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Effekter av manglende Lorentz-invarians på Nambu-Goldstone bosoner

Fredrik Nicolai Krohg

Master i fysikk og matematikk
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Hovedveileder: Jens Oluf Andersen, IFY

Norges teknisk-naturvitenskapelige universitet
Institutt for fysikk

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Abstract

In this thesis we discuss symmetry considerations of field theories that are both Lorentz invariant and Lorentz noninvariant. We develop an example of a free Lorentz noninvariant field theory, invariant under $E(2)$, whose associated Euler-Lagrange equation is the Schrödinger equation. After quantization, the ground state of the theory is found to be ambiguous, hence the theory exhibits spontaneous symmetry breaking. Introduction of a chemical potential, taking the thermodynamic limit and adiabatically removing said potential reveals that the theory admits a single type-II Nambu-Goldstone boson. Goldstone's theorem is proved for classical field theories under the assumption of Lorentz invariance, and a short discussion on quantum corrections is given. The application of this theorem is illustrated in the context of two Lorentz-invariant field theories. It is noted that the counting of NG-bosons as the number of broken generators does not apply to Lorentz noninvariant field theories such as the Schrödinger field theory, but another method of counting them developed by Watanabe does hold. We then discuss a Lorentz invariant theory that also is invariant under $SU(2) \times SU(2)$. The Lorentz invariance is broken by introducing a chemical potential which also breaks the $SU(2) \times SU(2)$ invariance down to an invariance under $U(2)$. Spontaneous symmetry breaking of this theory is then proved to imply one type-I and one type-II NG-boson. Again, the number of NG-bosons is seen not to equal the number of broken generators but depends on them in a more complicated way given by Watanabe's formula.

Sammendrag

I denne avhandlingen diskuterer vi symmetrier av feltteorier både med og uten Lorentz-invarians. Vi konstruerer en feltteori fri for interaksjoner som er invariant under den Euklidske gruppen $E(2)$, har tilhørende Euler-Lagrange likning gitt av Schrödingerlikningen og som ikke er invariant under Lorentztransformasjoner. Etter å ha kvantisert denne feltteorien finner vi at grunntilstanden ikke er entydig slik at symmetriene blir spontant brutt. Teorien blir så modifisert ved å introdusere et kjemisk potensial som gjør gunntilstanden entydig. Ved å ta den termodynamiske grensen og fjerne det kjemiske potensialet konkluderer vi med at teorien impliserer ett enkelt type-II Nambu-Goldstone boson. Vi gir et bevis for Goldstones teorem for klassiske Lorentz-invariante feltteorier og diskuterer kort hvilken effekt kvantisering har på validiteten av dette teoremet. Teoremet blir så illustrert for to Lorentz-invariante feltteorier. Ved å se tilbake på den $E(2)$-invariante feltteorien blir det illustrert hvordan antallet Nambu-Goldstone bosoner ikke er lik antallet brukne generatorer hvis teorien ikke er Lorentz-invariant. På den annen side viser vi at en metode for telling av disse bosonene som nylig er blitt utviklet av Watanabe holder også i dette tilfellet. Til sist tar vi for oss et eksempel med en Lorentz-invariant feltteori som også er invariant under gruppen $SU(2) \times SU(2)$. Vi bryter Lorentz-invariansen med et kjemisk potensial som også bryter $SU(2) \times SU(2)$ ned til en $U(2)$ invarians. Denne teorien viser seg å implisere et type-I- og et type-II Nambu-Goldstone boson. Antallet bosoner er igjen ikke lik antallet brukne generatorer, men avhenger av dem via den nevnte nylig, utviklede metoden som gjør bruk av deres kommutatorer.

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Notation

Some abbreviations and clarification on the notation and conventions used in this thesis.

- SSB: Spontaneous symmetry breaking.
- NG-boson: Nambu-Goldstone boson.
- n_{BG}: Number of broken symmetry generators.
- n_{NGB}: Number of NG-bosons.
- We use the Minkowski metric of the form familiar from particle physics $n_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.
- The term *map* is a shortening of *mapping* which is synonymous to *function*.
- The notation for a mapping is written as $f : D \to V, \ a \mapsto b$. In this notation D is the domain of the map f and V is its codomain. The notation means that the map f takes an element $a \in D$ and maps it to an element $b \in V$.
- In equations, the symbol ';' should be read as *such that*.
- For the set of all $m \times n$ matrices with entries in the field $\mathbb{F}$, the symbol $M_{m\times n}(\mathbb{F})$ will be used. If $n = m$, then m will be omitted, and the set is thus written $M_n(\mathbb{F})$. For example set of all 2×2 complex matrices is written $M_2(\mathbb{C})$.
- For a complex matrix or scalar $A \in M_{m\times n}(\mathbb{C})$, the notation A^* means complex conjugate, A^{T} is the transpose and $A^\dagger$ is the conjugate transpose of the matrix.
- Propositions & theorems: we use the term proposition for theorems that are less important and or have shorter proofs than theorems.
- The end of proofs are marked with the *quod erat demonstrandum* symbol $\blacksquare$.
- **Bold text** is used in defining new terms.

Chapter 1

Introduction

> *The notion of a symmetry is central in Quantum Mechanics, there's also in Classical Mechanics, but it comes up and really hits you over the head in Quantum Mechanics.*
>
> LEONARD SUSSKIND (2013)
> *Advanced Quantum Mechanics Lecture 1*

Over the history of advancements in physics, symmetries have been used with varying degrees of overtness to simplify problems and even draw conclusions when the explicit solution of a problem is yet unknown. One such example is when considering a system with a spherically symmetric potential, may it be a hydrogen atom or a classical two body system. Then we can utilize the inherent symmetry to simplify the problem considerably by using spherical instead of Cartesian coordinates. The dynamical implications of symmetries were first elucidated in 1917 when Emmy Noether proved the famous theorem that now bears her name [1]. The details of Noether's theorem will be presented later in this thesis but quickly summarized; it shows that for every continuous symmetry of Nature there is a corresponding conservation law.

1.1 Symmetry in particle physics

Before formulating the details of the Standard Model what is called internal symmetries played an important role in classifying the numerous "elementary particles" discovered through experiments with cosmic rays and particle accelerators from the 1920s to the 1960s. Historically, the idea of isospin symmetry among the elementary particles was discovered after Heisenberg noticed that the masses of the proton and neutron are nearly the same. Based on this observation he claimed that the force that held these particles together, the *strong nuclear force*, did not distinguish between them [2, p. 129]. This was thus a symmetry where, barring the effects of electromagnetism, you could exchange a proton for a neutron without changing the behaviour of the system. This kind of transformation of a system is called a symmetry transformation. Specifically these are transformations you can do (at least conceptually) to a system such that the answers to questions about the physical behaviour or properties of the system remain the same. A set of transformations satisfying certain properties, like the property that combining two transformations in the set results in a transformation that is also in the set can then be described by a group and we say that the system is invariant under a group when it is invariant under all the transformations of that group. To see how this works in practice consider the case of the proton and the neutron. In modern terminology, Heisenberg proposed that for this system the strong force is symmetric under the group $SU(2)$. Any element in this group can be expressed as a 2×2 complex matrix such that if the state of a nucleon is described by a vector $N = (\alpha,\ \beta)^{\mathrm{T}} \in \mathbb{C}^2$ where the proton state is given by the vector $(1,\ 0)^{\mathrm{T}}$ and the neutron state is given by the vector $(0,\ 1)^{\mathrm{T}}$, then a symmetry transformation A of $SU(2)$ can be described by

$$N \mapsto AN.$$

Since $SU(2)$ is a continuous group giving three independent symmetry transformations, then Noether's theorem implies three conservation laws for three conserved quantities. Collecting these three quantities in a vector we call isospin we can say that isospin is conserved by the strong nuclear force.

In 1961 Murray Gell-Mann and Yuval Ne'eman independently came up with the idea of using an extension of this symmetry called $SU(3)$ to classify the baryons and mesons known at the time [3, 4]. Gell-Mann called this classification the *Eightfold Way*, in which particles are placed at different positions in different geometrical patterns according to the values of the conserved quantities. These patterns then correspond to different dimensional representations of the symmetry group $SU(3)$. For example the baryon decuplet places 10 different heavy baryons in a triangular array and on this array one can then act with a 10-dimensional representation of $SU(3)$. Incidentally this particular pattern was used to predict the existence of a new particle called the Ω^- [5].

1.2 Symmetry breaking and the Higgs mechanism

In this thesis we focus on the situation where symmetries are not exact, but broken in a particular way. This situation happens all the time in Nature and the mathematical framework by which we understand this phenomenon plays an integral part in our understanding of, among many others; supersymmetry, phase transitions, superconductivity and the Higgs mechanism. The latter is perhaps currently the most popular because of the recent discovery of the Higgs boson in 2012 [6, 7]. In the context of Quantum Field Theory the Higgs boson, as well as all other particles, is understood as an oscillation of a field, in this case the Higgs field. What is special about the Higgs field is that it has a non-zero vacuum expectation value, meaning that in a vacuum, where no particles exist, the field still prefers to have some non-zero value. This is the hallmark of spontaneous symmetry breaking. Historically this field was needed in the Standard Model of elementary particle physics to explain why the weak and strong nuclear force are confined to act over very short distances. It was then realized that the existence of such a field could also explain how the particles responsible for some of these forces obtain masses. The mechanism by which they did this was dubbed 'the Higgs mechanism'. Informally speaking, it can be explained by first postulating that a force of Nature, like the strong Nuclear force, is invariant under a continuous symmetry, like $SU(2)$. If now this symmetry is not shared by the ground state of the system then Goldstone's theorem, which we will discuss later in this thesis, tells us that there must exist a certain number of massless particles called Nambu-Goldstone bosons. If the symmetry however is a *gauge symmetry*, which means that the symmetry transformations can vary at different points in spacetime, then instead of massless particles appearing, the lack of the symmetry in the ground state gives rise to masses of the particles responsible for the force considered. In a cute "dogma" of particle physics, one says that the gauge bosons eat the Nambu-Goldstone boson and thus gains a mass. In the case of weak interactions the gauge bosons that acquires a mass in this fashion are the W and Z bosons.

1.3 The structure of this thesis

We follow a textbook-esque approach where first the mathematical details are presented and then they are applied to example field theories in order to explore the effects of spontaneous symmetry breaking. First we will give a brief review of the mathematical ideas concerning symmetries called group theory. This is divided into sections about basic group theory, Lie groups which are basically continuous groups which have infinitely many elements, Lie algebras; a special type of algebra usually connected with Lie groups, and a section about representations of groups which is what we actually use to do transformations on a system. Then we introduce the language of field theory and through it present and prove Noether's theorem. Our main example of a non-relativistic field theory is then discussed by first quantizing a classical field theory related to the Schrödinger equation, calculating the energy spectrum of this theory and then examining how the vacuum in this theory is degenerate because of a symmetry. The degeneracy of the ground state is lifted by introducing a chemical potential which singles out a ground state from which we build the Hilbert space of this theory. In the last chapter Goldstone's theorem is presented and proved for classical field theories. We then take a look at how this theorem is applied to some relativistic theories and give a short discussion of how well the theorem holds for quantum theories where one has to worry about loop corrections. Finally, we look at a last example of a field theory invariant under the group $SU(2) \times SU(2)$ which we modify by introducing a chemical potential like before and find the Goldstone bosons of this modified theory.

Chapter 2

Mathematical preface

To understand Goldstone's theorem and its applications in quantum field theory it is imperative to first have a solid background in the theory of groups and their representations. In this chapter we briefly review the most pertinent definitions and results of these mathematical notions. For a more thorough understanding and rigorous proofs the reader is referred to [8, 9, 10].

2.1 Groups

Definition 2.1. A **group** $\langle G, * \rangle$ is a set G, closed under a binary operation $* : \; G \times G \to G$, such that the following axioms are satisfied [10]:

(a) Associativity: $\forall a, b, c, \in G$
$(ab)c = a(bc)$.

(b) Identity element: $\exists e \in G; \; \forall x \in G$
$ex = xe = x$

(c) Inverse: $\forall a \in G, \; \exists a^{-1} \in G;$
$aa^{-1} = a^{-1}a = e$

In the remainder of this report the $\langle\rangle$ and $*$ may be omitted when referring to a group $\langle G, * \rangle$ so that groups will only be written with the symbol G, as it will be clear from the context whether the symbol represents a group or some other mathematical object.

Example 2.1. An important example of a group is the group G of all invertible $n \times n$ matrices with real elements, where the binary operation $\times$ is defined as matrix multiplication. From the rules of matrix multiplication, associativity follows. The identity element is given by the identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

because for any matrix with $n \times n$ real elements (i. e. $A \in M_n(\mathbb{R})$), multiplication with the identity matrix gives $AI = IA = A$. Finally, existence of the inverse follows from the requirement that all matrices in G should have an inverse. Because $I \in G$, G is nonempty. The group described here is called the **general linear group of degree** n, usually denoted by $GL(n, \mathbb{R})$. Note that the complex field $\mathbb{C}$ could just as well have been used in the construction of the group [9]. In this case the group is called the **complex general linear group** $GL(n, \mathbb{C})$.

Example 2.2. A subset of $GL(n, \mathbb{R})$ consisting of all invertible $n \times n$ matrices A where $A^{\mathrm{T}} = A^{-1}$, gives a group of its own called the **orthogonal group** $O(n)$. To see that this group is closed under matrix multiplication let $A, B \in O(n)$. From the definition of $O(n)$, $A^{-1} = A^{\mathrm{T}}$ and $B^{-1} = B^{\mathrm{T}}$. Then the inverse of the product is also the transpose of the product because

$$(AB)^{-1} = B^{-1}A^{-1} = B^{\mathrm{T}}A^{\mathrm{T}} = (AB)^{\mathrm{T}},$$

which means that $AB \in O(n)$ and thus $O(n)$ is closed under matrix multiplication. Associativity follows from matrix multiplication. The identity matrix $I = I^{-1} = I^{\mathrm{T}}$ and is thus in $O(n)$. The inverse of any element $A \in O(n)$ is also in $O(n)$ because $(A^{-1})^{-1} = A = (A^{\mathrm{T}})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}$.

Definition 2.2. Let G be a group, and let there be a subset $H \subseteq G$. Then H is a **subgroup** of G if H itself is a group under the binary operator in G. This is denoted $H \leq G$, or $H < G$ if $H \neq G$.

Example 2.3. The group $O(n)$ introduced in Example 2.2 is a subgroup of $GL(n, \mathbb{R})$.

Theorem 2.1. *Let $H \subseteq G$ with the same binary operation as G. Then H is a subgroup of G if and only if all the following statements are true:*

(a) H is closed under the binary operation, i. e. $\forall a, b \in H,\ ab \in H$

(b) The identity element e of G is in H.

(c) $\forall a \in H,\ a^{-1} \in H$.

For a proof of this theorem see Appendix A.1.

Example 2.4. Define the subset $U(n) \subseteq GL(n, \mathbb{C})$ as all complex invertible matrices that are **unitary**, meaning that:

$$A^\dagger = A^{-1} \ \ \forall A \in U(n).$$

For complex matrices $A^\dagger = (A^{\mathrm{T}})^*$ where A^* is the complex-conjugate of the matrix. $U(n)$ is a subgroup of $GL(n, \mathbb{C})$ called the **unitary group**. Showing that this is indeed a group, first it is checked that $U(n)$ is closed under the binary operation in $GL(n, \mathbb{C})$ (matrix multiplication): Let $A, B \in U(n)$. Then

$$\begin{aligned}(AB)^\dagger &= ((AB)^{\mathrm{T}})^* = (B^{\mathrm{T}}A^{\mathrm{T}})^* = (B^{\mathrm{T}})^*(A^{\mathrm{T}})^* = B^\dagger A^\dagger = B^{-1}A^{-1} = (AB)^{-1}\\ &\Rightarrow \ AB \in U(n)\end{aligned}$$

Then because trivially the identity matrix $I^\dagger = (I^{\mathrm{T}})^* = I = I^{-1} \ \ \Rightarrow \ \ I \in U(n)$, the identity element in $GL(n, \mathbb{C})$ is also in $U(n)$. Finally the inverse matrices exists because if $A \in U(n)$ then the conjugate transpose of $A^{-1} \in GL(n, \mathbb{C})$ is

$$(A^{-1})^\dagger = (A^\dagger)^\dagger = A = (A^{-1})^{-1} \ \Rightarrow \ A^{-1} \in U(n).$$

By Theorem 2.1 then, $U(n) \leq GL(n, \mathbb{C})$.

Definition 2.3. The **special orthogonal** group $SO(n)$ is defined as the subset of $O(n)$ such that

$$SO(n) = \{R \in O(n) \,|\, \det R = 1\}$$

with matrix multiplication as the binary operation.

The fact that $SO(n)$ actually is a subgroup of $O(n)$ can be proved through Theorem 2.1. By definition $SO(n) \subseteq O(n)$. $SO(n)$ is closed because given $A, B \in SO(n)$,

$$\det(AB) = \det A \det B = 1.$$

$\det I = 1$ which implies that $I \in SO(n)$. Finally $\det R^{-1} = \frac{1}{\det R} = 1$, thus for any $R \in SO(n)$, R^{-1} is also an element in $SO(n)$. By Theorem 2.1 this implies that $SO(n) \leq O(n)$.

Definition 2.4. The **special unitary** group $SU(n)$ is defined as a subset of $U(n)$ such that

$$SU(n) = \{A \in U(n) \,|\, \det A = 1\}$$

with matrix multiplication as binary operation.

Because the extra condition to be an element in $SU(n)$ is the same as for $SO(n)$, given that $SU(n) \subseteq U(n)$, the proof that $SU(n)$ is a subgroup of $U(n)$ is completely analogous to the argument that $SO(n) \leq O(n)$ and is therefore omitted.

Example 2.5. To find the general form of a matrix in $SU(2)$ let $A \in M_2(\mathbb{C})$

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Applying the condition $A^\dagger = A^{-1}$ implies that $\gamma = -\beta^*$ and $\delta = \alpha^*$. If we then enforce $\det A = 1$ we see that A takes the form

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}, \quad \alpha, \beta \in \mathbb{C}; \quad |\alpha|^2 + |\beta|^2 = 1.$$

2.2 Lie groups

To define Lie groups rigorously one must describe them using the language of differential manifolds, however for virtually all physical applications it is sufficient to only consider a special class of Lie groups called *matrix Lie groups* or sometimes *linear Lie groups*, hence we mostly concern ourselves with matrix Lie groups in this thesis.

Definition 2.5. A **matrix Lie group** G is a subgroup $G \leq GL(n;\mathbb{C})$ such that any sequence of matrices $\{A_i\} \subset G$ either converges to a matrix $A \in G$ or to a matrix $A \notin GL(n;\mathbb{C})$ (i. e. a non-invertible matrix) if it converges at all.

The above definition implies that the group fulfills the criteria for being a differentiable manifold in a topological sense [8], which roughly means that G looks locally (in the neighborhood of an element) like a piece of $\mathbb{R}^n$, but the proof of this will not be presented here. The convergence of the series $\{A_i\}$ to a matrix A means that every entry should converge to the corresponding entry in A. So if $(A_i)_{jk}$ is the jkth entry in the matrix A_i then

$$\lim_{i\to\infty} A_i = A \;\Leftrightarrow\; \forall j,k \;\lim_{i\to\infty} (A_i)_{jk} = (A)_{jk}.$$

Example 2.6. The general linear groups $GL(n,\mathbb{C})$ and $GL(n,\mathbb{R})$ are both matrix Lie groups. Here this is only proved for $GL(n,\mathbb{C})$, but the proof for $GL(n,\mathbb{R})$ is analogous. Take any convergent sequence of matrices $\{A_i\};\ A_i \in GL(n,\mathbb{R})\ \forall i$, and let its limit be denoted A. Then A must be a matrix with complex (or real) entries, thus by Definition 2.5, $GL(n,\mathbb{C})$ is trivially a matrix Lie group since either $A \in GL(n,\mathbb{C})$ or $A \notin GL(n,\mathbb{C})$.

The previously defined groups $O(N)$, $SO(N)$, $U(N)$ and $SU(N)$ also satisfy the criteria for being matrix Lie groups [8].

2.3 Lie algebras

Definition 2.6. The **Lie algebra $\mathfrak{g}$ of a matrix Lie group** G is defined as

$$\mathfrak{g} = \{X \in M_n(\mathbb{C}) \mid \forall t \in \mathbb{R}\ e^{tX} \in G\}$$

Example 2.7. The Lie algebra of $GL(n, \mathbb{C})$
This Lie algebra is usually denoted $\mathfrak{gl}(n, \mathbb{C})$. $X \in \mathfrak{gl}(n, \mathbb{C}) \;\Leftrightarrow\; e^{tX}$ is invertible for all real t. However for any $X \in M_n(\mathbb{C})$

$$\forall t \in \mathbb{R} \quad e^{tX}e^{-tX} = I = e^{-tX}e^{tX},$$

thus e^{tX} is invertible and the condition $e^{tX} \in GL(n, \mathbb{C})\ \forall t \in \mathbb{R}$ is trivially satisfied $\Rightarrow\ \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$.

Proposition 2.2. *If $G \leq GL(n, \mathbb{R})$ is a matrix Lie group, then its Lie algebra $\mathfrak{g}$ must consist of only real matrices.*

Proof. $G \leq GL(n, \mathbb{R}) \;\Rightarrow\; (A \in G \;\Leftrightarrow\; A$ is real). Then all the entries of e^{tX} are real for any real t if X is in the Lie algebra $\mathfrak{g}$. All these entries can then be thought of as real functions of the real variable t. Because the derivative of a real function must be real then $\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{tX}\right)_{ij}$ must be real for the ijth entry. Taking the derivative and evaluating at $t = 0$ yields

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{tX}\bigg|_{t=0}\right)_{ij} = \left(Xe^{0X}\right)_{ij} = (XI)_{ij} = X_{ij}.$$

Thus X_{ij} must be real and because the ij are arbitrary, X must be real. ∎

Example 2.8. The Lie algebra of $SO(n)$
Denote this Lie algebra as $\mathfrak{so}(n)$. From Proposition 2.2 $\mathfrak{so}(n)$ must consist of real $n \times n$ matrices because $SO(n) \leq GL(n, \mathbb{R})$. Now $X \in \mathfrak{so}(n) \;\Rightarrow\; \forall t \in \mathbb{R},\ e^{tX} \in SO(n) \;\Rightarrow\; \left(e^{tX}\right)^{\mathrm{T}} = \left(e^{tX}\right)^{-1} \;\Leftrightarrow\; e^{tX^{\mathrm{T}}} = e^{-tX},\ \forall t \in \mathbb{R}$. Taking the derivative at $t = 0$, $e^{tX^{\mathrm{T}}} = e^{-tX}\ \forall t \in \mathbb{R} \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t}e^{tX^{\mathrm{T}}}\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}e^{-tX}\Big|_{t=0} \;\Leftrightarrow$

$$X^{\mathrm{T}} = -X. \tag{2.1}$$

This proves that $X \in \mathfrak{so}(n) \;\Rightarrow\; X^{\mathrm{T}} = -X$. Conversely assume Eq. (2.1) for a matrix $X \in M_n(\mathbb{R})$. Note that Eq. (2.1) implies that $X_{ii} = -X_{ii} \;\Leftrightarrow\; X_{ii} = 0 \;\Rightarrow\; \mathrm{Tr}(X) = 0$ where X_{ii} are the diagonal elements of X. Then for any $t \in \mathbb{R}$ $\left(e^{tX}\right)^{\mathrm{T}} = e^{tX^{\mathrm{T}}} = e^{-tX} = \left(e^{tX}\right)^{-1}$, and $\det e^{tX} = e^{t\,\mathrm{Tr}(X)} = e^0 = 1$. These two properties of e^{tX} makes it an element of $SO(n)$. Because t is arbitrary then $e^{tX} \in SO(n)\ \forall t \in \mathbb{R} \ \Rightarrow\ X \in \mathfrak{so}(n)$. In conclusion

$$\mathfrak{so}(n) = \left\{ X \in M_n(\mathbb{R}) \mid X^{\mathrm{T}} = -X \right\}.$$

It is worth noting that the extra condition that is put on e^{tX} to make X an element of $\mathfrak{so}(n)$ as opposed to $\mathfrak{o}(n)$ (the Lie algebra of $O(n)$), is that $\forall t \in \mathbb{R}\ \ \det e^{tX} = 1$. However it was proved that this followed automatically from the condition $X^{\mathrm{T}} = -X$, thus $\mathfrak{so}(n) = \mathfrak{o}(n)$.

