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Wiener and Gamma Processes Overview for Degradation Modelling and Prognostic

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Preface

This master thesis was written during the spring semester 2015 at the Department of Production and Quality Engineering, Faculty of Engineering, Science and Technology, Norwegian University of Science and Technology. The thesis is part of the two-year international master's program Reliability, Availability, Maintainability and Safety (RAMS).

The topic, which brought up in October 2014, was initally condition-based maintenance for stochastically deteiorating system. It was later slightly narrowed to become a study on Gamma process in degradation modeling and related maintenance policies. The title of this thesis is "Wiener and Gamma Processes Overview for Degradation Modelling and Prognostic" and is written with the guidance of my supervisor professor Anne Barros at the Department of Production and Quality Engineering.

This report is partly based on the theory and methods in TPK4120 Safety and Reliability Analysis, TMA4275 Lifetime Analysis and in TPK5170 RAMS Assessment and Optimization. So it is written for readers with basic knowledge in probability, reliability analysis, stochastic process theory, and preferably maintenance optimization.

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Yun Zhang

Acknowledgment

I would like to express my deepest gratitude to my supervisor, Professor Anne Barros, Department of Production and Quality Engineering at NTNU, for all her excellent guidance and support throughout my master thesis. She provided me valuable advices and much of the information needed to write this thesis.

Y.Z.

Summary

High reliability is an indispensable requirement for the operation of technical systems and infrastructure (like roads, railways, buildings, bridges and industrial plants). Failures in these areas can result in high costs and great hazards to humans and the environment. Many failure mechanisms can be traced to an underlying degradation process. Therefore, inspections and condition-based maintenance (CBM) are undertaken to monitor deterioration and to prevent damage and future failures.

Prognostic of component lifetime is important for CBM in many application domains where safety, reliability and availability are considered of first importance. However, one of the condition indexes - the remaining useful lifetime (RUL) prediction is seldom taken into account in the decision making and maintenance planning in practice. To make up for it, the thesis focuses on a stochastic degradation process for RUL estimation and prognostic use.

The thesis starts with reviewing some of the degradation models. The merits, limitations of each model are presented. By aggregating the information of each model, this paper provides the key information about circumstances for choosing suitable deterioration models in the context of maintenance optimization.

Two stochastic process - Brownian motion and Gamma process are discussed in detail. Their statistical properties, methods for estimation, and simulation of are systematically reviewed with numerical examples. Also, each of the model is associated with the component's uncertainties in the observations while estimating its RUL.

Since Gamma process has proven to be very useful in modeling degradation paths, determining optimal inspection and maintenance decisions, the paper then investigates the application of Gamma degradation process in maintenance policies. Two RUL related maintenance policies are proposed and compared with a traditional degradation level-based policy. The performances of the proposed policies are evaluated through numerical examples in previous papers and their advantages are stated in the end.

Contents

Chapter 1

Introduction

This chapter presents a description of the background for the project thesis first. The thesis objectives are then defined, along with its scope, limitations and approach. Structure of the report is provided at the end of this chapter.

1.1 Background

Many technical systems and engineering infrastructures are subject to failures resulting from deterioration with usage and age. Power plants, railway tracks, oil platforms, cutting tools, bridges, dike and pipelines are some examples to illustrate such degradation processes (Jeang (1999)). The degradation and failures of these systems could incur high costs (e.g. due to down-time production losses and unplanned maintenance actions on the systems) and pose hazards to people and the environment. In order to reduce the costs, practices like inspections and maintenance models are increasingly applied in different fields.

An important feature of maintenance optimization is that decision often must be made under uncertainty, such as in degradation and maintenance cost (Van Noortwijk (2009)). If the evolution of a certain component or system degradation is explicitly known or even adequately predicted, optimum scheme and better decisions could be carried out to maintain the degraded system before failures. For this reason, a growing interest has been drawn to model the degradation under uncertainty for precise prognostics and wise maintenance measurements.

According to Jardine et al. (2006), there are two main prediction types in prognostics. One is

to predict how much time is left before a failure occurs given the current subject condition and history operation profile. The length from the current time to the end of the useful life is called remaining useful life (RUL). Another type is to predict the chance that a subject operates without a fault or a failure up to some future time given the history and current status. Although there are more and more papers (see for example Jardine et al. (2006), Welte et al. (2006),Van Noortwijk (2009), Liu et al. (2013))covering many aspects of prognostic problems on RUL over the last few decades, most methods exhibit limits in cases with uncertainty from the inner states or the external operating conditions of systems.

Zhang et al. (2015) points out that such kind of uncertainty (in his words heterogeneity) is wide spread in both the inner states of the system and the related working environments. For instance, an aircraft engine may experience various operational modes, such as lift, drag and take off etc. Also, the engine may perform differently under different workloads. Even components from the same category may exhibit various degradation paths in the same environment. Therefore, in order to achieve a more accurate RUL estimation, we need to incorporate the above heterogeneity into degradation modeling. Towards this point, it is desirable to classify heterogeneity into three categories: the unit-to-unit variability for components from the same category, the variability in time-varying operating conditions, and the diversity of tasks and workloads of systems during their life cycles. The taxonomy of RUL estimation approaches for system with heterogeneity is illustrated in Fig.1.1.

From the figure we can see that RUL and the associated degradation modeling techniques are composed of large branches of situations where their uncertainties are quite different from each other. Adding one kind of heterogeneity to another could increase the dimension of complexity in degradation modeling. However, the combination of heterogeneity in RUL estimation is relatively novel and unexplored. There are few comprehensive papers on combined approaches for prognostics with RUL estimation. Therefore, the writer aims to improve the knowledge in each kind of heterogeneity and propose a hybrid solution for degradation modeling based RUL estimation for systems with three heterogeneity. As a starting point, this master thesis focuses on the first kind of heterogeneity (i.e. methods considering unit-to-unit variability) and only studies RUL estimation from the statistical point of view (i.e. stochastic process with random effects).

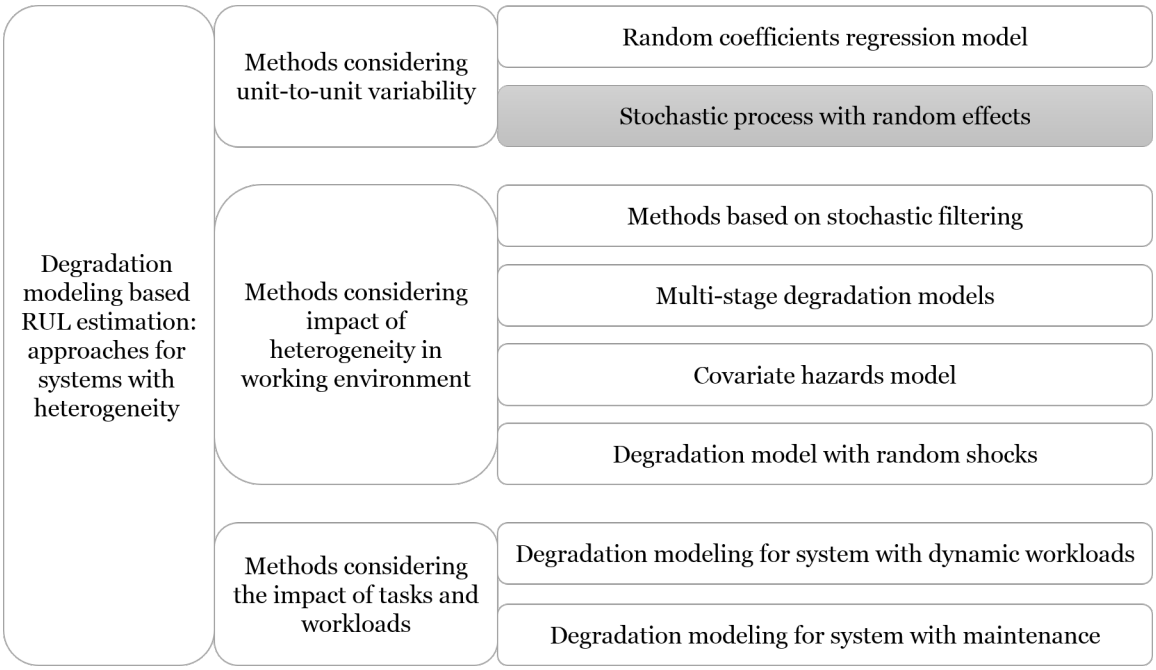

Figure 1.1: Taxonomy of RUL estimation approaches for system under heterogeneity (adopted from Zhang et al. (2015)).

The concept of RUL has been in various fields ranging from material science, biostatistics and econometrics (Si et al. (2011)). The word "useful" is normally economic related and defined upon individual explanations within specific context and operational characteristics. In this thesis, the writer assume that the definition of the useful life is know to the owner of the asset and the interest is to investigate the methods for degradation modeling related to RUL estimation given condition information.

Prognostic based on RUL estimation is one of the key factors in condition based maintenance, and prognostics and health management (Cui et al. (2004), Lee et al. (2006), Wang (2007b), Wang (2007a)). The estimation results could have great impacts on the planning of maintenance activities, spare parts provision, operational performance and the profitability of the owner of an asset(Jardine et al. (2006), Altay and Green (2006), Elwany and Gebraeel (2008), Papakostas et al. (2010)). For example, one may need to estimate the remaining technical life time of an industrial machine without data on failures. In other cases, one may want to schedule future repairs and maintenance on when to replace the machine, etc. RUL estimation has also an important role in the reusing and recycling products which , as a result, has a strategic impact on energy consumption, raw material use, pollution, and landfill (Mazhar et al. (2007)).

Therefore, it is very important to estimate RUL beyond CBM and prognostics.

1.2 Objectives

The thesis focuses on statistical-based approaches for degradation modeling and prognostic considering only unit-to-unit variability. The approaches rely on available past monitoring data and stochastic models to estimate the RUL in a probabilistic way. Such method is based on the fact that condition monitoring data and extracted features vary with the development of either the initiation and propagation process or the degradation process (Ahmadzadeh and Lundberg (2014)). So it requires no special product knowledge, physics or engineering principles. The method also have certain advantages as some nice mathematical properties can be analyzed regarding to the estimated RUL. The investigation result is intended to serve as a basis for maintenance optimization combining the merits of stochastic models (like Gamma process). This master thesis can be led with some connection with SUBPRO project. To achieve all these goals, the following tasks are identified:

1. Make a brief survey about degradation modeling to complement the work achieved during the specialized project.

2. Improve the knowledge to be able to implement the whole modeling process on some case studies published in previous works.

3. Specially focus on parameter estimation step when using degradation models considering stochastic process with random effects (e.g. Gamma process), in case of perfect and imperfect observations (e.g. white noise).

4. Study the Remaining Useful Lifetime (i.e. calculation of its law and estimation of the relevant parameters) in case of monotone and non-monotone degradation models.

5. Transport the merits of degradation modeling to maintenance optimization.

1.3 Scope and Limitations

This thesis is directed towards students, professors and researchers and other people who have basic knowledge of RAMS engineering and carry out required analysis in the condition-based maintenance and prognostic. The thesis is mainly an investigation and also a summary on degradation modeling with a focus on stochastic process. It serves as a theoretical preparation for related maintenance optimization policy. Most parts of this thesis involves mathematical models and statistical properties, and the development of an optimal maintenance policy is not quantitatively discussed here due to limited amount of time. However, it should be emphasized that the interest of this thesis is not only in mathematics, but more importantly, in applied statistics and RAMS optimization.

1.4 Approach

A large proportion of the thesis is established based on literature survey and text books. Simulation are implemented through out the thesis to demonstrate and illustrate the modeling processes, and it enhances the understanding of stochastic process and degradation modeling. Objective 1 will be established through the literature review from many RUL related papers scattered among operational research, reliability modeling, optimal maintenance, fault diagnosis and prognosis, and survival analysis literature. Objective 2,3,4 will be approached according to books and articles about stochastic process, lifetime analysis and applied statistics from web pages and libraries. Objective 5 will be realized by studying the handouts in the course of RAMS Assessment and Optimisation at NTNU. Besides, supplemented articles and useful insight from professor Anne Barros has contributed valuable inputs in the identification and analysis of the problems in the thesis.

1.5 Structure of the Report

The outline of this report is as follows. Chapter 2 presents a brief survey of the review papers related to prognostic with RUL. Modeling approaches like regression, Brownian motion, and Gamma process are briefly reviewed, advantages and limitations are stated in each section.

Chapter 3 and 4 presents the most important statistical properties of two stochastic process: Brownian motion and Gamma process. Parameter estimation processes are implemented based on testing dataset and RUL estimation in degradation models are also presented. Chapter 5 studies different monitoring methods and maintenance policies, and discusses how stochastic process can be used to optimize condition-based maintenance. Chapter 6 summarizes what has been done in this thesis and recommendations of future work.

Chapter 2

Review of the Problem

Maintenance decision making is a very critical step in condition-based maintenance. In order to support maintenance decision in a CBM program, two kinds of techniques can be involved: diagnostics and prognostics. As Jardine et al. (2006) states, diagnostics focuses on detection, isolation and identification of faults when they occur, while prognostic, on the other hand, attempts to predict faults or failures before they occur. From the definitions, we can see that prognostic is a more proactive approach as we can grossly predict and prevent upcoming failures and be prepare for the problems, and thus save extra unplanned maintenance cost. Generally, deterioration is monitored during working life of the subject so as to anticipate any faults in time. In the following sections, the writer will review articles that related to degradation modeling and prognostic in CBM decision support.

2.1 Literature Review on Degradation Modeling and Prognostic Approaches

With the rapid advance in prognostic, several review papers specifically on degradation modeling and maintenance issues have appeared, see for example (Jardine et al. (2006), Si et al. (2011), Ahmadzadeh and Lundberg (2014)). Among these papers, prognostic models can be grossly divided into four categories: physical, experimental, data-driven and hybrid. Most papers on machine prognostics discuss only data driven and model based approaches, and few address

the experience based approach. In this section, the review will be organized according to these categories.

Physical models assume that an accurate mathematical model can be constructed from first principles. They build technically comprehensive theoretical models to describe the physics of the system and failure modes, like crack propagation, wear corrosion and spall growth (Ahmadzadeh and Lundberg (2014)). Li et al. (1999) relates rolling element bearing defect growth rate to the instantaneous defect area size and material constants. They proposes a remaining life adaptation methodology based on mechanistic modeling and parameter tuning. Li and Lee (2005) presents a model-based method to forecast the RUL based on the estimated crack size and dynamic load. Watson et al. (2005) presents RUL prediction of highly dynamic, high power dry clutch system by combining physics-based simulation and wear prediction models. In these paper, the physical models must be configured specifically for the systems being monitored in order to give accurate prediction. The limitations are their high costs and component specialty. In other words, they cannot be directly applied to other types of components. Additionally, it could be difficult or even impossible to catch some system's behavior by a mathematical model.

Experimental based methodology utilizes experiments to improve the understanding of the lifetime of components. Given the data and knowledge accumulated from experience, this approach applies probabilistic or stochastic models of degradation of an subject. Lu and Meeker (1993) use fatigue-crack-growth data to develop statistical methods and use Monte Carlo simulation to obtain point estimates and confidence intervals for reliability assessment. Sutrisno et al. (2012) selects an experimental data set from 17 ball bearings from FEMTO-ST Institute. They develops prognostic algorithms to estimate RUL of the test bearings. Similarly, Le Son et al. (2013) presents a probabilistic method for prognostic applied to the 2008 PHM Conference Challenge data. A stochastic process is proposed to model the deterioration of the components and to estimate the RUL on a case study. Normally, experimental test are designed and set up in order to simulate real working conditions for studied subjects. Since a critical events could be catastrophic, we need to perform further research and combine this method with other approaches to secure the reliability of subjects.

Data-driven approaches use real data (like online gathered data with sensors or operator measures). They approximate and track features which reveal the degradation of components

and then forecast the global behavior of a system. They can transform high-dimensional noisy data into lower dimensional logical information, and thus easier to be applied in practice. Chinnam and Baruah (2003) presents a novel method for employing Hidden Markov Model for autonomous diagnostics as well as prognostics, as it exploits competitive learning to achieve HMM-based clustering. Gebraeel and Lawley (2008) focuses on the development of a neural network-based degradation model that utilizes condition-based sensory signals to compute and continuously update residual life distribution of partially degraded components. Si et al. (2013) presents a degradation path-dependent approach for RUL estimation through the combination of Bayesian updating and expectation maximization algorithm. However, the drawbacks of data driven approaches are that they highly depend on the quantity and quality of system operational data.

Hybrid approaches combine two or more prognostic methods for the formulation of degradation models and data analysis. Such combination could help to extract degradation data, greatly reduce the complexity of calculation or improve precision in predicting RUL. For instance, Banjevic and Jardine (2006) calculates the RUL by considering the hazard rate function and Markov property as a stochastic covariate process. Illustration of the main concepts is given using field data from a transmission's oil analysis histories. Mazhar et al. (2007) proposes a comprehensive two-step approach for RUL estimation of used components in consumer products. In the first stage, Weibull analysis is applied to the time-to-failure data to assess the mean life of components. In the second stage, the degradation and condition monitoring data are analyzed by developing an artificial network model. Satish and Sarma (2005) combines neural networks and fuzzy logic and forming a fuzzy back propagation network for identifying the present condition of the bearing and estimate the RUL of the motor. Yan and Lee (2007) presents a hybrid method for online assessment and performance prediction of RUL in drilling operations based on vibration signals. Logistic regression analysis combined with maximum likelihood technique is employed to evaluate tool wear condition based on features extracted from vibration signals using wavelet packet decomposition technique. Auto regressive moving average model is then applied to predict RUL.

2.2 Degradation Modeling

In Fig.1.1 in Chapter 1, the writer have already introduced the framework of degradation modeling based on RUL estimation. As discussed before, the topic of this thesis would be narrowed down to the first category of heterogeneity (unit-to-unit variability). Variability in the inner structures of the considered systems and the diversity in their working conditions would both contribute to unit-to-unit variability in performed degradation. So it is very easy to think of including random variables to capture such variability, and leaving the rest of parameters as constants describing the homogeneity in system degradation from the same type when modeling degradation process and estimate RUL. So, in this section, the writer would briefly review both two subcategories - random coefficients regression model and stochastic process with random effects. The writer only considers Brownian motion and Gamma processes in stochastic process sections as a study preparation for a further investigation in the following chapters.

