Maintenance analysis of a two-component load-sharing system

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Abstract

This paper presents a two-component load-sharing system. The failure rates of the two components are time dependent and load dependent. Whenever one fails, it is imperfectly repaired with a time delay during which the failure rate of the survival component increases because of the resulting overload. Three maintenance policies are proposed considering imperfect preventive maintenance and system replacement. The optimal average costs in the long run under different maintenance policies are derived from the theoretical propositions. Sensitivity analyses through numerical examples are carried out.

Keywords: Load-sharing system, multi-component systems, failure interaction, age replacement, imperfect repair, maintenance policy optimization, hoisting problem.

Notation

X_i	lifetime of component i, $i = 1, 2$.
$h_i(t)$	failure rate of component i at time t, $i = 1, 2$.
$l_i(t)$	load undertaken by component i at time t
$\beta_i(t)$	nominal failure rate of component i at time t in absence of the load
τ_0	duration of one mission
τ	duration between two consecutive imperfect repair in policy 1, $\tau = k_2\tau_0, k_2 \in \mathbb{N}^*$

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$\tau^{'}$	duration between two consecutive imperfect repair of the long-lived component after the failure of its counterpart in policy 3, $\tau^{'} = k_2^{'}\tau_0 < \tau$
λ	failure rate reduction factor, $0 < \lambda < 1$
$\lfloor x \rfloor$	maximum integer not greater then x
$\lceil x \rceil$	minimum integer not smaller then x
$r_1(t)$	failure rate of the short-lived component under policy 1
$r_i(t)$	failure rate of the long-lived component under policy i, $i = 2, 3$
T	system preventive replacement time under each policy, $T = k_3\tau$, $k_3 \in \mathbb{N}^*$
p_k	probability that the system is replaced at $k\tau_0$ in policy 1
q_k	probability that both components fail in the period $((k-1)\tau_0, k\tau_0]$ in policy 1
$F(x)$	lifetime distribution of component i when both of them are functional, $i = 1, 2$.
n_x	$\lfloor \frac{x}{\tau} \rfloor$
m_x	$\lfloor \frac{x}{\tau_0} \rfloor$
M_x	$\lceil \frac{x}{\tau_0} \rceil$
$\hat{n}_j$:	$\lfloor \frac{j}{k_2} \rfloor$, $j \in \mathbb{N}^*$
$\tilde{n}_j$	$\lfloor \frac{j}{k_2^{'}} \rfloor$, $j \in \mathbb{N}^*$
$p_{i,k}$	probability that the two components fail in $((i-1)\tau_0, i\tau_0]$, $((k-1)\tau_0, k\tau_0]$ respectively before the system preventive replacement under policy 2
P_{ik_3}	probability that one component survives at $k_3\tau_0$ while one fails in $((i-1)\tau_0, i\tau_0]$, $1 \leq i \leq k_3$ under policy 2.
$p_{i,k}^{(3)}$	probability that the two components fail in $((i-1)\tau_0, i\tau_0]$, $((k-1)\tau_0, k\tau_0]$ respectively before the system preventive replacement under policy 3
$P_{ik_3}^{(3)}$	probability that one component survives at $k_3\tau_0$ while one fails in $((i-1)\tau_0, i\tau_0]$, $1 \leq i \leq k_3$ under policy 3.

1. Introduction

In recent decades, maintenance study of multi-component systems are becoming more and more intractable. In fact, the reliability and optimal maintenance policies of multi-component systems can be obtained by similar approaches as when dealing with single unit systems only if components are totally independent. However, such assumption is often unrealistic because of the internal complex structures of the system, the set up costs, the common cause failures to the components, etc. For instance, in a brake system, serious damages are induced to the disc rotor if the brake pad is worn out and is not replaced [1]; in a high voltage system, a transmitter have to undertake more range if some of its counterparts fail [2]. Therefore, it is difficult to model the system or to calculate the system reliability because of the complex structure of the system and the stochastic correlation of failure times between them. Besides, component interactions complicate the maintenance policies as it provides the opportunity of group maintenance (also called opportunistic replacement) which can be more economical [3]. In the literature, Thomas [4] classified the interactions between components of the multi-component system into three types. Firstly, the economic interaction indicates that the system maintenance cost is related to the components [5, 6]. For example, it may be cost saving to repair several components together comparing to repair them individually. Structural dependence [7] implies that one has to remove or even to replace the non-failed components to repair the failed one. Stochastic dependence, also known as failure interaction means that the failures of components can affect the state (the deterioration level, the failure rate, etc.) of some working components [8, 9].