Example 2.9. The Lie algebra of $SU(n)$
This Lie algebra is denoted $\mathfrak{su}(n)$. $X \in \mathfrak{su}(n) \ \Leftrightarrow\ \forall t \in \mathbb{R}:\ e^{tX} \in SU(n) \ \Leftrightarrow\ (e^{tX})^\dagger = (e^{tX})^{-1} \ \wedge\ \det\left(e^{tX}\right) = 1 \ \Leftrightarrow\ e^{tX^\dagger} = e^{-tX} \ \wedge\ e^{t\,\mathrm{Tr}(X)} = 1$. Because this has to hold for all t, then the derivatives of both sides of $e^{tX^\dagger} = e^{-tX}$ evaluated at $t = 0$ must be equal, thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} e^{tX^\dagger}\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t} e^{-tX}\right|_{t=0} \ \Leftrightarrow\ X^\dagger = -X.$$

Also the derivatives of both sides of $e^{t\,\mathrm{Tr}(X)} = 1$ evaluated at $t = 0$ have to be equal. Thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} e^{t\,\mathrm{Tr}(X)}\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t} 1\right|_{t=0} \ \Leftrightarrow\ \mathrm{Tr}(X) = 0.$$

Note that in the last equation the derivative was of a normal exponential, whilst in the previous, it was of the matrix exponential.

Conversely assuming an $X \in M_n(\mathbb{C})$ satisfies $X^\dagger = -X$ and $\mathrm{Tr}(X) = 0$, then $\forall t \in \mathbb{R}\ \left(e^{tX}\right)^\dagger = e^{tX^\dagger} = e^{-tX} = \left(e^{tX}\right)^{-1}$ and $\det(e^{tX}) = e^{t\,\mathrm{Tr}(X)} = e^0 = 1 \ \Rightarrow\ \forall t \in \mathbb{R}:\ e^{tX} \in SU(n) \ \Rightarrow\ X \in \mathfrak{su}(n)$.

Consequently

$$\mathfrak{su}(n) = \left\{ X \in M_n(\mathbb{C}) \mid X^\dagger = -X \ \wedge\ \mathrm{Tr}(X) = 0 \right\}.$$

Theorem 2.3. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$, X and Y any two elements in $\mathfrak{g}$ and A any element in G. Then $\mathfrak{g}$ is a real vector space and the following properties hold:*

(a) $AXA^{-1} \in \mathfrak{g}$,
(b) $\forall r \in \mathbb{R},\ rX \in \mathfrak{g}$,
(c) $X + Y \in \mathfrak{g}$,
(d) $[X, Y] = XY - YX \in \mathfrak{g}$.

A proof of this theorem is included in Appendix A.2.

2.4 Representations

Definition 2.7. A finite-dimensional vector space V and a mapping Π constitute a real (complex) **finite-dimensional representation** of a matrix Lie group G if and only if V is a real (complex) vector space and and Π is a Lie group homomorphism from G into the group of linear invertible transformations on V:

$$\Pi\colon\quad G \to GL(V).$$

Because the vector space V is implied by the definition of Π, sometimes the vector space is omitted, and the representation is referred to only as Π. The reason for the name is that for a $g \in G$, $\Pi(g)$ can be thought of as *representing* the group element g as a linear transformation on a vector space. If Π is a representation as in the definition, then the convention is to write that Π *is a representation of G acting on V*. If V is an inner-product space and $\Pi(g)$ preserves this inner product for all g, then the representation is said to be **unitary**.

Since we have defined representations in terms of Lie group homomorphisms we also need to define these.

Definition 2.8. A **Lie group homomorphism** is defined as a map $\Pi\colon\ G \to G'$ between Lie groups G and G' such that

$$\forall a, b \in G \quad \Pi(ab) = \Pi(a)\Pi(b), \tag{2.2}$$

and Π is continuous. Eq. (2.2) is called the **homomorphism property**. If Π is bijective, meaning that it has an inverse, then Π is called an **isomorphism**, and the groups G and G' are said to be **isomorphic**. This is denoted $G \simeq G'$ and implies that the structure of the groups is the same, and only the names of the elements differ.

Example 2.10. The fundamental representation of a matrix Lie group G (sometimes also called the standard representation) is as the name suggests a rather intuitive representation. Because by definition $G \subseteq GL(n, \mathbb{C})$, the elements of G are matrices. Then it is natural to say that the group elements are represented by the $n \times n$ matrices that define them, but now interpreting these matrices as linear transformations on the n-dimensional vector space $\mathbb{C}^n$. The definition of this representation is thus that

$$\Pi\colon\quad G \to GL(\mathbb{C}^n),\ A \to \Pi(A) = A. \tag{2.3}$$

To prove that this map satisfies the homomorphism property is trivial[1] and because $G \subseteq GL(n, \mathbb{C})$ then $\Pi(A) \in GL(n, \mathbb{C}) = GL(\mathbb{C}^n)$ for any $A \in G$. Since $\mathbb{C}^n$ is a complex vector space, Π is a complex representation of G.

[1] $\Pi(AB) = AB = \Pi(A)\Pi(B)$

If G happens to be a subset of $GL(n, \mathbb{R})$ as in the case of $O(n)$, then Π might be considered to be a real representation mapping any element of G into $GL(n, \mathbb{R})$. In this case the representation acts on the vector space $\mathbb{R}^n$.

The fundamental representation of common Lie groups have their own names. As a rule of thumb these depend on which vector space the representation acts on. If $V = \mathbb{R}^n$ for some n, then the fundamental representation is usually called the vector representation, while if $V = \mathbb{C}^n$ it is called the spinor representation [9]. Thus the fundamental representation of $O(3)$ or $SO(3)$ is called the vector representation because it acts on $\mathbb{R}^3$, while the fundamental representation of $SU(2)$ is called the spinor representation of $SU(2)$ because it acts on $\mathbb{C}^2$.

Example 2.11. Another easy example of a representation is the **trivial representation**. This representation results from the trivial homomorphism which for any group G (it does not have to be a matrix Lie group) maps each element g in the group to the identity in the group in the codomain. In the case of the trivial representation the codomain is the group $GL(V)$. This group is well defined for any finite-dimensional vector space V, thus given any such V the trivial representation Π is defined as

$$\Pi\colon \quad G \to GL(V), g \mapsto I.$$

Π is (trivially) a homomorphism because for any $g_1, g_2 \in G$, $\Pi(g_1 g_2) = I = II = \Pi(g_1)\Pi(g_2)$.

Because $\Pi(g) = I$ for any $g \in G$, then for any $v \in V : \Pi(g)v = Iv = v$ and the trivial representation of the group “does nothing” to the elements of the vector space.

Definition 2.9. A **Lie algebra representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\pi\colon \; \mathfrak{g} \to \mathfrak{gl}(V)$ where V is any vector space.

Analogous to (group) representations, Lie algebra representations are said to be finite dimensional, and real or complex if the vector space V enjoys these properties. Since we have defined Lie algebra representations in terms of Lie algebra homomorphisms we need to explain what we mean by the latter.

Definition 2.10. A **Lie algebra homomorphism** is a linear map $\phi\colon \; \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ such that for all $X, Y \in \mathfrak{g}$

$$[\phi(X), \phi(Y)] = \phi\left([X, Y]\right),$$

where the brackets signify the commutator in $\mathfrak{h}$ on the left side of the equation and the commutator in $\mathfrak{g}$ on the right side.

If a Lie algebra homomorphism should happen to be bijective, then it is a **Lie algebra isomorphism** which means that the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ should be thought of as the same mathematical objects.

In many ways the Lie algebra representations are easier to work with and to visualize than group representations because for a finite dimensional Lie algebra they are fully determined by how they act on the basis vectors of the Lie algebra. This follows since Lie algebra representations are linear maps, and for any linear map $f\colon V \to W$ between two vector spaces V and W the range of f is a subspace of W. To see this assume $\{e_i\}_i$ is a basis on V. Then any vector in the range of f can be written as $f(v)$ for some $v \in V$. Then expanding this v in the basis using Einsteins summation convention $f(v) = f(v^i e_i) = v^i f(e_i)$. Thus any v in the range of f can be written as a linear combination of the vectors $f(e_i) \in W$ and the set of all linear combinations of these vectors, written $\operatorname{span}\big(f(e_i)\big)$ is thus the range of f. Let w_1 and w_2 be any two vectors in $\operatorname{span}\big(f(e_i)\big)$. Then $w_1 + w_2 = w_1^i f(e_i) + w_2^i f(e_i) = (w_1 + w_2)^i f(e_i) \;\Rightarrow\; w_1 + w_2 \in \operatorname{span}\big(f(e_i)\big)$. Also for any scalar λ in the field of V, $\lambda\, w_1 = \lambda\, w_1^i f(e_i) = (\lambda w_1)^i f(e_i) \;\Rightarrow\; \lambda w_1 \in \operatorname{span}\big(f(e_i)\big)$. Because $\operatorname{span}\big(f(e_i)\big)$ is a subset of W, this means that $\operatorname{span}\big(f(e_i)\big) = \operatorname{ran} f$ is a subspace of W. For the Lie algebra homomorphism π from the Lie algebra $\mathfrak{g}$ with basis $\{e_i\}_i$ this means that the range of π is the space spanned by the vectors $\pi(e_i)$, which must be a subspace of $\mathfrak{gl}(V)$. Also remember that if V is n-dimensional, then $\mathfrak{gl}(V)$ is the set of all $n \times n$ matrices c.f. Example 2.7, which can be interpreted as linear operators on vectors in V by expanding these vectors in a basis.

Chapter 3

Field theory

3.1 Noether's theorem for fields

Definition 3.1. In Minkowski space the action S of a field configuration $\phi(x)$ is given by

$$S[\phi] = \int_\Omega \mathrm{d}^4x \, \mathcal{L},$$

where $\mathcal{L}$ is the Lagrangian density that depends on the fields $\phi(x)$, their derivatives and possibly explicitly on the coordinates x themselves.

From this point on, we may use the shorthand notation of commas representing derivatives of the fields, e.g.

$$\phi^i_{,\mu} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\phi^i.$$

When total derivatives are used in field equations this implies that the derivative is also to take into account any implicit dependence in the fields, e.g.

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}(x^\nu \phi_{,\nu}) = \phi_{,\mu} + x^\nu \phi_{,\nu\mu}.$$

Proposition 3.1. *Let $\{\phi^i\}$ be an arbitrary collection of fields and let $\mathcal{L}$ be allowed to depend on x^μ, ϕ^i and $\phi^i_{,\mu}$ for any i and μ. Then the field configuration extremizes the action if and only if the fields satisfy the equations*

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}}\right) = 0,$$

*henceforth called the **Euler-Lagrange field equations**.*

Proof. The Euler-Lagrange field equations are derived by insisting on the action obtained from the Lagrangian density being stationary. Let $\mathcal{L}$ be a Lagrangian density as described above. Consider a small deviation in the fields $\phi^i(x) \mapsto \tilde{\phi}^i(x) = \phi^i(x) + \delta\phi^i(x)$ where the deviation is zero on the boundary of the spacetime region Ω over which the Lagrangian density is integrated to obtain the action. Inserting this into Definition 3.1 and expanding to first order in the deviation yields

$$\delta S = \int_\Omega \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi^i} \delta\phi^i + \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \delta\phi^i_{,\mu} \right). \tag{3.1}$$

To proceed we write the second term as a total divergence and use Gauss' law such that

$$\begin{aligned}
\int_\Omega \mathrm{d}^4x \, \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \, \delta\phi^i_{,\mu} &= \int_\Omega \mathrm{d}^4x \, \frac{\mathrm{d}}{\mathrm{d}x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \, \delta\phi^i \right) - \int_\Omega \mathrm{d}^4x \, \frac{\mathrm{d}}{\mathrm{d}x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \right) \delta\phi^i \\
&= \int_{\partial\Omega} \mathrm{d}^3S_\mu \, \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \delta\phi^i - \int_\Omega \mathrm{d}^4x \, \frac{\mathrm{d}}{\mathrm{d}x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \right) \delta\phi^i \\
&= - \int_\Omega \mathrm{d}^4x \, \frac{\mathrm{d}}{\mathrm{d}x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \right) \delta\phi^i,
\end{aligned}$$

by the assumption that the variation in the fields $\delta\phi^i$ vanish on the boundary $\partial\Omega$. Inserting this back into Eq. (3.1) and requiring that $\delta S = 0$ for an extremum when expanding to first order, we find

$$\begin{aligned}
\delta S = \int_\Omega \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \right) \right) \delta\phi^i = 0 \\
\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \right) = 0,
\end{aligned} \tag{3.2}$$

because the infinitesimal deviation in the fields $\delta\phi^i$ is arbitrary. This proves then that $\tilde{\phi}$ being an extremum of S is equivalent to the resulting $\mathcal{L}$ satisfying Eq. (3.2) because S was expanded to first order.

■

Proposition 3.2. *Let $\{\phi^i\}_i$ be a collection of fields that transforms as $\tilde{\phi}^i(\tilde{x}) = (\delta^i_j + \epsilon a(x)^i_j)\phi^j(x)$ under an infinitesimal coordinate transformation*

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \epsilon\xi^\mu(x).$$

Then the new fields can be written in terms of the old fields as

$$\tilde{\phi}^i(\tilde{x}) = \phi^i(\tilde{x}) + \epsilon(a(\tilde{x})^i_j\phi^j(\tilde{x}) - \xi^\mu\phi^i(\tilde{x})_{,\mu}),$$

where $\phi^i(\tilde{x})$ are the old fields but now denoting the variables $\tilde{x}^\mu$ instead of x^μ, and $a(\tilde{x})^i_j$ is a matrix that depends on the nature of the fields $\phi^i(x)$.

Proof. Assume an infinitesimal coordinate transformation is given by

$$\tilde{x}^\mu = x^\mu + \epsilon\xi^\mu(x).$$

Because ϵ is infinitesimal we can neglect all but first order terms, thus $\epsilon\xi^\mu(x) = \epsilon\xi^\mu(x + \epsilon\xi(x)) = \epsilon\xi^\mu(\tilde{x})$. This implies that the coordinate transformation is invertible through the inverse

$$x^\mu = \tilde{x}^\mu - \epsilon\xi^\mu(x) = \tilde{x}^\mu - \epsilon\xi^\mu(\tilde{x}).$$

Now let $\{\phi^i(x)\}$ be a collection of fields as defined in the proposition, such that the transformed fields can be written as

$$\tilde{\phi}^i(\tilde{x}) = A(x)^i_j\phi^j(x) \tag{3.3}$$

for coefficients $A(x)^i_j = \delta^i_j + \epsilon a(x)^i_j$. Inserting the inverse of the coordinate transformation in this equation yields

$$\tilde{\phi}^i(\tilde{x}) = A(\tilde{x} - \epsilon\xi(\tilde{x}))^i_j\phi^j(\tilde{x} - \epsilon\xi(\tilde{x})). \tag{3.4}$$

Expanding this to first order in ϵ we obtain

$$\begin{aligned}\tilde{\phi}^i(\tilde{x}) &= \left(\delta^i_j + \epsilon a(\tilde{x})^i_j\right)\left(\phi^j(\tilde{x}) - \epsilon\xi^\mu\phi^j(\tilde{x})_{,\mu}\right)\\ &= \phi^i(\tilde{x}) + \epsilon\left(a(\tilde{x})^i_j\phi^j(\tilde{x}) - \xi^\mu\phi^i(\tilde{x})_{,\mu}\right),\end{aligned}$$

which is exactly the expression for $\tilde{\phi}^i(\tilde{x})$ we wanted to prove.

■

Proposition 3.3. *If the original fields $\phi^i(x)$ satisfy the Euler-Lagrange field equations then any infinitesimal field transformation $\phi^i(x) \mapsto \phi^i(x) + \delta\phi^i(x)$ implies a change in the Lagrangian density $\mathcal{L}$ given by*

$$\delta\mathcal{L}(\phi^i) = \mathcal{L}(\phi^i + \delta\phi^i) - \mathcal{L}(\phi^i) = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\mathcal{M}^\mu$$

with

$$\mathcal{M}^\mu = \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i.$$

Proof. Assume that an infinitesimal coordinate transformation is given by $\phi^i(x) \mapsto \tilde{\phi}^i(x) = \phi^i(x) + \delta\phi^i(x)$. Then the new Lagrangian density resulting from these new fields is

$$\tilde{\mathcal{L}} = \mathcal{L}\big(\tilde{\phi}^i, \tilde{\phi}^i_{,\mu}, x^\mu\big),$$

where the new fields have been inserted in the old Lagrangian density. Since the field transformation is infinitesimal, $\tilde{\mathcal{L}}$ can be expanded in terms of ϕ^i and $\phi^i_{,\mu}$ as

$$\tilde{\mathcal{L}} = \mathcal{L}\big(\phi^i + \delta\phi^i, \phi^i_{,\mu} + \delta\phi^i_{,\mu}, x^\mu\big) = \mathcal{L}(\phi^i, \phi^i_{,\mu}, x^\mu) + \frac{\partial\mathcal{L}}{\partial\phi^i}\delta\phi^i + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i_{,\mu},$$

thus $\tilde{\mathcal{L}}$ can be written as $\tilde{\mathcal{L}} = \mathcal{L} + \delta\mathcal{L}$ with

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^i}\delta\phi^i + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i_{,\mu}.$$

Since the original fields ϕ^i satisfy the Euler-Lagrange field equations, $\delta\mathcal{L}$ can be written as

$$\begin{aligned}\delta\mathcal{L} &= \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\right)\delta\phi^i + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i_{,\mu} \\ &= \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i\right).\end{aligned}$$

By defining

$$\mathcal{M}^\mu = \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i,$$

$\delta\mathcal{L}$ can be written

$$\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\mathcal{M}^\mu.$$

■

Theorem 3.4. ***Noether's theorem for fields***
Given an infinitesimal field transformation $\phi^i \mapsto \phi^i + \delta\phi^i$, where ϕ^i satisfy the Euler-Lagrange field equations, that results in a change in the Lagrangian density given by $\delta\mathcal{L} = \frac{\mathrm{d}\mathcal{M}^\mu}{\mathrm{d}x^\mu}$, then

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i - \mathcal{M}^\mu$$

is a conserved current, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}j^\mu = 0.$$

Proof. In Proposition 3.3 it was proved that if the original fields satisfy the Euler-Lagrange field equations, then an infinitesimal field transformation leads to a change in the Lagrangian density given by

$$\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i\right).$$

Assuming that the change in the Lagrangian density caused by the field transformation can also be expressed as

$$\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\mathcal{M}^\mu$$

for an $\mathcal{M}^\mu$ not necessarily different from $\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i$, then

$$0 = \delta\mathcal{L} - \delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i - \mathcal{M}^\mu\right).$$

■

Remark. To get a non-trivial conserved current however, one usually wants

$$\mathcal{M}^\mu \neq \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \delta\phi^i.$$

Typically the transformation leaves $\mathcal{L}$ invariant such that $\delta\mathcal{L} = 0$. We can write this as a total divergence because

$$\delta\mathcal{L} = 0 = \sum_\mu \frac{\mathrm{d}}{\mathrm{d}x^\mu} 0 = \frac{\mathrm{d}}{\mathrm{d}x^\mu} \mathcal{M}^\mu$$

if we define $\mathcal{M}^\mu = 0$. Inserting this into Noether's theorem yields the conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \delta\phi^i,$$

as is immediately obvious from Proposition 3.3.

Proposition 3.5. *Given a conserved current j^μ one can obtain a family of conserved currents by*

$$\tilde{j}^\mu = j^\mu + \partial_\nu T^{\mu\nu}$$

for any antisymmetric tensor $T^{\mu\nu}$.

Proof. The new current $\tilde{j}^\mu$ is conserved because

$$\partial_\mu \tilde{j}^\mu = \partial_\mu j^\mu + \partial_\mu \partial_\nu T^{\mu\nu} = \partial_\mu \partial_\nu T^{\mu\nu} = 0.$$

In the last equality we used that the contraction of a symmetric tensor with an antisymmetric tensor is zero. ■

Corollary 3.6. *Given any conserved current j^μ whose spatial components vanish at infinity, then*

$$Q = \int_\mathcal{V} \mathrm{d}^3x\, j^0,$$

where $\mathcal{V}$ is all of space, is a globally conserved quantity.

Proof. That j^μ is a conserved current means

$$\partial_\mu j^\mu = \partial_t j^0 + \nabla \cdot \mathbf{j} = 0 \quad \Leftrightarrow \quad \partial_t j^0 = -\nabla \cdot \mathbf{j}.$$

Thus

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\mathcal{V} \mathrm{d}^3x\, j^0 = \int_\mathcal{V} \mathrm{d}^3x\, \partial_t j^0 = -\int_\mathcal{V} \mathrm{d}^3x\, \nabla \cdot \mathbf{j} = -\int_{\partial\mathcal{V}} \mathrm{d}A\, \hat{\mathbf{n}} \cdot \mathbf{j} = 0.$$

In the last equality we used the assumption the $\mathbf{j}$ vanishes at infinity. ■

Example 3.1. Let $\mathcal{L}$ be a Lagrangian density such that

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0,$$

$\phi^i(x)$ be fields that transform as scalars

$$\phi^i(x) \mapsto \tilde{\phi}^i(\tilde{x}) = \phi^i(x)$$

under a global coordinate transformation

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \xi^\mu,$$

and let the fields satisfy the Euler-Lagrange field equations. Consider letting ξ^μ be infinitesimal. By Proposition 3.2 the transformed fields resulting from this coordinate transformation can be written in terms of the old fields as

$$\tilde{\phi}^i = \phi^i - \xi^\mu \phi^i_{,\mu} \quad \Leftrightarrow \quad \delta\phi^i = -\xi^\mu \phi^i_{,\mu}, \tag{3.5}$$

suppressing the argument $\tilde{x}^\mu$.