2.2.1 Random coefficient regression Models

Lu and Meeker (1993) first described a general nonlinear regression model to characterize the degradation of a population of units as

$$Y(t) = D(t;\phi,\theta) + \varepsilon(t) \tag{2.1}$$

where $D(t;\phi,\theta)$ is the actual degradation path at time t, ϕ is the fixed regression coefficients for all units, θ is the random effect for individual unit, and $\varepsilon(t)$ is the random noise described by $N(0,\sigma_\varepsilon)$. Notice that θ and ε are usually assumed to be independent of each other. With this model, the RUL can be defined as

$$\text{RUL}_{t_i} = \{h_{t_i} : D(t_i + h_{t_i};\phi,\theta) \geq L | D(t_i;\phi,\theta) < L\} \tag{2.2}$$

where L is a predefined threshold, h_{t_i} is the residual useful lifetime given the current component status at time t_i. Notice that the underlying assumptions of this model include (Wang et al. (2000)):

1. the condition of the component deteriorates with operating time and the level of degra-

dation can be observed at any time;

2. the device being monitored comes from a population of devices, each of which exhibits the same degradation form;

3. the distribution of the random variable across the population of components is known.

Based on these assumptions, researchers have extended and developed the model presented by Lu and Meeker (1993) by incorporating some differences. For example, Gebraeel et al. (2005) develop a Bayesian updating methods to construct a closed-form residual-life distribution for a monitored device using sensor based monitoring signals. The focus on only single operating device is quite different from methods of computing RUL for a population of components. Some papers also replace Brownian motion for normally distributed noise variables in the degradation model, which enhance the dynamics of the original regression models.

The random deterioration path model is a very simple model to study, and it is directly related to statistical analysis of deterioration data. However, there are several limitations with this model. First, The fundamental assumption of random deterioration path models about the sample space and sample function of the deterioration process is restrictive when the patterns of some sample deterioration paths are inconsistent with the others due to slight or intensive variations in the environment that an individual asset operates. Also, the assumptions of the the common use of an independent and identical normal noise term are quite restrictive in capturing the temporal uncertainty. Another problem is that one single inspection is sufficient to determine the sample path in the linear random variable degradation and thus having no first hitting time (FHT) motivation. So it cannot model the temporal variability in RUL estimation as argued by Pandey and Yuan (2006).

2.2.2 Brownian Motion

Continuous-time Markov process with independent increments such as Brownian motion with drift is also termed as the Gaussian or Wiener process. It is a continuous-time stochastic process with drift parameter μ and variance parameter σ. The Brownian motion model has an additive

effect on degradation process and can be expressed as follows,

$$X(t) = X(0) + \mu t + \sigma B(t) \tag{2.3}$$

where $X(0)$ is the initial degradation value, μt is the trend and σ^2 is constant diffusion parameter. With this model, the RUL can be defined as

$$\text{RUL}_{t_i} = \inf\{h_{t_i} : Y(t_i + h_{t_i}) \geq L | Y(t_i) < L\} \tag{2.4}$$

The PDF of the FHT of the Brownian motion is the inverse Gaussian distribution. So explicit equations for calculation is available (more discussion in Chapter 3).

Improvements have been made for the application of Brownian motion model in both reliability engineering and biostatistics in relation to estimating the lifetime. For instance, Tseng and Peng (2004) remedies the weakness of Markov property in Wiener process and proposes an integrated Wiener process to model the cumulative degradation path of the product's quality characteristic. Peng and Tseng (2009) incorporates the random effect of a drift coefficient and measurement errors into a Wiener process-based degradation process for lifetime analysis.

However, the limitation of Brownian motion deserves a few comments. First, the property of either increasing or decreasing in the context of reliability is quite restrictive, and thus inadequate in modeling degradation which is monotone. Second, a Brownian motion is a time homogeneous process, but it may not apply to all degradation processes, like fatigue crack which the length grows faster during the course of crack propagation and thus producing time heterogeneity. Additionally, the variance of the noise term in the Brownian motion is proportional to the time interval, which is a strong requirement few state processes can satisfied.

2.2.3 Gamma Process

Since degradation processes are generally monotonic, it is a nature choice to model them by Gamma process, where degradation occurs gradually over time in a sequence of tiny positive increments. With this model, RUL can be can be defined in a similar way as in the previous

sections,

$$\mathrm{RUL}_{t_i} = \{h_{t_i} : Y(t_i + h_{t_i}) \geq L | Y(t_i) < L\} \tag{2.5}$$

Notice that, the sum of Gamma distributed increments is still a Gamma process. Obviously, the calculation is straightforward and can be figured out using the properties of Gamma process (see more discussion in Chapter 4). It is very useful in providing optimal maintenance decisions. Some extensions were for example Lawless and Crowder (2004) constructs a tractable gamma process model incorporating a random effect to characterize different degradation rates among individual components. Kuniewski et al. (2009) presents a method to model degradation process which is a combination of two stochastic processes, namely the process of defect initiation (non-homogeneous Poisson process) and the process of defect growth (Gamma process). In a word, Gamma process is chosen for its relatively straightforward calculation and ability to capture temporal variability.

But it should be remembered that Gamma process appears only appropriate to model degradation by a strictly monotonic process. Also, it is a challenge to reasonably choose random parameters and their distributions which both capture the unit-to-unit variability and simplify RUL calculation. Given these reasons, some modified Gamma processes like semi-parametric gamma process should be further studied to improve their modeling ability.

Chapter 3

Brownian Motion Model

Brownian motion is a continuous stochastic process that is widely used for modeling random behavior evolves over time. Today, the process and its many generalizations and extensions occur in diverse areas of both pure and applied science, such as mathematical statistics, management science,economics, communication theory and biology etc.

The purpose of this section is to review the some mathematical properties and understand the advantages of Brownian motion. The merits could possibly be extended to the understanding and application of other Markov stochastic processes, such as Gamma process. This chapter would first present some mathematical properties of standard Brownian motion (i.e. Wiener process), and then move to Brownian motion with drift. Besides, methods for Brownian motion path simulation, parameter estimation and RUL distribution estimation are presented in the following sections.

3.1 Standard Brownian Motion - Wiener Process

3.1.1 Introduction and Definition

Consider a symmetric random walk, which in each time unit is equally likely to take a unit step either up or down . If this process is speed up by taking smaller and smaller steps in smaller and smaller time intervals, we would obtain Brownian motion once we go the limit in the right manner.

For ease, we start with the assumption $X(0) = 0$. We also assume that there is no "drift" to the process, i.e. $\mathbb{E}(X(t)) = 0$. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed (i.i.d) random variables such that

$$\begin{aligned} \mathbb{E}(\xi_n) &= 0, \\ \mathrm{Var}(\xi_n) = (\xi_n^2) &= 1, \end{aligned} \tag{3.1}$$

Notice that ξ_n satisfies

$$\xi_n = \begin{cases} +1, & \text{if the } i\text{th step of length } \xi_n \text{ is up} \\ -1, & \text{if it is down} \end{cases} \tag{3.2}$$

Then $X(n)$ is the instantaneous position of a random walk on the integers $\mathbb{Z}$. It can be viewed as a function of the discrete time n as follows

$$\begin{aligned} X(0) &= 0, \\ X(n) &= \sum_{i=1}^{n} \xi_i, \end{aligned} \tag{3.3}$$

We will then rescale both time and space in order to construct a random function on $t \in [0, +\infty)$ and taking values in $\mathbb{R}$. Recall that the Central Limit Theorem asserts that

$$\frac{X(N)}{\sqrt{N}} \to N(0,1) \tag{3.4}$$

in distribution as $N \to \infty$. This suggest to rescale $X(n)$ and construct a constant random function $X^N(t)$ on $t \in [0, +\infty)$ by letting

$$X^N(t) = \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}} \tag{3.5}$$

where $\lfloor Nt \rfloor$ represents the largest integer less than Nt.

The function is illustrated in Fig. 3.1. for different resolution N. From the figure, we can see that when N increases, the discrete stair process will approach a continuous-time and space process. By the Central Limit Theorem, the distribution of $X^N(1) = \frac{S_N}{\sqrt{N}}$ approaches a normal

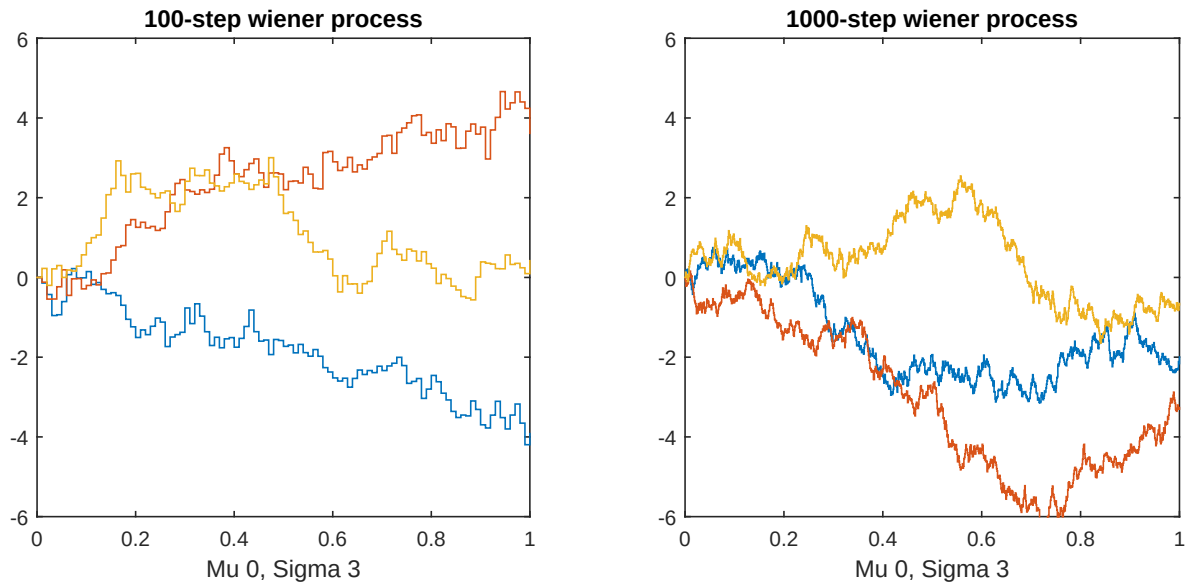

Figure 3.1: Realizations of $X^N(t)$ (i.e. $\mathbb{E}(X(t)) = 0$) for $N = 100$(Left), $N = 1000$(Right).

distribution with mean 0 and variance 1. Similarly, the distribution of $X^N(t)$ approaches a normal distribution with mean 0 and variance t. It can be shown that as $N \to \infty, X^N(t)$ converges in distribution to Brownian motion, that is $X^N \xrightarrow{d} X$, where $X(\cdot)$ is the Brownian motion. d denotes that the random process on both sides of the equality have the same distribution. The proof of such theorem is not presented here and we just study its properties in the following sections.

Recall Eq.3.4, we can construct Eq.3.5 like this

$$\begin{aligned} X(t) = X^N(t) &= \frac{S\lfloor Nt \rfloor}{\sqrt{N}} \\ &= \frac{S\lfloor Nt \rfloor}{\sqrt{N}} \frac{\sqrt{\lfloor Nt \rfloor}}{\sqrt{n}} \\ &\xrightarrow{d} \sqrt{t} N(0,1) \overset{d}{=} N(0,t) \end{aligned} \tag{3.6}$$

Besides, it is easy to prove that increments of $X(t)$ are normally distributed if we take each Δt time unit from the above equation. Standard Brownian motion is a Brownian motion with $\mu = 0, \sigma^2 = 1$. We can also speak of a general Brownian motion with $\mu \neq 0, \sigma^2 \neq 1$ based on the same idea. The expressions can be referred to Summary.

In a word, the Brownian motion $X(t)$ is essentially the accumulation of a series of normally distributed random variables. Between each time interval, the increment of the motion are nor-

mally distributed with mean equals 0 and variance equals $\sigma^2\Delta t$. As time goes, the variances of these normally distributed random variables increase meaning that it is more uncertain to predict the value of the process after a longer period.

Summary

For $\Delta X = X(t+\Delta t) - X(t) \sim \sigma\sqrt{\Delta t}N(0,1)$, we have

$$\begin{cases} \mathbb{E}(\Delta X) &= 0 \\ \text{Var}(\Delta X) &= \sigma^2\Delta t \end{cases} \tag{3.7}$$

$$\begin{cases} \mathbb{E}(X(t)) &= \mathbb{E}(X(t) - X(0)) = 0 \\ \text{Var}(X(t)) &= \text{Var(X(t) - X(0))} = n\cdot\sigma^2\Delta t = \sigma^2 t \end{cases} \tag{3.8}$$

Notice here the variance is additive because any pair of $\Delta X(t)_i$ are assumed to be independent. As $n \to \infty$, Δt converges to 0 and is normally denoted as dt, which means an infinitesimal time interval. Correspondingly, ΔX is re-denoted as dX. However, it is not very precise in mathematical sense to denote dt and dX in this way here. We mention it as a notation just to simplify the process of turning discrete stochastic process to a series of continuous normal distributions, i.e., a Brownian motion.

3.1.2 Properties of Standard Brownian Motion

When we speak of a Brownian motion with $\sigma^2 = 1$, and $X(0) = 0$, it turns to be a standard Brownian motion; otherwise, it is a general Brownian motion. Therefore, a standard Brownian motion $X(t)$ is a stochastic process with the following properties (Lawler (2006)).

1. **Initial condition** $X(0) = 0$, the process start at 0;

2. **Independence of increments** For any $s_1 \le t_1 \le s_2 \le t_2 \le \cdots \le s_n \le t_n$, the random variables $X_{t_1} - X_{s_1}, \ldots, X_{t_n} - X_{s_n}$ are independent;

3. **Normal increments** For any $s < t$, the random variable $X_t - X_s$ has a normal distribution with mean 0 and variance $t - s$. If we make $s = 0$, then $X(t) - X(0)$ has $N(0,t)$ distribution;

4. **Continuity of paths** $X(t), t \geq 0$ are continuous functions of t.

3.1.3 Properties of Standard Brownian Path

Given stochastic differential equations (SDEs)

$$dy(t) = a[y(t), t]dt + b[y(t), t]dx(t) \tag{3.9}$$

a general representation of Brownian sample path can be constructed as follows,

$$dy(t) = \mu dt + \sigma dx(t) \tag{3.10}$$

where $dx(t)$ is a standard Brownian motion (i.e. Wiener process). With initial condition $X(0) = 0$, scale parameter $\mu = 0$ and shape parameter $\sigma = 1$, Eq.3.10 describes the evolution of a standard Brownian motion with mean $\mu t = 0$ and variance $\sigma^2 t = t$. So the standard Brownian motion path can also be written as

$$dy(t) = dx(t) \tag{3.11}$$

$$Y(t) = X(t) = X(0) + \mu t + \sigma(X(t) - X(0)) \tag{3.12}$$

The density function of $Y(t)$ given that $Y(0) = 0$ is the transition probability density function of Brownian motion. Let $p_t(x, y)$ denote the transition densities from x to y,

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(y-x)^2}{2t}} \tag{3.13}$$

According to Klebaner et al. (2005), $X(t)$ as functions of t have the following properties. Almost every sample path $X(t), 0 \leq t \leq T$

1. is a continuous function of t;

2. is not monotone in any interval, no matter how small the interval is;

3. is not differentiable at any point;

4. has infinite variation on any interval, no matter how small it is;

5. has quadratic variation on $[0,t]$ equal to t, for any t.

The properties are not proved here because they can be found in many books on stochastic processes, like Lawler (2006) and Klebaner et al. (2005).

Summary of Standard Brownian Motion

For $\Delta X = X(t+\Delta t) - X(t) \sim \sqrt{\Delta t}N(0,1)$, we have

$$\begin{cases} \mathbb{E}(\Delta X) &= 0 \\ \text{Var}(\Delta X) &= \Delta t \end{cases} \tag{3.14}$$

$$\begin{cases} \mathbb{E}(X(t)) &= \mathbb{E}(X(t) - X(0)) = 0 \\ \text{Var}(X(t)) &= \text{Var(X(t) - X(0))} = n \cdot \Delta t = t \end{cases} \tag{3.15}$$

3.2 Brownian Motion with Linear Drift

3.2.1 Properties of Brownian Motion with Linear Drift

Similar to what have been presented in Section 3.1.2, Brownian motion with Linear drift $Y(t)$ possesses the following properties

1. **Initial condition** $Y(0) = x$, the process start at x;

2. **Independence of increments** For any $s_1 \le t_1 \le s_2 \le t_2 \le \cdots \le s_n \le t_n$, the random variables $Y_{t_1} - Y_{s_1}, \ldots, Y_{t_n} - Y_{s_n}$ are independent;

3. **Normal increments** For any $s < t$, the random variable $Y_t - Y_s$ has a normal distribution with mean $(t-s)\mu$ and variance $(t-s)\sigma^2$. If we make $s = 0$, then $X(t) - X(0)$ has $N(\mu t, \sigma^2 t)$ distribution;

4. **Continuity of paths** $Y(t), t \ge 0$ are continuous functions of t.

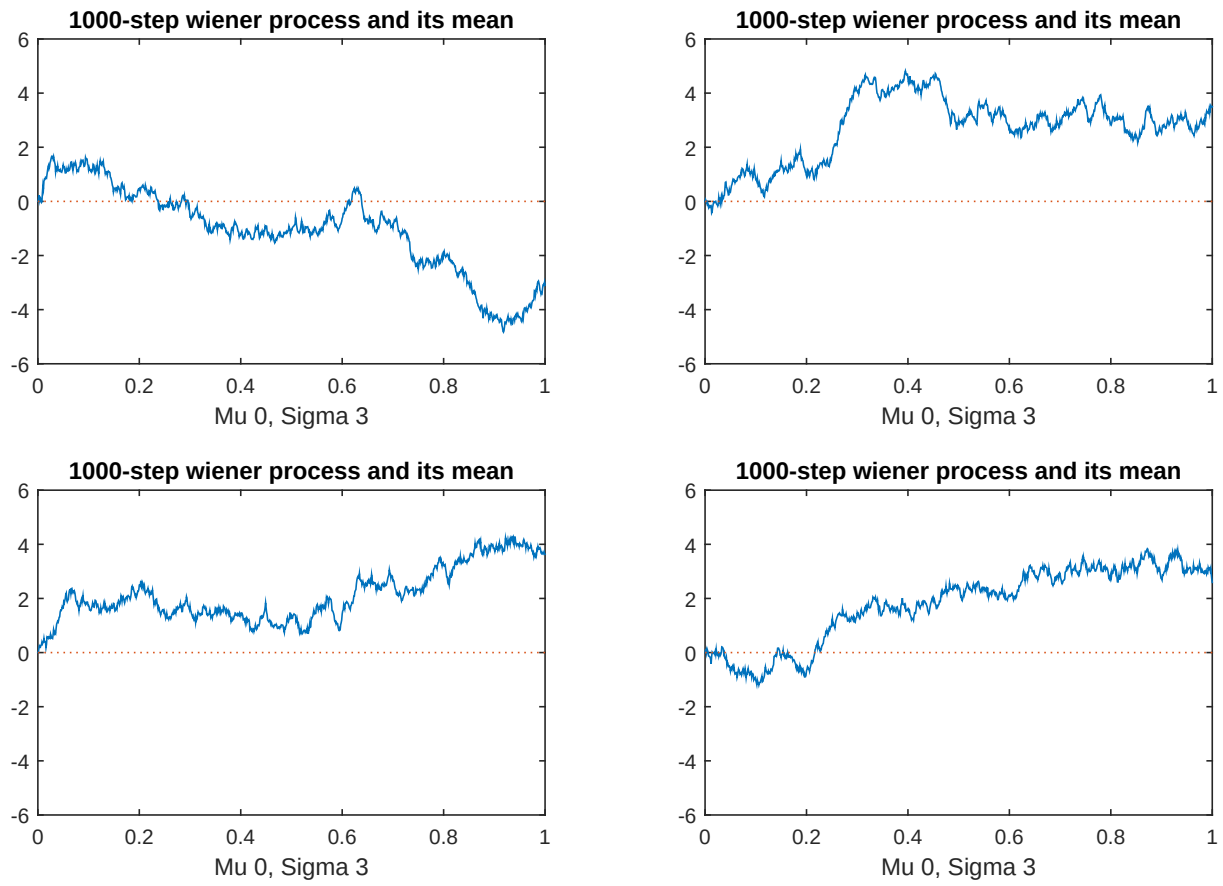

Figure 3.2: Four realizations of standard Brownian paths.

