Z}_{\mathbf{H}}(t,f)$ are just two special cases of a family of time-varying transfer functions known as *generalized Weyl symbol* [23], [25] and defined as

$$L^{(\alpha)}_{\mathbf{H}}(t,f) = \int_\tau H^{(\alpha)}(t,\tau)e^{-j2\pi f\tau}\, d\tau$$

where $H^{(\alpha)}(t,\tau)$ has been defined in (20). The transfer functions underlying the ES and the TES are reobtained with $\alpha = 1/2$ and $\alpha = -1/2$, respectively, i.e.,

$$Z_{\mathbf{H}}(t,f) = L^{(1/2)}_{\mathbf{H}}(t,f) \quad \text{and} \quad \tilde{Z}_{\mathbf{H}}(t,f) = L^{(-1/2)}_{\mathbf{H}}(t,f).$$

We now introduce the *generalized evolutionary spectrum* (GES) as [21]

$$\mathrm{GES}^{(\alpha)}_x(t,f) \triangleq |L^{(\alpha)}_{\mathbf{H}}(t,f)|^2. \tag{21}$$

The GES comprises the ES and TES as special cases with $\alpha = 1/2$ and $\alpha = -1/2$, respectively:

$$\begin{aligned}\mathrm{ES}_x(t,f) &= \mathrm{GES}^{(1/2)}_x(t,f) \\ \mathrm{TES}_x(t,f) &= \mathrm{GES}^{(-1/2)}_x(t,f).\end{aligned}$$

The definition of the GES in (21) contains a twofold ambiguity corresponding to the choice of the innovations system $\mathbf{H}$ and of the parameter α. This will be discussed in Sections VI and VII.

For a strictly underspread process, one can always find an innovations system $\mathbf{H}$ that is strictly underspread (see Section VI); here, the primary effect of $\mathbf{H}$ is a TF weighting, or equivalently, the TF shifts caused by $\mathbf{H}$ are small. Let us for the moment assume that the GES is based on this $\mathbf{H}$.

The average energy content of a process $x(t)$ around a TF analysis point (t,f) can be measured by the *physical spectrum* [4] $P_x(t,f) = \mathrm{E}\{|\langle x, g^{(t,f)}\rangle|^2\}$, where $g^{(t,f)}(t') = g(t'-t)e^{j2\pi ft'}$ with $g(t')$ being a normalized "window" or "test function" that is real-valued, even, and concentrated about the origin of the TF plane (note that $g^{(t,f)}(t')$ is then normalized too and is properly concentrated about the analysis TF point (t,f)). For a *strictly underspread* innovations system $\mathbf{H}$, it is proved in Appendix A that

$$\mathrm{GES}^{(\alpha)}_x(t,f) \approx P_x(t,f) \tag{22}$$

which shows that the GES is here physically meaningful. For $\alpha = 0$, (22) holds even when $\mathbf{H}$ is (weakly) underspread. Hence, the GES with $\alpha = 0$ is meaningful for a wider class of processes (see Section VII).

From the fact that the right-hand side of the approximation (22) is independent of α, it follows that the GES based on a strictly underspread innovations system is approximately independent of α, i.e., $\mathrm{GES}^{(\alpha_1)}_x(t,f) \approx \mathrm{GES}^{(\alpha_2)}_x(t,f)$. Let us compare two GES that are based on the same (strictly underspread) innovations system $\mathbf{H}$ but have different α values. It can then be shown (see Appendix B) that the difference $\Delta(t,f) = \mathrm{GES}^{(\alpha_1)}_x(t,f) - \mathrm{GES}^{(\alpha_2)}_x(t,f)$ is bounded both in a pointwise and an L_2-norm sense:

$$\begin{aligned}|\Delta(t,f)| &\le \pi|\Delta\alpha|(4\tau_{\mathbf{H}}\nu_{\mathbf{H}})^3\, \mathrm{tr}^2\{\mathbf{R}^{1/2}_x\} \\ \|\Delta\| &\le 8\pi|\Delta\alpha|(4\tau_{\mathbf{H}}\nu_{\mathbf{H}})^{3/2}\, \mathrm{tr}\{\mathbf{R}_x\}\end{aligned} \tag{23}$$

where $\Delta\alpha = \alpha_1 - \alpha_2, \tau_{\mathbf{H}}$ and $\nu_{\mathbf{H}}$ have been defined in Fig. 1, $\mathrm{tr}\{\cdot\}$ denotes the trace of an operator [27], and $\mathbf{R}^{1/2}_x$ is the positive semidefinite, self-adjoint operator square root of $\mathbf{R}_x$. Hence, for a strictly underspread innovations system, the choice of the GES parameter α is not critical. This is not true for weakly underspread processes since here no innovations system with small $\tau_{\mathbf{H}}\nu_{\mathbf{H}}$ exists.

The generalized spreading function introduced in Section IV can be shown [23], [25] to be the 2-D Fourier transform of the generalized Weyl symbol underlying the definition of the GES:

$$S^{(\alpha)}_{\mathbf{H}}(\tau,\nu) = \int_t \int_f L^{(\alpha)}_{\mathbf{H}}(t,f)e^{-j2\pi(\nu t-\tau f)}\, dt\, df. \tag{24}$$

For an underspread $\mathbf{H}$, $S^{(\alpha)}_{\mathbf{H}}(\tau,\nu)$ is concentrated about the origin of the (τ,ν) plane, and thus (24) implies that $L^{(\alpha)}_{\mathbf{H}}(t,f)$, and hence $\mathrm{GES}_x^{(\alpha)}(t,f)$, is a 2-D lowpass function (i.e., a *smooth* function). In particular, a small spread of $\mathbf{H}$ in the ν direction implies that $L_{\mathbf{H}}^{(\alpha)}(t,f)$ and $\mathrm{GES}_x^{(\alpha)}(t,f)$ are smooth with respect to t, whereas a small spread in the τ direction implies that $L_{\mathbf{H}}^{(\alpha)}(t,f)$ and $\mathrm{GES}_x^{(\alpha)}(t,f)$ are smooth with respect to f. These are the two situations allowing a meaningful interpretation of the ES and TES, respectively. This is consistent, as the response of a quasi-LTI system to stationary white noise is a quasistationary process, and the response of a quasi-LFI system to stationary white noise is a quasiwhite process.

B. Properties

In the following, we discuss some important properties of the GES.

1) Consistency: For a stationary process, the GES can be shown to reduce to the PSD and to be independent of $t, \mathrm{GES}_x^{(\alpha)}(t,f) \equiv S_x(f)$. For a nonstationary white process, the GES reduces to the average instantaneous intensity and is independent of $f, \mathrm{GES}_x^{(\alpha)}(t,f) \equiv q_x(t)$. For a stationary white process with $S_x(f) = q_x(t) \equiv N_0$, the GES is constant over the entire TF plane, $\mathrm{GES}_x^{(\alpha)}(t,f) \equiv N_0$.

2) Positivity: The GES is real-valued and nonnegative $\mathrm{GES}_x^{(\alpha)}(t,f) \geq 0$.

3) Self-Adjoint Innovations System: It can be shown that $L_{\mathbf{H}}^{(-\alpha)}(t,f) = L_{\mathbf{H}^+}^{(\alpha)*}(t,f)$, where $\mathbf{H}^+$ is the adjoint of $\mathbf{H}$. Hence, the GES with parameters α and $-\alpha$ are identical for a self-adjoint innovations system $\mathbf{H}$:

$$\mathbf{H} = \mathbf{H}^+ \implies \mathrm{GES}_x^{(-\alpha)}(t,f) = \mathrm{GES}_x^{(\alpha)}(t,f).$$

In particular, $\mathbf{H} = \mathbf{H}^+$ implies $\mathrm{ES}_x(t,f) = \mathrm{TES}_x(t,f)$. Thus, even though the ES and TES are based on different expansion models, they will produce identical results if they are based on a self-adjoint innovations system (which can always be found; see Section VI).

4) Marginals: The ES has correct time marginals

$$\int_f \mathrm{GES}_x^{(1/2)}(t,f)\, df = \mathrm{E}\{|x(t)|^2\}$$

while the TES has correct frequency marginals

$$\int_t \mathrm{GES}_x^{(-1/2)}(t,f)\, dt = \mathrm{E}\{|X(f)|^2\}.$$

If (and only if) the innovations system is *normal*, $\mathbf{H}\mathbf{H}^+ = \mathbf{H}^+\mathbf{H}$, then both the ES and the TES will satisfy both marginal properties, i.e., we also have

$$\int_t \mathrm{GES}_x^{(1/2)}(t,f)\, dt = \mathrm{E}\{|X(f)|^2\} \quad \text{and}$$
$$\int_f \mathrm{GES}_x^{(-1/2)}(t,f)\, df = \mathrm{E}\{|x(t)|^2\}.$$

It can be shown that underspread systems are approximately normal [23].