Since the old fields satisfy the Euler-Lagrange field equations, we want to use Noether's theorem to find a conserved current for this field transformation symmetry. To avoid a trivial result we need to find a new way of writing $\delta\mathcal{L}$. By expanding the new Lagrangian density $\tilde{\mathcal{L}}$ and inserting Eq. (3.5) we find

$$\begin{aligned}
\delta\mathcal{L} &= \tilde{\mathcal{L}}(\tilde{\phi}^i, \tilde{\phi}^i_{,\mu}) - \mathcal{L}(\phi^i, \phi^i_{,\mu}) = \mathcal{L}(\tilde{\phi}^i, \tilde{\phi}^i_{,\mu}) - \mathcal{L}(\phi^i, \phi^i_{,\mu}) \\
&= \frac{\partial \mathcal{L}}{\partial \phi^i}\delta\phi^i + \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}}\delta\phi^i_{,\mu} = -\xi^\nu \left(\frac{\partial \mathcal{L}}{\partial \phi^i}\phi^i_{,\nu} + \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}}\phi^i_{,\nu\mu} \right) \\
&= -\xi^\nu \left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tilde{x}^\nu} - \frac{\partial \mathcal{L}}{\partial \tilde{x}^\nu} \right) = -\xi^\nu \left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\tilde{x}^\nu} - \frac{\partial \mathcal{L}}{\partial x^\nu} \right) = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}^\nu}\left(-\xi^\nu \mathcal{L}\right)
\end{aligned}$$

where in the last equation we have used the assumption that $\mathcal{L}$ does not have an explicit x-dependence. Inserting this into Noether's theorem yields the conserved current

$$j^\mu = -\xi_\nu \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \partial^\nu \phi^i - \eta^{\mu\nu}\mathcal{L} \right) = -\xi_\nu T^{\mu\nu},$$

where we have defined the canonical energy momentum stress tensor $T^{\mu\nu}$.

Since ξ^μ are independent constants this results in four independent translations and thus the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{x}^\mu} T^{\mu\nu} = 0.$$

By Corollary 3.6 this implies the four globally conserved charges

$$Q^\nu = \int \mathrm{d}^3x\, T^{0\nu} = \int \mathrm{d}^3x \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,0}} \partial^\nu \phi^i - \eta^{0\nu}\mathcal{L} \right).$$

To understand what these charges are we consider the zeroth component of Q_μ, which is obtained from Q^ν by $Q_\mu = \eta_{\mu\nu} Q^\nu$:

$$Q_0 = \int \mathrm{d}^3x \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,0}} \phi^i_{,0} - \delta^0_0 \mathcal{L} \right) = \int \mathrm{d}^3x \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,0}} \phi^i_{,0} - \mathcal{L} \right) = \int \mathrm{d}^3x \, \mathcal{H},$$

where we have identified the Hamiltonian density $\mathcal{H}$. Thus Q_0 is nothing more than the energy connected with the fields. By changing frame of reference we thus identify the spatial components of Q_μ as the components of a spatial momentum $\mathbf{p}$. Thus Q_μ is identified with a four-momentum p_μ.

3.2 The Schrödinger field

Consider the Lagrangian density

$$\mathcal{L} = i\hbar \psi^* \partial_t \psi - \frac{\hbar^2}{2m} \nabla \psi^* \cdot \nabla \psi, \tag{3.6}$$

where ψ is a complex field and ψ^* is the complex conjugate of this field.

$$\psi = \psi_1 + i\psi_2$$

and thus depends on the two real fields ψ_1 and ψ_2. Because ψ_1 and ψ_2 can be solved in terms of ψ and ψ^*, we can vary ψ and ψ^* as if they were the independent fields to obtain the Euler-Lagrange equations instead of varying ψ_1 and ψ_2. Varying ψ and requiring that the action be stationary results in the Euler-Lagrange field equation

$$i\hbar \frac{\partial \psi^*}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 \psi^*$$

which is equivalent to the complex conjugate of the Schrödinger equation. Varying ψ^* results in the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi \tag{3.7}$$

directly.

We now want to consider if the Euclidean group $E(2)$ is a symmetry group of this system in the sense that representations of its Lie algebra yields field transformations under which the action is invariant. Any element of the Euclidean group can be decomposed in a translation and an element of $O(2)$ [11]. First we consider the infinitesimal translations

$$\begin{aligned} \psi_1 &\mapsto \psi_1 + \epsilon \xi_1, \\ \psi_2 &\mapsto \psi_2 + \epsilon \xi_2. \end{aligned}$$

In terms of the fields ψ and ψ^*, which we have shown it is useful to work with, these transformations become

$$\begin{aligned} \psi &\mapsto \psi + \epsilon \xi \quad &\Rightarrow \quad \delta\psi &= \epsilon \xi, \\ \psi^* &\mapsto \psi^* + \epsilon \xi^* \quad &\Rightarrow \quad \delta\psi^* &= \epsilon \xi^*, \end{aligned}$$

where we have defined the complex number $\xi = \xi_1 + i\xi_2$. Inserting this into the Lagrangian density we get a new Lagrangian density $\tilde{\mathcal{L}}$ such that the change in the Lagrangian density is

$$\delta\mathcal{L} = i\hbar\epsilon\xi^*\partial_t\psi = \delta\psi^*\frac{\partial\mathcal{L}}{\partial\psi^*},$$

where we have noticed that $i\hbar\partial_t\psi$ is equal to the partial derivative of the Lagrangian density with respect to ψ^* and substituted $\delta\psi^*$ for $\epsilon\xi^*$. Since ψ satisfies the Schödinger equation, the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\right)$$

is satisfied. Thus $\delta\mathcal{L}$ can be written as

$$\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\delta\psi^*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\right)$$

and we have succeeded in writing the change in the Lagrange density as a total derivative. Note that we could take $\delta\psi^*$ inside the derivative because it is a constant. Since the transformations are infinitesimal and the original fields ψ and ψ^* satisfy the Euler-Lagrange equations, Proposition 3.3 applies and gives

$$\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\delta\psi^*\right).$$

Subtracting these two expressions for $\delta\mathcal{L}$ yields the equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi\right) = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\epsilon\xi\right) \quad\Leftrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = 0.$$

Substituting Eq. (3.6) for $\mathcal{L}$ and taking the complex conjugate, this equation becomes

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

which is the Schrödinger equation. Inserting ψ_1 and ψ_2 into ψ and dividing both sides by $i\hbar$ we get

$$\left(\frac{\partial\psi_1}{\partial t} + \nabla\cdot\left(\frac{\hbar}{2m}\nabla\psi_2\right)\right) + i\left(\frac{\partial\psi_2}{\partial t} + \nabla\cdot\left(-\frac{\hbar}{2m}\nabla\psi_1\right)\right) = 0.$$

Since ψ_1 and ψ_2 are real, each of the outer parentheses have to be independently zero, so we can read off a conserved current from each of these. For the first parentheses we get the conserved current

$$j_1^0 = \psi_1, \quad \mathbf{j}_1 = \frac{\hbar}{2m}\nabla\psi_2,$$

and from the second we get

$$j_2^0 = \psi_2, \quad \mathbf{j}_2 = -\frac{\hbar}{2m}\nabla\psi_1.$$

Now writing ψ_1 and ψ_2 in terms of ψ and ψ^*, these conserved currents are equivalent to the currents

$$j_1^0 = \psi + \psi^*, \quad \mathbf{j}_1 = -\frac{i\hbar}{2m}\nabla(\psi - \psi^*)$$
$$j_2^0 = i(\psi - \psi^*), \quad \mathbf{j}_2 = \frac{\hbar}{2m}\nabla(\psi + \psi^*).$$

By Corollary 3.6, these conserved currents imply globally conserved charges

$$Q_1 = \int \mathrm{d}^3x\, \psi + \psi^*$$
$$Q_2 = i\int \mathrm{d}^3x\, \psi - \psi^*.$$

as long as $\nabla\psi$ vanishes at infinity. Note that we did not assume anything particular for ξ or ξ^* in the derivation, thus independent of what these are, we will always end up with conserved currents that are equivalent to the ones shown as long as the current obtained is non-trivial.

Now we consider the $O(2)$ subgroup of $E(2)$. Using the fundamental representation of $O(2)$ acting on the vector consisting of ψ_1 and ψ_2 we get the field transformations

$$\begin{bmatrix}\tilde{\psi}_1\\ \tilde{\psi}_2\end{bmatrix} = r\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$$

for $r = \pm 1$, which in terms of ψ and ψ^* becomes

$$\tilde{\psi} = re^{-i\theta}\psi, \quad \tilde{\psi}^* = re^{i\theta}\psi^*. \tag{3.8}$$

Inserting these new fields into $\mathcal{L}$ shows that $\delta\mathcal{L} = 0$. Thus any element of $O(2)$ leaves the Lagrangian density invariant. To find the generators of this group representation we consider transformations smoothly connected and infinitesimally close to the identity. This implies that $r = 1$ and $\theta \to \epsilon\theta$. Substituting this into Eq. (3.8) yields the changes in the fields

$$\delta\psi = -i\epsilon\theta\psi, \quad \delta\psi^* = i\epsilon\theta\psi^*.$$

Because we are considering infinitesimal field transformations and the original fields ψ and ψ^* satisfies the Euler-Lagrange equations, Proposition 3.3 applies. Additionally the transformation of the fields is an element of the fundamental representation of $O(2)$ and hence leaves the Lagrangian density invariant as shown above. Thus Proposition 3.3 implies the equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\delta\psi^*\right).$$

Inserting for the Lagrangian density and the changes in the fields, this equation can be written as

$$\frac{\partial}{\partial t}\psi^*\psi + \nabla\cdot\left(\frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)\right) = 0,$$

whence we can read off the conserved current

$$j^0 = \psi^*\psi, \quad \mathbf{j} = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) = \mathrm{Re}\left(\psi^*\frac{\hbar}{im}\nabla\psi\right),$$

which we recognize as the equation for conservation of probability in quantum mechanics. The conserved charge obtained by Corollary 3.6 is

$$Q = \int \mathrm{d}^3x\, \psi^*\psi,$$

which is usually normalized to 1 and interpreted as the probability that the particle, ψ represents exists somewhere in three dimensional space, i.e. Q is the number of particles we are considering.

3.3 Second quantization of the Schrödinger field

We next quantize the field theory discussed in the last example. To this end we first find a complete set of states in which any solution can be expanded. In any finite-dimensional vector space the eigenvectors of a Hermitian operator gives such a set. The momentum operator

$$\hat{\mathbf{p}} = \frac{\hbar}{i}\nabla$$

is Hermitian and thus furnishes a complete set of states. In addition to choosing this as the Hermitian operator we restrict the system to a quantization volume $\mathcal{V}$ consisting of a cubic box with sides $L = \sqrt[3]{\mathcal{V}}$. We assume periodic boundary conditions on this system such that for any wave function ψ existing in it, we must have

$$\psi(x^j + L) = \psi(x^j)$$

for any $j \in \{1, 2, 3\}$. The eigenfunctions of the momentum operator are especially well suited for this kind of system because translational invariance means that momentum is conserved, thus $\hat{\mathbf{p}}$ commutes with the Hamiltonian and we can find simultaneous eigenfunctions of the Hamiltonian and $\hat{\mathbf{p}}$. As we will see, we can easily satisfy the boundary conditions with these eigenfunctions by restricting $\mathbf{k}$ to have certain values.

Using separation of variables, we find that the solutions of the eigenvalue equation

$$\hat{\mathbf{p}}\psi_{\mathbf{k}} = \frac{\hbar}{i}\nabla\psi_{\mathbf{k}} = \hbar\mathbf{k}\psi_{\mathbf{k}} \tag{3.9}$$

are

$$\psi_{\mathbf{k}} = Ce^{i(\mathbf{k}\cdot\mathbf{r})}$$

for some normalization constant C. Normalizing the states such that

$$\int \mathrm{d}^3x\, \psi^*_{\mathbf{k}}\psi_{\mathbf{k}} = 1$$

and choosing C real, the solutions are

$$\psi_{\mathbf{k}} = \frac{1}{\sqrt{\mathcal{V}}} e^{i(\mathbf{k}\cdot\mathbf{r})}.$$

The periodic boundary conditions restrict $\mathbf{k}$ to be of the form

$$\mathbf{k} = \frac{2\pi}{L} \sum_{j=1}^{3} n_j \mathbf{e}_j \tag{3.10}$$

for $n_j \in \mathbb{Z}$. Eigenvectors of Hermitian operators with different eigenvalues are orthogonal, thus with the normalization above, the wave functions $\psi_{\mathbf{k}}$ satisfy the orthonormality relation

$$\int \mathrm{d}^3x\, \psi^*_{\mathbf{k}}(\mathbf{r}) \psi_{\mathbf{k}'}(\mathbf{r}) = \delta_{\mathbf{k}\mathbf{k}'}. \tag{3.11}$$

At any point in time, any state ψ can be expanded in this basis set

$$\psi(\mathbf{r}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r})$$

for some coefficients $a_{\mathbf{k}}$.

Now we consider how the solutions evolve in time. This time evolution is determined by the time-dependent Schrödinger equation in Eq. (3.7). Written in terms of the momentum operator it says that

$$i\hbar \frac{\partial \psi_{\mathbf{k}}(\mathbf{r},t)}{\partial t} = \frac{\hat{\mathbf{p}}^2}{2m} \psi_{\mathbf{k}}(\mathbf{r},t) = \frac{\hbar^2 \mathbf{k}^2}{2m} \psi_{\mathbf{k}}(\mathbf{r},t) \quad \Rightarrow \quad \psi_{\mathbf{k}}(\mathbf{r},t) = e^{-i\omega_{\mathbf{k}} t} \psi_{\mathbf{k}}(\mathbf{r}),$$

where we have used Eq. (3.9) to evaluate $\hat{\mathbf{p}}\psi_{\mathbf{k}}(\mathbf{r},t)$ twice, solved the resulting first order separable equation and defined the frequency

$$\omega_{\mathbf{k}} = \frac{E_{\mathbf{k}}}{\hbar} = \frac{\hbar \mathbf{k}^2}{2m}.$$

It follows from Eq. (3.11) that

$$\int \mathrm{d}^3x\, \psi^*_{\mathbf{k}}(\mathbf{r},t) \psi_{\mathbf{k}'}(\mathbf{r},t) = \delta_{\mathbf{k}\mathbf{k}'}.$$

Thus as time evolves, the wave function that was initially expanded in the momentum eigenfunction basis becomes

$$\psi(\mathbf{r},t) = \sum_{\mathbf{k}} a_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r},t). \tag{3.12}$$

Taking the complex conjugate of this equation, we obtain the expansion of ψ^*:

$$\psi^*(\mathbf{r},t) = \sum_{\mathbf{k}} a^*_{\mathbf{k}} \psi^*_{\mathbf{k}}(\mathbf{r},t). \tag{3.13}$$

To quantize this field theory we promote the functions ψ and ψ^* to field operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ called the annihilation and creation field operators respectively. These operators are assumed to obey the canonical harmonic oscillator commutation relations for a continuous spectrum, i.e.

$$[\hat{\psi}(\mathbf{r},t),\hat{\psi}^\dagger(\mathbf{r}',t)] = \delta(\mathbf{r}-\mathbf{r}'), \quad [\hat{\psi}(\mathbf{r},t),\hat{\psi}(\mathbf{r}',t)] = 0, \quad [\hat{\psi}^\dagger(\mathbf{r},t),\hat{\psi}^\dagger(\mathbf{r}',t)] = 0.$$

When these operators act on a quantum many-particle state, they are interpreted as creating and annihilating a particle in position eigenstate $\mathbf{r}$. To see what this means for the coefficients in the Fourier expansion in Eq. (3.12) and Eq. (3.13), we multiply both of these equations by the function $\psi_{\mathbf{k}'}(\mathbf{r},t)$ and integrate over all $\mathbf{r}$. Since $\psi_{\mathbf{k}}(\mathbf{r},t)$ and $\psi_{\mathbf{k}'}(\mathbf{r},t)$ are orthogonal, we obtain

$$\hat{a}_{\mathbf{k}} = \int \mathrm{d}^3x\, \hat{\psi}(\mathbf{r},t)\psi_{\mathbf{k}}(\mathbf{r},t), \tag{3.14}$$

$$\hat{a}_{\mathbf{k}}^\dagger = \int \mathrm{d}^3x\, \hat{\psi}^\dagger(\mathbf{r},t)\psi_{\mathbf{k}}(\mathbf{r},t), \tag{3.15}$$

after promoting the functions to operators. These operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ create and annihilate a particle in momentum eigenstate $\mathbf{k}$. Using the commutation relations for $\hat{\psi}$ and $\hat{\psi}^\dagger$, $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ can be shown to satisfy the commutation relations

$$[\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}] = 0, \quad [\hat{a}_{\mathbf{k}}^\dagger,\hat{a}_{\mathbf{k}'}^\dagger] = 0, \quad [\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}, \tag{3.16}$$

by inserting Eq. (3.14) and Eq. (3.15) into the commutators in Eq. (3.16).

Promoting ψ and ψ^* to operators, the zeroth components of the conserved currents found in Example 3.2 are given by

$$\hat{\rho}_1 = i(\hat{\psi}-\hat{\psi}^\dagger), \quad \hat{\rho}_2 = \hat{\psi}+\hat{\psi}^\dagger, \quad \hat{\rho}_3 = \hat{\psi}^\dagger\hat{\psi}.$$

The equal time commutation relations between these operators follows from the field operator commutators such that

$$\begin{aligned}
[\hat{\rho}_1(\mathbf{r},t),\hat{\rho}_2(\mathbf{r}',t)] &= i2\delta(\mathbf{r}-\mathbf{r}'),\\
[\hat{\rho}_3(\mathbf{r},t),\hat{\rho}_1(\mathbf{r}',t)] &= -i\delta(\mathbf{r}-\mathbf{r}')\hat{\rho}_2(\mathbf{r},t),\\
[\hat{\rho}_2(\mathbf{r},t),\hat{\rho}_3(\mathbf{r}',t)] &= -i\delta(\mathbf{r}-\mathbf{r}')\hat{\rho}_1(\mathbf{r},t).
\end{aligned}$$

Since these operators are the zeroth components of conserved currents, their integrals yield conserved charges by Corollary 3.6. Defining the operators

$$\hat{Q}_i = \int \mathrm{d}^3x\, \hat{\rho}_i$$

such that

$$\hat{Q}_1 = i\sqrt{\mathcal{V}}\left(\hat{a}_{\mathbf{0}}-\hat{a}_{\mathbf{0}}^\dagger\right), \quad \hat{Q}_2 = \sqrt{\mathcal{V}}\left(\hat{a}_{\mathbf{0}}+\hat{a}_{\mathbf{0}}^\dagger\right), \quad \hat{Q}_3 = \sum_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}}. \tag{3.17}$$

In deriving these identities we have used the evaluation of the integral over $\psi_{\mathbf{k}}(\mathbf{r}, t)$

$$\int \mathrm{d}^3x\, \psi_{\mathbf{k}}(\mathbf{r}, t) = \sqrt{\mathcal{V}}\delta_{\mathbf{k}\mathbf{0}} \tag{3.18}$$

which implies

$$\int \mathrm{d}^3x\, \psi^*_{\mathbf{k}}(\mathbf{r}, t) = \sqrt{\mathcal{V}}\delta_{\mathbf{k}\mathbf{0}}. \tag{3.19}$$

The commutation relations between the $\hat{Q}_i$ can be obtained directly by integrating the commutation relations for $\hat{\rho}_i$ with respect to $\mathbf{r}$ and $\mathbf{r}'$;

$$[\hat{Q}_1, \hat{Q}_2] = 2i\mathcal{V}\hat{I}, \quad [\hat{Q}_3, \hat{Q}_1] = -i\hat{Q}_2, \quad [\hat{Q}_2, \hat{Q}_3] = -i\hat{Q}_1, \tag{3.20}$$

thus it follows that the set $\{i\hat{Q}_1, i\hat{Q}_2, i\hat{Q}_3, i\hat{I}\}$ is a Lie algebra. If the Lagrangian density is invariant under a transformation, then in the canonical quantization, the conserved charges generate the transformations by commuting with the fields [12]. In this case however, the Lagrangian density is not invariant, and the $\hat{Q}_i$ do not yield a Lie algebra representation of the Euclidean group. This can be seen by computing the structure constants of the Lie algebra of the Euclidean group, which stem from the commutation relations of its basis vectors. This Lie algebra is spanned by the matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose commutation relations are

$$[A_1, A_2] = 0, \quad [A_3, A_1] = -A_2, \quad [A_2, A_3] = -A_1.$$

Hence the nonvanishing structure constants are given by

$$a_{312} = -1, \quad a_{132} = 1, \quad a_{231} = -1, \quad a_{321} = 1.$$

Out of the $4^3 = 64$ structure constants pertaining to the $\hat{Q}_i$ and $\hat{I}$, the only nonvanishing ones are

$$q_{124} = -2\mathcal{V}, \quad q_{214} = 2\mathcal{V}, \quad q_{312} = 1, \quad q_{132} = -1, \quad q_{231} = 1, \quad q_{321} = -1,$$

which follows from Eq. (3.20) by denoting $i\hat{I}$ as the fourth operator. For a set of operators to be a Lie algebra representation, it has to have an equivalent set of structure constants to that of its Lie algebra [13], which the set of $i\hat{Q}_i$ and $i\hat{I}$ thus do not possess. The problem arises from the commutator between $\hat{Q}_1$ and $\hat{Q}_2$ which is proportional to the identity. Such a proportionality factor is called a central charge and comes from the generators being related to a projective representation of the group, rather than a representation [14]. In the context of quantum mechanics, a projective representation is the same as a representation up to a phase, which means that when applying two unitary operators corresponding to two different symmetry transformations to a state, this might differ by a phase from applying the unitary operator corresponding to the combined transformation of the two symmetry transformations [14]. This is indeed the case for the $\hat{Q}_i$ since a unitary operator can be obtained by exponentiation and

$$
\begin{aligned}
e^{i\theta_1\hat{Q}_1}e^{i\theta_2\hat{Q}_2}\left|\psi\right\rangle &= \exp\left[i(\theta_1\hat{Q}_1+\theta_2\hat{Q}_2)-\frac{\theta_1\theta_2}{2}2i\mathcal{V}I\right]\left|\psi\right\rangle \\
&= e^{i\theta_1\theta_2\mathcal{V}}\exp\left[i(\theta_1\hat{Q}_2+\theta_2\hat{Q}_2)\right]\left|\psi\right\rangle .
\end{aligned}
$$

We observe that the $\hat{Q}_i$ are Hermitian and thus correspond to observable quantities. To check that these quantities are conserved we need to see if the operators commute with the Hamiltonian of the system. The Hamiltonian can be derived from the Lagrangian density in Eq. (3.6) by a Legendre transformation and subsequent integration, i.e.

$$
H=\int \mathrm{d}^3x\,\mathcal{H}=\int \mathrm{d}^3x\,\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\partial_t\psi+\frac{\partial\mathcal{L}}{\partial(\partial_t\psi^*)}\partial_t\psi^*-\mathcal{L}\right)=\frac{\hbar^2}{2m}\int \mathrm{d}^3x\,\nabla\psi^*\cdot\nabla\psi .
$$

Now promoting ψ and ψ^* to operators and inserting their expansion in terms of $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ yields the second-quantized Hamiltonian

$$
\hat{H}=\sum_{\mathbf{k}}\frac{\hbar^2\mathbf{k}^2}{2m}\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}}. \tag{3.21}
$$

We can then use this Hamiltonian to evaluate the commutators $[\hat{H},\hat{Q}_i]$ which vanish as advertised.