Refer to Eq.3.10, the Brownian motion path with drift is

$$dy(t) = \mu dt + \sigma dx(t) \tag{3.16}$$

where initial condition $Y(0) = x$, and it describes the path of a Brownian motion with mean μt and variance $\sigma^2 t$. Therefore, it can also be expresses as

$$Y(t) = x + \mu t + X(t) \tag{3.17}$$

where $X(t)$ is a zero drift Brownian motion with variance σ^2 starting at 0.

Notice that the motion $Y(t)$ consists of a linear motion in the direction μ with random fluctuations related to σ.

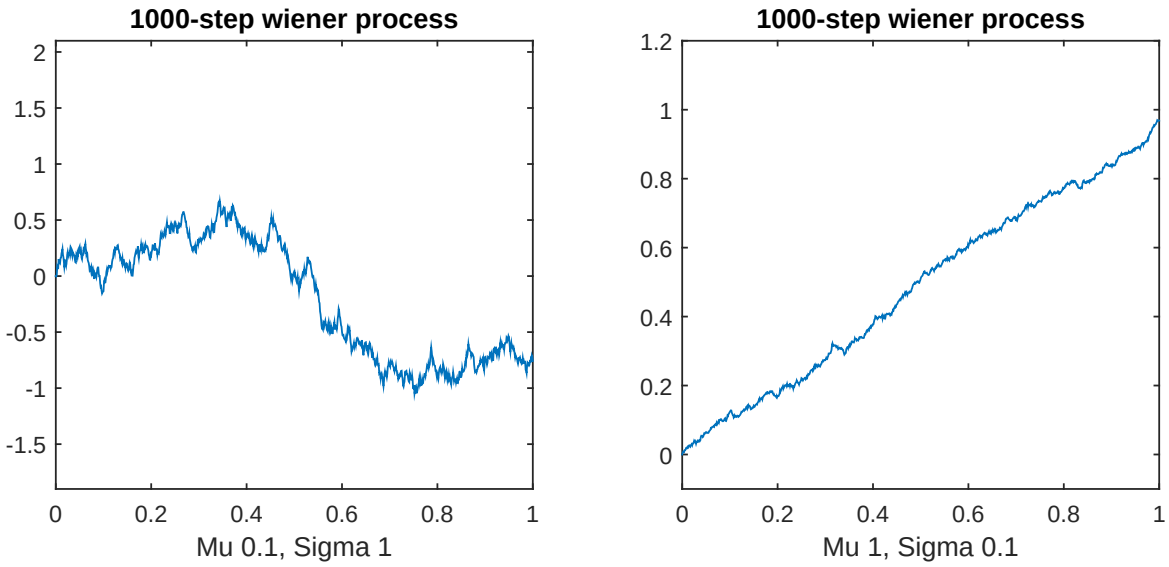

Figure 3.3: Realizations of different μ, σ in Brownian paths.

The unconditional probability density function at a fixed time t given $Y(0) = x$ is

$$p_t(x,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{\frac{-|y-x-\mu t|^2}{2\sigma^2 t}} \tag{3.18}$$

Summary of Brownian Motion with Linear Drift

For $\Delta Y = Y(t+\Delta t) - Y(t) \sim \sqrt{\Delta t} N(\mu, \sigma^2)$, we have

$$\begin{cases} \mathbb{E}(\Delta Y) &= \mu \Delta t \\ \text{Var}(\Delta Y) &= \sigma^2 \Delta t \end{cases} \tag{3.19}$$

$$\begin{cases} \mathbb{E}(Y(t)) &= \mathbb{E}(Y(t) - Y(0)) = x + \mu t \\ \text{Var}(Y(t)) &= \text{Var(Y(t) - Y(0))} = n \cdot \sigma^2 \Delta t = \sigma^2 t \end{cases} \tag{3.20}$$

3.2.2 Some Graphs and Discussions

Graphs of some examples of Brownian motion are given in order to visualize the processes and the equations presented in the Summary.

Example 1. Randomness Suppose we simulate a Brownian motion process by Matlab with mean $\mu = 0$ and variance $\sigma = 0.1$. The overall time is 1 and there are 1000 steps. Initial condition

is $Y(0) = x = 0$. Four realizations of such Brownian motion are shown in Fig.3.2. We set $\mathbb{E}(Y(t)) = 0$ which represents a process with no drift (i.e. zero mean) and governed by pure randomness (White Noise). However, it appears that the paths has regions where motions looks like they has "trends".

Example 2. Drift and Noise Suppose we simulate two Brownian motion paths with initial condition $Y(0) = x = 0$. The overall time scale is 1 and there are 1000 steps. We suppose the first path is with parameter $\mu = 0.1, \sigma = 1$ (See Fig.3.3, left), while the second path is with $\mu = 1, \sigma = 0.1$(See Fig.3.3, right). These figures demonstrate that when scale parameter μ is small in comparison with shape parameter σ, drift has a greater impact on the Brownian process; if shape parameter σ is small in comparison with scale parameter μ, then noise dominates in the behaviour of the Brownian process. Recall from Eq. 3.17, $X(t)$ can be considered as "Pure Noise", while $(x + \mu t)$ is the "Actual Degradation Path" of interest.

Example 3. Effect of μ and σ Suppose we simulate four Brownian motion paths with initial condition $Y(0) = x = 0$. The overall time scale is 1 and there are 1000 steps. We first keep the value of $\sigma = 1$ in all four realizations and only change μ to see its impact on process. The paths are with parameters $\mu = 1, 2, 3, 4$. Fig. 3.4 shows 200 realizations and their means (blue lines) in each graph. Recall Eq.3.20 and we can see from the figures that the slope $\mathbb{E}(Y(t))$ increases when μ increases. It is not clear from the figure whether the processes are more concentrated, spread out or just keep the same, but from Eq. 3.20, the variance would not be affected by changes in μ and keep all the same.

Then, we keep the value of $\mu = 1$ in all four realizations and only change σ to see its impact on process. The paths are with parameters $\mu = 1, 2, 3, 4$. Fig. 3.5 shows 200 realizations and their means(blue lines) in each graph. Recall Eq.3.20 and we can see from the figures that the slope $(\mathbb{E}(Y(t)))$ keeps unchanged when μ increases. But it is very obvious from the figure that the processes are more spread out. It demonstrates Eq. 3.20 that the variance of process increases when σ increases.

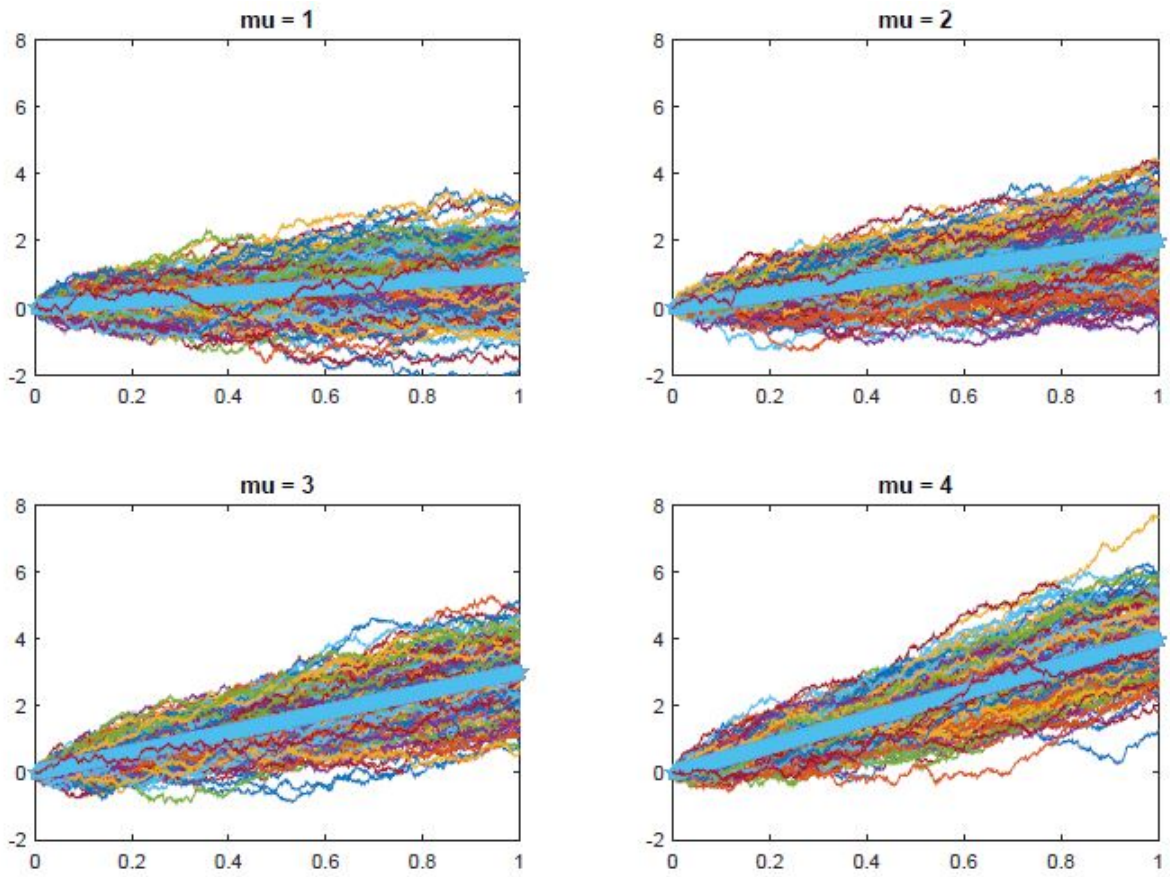

Figure 3.4: Four groups of realizations with increasing μ.

3.2.3 First Hitting Time and RUL Distribution

One of the purposes of studying degradation model is to estimate the residual useful lifetime and its distribution function. If previous data before a monitoring time t are available, how much time would the component still have to be useful? The first hitting time is one of the challenges when we transfer this situation into Brownian motion models. From the previous definitions and figures, we can easily find out that Brownian motion is a non-monotone process. Given a level L, the time of hitting this level could be more than once due to the random nature of Brownian motion process. The following contents provide the general computation of a first passage time hits level L, hits time T_L and RUL distribution.

Hitting Level L Suppose T_L is the first time Brownian motion $Y(t)$ hits level L. Let

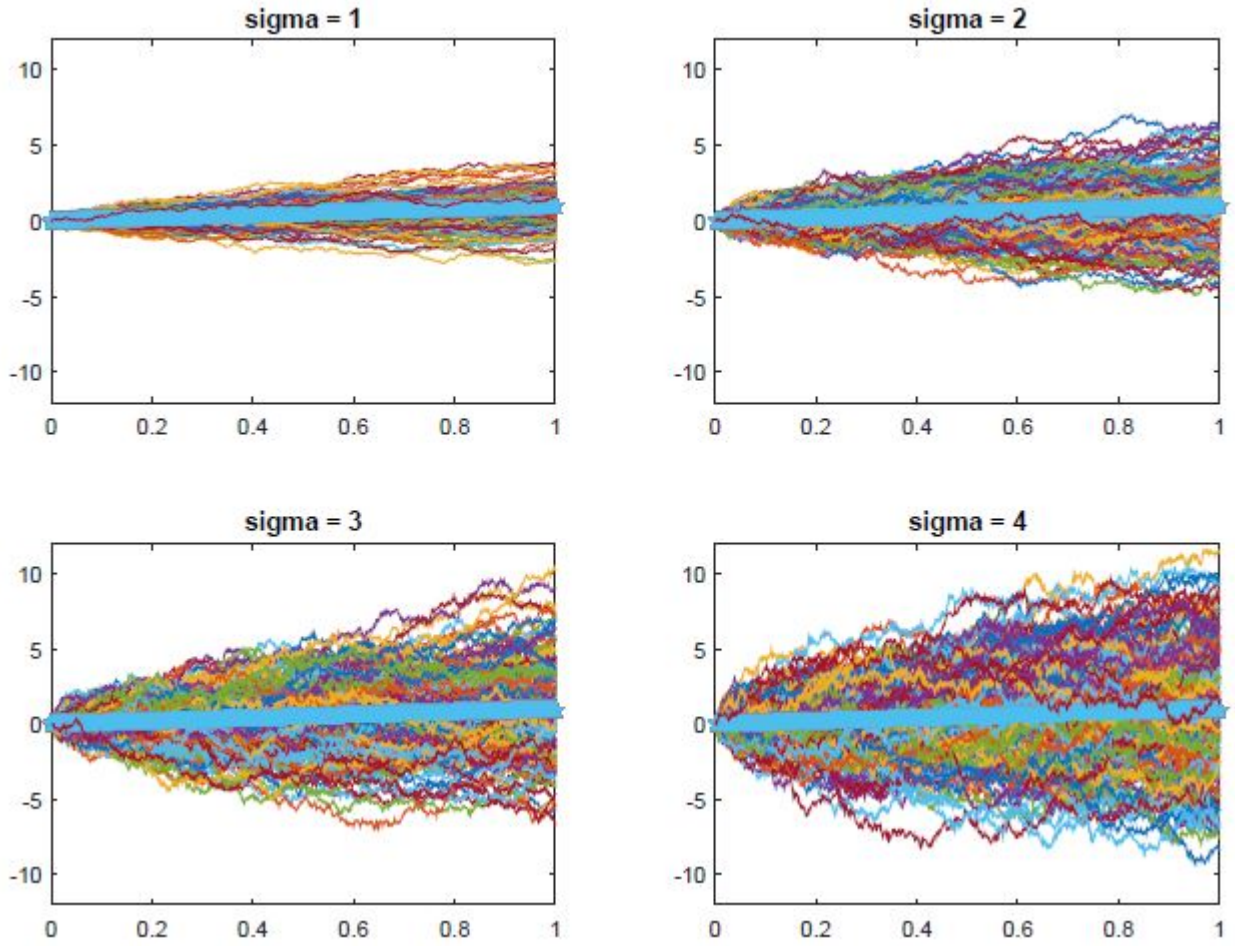

Figure 3.5: Four groups of realizations with increasing σ.

$$M(t) = \max_{0 \le s \le t} Y(s) \tag{3.21}$$

$$T_L = \inf\{t > 0 : Y(t) \ge L\} \tag{3.22}$$

If the maximum at time t is greater than L, then the Brownian motion took value L at some time before t. Meanwhile, if Brownian motion took value L at a certain time before t, then the maximum value will be at least L. In mathematical form, the above argument could be written as follows,

$$\{Y(t) \ge L\} \subset \{T_L \le t\} \tag{3.23}$$

Therefore, it is nature to say that

$$\mathbb{P}(Y(t) \geq L) = \mathbb{P}(Y(t) \geq L, T_L \leq t) \tag{3.24}$$

Since $Y(T_L) = L$,

$$\mathbb{P}(Y(t) \geq L) = \mathbb{P}(T_L \leq t, Y(T_L + (t - T_L)) - Y(T_L) \geq 0) \tag{3.25}$$

By Theorem B.1 in appendix, T_L is a finite stopping time, and by the strong Markov property (B.2), the random variable $\hat{Y}(s) = Y(T_L + s) - Y(T_L)$ is independent of $\mathscr{F}_{T_L}$ and has a Normal distribution, so

$$\mathbb{P}(Y(t) \geq L) = \mathbb{P}(T_L \leq t, \hat{Y}(t - T_L) \geq 0) \tag{3.26}$$

It is easy to show that $T_L \leq t$ and $M(t) \geq L$ are two equal events. If s is independent of T_L, then

$$\begin{aligned} \mathbb{P}(T_L \leq t, \hat{Y}(s) \geq 0) &= \mathbb{P}(T_L \leq t)\mathbb{P}(\hat{Y}(s \geq 0)) \\ &= \mathbb{P}(T_L \leq t)\frac{1}{2} \\ &= \mathbb{P}(M(t) \geq L)\frac{1}{2} \end{aligned} \tag{3.27}$$

That is to say for any $L > 0$,

$$\begin{aligned} \mathbb{P}_x(M(t) \geq L) &= 2\mathbb{P}_0(Y(t) \geq L) \\ &= 2\left(1 - \Phi\left(\frac{L - x - \mu t}{\sigma\sqrt{t}}\right)\right) \end{aligned} \tag{3.28}$$

where $\mathbb{P}_x$ denotes the probability of events when process starts at x, $\Phi(\cdot)$ stands for the standard Normal distribution function. The last part of equation can be explained by central limit theorem. But in Eq.3.26 $s = t - T_L$, and is clearly dependent on T_L. It is not easy to transfer Eq.3.26 to Eq. 3.28. Detail proof is not discussed here, but can be found in other books.

First Passage Time T_L To derive the distribution of first passage time T_L, we should first mention inverse Gaussian distribution (IG). It is a two-parameter continuous distribution given by its density function

$$f(x;\mu,\lambda) = \left[\frac{\lambda}{2\pi x^3}\right]^{1/2} \exp\frac{-\lambda(x-\mu)^2}{2\mu^2 x} \tag{3.29}$$

for $x > 0$, where $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter. For a random variable X with inverse Gaussian distribution we write $X \sim IG(\mu,\lambda)$.