For $\alpha \neq \pm 1/2$, the marginal properties will *not* be satisfied exactly, but they will be approximately satisfied for strictly underspread innovations systems. Indeed, the deviation between the time marginal of the GES and the expected instantaneous power $\Delta_1(t) \triangleq \int_f \mathrm{GES}_x^{(\alpha)}(t,f)\, df - \mathrm{E}\{|x(t)|^2\}$ can be bounded as

$$|\Delta_1(t)| \leq 4\pi|\alpha - \tfrac{1}{2}|\, \nu_{\mathbf{H}}\, (4\tau_{\mathbf{H}}\nu_{\mathbf{H}})^2 \operatorname{tr}^2\{\mathbf{R}_x^{1/2}\}$$
$$\|\Delta_1\| \leq 2\pi|\alpha - \tfrac{1}{2}|\, \sqrt{\nu_{\mathbf{H}}}\, (4\tau_{\mathbf{H}}\nu_{\mathbf{H}}) \operatorname{tr}\{\mathbf{R}_x\}$$

where $\tau_{\mathbf{H}}$ and $\nu_{\mathbf{H}}$ have been defined in Fig. 1. Similarly, the deviation between the frequency marginal of the GES and the expected spectral energy density $\Delta_2(f) \triangleq \int_t \mathrm{GES}_x^{(\alpha)}(t,f)\, dt - \mathrm{E}\{|X(f)|^2\}$ can be bounded as

$$|\Delta_2(f)| \leq 4\pi|\alpha + \tfrac{1}{2}|\tau_{\mathbf{H}}(4\tau_{\mathbf{H}}\nu_{\mathbf{H}})^2 \operatorname{tr}^2\{\mathbf{R}_x^{1/2}\}$$
$$\|\Delta_2\| \leq 2\pi|\alpha + \tfrac{1}{2}|\sqrt{\tau_{\mathbf{H}}}(4\tau_{\mathbf{H}}\nu_{\mathbf{H}}) \operatorname{tr}\{\mathbf{R}_x\}.$$

Note that the above bounds correctly reflect the fact that the ES ($\alpha = 1/2$) has correct time marginals, while the TES ($\alpha = -1/2$) has correct frequency marginals.

5) Finite Support: From the marginal and positivity properties, it follows that the GES with $\alpha = \pm 1/2$ and normal $\mathbf{H}$ satisfies the following "strong" finite support properties:

$$\mathrm{E}\{|x(t_0)|^2\} = 0 \implies \mathrm{GES}_x^{(\pm 1/2)}(t_0,f) = 0$$
$$\mathrm{E}\{|X(f_0)|^2\} = 0 \implies \mathrm{GES}_x^{(\pm 1/2)}(t,f_0) = 0.$$

For $|\alpha| \leq 1/2$ and normal $\mathbf{H}$, the GES satisfies the following "weak" finite support properties:

$$\mathrm{E}\{|x(t)|^2\} = 0,\ t \notin [t_1,t_2] \implies \mathrm{GES}_x^{(\alpha)}(t,f) = 0,\ t \notin [t_1,t_2]$$
$$\mathrm{E}\{|X(f)|^2\} = 0,\ f \notin [f_1,f_2] \implies \mathrm{GES}_x^{(\alpha)}(t,f) = 0,\ f \notin [f_1,f_2].$$

6) TF Shift and Scaling Covariance: Let $\mathrm{GES}_x^{(\alpha)}(t,f)$ be based on an innovations system $\mathbf{H}_x$. If $x(t)$ is shifted in time by τ and in frequency by ν, i.e., $\tilde{x}(t) = (\mathbf{S}_{\tau,\nu}x)(t) = x(t-\tau)e^{j2\pi\nu t}$, then the correlation operator of the shifted process $\tilde{x}(t)$ is $\mathbf{R}_{\tilde{x}} = \mathbf{S}_{\tau,\nu}\mathbf{R}_x\mathbf{S}_{\tau,\nu}^+$, and a *specific* innovations system of $\tilde{x}(t)$ is $\hat{\mathbf{H}}_{\tilde{x}} = \mathbf{S}_{\tau,\nu}\mathbf{H}_x\mathbf{S}_{\tau,\nu}^+$. If (and only if) the innovations system used for calculating the GES of $\tilde{x}(t)$ is chosen as $\mathbf{H}_{\tilde{x}} = \hat{\mathbf{H}}_{\tilde{x}}$, then the GES of the shifted process is an appropriately shifted GES

$$\mathrm{GES}_{\tilde{x}}^{(\alpha)}(t,f) = \mathrm{GES}_x^{(\alpha)}(t-\tau, f-\nu).$$

By choosing the respective *positive semidefinite* innovations systems for both $x(t)$ and $\tilde{x}(t)$ (see Section VI), it is guaranteed that $\mathbf{H}_{\tilde{x}} = \hat{\mathbf{H}}_{\tilde{x}}$ so that the above "covariance property" will always be satisfied. In a similar manner, it can be shown that the GES will satisfy the covariance property with respect to a TF scaling

$$\tilde{x}(t) = \sqrt{|a|}x(at) \implies \mathrm{GES}_{\tilde{x}}^{(\alpha)}(t,f) = \mathrm{GES}_x^{(\alpha)}\left(at, \frac{f}{a}\right)$$

if it is based on the positive semidefinite innovations system.

7) LTV System: If $x(t)$ is transformed by a positive semidefinite LTV system $\mathbf{K}$ with kernel $K(t,t')$

$$y(t) = (\mathbf{K}x)(t) = \int_{t'} K(t,t')x(t')\,dt'$$

then the correlation operator of the transformed process $y(t)$ is $\mathbf{R}_y = \mathbf{K}\mathbf{R}_x\mathbf{K} = \mathbf{K}\mathbf{H}_x\mathbf{H}_x^+\mathbf{K}$, where $\mathbf{H}_x$ is an innovations system of $x(t)$. For reasons to be explained in Section VI, we choose $\mathbf{H}_x$ to be the *positive semidefinite* innovations system, and we look for the positive semidefinite innovations system $\mathbf{H}_y$ of $y(t)$. Let us assume that $\mathbf{K}$ and $\mathbf{H}_x$ are *jointly* strictly underspread in the sense that the effective supports of their spreading functions are both bounded by the *same* rectangular region that is parallel to the τ and ν axes and whose area is much less then 1 (this requires that both systems $\mathbf{K}$ and $\mathbf{H}_x$ are individually strictly underspread, i.e., $4\tau_{\mathbf{H}_x}\nu_{\mathbf{H}_x} \ll 1$ and $4\tau_{\mathbf{K}}\nu_{\mathbf{K}} \ll 1$). It can then be shown that $\mathbf{H}_y \approx \mathbf{K}^{1/2}\mathbf{H}_x\mathbf{K}^{1/2}$ and $L^{(\alpha)}_{\mathbf{K}^{1/2}\mathbf{H}_x\mathbf{K}^{1/2}}(t,f) \approx L^{(\alpha)}_{\mathbf{K}^{1/2}}(t,f)L^{(\alpha)}_{\mathbf{H}_x}(t,f)L^{(\alpha)}_{\mathbf{K}^{1/2}}(t,f) \approx L^{(\alpha)}_{\mathbf{K}}(t,f)L^{(\alpha)}_{\mathbf{H}_x}(t,f)$ [23], so that the GES of $y(t)$ is

$$\begin{aligned}\mathrm{GES}_y^{(\alpha)}(t,f) &\approx |L^{(\alpha)}_{\mathbf{K}^{1/2}\mathbf{H}_x\mathbf{K}^{1/2}}(t,f)|^2\\ &\approx |L^{(\alpha)}_{\mathbf{K}}(t,f)L^{(\alpha)}_{\mathbf{H}_x}(t,f)|^2\\ &= |L^{(\alpha)}_{\mathbf{K}}(t,f)|^2\,\mathrm{GES}_x^{(\alpha)}(t,f).\end{aligned} \tag{25}$$

Thus, the GES of the output process is approximately equal to the GES of the input process multiplied by the squared magnitude of the generalized Weyl symbol of the LTV system $\mathbf{K}$. This relation suggests to interpret the effect of $\mathbf{K}$ as a TF weighting characterized by $|L^{(\alpha)}_{\mathbf{K}}(t,f)|^2$. It generalizes the relation $S_y(f) = |G(f)|^2S_x(f)$ obtained when a stationary process is transformed by an LTI system with transfer function $G(f)$ and the relation $q_y(t) = |m(t)|^2q_x(t)$ obtained when a nonstationary white process is transformed by an LFI system with multiplier function $m(t)$.