3.4 Degeneracy of the vacuum state

Next we consider the state resulting from a transformation of the vacuum state by an element of the projective representation of the Euclidean group generated by the conserved charges $\hat{Q}_1$ and $\hat{Q}_2$:

$$
\left|z\right\rangle = e^{i\left(\theta_1\hat{Q}_1+\theta_2\hat{Q}_2\right)}\left|0\right\rangle , \tag{3.22}
$$

where $\left|0\right\rangle$ denotes the vacuum state defined by $\hat{a}_{\mathbf{k}}\left|0\right\rangle = 0$, i.e. the state without any particles. The state $\left|z\right\rangle$ is thus, up to a phase, the result of translating the fields in field space, starting from the vacuum state. Inserting Eq. (3.17) for $\hat{Q}_1$ and $\hat{Q}_2$, and defining the complex number $z=\theta_1+i\theta_2$ yields that

$$
\left|z\right\rangle = e^{\sqrt{\mathcal{V}}\left(z\hat{a}_{\mathbf{0}}^\dagger - z^*\hat{a}_{\mathbf{0}}\right)}\left|0\right\rangle .
$$

To calculate

$$
\left\langle z'|z\right\rangle = \left\langle 0\right| e^{\sqrt{\mathcal{V}}\left(z'^*\hat{a}_{\mathbf{0}}-z'\hat{a}_{\mathbf{0}}^\dagger\right)}e^{\sqrt{\mathcal{V}}\left(z\hat{a}_{\mathbf{0}}^\dagger-z^*\hat{a}_{\mathbf{0}}\right)}\left|0\right\rangle , \tag{3.23}
$$

we make use of the Baker-Campbell-Hausdorff formula [8]

$$
e^Xe^Y=\exp\left(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\dots\right)
$$

for any two linear operators X and Y, to express the two exponentials as

$$e^{i\mathcal{V}\,\mathrm{Im}(zz'^*)}\exp\left[\sqrt{\mathcal{V}}\Big(\hat{a}_{\mathbf{0}}(z'-z)^*-\hat{a}_{\mathbf{0}}^\dagger(z'-z)\Big)\right].$$

Using the formula in reverse, the last exponential in the above expression can be written as

$$e^{-\sqrt{\mathcal{V}}\hat{a}_{\mathbf{0}}^\dagger(z'-z)}e^{\sqrt{\mathcal{V}}\hat{a}_{\mathbf{0}}(z'-z)^*}e^{-\frac{1}{2}\mathcal{V}|z'-z|^2}.$$

Substituting this back into Eq. (3.23) we get

$$\begin{aligned}\langle z'|z\rangle &= e^{i\mathcal{V}\,\mathrm{Im}(zz'^*)}e^{-\frac{1}{2}\mathcal{V}|z'-z|^2}\langle 0|\,e^{-\sqrt{\mathcal{V}}\hat{a}_{\mathbf{0}}^\dagger(z'-z)}e^{\sqrt{\mathcal{V}}\hat{a}_{\mathbf{0}}(z'-z)^*}\,|0\rangle\\ &= e^{i\mathcal{V}\,\mathrm{Im}(zz'^*)}e^{-\frac{1}{2}\mathcal{V}|z'-z|^2}\langle 0|0\rangle\,,\end{aligned}$$

whence it follows that

$$|\langle z'|z\rangle| = e^{-\frac{1}{2}\mathcal{V}|z'-z|^2} \tag{3.24}$$

and

$$\langle z|z\rangle = 1$$

assuming the vacuum state $|0\rangle$ is normalized to unity. Now we consider acting on $|z\rangle$ with the field operator $\hat{\psi}(\mathbf{r},t)$:

$$\begin{aligned}\hat{\psi}(\mathbf{r},t)\,|z\rangle &= \sum_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{r},t)\hat{a}_{\mathbf{k}}e^{\sqrt{\mathcal{V}}(z\hat{a}_{\mathbf{0}}^\dagger-z^*\hat{a}_{\mathbf{0}})}\,|0\rangle\\ &= e^{-\frac{1}{2}\mathcal{V}|z|^2}\sum_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{r},t)\hat{a}_{\mathbf{k}}e^{\sqrt{\mathcal{V}}z\hat{a}_{\mathbf{0}}^\dagger}e^{-\sqrt{\mathcal{V}}z^*\hat{a}_{\mathbf{0}}}\,|0\rangle\\ &= e^{-\frac{1}{2}\mathcal{V}|z|^2}\sum_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{r},t)\sum_{n=1}^{\infty}\hat{a}_{\mathbf{k}}\frac{(z\sqrt{\mathcal{V}})^n}{n!}(\hat{a}_{\mathbf{0}}^\dagger)^n\,|0\rangle\\ &= e^{-\frac{1}{2}\mathcal{V}|z|^2}\psi_{\mathbf{0}}(\mathbf{r},t)\sum_{n=1}^{\infty}\frac{(z\sqrt{\mathcal{V}})^n}{n!}\hat{a}_{\mathbf{0}}(\hat{a}_{\mathbf{0}}^\dagger)^n\,|0\rangle\,,\end{aligned}$$

where we first have used the Baker-Campbell-Hausdorff formula in reverse to express the exponent in $|z\rangle$ as three separate exponents, then we have used that $e^{\alpha\hat{a}_{\mathbf{k}}}\,|0\rangle = |0\rangle$ because $\hat{a}_{\mathbf{k}}\,|0\rangle = 0$ and finally we have used that for all $\mathbf{k}\neq\mathbf{0}$, $\hat{a}_{\mathbf{k}}(\hat{a}_{\mathbf{0}}^\dagger)^n\,|0\rangle = 0$ because $[\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{0}}^\dagger] = \delta_{\mathbf{0}\mathbf{k}}$. This commutation relation can also be used to show that

$$\hat{a}_{\mathbf{0}}(\hat{a}_{\mathbf{0}}^\dagger)^n\,|0\rangle = n(\hat{a}_{\mathbf{0}}^\dagger)^{n-1}\,|0\rangle\,.$$

Inserting this and the fact that $\psi_{\mathbf{0}}(\mathbf{r},t) = \frac{1}{\sqrt{\mathcal{V}}}$

$$\begin{aligned}\hat{\psi}(\mathbf{r},t)\,|z\rangle &= e^{-\frac{1}{2}\mathcal{V}|z|^2}\frac{1}{\sqrt{\mathcal{V}}}z\sqrt{\mathcal{V}}\sum_{n=1}^{\infty}\frac{(z\sqrt{\mathcal{V}})^{(n-1)}}{(n-1)!}(\hat{a}_{\mathbf{0}}^\dagger)^{(n-1)}\,|0\rangle\\ &= e^{-\frac{1}{2}\mathcal{V}|z|^2}z\sum_{m=0}^{\infty}\frac{(z\sqrt{\mathcal{V}})^m}{m!}(\hat{a}_{\mathbf{0}}^\dagger)^m\,|0\rangle = e^{-\frac{1}{2}\mathcal{V}|z|^2}ze^{z\sqrt{\mathcal{V}}\hat{a}_{\mathbf{0}}^\dagger}\,|0\rangle\\ &= e^{-\frac{1}{2}\mathcal{V}|z|^2}ze^{z\sqrt{\mathcal{V}}\hat{a}_{\mathbf{0}}^\dagger}e^{-\sqrt{\mathcal{V}}z^*\hat{a}_{\mathbf{0}}}\,|0\rangle = ze^{\sqrt{\mathcal{V}}(z\hat{a}_{\mathbf{0}}^\dagger-z^*\hat{a}_{\mathbf{0}})}\,|0\rangle = z\,|z\rangle\,.\end{aligned} \tag{3.25}$$

Because $\hat{\psi}$ acts as a lowering operator on the number of particles, Eq. (3.25) implies that $|z\rangle$ is a coherent state by the definition in [15]. Taking the adjoint of Eq. (3.25) yields

$$\langle z| \hat{\psi}^\dagger(\mathbf{r}, t) = \langle z| z^*.$$

With these two relations it is trivial to show that

$$\begin{aligned}\langle z| \hat{\psi}(\mathbf{r}, t) |z\rangle &= z \langle z|z\rangle = z, \\ \langle z| \hat{\psi}^\dagger(\mathbf{r}, t)\hat{\psi}(\mathbf{r}, t) |z\rangle &= |z|^2,\end{aligned}$$

and indeed in general

$$\langle z| (\hat{\psi}^\dagger(\mathbf{r}, t))^n(\hat{\psi}(\mathbf{r}, t))^m |z\rangle = (z^*)^n z^m.$$

By the property of being a coherent state, we expect the state to give a minimum uncertainty in measuring certain incompatible observables. In this case the observables are given by the Hermitian operators $\hat{Q}_1$ and $\hat{Q}_2$. According to the general uncertainty principle [16] the variance in measuring these two quantities is

$$\sigma^2_{\hat{Q}_1}\sigma^2_{\hat{Q}_2} \geq \left(\frac{1}{2i}\left\langle[\hat{Q}_1, \hat{Q}_2]\right\rangle\right)^2 = \left(\frac{1}{2i}2i\mathcal{V}\right)^2 = \mathcal{V}^2,$$

independently of what state the system is in. To evaluate the actual variance in measurements when the systems is in a state $|z\rangle$, we first have to compute the expectation values

$$\begin{aligned}\left\langle\hat{Q}_1\right\rangle_z &= \langle z| \int \mathrm{d}^3x\, i\left(\hat{\psi}(\mathbf{r}, t) - \hat{\psi}^\dagger(\mathbf{r}, t)\right)|z\rangle \\ &= \int \mathrm{d}^3x\, i(z - z^*) = -2\mathcal{V}\,\mathrm{Im}(z), \\ \left\langle\hat{Q}_1{}^2\right\rangle_z &= \langle z| \int \mathrm{d}^3x \int \mathrm{d}^3x'\, i\left(\hat{\psi}(\mathbf{r}, t) - \hat{\psi}^\dagger(\mathbf{r}, t)\right) i\left(\hat{\psi}(\mathbf{r}', t) - \hat{\psi}^\dagger(\mathbf{r}', t)\right) \\ &= -\int \mathrm{d}^3x \int \mathrm{d}^3x'\left(z^2 - \delta(\mathbf{r} - \mathbf{r}') - 2|z|^2 + (z^*)^2\right) \\ &= \mathcal{V} - \mathcal{V}^2(z - z^*)^2 = \mathcal{V} + (2\mathcal{V}\,\mathrm{Im}(z))^2 = \mathcal{V} + \left\langle\hat{Q}_1\right\rangle_z^2, \\ \left\langle\hat{Q}_2\right\rangle_z &= \langle z| \int \mathrm{d}^3x\, \hat{\psi}(\mathbf{r}, t) + \hat{\psi}^\dagger(\mathbf{r}, t)|z\rangle = \int \mathrm{d}^3x\, z + z^* = 2\mathcal{V}\,\mathrm{Re}(z), \\ \left\langle\hat{Q}_2{}^2\right\rangle_z &= \mathcal{V} + \left\langle\hat{Q}_2\right\rangle_z^2,\end{aligned} \tag{3.26}$$

where the last equation was obtained through a similar calculation to that of $\left\langle\hat{Q}_1{}^2\right\rangle_z$. Thus in the state $|z\rangle$, the variance is given by

$$\sigma^2_{\hat{Q}_1}\sigma^2_{\hat{Q}_2} = \left(\left\langle\hat{Q}_1{}^2\right\rangle_z - \left\langle\hat{Q}_1\right\rangle_z^2\right)\left(\left\langle\hat{Q}_2{}^2\right\rangle_z - \left\langle\hat{Q}_2\right\rangle_z^2\right) = \mathcal{V}^2,$$

hence the uncertainty in measuring the observables connected with $\hat{Q}_1$ and $\hat{Q}_2$ is at a minimum. Another consequence of the expectation values in Eq. (3.26) is that the states

$|z\rangle$ and $|z'\rangle$ represent different physical states of the system for $z \neq z'$ since they will result in different expectation values of these Hermitian operators which correspond to physical observables.

Whilst exploring the properties of the $|z\rangle$ states, the energy is, as usual, of indispensable significance. We begin by translating the eigenvalue equation in Eq. (3.25) into the context of the momentum annihilation operators $\hat{a}_{\mathbf{k}}$. By relating the momentum annihilation field operators to the position annihilation field operator through Eq. (3.14) then

$$\hat{a}_{\mathbf{k}} |z\rangle = \int \mathrm{d}^3x\, \psi_{\mathbf{k}}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\,|z\rangle = z\int \mathrm{d}^3x\, \psi_{\mathbf{k}}(\mathbf{r},t) = z\sqrt{\mathcal{V}}\delta_{\mathbf{k0}}, \tag{3.27}$$

follows from Eq. (3.25) and Eq. (3.18). Taking the adjoint of this equation yields

$$\langle z|\,\hat{a}^{\dagger}_{\mathbf{k}} = z^{*}\sqrt{\mathcal{V}}\delta_{\mathbf{k0}}. \tag{3.28}$$

Inserting Eq. (3.21) for $\hat{H}$ we get

$$\langle \hat{H}\rangle_z = \sum_{\mathbf{k}} \frac{\hbar^2\mathbf{k}^2}{2m}\,\langle z|\,\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,|z\rangle = \sum_{\mathbf{k}} \frac{\hbar^2\mathbf{k}^2}{2m}\mathcal{V}\,|z|^2\,\delta_{\mathbf{k0}} = 0,$$

thus the energy vanishes for all $|z\rangle$ including the state resulting from the choice $z = 0$. From the above discussion we recall that different zs yield physically distinct states. Since there are infinitely many choices for z, we can say that the vacuum state is infinitely degenerate because of the field translation symmetry in the Lagrangian density.

3.4.1 Lifting the degeneracy

Since the degeneracy prevents us from singling out a unique ground state which we can use in building the Hilbert space, we would like to lift this degeneracy. We do this by introducing a term in the Hamiltonian that depends on some adjustable continuous parameter such that the degeneracy can be turned on and off by taking a limit in this parameter. Let the new Hamiltonian be defined by

$$\hat{H}_{\mu_3} = \hat{H} - \mu_3\hat{Q}_3, \tag{3.29}$$

where $\hat{H}$ is the old Hamiltonian and μ_3 is a real scalar. Since $\hat{Q}_3$ is the number operator, μ_3 can be regarded as a chemical potential by comparison with the exponent in the grand canonical partition function of $\hat{H}$. In the state $|z\rangle$ the expectation value of $\hat{Q}_3$ is

$$\langle \hat{Q}_3\rangle_z = \sum_{\mathbf{k}} \langle z|\,\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,|z\rangle = \sum_{\mathbf{k}} \mathcal{V}\,|z|^2\,\delta_{\mathbf{k0}} = \mathcal{V}\,|z|^2\,, \tag{3.30}$$

where we again have used Eq. (3.27) and Eq. (3.28) in evaluating the expression $\langle z|\,\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,|z\rangle$. With the new Hamiltonian, the expected energy of the $|z\rangle$ state becomes

$$\langle \hat{H}_{\mu_3}\rangle_z = \langle \hat{H} - \mu_3\hat{Q}_3\rangle_z = \langle \hat{H}\rangle_z - \mu_3\langle \hat{Q}_3\rangle_z = -\mu_3\mathcal{V}\,|z|^2\,, \tag{3.31}$$

thus we must require that $\mu_3 \leq 0$ in order to avoid the energy becoming unbounded from below. If $\mu_3 = 0$, $\hat{H}_{\mu_3}$ is equivalent to $\hat{H}$ and again the ground state will be degenerate so that any value of z would yield a ground state. Finally if $\mu < 0$ and we consider the thermodynamic limit where $\mathcal{V} \to \infty$ then $z = 0$ is singled out as the ground state and every other choice for z will be prohibited by its diverging energy eigenvalue.

The thermodynamic limit also has other interesting consequences. One of them is that in this limit the generators $\hat{Q}_1$ and $\hat{Q}_2$ are not well defined as can be seen from Eq. (3.17). This is an example of a general result saying that all broken generators are ill defined in the thermodynamic limit [17], broken generators being generators that do not annihilate the vacuum state. If the vacuum is assumed to be translationally invariant then $\hat{\rho}(\mathbf{r}, t)\,|0\rangle = \hat{\rho}(\mathbf{0}, t)\,|0\rangle$ for a charge density ρ. Thus considering the inner product of the state $\hat{Q}(t)\,|0\rangle$ with itself yields

$$\langle 0|\,\hat{Q}(t)\hat{Q}(t)\,|0\rangle = \int \mathrm{d}^3x\ \langle 0|\,\hat{Q}(t)\rho(\mathbf{r}, t)\,|0\rangle = \langle 0|\,\hat{Q}(t)\rho(\mathbf{0}, t)\,|0\rangle \int \mathrm{d}^3x,$$

which diverges unless $Q(t)\,|0\rangle = 0$. The definition of being an unbroken generator is that $Q(t)\,|0\rangle \neq 0$ thus strictly speaking unbroken generators are not well defined in the thermodynamic limit.

It is also worth noting that the $|z\rangle$ states have important physical consequences. From Eq. (3.30) we see that the expectation value of $\hat{Q}_3$ diverges in the thermodynamic limit, however the density remains finite such that $|z\rangle$ can be interpreted as the states where all particles are in the ground state giving a homogeneous number density. Such states are relevant in the study of Bose-Einstein condensates where a macroscopic number of bosons are in a single quantum state [18].

Finally we would like to justify the method that has been used in singling out an unambiguous ground state. In this approach we have used the operator $\hat{Q}_3$ governed by the parameter μ_3 to explicitly break the symmetry of the original Hamiltonian. By then considering the thermodynamic limit we were led to a unique ground state from which, as will be shown in the next section, we may build the Hilbert space of excited states. We can then adiabatically turn off the modification to the system by letting $\mu_3 \to 0$. The reason why we need to choose a specific $|z\rangle$ as our starting point is tied to the large size of the system in the thermodynamic limit. Without any modification it might be natural to believe that the true ground state of the system would be a superposition of $|z\rangle$ such that this superposition retained the symmetry of $\hat{H}$. In the thermodynamic limit however, any off diagonal elements $\langle z|\,\hat{H}\,|z'\rangle\,,\ z \neq z'$ are exponentially suppressed and so are the off-diagonal elements of any small potentially symmetry-breaking perturbation $\hat{H}'$ to the Hamiltonian, as long as this perturbation depends on local Hermitian operators [19]. Thus for any such small perturbation, the diagonal elements of the resulting Hamiltonian will differ much more than the off-diagonal elements such that the resulting ground state will be very close to one of the $|z\rangle$ and not to a superposition of them. Exactly which $|z\rangle$, depends on the perturbation. In Nature we will always have minuscule perturbations to the system and thus the ground state will always collapse to a specific $|z\rangle$. In the case of our small μ_3-dependent perturbation, the ground state collapses to the choice $z = 0$. From Eq. (3.24) it is clear that off-diagonal elements are suppressed since for $z \neq z'$, $\langle z|z'\rangle \sim e^{-\mathcal{V}}$.

This factor is analogous to the tunneling amplitude when going from one well to the other in a double-well potential in quantum mechanics [13]. If we regard a quantum field theory as an infinite set of oscillators where each is in such a double-well potential, then there is a tunneling amplitude for each of them to switch to the other well, but for the field theoretic state $|z\rangle$ to turn into $|z'\rangle$, all the oscillators have to tunnel, the probability of which is thus given by the probability of a single tunneling event to the power of the number of oscillators, which scales as the volume and thus is zero in the thermodynamic limit.

The mathematical consequence of Eq. (3.24) is that in the thermodynamic limit the set of $|z\rangle$ cannot be contained in a single Hilbert space, but rather every choice of z forms a separate Hilbert space. Each of these Hilbert spaces however gives a physically equivalent description of the system [20].

3.4.2 Building the Hilbert space

The excited states of this Hilbert space can now be found by repeated application of the creation operator on $|0\rangle$. Since the set of possible $\mathbf{k}$ from Eq. (3.10) is discrete we can label possible $\mathbf{k}$, $\mathbf{k}_i$. Let n_i denote the number of excited $\mathbf{k}_i$ modes such that

$$|n_1, n_2, \dots\rangle = \prod_i \frac{\left(\hat{a}^\dagger_{\mathbf{k}_i}\right)^{n_i}}{\sqrt{n_i!}} |0\rangle ,$$

are states with n_1 modes in a $\mathbf{k}_1$ eigenstate, n_2 modes in a $\mathbf{k}_2$ eigenstate, etc. These states are both orthonormal and complete in what is called the Fock space. To prove that this set is orthonormal we see that in general

$$\hat{a}_{\mathbf{k}'} \left(\hat{a}^\dagger_{\mathbf{k}}\right)^n |0\rangle = n\, \delta_{\mathbf{k}\mathbf{k}'} \left(\hat{a}^\dagger_{\mathbf{k}}\right)^{n-1} |0\rangle ,$$

and consequently

$$\left(\hat{a}_{\mathbf{k}'}\right)^{n'} \left(\hat{a}^\dagger_{\mathbf{k}}\right)^n |0\rangle = \begin{cases} \frac{n!}{(n-n')!} \delta_{\mathbf{k}\mathbf{k}'} \left(\hat{a}^\dagger_{\mathbf{k}}\right)^{n-n'} |0\rangle , & n \geq n' \\ 0, & n < n'. \end{cases} \tag{3.32}$$

Using this and the fact that the $\hat{a}$s commute for different $\mathbf{k}$, then for two different sets $\{n'_j\}$ and $\{n_j\}$

$$\begin{aligned} \langle n'_1, n'_2, \dots | n_1, n_2, \dots \rangle &= \langle 0| \prod_i \frac{\left(\hat{a}_{\mathbf{k}_i}\right)^{n'_i}}{\sqrt{n'_i!}} \frac{\left(\hat{a}^\dagger_{\mathbf{k}_i}\right)^{n_i}}{\sqrt{n_i!}} |0\rangle \\ &= \langle 0| \prod_i \delta_{n'_i n_i} \frac{1}{\sqrt{n_i! n'_i!}} \frac{n_i!}{0!} |0\rangle = \prod_i \delta_{n'_i n_i} . \end{aligned}$$

On the last line we used Eq. (3.32) to see that the expression would vanish if $n_i < n'_i$ for any i and then further used the first case to argue that if $n_i > n'_i$ then there would be a remaining $\hat{a}^\dagger_{\mathbf{k}_i}$ which would annihilate the left vacuum state, thus we get $\delta_{n_i n'_i}$. This proves that these states are orthonormal.