According to Pärna (2014), the inverse Gaussian distribution describes the distribution of the time a Brownian motion with positive drift takes to reach a given positive level. As what has been denoted before, let T_L be the first passage time for a fixed level $L > 0$ by Y_t. Then T_L has inverse Gaussian distribution, $T_L \sim IG(\frac{L}{\mu}, \frac{L^2}{\sigma^2})$. So put these parameters into Eq.3.29, we have probability density function

$$f(x;\mu,\sigma) = \frac{L}{\sqrt{2\pi\mu^3 x^2}} \exp\left(-\frac{(L-\mu x)^2}{2\sigma^2 x}\right) \tag{3.30}$$

Therefore, the first passage time T_L satisfies the following function

$$F(x;\mu,\sigma) = \mathbb{P}(T_L \le t) = \int_0^t \frac{L}{\sqrt{2\pi\mu^3 x^2}} \exp\left(-\frac{(L-\mu x)^2}{2\sigma^2 x}\right) dx \tag{3.31}$$

RUL Distribution Given Eq.3.30, it is easy to obtain residual useful liftime distribution. Observing the process at time t is at position $y(t)$, the probability that residual useful lifetime is less than a predefined period h is

$$\mathbb{P}(RUL_{y(t)} \le h) = \int_0^h \frac{L-y(t)}{\sqrt{2\pi\mu^3 x^2}} \exp\left(-\frac{(L-y(t)-\mu x)^2}{2\sigma^2 x}\right) dx \tag{3.32}$$

3.3 Philosophy of RUL Estimation

The purpose of this section to estimate remaining useful lifetime (RUL) using Brownian motion process to model the degradation. First, assumptions and an example training data set are presented. Then, the parameters in the degradation model are estimated with the training data set. And finally the RUL distribution is estimated proposed for the units of testing data.

3.3.1 Assumptions and Training Data Set

In this example, the Brownian motion process is used for degradation modeling corresponding to training data set. Assume that Y is an degradation indicator for $N = 100$ independent identical tested items, and $Y_{i,j}$ denotes the degradation measurements of the ith items at time j, where $i = 1,2,\ldots N$ and $j = 1,2,\ldots r$, r is the last observation time. All the degradation realizations are based on homogeneous Brownian motion with drift with two parameters (μ,σ), and that are the same for all the items. According to Eq.3.19, each increment of degradation $\Delta Y_{i,j} = Y_{i,j+1} - Y_{i,j}$ of each items follows a normal distribution $N(\mu\Delta t_{i,j},\sigma^2\Delta t_{i,j})$, where $\mu = 6,\sigma = 4$. There are 1000 observations within the time frame $T = 1$, and the initial condition is $Y(0) = x = 1$. Let the time step equals $1/1000$. The degradation paths for all $N = 100$ units and certain units are illustrated in Fig.3.6. The following parameter estimation, process verification and RUL distribution are all based on these built testing dataset.

3.3.2 Parameter Estimation

Given degradation path Y_{ij} based on Brownian motion process, each increment is $\Delta Y_{i,j} = Y_{i,j+1} - Y_{i,j} \sim N(\mu\Delta t_{i,j},\sigma^2\Delta t_{i,j})$ is normally distributed for all identically and independent components. Recall the transition density function of Brownian motion process in Eq. 3.18, then

$$f_{(\mu\Delta t_{i,j},\sigma^2\Delta t_{i,j})}(\Delta Y_{i,j}) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t_{i,j}}} e^{-\frac{(\Delta Y_{i,j}-\mu\Delta t_{i,j})^2}{2\sigma^2\Delta t_{i,j}}} \tag{3.33}$$

Since Y is Markovian, the maximum likelihood estimator (MLE) of $\theta = (\mu,\sigma)$ can be calculated once transition density function of Y are known. Consider the degradation measurements for item i, $\Delta Y_i = (\Delta Y_{i,1},\Delta Y_{i,2},\ldots,\Delta Y_{i,r})$. For item i, the likelihood function for item i is

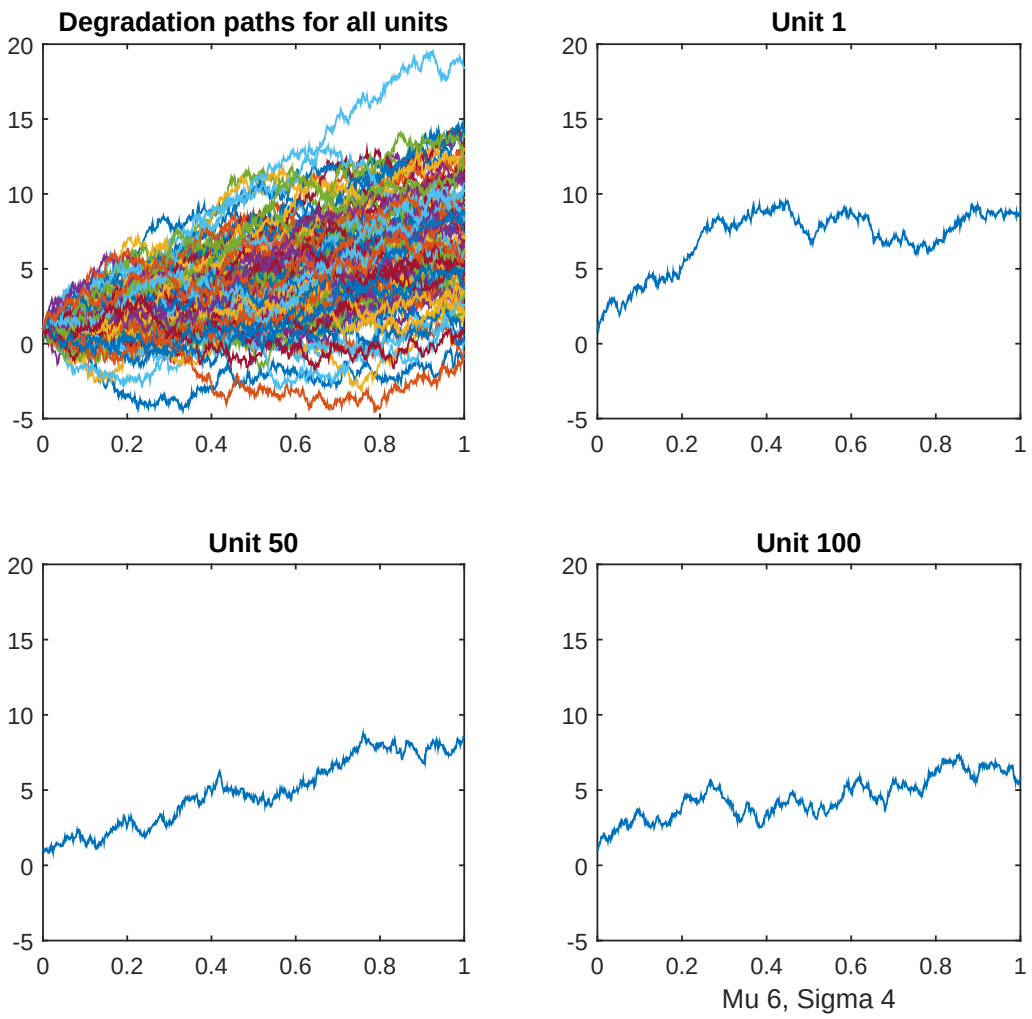

Figure 3.6: Degradation paths of all units and certain units.

$$
\begin{aligned}
L_i(\theta) &= f_i(\Delta Y_i) = f_i(\Delta Y_{i,1}, \Delta Y_{i,2}, \ldots, \Delta Y_{i,r} | \mu, \sigma) \\
&= \prod_{j=1}^{r} f_i(\Delta Y_{i,j} | \mu, \sigma) \\
&= \prod_{j=1}^{r} \frac{1}{\sqrt{2\pi\sigma^2 \Delta t_{i,j}}} e^{-\frac{(\Delta Y_{i,j} - \mu \Delta t_{i,j})^2}{2\sigma^2 \Delta t_{i,j}}}
\end{aligned} \tag{3.34}
$$

Then for the ith item, the log-likelihood is given by

$$l_i(\theta) = \ln L_i(\theta) = \ln\left(\prod_{j=1}^{r} \frac{1}{\sqrt{2\pi\sigma^2\Delta t_{i,j}}} e^{-\frac{(\Delta Y_{i,j}-\mu\Delta t_{i,j})^2}{2\sigma^2\Delta t_{i,j}}}\right) \tag{3.35}$$

Since the measurements $Y_{i,j}$ are independent

$$\begin{aligned} l(\theta) &= \ln(\Delta Y_1, \Delta Y_2, \ldots, \Delta Y_N) \\ &= \sum_{i=1}^{N} \ln(f_i(\Delta Y_{i,1}, \Delta Y_{i,2}, \ldots, \Delta Y_{i,r})) \\ &= \sum_{i=1}^{N} \ln\left(\prod_{j=1}^{r} \frac{1}{\sqrt{2\pi\sigma^2\Delta t_{i,j}}} e^{-\frac{(\Delta Y_{i,j}-\mu\Delta t_{i,j})^2}{2\sigma^2\Delta t_{i,j}}}\right) \end{aligned} \tag{3.36}$$

that is to say $l(\theta) = \sum_{i=1}^{N} l_i(\theta)$, where $f_i / l_i(\theta)$ is the probability density function /log-likelihood of increments corresponding to each item, and $f / l(\theta)$ is the probability density function /log-likelihood of all increments.

The maximum likelihood estimator $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$ are found by maximizing $l(\theta)$. In practice it is done by taking the partial derivative of the log-likelihood function of Eq. 3.36 with respect to μ and σ. This gives equations

$$\frac{\partial l(\theta)}{\partial \mu} = \sum_{i=1}^{N}\sum_{j=1}^{r} \frac{\Delta Y_{i,j} - \mu\Delta t_{i,j}}{\sigma^2} = 0 \tag{3.37}$$

$$\frac{\partial l(\theta)}{\partial \sigma} = -\frac{rN}{\sigma} + \sum_{i=1}^{N}\sum_{j=1}^{r} \frac{(\Delta Y_{i,j} - \mu\Delta t_{i,j})^2}{\sigma^3\Delta t_{i,j}} = 0 \tag{3.38}$$

So the maximum likelihood estimator for $\theta = (\mu, \sigma)$ is

$$\hat{\mu} = \frac{\sum_{i=1}^{N}\sum_{j=1}^{r} \Delta Y_{i,j}}{\sum_{i=1}^{N}\sum_{j=1}^{r} \Delta t_{i,j}} \tag{3.39}$$

$$\hat{\sigma} = \sqrt{\frac{1}{rN}\sum_{i=1}^{N}\sum_{j=1}^{r} \frac{(\Delta Y_{i,j} - \mu\Delta t_{i,j})^2}{\Delta t_{i,j}}} \tag{3.40}$$

Eq. 3.40 can be substituted and solved by Eq. 3.39.

To illustrate the process, the testing dataset was analyzed using MATLAB (see code in Appendix). The estimated parameters corresponding to the testing dataset are obtained as follows:

$$\hat{\mu} = 5.9923, \hat{\sigma} = 4.0052$$

3.3.3 Likelihood Confidence Interval and Likelihood Test

Likelihood Confidence Interval The parameter estimation are based on the maximization of an approximation of the likelihood function. Thus, the obtained MLE of $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$ is asymptotically normally distributed (approximated unbiased), with a standard deviation which can be estimated by the inverse of expected Fisher information matrix. According to Lindqvist (2013), we consider the observed Fisher information in place of the expected Fisher information, since the first one if unkown and the latter one makes little difference numerically. In the case of two parameters μ and σ we defined observed information matrix to be

$$I(\hat{\mu}, \hat{\sigma}) = def \left[\begin{array}{cc} -\frac{\partial^2 l(\mu,\sigma)}{\partial \mu^2} & -\frac{\partial^2 l(\mu,\sigma)}{\partial \mu \partial \sigma} \\ -\frac{\partial^2 l(\mu,\sigma)}{\partial \sigma \partial \mu} & -\frac{\partial^2 l(\mu,\sigma)}{\partial \sigma^2} \end{array} \right]_{\mu=\hat{\mu},\sigma=\hat{\sigma}} \tag{3.41}$$

It is also called Hessian matrix, which is the matrix of second derivatives of $l(\theta) = l(\mu, \sigma)$ with respect to μ, σ. The theory of maximum likelihood also shows that

$$I(\hat{\mu}, \hat{\sigma})^{-1} = def \left[\begin{array}{cc} \widehat{Var(\hat{\mu})} & \widehat{Cov(\hat{\mu}, \hat{\sigma})} \\ \widehat{Cov(\hat{\sigma}, \hat{\mu})} & \widehat{Var(\hat{\sigma})} \end{array} \right] \tag{3.42}$$

which means that by inverting the observed information matrix we get a matrix with estimated variances of the parameters on the diagonal. We furthermore get estimated covariances outside the diagonal. These are used for computation of estimated variances of functions of both μ and σ. Therefore, the Hessian matrix and its inverse for testing dataset are calculated by MATLAB as

$$I(\hat{\mu}, \hat{\sigma}) = def \left[\begin{array}{cc} \sum_{i=1}^{N} \sum_{j=1}^{r} \frac{\Delta t_{i,j}}{\sigma^2} & 2\sum_{i=1}^{N} \sum_{j=1}^{r} \frac{\Delta Y_{i,j} - \mu \Delta t_{i,j}}{\sigma^3} \\ 2\sum_{i=1}^{N} \sum_{j=1}^{r} \frac{\Delta Y_{i,j} - \mu \Delta t_{i,j}}{\sigma^3} & -\frac{rN}{\sigma^2} + 3\sum_{i=1}^{N} \sum_{j=1}^{r} \frac{(\Delta Y_{i,j} - \mu \Delta t_{i,j})^2}{\sigma^4 \Delta t_{i,j}} \end{array} \right]_{\mu=\hat{\mu},\sigma=\hat{\sigma}} \tag{3.43}$$

$$I(5.9923, 4.0052) = def \begin{bmatrix} 6.2337 & 1.1726 \times 10^{-13} \\ -1.1726 \times 10^{-13} & 1.2467 \times 10^{4} \end{bmatrix} \tag{3.44}$$

$$I(5.9923, 4.0052)^{-1} = def \begin{bmatrix} 0.1604 & -1.5089 \times 10^{-18} \\ -1.5089 \times 10^{-18} & 8.0210 \times 10^{-5} \end{bmatrix} \tag{3.45}$$

From the estimated variance we compute standard errors by taking square roots. The standard 95% confidence interval for positive parameters μ, σ are given as follows,

$$\begin{aligned} \hat{\mu} e^{\pm 1.96 \frac{\widehat{\mathrm{SD}(\mu)}}{\hat{\mu}}} &= [5,2566, 6.8310] \\ \hat{\sigma} e^{\pm 1.96 \frac{\widehat{\mathrm{SD}(\sigma)}}{\hat{\sigma}}} &= [3.9877, 4.0228] \end{aligned} \tag{3.46}$$

Therefore, within these interval, there are 95% possibility to capture the true value of μ and σ.

Likelihood Test The likelihood shows that

$$W(\theta) = 2(l(\hat{\theta}) - l(\theta)) \approx \chi_1^2 \tag{3.47}$$

here $\theta = (\mu, \sigma)$ is the true value behind testing dataset, and χ_1^2 means chi-square distribution with 1 degree of freedom. This equation can be used to test $\theta = (\mu, \sigma)$ has prespecified value, i.e. $\mu = 6, \sigma = 4$ in our testing dataset.

For a formal check of this, one wants to test the hypothesis

$$H_0 : \mu = 6, \sigma = 4 \quad vs. \quad H_1 : \mu \neq 6, \sigma \neq 4$$

χ^2 with one degree of freedom is $\mathbb{P}(Z > z_{0.05,1}) = 0.05, \chi^2_{0.05,1} = 3.84$. So the null hypothesis could be rejected at 5% significance level if $W(6,4) > 3.84$.

Use Eq.3.46 and fit the testing dataset into the equation by MATLAB,

$$\begin{aligned} W(\mu,\sigma) &= 2(l(\hat{\mu},\hat{\sigma}) - l(\mu,\sigma)) \\ &= 2(l(5.9923, 4.0052) - l(6,4)) \\ &= 0.3089 < 3.84 \end{aligned} \tag{3.48}$$

which is too small to reject the null hypothesis. So there is not enough evidence to conclude that the testing dataset does not have parameters $\mu = 6, \sigma = 4$.

3.3.4 RUL Distribution

In order to verify that the proposed Brownian motion path model is good enough to capture the true degradation process, a comparison is performed. We apply Monte Carlo simulation to our predefined Brownian motion parameters($\mu = 6, \sigma = 4$). Suppose we want to figure out the time when the process first cross degradation level $L = 5$. 10^4 random samples are generated, and the distributions of first hitting time (FHT) obtained by Monte Carlo simulations are compared with analytical function of FHT in Eq.3.30. Fig.3.7 shows that similar results can be obtained by analytical method. The most likely first hitting is at around 5, which almost reflect the situation in Degradation paths for all units if a horizontal line is drawn at 5 in Fig.3.6.

Therefore, we could say that the analytical function of FHT based on Brownian motion roughly captures the true value in degradation process. However, it is not very reasonable to make such deterministic conclusion only according to figures. An advanced evaluation need to be carry out to judge the accuracy of the proposed Brownian motion method. In this paper, real dataset is not available at the moment and only Monte Carlo simulations are uses. So this evaluation process is postponed to future work.

Suppose we want to find out the mean residual useful lifetime when the unit condition hits critical boundary $L = 5$. Given the estimated parameters where $\hat{\mu} = 6.1516, \hat{\sigma} = 4.0013$, the analytical result of RUL distribution can be given based on Eq. 3.32. Correspondingly, we obtain the PDFs of the RUL at different observation time as shown in Fig.3.8. The first curve in the figure is the RUL distribution when monitoring time is 0.2 time unit, and the last curve shows RUL when monitoring time is 3.4 time unit which is more approaching the mean FHT=5. We draw a new

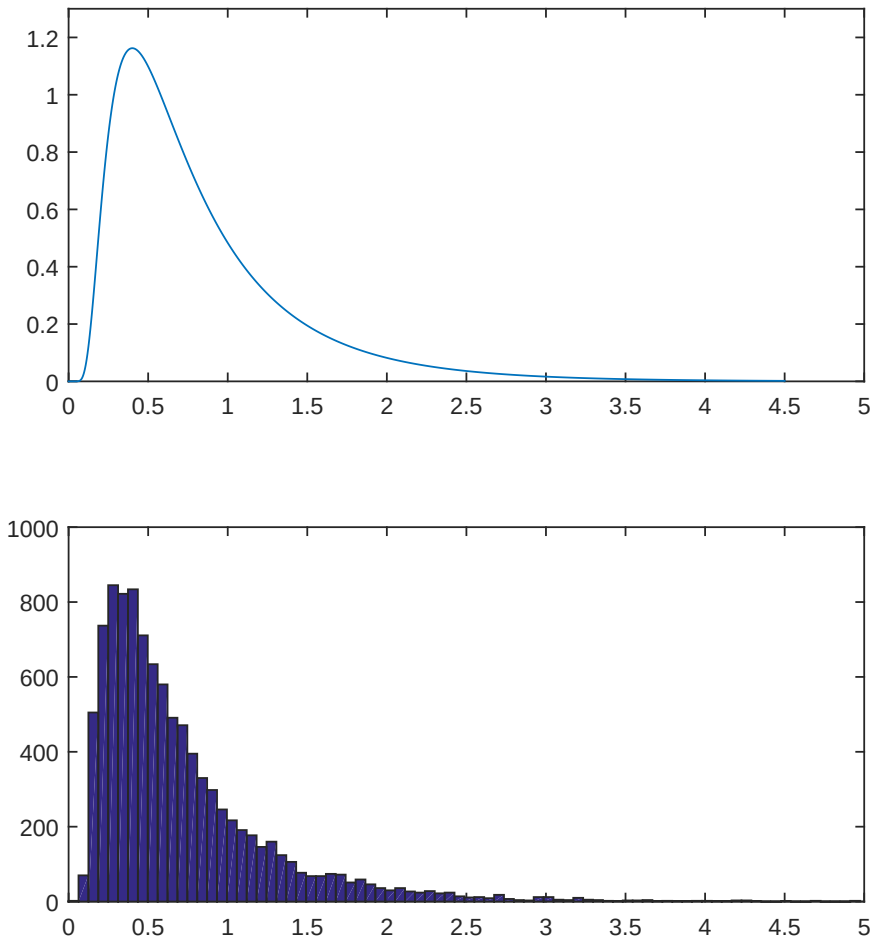

Figure 3.7: Comparison of the Monte Carlo simulation with the proposed FHT density.