The main goal of the present paper is to focus on a specific case of stochastic dependency (load sharing) and to propose optimal maintenance decision rules in this context. The system under study is supposed to be described by two main equivalent and interacting sub-systems.

Different classification of stochastic dependencies are introduced in literature. Murthy and Nguyen [10, 11] presented two types of stochastic dependence of two-component systems and multi-component systems respectively. For two-component systems, type 1 failure interaction implies that the failure of component 1 may induce the failure of component 2 with a given probability (constant or time dependent) and vice versa. While type 2 failure interaction indicates that the failure of a component can affect the failure rates of the others. Further, Nakagawa and Murthy [12] introduced shock damage interaction which has been intensively studied [13, 14, 15]. Such type of interactions implies that the failure of a component induces a random damage to the non-failed component.

It is worth mentioning that the type 2 failure interaction between components is common in the load-sharing systems [16, 17, 18, 19]. In material testing, software reliability, population sampling, mechanical engineering the load can strongly impact the component state [20, 21, 22, 23] (failure rate, reliability, availability, damage level etc.). Because in a load-sharing system, when a component fails, the static or time-varying workload [24] is undertaken by the non-failed components. Therefore the state of the survival components are affected by the increased load they bear.

Load-sharing systems have been extensively studied in the framework of statistic inference and reliability characteristics. Kim et al. [25] proposed the classical maximum-likelihood estimation of the system parameters where all components have identical exponential distribution. Park et al. [26] extended the model of Kim et al. [25] by considering the parallel system with Weibull distributed lifetime distribution. The closed-form Maximum Likelihood Estimator and conditional Best Unbiased Estimator of the Weibull rate parameters are derived. Singh et al. [27] developed the reliability of a parallel system with Lindy lifetime distribution components and the system parameters estimation are presented. Jain and Gupta [28] obtained the reliability and mean time to failure of a load sharing M-out-of-N system with non-identical and non-repairable components. Amari et al. [29] gave an overview of the load-sharing systems. More recent papers have considered load sharing systems for instance [30, 31, 32, 33].

Most part of the aforementioned works deal with statistic inference and reliability study and very few are focused on maintenance optimization [32, 34]. Almost none are looking at dedicated maintenance strategies to handle the load sharing problems. In this paper, a two-component system is studied to investigate how preventive maintenance actions can compensate for the negative effects of the load sharing. The remainder of the paper is organized as follows. In section 2, model assumptions and different maintenance policies are presented. Section 3 is devoted to the maintenance optimization problem. The maintenance policy performances are analyzed through numerical calculations and Monte Carlo simulations in section 6. The advantages and weakness of the policies are pointed out and discussed. Finally, the conclusion and further research aspects are indicated.

2. Problem statement

The main goal is to investigate how maintenance actions can compensate for the negative effects of the load sharing. A first strategy (Policy 1) is proposed which is very conservative and very simple to implement: inspections and preventive actions are performed for each components, and the whole system is either correctively renewed if one sub-system fails or preventively renewed at a pre-determined age T. Hence, the effect of the load sharing is assumed to be so risky that the whole system is replaced as soon as possible after the first failure. The second proposed strategy (Policy 2) tends to be less conservative: the whole system is renewed only after a given age (i.e. time in operation) or after a short delay of the system failure. At last, a third strategy (Policy 3) will add the possibility to increase the inspections and preventive actions periodicity after the first failure of a component. In each situation, the preventive maintenance age is optimized to balance between the risk of total failure and the cost of over-renewal. Hence, the effects of the load sharing are carefully taken into account from two main perspectives:

- how long the load sharing system is supposed to be in usage?
- what is the optimal inspection periodicity before and after the first failure of one of the redundant sub-system?

2.1. Model description for the load sharing

In the work of Birnbaum et al. [22], they assumed that the system failure is caused by two different causes: the system load and deterioration factors independent of the load. Under these hypotheses, the example of lifetime estimation of the 6061-T6 aluminum sheeting is addressed. Moreover, in [23], the author considered a system where the failure rate depends on the load and a constant deterioration.