VI. The Factorization Problem

For given correlation operator $\mathbf{R}_x$, the innovations systems $\mathbf{H}$ are defined by (11) or equivalently

$$\mathbf{H}\mathbf{H}^+ = \mathbf{R}_x. \tag{26}$$

The solution to this *factorization problem* is not unique. Indeed, if $\mathbf{H}$ is a valid innovations system satisfying (26), and if $\mathbf{U}$ is an arbitrary unitary operator (satisfying $\mathbf{U}\mathbf{U}^+ = \mathbf{I}$), then $\tilde{\mathbf{H}} = \mathbf{H}\mathbf{U}$ is an innovations system as well: $\tilde{\mathbf{H}}\tilde{\mathbf{H}}^+ = \mathbf{H}\mathbf{U}\mathbf{U}^+\mathbf{H}^+ = \mathbf{H}\mathbf{H}^+ = \mathbf{R}_x$.

In the stationary case, the innovations systems $\mathbf{H}$ are time invariant and have identical transfer function magnitude. A similar situation exists in the nonstationary white case. However, in the general nonstationary case, different choices of $\mathbf{H}$ will lead to different generalized Weyl symbol magnitudes and, hence, to different GES results. This ambiguity of the GES definition can be resolved by imposing certain constraints on $\mathbf{H}$. For example, the *Wold-Cramér ES* [9], [10] is obtained with a *causal* $\mathbf{H}$. In this section, however, we will discuss the advantages of the *positive semidefinite* $\mathbf{H}$.

It is reasonable to adopt the "maximally underspread" innovations system $\mathbf{H}$ for which *TF displacement effects* are minimized (see Section IV). This system primarily produces a TF weighting that can be described by the squared magnitude of the generalized Weyl symbol, which is the GES. This permits the interpretation of the GES as an average TF energy distribution and is also consistent with the conditions that $Z_{\mathbf{H}}(t,f)$ and $\tilde{Z}_{\mathbf{H}}(t,f)$ be smooth with respect to t and f, respectively. In the following, the maximally underspread $\mathbf{H}$ will be defined as the $\mathbf{H}$ minimizing the *TF displacement radius*

$$\rho_{\mathbf{H}}^2 \triangleq \frac{\overline{\tau_{\mathbf{H}}^2}}{T^2} + T^2\overline{\nu_{\mathbf{H}}^2}$$

where T is an arbitrary normalization time constant and

$$\overline{\tau_{\mathbf{H}}^2} \triangleq \frac{\int_\tau\int_\nu \tau^2|S_{\mathbf{H}}(\tau,\nu)|^2\,d\tau\,d\nu}{\int_\tau\int_\nu |S_{\mathbf{H}}(\tau,\nu)|^2\,d\tau\,d\nu}$$

$$\overline{\nu_{\mathbf{H}}^2} \triangleq \frac{\int_\tau\int_\nu \nu^2|S_{\mathbf{H}}(\tau,\nu)|^2\,d\tau\,d\nu}{\int_\tau\int_\nu |S_{\mathbf{H}}(\tau,\nu)|^2\,d\tau\,d\nu} \tag{27}$$

measure the extension of the spreading function in the τ and ν direction, respectively. The minimization of $\rho_{\mathbf{H}}$ will be performed within the class of *normal* $\mathbf{H}$ (satisfying $\mathbf{H}\mathbf{H}^+ = \mathbf{H}^+\mathbf{H}$). A normal system is advantageous since here *both* marginal properties are satisfied by both the ES and the TES (see Section V-B). Any normal innovations system $\mathbf{H}$ allows a *polar decomposition* [27]

$$\mathbf{H} = \mathbf{H}_p\mathbf{U} \tag{28}$$

where $\mathbf{H}_p$ (satisfying $\mathbf{H}_p\mathbf{H}_p^+ = \mathbf{R}_x$) is the positive semidefinite, self-adjoint operator square root of $\mathbf{R}_x, \mathbf{H}_p = \mathbf{R}_x^{1/2}$, and $\mathbf{U}$ is a unitary operator constrained by the normality of $\mathbf{H}$. The kernel of $\mathbf{H}_p$ is

$$H_p(t,t') = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,u_k(t)u_k^*(t')$$

where $\lambda_k \geq 0$ are the eigenvalues and $u_k(t)$ are the eigenfunctions of the correlation operator $\mathbf{R}_x$.

Since $\mathbf{H}_p$ is fixed, the unitary factor $\mathbf{U}$ minimizing the TF displacement radius $\rho_{\mathbf{H}}^2$ remains to be chosen. It is shown in Appendix C that the solution to this problem is the identity operator up to a trivial phase factor that will be set to 1 in the following, i.e., $\mathbf{U}_{\mathrm{opt}} = \mathbf{I}$ [30]. Thus, the innovations system with minimum TF displacement radius is the positive semidefinite root of $\mathbf{R}_x$:

$$\mathbf{H}_{\mathrm{opt}} = \mathbf{H}_p = \mathbf{R}_x^{1/2}.$$

We note that this maximally underspread innovations system will lead to a generalized Weyl symbol and, in turn, a GES that is maximally smooth (cf. Section V-A).

Using the positive semidefinite root of $\mathbf{R}_x$ as innovations system has another important advantage. Consider a unitary transformation of the process, i.e., $\tilde{x}(t) = (\mathbf{U}x)(t)$, where $\mathbf{U}\mathbf{U}^{+} = \mathbf{I}$. The correlation operator of the new process $\tilde{x}(t)$ is $\mathbf{R}_{\tilde{x}} = \mathbf{U}\mathbf{R}_x\mathbf{U}^{+}$, and its positive semidefinite root is given by $\tilde{\mathbf{H}}_p = \mathbf{U}\mathbf{H}_p\mathbf{U}^{+}$, where $\mathbf{H}_p$ is the positive semidefinite root of $\mathbf{R}_x$. This relation between the innovations systems of $x(t)$ and $\tilde{x}(t)$ was seen in Section V-B to guarantee the shift covariance and scaling covariance properties of the GES when $\mathbf{U}$ is a TF shift operator and TF scaling operator, respectively. Positive semidefinite factorization is thus consistent with unitary signal transformations. This is not generally true for other types of factorizations.

VII. The Weyl Spectrum

A. Definition and Interpretation

The definition of the GES in Section V contained a twofold ambiguity, namely, the choice of the parameter α and that of the innovations system $\mathbf{H}$. We now consider the case $\alpha = 0$, where the generalized Weyl symbol reduces to the *Weyl symbol*, $L_{\mathbf{H}}(t,f) = L_{\mathbf{H}}^{(0)}(t,f)$ [31]–[34]. Furthermore, for reasons explained in the previous section, we adopt the positive semidefinite root $\mathbf{H}_p = \mathbf{R}_x^{1/2}$ of the correlation operator $\mathbf{R}_x$ as the innovations system. These two choices result in a member of the GES family given by[3]

$$\mathrm{WS}_x(t,f) \triangleq \mathrm{GES}_x^{(0)}(t,f)\Big|_{\mathbf{H}=\mathbf{H}_p} = L_{\mathbf{H}_p}^2(t,f) = \left[\int_\tau H_p\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,d\tau\right]^2.$$

This time-varying power spectrum will be called the *Weyl spectrum* (WS) [21]. Because it is the GES with $\alpha = 0$ that uses the positive semidefinite innovations system, the WS is uniquely defined for given $\mathbf{R}_x$. It has important advantages over all other GES members obtained for $\alpha \neq 0$ and/or $\mathbf{H} \neq \mathbf{R}_x^{1/2}$:

- The approximation $\mathrm{GES}_x^{(\alpha)}(t,f) \approx \mathrm{E}\{|\langle x, g^{(t,f)}\rangle|^2\}$ (see Section V-A) imparting an energetic interpretation to the GES holds for $\alpha \neq 0$ only if the process $x(t)$ is *strictly* underspread. In contrast, the WS will satisfy the above approximation even if $x(t)$ is merely *weakly* underspread, i.e., if $\sigma_x \ll 1$ but not $4\tau_x\nu_x \ll 1$. Hence, the WS is physically meaningful for a broader class of processes. In particular, the WS is much better suited to processes with "chirp components" (appearing as slanted structures in the TF plane) than is the ES or the TES. Some examples will be shown in Section VIII.
- The WS is based on the positive semidefinite innovations system that introduces minimal TF displacement effects. This favors the interpretation of the WS as a proper time-varying power spectrum. In addition, the use of the positive semidefinite innovations system is a prerequisite for covariance properties with respect to unitary signal transformations such as TF shifts or TF scalings (see Section VI).
- The WS is based on the Weyl symbol (generalized Weyl symbol with $\alpha = 0$) whose symmetric structure leads to important advantages over generalized Weyl symbols with $\alpha \neq 0$. This entails corresponding advantages of the WS over other members of the GES family. In particular, the WS satisfies certain covariance properties that are not satisfied by the GES with $\alpha \neq 0$, as detailed further below.