RUL distribution curve every 0.2 time unit. Clearly, the later the observing time, the sooner that component would fail with an increasing higher possibility. Similarly, we can estimate the mean RUL as an useful input for maintenance in the future.

3.4 Discussion about Model Relevance

A characteristic feature of Brownian motion is that a component's degradation level alternately increases and decreases. This is contradict from our understanding as degradation should be always monotone. However, studying Brownian motion model is not without merit.

The mechanism of degradation is complex and like a black box. Most efforts have been done and would be done are to find out a model which could be more precise to describe the true

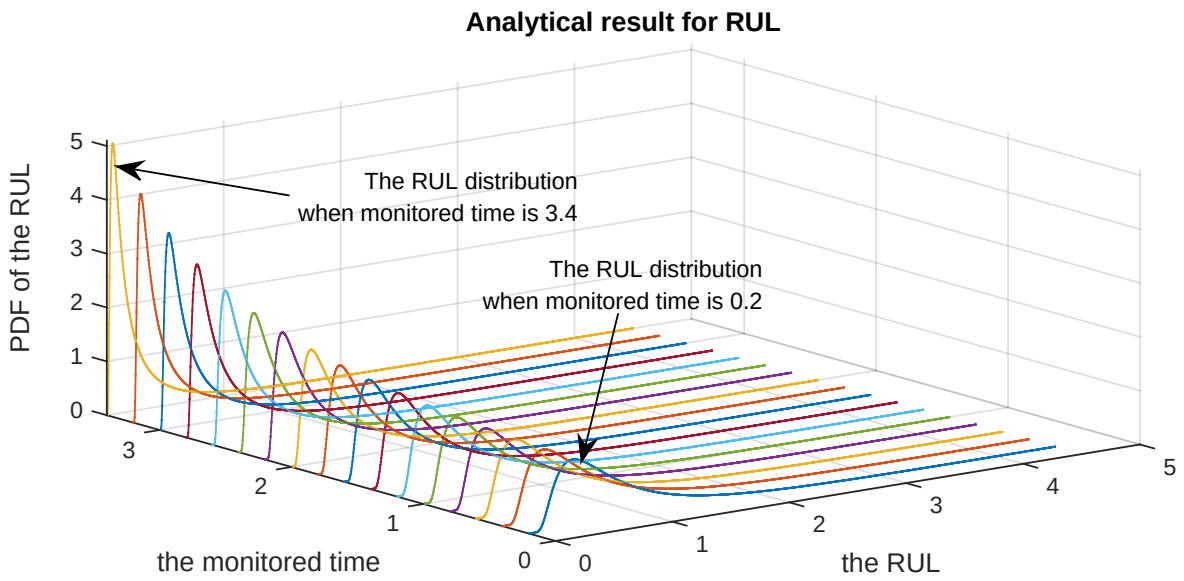

Figure 3.8: PDFs of the RULs at different observation time.

status of components. An important but difficult engineering challenge of degradation modeling is to find indicators that are closely related to degradation level and develop methods for accurately measuring these variables. Sometimes, the limited amount of information in such degradation measurements can be the results of monitoring a performance variable (Escobar and Meeker (1998))(e.g temperature, output voltage) rather than actual physical degradation (e.g. amount of material displaced by erosion). The sudden increase of these performance indicators do not necessarily result from increase of degradation. Possible reasons for them could be manufacturing defects, a different operational mode, or shocks to components. Also, the components might experience self-healing process when deteriorating. Therefore, it is fully possible that performance indicators either increases and decreases. Since we agree on the fact that degradation process is generally monotone, we assume that in a Brownian motion model, all the factors that contributes to non-monotone effects are pure noise. This assumption is very important, as many degradation data are more or less mixed with noise, and finding out pure degradation data from noise is necessary. So, it is fairly justified that Brownian motion model has its practical significance and advantages.

Chapter 4

Gamma Process Model

Gamma process is a stochastic process with independent, non-negative increments having a gamma distribution with an identical scale parameter (Van Noortwijk (2009)). It is ideally suited to model gradual deterioration that monotonically accumulates over time, such as wear, corrosion, erosion, and cracks of materials, which are common causes of failure of engineering components.

The purpose of this chapter is to present the most important mathematical properties of the gamma, i.e. the definition and properties of gamma distribution, and homogeneous gamma process. Besides, methods for Gamma process simulation, parameter estimation and RUL distribution estimation are presented in the following sections. The writer want to explore its use as a degradation model.

4.1 Homogeneous Gamma Process

4.1.1 Introduction and Definition

Before discussing Gamma process, it is necessary to recall some properties of Gamma distribution as basis for further analysis.

The Gamma function is equal to

$$\Gamma(x) = \int_0^{+\infty} u^{x-1} e^{-u} du, x > 0 \tag{4.1}$$

The Gamma distribution $\Gamma(a,b)$ has the probability distribution function

$$f_{a,b}(x) = \frac{1}{\Gamma(a)} b^a x^{a-1} e^{-bx} \mathbf{1}_{\{x>0\}} \tag{4.2}$$

with shape parameter $a > 0$ and scale parameter $b > 0$. If X is Gamma distributed with $X \sim \Gamma(a,b)$, then the expectation and variance are

$$\mathbb{E}(X) = \frac{a}{b}, \mathrm{Var}(X) = \frac{a}{b^2} \tag{4.3}$$

The Laplace transform for $s > 0$ is

$$\begin{aligned} \mathbb{E}(e^{-sX}) &= \frac{1}{\Gamma(a)} b^a \int_0^{+\infty} x^{a-1} e^{-(s+b)x} dx \\ &= \left(\frac{b}{s+b}\right)^a = \left(\frac{1}{\frac{s}{b}+1}\right)^a \\ &= \mathbb{E}(e^{-\frac{s}{b}aX}) \end{aligned} \tag{4.4}$$

If $X_1 \sim \Gamma(a_1,b)$ and $X_1 \sim \Gamma(a_2,b)$ with X_1, X_2 being independent, then $X_1 + X_2 \sim \Gamma(a_1+a_2,b)$. Indeed for $s > 0$,

$$\mathbb{E}(e^{-s(X_1+X_2)}) = \mathbb{E}(e^{-sX_1})\mathbb{E}(e^{-sX_2}) = \left(\frac{b}{s+b}\right)^{a_1+a_2} \tag{4.5}$$

4.1.2 Properties of Homogeneous Gamma Process

A gamma process is a random process with independent gamma distributed increments. Often written as $\Gamma(t;a,b)$. It is a pure-jump increasing Lévy process. An homogeneous gamma process with shape parameter a and scale parameter b is a stochastic process $X(t), t \geq 0$ on $\mathbb{R}_+$ such that:

1. $X(0) = 0$,
2. $X(t), t \geq 0$ is a stochastic process with independent increments,
3. For $0 \leq s < t$, the distribution of the random variable $X(t) - X(s)$ is the gamma distribution

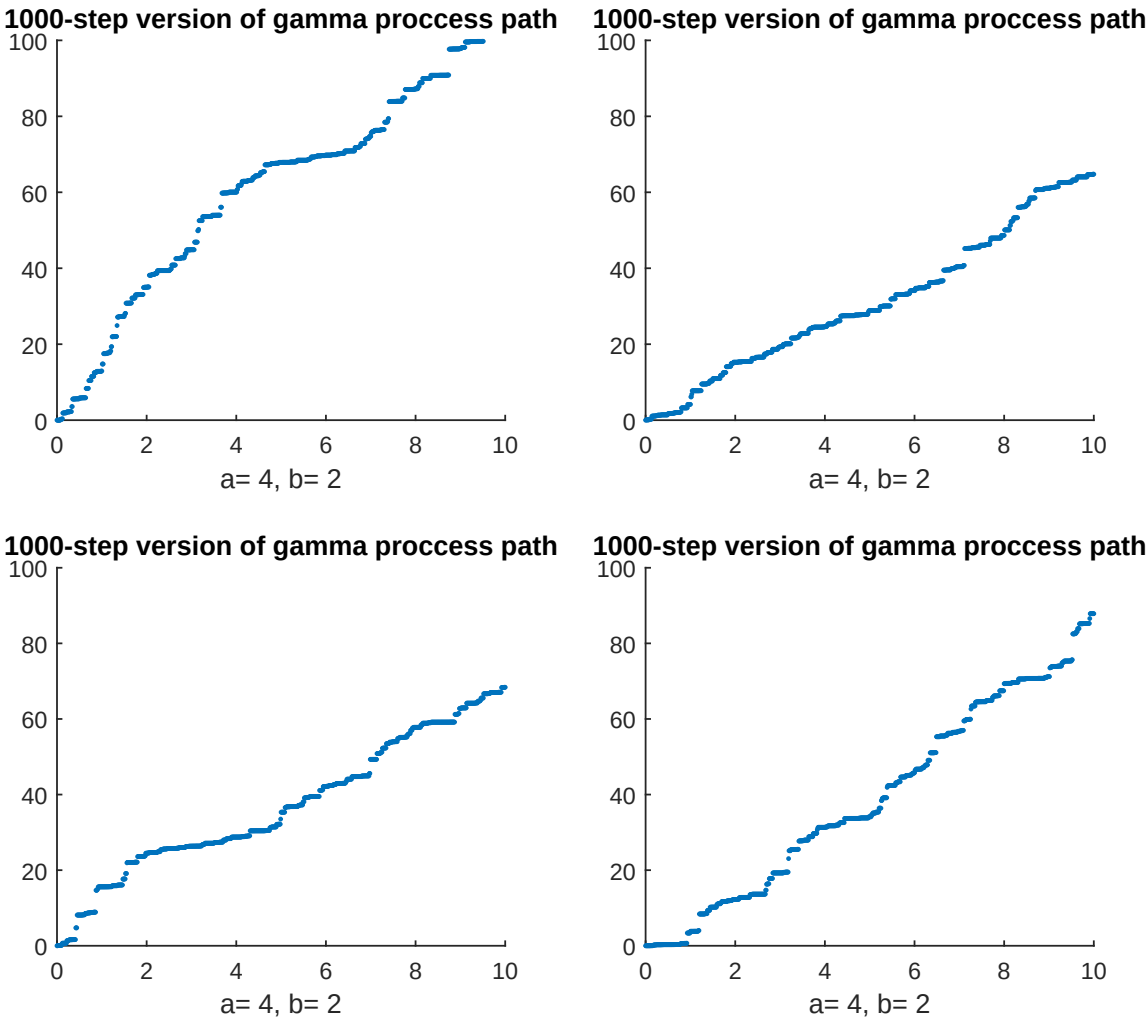

Figure 4.1: Four realizations of sample path of gamma process

$\Gamma(a(t-s),b)$ as in Eq.4.6, therefore it only depends on $t-s$.

$$X(t)-X(s)\sim\Gamma(a(t-s),b)=f_{a(t-s),b}(x)=\frac{1}{\Gamma(a(t-s))}b^{a(t-s)}x^{a(t-s)-1}e^{-bx}\mathbf{1}_{\{x\geq 0\}} \tag{4.6}$$

According to these properties, is it easy to conclude that $X(t), t\geq 0$ is a non-decreasing, coherent process. For all $t\geq 0$,

$$\mathbb{E}(X)=\frac{a}{b}t, \mathrm{Var}(X)=\frac{a}{b^2}t \tag{4.7}$$

The mean value and variance of the process increases linearly. However, gamma process introduces more intrinsic randomness. In other words, at each time t_j, we introduce a new random variable which is not reduced to the noise.

4.1.3 Some Graphs and Discussions

The choice of the homogeneous gamma process in application is motivated by the fact that it has independent increments, homogeneous in time, and monotone increasing trajectories. Furthermore, independent increments makes the subsequent mathematical treatment quite tractable (Lawless and Crowder (2004)). We consider that the realizations of the gamma process here consist of a countably infinite number of jumps of a finite interval. The summation of those increments form homogeneous gamma process paths.

Fig.4.1 shows the sample paths for realizations of gamma processes with shape parameter $a = 4$ and scale parameter $b = 2$. From the figure, we can see that the process is a rough upward trend with occasional large shifts. Gamma process has increments which are strictly positive whatever small time interval has elapsed.

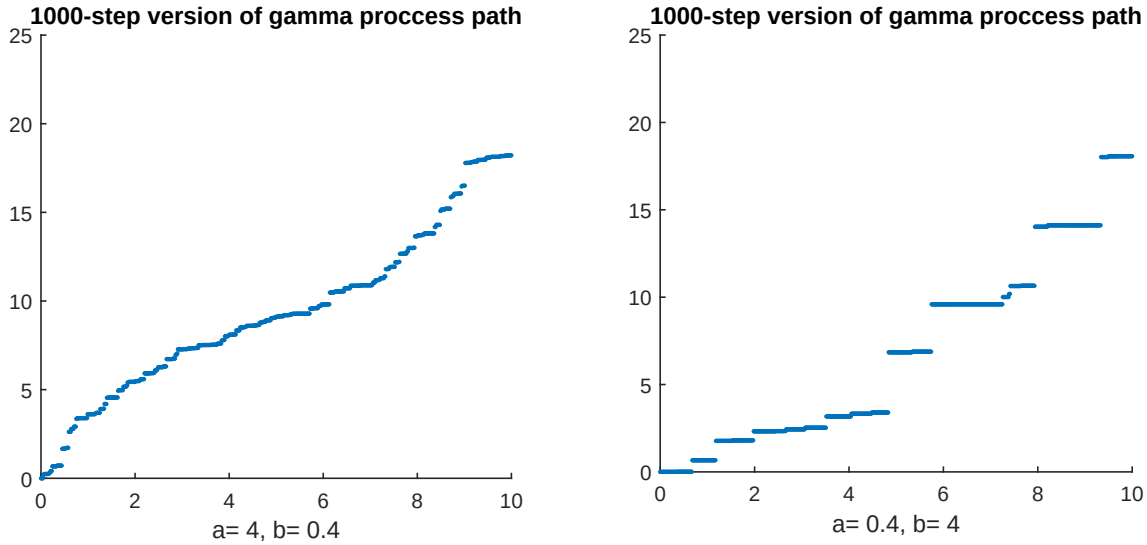

Figure 4.2: Realizations of different a, b in gamma process paths

Example 1. Rate and Size Suppose we simulate two gamma process paths with initial condition 0 when $t = 0$. The overall time scale is 10 and there are 1000 steps. We suppose the first path is with parameter $a = 4, b = 0.4$ (See Fig.4.2, left), while the second path is with $a = 0.4, b = 4$(See Fig.4.2, right). These figures demonstrate that parameter a controls the rate of jump arrivals and the scaling parameter b inversely controls the jump size. When scale parameter b is small in comparison with shape parameter a, the path of the process is more contin-

uous(but not continuous anywhere), and the rate of jump has a greater impact on the gamma process path; if shape parameter a is small in comparison with scale parameter b, then jump size dominates in the behaviour of the gamma process path as more large shifts occur in the right figure. However, both parameters would limit the cumulated amount of increments if we either decrease the shape parameter or scale parameter.

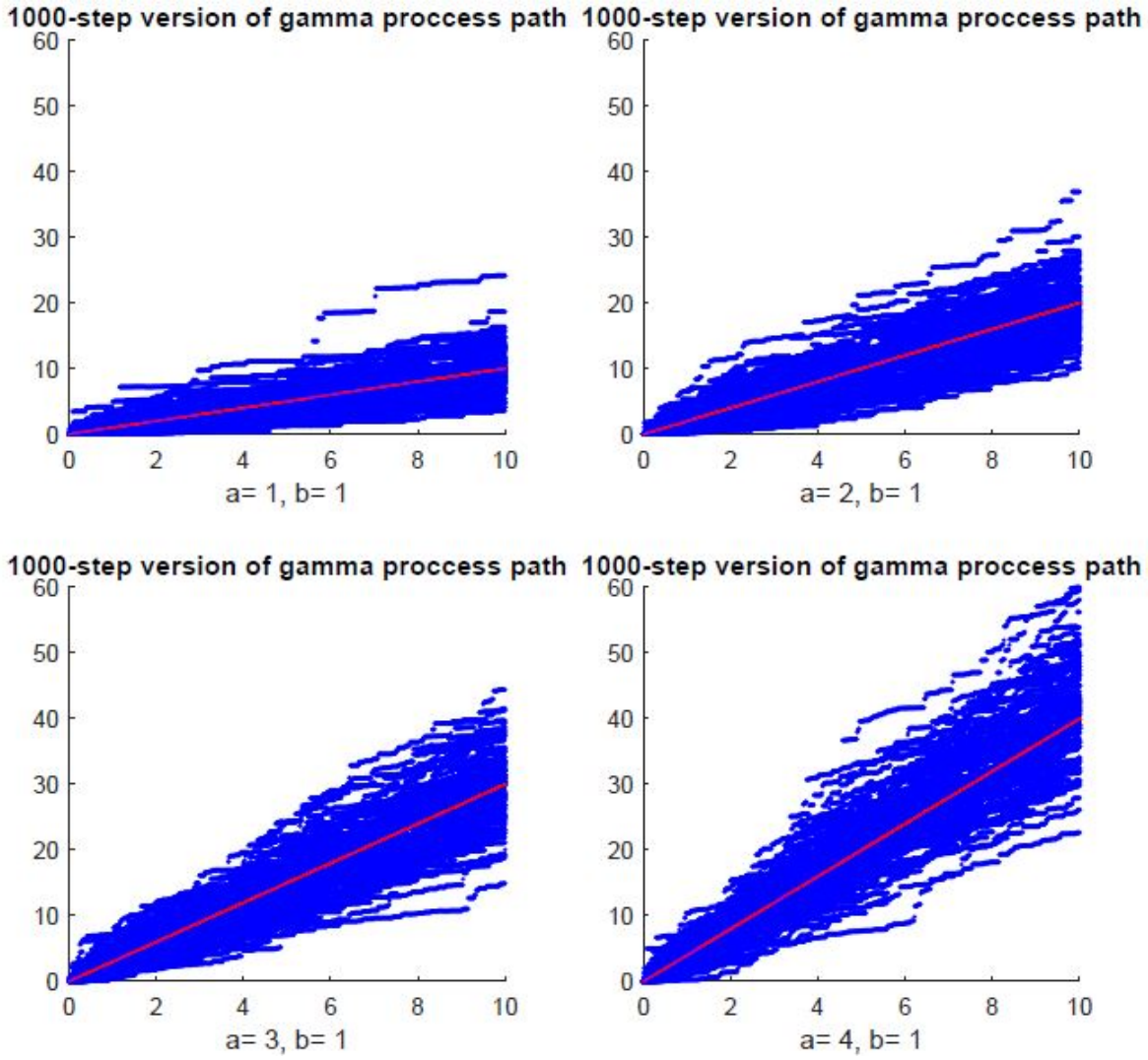

Figure 4.3: Four groups of realizations with increasing a.