Similarly to Birnbaum et al. [22], in our study, we consider that the failure rate depends on the load and a deterioration factor. On contrary to [23], we consider that the deterioration is time dependent. More precisely, the failure rate of component i at time t is defined as follows:

$$h_i(t) = \beta_i(t) l_i(t)$$

where $l_i(t)$ is the load it undertakes at time t, $\beta_i(t)$ is the nominal failure rate in absence of load representing the deterioration or corrosion related factor of component i at time t, $i = 1, 2..$

Furthermore, it is assumed that in absence of load, according to the number of survival components, the lifetime of component i follows a Weibull distribution with a scale parameter equal to one. In other words,

$$\beta_i(t) = \begin{cases} at^{a-1} & \text{if both components are operational at time } t \\ a_1 t^{a_1-1} & \text{if component } 3-i \text{ fails before time } t \end{cases}$$

where, $i = 1, 2$ and $a_1 \geq a \geq 1$. So the component deterioration has a positive dependence on the load it bears. The system load $2l$ is shared uniformly by both components when they operate. If one component fails at time t, the system is still functional with the survival one who takes the whole load.

The duration of one mission of the system is noted as τ_0. During a mission, that is to say within a time horizon $((k\tau_0, (k+1)\tau_0]$ for any $k \in \mathbb{N}^*$, the system cannot be maintained if failure occurs. Therefore, the maintenance operations can be carried out only at the end of missions at time $\tau_0, 2\tau_0, \ldots$.

To avoid failure and therefore a period of unavailability and loss of production, different maintenance operations are carried out. We propose and analysis three policies in our study. The maintenance operations, their impacts and scheduling are described as follows.

2.2. Policy 1: component based policy

The system undergoes

- preventive imperfect repairs after each k_2 missions. In other words, preventive imperfect repairs are carried out at age $\tau, 2\tau, \cdots$ where $\tau = k_2\tau_0, k_2 \in \mathbb{N}^*$. The approach of Arithmetic Reduction of Intensity with memory one (ARI_1) [35] is carried out to describe the imperfect repair action which yields

  $$\beta_i((j\tau)^+) = \beta_i(j\tau) - \lambda[\beta_i(j\tau) - \beta_i((j-1)\tau)^+]$$

  where $(j\tau)^+$ is the right limit of $j\tau$, $j \in \mathbb{N}^*$, $0 < \lambda < 1$, $i = 1, 2$. It is also assumed that the imperfect repair has no effect on the wear-out speed of the system.

- preventive replacements at system age T or at the end of mission after the first component failure which occurs first. In other words, The system is preventively replaced at $T = k_3\tau_0$ or at $i\tau_0$, where the **first component** failure occurs in $((i-1)\tau_0, i\tau_0]$, $i = 1, 2, \cdots, (k_3-1)$, which comes first. The constant k_3 is a decision parameter.

- The replacement after failure is not instantaneous. More precisely, there is a delay $i\tau_0 - t_f$ when a component fails at $t_f \in ((i-1)\tau_0, i\tau_0]$, $i = 1, 2, \cdots, (k_3 - 1)$.

- The cost of system imperfect repair is c_2 ($\frac{c_2}{2}$ for each component), and the cost for renewing system is c_r each time. Besides, there is a penalty c_p when both components fail in the same time period $((i-1)\tau_0, i\tau_0]$, $i = 1, 2, \cdots, k_3$.

An example of such policy is depicted in Figure 1.

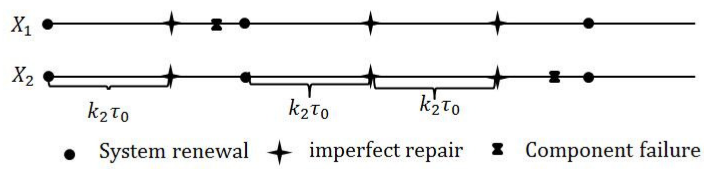

Figure 1: A possible operational process of the system under policy 1

2.3. *Policy 2: system based policy*

- The system is imperfectly repaired as in policy 1.

- The system is replaced at age T ($T = k_3\tau_0$) or at $k\tau_0$ when **the system** fails in $((k-1)\tau_0, k\tau_0]$ which occurs first with cost c_r , $k = 1, 2, \cdots, (k_3 - 1)$.

- There is a penalty c_p if both the two components fail by time T.

Comparing to policy 1, under which the system is replaced with a time delay when the short-lived component fails, in policy 2, the system keeps operating with the long-lived component until it fails. Since the replacement is not instantaneous, there is a period of unavailability. An example of policy 2 is depicted in Figure 2.

2.4. *Policy 3: component based policy variant*

- When both components are operational as in policy 1, preventive imperfect repairs are carried out at age $\tau, 2\tau, \cdots$. After the first component failure, the survival component is imperfectly repaired as in policy 1 but more frequently at intervals of $\tau' = k_2'\tau_0$, $k_2' < k_2$. An imperfect repair for each component incurs a cost $\frac{c_2}{2}$.