B. Properties

We now discuss the properties of the WS in more detail. Since the general properties of the GES have been discussed in Section V-B, we concentrate on WS properties that are *not* satisfied by other GES members.

1) TF Coordinate Transforms: There exist a class of unitary signal transformations $\mathbf{U}$ corresponding to area-preserving, affine TF coordinate transforms $(t,f) \to (at+bf-\tau, ct+df-\nu)$ with $\det\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = ad - bc = 1$. Specific signal transformations $\mathbf{U}$ depend on the TF coordinate transform parameters a, b, c, d, τ, and ν [35]. The WS satisfies the following general covariance property with respect to the unitary transformations $\mathbf{U}$:

$$\tilde{x}(t) = (\mathbf{U}x)(t) \implies \mathrm{WS}_{\tilde{x}}(t,f) = \mathrm{WS}_x(at+bf-\tau, ct+df-\nu). \quad (29)$$

Important special cases are listed in the following. For any set of parameters a, b, c, d, τ, and ν with $ad - bc = 1$, the corresponding signal transformation $\mathbf{U}$ can be composed of some of these special transformations.

- *TF Shifts*:
$$\tilde{x}(t) = (\mathbf{S}_{\tau,\nu}x)(t) = x(t-\tau)e^{j2\pi\nu t} \Rightarrow \mathrm{WS}_{\tilde{x}}(t,f) = \mathrm{WS}_x(t-\tau, f-\nu).$$
- *TF Scalings*:
$$\tilde{x}(t) = \sqrt{|a|}x(at) \Rightarrow \mathrm{WS}_{\tilde{x}}(t,f) = \mathrm{WS}_x\left(at, \frac{f}{a}\right).$$
- *Chirp Multiplication*:
$$\tilde{x}(t) = e^{j\pi ct^2}x(t) \Rightarrow \mathrm{WS}_{\tilde{x}}(t,f) = \mathrm{WS}_x(t, f-ct).$$
- *Chirp Convolution*:
$$\tilde{x}(t) = \sqrt{|c|}e^{j\pi ct^2} * x(t) \Rightarrow \mathrm{WS}_{\tilde{x}}(t,f) = \mathrm{WS}_x\left(t-\frac{f}{c}, f\right).$$
- *Fourier Transform*:
$$\tilde{x}(t) = \sqrt{|c|}X(ct) \Rightarrow \mathrm{WS}_{\tilde{x}}(t,f) = \mathrm{WS}_x\left(-\frac{f}{c}, ct\right).$$

We emphasize that the GES with $\alpha \neq 0$ satisfies only the covariance properties with respect to TF shifts and TF scalings provided that the innovations systems of $x(t)$ and $\tilde{x}(t)$ are related as $\mathbf{H}_{\tilde{x}} = \mathbf{U}\mathbf{H}_x\mathbf{U}^{+}$ (see Section V-B). The general covariance property (29) will not be satisfied for $\alpha \neq 0$.

[3]Note that the Weyl symbol of self-adjoint operators is real-valued so that $|L_{\mathbf{H}_p}(t,f)|^2 = L_{\mathbf{H}_p}^2(t,f)$.

2) Marginals: For the important class of (weakly) underspread processes, the marginal properties will be satisfied by the WS in an approximate manner. Specifically, the deviation between the time marginal of the WS and the expected instantaneous power $\Delta_1(t) \triangleq \int_f \mathrm{WS}_x(t,f)\, df - \mathrm{E}\{|x(t)|^2\}$ can be bounded as

$$|\Delta_1(t)| \le 8\pi\Delta\nu\, \sigma^2_{\mathbf{R}_x^{1/2}}\, \mathrm{tr}^2\{\mathbf{R}_x^{1/2}\}$$
$$\|\Delta_1\| \le 2\pi\sqrt{\Delta\nu}\, \sigma_{\mathbf{R}_x^{1/2}}\, \mathrm{tr}\{\mathbf{R}_x\}$$

and the deviation between the frequency marginal of the WS and the expected spectral energy density $\Delta_2(f) \triangleq \int_t \mathrm{WS}_x(t,f)\, dt - \mathrm{E}\{|X(f)|^2\}$ can be bounded as

$$|\Delta_2(f)| \le 8\pi\Delta\tau\, \sigma^2_{\mathbf{R}_x^{1/2}}\, \mathrm{tr}^2\{\mathbf{R}_x^{1/2}\}$$
$$\|\Delta_2\| \le 2\pi\sqrt{\Delta\tau}\, \sigma_{\mathbf{R}_x^{1/2}}\, \mathrm{tr}\{\mathbf{R}_x\}.$$

Here, $\Delta\nu$ and $\Delta\tau$ have been defined in Fig. 1. Since $\sigma_{\mathbf{R}_x^{1/2}} \ll 1$ for underspread processes, these bounds imply the approximate validity of the marginal properties. Note that for the GES with $\alpha \neq 0$, approximate validity of the marginal properties required $x(t)$ to be *strictly* underspread in general (see Section V-B).

3) Superposition Law: Let

$$x(t) = \sum_{k=1}^{N} x_k(t)$$

be the sum of N uncorrelated, zero-mean processes $x_k(t)$. Since $R_{x_k,x_l}(t,t') = E\{x_k(t)x_l^*(t')\} = 0$ for $k \neq l$, one has $\mathbf{R}_x = \sum_{k=1}^N \mathbf{R}_{x_k}$. In general, there is no simple way to express an innovations system of $x(t)$ in terms of innovations systems of the component processes $x_k(t)$. However, if the realizations of the $x_k(t)$ belong to orthogonal signal spaces, then it can be shown that the positive semidefinite root of $\mathbf{R}_x$ (cf. Section VI) is equal to the sum of the positive semidefinite roots of the $\mathbf{R}_{x_k}$, i.e., $\mathbf{H}_{p,x} = \sum_{k=1}^N \mathbf{H}_{p,x_k}$. By the linearity of the Weyl symbol, we then obtain

$$\begin{aligned}\mathrm{WS}_x(t,f) &= \left[\sum_{k=1}^{N} L_{\mathbf{H}_{p,x_k}}(t,f)\right]^2\\ &= \sum_{k=1}^{N} \mathrm{WS}_{x_k}(t,f)\\ &\quad + 2\sum_{k=1}^{N}\sum_{\substack{l=1\\k\neq l}}^{N} L_{\mathbf{H}_{p,x_k}}(t,f)L_{\mathbf{H}_{p,x_l}}(t,f).\end{aligned}$$

With the assumption that the realizations of the processes $x_k(t)$ are *TF disjoint* (which then also implies that they belong to orthogonal signal spaces [36]), the cross terms $2L_{\mathbf{H}_{p,x_k}}(t,f)L_{\mathbf{H}_{p,x_l}}(t,f)$ with $k \neq l$ vanish since the $L_{\mathbf{H}_{p,x_k}}(t,f)$ do not overlap. We then obtain the *superposition law*

$$\mathrm{WS}_x(t,f) = \sum_{k=1}^{N} \mathrm{WS}_{x_k}(t,f). \tag{30}$$

In practice, the $x_k(t)$ will typically be *effectively* TF disjoint rather than exactly TF disjoint, in which case (30) is valid in an approximate sense.

4) Deterministic Signal Components: Let us now assume that $x_k(t) = \alpha_k s_k(t)$, i.e.,

$$x(t) = \sum_{k=1}^{N} \alpha_k s_k(t) \tag{31}$$

with deterministic, orthonormal signals $s_k(t)$ and uncorrelated, zero-mean random factors α_k with powers $\rho_k^2 = \mathrm{E}\{|\alpha_k|^2\}$. The positive semidefinite innovations system of $x_k(t)$ is given by $H_{p,x_k}(t,t') = \rho_k s_k(t)s_k^*(t')$. If the $s_k(t)$ are TF disjoint in the sense that their Wigner distributions $W_{s_k}(t,f)$ [36]–[40] do not overlap, then we obtain with (30)

$$\mathrm{WS}_x(t,f) = \sum_{k=1}^{N} \rho_k^2 W_{s_k}^2(t,f). \tag{32}$$

5) Chirp Processes: The WS features superior TF concentration for "chirp processes" corresponding to slanted structures in the TF plane. Let us consider a chirp process $x(t) = \alpha w(t)e^{j\pi ct^2}$ with zero-mean random factor α and deterministic envelope $w(t)$. The WS is obtained as

$$\mathrm{WS}_x(t,f) = \rho_\alpha^2 W_w^2(t, f - ct)$$

where $W_w(t,f)$ is the Wigner distribution of $w(t)$. This result shows that the WS of a chirp process is well concentrated along the instantaneous frequency $f_x(t) = ct$. This can be generalized to multicomponent chirp signals whose components are approximately nonoverlapping in the TF plane [see (31)]. Numerical examples illustrating these results are provided in Section VIII.