Example 2. Effect of a and b Suppose we simulate four Brownian motion paths with initial condition 0 when $t = 0$. The overall time scale is 10 and there are 1000 steps. We first keep the value of $b = 1$ in all four realizations and only change shape parameter a to see its impact on process. The paths are with parameters $a = 1, 2, 3, 4$. Fig.4.3 shows 100 realizations and their

expectations (red lines) in each graph. Recall Eq.4.7 and we can see from the figures that the slope $\mathbb{E}(X)$ increases when a increases. It is not clear from the figure that the processes are more spread out, but from Eq.4.7, the variance should increase linearly when shape parameter a increases.

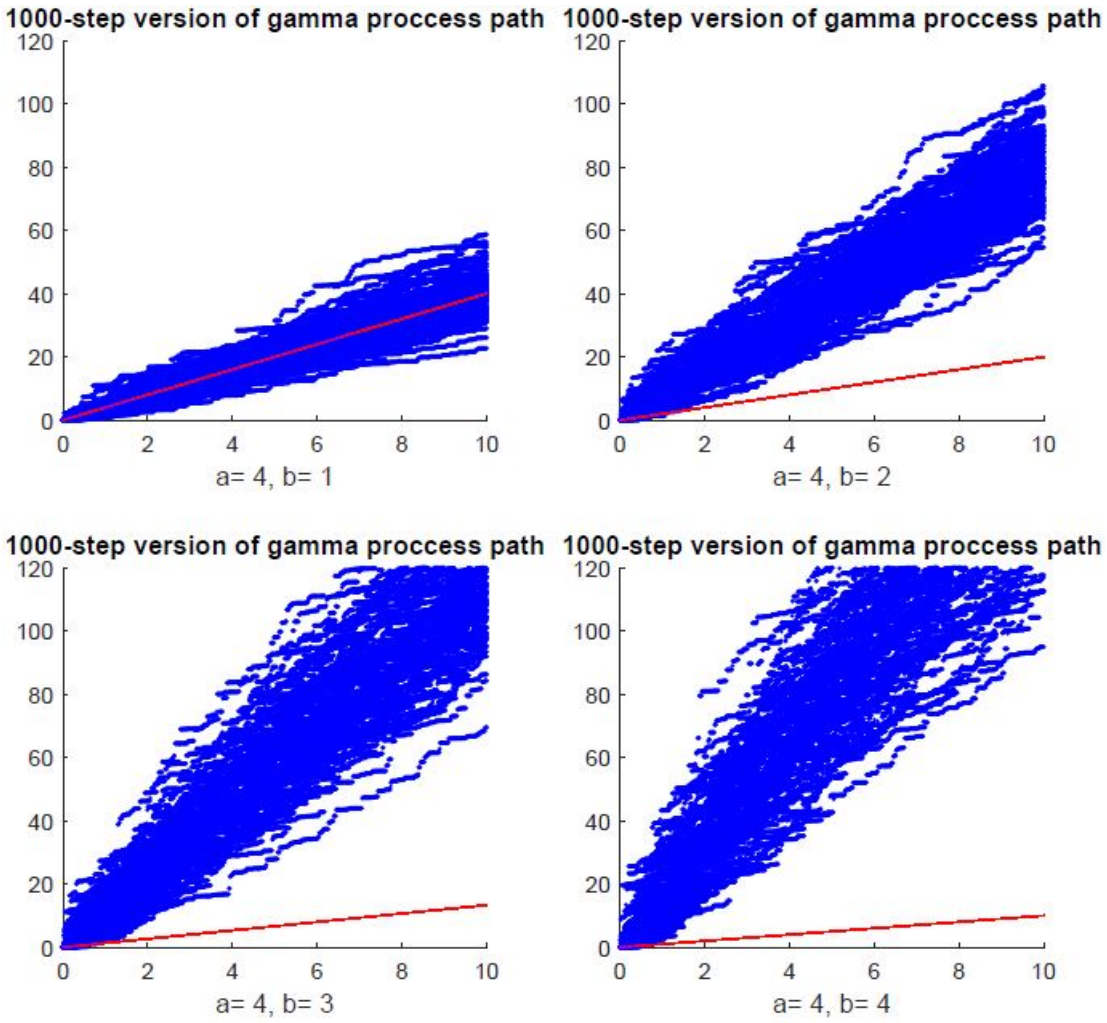

Figure 4.4: Four groups of realizations with increasing b.

Similarly, we keep the value of $a = 4$ in all four realizations and only change scale parameter b to see its impact on process. The paths are with parameters $b = 1,2,3,4$. Fig. 4.4 shows 100 realizations and their expectations(red lines) in each graph. Recall Eq.3.20 and we expect that the slope ($\mathbb{E}(Y(t))$) decreases and processes more narrow when b increases , but figures the opposite situations. This may be due to the property of gamma distribution. It is actually not possible to calculate true values in gamma distribution, and normally what we do is to use other

numerical methods to generally capture the true values. However, such approximation has limited interval. The approximated values is thus nonsense if we cross such boundary in extreme cases. So the confusing situations in Fig.4.4 might because of foolish parameters we defined in gamma process. However, these figures are not useless. At lease we can conclude that changing either of shape or scale parameter would affect mean and variance of gamma process at the same time. And such properties would make it more difficult if we need to control only mean (or variance) in our settlement.

4.1.4 First Hitting time and RUL distribution

First Passage Time T_L Typically, a Gamma process $X(t)$ model the degradation of a component at time t. The failure time T_L for a unit is defined as the time at which its degradation path first crosses a threshold value L. For the Gamma process model considered here the sample paths are monotonic and the failure time distribution is easily obtained as in Eq.4.8. The component works until the deterioration reaches a failure threshold L. The failure time (or so-called first hitting time)of the component is then

$$T_L = \inf(t > 0 : X(t) \geq L) \tag{4.8}$$

Therefore, the first passage time T_L satisfies the following function

$$\mathrm{F}(x; a, b) = \mathbb{P}(T_L \leq t) = \mathbb{P}(X(T_L) \geq L) = 1 - \int_0^L \frac{1}{\Gamma(at)} b^{at} x^{at-1} e^{-bx} dx \tag{4.9}$$

RUL Distribution Suppose that a unit is unfailed at time t_j, we are interested in whether it will survive beyond time t. If we do not know the current degradation level $x(t_j)$, the conditional

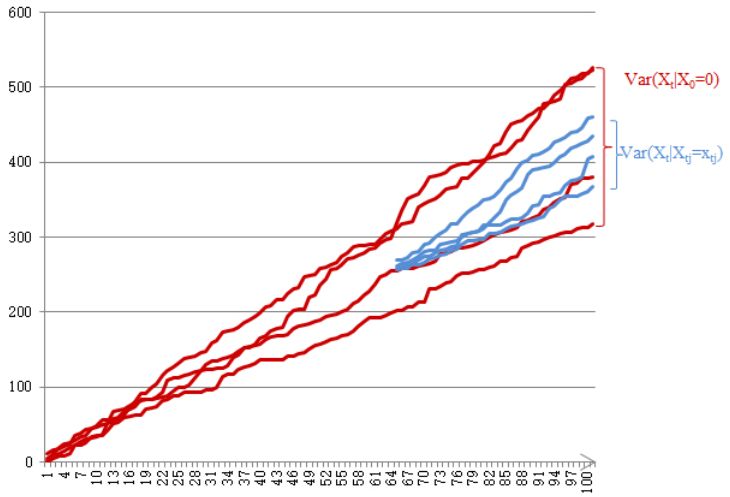

Figure 4.5: Realizations for deterioration given different information.

reliability is

$$\begin{aligned} R_i(t|T_i > t_j) &= \mathbb{P}(T_i \geq t|T_i > t_j) \\ &= \mathbb{P}(X_i(t) \leq L|T_i > t_j) \\ &= \frac{\mathbb{P}(X_i(t) - X_i(0)) \leq L}{\mathbb{P}(X_i(t_j) - X_i(0)) \leq L} \\ &= \frac{\int_0^L f_{at,b}(x)dx}{\int_0^L f_{at_j,b}(x)dx} \end{aligned} \tag{4.10}$$

However, if we do know the current degradation level $x(t_j)$, the conditioning on it would be expected to give more precise predictions than Eq.4.10.

One of the advantages of measured degradation is its use in predicting time to failure. Fig. 4.5 illustrates the effect of knowing $X(t_j)$ in a deterioration process. If $t_j = 0$, there are plenty of realizations for deterioration (as shown in red paths), and the variance of RL is very uncertain. If we move t_j close to t, then the variance is significantly reduced (in blue paths). Such property can also be seen in the probability density plot, where variance becomes narrower as the inspection time t_j occurs later. That means, if frequent inspection is possible, especially at the later stage of deterioration, variance of residual useful lifetime is substantially narrowed, the less uncertain the deterioration process would be. Consequently, better maintenance management strategies could be implemented based on such information.

Observing the process at time t is at position $x(t)$, the probability that residual life is less than a predefined period h is

$$\begin{aligned}
\mathbb{P}(RUL_{X(t)} \leq h) &= 1 - \mathbb{P}(RUL_{X(t)} > h) \\
&= 1 - \mathbb{P}(X(t+h) \leq L | X(t) = x(t)) \\
&= 1 - \frac{\mathbb{P}(X(t+h) - x(t) \leq L - x(t))}{\mathbb{P}(X(t) \leq L)} \\
&= 1 - \frac{\int_0^{L-x(t)} f_{ah,b}(u)\,du}{\int_0^L f_{at,b}(v)\,dv} \\
&= 1 - \frac{\int_0^{L-x(t)} \left(b^{ah} u^{ah-1} e^{-bu}/\Gamma(ah)\right) du}{\int_0^L \left(b^{at} v^{at-1} e^{-bv}/\Gamma(at)\right) dv}
\end{aligned} \tag{4.11}$$

4.2 Philosophy of RUL Estimation

The purpose of this section to estimate remaining useful lifetime (RUL) using gamma process to model the degradation. First, assumptions and an example training data set are presented. Then, the parameters in the degradation model are estimated with the training data set. And finally the RUL distribution is estimated proposed for the units of testing data.

4.2.1 Assumptions and Training Data Set

Assume that X is an degradation indicator for $N = 100$ independent identical tested items, and $X_{i,j}$ denotes the degradation measurements of the ith items at time j, where $i = 1, 2, \ldots N$ and $j = 1, 2, \ldots r$, r is the last observation time. All the degradation realizations are based on homogeneous gamma process with two parameters (a, b), and that are the same for all the items. According to Eq. 4.6, each increment of degradation $\Delta X_{i,j} = X_{i,j+1} - X_{i,j}$ of each items follows a gamma distribution $N(a\Delta t_{i,j}, b)$, where $a = 4, b = 1$. There are 1000 observations within the time frame $T = 10$, and the initial condition is $X(0) = 0$. Let the time step equals $1/1000$. The degradation paths for all $N = 100$ units and certain units are illustrated in Fig.4.6. The following parameter estimation, process verification and RUL distribution are all based on these built testing dataset.

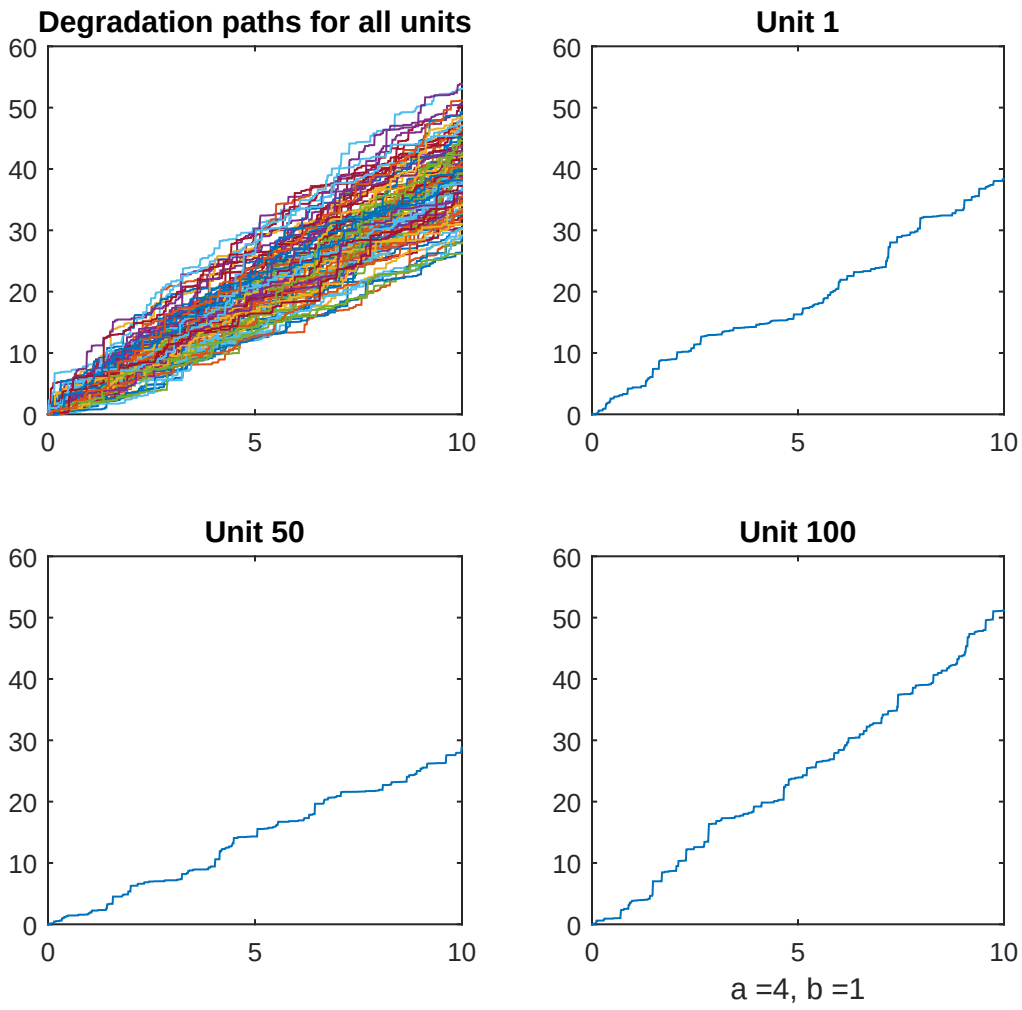

Figure 4.6: Degradation paths of all units and certain units.

4.2.2 Parameter Estimation

Given degradation path X_{ij} based on Brownian motion process, each increment is $\Delta X_{i,j} = X_{i,j+1} - X_{i,j} \sim \Gamma(a\Delta t_{i,j}, b)$ is gamma distributed for all identically and independent components. Recall the probability density function of gamma process in Eq. 4.6, then

$$f_{a\Delta t_{i,j},b}(\Delta X_{i,j}) = \frac{1}{\Gamma(a\Delta t_{i,j})} b^{a\Delta t_{i,j}} (\Delta X_{i,j})^{a\Delta t_{i,j}-1} e^{-b\Delta X_{i,j}} \tag{4.12}$$

It is also possible to use maximum likelihood estimators as in Chapter 3. The random variables $\Delta X_{i,j}$ are independent. For item i, the likelihood function for item i is

$$\begin{aligned} L_i(a,b) &= f_i(\Delta X_i) = f_i(\Delta X_{i,1}, \Delta X_{i,2}, \ldots, \Delta X_{i,r}|a,b) \\ &= \prod_{j=1}^{r} f_i(\Delta X_{i,j}|a,b) \\ &= \prod_{j=1}^{r} \frac{1}{\Gamma(a\Delta t_{i,j})} b^{a\Delta t_{i,j}} (\Delta X_{i,j})^{a\Delta t_{i,j}-1} e^{-b\Delta X_{i,j}} \end{aligned} \tag{4.13}$$

Then for the ith item, the log-likelihood is given by

$$l_i(a,b) = \ln L_i(a,b) = \ln\left(\prod_{j=1}^{r} \frac{1}{\Gamma(a\Delta t_{i,j})} b^{a\Delta t_{i,j}} (\Delta X_{i,j})^{a\Delta t_{i,j}-1} e^{-b\Delta X_{i,j}} \right) \tag{4.14}$$

Since the measurements $X_{i,j}$ are independent

$$\begin{aligned} l(a,b) &= \ln(\Delta X_1, \Delta X_2, \ldots, \Delta X_N) \\ &= \sum_{i=1}^{N} \ln(f_i(\Delta X_{i,1}, \Delta X_{i,2}, \ldots, \Delta X_{i,r})) \\ &= \sum_{i=1}^{N} \ln\left(\prod_{j=1}^{r} \frac{1}{\Gamma(a\Delta t_{i,j})} b^{a\Delta t_{i,j}} (\Delta X_{i,j})^{a\Delta t_{i,j}-1} e^{-b\Delta X_{i,j}} \right) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{r} \left(a\Delta t_{i,j} \ln b - \ln(\Gamma(a\Delta t_{i,j})) + (a\Delta t_{i,j} - 1)\ln(\Delta X_{i,j}) - b\Delta X_{i,j} \right) \end{aligned} \tag{4.15}$$

The maximum likelihood estimator $(\hat{a}, \hat{b})$ are found by maximizing $l(a,b)$. In practice it is done by taking the partial derivative of the log-likelihood function of Eq. 4.15 with respect to a and b. This gives equations

$$\frac{\partial l(a,b)}{\partial a} = \sum_{i=1}^{N} \sum_{j=1}^{r} \left(\Delta t_{i,j} \ln b - \Delta t_{i,j} \frac{\Gamma'(a\Delta t_{i,j})}{\Gamma(a\Delta t_{i,j})} + \Delta t_{i,j} \ln(\Delta X_{i,j}) \right) = 0 \tag{4.16}$$

$$\frac{\partial l(a,b)}{\partial b} = \sum_{i=1}^{N} \sum_{j=1}^{r} \left(\frac{a\Delta t_{i,j}}{b} - \Delta X_{i,j} \right) = 0 \tag{4.17}$$

From Eq.4.17,

$$b = a \frac{\sum_{i=1}^{N} \sum_{j=1}^{r} \Delta t_{i,j}}{\sum_{i=1}^{N} \sum_{j=1}^{r} \Delta X_{i,j}} \tag{4.18}$$

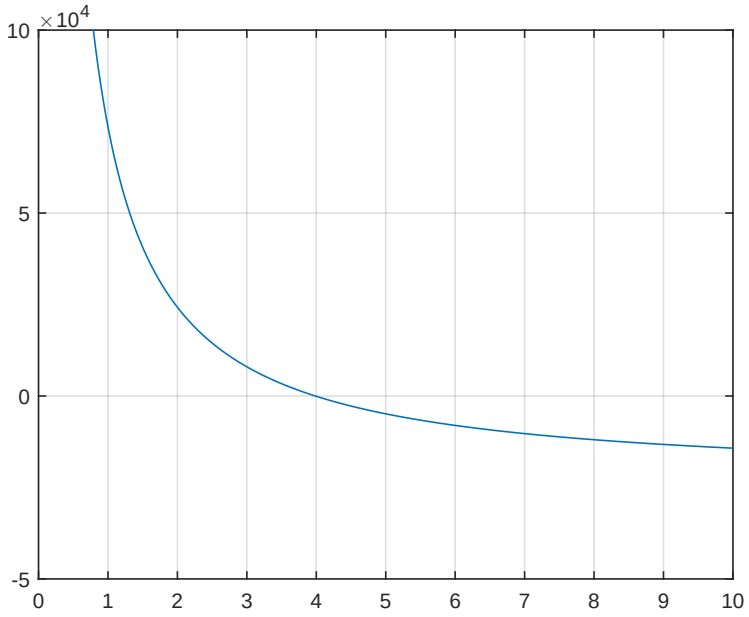

Figure 4.7: Plot of function g(a).