The states given by $|n_1, n_2, \ldots\rangle$ are also eigenstates of the Hamiltonian because

$$\begin{aligned}
\hat{H}\,|n_1, n_2, \ldots\rangle &= \sum_j \frac{\hbar^2 \mathbf{k}_j^2}{2m} \hat{a}^\dagger_{\mathbf{k}_j} \hat{a}_{\mathbf{k}_j} \prod_i \frac{\left(\hat{a}^\dagger_{\mathbf{k}_i}\right)^{n_i}}{\sqrt{n_i!}} |0\rangle \\
&= \sum_j \frac{\hbar^2 \mathbf{k}_j^2}{2m} \prod_{i \neq j} \frac{\left(\hat{a}^\dagger_{\mathbf{k}_i}\right)^{n_i}}{\sqrt{n_i!}} \hat{a}^\dagger_{\mathbf{k}_j} \hat{a}_{\mathbf{k}_j} \frac{\left(\hat{a}^\dagger_{\mathbf{k}_j}\right)^{n_j}}{\sqrt{n_j!}} |0\rangle \\
&= \sum_j \frac{\hbar^2 \mathbf{k}_j^2}{2m} \prod_{i \neq j} \frac{\left(\hat{a}^\dagger_{\mathbf{k}_i}\right)^{n_i}}{\sqrt{n_i!}} \hat{a}^\dagger_{\mathbf{k}_j} \frac{n_j}{\sqrt{n_j!}} \left(\hat{a}^\dagger_{\mathbf{k}_j}\right)^{n_j - 1} |0\rangle \\
&= \left(\sum_j \frac{\hbar^2 \mathbf{k}_j^2}{2m} n_j \right) |n_1, n_2, \ldots\rangle\,.
\end{aligned}$$

A similar computation shows that

$$\hat{Q}_3\,|n_1, n_2, \ldots\rangle = \left(\sum_j n_j \right) |n_1, n_2, \ldots\rangle\,,$$

thus

$$\hat{H}_\mu\,|n_1, n_2, \ldots\rangle = \left[\sum_j \left(\frac{\hbar^2 \mathbf{k}_j^2}{2m} - \mu \right) n_j \right] |n_1, n_2, \ldots\rangle\,.$$

By now adiabatically turning off the modification to $\hat{H}$ by letting $\mu \to 0$ we see from the above equation that this is a theory of single particles with momentum eigenstates given by

$$|\mathbf{k}\rangle = \hat{a}^\dagger_{\mathbf{k}}\,|0\rangle$$

and energy in these states given by

$$E_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m}. \tag{3.33}$$

When $\mathbf{k} \to \mathbf{0}$, the energy vanishes. This means that these modes are massless and thus examples of Nambu-Goldstone bosons which will be discussed in the next chapter.

3.5 Covariant derivatives in a $U(1)$ invariant field theory

When including a term like $\mu\hat{Q}_3$ in $\hat{H}_\mu$ in an originally Lorentz invariant theory, the resulting theory is generally not Lorentz invariant. To see this we consider a Lorentz invariant theory with a Lagrangian density

$$\mathcal{L} = \partial_\mu \phi\, \partial^\mu \phi^* - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2, \tag{3.34}$$

with a complex scalar field ϕ. Since ϕ is a scalar, the potential terms are Lorentz invariant and since the Lorentz indices are contracted in the kinetic energy term this term is Lorentz invariant as well. This scalar complex field can be written in terms of two scalar fields $\phi = 1/\sqrt{2}(\phi_1 + i\phi_2)$ which upon evaluating the Euler-Lagrange equations satisfies the Klein-Gordon equations with an interaction term due to λ. By considering ϕ and ϕ^* as two independent fields we get the two conjugate momenta

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \dot{\phi}^*, \quad \pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \dot{\phi}.$$

Thus the Hamiltonian density becomes

$$\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} = \pi\pi^* + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi + \lambda(\phi^*\phi)^2.$$

From Eq. (3.34) we see that $\mathcal{L}$ is left invariant by a $U(1)$ transformation

$$\phi \mapsto e^{i\alpha}\phi \quad \Rightarrow \quad \delta\phi = i\alpha\phi, \quad \delta\phi^* = -i\alpha\phi^*.$$

By Noether's theorem this implies the conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\delta\phi^* = \alpha \operatorname{Im}\left(\phi^* \overleftrightarrow{\partial}^\mu \phi\right),$$

where we have used the notation

$$A \overleftrightarrow{\partial}^\mu B = A\left(\partial^\mu B\right) - \left(\partial^\mu A\right) B.$$

α can be scaled out of this current so that we can redefine j^0 to be

$$j^0 = \operatorname{Im}\left(\phi^* \overleftrightarrow{\partial}^0 \phi\right) = i\left(\dot{\phi}^*\phi - \dot{\phi}\phi^*\right),$$

which yields the charge

$$Q = i\int \mathrm{d}^3x \left(\dot{\phi}^*\phi - \dot{\phi}\phi^*\right).$$

Next we modify $\mathcal{H}$ as in Eq. (3.29) by defining

$$\begin{aligned}
\mathcal{H}_{\mu_3} &= \mathcal{H} + \mu_3 j^0 \\
&= \pi\pi^* + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2 + i\mu_3\left(\pi\phi - \pi^*\phi^*\right).
\end{aligned}$$

for a scalar μ_3, where we have inserted for the definitions of π and π^* to get the Hamiltonian density in terms of only the conjugate momenta and fields. To obtain the related Lagrange density we need to do another Legendre transformation, but since $\mathcal{H}$ has changed we can not use the old relationships between the time derivatives of the fields and the conjugate momenta. We use Hamilton's equations to obtain

$$\begin{aligned}
\dot{\phi} = \frac{\partial \mathcal{H}_{\mu_3}}{\partial \pi} = \pi^* + i\mu_3\phi \quad &\Leftrightarrow \quad \pi^* = \dot{\phi} - i\mu_3\phi, \\
&\Leftrightarrow \quad \pi \;= \dot{\phi}^* + i\mu_3\phi^*.
\end{aligned}$$

Inserting this for π and π^* we get that

$$\begin{aligned}\mathcal{L}_{\mu_3} &= \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{H}_{\mu_3} \\ &= \left(\partial_\mu + i\mu_3\delta_{\mu 0}\right)\phi^* \left(\partial^\mu - i\mu_3\delta^{\mu 0}\right)\phi - m^2\phi^*\phi - \lambda(\phi^*\phi)^2,\end{aligned} \tag{3.35}$$

which we immediately see is not Lorentz invariant since it treats derivatives in space and time differently. Thus modifying the theory with the term $\mu_3 j^0$ has broken the Lorentz invariance of $\mathcal{L}$.

Chapter 4

Goldstone's theorem

4.1 Classical version

Theorem 4.1. *Given a Lorentz invariant Lagrangian density $\mathcal{L}$ of the form*

$$\mathcal{L} = (\textit{derivatives of } \phi) - V(\phi)$$

*for some arbitrary collection of fields $\{\phi^i\}$ that is invariant under a compact Lie group G and whose ground state is invariant under a subgroup $H \subseteq G$, then when expanding about the ground state a number of massless modes called **Nambu-Goldstone bosons** appear. The number n_{NGB} of modes equals the dimension of the coset space G/H.*

Proof. First assume a Lagrangian density that is of the form in the theorem and denote the configuration of constant fields that minimizes the potential $\{\phi_0^i\}$ or simply ϕ_0. We identify this configuration of fields with the ground state of the system. In quantum theory where the fields are operators, the ground state configuration gives the vacuum expectation value of the fields [21], but in the quantum case the ground state will be determined by a different potential that includes all effects from loop corrections. Expanding about the ground state gives to second order in the fields

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^i(\phi - \phi_0)^j M_{ij},$$

where we have defined the mass matrix

$$M_{ij} = \left. \frac{\partial^2 V}{\partial \phi^i \, \partial \phi^j} \right|_{\phi_0}.$$

Since ϕ_0 is a minimum of the potential V, then M_{ij} is a positive semi-definite matrix meaning it only has positive or vanishing eigenvalues. Since the Lagrangian density is Lorentz invariant, we can interpret oscillations of the fields about the ground state configuration as particles whose masses are determined by mass matrix M. Specifically the eigenvalues of this matrix gives the square of the masses of the particles, whilst the particles are interpreted as oscillations of the fields in the corresponding eigenvector directions. Since ϕ_0 is a constant field configuration, the derivatives in $\mathcal{L}$ vanish and thus, since $\mathcal{L}$ is invariant under G this means V must be invariant under G as well. Given a representation U of this Lie group[1] then for an infinitesimal transformation $g \in G$

$$V(\phi_0) = V(U(g)\phi_0) = V(\phi_0) + \frac{1}{2}\delta\phi^i\delta\phi^j M_{ij}, \tag{4.1}$$

for

$$\delta\phi^i = U(g)_{ij}\phi_0^j - \phi_0^i = i\theta^a(T^a)_{ij}\phi_0^j, \tag{4.2}$$

where T^a are the generators of G such that $U(g) = e^{i\theta^a T^a}$. From Eq. (4.1) it follows that

$$M_{ij}\delta\phi^i\delta\phi^j = 0 \tag{4.3}$$

and it is this equation which we will see implies the massless eigenvalues and hence the Goldstone bosons. To evaluate this equation for all infinitesimal transformations $g \in G$ it suffices to consider only one generator at the time meaning to set $U(g) = e^{i\theta T^a}$. For each generator, either the corresponding $g \in H$ or $g \notin H$.

(a) If $g \in H$ then from Eq. (4.2) we see that

$$U(g)\phi_0 = \phi_0 \quad \Leftrightarrow \quad \delta\phi^i = i\theta(T^a)_{ij}\phi_0^j = 0 \quad \Leftrightarrow \quad T^a\phi_0 = 0.$$

In this case we call T^a an **unbroken generator**. Since $\delta\phi^i = 0$, Eq. (4.3) is trivially satisfied and thus does not imply a vanishing eigenvalue of the mass matrix.

(b) However if $g \notin H$, then $\delta\phi^i \neq 0$ and $T^a\phi_0 \neq 0$. We say that the symmetry associated with T^a is spontaneously broken by the ground state. T^a itself is called a **broken generator**. By looking at $\delta\phi^i$ as elements in a vector $\boldsymbol{\delta\phi}$

$$M_{ij}\delta\phi^i\delta\phi^j = 0 \quad \Leftrightarrow \quad \boldsymbol{\delta\phi}^{\mathrm{T}} M \boldsymbol{\delta\phi} = 0 \quad \Leftrightarrow \quad M\boldsymbol{\delta\phi} = 0 = 0\,\boldsymbol{\delta\phi}.$$

This equation says that $\boldsymbol{\delta\phi}$ is a non-zero eigenvector with zero eigenvalue of the mass matrix M. In the classical case $\boldsymbol{\delta\phi}$ is called a **flat direction**. Interpreting oscillations about the ground state as particles, then oscillations in a flat direction gives a massless particle called a **Nambu-Goldstone boson**.

From these two cases we see that the number of Nambu-Goldstone bosons n_{NGB} is given by the number of broken generators n_{BG}. Since the unbroken generators generate the subgroup H, their number is given by the dimension of the Lie algebra of H, thus

$$n_{\text{NGB}} = n_{\text{BG}} = \dim(\mathfrak{g}) - \dim(\mathfrak{h}),$$

or equivalently the dimensionality of the left coset space G/H.

■

[1] U must be unitary to preserve probabilities in quantum mechanics.

4.1.1 SSB in a Lorentz and $U(1)$ invariant field theory

To illustrate this theorem consider the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi^* + m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2, \quad m, \lambda > 0 \tag{4.4}$$

for a complex scalar field ϕ. Identifying $V(\phi)$ as the negative of the right part of $\mathcal{L}$ this can be written as

$$V(\phi) = \frac{\lambda}{4}\left[(\phi^*\phi - v)^2 - v^2\right],$$

where we have introduced the constant

$$v = \frac{2m^2}{\lambda}.$$

To find the ground state configuration of the field we want the constant field that minimizes $V(\phi)$. Since $\phi \in \mathbb{C}$ we can write it as $\phi(x) = \rho(x)e^{i\theta(x)}$ for two real fields $\rho(x)$ and $\theta(x)$. Then

$$V(\phi) = V(\rho, \theta) = \frac{\lambda}{4}\left[\left(\rho^2 - v\right)^2 - v^2\right] = V(\rho)$$

so V is a function of one real variable ρ. Thus we can find the points where V is stationary by requiring that

$$\frac{\mathrm{d}V}{\mathrm{d}\rho} = \lambda\left(\rho^2 - v\right)\rho = 0 \quad \wedge \quad \frac{\mathrm{d}^2V}{\mathrm{d}\rho^2} = \lambda\left(3\rho^2 - v\right) > 0.$$

Only $\rho = \sqrt{v}$ satisfies both equations thus we have a continuum of ground state configurations given by

$$\phi_0 = \sqrt{v}e^{i\theta}.$$

This is illustrated by the lower dashed circle in Figure 4.1.

Next we claim that Eq. (4.4) is invariant under the Lie group $U(1)$. To prove this recall from Example 2.4 that $U(N)$ is defined as the set of all $N \times N$ invertible matrices A that satisfies the equation $A^{-1} = A^\dagger$. Thus $U(1)$ is the set of all complex numbers $a \in \mathbb{C}\setminus\{0\}$ with the property $a^* = a^{-1}$. Parameterizing a as $a = re^{i\omega}$ this condition becomes $r = 1$ such that any element $a \in U(1)$ can be written $e^{i\omega}$. Since ϕ is a scalar field it furnishes the fundamental representation of $U(1)$ such that a group element acting on an element of the vector field $\mathbb{C}$ is defined by normal multiplication which is to say

$$U(a)\phi(x) = e^{i\omega}\phi(x).$$

Also note that this representation is unitary as required by conservation of probability in quantum mechanics. By inspection we see that transforming the field to $\tilde{\phi} = e^{i\omega}\phi$ leaves Eq. (4.4) invariant for any ω which proves our claim. Thus we have identified the symmetry group $G = U(1)$.

By expanding the exponential $e^{i\omega} = 1 + i\omega$ for small ω we see that the only generator of $U(1)$ is the number 1. Since $1\phi_0 = 1\sqrt{v}e^{i\theta} \neq 0$, this generator is broken and since the theory is Lorentz invariant we can apply Goldstone's theorem to conclude that there

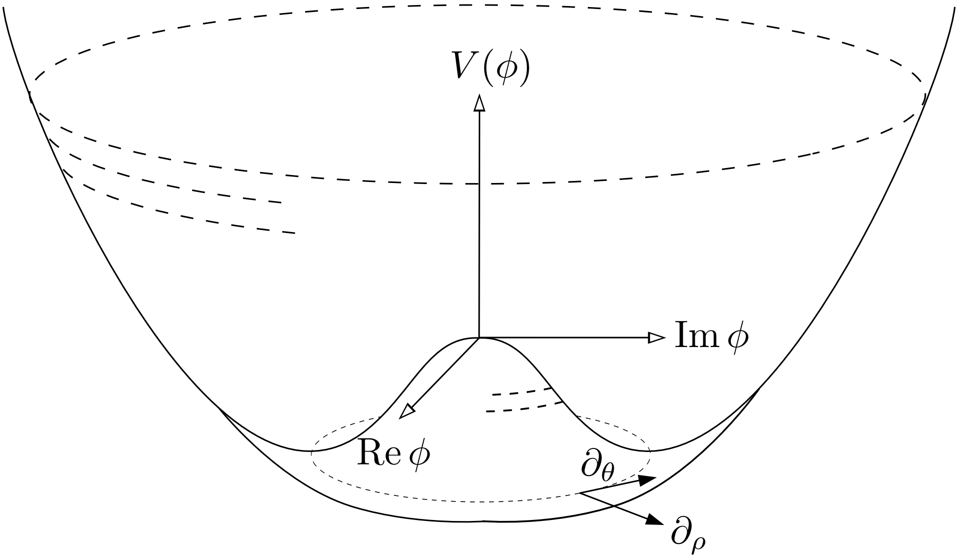

Figure 4.1: A geometric illustration of the potential $V(\phi) = \lambda/4\left[(\phi^*\phi - v)^2 - v^2\right]$ often called the "Mexican hat" potential due to its characteristic shape.

must be one Nambu-Goldstone boson. In other words: since the ground state is a point on the circle with radius $\sqrt{v}$ in the complex plane, and acting on ϕ_0 with $e^{i\omega}$ rotates the configuration about the origin, any element in $U(1)$ other than the identity leads to a different point, thus the invariant subgroup $H = I$ and

$$n_{\text{NGB}} = \dim(G/H) = \dim(G) = 1.$$

To see how the Nambu-Goldstone boson comes about explicitly consider the field χ defined by

$$\chi(x) = \sqrt{2}(\rho(x) - \sqrt{v}).$$

Since the ground state is $\rho = \sqrt{v}$, the field χ has vanishing vacuum expectation value in the quantum theory. Written in terms of $\chi(x)$ and the field $\theta(x)$ the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi + \frac{1}{2}(\chi + \sqrt{2v})^2\partial_\mu\theta\,\partial^\mu\theta - \frac{\lambda}{4}\left(\frac{\chi^4}{4} + \sqrt{2v}\chi^3 + 2v\chi^2 - v^2\right).$$

From the coefficient of the χ^2 term we can identify the mass of this field as

$$m_\chi^2 = \lambda v = 2m^2.$$

More importantly there is no θ^2 term which implies that this is a massless field. Thus $\theta(x)$ parameterizes the flat direction of the potential as can be easily seen in Figure 4.1. This is the Nambu-Goldstone boson!

4.1.2 SSB in a Lorentz and $SO(N)$ invariant field theory

The preceding example was of spontaneous symmetry breaking of the abelian symmetry group $U(1)$. An example of symmetry breaking of a non-abelian group is given by the Lagrange density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}m^2 \phi^i \phi^i - \frac{\lambda}{4!}(\phi^i\phi^i)^2, \quad m, \lambda > 0, \tag{4.5}$$

where repeated indices are implicitly summed from 1 to N for the N real scalar fields ϕ^i. This Lagrangian density is invariant under the symmetry group $SO(N)$; the group of rotations in N dimensions. Recall from Definition 2.3 that $SO(N)$ is given by the set

$$SO(N) = \left\{R \in GL(\mathbb{R}, N) \mid R^{-1} = R^{\mathrm{T}},\ \det R = 1\right\},$$

where $GL(N, \mathbb{R})$ is the general linear group of invertible $N \times N$ matrices with real entries. A particular set of fields $\{\phi^i\}$ can be considered a vector in a real N-dimensional vector space, often called the field space, since the fields take on only real values. We can thus choose the fundamental representation of $SO(N)$ by letting group elements act on vectors in field space through matrix multiplication such that the components of a transformed configuration of fields $\tilde{\phi}^i$ are given by

$$\tilde{\phi}^i = (U(R)\phi)^i = R_{ij}\phi^j,$$

for an element $R \in SO(N)$ and an initial set of fields denoted ϕ. To prove that the Lagrangian density is invariant under such a transformation we calculate

$$\begin{aligned}\tilde{\phi}^i\tilde{\phi}^i &= R_{ij}\phi^j R_{ik}\phi^k = \left(R^{\mathrm{T}}\right)_{ji} R_{ik}\phi^j\phi^k = \left(R^{\mathrm{T}}R\right)_{jk}\phi^j\phi^k \\ &= \delta_{jk}\phi^j\phi^k = \phi^i\phi^i,\end{aligned}$$

where in the last expression we renamed the summation index i. This proves that the second and third term in Eq. (4.5) are invariant. Since R_{ij} is a constant matrix, the derivatives in the first term have no effect on the calculation which proves that this term is invariant as well, thus the Lagrangian density is invariant.