C. Comparison with the Wigner–Ville Spectrum

The *Wigner–Ville spectrum* (WVS) [1]–[3] is an important time-varying spectrum defined as the Weyl symbol of the correlation operator

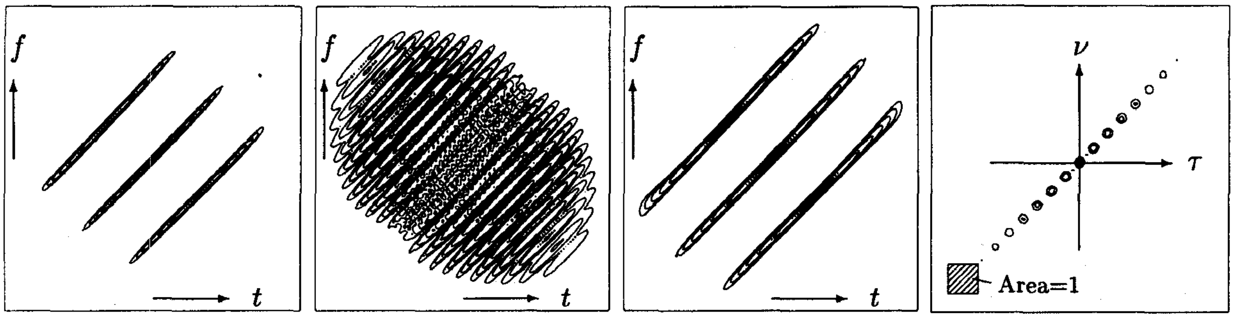
$$\overline{W}_x(t,f) = L_{\mathbf{R}_x}(t,f) = \int_\tau R_x\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\, d\tau.$$

The WVS satisfies many desirable properties; in particular, it is unitarily related to the correlation function $R_x(t_1,t_2)$, and it satisfies both marginal properties. However, it may assume negative values [41].

In the case of an *underspread* process $x(t)$, the WS and WVS yield very similar results. Indeed, for $\mathbf{R}_x$ underspread, it can be shown (using techniques similar to those used in Appendices A and B) that $L^2_{\mathbf{R}_x^{1/2}}(t,f) \approx L_{\mathbf{R}_x}(t,f)$, i.e., taking the square root of the correlation operator is approximately compensated by taking the square of the resulting Weyl symbol. Hence

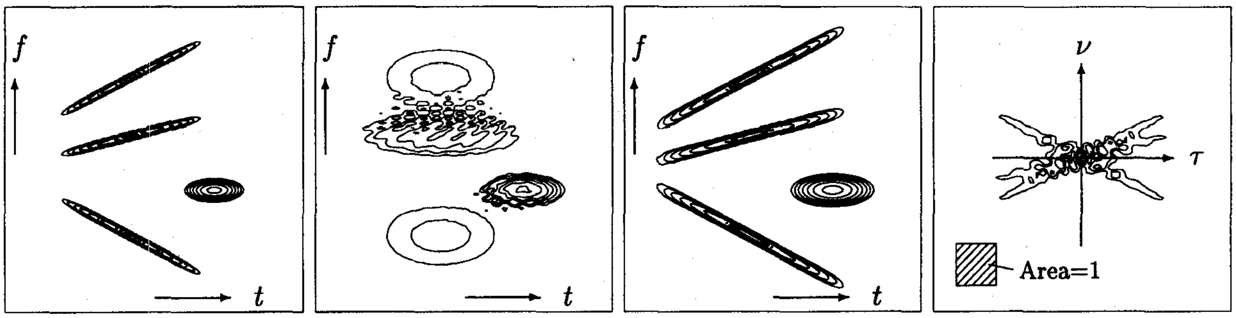
$$\mathrm{WS}_x(t,f) = L^2_{\mathbf{R}_x^{1/2}}(t,f) \approx L_{\mathbf{R}_x}(t,f) = \overline{W}_x(t,f).$$

Note that the approximate equivalence of the WS and WVS for an underspread process implies that the WVS of an underspread process is approximately nonnegative.



(a) (b) (c) (d)

Fig. 2. Synthetic process consisting of three "parallel" chirp signals. (a) WS. (b) ES/TES with positive semidefinite innovations system. (c) WVS. (d) Magnitude of EAF (a hatched square of area 1 is included in this and subsequent plots to allow an assessment of the process's underspread property).



(a) (b) (c) (d)

Fig. 3. Synthetic process consisting of three "nonparallel" chirp signals and a Gaussian signal. (a) WS. (b) ES/TES with positive semidefinite innovations system. (c) WVS. (d) EAF magnitude.

Although the WS and WVS are similar for underspread processes, they may be quite different otherwise. Let us reconsider the multicomponent process in (31) consisting of deterministic, TF disjoint signals $s_k(t)$ with statistically independent random factors α_k. WS and WVS are here obtained as [cf. (32)]

$$\mathrm{WS}_x(t,f) = \sum_{k=1}^{N} \rho_k^2 W_{s_k}^2(t,f)$$

$$\overline{W}_x(t,f) = \sum_{k=1}^{N} \rho_k^2 W_{s_k}(t,f).$$

We see that the WVS is given by a weighted superposition of the Wigner distributions of the individual components, whereas in the WS, these Wigner distributions are squared. This squaring entails a sharper representation of the process components in the WS (see Section VIII).

VIII. Numerical Simulations

We now apply the WS and GES to the TF analysis of synthetic and real-data processes. The duration of all processes considered is 128 samples. Our first example, which is shown in Fig. 2, illustrates the superiority of the WS over other members of the GES family in the case of chirp processes. The (synthetic) random process under analysis is of the type (31); it consists of three TF-shifted windowed "parallel" chirp signals $s_k(t) = w(t-t_k)e^{j\pi c(t-t_k)^2}e^{j2\pi f_k t}$ with identical chirp rates c and statistically independent amplitude factors α_k with equal average powers. The EAF in part (d) shows that this process is reasonably underspread but not strictly underspread. As a consequence, the WS performs satisfactorily, whereas the ES (simultaneously the TES due to the use of the positive semidefinite innovations system) totally fails to resolve the three chirp components. The WVS, which is shown for comparison, performs satisfactorily as well. Fig. 3 shows that the good performance of the WS extends to the case where the overall process is not underspread but all process components are TF disjoint and individually underspread. The process underlying Fig. 3 consists of three windowed "nonparallel" chirp signals $s_k(t) = w(t)e^{j\pi c_k t^2}$ (with different chirp rates c_k) and a Gaussian signal, again with statistically independent amplitude factors α_k. Note that the ES/TES does not correctly indicate the frequency modulation of the three chirp components.

Whereas the WS and the ES/TES yielded dramatically different results in Figs. 2 and 3, Fig. 4 shows that these spectra become very similar for *strictly* underspread processes. The process under analysis, whose correlation function was constructed using the TF synthesis method proposed in [42], consists of three uncorrelated random components appearing as smooth structures in the TF plane. The EAF shows that the process is indeed strictly underspread. The strong similarity of the WS and the ES/TES corroborates the approximate α-invariance of the GES in the case of strictly underspread processes (see Section V-A). The WS and ES/TES are also very similar to the WVS, as predicted in Section VII-C.

Fig. 5 corroborates the approximation (25) for the GES of a filtered process. The three-component process $x(t)$ from Fig. 4

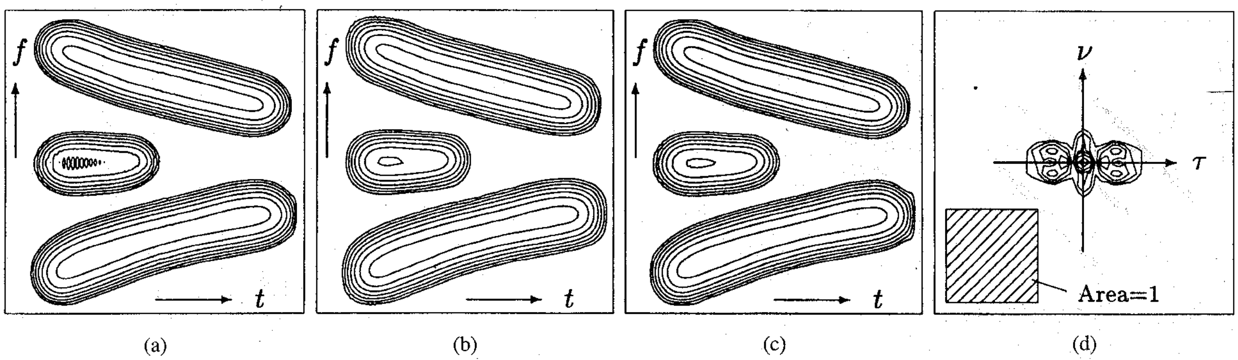


Fig. 4. Strictly underspread synthetic process. (a) WS. (b) ES/TES with positive semidefinite innovations system. (c) WVS. (d) EAF magnitude.