Eq. 4.17 can be substituted as

$$\frac{\partial l(a,b)}{\partial a} = \sum_{i=1}^{N}\sum_{j=1}^{r}\left(\Delta t_{i,j}\ln\left(a\frac{\sum_{i=1}^{N}\sum_{j=1}^{r}\Delta t_{i,j}}{\sum_{i=1}^{N}\sum_{j=1}^{r}\Delta X_{i,j}}\right) - \Delta t_{i,j}\frac{\Gamma'(a\Delta t_{i,j})}{\Gamma(a\Delta t_{i,j})} + \Delta t_{i,j}\ln(\Delta X_{i,j})\right) = 0 \tag{4.19}$$

This value of a in the equation can be solved numerically for instance with the Newton method. It is a method for finding successively better approximations to the roots(zeros) of a real-valued function. The basic idea is shown as follows. Give a function g defined over the real a and its derivative g', we begin with a first guess a_0 for a root of the function. A better approximation is repeated as $a_{n+1} = a_n - \frac{g(a_n)}{g'(a_n)}$ until a sufficiently accurate value is reached.

In our case, $g(a)$ equals Eq. 4.19 and

$$g'(a) = \frac{\sum_{i=1}^{N}\sum_{j=1}^{r}\Delta t_{i,j}}{a} - \sum_{i=1}^{N}\sum_{j=1}^{r}\Delta t_{i,j}^2\frac{\Gamma''(a\Delta t_{i,j})\Gamma(a\Delta t_{i,j}) - (\Gamma'(a\Delta t_{i,j}))^2}{(\Gamma(a\Delta t_{i,j}))^2} \tag{4.20}$$

To check out in which range the root is, we first plot $g(a)$ in the range $a \in [1,10]$. Fig.4.7 indicates that there is a zero in the range around $a \in [3.8,4.2]$. Using the Newton's iteration with

a starting value $a = 3.8$. In that range to approximate the root, we are satisfied if the difference between successive iterates is smaller than 10^{-4}. Also, we restrict the number of iteration to $n \leq 200$. To illustrate the process, the testing dataset was analyzed using MATLAB (see code in Appendix). The estimated parameters corresponding to the testing dataset are obtained as follows:

$$\hat{a} = 4.0740, \hat{b} = 1.0318$$

4.2.3 Likelihood Confidence Interval

Likelihood Confidence Interval Similar to what have been done in Chapter 3, estimated variances of the parameters can be calculated by Hessian matrix with given test dataset.

$$I(\hat{a},\hat{b}) = def \left[\begin{array}{cc} \sum_{i=1}^{N}\sum_{j=1}^{r}\Delta t_{i,j}^2 \frac{\Gamma''(a\Delta t_{i,j})\Gamma(a\Delta t_{i,j})-(\Gamma'(a\Delta t_{i,j}))^2}{(\Gamma(a\Delta t_{i,j}))^2} & -\sum_{i=1}^{N}\sum_{j=1}^{r}\frac{\Delta t_{i,j}}{b} \\ -\sum_{i=1}^{N}\sum_{j=1}^{r}\frac{\Delta t_{i,j}}{b} & \sum_{i=1}^{N}\sum_{j=1}^{r}\frac{a\Delta t_{i,j}}{b^2} \end{array} \right]_{a=\hat{a},b=\hat{b}} \tag{4.21}$$

$$I(4.0740, 1.0318) = def \left[\begin{array}{cc} 6.0415\times10^{3} & -969.1499 \\ -969.1499 & 3.8265\times10^{3} \end{array} \right] \tag{4.22}$$

$$I(4.0740, 1.0318)^{-1} = def \left[\begin{array}{cc} 1.7253\times10^{-4} & 4.3697\times10^{-5} \\ 4.3697\times10^{-5} & 2.7240\times10^{-4} \end{array} \right] \tag{4.23}$$

From the estimated variance we compute standard errors by taking square roots. The standard 95% confidence interval for positive parameters a, b are given as follows,

$$\begin{aligned} \hat{a}e^{\pm1.96\frac{\widehat{SD(\hat{a})}}{\hat{a}}} &= [3.9743, 4.0258] \\ \hat{b}e^{\pm1.96\frac{\widehat{SD(\hat{b})}}{\hat{b}}} &= [0.9682, 1.0329] \end{aligned} \tag{4.24}$$

4.2.4 RUL Distribution

Noticing that the identical procedures of estimating RUL distribution and also the difficulty of presenting all of units' RUL, the writer just gives an example of estimating the RUL in a specific situation. Suppose the degradation path of a unit follows gamma process model where shape

parameter $a = 4.0740$ and scale parameter $b = 1.0318$ as we estimated before. The prognostic degradation path is illustrated in Fig.4.8. Critical boundary is defined as $L = 5$. We know that the condition of this unit is $x(t) = 4.387$ at time $t = 2$. The aim is to find out the probability that a unit would fail within a predefined period h given the current degradation status.

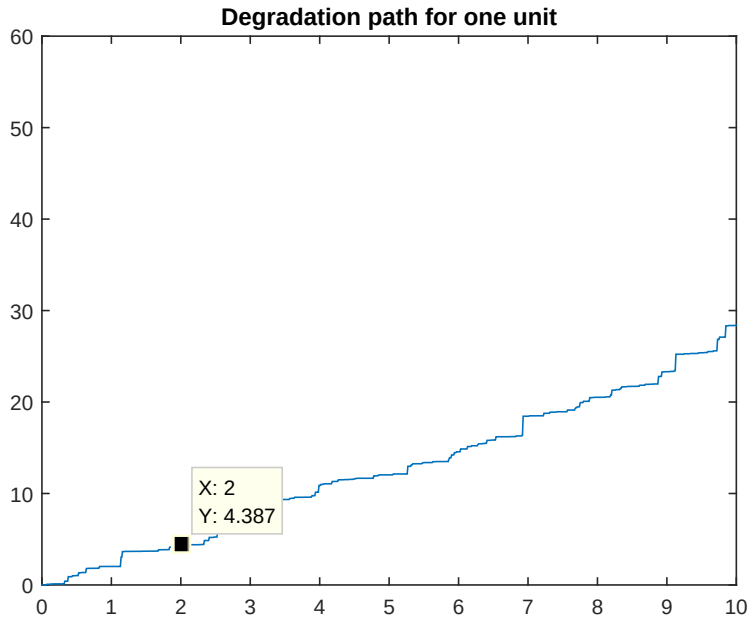

Figure 4.8: Degradation path of one unit.

We select several predefined period h and calculate the cumulative RUL distribution in each situation shown in Fig.4.9. The right curve in the figure is the RUL distribution when monitoring time is 2 time unit, and the left curve shows RUL when monitoring time is 7 time unit. We draw a new RUL distribution curve every 3 time unit. As we can see, the later the observing time, the higher possibility that the component would fail within the predefined time period.

4.3 Discussion about Model Relevance

Gamma process model is appropriate in a lot of degradation problems, especially in modeling with uncertainties. Pandey and Yuan (2006) points out that there are two kinds of variables: sampling and temporal uncertainties. Sampling uncertainty refers to the variability of deterioration from sample to sample. As an epistemic uncertainty, it can be reduced by additional

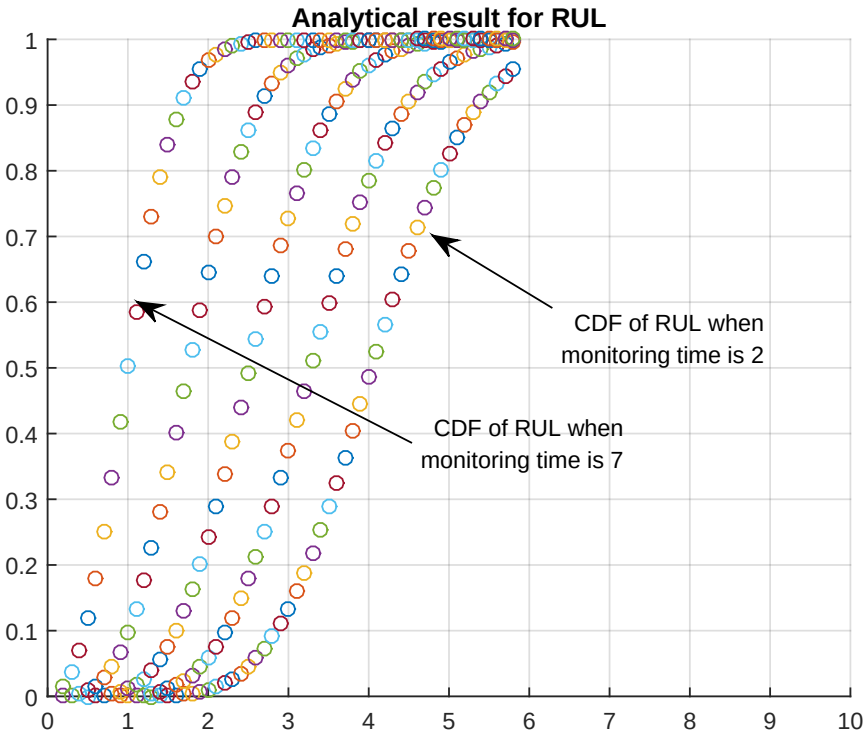

Figure 4.9: CDFs of the RULs at different observation time.

inspections and more precise observations. Temporal uncertainty, on the other hand, is uncertainty inherent with the progression of degradation over time. It is aleatory in nature so that it cannot be eliminated completely by increasing inspections. As discussed in the above sections, gamma process model can incorporate unit-specific random effects which is different from noise. Thus, we have the possibility to capture the substantial heterogeneity between degradation paths of different components, even though the same treatment or environment are present.

Chapter 5

Condition-based Maintenance with Gamma-Process Deterioration

Condition-based maintenance is maintenance when need arises. The maintenance is based on using real-time condition monitoring data to prioritize and optimize maintenance resources (see http://en.wikipedia.org/wiki/Condition-based_maintenance). It answers two questions: The first one is "When should the inspections be, at fixed time interval or at random conditioned times, or continuously"; the second question is "What action to take, Preventive Maintenance (PM) or Corrective Maintenance (CM)". By answering these two questions, a system will determine the equipment's health, and act only when maintenance is actually necessary, which in return, minimizing spare parts cost,maintenance labors, system downtime and time spent on maintenance. In order to improve the maintenance performance under budget and resources constraints and to gain competitive advantage, a lot of CBM models have been proposed. Among these models, control limit policy dominates most maintenance strategies. Usually there is a critical maintenance level called control level, and preventive maintenance is carried out if the system states reach or exceed this threshold. Another decision variable is the inspection interval where the optimal periodic interval needed to be chosen to improve the reliability of system or minimizing long term average cost per unit time. Therefore, determination of these two condition index could greatly influence the performance of a CBM strategy.

Degradation level is the most common condition index for CBM decision of a degrading system (see Huynh et al. (2014)). The failure occurs when the level of degradation crosses a

specified threshold is called soft failure. But we should notice that system failure can be due to mechanisms other than degradation, for example random shocks. This is called hard failures which failure times will not correspond with a particular threshold. The degradation level appears insufficient to fully describe the component health state. It is then interesting to construct a more relevant condition index for CBM decision. RUL is one possibility. As discussed in the previous chapters, RUL makes up for such weakness by providing a probabilistic solution for expected time remaining to component failure, and it may be then a good alternative to describe a component with multiple failure modes and types. So in this chapter, one traditional degradation level based and two RUL based maintenance model for the component are briefly introduced. Their performance are compared and summarized at the end of this chapter.

5.1 Assumptions and Performance Assessment Criterion

Assumptions The writer consider the maintenance of a technical device subject to a continuous random degradation. In our model, the condition of the component can be characterized by an degradation stochastic process $Y(t)$. Suppose that in the absence of maintenance actions, the degradation variable evolves like a stochastic process for example Gamma process with shape parameter μ and σ. At time 0 the component is new and $Y(0) = 0$. When the component condition reaches a threshold level L, a failure happens. The writer wants to first present a maintenance policy depends on the degradation level of the component. And then two other maintenance policies based on RUL are presented.

Periodic Inspection All the strategies are based on periodic inspection scheme at times $T_1, T_2, \ldots$ where $T_k = kT$ with $k \in \mathbb{N}$ and $T \in \mathbb{R}$. The periodic assumption is reasonable in economic and safety sense, as implementing continuous monitoring in an industrial context like subsea or offshore is quite difficult. The maintenance policy is driven by the knowledge of system state at times of inspection and an inspection cost C_i is generated. At every inspection epoch two maintenance actions are possible: a preventive replacement with cost C_p and a corrective replacement with cost Cc. $C_c > C_p > C_i$ because CM is unplanned and it has to be performed on a more deteriorating system, thus very likely to be more expensive. After each

replacement (CM or PM) the system is brought back to its initial state $Y(0) = 0$ and its evolution does not depend on he past events. The CM is carried out as soon as a failure is detected. Also, there is a system unavailability duration after a failure, and an additional cost rate C_d is incurred from the failure time until the next replacement time.

Cost Model Formulation To evaluate the performance of these three maintenance policies, the writer focus on the expected maintenance cost per unit time over an long term time span as cost criterion:

$$C_\infty = \lim_{t\to\infty} \frac{C(t)}{t} = \frac{\mathbb{E}[C(S)]}{\mathbb{E}[S]} \tag{5.1}$$

where S is the length of a renewal cycle and $C(.)$ is the cumulative maintenance cost at time t where (see Huynh et al. (2014))

$$C(t) = C_i N_i(t) + C_p N_p(t) + C_c N_c(t) + C_d d_d(t) \tag{5.2}$$

with $N_p(t)$ is the number of preventive maintenance before t, $N_c(t)$ is the number of corrective maintenance before t, $d_d(t)$ the cumulative unavailability duration of the system before t and $N_i(t)$the number of inspections before t. More explanation on cost components, refer to Huynh et al. (2011).

5.2 Degradation-based Maintenance Model (T, M) Policy

Recall that Y_{kT} is the degradation level at the inspection time kT. In the framework of the degradation-based maintenance policy a fixed threshold M of the degradation level is defined. L is a critical degradation level where the component fails above this level. The following decision frame is adopted:

- If $Y_{kT} \leq M$, the nothing is done and the decision is postponed until next inspection.
- If $M \leq Y_{kT} \leq L$, the component is preventively replaced with a cost C_p.
- If $Y_{kT} > L$, a corrective maintenance is carried out with a cost C_c.

The inspection period T and the PM threshold M are two parameters we need to optimized and we call this (T,M) policy. Recall 5.2, the long term expected maintenance cost rate of (T,M) is

$$C_\infty(T,M) = \frac{C_p\mathbb{P}_p(T,M) + C_c\mathbb{P}_c(T,M) + C_i\mathbb{E}[N_i(T,M)] + C_d\mathbb{E}[d_d(T,M)]}{\mathbb{E}[S(T,M)]} \tag{5.3}$$

where $\mathbb{E}[S(T,M)]$ is the expected length of a replacement cycle under (T,M) policy, $\mathbb{P}_p(T,M)$, $\mathbb{P}_c(T,M)$, $\mathbb{E}[N_i(T,M)]$ and $\mathbb{E}[d_d(T,M)]$ are respectively, the probability of a PM, the probability of a CM, the expected number of inspections, and the expected cumulative down time of the system in a maintenance cycle. The optimal values of T and M are obtained by the following expression:

$$C_\infty(T_{opt},M_{opt}) = \min_{T,M}\{C_\infty(T,M), T>0, 0\le M\le L\} \tag{5.4}$$

More discussion on cost component refer to Huynh et al. (2011).

5.3 CDF RUL-based Maintenance Model (T,Q) Policy

Recall that the $\mathbb{P}(RUL(kT))$ and Y_{kT} are respectively cumulative distribution function (CDF) of RUL and the degradation level at the inspection time kT. In the framework of CDF RUL-based maintenance policy, a percentile Q is settled. The following decision frame is adopted:

- If $Y_{kT}\le L$, and $\mathbb{P}(RUL(kT)<T)\le Q$, nothing is done and the decision is postponed until next inspection.
- If $Y_{kT}\le L$, and $\mathbb{P}(RUL(kT)<T)>Q$, the component is preventively replaced with a cost C_p.
- If $Y_{kT}>L$, a corrective maintenance is carried out with a cost C_c.