Now that we know $\mathcal{L}$ is invariant under $SO(N)$ we would like to know how many generators there are. To this end consider the Lie algebra of $SO(N)$ discussed in Example 2.8. There we proved that

$$X \in \mathfrak{so}(N) \quad \Leftrightarrow \quad X = -X^{\mathrm{T}}.$$

From this condition we see that the diagonal elements vanish and that the lower diagonal elements are given as the negatives of the upper diagonal elements, thus only the upper diagonal elements are independent. We can thus form a basis for this Lie algebra consisting of matrices with 1 in an upper diagonal entry, -1 in the symmetric lower diagonal entry and 0 in all other entries. The dimension of this vector space is thus equal to the number of upper diagonal elements in an $N \times N$ matrix. Counting from the top row and down there are $N - 1 + (N - 2) + \ldots + 1 = \frac{1}{2}N(N-1)$ upper diagonal elements in an $N \times N$ matrix, thus

$$\dim \mathfrak{so}(N) = \frac{1}{2}N(N-1). \tag{4.6}$$

To apply Goldstone's theorem we first need to choose a ground state for this theory. From Eq. (4.5) we get the potential $V(\phi)$ which similarly to Example 4.1.1 can be written as

$$V(\phi) = \frac{\lambda}{4!}\left[\left(\phi^i\phi^i - v\right)^2 - v^2\right], \tag{4.7}$$

for the constant $v = 4!\,m^2/(4\lambda)$. Requiring that

$$\frac{\partial V}{\partial \phi^j} = \frac{4\lambda}{4!}\left(\phi^i\phi^i - v\right)\phi^j = 0 \quad \forall j,$$

for a stationary point in field space we get two distinct possibilities: either $\phi^j = 0\ \forall j$ or $\phi^i\phi^i = v$. The second derivative of the potential is

$$\frac{\partial^2 V}{\partial \phi^j\,\partial \phi^k} = \frac{4\lambda}{4!}\left[(\phi^i\phi^i - v)\delta^{jk} + 2\phi^j\phi^k\right],$$

and evaluating it at $\phi^j = 0$ we get the matrix

$$\left.\frac{\partial^2 V}{\partial \phi^j\,\partial \phi^k}\right|_0 = -\frac{4\lambda}{4!}v\delta^{jk},$$

which is a diagonal matrix. Thus we see directly that the non-zero eigenvalues are all negative, hence this is a local maximum. Evaluating for $\phi^i\phi^i = v$, we get

$$\left.\frac{\partial^2 V}{\partial \phi^j\,\partial \phi^k}\right|_{\phi^i\phi^i = v} = 2\frac{4\lambda}{4!}\phi^j\phi^k.$$

This matrix is not diagonal so to obtain its eigenvalues we need to solve the equation

$$\det\left(\left.\frac{\partial^2 V}{\partial \phi^j\,\partial \phi^k}\right|_{\phi^i\phi^i = v} - \Lambda\delta^{jk}\right) = \left(2\frac{4\lambda}{4!}\right)^N \det\left(\phi^j\phi^k - \xi\delta^{jk}\right) = 0,$$

where we have used the notation that the determinant of an expression with two free indices means the determinant of the $N \times N$ matrix that results when evaluating the expression for different indices. We have also defined the variable $\xi = 4!/(8\lambda)\,\Lambda$. It is proved in Appendix A.3 that

$$\det\left(\phi^j\phi^k - \xi\delta^{jk}\right) = (-\xi)^{N-1}(\phi^i\phi^i - \xi), \tag{4.8}$$

thus the eigenvalue equation becomes

$$(-\xi)^{N-1}(\phi^i\phi^i - \xi) = 0.$$

This gives two different eigenvalues; $\Lambda = 0$ or

$$\Lambda = 2\frac{4\lambda}{4!}\xi = 2\frac{4\lambda}{4!}\phi^i\phi^i = 2\frac{4\lambda}{4!}v = 2m^2 > 0,$$

which shows that if $\phi^i\phi^i = v$ then this configuration is a minimum of $V(\phi)$. Since $\mathcal{L}$ is rotationally invariant we can without loss of generality pick the ground state configuration

$\phi_0^i = \sqrt{v}\delta^{i1}$. We now want to find the subgroup H of $SO(N)$ under which the ground state is invariant. By transforming the ground state with an arbitrary element R of $SO(N)$ the condition for invariance becomes

$$\tilde{\phi_0}^i = R_{ij}\phi_0^j = R_{ij}\sqrt{v}\delta^{j1} = \sqrt{v}R_{i1} = \phi_0^i \quad \Leftrightarrow \quad R_{i1} = \delta^{i1},$$

thus R is of the form

$$R = \begin{bmatrix} 1 & \omega^{\mathrm{T}} \\ 0 & \\ \vdots & A \\ 0 & \end{bmatrix}$$

for an $(N-1)\times(N-1)$ matrix A and vector ω with $N-1$ components. Since $R \in SO(N)$ we must require that $R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$ which implies that $\omega = 0$ and $A^{\mathrm{T}}A = AA^{\mathrm{T}} = I$. By requiring that $\det R = 1$ then $\det A = 1$ such that $A \in SO(N-1)$. From this we conclude that ϕ_0 is invariant under the group $SO(N-1)$, where the representation of $SO(N-1)$ on the N-dimensional field space is given by

$$A \in SO(N-1) \mapsto \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{bmatrix}.$$

Now since $\mathcal{L}$ is Lorentz invariant we can invoke Goldstone's theorem to conclude that since $G = SO(N)$ and $H = SO(N-1)$ there must be

$$n_{\mathrm{NGB}} = \dim \mathfrak{g} - \dim \mathfrak{h} = \frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1$$

Nambu-Goldstone bosons. Here we used Eq. (4.6) to evaluate the dimensions of the Lie algebras. To see directly where these bosons comes from, we shift the fields such that they will have a vanishing expectation value in the quantum theory. For the ϕ_0 that was picked, we do this by defining the field

$$\chi = \phi^1 - \sqrt{v}.$$

Inserting this into Eq. (4.5) we get

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\sum_{i=2}^{N}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{\lambda v^2}{4!} \\ & - \frac{\lambda}{4!}\left[\chi^4 + 4\sqrt{v}\chi^3 + \left(4v + 2\sum_{i=2}^{N}\phi^i\phi^i\right)\chi^2 + 4\sqrt{v}\chi\sum_{i=2}^{N}\phi^i\phi^i + \left(\sum_{i=2}^{N}\phi^i\phi^i\right)^2\right]\end{aligned}$$

From this equation we easily see that $\mathcal{L}$ is still invariant under rotations in $SO(N-1)$. Furthermore we see from the coefficient of the χ^2 term that

$$m_\chi^2 = 2\frac{4\lambda}{4!}v = 2m^2,$$

which incidentally is the non-zero eigenvalue of the second derivative matrix of the potential. There are on the other hand no square terms for the other $N-1$ fields and these are thus massless and hence the Nambu-Goldstone bosons of this theory.

4.1.3 Generators and currents in a $SO(4)$ invariant field theory

We now specialize to the case where $N = 4$ and consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - \frac{m^2}{2}(\phi^i \phi^i) - \frac{\lambda}{4}(\phi^i \phi^i)^2,$$

where repeated indices are implicitly summed from 1 to 4 and ϕ^i are real scalar fields. This theory is invariant under $SO(4)$ and we would like to find the generators of these transformations. Following the discussion in Example 4.1.2 the group elements act through the fundamental representation of $SO(4)$ which means we can obtain their generators by considering the Lie algebra $\mathfrak{so}(4)$. From the discussion of $\mathfrak{so}(N)$ in Example 4.1.2 we conclude that a basis for this Lie algebra can be constructed by inserting a 1 and -1 such that $B_{ij} = -B_{ji}$ into a 4×4 matrix B with zeroes in the remaining entries. If B is a matrix in this basis, then its corresponding generator T is given by $T = -iB$. From this we get the generators

$$L_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_y = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_z = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_x = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad K_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \quad K_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}.$$

Thus any transformation in $SO(4)$ continuously connected to the identity can be written as $e^{i(\mathbf{n}\cdot\mathbf{L}+\mathbf{k}\cdot\mathbf{K})}$ for the vectors of operators $\mathbf{L} = (L_x, L_y, L_z)$ and $\mathbf{K} = (K_x, K_y, K_z)$, and the vectors of scalars $\mathbf{n}$ and $\mathbf{k}$. Since $\mathcal{L}$ is invariant under any transformation in $SO(4)$, Noether's theorem implies the existence of a conserved current for each independent transformation given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\delta\phi^i = \left(\partial^\mu \phi^i\right)\delta\phi^i,$$

where $\delta\phi^i$ is the change in the fields under such a transformation. Each generator yields a different current derived by considering infinitesimal transformations $e^{i\epsilon L_x}$, $e^{i\epsilon L_y}$, etc., which yields infinitesimal changes in the fields by

$$\delta\phi^i = i\epsilon L_x.$$

Denoting the current obtained from the generator L_x by $j^\mu_{L_x}$ and likewise for the other generators we get

$$j^\mu_{L_x} = \phi^3 \overleftrightarrow{\partial}^\mu \phi^1, \quad j^\mu_{L_y} = \phi^1 \overleftrightarrow{\partial}^\mu \phi^3, \quad j^\mu_{L_z} = \phi^2 \overleftrightarrow{\partial}^\mu \phi^1,$$
$$j^\mu_{K_x} = \phi^4 \overleftrightarrow{\partial}^\mu \phi^1, \quad j^\mu_{K_y} = \phi^4 \overleftrightarrow{\partial}^\mu \phi^2, \quad j^\mu_{K_z} = \phi^4 \overleftrightarrow{\partial}^\mu \phi^3,$$

where we have used the $\overleftrightarrow{\partial}^\mu$ notation introduced in Example 3.5.

This theory can instead of four real fields be expressed in terms of two complex fields. If we define the complex scalar fields α and β as

$$\alpha = \frac{1}{\sqrt{2}}\left(\phi^1 + i\phi^2\right), \quad \beta = \frac{1}{\sqrt{2}}\left(\phi^3 + i\phi^4\right),$$

the Lagrangian density takes the form

$$\mathcal{L} = \left(\partial_\mu \alpha\right)\left(\partial^\mu \alpha^*\right) + (\partial_\mu \beta)(\partial^\mu \beta^*) - m^2\left(\alpha\alpha^* + \beta\beta^*\right) - \lambda\left(\alpha\alpha^* + \beta\beta^*\right)^2 .$$

Expressed in terms of these new complex fields, the conserved currents become

$$\begin{aligned}
j^\mu_{L_x} &= -\operatorname{Im}\left(\alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta + \alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta^*\right), & j^\mu_{K_x} &= -\operatorname{Im}\left(\alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta + \alpha^*\overset{\leftrightarrow}{\partial}{}^\mu\beta\right), \\
j^\mu_{L_y} &= \operatorname{Re}\left(\alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta + \alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta^*\right), & j^\mu_{K_y} &= \operatorname{Re}\left(\alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta + \beta\overset{\leftrightarrow}{\partial}{}^\mu\alpha^*\right), \\
j^\mu_{L_z} &= i\alpha^*\overset{\leftrightarrow}{\partial}{}^\mu\alpha, & j^\mu_{K_z} &= i\beta^*\overset{\leftrightarrow}{\partial}{}^\mu\beta.
\end{aligned}$$

Since the generators form a basis of a finite-dimensional vector space, any set of linear combinations of the generators can form an equivalent basis for this vector space as long as the elements in this set are linearly independent and the set has the same number of elements as there are generators. Thus we can define an equivalent set of generators by

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{L} + \mathbf{K}\right), \quad \mathbf{B} = \frac{1}{2}\left(\mathbf{L} - \mathbf{K}\right),$$

and use these in the same manner as above to find an equivalent set of conserved currents. To check that the set of currents really is equivalent we do an explicit calculation, denoting them $j^\mu_{A_1}$, $j^\mu_{A_2}$, etc.:

$$\begin{aligned}
j^\mu_{A_1} &= \frac{1}{2}\left(\phi^4\overset{\leftrightarrow}{\partial}{}^\mu\phi^1 + \phi^3\overset{\leftrightarrow}{\partial}{}^\mu\phi^2\right) = -2\operatorname{Im}\left(\alpha\partial^\mu\beta\right), \\
j^\mu_{A_2} &= \frac{1}{2}\left(\phi^4\overset{\leftrightarrow}{\partial}{}^\mu\phi^2 + \phi^1\overset{\leftrightarrow}{\partial}{}^\mu\phi^3\right) = \operatorname{Re}\left(\alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta\right), \\
j^\mu_{A_3} &= \frac{1}{2}\left(\phi^2\overset{\leftrightarrow}{\partial}{}^\mu\phi^1 + \phi^4\overset{\leftrightarrow}{\partial}{}^\mu\phi^3\right) = -\operatorname{Im}\left(\alpha^*\partial^\mu\alpha + \beta^*\partial^\mu\beta\right), \\
j^\mu_{B_1} &= \frac{1}{2}\left(\phi^1\overset{\leftrightarrow}{\partial}{}^\mu\phi^4 + \phi^3\overset{\leftrightarrow}{\partial}{}^\mu\phi^2\right) = -\operatorname{Im}\left(\alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta^*\right), \\
j^\mu_{B_2} &= \frac{1}{2}\left(\phi^2\overset{\leftrightarrow}{\partial}{}^\mu\phi^4 + \phi^1\overset{\leftrightarrow}{\partial}{}^\mu\phi^3\right) = \operatorname{Re}\left(\alpha\overset{\leftrightarrow}{\partial}{}^\mu\beta^*\right), \\
j^\mu_{B_3} &= \frac{1}{2}\left(\phi^2\overset{\leftrightarrow}{\partial}{}^\mu\phi^1 + \phi^3\overset{\leftrightarrow}{\partial}{}^\mu\phi^4\right) = \frac{i}{2}\left(\alpha^*\overset{\leftrightarrow}{\partial}{}^\mu\alpha + \beta\overset{\leftrightarrow}{\partial}{}^\mu\beta^*\right).
\end{aligned}$$

As expected from the definition of $\mathbf{B}$ we get that

$$j^\mu_{B_3} = \frac{1}{2}\left(j^\mu_{L_z} - j^\mu_{K_z}\right),$$

and similarly for the other currents, which proves that the new set of currents is equivalent to the original.

For future reference we work out the commutation relations for the A_i and B_i. The result of this exercise can be summarized as

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0. \tag{4.9}$$

Note that the set of generators A_i have the same commutation relations as the generators of $SU(2)$, and so does the set of B_i. This will be of importance when we establish the homomorphism between $SO(4)$ and $SU(2) \times SU(2)$ in Section 4.4.2.

4.2 Quantum theory

One might be concerned that the masslessness of the Nambu-Goldstone bosons predicted by the classical theory be lost due to loop corrections in the quantum theory. This issue can be addressed by considering the quantum action[2] denoted $\Gamma(\phi)$ which encapsulates all loop corrections [13]. By expanding in terms of derivatives, $\Gamma[\phi]$ can be written as

$$\Gamma[\phi] = \int \mathrm{d}^d x \left[-\mathcal{U}(\phi) + \frac{1}{2}\mathcal{Z}(\phi)\partial^\mu\phi\partial_\mu\phi + \dots\right],$$

where $\mathcal{U}$ is called the effective potential. We now assume that the (normal) action $S[\phi]$ and the integration measure in the path integral in the quantum theory is invariant under a Lie group. Considering the generators of the Lie group separately, an infinitesimal transformation due to one of the generators acting on the configuration of fields can be written

$$\tilde{\phi}^i(x) = \phi^i(x) + i\epsilon T^a_{ij}\phi^j$$

for the generator T^a. Then by using functional derivatives

$$\delta\Gamma = \int \mathrm{d}^d x \, \frac{\delta\Gamma}{\delta\phi^i}\delta\phi^i = i\epsilon \int \mathrm{d}^d x \, \frac{\delta\Gamma}{\delta\phi^i}T^a_{ij}\phi^j = 0. \tag{4.10}$$

Specifying to constant fields the derivatives in the derivative expansion of Γ vanish such that

$$\int \mathrm{d}^d x \, \frac{\delta\Gamma}{\delta\phi^i} = \int \mathrm{d}^d x \int \mathrm{d}^d y \left(-\frac{\partial\mathcal{U}}{\partial\phi^k}\delta^{ik}\delta(x-y)\right) = -\mathcal{V}\frac{\partial\mathcal{U}}{\partial\phi^i}.$$

Inserting this into Eq. (4.10) and dividing out constants we get

$$\frac{\partial\mathcal{U}}{\partial\phi^i}T^a_{ik}\phi^k = 0.$$

Taking the partial derivative of this equation with respect to ϕ^j we get

$$\frac{\partial^2\mathcal{U}}{\partial\phi^i\,\partial\phi^j}T^a_{ik}\phi^k + \frac{\partial\mathcal{U}}{\partial\phi^i}T^a_{ik}\delta^{kj} = 0. \tag{4.11}$$

[2]The quantum action is also called the *effective action* or the *quantum effective action*.

The minimum of $\mathcal{U}$ gives the vacuum expectation value of the fields [22]. Evaluating Eq. (4.11) in this configuration the derivative in the second term vanishes such that the equation simplifies to

$$\left.\frac{\partial^2 \mathcal{U}}{\partial \phi^i\, \partial \phi^j} T^a_{ik} \phi^k \right|_{\phi_0} = \mathcal{M}_{ij} T^a_{ik} \phi^k \big|_{\phi_0} = 0,$$

where we have denoted the second derivative of the potential $\mathcal{M}_{ij}$. This is can be considered as a matrix equation where the matrix $\mathcal{M}$ is multiplied by a vector $T^a \phi_0$. If $T^a \phi_0 \neq 0$, i.e. T^a is a broken generator, then $T^a \phi_0$ is an eigenvector of $\mathcal{M}$ with eigenvalue 0. Thus every broken generator implies a separate zero eigenvalue of $\mathcal{M}$. It can be shown [19] that if the theory is Lorentz invariant then a zero eigenvalue of $\mathcal{M}$ implies that one of the fields are massless. From this point the argument parallels that of the classical case resulting in the total number of

$$n_{\text{NGB}} = \dim \mathfrak{g} - \dim \mathfrak{h}$$

massless fields where $\mathfrak{g}$ is the Lie algebra of the original symmetry group and $\mathfrak{h}$ is the Lie algebra of the subgroup under which the ground state is invariant.

4.3 Revisiting Section 3.2 and counting NG-bosons

In 1975 Nielsen and Chadha showed that it is useful to classify NG-bosons according to the behaviour of their dispersion relations in the long-wavelength limit [23]. An NG-boson with an odd dependence on $|\mathbf{k}|$ is classified as a type-I NG-boson, while if the dependence is even, it is called a type-II NG-boson. If we denote the number of type-I NG-bosons n_{I} and the number of type-II NG-bosons n_{II}, Nielsen and Chadha proved that

$$n_{\text{I}} + 2n_{\text{II}} \geq n_{\text{BG}}, \tag{4.12}$$

where n_{BG} is the number of broken generators of the symmetry of the theory, holds for Lorentz invariant as well as Lorentz noninvariant field theories.

In the field theory in Section 3.2 we have 3 generators of the $E(2)$ symmetry given by

$$\hat{Q}_1 = i\sqrt{\mathcal{V}}\left(\hat{a}_{\mathbf{0}} - \hat{a}^\dagger_{\mathbf{0}}\right), \quad \hat{Q}_2 = \sqrt{\mathcal{V}}\left(\hat{a}_{\mathbf{0}} + \hat{a}^\dagger_{\mathbf{0}}\right), \quad \hat{Q}_3 = \sum_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}},$$

with commutation relations

$$[\hat{Q}_1, \hat{Q}_2] = 2i\mathcal{V}\hat{I}, \quad [\hat{Q}_3, \hat{Q}_1] = -i\hat{Q}_2, \quad [\hat{Q}_2, \hat{Q}_3] = -i\hat{Q}_1.$$

By the definition of the generators

$$\begin{aligned}
\hat{Q}_1 |0\rangle &= i\sqrt{\mathcal{V}}\hat{a}^\dagger_{\mathbf{0}} |0\rangle \neq 0, \\
\hat{Q}_2 |0\rangle &= \sqrt{\mathcal{V}}\hat{a}^\dagger_{\mathbf{0}} |0\rangle \neq 0, \\
\hat{Q}_3 |0\rangle &= \sum_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} |0\rangle = 0.
\end{aligned}$$

Thus $n_{\mathrm{BG}} = 2$. From Eq. (3.33):

$$E_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m},$$

we see that in this field theory there is a single type-II NG-boson such that $n_{\mathrm{I}} = 0$ and $n_{\mathrm{II}} = 1$. Inserting this into Eq. (4.12) we get that $2 \geq n_{\mathrm{BG}}$ which in this case is an equality since $n_{\mathrm{BG}} = 2$.

Recent progress has been made in the understanding of Nambu-Goldstone bosons in Lorentz noninvariant field theories. In 2011 Watanabe and Brauner conjectured the formula [24]

$$n_{\mathrm{BG}} - n_{\mathrm{NGB}} = \frac{1}{2} \operatorname{rank} \rho, \tag{4.13}$$

where ρ is the matrix given by

$$\rho_{ij} = \lim_{\mathcal{V} \to \infty} \frac{-i}{\mathcal{V}} \langle 0| [\hat{Q}_i, \hat{Q}_j] |0\rangle ,$$

and $\hat{Q}_i$ are the generators of the symmetry, $\mathcal{V}$ is the spatial volume of the system and n_{NGB} is the total number of Nambu-Goldstone bosons. This conjecture was proved by Watanabe in 2012 [25]. Here we make no attempt at independently proving this formula but merely content ourselves with showing its validity for the free non-relativistic field theory discussed in Section 3.2

Using the commutators and definitions of the generators of the $E(2)$ symmetry given above we can calculate the elements of ρ as

$$\rho_{12} = \lim_{\mathcal{V} \to \infty} \frac{-i}{\mathcal{V}} \langle 0| [\hat{Q}_1, \hat{Q}_2] |0\rangle = \lim_{\mathcal{V} \to \infty} \frac{-i}{\mathcal{V}} \langle 0| 2i\mathcal{V} |0\rangle = 2,$$

etc., so that we get the matrix

$$\rho = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Acting on a vector $\mathbf{v} \in \mathbb{R}^3$ we see that

$$\rho \mathbf{v} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2 \begin{bmatrix} v_2 \\ -v_1 \\ 0 \end{bmatrix},$$

thus the dimension of the range of ρ, which is its rank, is 2. According to the Watanabe formula in Eq. (4.13) then

$$n_{\mathrm{NGB}} = n_{\mathrm{BG}} - \frac{1}{2} \operatorname{rank} \rho = 2 - \frac{1}{2} 2 = 1,$$

which agrees with our previous results since we have shown that the theory has a single type-II Nambu-Goldstone boson.

4.4 Breaking an $SU(2) \times SU(2)$ invariant field theory

In this section we want to investigate a theory that we posit is symmetric under the group $SU(2) \times SU(2)$. First we discuss the construction of $SU(2) \times SU(2)$ as a direct product of two $SU(2)$ groups and justify our claim by establishing a homomorphism between $SU(2) \times SU(2)$ and $SO(4)$. We then attempt to construct representations of this group on $\mathbb{C}^2$ to try to prove our postulate.