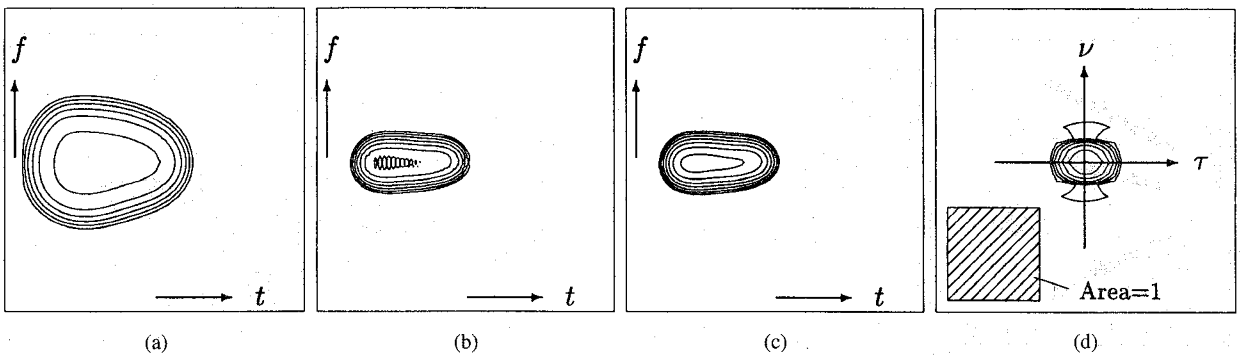


Fig. 5. LTV filtering of the strictly underspread process from Fig. 4. (a) Squared Weyl symbol of LTV filter $\mathbf{K}$. (b) WS of input process [see Fig. 4(a)] multiplied by squared Weyl symbol of $\mathbf{K}$ (approximation to WS of filter output process). (c) Exact WS of filter output process. (d) Magnitude of spreading function of $\mathbf{K}$.

was filtered by an LTV system $\mathbf{K}$ in order to isolate the middle component. Comparing the spreading function of $\mathbf{K}$ with the EAF of $x(t)$ [which is shown in Fig. 4(d)], we see that $x(t)$ and $\mathbf{K}$ are *jointly underspread*, which is the condition for the approximation (25). Fig. 5 shows that the WS of the output process $(\mathbf{K}x)(t)$ is indeed approximately equal to the WS of the input process $x(t)$ multiplied by the squared Weyl symbol of the LTV system $\mathbf{K}$. Similar results (which are not shown) are obtained for the ES/TES.

We finally applied the WS, ES, and TES to cylinder pressure signals measured in the course of combustion cycles in a car engine[4] [43], [44]. This process is well described by the multicomponent process model discussed in Section VII-B [see (31)]. The signal corresponding to a given combustion cycle consists of several resonant components (due to knocking). Within one cycle, the resonance frequencies decrease with time due to the decreasing gas temperature. All spectra shown are based on an estimate of the process' correlation function that was derived from 149 realizations corresponding to 149 different combustion cycles. Fig. 6 shows that the resulting WS is considerably more concentrated than the ES/TES. In particular, the ES/TES does not clearly indicate the decrease of the resonance frequencies. Fig. 6 also shows that the results obtained with the positive semidefinite innovations system are much better than those obtained with the causal innovations system. Finally, it is seen that the WS shows better TF concentration and contains smaller interference terms than the WVS.

[4] We are grateful to D. König and J. F. Böhme and to Volkswagen for making these data accessible to us.

IX. Conclusions

We have introduced and studied a family of time-varying spectra called generalized evolutionary spectrum (GES). While two prominent special cases of the GES are the classical evolutionary spectrum and the recently introduced transitory evolutionary spectrum, we have shown that another special case of the GES—the novel Weyl spectrum (WS)—features significant advantages over all other GES members.

Based on the definition of the GES in terms of an innovations system of the process under analysis, we have furthermore shown the importance of an *underspread* property for a satisfactory interpretation of the GES as a time-varying spectrum. Here again, the WS is advantageous since it merely requires the process to be underspread, whereas the other GES members require the process to be strictly underspread.

We have also shown and verified by simulations that in the underspread case, the WS is approximately identical to the Wigner–Ville spectrum; for deterministic signal components, however, it is more concentrated than the Wigner–Ville spectrum.

Appendix A
Proof of Approximation (22)

In order to prove[5] (22), we first note that $\mathrm{E}\{|\langle x, g^{(t,f)}\rangle|^2\} = \mathrm{E}\{|\langle \mathbf{H}n, g^{(t,f)}\rangle|^2\}$ with $g^{(t,f)}(t') = g(t'-t)e^{j2\pi f t'}$ can be

[5] The basic technique of proof used here and also in Appendix B has been developed in [23]. We note that other bounds and approximations mentioned in this paper can be derived in a similar manner.

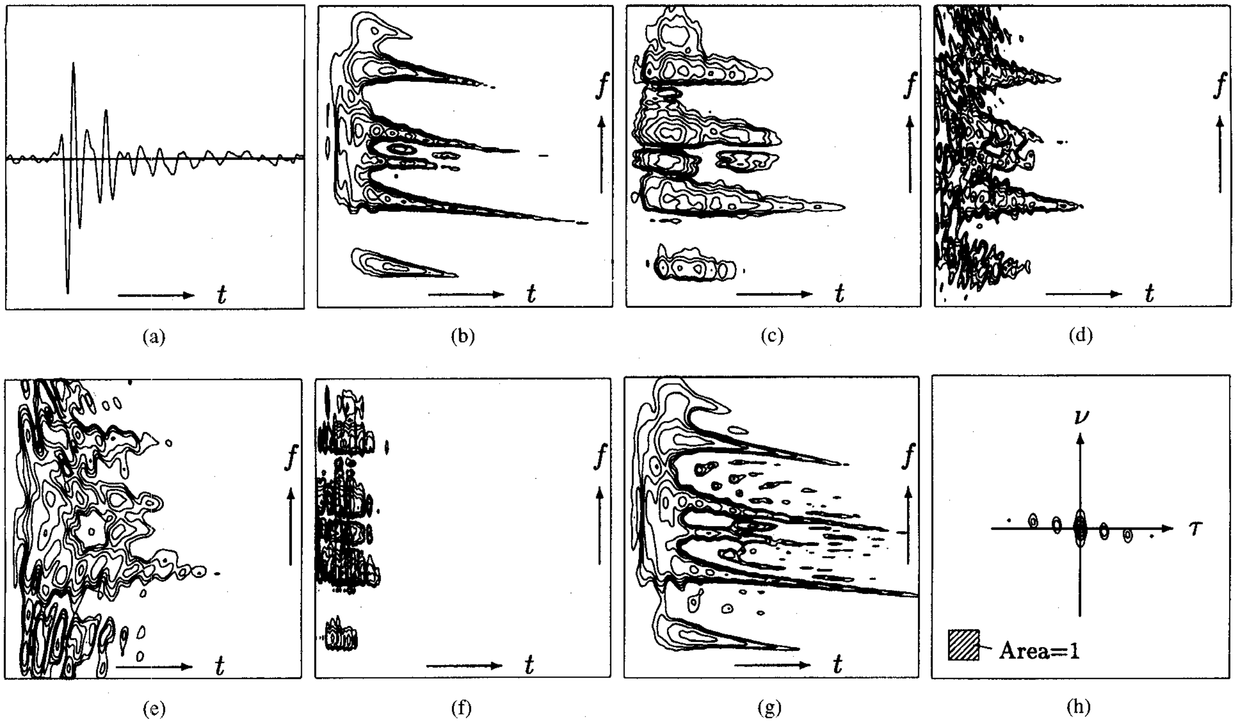


Fig. 6. Cylinder pressure process. (a) Typical process realization. (b) WS. (c) ES/TES with positive semidefinite innovations system. (d)–(f) GES with causal innovations system with (d) $\alpha = 0$, (e) $\alpha = 1/2$, and (f) $\alpha = -1/2$. (g) WVS. (h) EAF magnitude.

reformulated as

$$\begin{aligned}\mathrm{E}\{|\langle \mathbf{H}n, g^{(t,f)}\rangle|^2\} &= \int_\tau\int_\nu\int_{\tau'}\int_{\nu'} S_{\mathbf{H}}(\tau,\nu)S^*_{\mathbf{H}}(\tau',\nu') \\ &\quad\cdot A_g(\tau'-\tau,\nu'-\nu)e^{j2\pi((\tau'-\tau)f-(\nu'-\nu)t)} \\ &\quad\cdot e^{j2\pi(\tau'\nu-\tau\nu')}\,d\tau\,d\nu\,d\tau'\,d\nu'\end{aligned}$$

where $A_g(\tau,\nu)$ is the ambiguity function [40], [45] of $g(t)$. Similarly, using (24), the GES can be written as

$$\begin{aligned}\mathrm{GES}^{(\alpha)}_x(t,f) &= |L^{(\alpha)}_{\mathbf{H}}(t,f)|^2 \\ &= \int_\tau\int_\nu\int_{\tau'}\int_{\nu'} S_{\mathbf{H}}(\tau,\nu)S^*_{\mathbf{H}}(\tau',\nu') \\ &\quad\cdot e^{j2\pi((\tau'-\tau)f-(\nu'-\nu)t)} \\ &\quad\cdot e^{j2\pi\alpha(\tau'\nu'-\tau\nu)}\,d\tau\,d\nu\,d\tau'\,d\nu'.\end{aligned}$$