Similar to 5.3, the long term expected maintenance cost rate of (T,Q) is

$$C_\infty(T,M) = \frac{C_p\mathbb{P}_p(T,Q) + C_c\mathbb{P}_c(T,Q) + C_i\mathbb{E}[N_i(T,Q)] + C_d\mathbb{E}[d_d(T,Q)]}{\mathbb{E}[S(T,Q)]} \tag{5.5}$$

where $\mathbb{E}[S(T,Q)]$ is the expected length of a replacement cycle under (T,Q) policy, $\mathbb{P}_p(T,Q)$, $\mathbb{P}_c(T,Q)$, $\mathbb{E}[N_i(T,Q)]$ and $\mathbb{E}[d_d(T,Q)]$ are respectively, the probability of a PM, the probability

of a CM, the expected number of inspections, and the expected cumulative down time of the system in a maintenance cycle. The optimal values of T and Q are obtained by the following expression:

$$C_\infty(T_{opt}, Q_{opt}) = \min_{T,Q}\{C_\infty(T,Q), T > 0, 0 \le Q \le 1\} \tag{5.6}$$

More discussion on cost component refer to Huynh et al. (2014).

5.4 Mean RUL-based Maintenance Model (T, τ_m) Policy

Khanh Le Son shows that the expectation of $RUL(kT)$ is defined as follows:

$$\mathbb{E}(RUL(kT)) = \int h \cdot \frac{dF_{RUL(kT)}(h)}{dh} dh \tag{5.7}$$

where $F_{RUL(kT)}(h)$ is the CDF of RUL at time kT.

In the framework of Mean RUL-based maintenance policy, a fixed threshold τ_m is settled. The following decision frame is adopted:

- If $Y_{kT} \le L$, and $\mathbb{E}(RUL(kT)) > \tau_m$, that means the estimated mean RUL is still higher than the threshold τ_m. Nothing is done and the decision is postponed until next inspection.
- If $Y_{kT} \le L$, and $\mathbb{E}(RUL(kT)) \le \tau_m$, the component is preventively replaced with a cost C_p.
- If $Y_{kT} \ge L$, the system has already failed, and a corrective maintenance is carried out with a cost C_c.

Similar to 5.5, the long term expected maintenance cost rate of (T, τ_m) is

$$C_\infty(T, M) = \frac{C_p\mathbb{P}_p(T,\tau_m) + C_c\mathbb{P}_c(T,\tau_m) + C_i\mathbb{E}[N_i(T,\tau_m)] + C_d\mathbb{E}[d_d(T,\tau_m)]}{\mathbb{E}[S(T,\tau_m)]} \tag{5.8}$$

where $\mathbb{E}[S(T,\tau_m)]$ is the expected length of a replacement cycle under (T,τ_m) policy, $\mathbb{P}_p(T,\tau_m)$, $\mathbb{P}_c(T,\tau_m)$, $\mathbb{E}[N_i(T,\tau_m)]$ and $\mathbb{E}[d_d(T,\tau_m)]$ are respectively, the probability of a PM, the probability of a CM, the expected number of inspections, and the expected cumulative down time of the system in a maintenance cycle. The optimal values of T and τ_m are obtained by the following

expression:

$$C_\infty(T_{opt}, \tau_{m_{opt}}) = \min_{T,\tau_m}\{C_\infty(T,\tau_m), T>0, 0 \le \tau_m \le MTTF\} \tag{5.9}$$

More discussion on cost component refer to Huynh et al. (2014).

5.5 Comparison and Performance Assessment

Due to limited amount of time, the writer does not perform a detailed quantitative comparison between the above three maintenance policies. Such work can be further investigated in future works. However, the performances of these policies can be seen from some previous papers.

Huynh et al. (2011) and Huynh et al. (2014) has qualitatively and quantitatively assessed the efficiency of RUL application compared to degradation level in maintenance. Sensitivity analysis on various maintenance costs are carried out in his papers. Also he analyzes the equivalence, performance and the flexibility of the mentioned three policies. Several points can be drawn from his papers:

- RUL-based maintenance policies are always more profitable than degradation-based maintenance policy. Additionally, the benefits of (T,τ_m) policy is more obvious when the variance of degradation process becomes more important.
- The (T,Q) policy is more stable around the optimal value to the variation of T. Thus such policy is more robust when it is difficult to determine inspection interval T.
- The impact of the decision parameter error on the performance of the RUL-based maintenance policies are more negligible for a high variance degradatio process.
- Under some special situations, all three maintenance policies can lead to almost same costs. Therefore, it appears more natural to use degradation level based maintenance policy because of it simplicity.

From these key points, we can see that RUL is an promising indicator to ensure better performance of maintenance actions. But, such conclusion is based on a strong assumption includes perfect monitoring and stochastic degradation process etc. More numerical implementations should be carried out to clearly points out the limitations and advantages of RUL related policy.

Chapter 6

Summary

This chapter sums up what has been done and what the result shows. Some challenges are discussed and recommendations for future work are given as well.

6.1 Summary and Conclusions

The thesis focuses on prognostic maintenance policy based on a relatively novel condition index: the Remaining Useful Lifetime (RUL). The system degradation process is modeled by a homogeneous Gamma process which considering only unit-to-unit variability. Such method is advantageous as it requires no special product knowledge, physics or engineering principles, easy implementation and higher computational efficiency. Based on this degradation model, the maintenance cost of two proposed prognostic maintenance policies are briefly reviewed and compared with the traditional degradation level maintenance policy. The maintenance inspection is fixed. Previous papers show that RUL is a more effective condition indicator for Condition-based Maintenance decision making. However, the inspection period can be optimized by applying RUL estimation, and special attention should be paid on configurations if one decides to rely on RUL-based maintenance policies.

The thesis have 6 chapters and the objective of this thesis is to review, summarize, demonstrate and extend the formulas, methods and models mentioned above, explain how they can be integrated and used, and hopefully figure out their limitations and reasons.

Most of the tasks stated in Chapter 1 has been implemented. The thesis starts with introduc-

ing the background, reasons, objective, limitations of this thesis. Second, the writer presents a brief survey over existing review papers in relation to condition-based maintenance and prognostic methods. Given the taxonomies in Chapter 1, the writer narrows down the main topic into stochastic process for prognostic, and summarizes mathematical properties relevant for parameter estimation and RUL calculation. Brownian motion process is presented first as a preparation study for Gamma process afterwards. The writer also refrained from theory and methods which are too difficult to understand or require too much efforts to use, and includes some simulations to illustrate their most important properties. One numerical example is presented and analyzed in each of stochastic process to integrate the knowledge discussed before and to demonstrate the whole modeling process and their benefits. Among such, special focus have been paid on parameter estimation and RUL calculation. At last, two RUL-based maintenance policies are proposed and compared with degradation level-based maintenance policy. Some advantages of the prior policies are briefly stated.

6.2 Discussion and Recommendations for Further Work

In this thesis, the writer tempts to review, summarize and demonstrate the stochastic process model for estimating RUL and its use in condition based maintenance. Some algorithms, models and modeling techniques have been discussed with extensive literatures and illustrated by simulations. Although the numerical examples shows that the presented model in this thesis work better in current cases, the investigation is still very preliminary. There remain many open questions and practical challenges to be further studies before valid degradation models and CBM policies can be applied to practical systems (i.e subsea systems). Some of these include the following:

1. First, it is desirable to develop an integrated RUL estimation model involving more than one kind of heterogeneity (i.e. unit-to-unit variability, various working environment, and different workloads etc.). This will complicate the problem by considering the influence of both external variables such as temperature, shocks, or maintenance actions etc. and also internal variables like material, structure, functional modes etc. Those variables will together affect the observed condition indicators which in turn will influence degradation

modeling and RUL calculation.

2. The second challenge lies in data fusion. Different sources of condition indexes are available. But they are not in the same scale and may be dependent on each other. Few papers consider this problem now and it remains to be answered like how to effectively utilize these informations and contribute to the overall precision of degradation modeling.

3. The thesis focus on computing RUL distribution for a population of independent identical components. However, it is more common and interesting to estimate RUL for a single specific operating device using sensor based monitoring signals. A possible way is to add Bayesian network to the current stochastic process so that history data and online data of the individual components characterize the performance of the system.

4. The thesis presents a degradation model based homogeneous Gamma process which each increments of the process is identically distributed. In practice, however, such assumptions are quite restrictive and nonlinear non-homogeneous stochastic process are more common. For example, the length of crack propagation is non-linear in time, and a transformation in time scale is necessary. Therefore, some modifications (like includes covariates, noise, hybrid models)in Gamma process should be incorporated into degradation modeling and RUL estimation in future studies to enhance its performance in versatile situations.

5. The fourth challenge lies in parameter estimation when inspection interval are not uniform. Flexible inspection time is a common scenario in CM practice especially in maintenance grouping. For Gamma process, the observations of degradation index in this situation are independent random variables, but not identically distributed random variables. As a result, the calculation is more complex but is mathematically tractable as well.

6. The fifth challenge is to develop a degradation model based on very few or even no data situations. As Ahmadzadeh and Lundberg (2014) states this is typical for newly developed systems where no observed failure data and CM information exist. As a result, it is very difficult to identify a real maintenance case with sufficient recorded data. Alternative solutions may be to integrate physical based model with expert judgment from designers

and manufacturers. These require basic understanding of engineering aspects of the system.

In summary, most of the models in the thesis are based on theoretical equations without real application focuses, and a lot of works need to be done in order to justify the application of these policies.

Appendix A

Acronyms

CBM Condition-based Maintenance

CDF Cumulative Distribution Function

CM Corrective Maintenance

FHT First Hitting Time

IG Inverse Gaussian

MTTF Mean time to failure

PM Preventive Maintenance

RAMS Reliability, availability, maintainability, and safety

RUL Remaining Useful Lifetime

SDE Stochastic Differential Equation

Appendix B

Definitions and Theorems

B.1 Stopping Time

Definition 1. *A random time T is called a stopping time for $X(t), t \geq 0$, if for any t it is possible to decide whether T has occurred or not by observing $X(s), 0 \leq s \leq t$.*

B.2 Strong Markov Property

Theorem 1. *For any finite stopping time T the regular conditional distribution of $X(T+t), t \geq 0$ given $\mathscr{F}_T$ is $\mathbb{P}_{X(T)}$, that is,*

$$\mathbb{P}(X(T+t) \leq y | \mathscr{F}_T) = \mathbb{P}(X(T+t) \leq y | X(T)) \, a.s. \tag{B.1}$$

B.3 New Process

Corollary 1. *Let T be a finite stopping time. Define the new process in $t \geq 0$ by*

$$\hat{X}(t) = X(T+t) - X(T) \tag{B.2}$$

Then $\hat{X}(t)$ is a Brownian motion started at zero and independent of $\mathscr{F}_T$.

Appendix C

Matlab Code for Wiener Process Simulation in Chapter 3

C.1 Example 1

```
function [W] = wiener(N,mu,sigma,T)
dt = T/N; % dt is time step
dW = zeros(1,N); % preallocate arrays
W = zeros (1,N);
Y1 = zeros(1,N+1);

for pic = 1:1:4
t = 0:dt:T; % create a time vector with N steps
dW (1) = sqrt (dt) * randn; % first approximation outside loop...
W (1) = dW (1) ; % since W(0) is not allowed
    for step = 2: N
    dW(step) = sqrt(dt)*randn; % general increment
    W(step) = W(step-1) + dW(step);
    end
Y1 = mu * t + sigma * [0,W];
```

```
16 subplot(2,2,pic);plot(t, Y1); hold on % plot the path
17 subplot(2,2,pic);plot(t,mu * t,':'); hold on
18 axis([0 T min(-sigma,(mu-2*sigma)*T) max(sigma,(mu+2*sigma)*T)])
19 title([int2str(N) '-step wiener process and its mean'])
20 xlabel(['Mu ' num2str(mu) ', Sigma ' num2str(sigma)])
21 end
22 hold off
```

C.2 Example 2

```
1 function tempomu( X, N, mu, sigma, T, delta_mu, mu_steps )
2 % delta_mu is the step increment, mu_steps is the number of increments
3 for j = 1 : mu_steps
4     subplot( 2,2,j);
5     for i = 1 : X
6         wiener(N, mu, sigma, T);
7         hold on;
8     end
9     title( [ 'mu = ' num2str(mu) ]);
10 mu = mu + delta_mu;
11 end
```

C.3 Example 3

```
1 function temposigma( X, N, mu, sigma, T, delta_sigma, sigma_steps )
2 % delta_sigma is the step increment, sigma_steps is the number of
    increments
3 for j = 1 : sigma_steps
4     subplot( 2,2, j);
5     for i = 1 : X
6         wiener(N, mu, sigma, T);
```

```
7           hold on;
8       end
9       title( [ 'sigma = ' num2str(sigma)  ]);
10    sigma = sigma + delta_sigma;
11  end
```

C.4 Testing Data Generation and Parameter Estimation

```
1  %function [W] = wiener(x,N,mu,sigma,T)
2  %generate 100 paths as testing dataset
3  T = 1;   % set time interval [0,T]
4  N = 1000; % set number of steps to compute in [0,T]
5  dt = T/N; % dt is time step
6  mu = 6;
7  sigma = 3;
8  dW = zeros(1,N); % preallocate arrays
9  W = zeros (100,N);
10 Y = zeros(100,N+1);
11 x = ones (1,N+1);
12
13 for unit = 1:1:100
14 t = 0:dt:T; % create a time vector with N steps
15 dW (1) = sqrt (dt) * randn; % first approximation outside loop...
16 W (1) = dW (1) ; % since W(0) is not allowed
17      for step = 2: N
18          dW(unit,step) = sqrt(dt)*randn; % general increment
19          W(unit,step) = W(unit,step-1) + dW(unit,step);
20      end
21 Y(unit,1:N+1) = x + mu * t + sigma * [0,W(unit,:)];
22 subplot(2,2,1);plot(t, Y(:,1:N+1));
```

```
23 title('Degradation paths for all units');
24 axis([0 1 -5 20]);% plot all the paths
25 subplot(2,2,2);plot(t,Y(1,1:N+1));
26 title('Unit 1');
27 axis([0 1 -5 20]);
28 subplot(2,2,3);plot(t,Y(50,1:N+1));
29 title('Unit 50');
30 axis([0 1 -5 20]);
31 subplot(2,2,4);plot(t,Y(100,1:N+1));
32 title('Unit 100');
33 axis([0 1 -5 20]);
34 xlabel(['Mu ' num2str(mu) ', Sigma ' num2str(sigma)])
35 end
36
37 %parameter estimation,estimate mu
38 s = 0;
39 for row = 1:1:100;
40     s = s + Y(row,1001);
41 end
42 q = (s-100)/100;
43 %calculate Delta Y_ij table, estimate sigma
44 ss = 0;
45 for row = 1:1:100;
46     for column = 1:1:1000;
47     DeltaY(row, column)= (Y(row,column + 1)-Y(row,column));
48     v(row,column) = ((DeltaY(row,column) - q * dt)^2)/ dt;
49     ss = ss + v(row, column);
50     end
51 end
52 p = sqrt(ss/(N * 100));
```

Appendix D

Matlab Code for Gamma Process Simulation in Chapter 4

D.1 Realizations of gamma process path

```
a = 4; b = 1;
T = 10;
N = 1000; %1000 time steps
M = 100; %generate 100 paths as testing dataset
h = T/N;
t = (0: h :T);
g = zeros(M,N+1);
G = zeros(M,N+1);

for unit = 1:1:M
for i = 1:N
    g(unit,i) = gamrnd(a * h, b);
    G(unit,i+1) = G(unit,i) + g(unit,i);
end
subplot(2,2,1);
```

```
scatter(t,G(unit,:),'b','.');hold on
plot(t,a*t/b,'r');
axis([0 T 0 120])
title([int2str(N) '-step version of gamma proccess path'])
xlabel(['a= ' num2str(a) ', b= ' num2str(b)])
end
```

D.2 Testing Data Generation and Parameter Estimation

```
a = 4; b = 1;
T = 10;
N = 1000; %number of obs
M = 100; %number of units
h = T/N;
t = (0: h :T);
g = zeros(M,N+1);
G = zeros(M,N+1);

for unit = 1:1:M
    for i = 1:N
    g(unit,i) = gamrnd(a * h, b);
    G(unit,i+1) = G(unit,i) + g(unit,i);
    end
subplot(2,2,1);plot(t, G(1:M,1:N+1));
title('Degradation paths for all units');
axis([0 10 0 60]);% plot all the paths
subplot(2,2,2);plot(t,G(1,1:N+1));
title('Unit 1');
axis([0 10 0 60]);
subplot(2,2,3);plot(t,G(50,1:N+1));
```

```
22 title('Unit 50');
23 axis([0 10 0 60]);
24 subplot(2,2,4);plot(t,G(M,1:N+1));
25 title('Unit 100');
26 axis([0 10 0 60]);
27 xlabel(['a =' num2str(a) ', b =' num2str(b)])
28 end
29
30 % Newton method iteration
31 % Solves g(a)=0 by doing nmax steps of Newton's method starting at a =1
32 aa = 3.8; %set starting value
33 itermax = 200; %the max number of iterations
34 eps = 1; %initialize error bound eps
35 %avals = aa; %initialize array of iterates
36 iter = 0; %initialize iteration count
37 XSum = 0;
38 for  row = 1:1:M;
39     XSum = XSum + G(row,N+1);
40 end
41 d = 0;
42 dd = 0;
43 while eps>=1e-4 && iter<=itermax
44     o = gamma(aa * h);
45     y = gamma(aa * h)*psi(aa * h); % differentiation of gamma distribution
46     z = gamma(aa * h)* (psi(aa * h)^2) + gamma(aa * h)*psi(1, aa *h); % 2
          order diff of gamma
47     for row = 1:1:M;
48         for column = 1:1:N;
49             d = d + h .* (log(aa .* T .* M ./ XSum)) + h .* (log(g(row,
                  column))- psi(aa .* h));
```

```
50              dd = dd + h ./ aa - (h.^2) * (z * o-(y.^2))/ (o.^2);
51          end
52      end
53   u = aa - d./dd;
54   eps = abs(u-aa);
55   aa = u;
56   iter = iter + 1;
57  end
58  bb = aa .* T .* M ./ XSum;
59
60  %calculate a0, a1, a2, a3 in Hessian Matrix
61  a0 = 1;
62  o = gamma(aa * h);
63  y = gamma(aa * h)*psi(aa * h); % differentiation of gamma distribution
64  z = gamma(aa * h)* (psi(aa * h)^2) + gamma(aa * h)*psi(1, aa *h); % 2
      order diff of gamma
65      for row = 1:1:M;
66          for column = 1:1:N;
67              a0 = a0 + (h.^2) * (z * o-(y.^2))/ (o.^2);
68          end
69      end
70  a1 = -T .* M ./ bb;
71  a2 = a1;
72  a3 = aa * T * M / (bb^2);
73
74  I = [a0 a1; a2 a3];
75  I1 = I ^ -1; %hessian matrix with estimated parameters
```

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