4.4.1 Direct product of groups

Definition 4.1. The **direct product of two groups** G and G' is defined as the set

$$G \times G' = \{(g, g') \mid g \in G\ g' \in G'\},$$

which is the Cartesian product of the sets G and G', with binary operation defined as

$$(a_1, b_2)(a_2, b_2) = (a_1 a_2, b_1 b_2).$$

Proposition 4.2. *The direct product of two groups is also a group.*

Proof. Associativity follows from the associativity of the groups G and G' because if $a_i \in G$ and $b_i \in G'$ then

$$\begin{aligned}[(a_1, b_1)(a_2, b_2)]\,(a_3, b_3) &= (a_1 a_2, b_1 b_2)(a_3, b_3) = (a_1 a_2 a_3, b_1 b_2 b_3) \\ &= (a_1, b_1)(a_2 a_3, b_2 b_3) = (a_1, b_1)\,[(a_2, b_2)(a_3, b_3)]\,.\end{aligned}$$

The identity element of the group is given by (e, e') where e is the identity element in G and e' is the identity element in G' because

$$(e, e')(a_1, b_1) = (ea_1, e'b_1) = (a_1, b_1) = (a_1 e, b_1 e') = (a_1, b_1)(e, e').$$

Given any element (a, b) in $G \times G'$, the inverse is given by (a^{-1}, b^{-1}) where a^{-1} is the inverse of a in G, and b^{-1} is the inverse of b in G' because

$$(a, b)(a^{-1}, b^{-1}) = (aa^{-1}, bb^{-1}) = (e, e') = (a^{-1}a, b^{-1}b) = (a^{-1}, b^{-1})(a, b).$$

Thus $G \times G'$ has all the properties of a group. ■

4.4.2 Relationship between $SU(2) \times SU(2)$ and $SO(4)$

Proposition 4.3. *The groups* $SU(2) \times SU(2)$ *and* $SO(4)$ *are homomorphic.*

Proof. The group $SU(2) \times SU(2)$ is compact and thus its elements can be written as $(e^{iv^i \sigma_i}, e^{w^i \sigma_i})$ where σ_i are the Pauli matrices which (when multiplied by i) form a basis of the Lie algebra of $SU(2)$ and satisfy the commutation relations

$$[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k.$$

Now let A_i and B_i be the generators of $SO(4)$ discussed in Section 4.1.3. In that section we defined them as linear combinations $\mathbf{A} = \frac{1}{2}(\mathbf{L}+\mathbf{K})$ and $\mathbf{B} = \frac{1}{2}(\mathbf{L}-\mathbf{K})$ of generators L_i and K_i obtained by inserting i and $-i$ symmetrically about the diagonal in 4×4 matrices. Written out, these matrices are given by

$$A_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \qquad B_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \qquad B_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},$$

$$A_3 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \qquad B_3 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}.$$

Also recall from Eq. (4.9) that these matrices satisfy the commutation relations

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0.$$

Then for any v^i and w^i, the matrix $e^{iv^i A_i}e^{iw^i B_i}$ is in $SO(4)$ since the generators A_i and B_i commute. Now consider the map

$$\Pi\colon \quad SU(2) \times SU(2) \to SO(4), \quad (e^{iv^i\sigma_i}, e^{iw^i\sigma_i}) \mapsto e^{iv^i A_i}e^{iw^i B_i}.$$

Let $(e^{iv^i\sigma_i}, e^{iw^i\sigma_i})$ and $(e^{iv'^i\sigma_i}, e^{iw'^i\sigma_i})$ be two arbitrary elements in $SU(2) \times SU(2)$. By using the Baker-Campbell-Hausdorff formula we get that

$$e^{iv^i\sigma_i}e^{iv'^i\sigma_i} = \exp i\left(v^i\sigma_i + v'^i\sigma_i + \frac{1}{2i}v^i v'^j[\sigma_i, \sigma_j] + \dots\right).$$

By grouping together all the terms with σ_i we get a separate series for each σ_i. We denote these series as

$$\Sigma(v, v')^i = v^i + v'^i + \frac{1}{2}v^j v'^k \epsilon_{jki} + \dots .$$

Then we can write

$$e^{iv^i\sigma_i}e^{iv'^i\sigma_i} = e^{i\Sigma(v,v')^i\sigma_i}.$$

Defining the series $\Sigma(w, w')^i$ analogously for the element $e^{iw^i\sigma_i}e^{iw'^i\sigma_i}$ we get that

$$\begin{aligned}
\Pi\left((e^{iv^i\sigma_i}, e^{iw^i\sigma_i})(e^{iv'^i\sigma_i}, e^{iw'^i\sigma_i})\right) &= \Pi\left(e^{iv^i\sigma_i}e^{iv'^i\sigma_i}, e^{iw^i\sigma_i}e^{iw'^i\sigma_i}\right) \\
&= \Pi\left(e^{i\Sigma(v,v')^i\sigma_i}, e^{i\Sigma(w,w')^i\sigma_i}\right) \\
&= e^{i\Sigma(v,v')^iA_i}e^{i\Sigma(w,w')^iB_i} \\
&= e^{iv^iA_i}e^{iv'^iA_i}e^{iw^iB_i}e^{iw'^iB_i} \\
&= e^{iv^iA_i}e^{iw^iB_i}e^{iv'^iA_i}e^{iw'^iB_i} \\
&= \Pi\left(e^{iv^i\sigma_i}, e^{iw^i\sigma_i}\right)\Pi\left(e^{iv'^i\sigma_i}, e^{iw'^i\sigma_i}\right),
\end{aligned}$$

where in line 4 we used that A_i and B_i separately satisfy the same commutation relations as σ_i, and in line 5 we used that $[A_i, B_j] = 0$. Thus Π is a homomorphism and the groups $SU(2) \times SU(2)$ and $SO(4)$ are homomorphic. ■

4.4.3 An $SU(2) \times SU(2)$ invariant field theory

By promoting the fields ϕ and ϕ^* in Eq. (3.34) to complex vectors with two components we get the Lagrangian density

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2, \tag{4.14}$$

for $\Phi \in \mathbb{C}^2$. Since this is a two component complex vector it can be written in terms of four real field by

$$\Phi = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4.\end{bmatrix}.$$

If we rewrite the Lagrangian density in terms of these four real fields, we end up with the Lagrangian density in Section 4.1.3. As we proved in that section, $\mathcal{L}$ is invariant under $SO(4)$. Since we know that $SO(4)$ is homomorphic to $SU(2) \times SU(2)$ we might suspect that this theory is invariant under $SU(2) \times SU(2)$ as well.

This is however not a very convincing argument because any group is homomorphic to the trivial group. The trivial group is given by

$$G = \{e\},$$

where e is the identity element of the group. Now let $\mathcal{L}'$ be an arbitrary theory that depends on a single vector Φ' in some vector space V. Then we can construct a representation of G by

$$\mu\colon \quad e \mapsto I$$

where I is the identity element in the group $GL(V)$. This is a faithful representation and we see that $\mu(e)\Phi' = I\Phi' = \Phi'$. Thus $\mathcal{L}'$ is invariant under G. Also, it is easy to show that any group G' is homomorphic to G by the homomorphism

$$\Pi\colon G' \to G, \quad g \mapsto e.$$

Thus the fact that a group is homomorphic to a group that the theory is invariant under, does not necessarily imply that the theory is invariant under the original group.

A more convincing argument is that the groups $SU(2) \times SU(2)$ and $SO(4)$ are locally identical close to the identity element, because they have isomorphic Lie algebras. In fact $SU(2) \times SU(2)$ is the double cover of $SO(4)$, thus we might suspect it to be possible to find some representation of $SU(2) \times SU(2)$ under which $\mathcal{L}$ is invariant.

Our strategy is borrowed from the theory of QCD where it can be shown [19] that the Lagrangian density

$$\mathcal{L}_{\text{QCD}} = -\bar{u}\gamma^\mu D_\mu u - \bar{d}\gamma^\mu D_\mu d - \ldots,$$

is invariant under the transformation

$$\begin{bmatrix} u \\ d \end{bmatrix} \mapsto \exp i\left(v^i\sigma_i + \gamma_5 w^i \sigma_i\right) \begin{bmatrix} u \\ d \end{bmatrix},$$

where γ^μ are Dirac's gamma matrices satisfying the Clifford algebra given by the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$

and γ_5 is defined as

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

By separating the vector $[u, d]^{\mathrm{T}}$ into a left-handed and right handed part, this invariance means that $\mathcal{L}_{\text{QCD}}$ is invariant under the group $SU(2)_L \times SU(2)_R$, where elements of $SU(2)_L$ act only on the subspace of left-handed multiplets and vice versa for $SU(2)_R$.

We thus try to write Φ as

$$\Phi = \Phi_L + \Phi_R, \tag{4.15}$$

where Φ_L and Φ_R should be projections of Φ into some subspaces where separate $SU(2)$ representations should act. If we can make $\Phi^\dagger\Phi$ invariant under some representation

$$\Pi\colon \quad SU(2) \times SU(2) \to GL(\mathbb{C}^2), \quad (A, B) \mapsto \Pi_{(A,B)},$$

of $SU(2) \times SU(2)$, then $\mathcal{L}$ will also be invariant under this representation since the derivatives in the Lagrangian density commute with representations.

Recall that a representation has to be a linear map. Since any linear map on $\mathbb{C}^2$ can be expressed as a matrix we can, if we follow the strategy of Eq. (4.15), without loss of generality write the projections as $\Phi_L = P_L\Phi$ and $\Phi_R = P_R\Phi$ for two projection matrices P_L and P_R. Since we want these matrices to be projection operators, we require that they satisfy the properties $P_L + P_R = I$ and $P_{L/R}^2 = P_{L/R}$. With these projection operators at hand we can define the map

$$\Pi_{(A,B)}\colon \quad \Phi \mapsto \Phi' = A\Phi_L + B\Phi_R = (AP_L + BP_R)\Phi.$$

Since this is a linear operator on $\mathbb{C}^2$ we can write $\Pi(A, B) = \Pi_{(A,B)} = AP_L + BP_R$. For this to be a representation, then Π has to satisfy the homomorphism property. Thus we must ensure that

$$\Pi(A, B)\Pi(C, D) = \Pi\left((A, B)(C, D)\right) = \Pi(AC, BD).$$

First we calculate that

$$\Pi(AC, BD) = ACP_L + BDP_R.$$

Then we can show that we can write

$$\begin{aligned}\Pi(A,B)\Pi(C,D) &= ACP_L + BDP_R + A[P_L, C]P_L + A[P_L, D]P_R \\ &\quad + B[P_R, C]P_L + B[P_R, D]P_R \\ &= \Pi(AC, BD) + A[P_L, C]P_L + A[P_L, D]P_R \\ &\quad + B[P_R, C]P_L + B[P_R, D]P_R.\end{aligned}$$

From this we conclude that for Π to be a homomorphism, then $\forall A \in SU(2) : \; [P_{L/R}, A] = 0$. Since any matrix $A \in SU(2)$ can be written as $e^{iv^i \sigma_i}$, then

$$\forall A \in SU(2)[P_{L/R}, A] = 0 \quad \Leftrightarrow \quad \forall i : [P_{L/R}, \sigma_i] = 0.$$

An explicit calculation shows that this criterion implies that $P_{L/R}$ has to be proportional to the identity matrix. Thus we can write $P_L = xI$ and $P_R = yI$ for some complex scalars $x, y \in \mathbb{C}$. Requiring that $P_{L/R}$ be idempotent and that $P_L + P_R = I$, implies the two possibilities $x = 1 \wedge y = 0$ or $x = 0 \wedge y = 1$. This corresponds to the two irreducible representations of $SU(2) \times SU(2)$ on $\mathbb{C}^2$ which in the language of addition of angular momentum are denoted $({}^1\!/_2, 0)$ and $(0, {}^1\!/_2)$. Since this is not a faithful representation of $SU(2) \times SU(2)$ we can not conclude that $\mathcal{L}$ is invariant under this group. We have thus proved that any attempt at splitting Φ using projection operators acting on the spinor indices does not work. This begs the question why it does work in QCD. The answer is that there the projection operators act on different indices from the ones acted on by the $SU(2)$ matrices, thus there is more structure to the space of the multiplets $[u, d]^{\mathrm{T}}$ than that supplied simply by $\mathbb{C}^2$.

A strategy that does work is to define

$$\Phi = \begin{bmatrix} \Phi_L \\ \Phi_R \end{bmatrix}$$

for the two complex scalar fields $\Phi_L, \Phi_R \in \mathbb{C}$, and then to construct the matrix

$$U(\Phi) = \begin{bmatrix} \Phi_L & \Phi_R \\ -\Phi_R^* & \Phi_L^* \end{bmatrix}.$$

This matrix has the same form as a matrix in $SU(2)$ c.f. Example 2.5 without the restriction that the determinant be 1. Hence we get that

$$U(\Phi)^\dagger U(\Phi) = (\Phi_L^* \Phi_L + \Phi_R^* \Phi_R) I.$$

Because of this property we can now write the term $\Phi^\dagger \Phi$ in the Lagrangian density as

$$\Phi^\dagger \Phi = \Phi_L^* \Phi_L + \Phi_R^* \Phi_R = \frac{1}{2} \operatorname{Tr} \Big[U(\Phi)^\dagger U(\Phi) \Big].$$

This might seem like an unnecessary complication, but it simplifies the problem of constructing a representation of $SU(2) \times SU(2)$. The set of all such matrices forms a subspace $V \subset M_2(\mathbb{C})$ that is isomorphic to $\mathbb{C}^2$ and we can now define the representation of

$SU(2) \times SU(2)$ on this vector space by

$$\Pi\colon \quad SU(2) \times SU(2) \to GL(V), \quad (A, B) \mapsto \Pi_{(A,B)}$$

such that

$$\Pi_{(A,B)}\colon \quad V \to V, \quad U(\Phi) \mapsto AU(\Phi)B^{-1}.$$

To see that this satisfies the homomorphism property let (A, B) and (C, D) be two elements in $SU(2) \times SU(2)$. Then

$$\begin{aligned}
&\Pi\left((A, B)(C, D)\right) U(\Phi) = \Pi_{(AC,BD)}U(\Phi) = ACU(\Phi)(BD)^{-1} = ACU(\Phi)D^{-1}B^{-1},\\
&\Pi\left((A, B)\right) \Pi\left((C, D)\right) U(\Phi) = \Pi_{(A,B)}CU(\Phi)D^{-1} = ACU(\Phi)D^{-1}B^{-1},
\end{aligned}$$

thus Π satisfies the homomorphism property. The map $\Pi_{(A,B)}$ is also invertible by the inverse $\Pi_{(A^{-1},B^{-1})}$ and linearity follows from matrix multiplication. We also see that $\Pi_{(A,B)}$ maps elements in V to elements in V. This follows since $U(\Phi)$ has the same form as elements in $SU(2)$ as mentioned earlier. Thus the codomain of Π is $GL(V)$ and we have proved that Π is a representation. This is also a faithful representation of $SU(2) \times SU(2)$ since the two matrices act independently.

Now we can transform the elements of Φ by acting with this representation on $U(\Phi)$. Let $U'(\Phi) = \Pi_{(A,B)}U(\Phi)$ denote the transformed matrix in V and let Φ' be the vector in $\mathbb{C}^2$ we obtain by mapping the elements of $U'(\Phi)$ back to $\mathbb{C}^2$. Then

$$\begin{aligned}
\Phi'^{\dagger}\Phi' &= \frac{1}{2}\,\mathrm{Tr}\left[U'^{\dagger}(\Phi)U'(\Phi)\right] = \frac{1}{2}\,\mathrm{Tr}\left[BU^{\dagger}(\Phi)A^{\dagger}AU(\Phi)B^{\dagger}\right]\\
&= \frac{1}{2}\,\mathrm{Tr}\left[B^{\dagger}BU^{\dagger}(\Phi)U(\Phi)\right] = \frac{1}{2}\,\mathrm{Tr}\left[U^{\dagger}(\Phi)U(\Phi)\right] = \Phi^{\dagger}\Phi,
\end{aligned}$$

where we have used the cyclic property of the trace to move the matrix $B^{\dagger}$ to the left. From this we can conclude that the theory is invariant under the group $SU(2) \times SU(2)$.

We also note that this proof could have been accomplished by using quaternions since the group $SU(2)$ is isomorphic to the multiplicative group of quaternions with unit norm. In that case we would identify Φ with a quaternion q and let elements of $SU(2) \times SU(2)$ act on q by the representation

$$\Pi_{(A,B)}q = aqb^{-1},$$

where we have used the isomorphism between $SU(2)$ and the group of quaternions with unit norm to identify A with the quaternion a and B with the quaternion b.

4.4.4 Goldstone bosons of a $U(2)$ invariant field theory

We have seen that Eq. (4.14) is invariant under $SU(2) \times SU(2)$. If we now modify this theory by introducing a scalar μ_I in the same way as was done in Eq. (3.35) we get the Lagrangian density

$$\begin{aligned}
\mathcal{L} &= \left(\partial_\mu + i\mu_I\delta_{\mu 0}\right)\Phi^{\dagger}\left(\partial^\mu - i\mu_I\delta^{\mu 0}\right)\Phi - m^2\Phi^{\dagger}\Phi - \lambda\left(\Phi^{\dagger}\Phi\right)^2\\
&= \partial_\mu\Phi^{\dagger}\partial^\mu\Phi + i\mu_I\left(\Phi^{\dagger}\dot{\Phi} - \dot{\Phi}^{\dagger}\Phi\right) + (\mu_I^2 - m^2)\Phi^{\dagger}\Phi - \lambda\left(\Phi^{\dagger}\Phi\right)^2.
\end{aligned}$$

This theory is no longer invariant under $SU(2) \times SU(2)$. Furthermore it is not Lorentz invariant because of the way time derivatives are treated, thus the counting of NG-bosons by Goldstone's theorem is not applicable. By letting the fields be constant, the derivatives vanish and we obtain the potential for constant field configurations

$$V(\Phi_0) = (m^2 - \mu_I^2)\Phi_0^\dagger\Phi_0 + \lambda(\Phi_0^\dagger\Phi_0)^2 = \lambda\left[(\Phi_0^\dagger\Phi_0 - v)^2 - v^2\right],$$

where we have defined the constant

$$v = \frac{\mu^2 - m^2}{2\lambda}.$$

If we write the components of Φ_0 as

$$\Phi_0 = \begin{bmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{bmatrix},$$

we can write $V(\Phi_0)$ as

$$V(\Phi_0) = \lambda\left[\left(\sum_{i=1}^{4}\phi_i\phi_i - v\right)^2 - v^2\right],$$

which is of the same form as the potential for the $SO(N)$ field theory in Eq. (4.7). The discussion in Section 4.1.2 can thus be used to argue that the minimum is found for a constant field configuration $\Phi_0^\dagger\Phi_0 = v$ if $\mu_I^2 > m^2$. This gives a continuum of possible ground states and because of perturbations as discussed towards the end of Section 3.4.1, the true ground state will be either one of these. We choose the ground state

$$\Phi_0 = \begin{bmatrix}\sqrt{v} \\ 0\end{bmatrix}.$$

If we now were to naively apply Goldstone's theorem we would need to find the subgroup H under which the ground state is invariant. To find this we act on Φ_0 with a general matrix $U \in M_2(\mathbb{C})$ and require invariance. This implies that U can be written as

$$U = \begin{bmatrix}1 & \beta \\ 0 & \alpha\end{bmatrix}.$$

Next we require unitarity since we need H to be a subgroup of $U(2)$. This implies that $\beta = 0$ and $\alpha^*\alpha = 1$. Writing $\alpha = e^{i\theta}$ we thus get that

$$U(e^{i\theta}) = \begin{bmatrix}1 & 0 \\ 0 & e^{i\theta}\end{bmatrix}. \tag{4.16}$$

We recognize this as a faithful representation of $U(1)$ given by the map

$$U\colon \quad U(1) \to GL(\mathbb{C}^2), \quad e^{i\theta} \mapsto \begin{bmatrix}1 & 0 \\ 0 & e^{i\theta}\end{bmatrix}.$$

It is easily checked that this map satisfies the homomorphism property and gives a linear invertible operator on $\mathbb{C}^2$. Thus $H = U(1)$, and since the original theory was invariant under $U(2)$, which can be decomposed into $U(2) \simeq SU(2) \times U(1)$ we have 3 broken generators from $SU(2)$ and one unbroken generator from $U(1)$. Goldstone's theorem thus would imply 3 NG-bosons.

To get the fields corresponding to fluctuations about the ground state we define new fields $\tilde{\Phi}$ with respect to some constant background field $\langle\Phi\rangle_0$ which we set to be the chosen ground state:

$$\tilde{\Phi} = \Phi - \langle\Phi\rangle_0 = \Phi - \Phi_0.$$

We then rename $\tilde{\Phi} \to \Phi$ so that we can write the Lagrangian density as

$$\begin{aligned}
\mathcal{L} =& \partial_\mu \Phi^\dagger \partial^\mu \Phi + i\mu_I \left(\Phi^\dagger \dot{\Phi} + \langle\Phi\rangle_0^\dagger \dot{\Phi} - \dot{\Phi}^\dagger \Phi - \dot{\Phi}^\dagger \langle\Phi\rangle_0 \right) \\
&+ (\mu_I^2 - m^2) \left(\Phi^\dagger \Phi + \langle\Phi\rangle_0^\dagger \Phi + \Phi^\dagger \langle\Phi\rangle_0 + \langle\Phi\rangle_0^\dagger \langle\Phi\rangle_0 \right) \\
&- \lambda \left(\Phi^\dagger \Phi + \langle\Phi\rangle_0^\dagger \Phi + \Phi^\dagger \langle\Phi\rangle_0 + \langle\Phi\rangle_0^\dagger \langle\Phi\rangle_0 \right)^2,
\end{aligned}$$

in terms of the new fields. Specifying to a set of four real fields by writing

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{bmatrix},$$

and inserting for $\langle\Phi\rangle_0$, this becomes

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2} \left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2 + \dot{\phi}_4^2 - \nabla\phi_1 \cdot \nabla\phi_1 - \nabla\phi_2 \cdot \nabla\phi_2 - \nabla\phi_3 \cdot \nabla\phi_3 - \nabla\phi_4 \cdot \nabla\phi_4 \right) \\
&- \mu_I \left(\phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1 + \phi_3\dot{\phi}_4 - \phi_4\dot{\phi}_3 + \sqrt{2v}\dot{\phi}_2 \right) \\
&+ \frac{\mu_I^2 - m^2}{2} \left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 + 2v + 2\sqrt{2v}\phi_1 \right) \\
&- \frac{\lambda}{4} \Big(\phi_1^4 + \phi_2^4 + \phi_3^4 + \phi_4^4 + 4v^2 + 8v\phi_1^2 + 2\Big(\phi_1^2\phi_2^2 + \phi_1^2\phi_3^2 + \phi_1^2\phi_4^2 \\
&+ 2v\phi_1^2 + 2\sqrt{2v}\phi_1^3 + \phi_2^2\phi_3^2 + \phi_2^2\phi_4^2 + 2v\phi_2^2 + 2\sqrt{2v}\phi_1\phi_2^2 + \phi_3^2\phi_4^2 + 2v\phi_3^2 \\
&+ 2\sqrt{2v}\phi_1\phi_3^2 + 2v\phi_4^2 + 2\sqrt{2v}\phi_1\phi_4^2 + 4v\sqrt{2v}\phi_1 \Big) \Big).
\end{aligned}$$

Since we are only interested in small fluctuations about the ground state we neglect terms with fields of cubic power or higher and insert for v. After the algebraic dust settles, the resulting Lagrangian density becomes

$$\begin{aligned}
\mathcal{L}_{\text{free}} =& \frac{1}{2} \left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 + \partial_\mu\phi_3\partial^\mu\phi_3 + \partial_\mu\phi_4\partial^\mu\phi_4 \right) \\
&- \mu_I \left(\phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1 + \phi_3\dot{\phi}_4 - \phi_4\dot{\phi}_3 + \sqrt{2v}\dot{\phi}_2 \right) \\
&+ \frac{(\mu_I^2 - m^2)^2}{4\lambda} - (\mu^2 - m^2)\phi_1^2.
\end{aligned}$$