The difference $\Delta_0(t,f) = \mathrm{E}\{|\langle x, g^{(t,f)}\rangle|^2\} - \mathrm{GES}^{(\alpha)}_x(t,f)$ can hence be expressed as

$$\begin{aligned}\Delta_0(t,f) &= \int_\tau\int_\nu\int_{\tau'}\int_{\nu'} S_{\mathbf{H}}(\tau,\nu)S^*_{\mathbf{H}}(\tau+\tau',\nu+\nu') \\ &\quad\cdot e^{j2\pi(\tau'f-\nu't)}e^{j2\pi\alpha(\tau\nu'+\tau'\nu+\tau'\nu')} \\ &\quad\cdot [1-A_g(\tau',\nu')e^{j2\pi\phi^{(\alpha)}(\tau,\nu,\tau',\nu')}]\,d\tau\,d\nu\,d\tau'\,d\nu'\end{aligned}$$

where $\phi^{(\alpha)}(\tau,\nu,\tau',\nu') = \tau'\nu - \tau\nu' - \alpha(\tau\nu'+\tau'\nu+\tau'\nu')$ and the domains of integration are $(\tau,\nu) \in [-\tau_{\mathbf{H}},\tau_{\mathbf{H}}]\times[-\nu_{\mathbf{H}},\nu_{\mathbf{H}}]$ and $(\tau',\nu') \in [-2\tau_{\mathbf{H}},2\tau_{\mathbf{H}}]\times[-2\nu_{\mathbf{H}},2\nu_{\mathbf{H}}]$ since $\mathbf{H}$ is assumed to be strictly underspread. The magnitude of $\Delta_0(t,f)$ can now be bounded as

$$\begin{aligned}|\Delta_0(t,f)| &\le \int_\tau\int_\nu\int_{\tau'}\int_{\nu'} |S_{\mathbf{H}}(\tau,\nu)||S^*_{\mathbf{H}}(\tau+\tau',\nu+\nu')| \\ &\quad\cdot |1-A_g(\tau',\nu')e^{j2\pi\phi^{(\alpha)}(\tau,\nu,\tau',\nu')}|\,d\tau\,d\nu\,d\tau'\,d\nu' \\ &\le \mathrm{tr}^2\{\mathbf{R}_x^{1/2}\}\int_{-2\tau_{\mathbf{H}}}^{2\tau_{\mathbf{H}}}\int_{-2\nu_{\mathbf{H}}}^{2\nu_{\mathbf{H}}}\left[\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}}\right. \\ &\quad\left.\cdot |1-A_g(\tau',\nu')e^{j2\pi\phi^{(\alpha)}(\tau,\nu,\tau',\nu')}|\,d\tau\,d\nu\right]d\tau'\,d\nu'\end{aligned}$$

where we have used $|S_{\mathbf{H}}(\tau,\nu)| \le \mathrm{tr}\{\mathbf{R}_x^{1/2}\}$ [23]. Since the domain of integration is very concentrated around the origin, we use the approximation $A_g(\tau,\nu) \approx 1$ [a more rigorous but lengthier derivation could be given by using a higher order Taylor expansion of $A_g(\tau,\nu)$ around (0,0)]. Furthermore, using

$$\begin{aligned}|1-e^{j2\pi\phi^{(\alpha)}(\tau,\nu,\tau',\nu')}| &\le 2|\sin(\pi\phi^{(\alpha)}(\tau,\nu,\tau',\nu'))| \\ &\le 2\pi|\phi^{(\alpha)}(\tau,\nu,\tau',\nu')| \le 2\pi[(1+|\alpha|)(|\tau\nu'|+|\tau'\nu|) \\ &\quad+|\alpha||\tau'\nu'|] \le 2\pi(1+2|\alpha|)\sigma_{\mathbf{H}}\end{aligned}$$

(on the integration domain) leads, after integrating, to the final bound $|\Delta_0(t,f)| \le \epsilon$ with $\epsilon \approx 8\pi(1+2|\alpha|)\sigma^3_{\mathbf{H}}\ \mathrm{tr}^2\{\mathbf{R}_x^{1/2}\}$. This bound approaches zero with decreasing spread $\sigma_{\mathbf{H}}$ of the innovations system, which proves the approximation (22).

APPENDIX B
PROOF OF THE BOUNDS (23)

The difference between two GES based on the same innovations system $\mathbf{H}$ can be expressed as

$$\begin{aligned}\Delta(t,f) &= \mathrm{GES}_x^{(\alpha_1)}(t,f) - \mathrm{GES}_x^{(\alpha_2)}(t,f)\\ &= L_{\mathbf{H}}^{(\alpha_1)}(t,f)L_{\mathbf{H}}^{(\alpha_1)*}(t,f) - L_{\mathbf{H}}^{(\alpha_2)}(t,f)L_{\mathbf{H}}^{(\alpha_2)*}(t,f)\\ &= \int_\tau\int_\nu\int_{\tau'}\int_{\nu'} S_{\mathbf{H}}^{(\alpha_1)}(\tau,\nu)S_{\mathbf{H}}^{(\alpha_1)*}(\tau',\nu')\\ &\quad\cdot[1-e^{j2\pi\Delta\alpha(\tau'\nu'-\tau\nu)}]\\ &\quad\cdot e^{j2\pi[(\nu-\nu')t-(\tau-\tau')f]}\,d\tau\,d\nu\,d\tau'\,d\nu'\end{aligned}$$

where we have used (24) and the relation $S_{\mathbf{H}}^{(\alpha_2)}(\tau,\nu) = S_{\mathbf{H}}^{(\alpha_1)}(\tau,\nu)e^{j2\pi\tau\nu\Delta\alpha}$ with $\Delta\alpha = \alpha_1 - \alpha_2$. The magnitude of $\Delta(t,f)$ can then be bounded as

$$\begin{aligned}|\Delta(t,f)| &\le \int_\tau\int_\nu\int_{\tau'}\int_{\nu'} |S_{\mathbf{H}}(\tau,\nu)||S_{\mathbf{H}}(\tau',\nu')|\\ &\quad\cdot|1-e^{j2\pi\Delta\alpha(\tau'\nu'-\tau\nu)}|\,d\tau\,d\nu\,d\tau'\,d\nu'\\ &= 2\int_\tau\int_\nu\int_{\tau'}\int_{\nu'} |S_{\mathbf{H}}(\tau,\nu)||S_{\mathbf{H}}(\tau',\nu')|\\ &\quad\cdot|\sin[\pi\Delta\alpha(\tau'\nu'-\tau\nu)]|\,d\tau\,d\nu\,d\tau'\,d\nu'.\end{aligned}$$

We now assume that $\mathbf{H}$ is strictly underspread, which means that the (effective) support of $S_{\mathbf{H}}^{(\alpha)}(\tau,\nu)$ is enclosed by the rectangle $[-\tau_{\mathbf{H}},\tau_{\mathbf{H}}]\times[-\nu_{\mathbf{H}},\nu_{\mathbf{H}}]$, which limits the domain of integration accordingly. Using $|\sin x|\le|x|$ and the fact that $|\tau'\nu'-\tau\nu|\le 2\tau_{\mathbf{H}}\nu_{\mathbf{H}}$ within the domain of integration, we further obtain

$$\begin{aligned}|\Delta(t,f)| &\le \pi|\Delta\alpha|\,4\tau_{\mathbf{H}}\nu_{\mathbf{H}}\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}}|S_{\mathbf{H}}(\tau,\nu)|\,d\tau\,d\nu\\ &\quad\cdot\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}}|S_{\mathbf{H}}(\tau',\nu')|\,d\tau'\,d\nu'.\end{aligned}$$

Using $|S_{\mathbf{H}}(\tau,\nu)|\le \mathrm{tr}\{\mathbf{R}_x^{1/2}\}$ [23], it follows that

$$\begin{aligned}&\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}}|S_{\mathbf{H}}(\tau,\nu)|\,d\tau\,d\nu\\ &\quad\le \mathrm{tr}\{\mathbf{R}_x^{1/2}\}\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}}d\tau\,d\nu = \mathrm{tr}\{\mathbf{R}_x^{1/2}\}\,4\tau_{\mathbf{H}}\nu_{\mathbf{H}}.\end{aligned}$$

This finally yields the first bound in (23) $|\Delta(t,f)|\le \pi|\Delta\alpha|(4\tau_{\mathbf{H}}\nu_{\mathbf{H}})^3\mathrm{tr}^2\{\mathbf{R}_x^{1/2}\}$.