Despite this simplification, it is still not obvious how many massive or massless modes are in the theory from this expression because of the mixing of the fields in terms such as $-\mu_I \phi_1 \dot{\phi}_3$. Analogous to how one finds the normal modes of coupled oscillators in classical mechanics, we find find the dispersion relations for this field theory by first making the ansatz

$$\phi_j(\mathbf{r}, t) = \phi_j(p) e^{p_0 t - \mathbf{r} \cdot \mathbf{p}}$$

for plane wave solutions where p is a four vector. Then any term $\dot{\phi}_j = i p_0 \phi_j$ and any term $\nabla \phi_j = -i \mathbf{p} \phi_j$. Thus we can write the terms quadratic in the fields as a product of two vectors and a matrix

$$\frac{1}{2} \mathbf{y}^{\mathrm{T}} A \mathbf{y},$$

where $\mathbf{y}$ is

$$\mathbf{y}^{\mathrm{T}} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix},$$

and A is the matrix

$$A = \begin{bmatrix} -p_0^2 + \mathbf{p}^2 - 2(\mu_I^2 - m^2) & -2i\mu_I p_0 & 0 & 0 \\ 2i\mu_I p_0 & -p_0^2 + \mathbf{p}^2 & 0 & 0 \\ 0 & 0 & -p_0^2 + \mathbf{p}^2 & -2i\mu_I p_0 \\ 0 & 0 & 2i\mu_I p_0 & -p_0^2 + \mathbf{p}^2 \end{bmatrix}.$$

By requiring that the determinant of this matrix vanish we obtain the 8 dispersion relations

$$E_1^{\pm} = \pm\sqrt{\mathbf{p}^2 + \mu_I^2 + m^2 \pm \sqrt{(m^2 + \mu^2)^2 + 4\mu_I^2 \mathbf{p}^2}},$$
$$E_2^{\pm} = \pm\sqrt{\mathbf{p}^2 + 2\mu_I^2 \pm 2\mu_I \sqrt{\mathbf{p}^2 + \mu_I^2}},$$

where E_1^+ refers to a $+$ in front of the second square root etc. To see which modes are massive and massless we expand in the long wavelength limit given by $|\mathbf{p}|/\mu_I \ll 1$, whence we get that

$$E_1^+ \approx \pm\sqrt{2(m^2 + \mu_I^2)} \left[1 + \frac{\mathbf{p}^2(m^2 + 3\mu_I^2)}{4(m^2 + \mu_I^2)^2}\right],$$
$$E_1^- \approx \pm|\mathbf{p}|\sqrt{\frac{m^2 - \mu_I^2}{m^2 + \mu_I^2}},$$
$$E_2^+ \approx \pm 2\mu_I \left(1 + \frac{\mathbf{p}^2}{2\mu_I^2}\right),$$
$$E_2^- \approx \pm\frac{\mathbf{p}^2}{2\mu_I^2}.$$

Since we are considering real scalar fields we discard the negative energy modes as unphysical and hence see the above equations yield 4 different particles, two of whose energies approach zero in the long wavelength limit. These are thus the Nambu-Goldstone bosons of the theory. Classifying them as in Section 4.3 we see that we have one type-II NG-boson because of the quadratic $|\mathbf{p}|$ dependence, and one type-I NG-boson because of the linear $|\mathbf{p}|$ dependence. As expected, Goldstone's theorem which would have predicted 3 NG-bosons does not hold since the theory is not Lorentz invariant. However the Nielsen-Chadha formula in Eq. (4.12) holds since

$$n_{\mathrm{I}} + 2n_{\mathrm{II}} = 3,$$

and the broken subgroup of $U(2)$ is $SU(2)$ which has 3 generators given by the Pauli matrices, hence $n_{\mathrm{BG}} = 3$.

To see if Eq. (4.13) holds we first need to find the generator of the unbroken $U(1)$ subgroup. From the representation of elements in $U(1)$ given by Eq. (4.16) we get that the generator is given by

$$Q_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

since

$$e^{i\theta Q_4} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} = U(e^{i\theta}).$$

Since the theory has not yet been quantized we can not apply Eq. (4.13) directly, but by interpreting Φ_0 as $|0\rangle$, $\hat{Q}_i$ as σ_i for $i \in \{1,2,3\}$ and $\hat{Q}_4$ as Q_4, we can calculate the elements of a matrix analogous to ρ by

$$\tilde{\rho}_{12} = \Phi_0^\dagger[\sigma_1, \sigma_2]\Phi_0 = iv,$$
$$\tilde{\rho}_{14} = \Phi_0^\dagger[\sigma_1, Q_4]\Phi_0 = 0,$$

etc. Thus we get that

$$\tilde{\rho} = \begin{bmatrix} 0 & iv & 0 & 0 \\ -iv & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By letting $\tilde{\rho}$ act on a vector in $\mathbb{R}^4$ we see that $\operatorname{rank}\tilde{\rho} = 2$, thus our interpretation of Eq. (4.13) for this classical field theory yields that

$$n_{\mathrm{NGB}} = n_{\mathrm{BG}} - \frac{1}{2}\operatorname{rank}\tilde{\rho} = 3 - 1 = 2,$$

which agrees with our results.

Chapter 5

Conclusion and outlook

5.1 Conclusion

After presenting and defining relevant mathematical concepts relating to group theory, our first result was to derive the Euler-Lagrange field equations by considering the field configuration for which the action is stationary. This equipped us with the necessary tools for proving Noether's theorem from which we saw that every independent continuous symmetry leads to an independent conservation law.

We then considered a Lagrangian density with a single scalar field whose associated Euler-Lagrange field equation turned out to be the time dependent Schrödinger equation. This Lagrangian density was shown to be symmetric under the Euclidean group $E(2)$ which through Noether's theorem implied 3 conserved currents. Two currents tied to translation in field space, and one to rotation. The conserved quantities obtained from these currents gave a projective representation of $E(2)$. Using periodic boundary conditions on a finite volume we also found a complete set of solutions to the Schrödinger equation which we used to quantize the theory in terms of field operators. By transforming the Fock-vacuum state using generators in the projective representation of $E(2)$ we showed that the vacuum was infinitely degenerate with the vacuum states being orthogonal to each other in the thermodynamic limit, and individually coherent states of the field operator. We then lifted the degeneracy of the vacuum by introducing a chemical potential. This singled out the Fock-vacuum as the true vacuum-state from which we could build the Hilbert-space in terms of excited states. After adiabatically removing the chemical potential we saw that these excited states were excited states of a single massless particle which we claimed to be a NG-boson. This agreed with the newly found formula for counting Nambu-Goldstone bosons in theories that are not relativistically invariant. At the end of the chapter we showed how adding a chemical potential in the manner done for the previously discussed field theory breaks the Lorentz invariance of a theory that initially has this property by sending invariant derivatives to covariant derivatives.

In an effort to explain why the NG-boson appeared from a more general context we discussed Goldstone's theorem. We proved the theorem in the case of classical fields, from which we learned that for every broken independent symmetry there appears a NG-boson if the theory is also Lorentz invariant. By a broken symmetry we mean a transformation under which the Lagrangian density is symmetric, but the ground-state is not. We then presented two examples where we applied this theorem. The first was for a theory with a single complex scalar field invariant under the group $U(1)$. This group has only one generator which is broken by the ground state, thus one NG-boson appears by SSB. The next example consisted of N real scalar fields and was invariant under the group $SO(N)$, but the ground state was spontaneously broken to $SO(N-1)$. This left $N-1$ broken generators and hence this same number of NG-bosons. We specialized this example to the case of $N=4$ and found the conserved currents implied by the $SO(4)$ symmetry of the theory.

Next, we gave a short discussion on the state of Goldstone's theorem for quantum field theories. We saw that the theorem still holds in this case given that there are no quantum anomalies in the symmetries.

Finally we discussed a last example of a theory invariant under $SU(2) \times SU(2)$. This group was found to be described simply as a Cartesian product such that each element consisted of two separate $SU(2)$ matrices. This group was proved to be homomorphic to $SO(4)$ and the field theory that was initially shown to be invariant under $SO(4)$ was found to be invariant under $SU(2)\times SU(2)$ as well. Next this theory was modified by introducing a chemical potential μ_I as was done for the Schrödinger theory which broke the invariance under $SU(2) \times SU(2)$ and led to spontaneous symmetry breaking of the group $U(2)$ for $\mu_I^2 > m^2$. A ground state was chosen and the fields expanded in terms of fluctuations about this state. The dispersion relations for these fluctuating fields where found to admit one type-I Nambu-Goldstone boson and one type-II Nambu-Goldstone boson.

5.2 Outlook

In this thesis we have discussed Goldstone's theorem mostly in terms of classical fields and worked out some concrete examples. We have found that the appearance of NG-bosons is different for relativistic and non-relativistic field theories. In future work we would like to explore these differences in a more systematic manner. In relativistic theories, we can assume the effective Lagrangian to be invariant without loss of generality [26], however for non-relativistic theories as we have seen in the case of the Schrödinger equation the symmetries of the theories may change the Lagrangian density with a total derivative. This foils attempts at using the methods developed in high-energy physics for Lorentz invariant theories to construct effective Lagrangian densities. It would therefore be interesting to understand how one can construct such Lorentz noninvariant effective Lagrangian densities and under what conditions the methods for doing so applies.

Additionally, we would also like to check explicitly that Goldstone's theorem holds for quantum field theories by working out some loop corrections.

Finally, it would be interesting to look at some concrete applications of these theoretical concepts to current condensed matter research. One such application could be to study the pseudogap line occurring in high temperature cuprate superconductors. When the amount of doping and temperature is varied in such superconductors there is a change from the normal metallic phase to the strange metallic phase [27]. It is not yet clear if this is a bona fide phase transition resulting from a broken continuous symmetry, however recent experiments [28, 29] indicate that this is the case. It would therefore be interesting to see what consequences such a broken symmetry would have for the macroscopic physical properties of such superconductors.

Appendix A

Proofs

A.1 Proof of Theorem 2.1

Proof. Start by assuming that $H \leq G$. Then from the definition of a subgroup H is closed under the binary operation of G.
Next take $a \in H$ and try to solve the equation $ax = a$. Because H is a group $\exists (a^{-1})' \in H$;

$$\begin{aligned} ax &= a \\ (a^{-1})'(ax) &= (a^{-1})'(a) \\ ((a^{-1})'a)x &= (a^{-1})'a \\ e'x &= e' \\ x &= e', \end{aligned}$$

where e' is the identity element in H. But because $H \leq G$, $a \in H \Rightarrow a \in G$ and G also is a group, thus $\exists a^{-1} \in G$. Doing the same manipulation on $ax = a$ then yields $x = e$. Thus for identity element $e' \in H$ and identity element $e \in G$, $e' = x = e \;\Rightarrow\; e \in H$.
Showing that $a^{-1} \in H$ is done in the same way but now considering the equation

$$\begin{aligned} ax &= e' \in H, \text{where } a \in H \\ (a^{-1})'(ax) &= (a^{-1})'e' = (a^{-1})' \\ e'x &= x = (a^{-1})'. \end{aligned}$$

Then by considering the same equation but multiplying on the left side by a^{-1}, one finds that $x = a^{-1} \in G \;\Rightarrow\; (a^{-1})' = a^{-1} \in H$. Now showing the other implication direction, the properties listed are assumed. Closedness follows directly from the first property. From the second property it as guaranteed that there is a identity element $e' \in H$, namely the identity element $e' = e$ from G. Existence of the inverse similarly follows from the third property. To check associativity in H consider $(ab)c$ with $a, b, c \in H$. Because $H \subseteq G$, $a, b, c \in G$ as well. Because G is a group, using the binary operator in G yields

$(ab)c = a(bc)$, and since the same operator is used in H, this equation is also valid there, which guarantees associativity. Thus $H \leq G$ ■

A.2 Proof of Theorem 2.3

Proof. First we prove property *(a)*. To show that $AXA^{-1} \in \mathfrak{g}$, it has to satisfy that $\forall t \in \mathbb{R}$ $e^{tAXA^{-1}} \in G$. Using the series expansion of the exponential we see that $e^{tAXA^{-1}} = e^{AtXA^{-1}} = Ae^{tX}A^{-1}$. Since $X \in \mathfrak{g}$, $e^{tX} \in G$, so $Ae^{tX}A^{-1}$ is just a product of elements in G. Because G is closed under matrix multiplication $Ae^{tX}A^{-1} \in G, \Rightarrow e^{tAXA^{-1}} \in G$. This must hold for all t because the choice of t was arbitrary, which means that $AXA^{-1} \in \mathfrak{g}$.

Now proving property *(b)*. Because $X \in \mathfrak{g}$, $e^{tX} \in G \;\; \forall t \in \mathbb{R}$. Now for any $s \in \mathbb{R}$, $sr \in \mathbb{R}$, thus $\exists t \in \mathbb{R};\ t = st$. This proves that for any s, $e^{srX} = e^{tX} \in G$ which is the definition of rX to be in the Lie algebra.

Now proving property *(c)*. Using the Lie product formula, for any $t \in \mathbb{R}$

$$e^{t(X+Y)} = e^{tX+tY} = \lim_{n\to\infty} \left(e^{\frac{tX}{n}} e^{\frac{tY}{n}}\right)^n .$$

In this limit $n \in \mathbb{N} \subseteq \mathbb{R}$, so using property *(b)*, $e^{\frac{tX}{n}}, e^{\frac{tY}{n}} \in G$. Furthermore because G is closed under group multiplication $\left(e^{\frac{tX}{n}} e^{\frac{tY}{n}}\right)^n \in G$. Construct the sequence

$$\left\{\left(e^{\frac{tX}{n}} e^{\frac{tY}{n}}\right)^n\right\}_{n\in\mathbb{N}} .$$

From the above discussion this sequence only consists of elements in G, and has limit $e^{t(X+Y)}$. Because G is a matrix Lie group, the limit of any converging sequence of elements in G must either be in G itself, or be uninvertible. Since $e^{t(X+Y)}$ is invertible, this eliminates the second possibility, thus $e^{t(X+Y)} \in G$. Because t was arbitrary, this must hold for all t, which is the defining property of the matrix Lie algebra.

Because $\mathfrak{g} \subseteq M_n(\mathbb{C})$, it follows immediately from property *(b)* and *(c)* that $\mathfrak{g}$ is a real subspace of $M_n(\mathbb{C})$ and thus a real vector space.

The final proof is of property *(d)*. Consider the derivative of $e^{tY}Xe^{-tY}$ evaluated at 0. Using the product rule for matrix valued functions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tY}Xe^{tY}\right)\bigg|_{t=0} = -YX + XY = [X, Y].$$

Because $Y \in \mathfrak{g}$, $e^{\pm tY} \in G$, then *(a)* implies that $e^{-tY}Xe^{tY} \in G$ for all t. By the definition of the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tY}Xe^{tY}\right) = \lim_{h\to 0} \frac{e^{-(t+h)Y}Xe^{(t+h)Y} - e^{-tY}Xe^{tY}}{h}.$$

On the right side of the limit there is a sum and a scalar product of two elements in $\mathfrak{g}$. Using properties *(b)* and *(c)* then,

$$\frac{e^{-(t+h)Y}Xe^{(t+h)Y} - e^{-tY}Xe^{tY}}{h} \equiv Z(h) \in \mathfrak{g}$$

for any h. Because $\mathfrak{g}$ is a vector space, any limit point must be inside $\mathfrak{g}$ $\Rightarrow \lim_{h\to 0} Z(h) \in \mathfrak{g}$, which proves that

$$[X, Y] = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tY} X e^{tY}\right)\Big|_{t=0} \in \mathfrak{g}.$$

■

A.3 Inductive proof of Eq. (4.8)

Proof. To prove Eq. (4.8) which is repeated here for convenience:

$$\det\left(\phi^j\phi^k - \xi\delta^{jk}\right) = (-\xi)^{N-1}(\phi^i\phi^i - \xi),$$

first we see that matrices of the form

$$\begin{aligned}
&\det\begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^n & \phi^1\phi^{n+1} & \cdots & \phi^1\phi^{N+1} \\ \phi^n\phi^1 & \phi^n\phi^n & \phi^n\phi^{n+1} & \cdots & \phi^n\phi^{N+1} \\ \phi^{n+1}\phi^1 & \phi^{n+1}\phi^n & & & \\ \vdots & \vdots & & B & \\ \phi^{N+1}\phi^1 & \phi^{N+1}\phi^n & & & \end{bmatrix} \\
=&\det\begin{bmatrix} (\phi^1+\phi^n)\phi^1 & (\phi^1+\phi^n)\phi^n & (\phi^1+\phi^n)\phi^{n+1} & \cdots & (\phi^1+\phi^n)\phi^{N+1} \\ \phi^n\phi^1 & \phi^n\phi^n & \phi^n\phi^{n+1} & \cdots & \phi^n\phi^{N+1} \\ \phi^{n+1}\phi^1 & \phi^{n+1}\phi^n & & & \\ \vdots & \vdots & & B & \\ \phi^{N+1}\phi^1 & \phi^{N+1}\phi^n & & & \end{bmatrix} \qquad \text{(A.1)} \\
=&(\phi^1+\phi^n)\phi^n \det\begin{bmatrix} \phi^1 & \phi^n & \phi^{n+1} & \cdots & \phi^{N+1} \\ \phi^1 & \phi^n & \phi^{n+1} & \cdots & \phi^{N+1} \\ \phi^{n+1}\phi^1 & \phi^{n+1}\phi^n & & & \\ \vdots & \vdots & & B & \\ \phi^{N+1}\phi^1 & \phi^{N+1}\phi^n & & & \end{bmatrix} = 0,
\end{aligned}$$

where B is an arbitrary $(N-2)\times(N-2)$ matrix and repeated indices are no longer implicitly summed over.

Next we call the matrix in the determinant in Eq. (4.8) A_1 and generalize this definition such that

$$A_n = \begin{bmatrix} \phi^n\phi^n - \xi & \phi^n\phi^{n+1} & \cdots & \phi^n\phi^N \\ \phi^{n+1}\phi^n & \phi^{n+1}\phi^{n+1} - \xi & & \\ \vdots & & \ddots & \\ \phi^N\phi^n & \phi^N\phi^{n+1} & & \phi^N\phi^N - \xi \end{bmatrix}$$

given that A_1 is a $N\times N$ matrix. Then we can prove that

$$\det\begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^n & \cdots & \phi^1\phi^{N+1} \\ \phi^n\phi^1 & & & \\ \vdots & & A_n & \\ \phi^{N+1}\phi^1 & & & \end{bmatrix} = (-\xi)^{N-n+2}\phi^1\phi^1 \qquad \text{(A.2)}$$

by induction where we have taken A_1 to be an $(N+1)\times(N+1)$ matrix. Then the base case is for $n = N+1$ that

$$
\begin{aligned}
&\det\begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^{N+1} \\ \phi^{N+1}\phi^1 & A_{N+1} \end{bmatrix} = \det\begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^{N+1} \\ \phi^{N+1}\phi^1 & \phi^{N+1}\phi^{N+1} - \xi \end{bmatrix} \\
&= (-\xi)\phi^1\phi^1 = (-\xi)^{N-(N+1)+2}\phi^1\phi^1.
\end{aligned}
$$

The induction step is that given this equation is true for $n+1$ then

$$
\begin{aligned}
&\det\begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^n & \cdots & \phi^1\phi^{N+1} \\ \phi^n\phi^1 & & & \\ \vdots & & A_n & \\ \phi^{N+1}\phi^1 & & & \end{bmatrix} \\
=&\det\begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^n & \phi^1\phi^{n+1} & \cdots & \phi^1\phi^{N+1} \\ \phi^n\phi^1 & \phi^n\phi^n & \phi^n\phi^{n+1} & \cdots & \phi^n\phi^{N+1} \\ \phi^{n+1}\phi^1 & \phi^{n+1}\phi^n & & & \\ \vdots & \vdots & & A_{n+1} & \\ \phi^{N+1}\phi^1 & \phi^{N+1}\phi^n & & & \end{bmatrix} \\
&-\xi\det\begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^n & \phi^1\phi^{n+1} & \cdots & \phi^1\phi^{N+1} \\ 0 & 1 & 0 & \cdots & 0 \\ \phi^{n+1}\phi^1 & \phi^{n+1}\phi^n & & & \\ \vdots & \vdots & & A_{n+1} & \\ \phi^{N+1}\phi^1 & \phi^{N+1}\phi^n & & & \end{bmatrix} \\
=&(-\xi)\det\begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^{n+1} & \cdots & \phi^1\phi^{N+1} \\ \phi^{n+1}\phi^1 & & & \\ \vdots & & A_{n+1} & \\ \phi^{N+1}\phi^1 & & & \end{bmatrix} \\
=&(-\xi)(\xi)^{N-(n+1)+2}\phi^1\phi^1 = (-\xi)^{N-n+2}\phi^1\phi^1,
\end{aligned}
$$

where in the first line we have used the determinant as a sum of determinants, and in the second we used that the first determinant vanishes due to Eq. (A.1). In the fourth we used the induction hypothesis. Thus Eq. (A.2) is proved.

Finally we can prove Eq. (4.8) by induction. The base case is for $N = 2$

$$
\begin{aligned}
\det\begin{bmatrix} \phi^1\phi^1 - \xi & \phi^1\phi^2 \\ \phi^2\phi^1 & \phi^2\phi^2 - \xi \end{bmatrix} &= (-\xi)(\phi^1\phi^1 + \phi^2\phi^2) + (-\xi)^2 \\
&= (-\xi)^{N-1}\left(\sum_{i=1}^{2}\phi^i\phi^i - \xi\right).
\end{aligned}
$$

For the induction step we assume Eq. (4.8) is true for N and want to show it for $N+1$. For this calculation then A_1 is a $(N+1)\times(N+1)$ matrix (which is why we assumed this when proving Eq. (A.2)).

$$
\begin{aligned}
\det A_1 =& \det \begin{bmatrix} \phi^1\phi^1 & \phi^1\phi^2 & \cdots & \phi^1\phi^{N+1} \\ \phi^2\phi^1 & & & \\ \vdots & & A_2 & \\ \phi^{N+1}\phi^1 & & & \end{bmatrix} - \xi \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \phi^2\phi^1 & & & \\ \vdots & & A_2 & \\ \phi^{N+1}\phi^1 & & & \end{bmatrix} \\
=& (-\xi)^N \phi^1\phi^1 - \xi \det A_2 = (-\xi)^N \phi^1\phi^1 - \xi(-\xi)^{N-1}\left(\sum_{i=2}^{N+1} \phi^i\phi^i - \xi\right) \\
=& (-\xi)^{(N+1)-1}\left(\sum_{i=1}^{N+1} \phi^i\phi^i - \xi\right),
\end{aligned}
$$

where in the first line we have used the sum of determinants rule, then used Eq. (A.2) to evaluate the first determinant and expanded the second determinant along the top row. In the second line we used the induction hypothesis for A_2 which is an $N \times N$ matrix. Thus Eq. (4.8) is proved by induction.

■

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