We next consider the L_2-norm of $\Delta(t,f)$. With (24) and Schwarz' inequality, we obtain

$$\begin{aligned}\|\Delta\|^2 &= \int_t\int_f|\Delta(t,f)|^2\,dt\,df\\ &= \int_t\int_f|L_{\mathbf{H}}^{(\alpha_1)}(t,f)L_{\mathbf{H}}^{(\alpha_1)*}(t,f)\\ &\quad - L_{\mathbf{H}}^{(\alpha_2)}(t,f)L_{\mathbf{H}}^{(\alpha_2)*}(t,f)|^2\,dt\,df\\ &= \int_{-2\tau_{\mathbf{H}}}^{2\tau_{\mathbf{H}}}\int_{-2\nu_{\mathbf{H}}}^{2\nu_{\mathbf{H}}}\Bigg|\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}} S_{\mathbf{H}}^{(\alpha_1)}(\tau+\tau',\nu+\nu')\\ &\quad S_{\mathbf{H}}^{(\alpha_1)*}(\tau',\nu')\\ &\quad\cdot[1-e^{j2\pi\Delta\alpha(\tau\nu+\tau'\nu+\tau\nu')}]\,d\tau'\,d\nu'\Bigg|^2\,d\tau\,d\nu\\ &\le \int_{-2\tau_{\mathbf{H}}}^{2\tau_{\mathbf{H}}}\int_{-2\nu_{\mathbf{H}}}^{2\nu_{\mathbf{H}}}\Bigg[4\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}}|S_{\mathbf{H}}(\tau',\nu')|^2\\ &\quad\cdot\sin^2[\pi\Delta\alpha(\tau\nu+\tau'\nu+\tau\nu')]\,d\tau'\,d\nu'\Bigg]\\ &\quad\cdot\Bigg[\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}}|S_{\mathbf{H}}(\tau+\tau',\nu+\nu')|^2\,d\tau'\,d\nu'\Bigg]d\tau\,d\nu\end{aligned}$$

where the relation $S_{\mathbf{H}}^{(\alpha_2)}(\tau,\nu) = S_{\mathbf{H}}^{(\alpha_1)}(\tau,\nu)e^{j2\pi\tau\nu\Delta\alpha}$ has been used. With $\sin^2 x\le x^2$ and the fact that $|\tau\nu+\tau'\nu+\tau\nu'|\le 8\tau_{\mathbf{H}}\nu_{\mathbf{H}}$ within the domain of integration, we further obtain

$$\begin{aligned}\|\Delta\|^2 &\le 16\pi^2(\Delta\alpha)^2(4\tau_{\mathbf{H}}\nu_{\mathbf{H}})^2\int_{-2\tau_{\mathbf{H}}}^{2\tau_{\mathbf{H}}}\int_{-2\nu_{\mathbf{H}}}^{2\nu_{\mathbf{H}}}\\ &\quad\cdot\left[\int_{-\tau_{\mathbf{H}}}^{\tau_{\mathbf{H}}}\int_{-\nu_{\mathbf{H}}}^{\nu_{\mathbf{H}}}|S_{\mathbf{H}}(\tau',\nu')|^2\,d\tau'\,d\nu'\right]^2 d\tau\,d\nu\\ &= 64\pi^2(\Delta\alpha)^2(4\tau_{\mathbf{H}}\nu_{\mathbf{H}})^3\mathrm{tr}^2\{\mathbf{R}_x\}\end{aligned}$$

where we have used $\int_\tau\int_\nu|S_{\mathbf{H}}(\tau,\nu)|^2\,d\tau\,d\nu = \|\mathbf{H}\|^2 = \mathrm{tr}\{\mathbf{R}_x\}$.This proves the second bound in (23).

APPENDIX C
MINIMIZATION OF THE DISPLACEMENT RADIUS

Following [30], we shall minimize the squared TF displacement radius $\rho_{\mathbf{H}}^2 = \frac{\overline{\tau_{\mathbf{H}}^2}}{T^2}+T^2\overline{\nu_{\mathbf{H}}^2}$ with $\overline{\tau_{\mathbf{H}}^2}$ and $\overline{\nu_{\mathbf{H}}^2}$ defined in (27), under the side constraint of a normal innovations system $\mathbf{H}$:

$$\mathbf{H}_{\mathrm{opt}} \triangleq \underset{\mathbf{H}\mathbf{H}^+=\mathbf{H}^+\mathbf{H}}{\arg\min}\ \rho_{\mathbf{H}}^2.$$

We use the polar decomposition (28) $\mathbf{H} = \mathbf{H}_p\mathbf{U}$, where $\mathbf{H}_p = \mathbf{R}_x^{1/2}$. The kernel of $\mathbf{H}_p$ is

$$H_p(t,t') = \sum_{k=1}^{\infty}\sqrt{\lambda_k}u_k(t)u_k^*(t')$$

where λ_k and $u_k(t)$ are the (known) eigenvalues and eigenfunctions, respectively, of $\mathbf{R}_x$. Since $\mathbf{H}$ is assumed normal, the kernel of the unitary operator $\mathbf{U}$ must be of the form

$$U(t,t') = \sum_{k=1}^{\infty}e^{j\varphi_k}u_k(t)u_k^*(t')$$

with arbitrary φ_k. Hence, the kernel of the innovations system $\mathbf{H} = \mathbf{H}_p\mathbf{U}$ is

$$H(t,t') = \sum_{k=1}^{\infty}\sqrt{\lambda_k}e^{j\varphi_k}u_k(t)u_k^*(t'). \tag{33}$$

Since the λ_k and $u_k(t)$ are given, our minimization is only with respect to the eigenvalue phases φ_k.

We first consider the minimization of $\overline{\tau_{\mathbf{H}}^2}$. The denominator of $\overline{\tau_{\mathbf{H}}^2}$ [see (27)] is $\int_\tau\int_\nu|S_{\mathbf{H}}(\tau,\nu)|^2\,d\tau\,d\nu = \int_t\int_{t'}|H(t,t')|^2\,dt\,dt' = \sum_{k=1}^{\infty}\lambda_k$, which is independent of the φ_k. Hence, it suffices to minimize the numerator of

$\overline{\tau_{\mathbf{H}}^2}$, which can be rewritten as

$$\begin{aligned}\int_\tau \int_\nu \tau^2 |S_{\mathbf{H}}(\tau,\nu)|^2 \, d\tau \, d\nu &= \int_t \int_{t'} (t-t')^2 |H(t,t')|^2 \, dt \, dt' \\ &= \int_t \int_{t'} t^2 |H(t,t')|^2 \, dt \, dt' \\ &\quad - 2 \int_t \int_{t'} tt' |H(t,t')|^2 \, dt \, dt' \\ &\quad + \int_t \int_{t'} t'^2 |H(t,t')|^2 \, dt \, dt'.\end{aligned}$$

The first and the last term can again be shown to be independent of the φ_k. Thus, our minimization problem reduces to the maximization of the quantity

$$M = \int_t \int_{t'} tt' |H(t,t')|^2 \, dt \, dt'$$

which, after a few manipulations using the eigendecomposition (33), can be written as

$$M = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} m_{kl} \cos(\varphi_k - \varphi_l)$$

$$\text{with} \quad m_{kl} = \sqrt{\lambda_k \lambda_l} \left| \int_t t u_k(t) u_l^*(t) \, dt \right|^2 \geq 0.$$

Since $m_{kl} \geq 0$, M is maximized for $\cos(\varphi_k - \varphi_l) = 1$, which implies that all φ_k are identical, i.e., $\varphi_k = \varphi_0$, where φ_0 is arbitrary. Inserting in (33), we see that the optimum system minimizing $\overline{\tau_{\mathbf{H}}^2}$ is given by $\mathbf{H}_{\text{opt}} = e^{j\varphi_0} \mathbf{H}_p$. The same solution is obtained when $\overline{\nu_{\mathbf{H}}^2}$ is minimized instead of $\overline{\tau_{\mathbf{H}}^2}$, and thus, $\mathbf{H}_{\text{opt}}$ minimizes the squared TF displacement radius $\rho_{\mathbf{H}}^2 = \frac{\overline{\tau_{\mathbf{H}}^2}}{T^2} + T^2 \overline{\nu_{\mathbf{H}}^2}$ as well.

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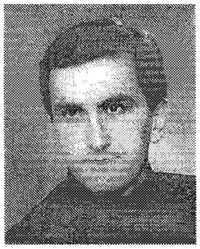

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