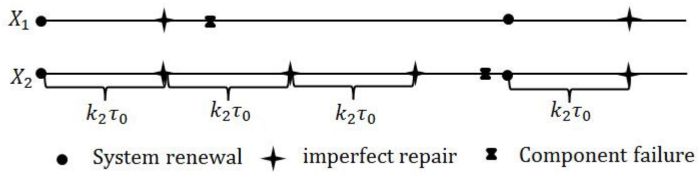

Figure 2: A possible operational process of the system under policy 2

- The replacement policies (both the corrective and the preventive replacement) are similar to policy 1.

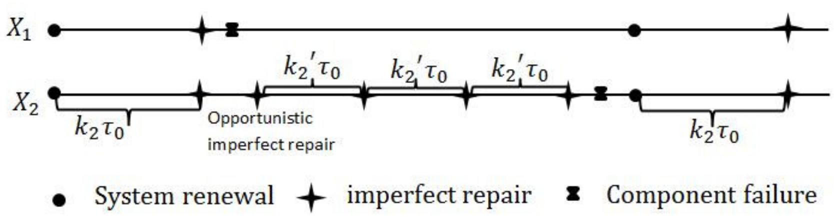

Figure 3: A possible operational process of the system under policy 3

Possible operational processes of the system policy 3 is depicted in Figure 3.

Let be $r_1(t)$ the failure rate of the short-lived component, $r_2(t)$ (resp. $r_3(t)$) be the failure rate of the long-lived component in policy 2 (resp. policy 3). Figure 4 gives an example of their variation tendencies where the red point represents the failure of the short-lived component.

3. Maintenance policy evaluation

In this section, the long run average maintenance costs under different maintenance policies are calculated.

3.1. Average cost evaluation under policy 1

3.1.1. The failure rate and the lifetime distribution

It is easily seen that the failure rate of the short-lived component under policy 1 can be given as

$$r_1(t) = at^{a-1}l - \lambda a(i\tau)^{a-1}l, i\tau < t \leq (i+1)\tau.$$

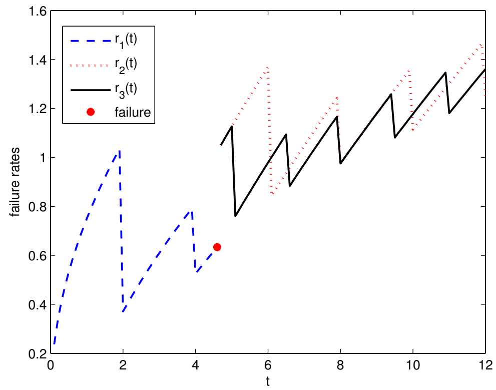

Figure 4: failure rates

Let $\overline{F}(x)$ be the survival function of the short-lived component. Denote $n_x := \lfloor \frac{x}{\tau} \rfloor$ for any $0 \leq x < T$, then we have:

$$\overline{F}(x) = \exp(-x^a l + \lambda l a \tau^{a-1} \sum_{i=1}^{n_x} z_i i^{a-1}) \tag{1}$$

where $z_i = x - n_x\tau$ if $i = n_x$ and τ otherwise.

3.1.2. The system replacement and failure probability

Denote p_k be the probability that the system is replaced at time $k\tau_0$, q_k be the probability that both components fail in the same period $((k-1)\tau_0, k\tau_0]$, $k = 1, 2, \cdots, k_3$. As

$$p_k = \mathbb{P}(X_1 > (k-1)\tau_0, X_2 > (k-1)\tau_0) - \mathbb{P}(X_1 > k\tau_0, X_2 > k\tau_0), \quad k = 1, 2, \cdots, (k_3-1).$$

one can deduce

$$p_k = \overline{F}^2((k-1)\tau_0) - \overline{F}^2(k\tau_0), k = 1, 2, \cdots, (k_3 - 1). \tag{2}$$

$$p_{k_3} = 1 - \sum_{k=1}^{k_3-1} p_k. \tag{3}$$

Similarly, we have q_k as follows:

$$
\begin{aligned}
q_k &= 2\mathbb{P}\Big((k-1)\tau_0 < X_1 \le X_2 < k\tau_0\Big) \qquad (4)\\
&= p_k - 2\mathbb{P}\Big((k-1)\tau_0 < X_1 < k\tau_0, X_2 > k\tau_0\Big)\\
&= p_k - 2\int_{(k-1)\tau_0}^{k\tau_0} f(x)\overline{F}(x)\exp\Big(-2l[(k\tau_0)^{a_1} - x^{a_1}] + \lambda la(n_x\tau)^{a-1}(k\tau_0 - x)\Big)\mathrm{d}x
\end{aligned}
$$

where $n(x) = \lfloor \frac{x}{\tau} \rfloor$, $F(x) = 1 - \overline{F}(x)$ and $f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x} = (lax^{a-1} - \lambda la(n_x\tau)^{a-1})\overline{F}(x)$ for $k = 1, 2, \cdots, k_3$.

3.1.3. Average long-run cost per unit time

The mean cost of one renewal cycle is:

$$\sum_{k=1}^{k=k_3} p_k(\hat{n}_{k-1}\ c_2 + c_r) + c_p \sum_{k=1}^{k=k_3} q_k$$

where $\hat{n}_{k-1} = \lfloor \frac{k-1}{k_2} \rfloor$. The average length of a lifetime cycle is defined:

$$\sum_{k=1}^{k_3} p_k k\tau_0$$

According to the renewal reward process, the long-run cost per unit time $C(k_3)$ under policy 1 can be given by

$$C(k_3) = \frac{c_r +\ c_2 \sum_{k=1}^{k=k_3} p_k\hat{n}_{k-1} + c_p \sum_{k=1}^{k=k_3} q_k}{\sum_{k=1}^{k_3} p_k k\tau_0} \qquad (5)$$

The average cost can be obtained by substituting equations (2), (3) and (4) into equation (5). By utilizing the similar method, the cost rate in the long run under policy 2 and 3 are derived in the following respectively.

3.2. Average cost evaluation under policy 2

3.2.1. The failure rate

Given that $min(X_1, X_2) = x$, the failure rate of the long-lived component $r_2(t)$ under policy 2 can be represented as

$$r_2(t) = \begin{cases} r_1(t) & t \le x \\ 2la_1t^{a_1-1} - \lambda la(n_x\tau)^{a-1} & x < t \le (n_x+1)\tau \\ 2la_1t^{a_1-1} - 2\lambda la_1((n_x+i)\tau)^{a_1-1} & (n_x+i)\tau < t \le (n_x+i+1)\tau, \end{cases}$$

for $i = 1, 2, \cdots$

3.2.2. The system replacement and failure probability

Denote by $p_{i,k}$ the probability that the two components fail in $((i-1)\tau_0, i\tau_0]$, $((k-1)\tau_0, k\tau_0]$ respectively, where $1 \leq i < k \leq k_3$, one can deduce:

$$\begin{aligned} p_{i,k} &= 2\mathbb{P}\Big((i-1)\tau_0 < X_1 \leq i\tau_0, (k-1)\tau_0 < X_2 \leq k\tau_0\Big) \qquad (6) \\ &= 2\int_{(i-1)\tau_0}^{i\tau_0} f(x)\overline{F}(x)\Big(\exp\big(-\int_x^{(k-1)\tau_0} r_2(t)\mathrm{d}t\big) - \exp\big(-\int_x^{k\tau_0} r_2(t)\mathrm{d}t\big)\Big)\mathrm{d}x \\ &= 2\int_{(i-1)\tau_0}^{i\tau_0} f(x)\overline{F}(x)\big(K_{(k-1)\tau_0}(x) - K_{k\tau_0}(x)\big)\mathrm{d}x \end{aligned}$$

where

$$K_t(x) = \begin{cases} \exp\big[-2l\big(t^{a_1} - x^{a_1}\big) + \lambda la(n_x\tau)^{a-1}(t-x)\big], & n_x = n_t \\ \exp\big[-2l\big(t^{a_1} - x^{a_1}\big) + \lambda la(n_x\tau)^{a-1}((n_x+1)\tau - x) & \\ \quad +2l\lambda\sum_{j=1}^{n_t-n_x} z_j a_1((n_x+j)\tau)^{a_1-1}\big], & \text{otherwise} \end{cases} \qquad (7)$$

where $n_(x) = \lfloor \frac{x}{\tau} \rfloor$, $F(\cdot)$ is the system lifetime distribution defined in equation (1) and $f(\cdot)$ is its intensity function. $z_j = t - n_t\tau$ when $j = n_t - n_x$ and τ otherwise.

Similarly, denote P_{ik_3} be the probability that one component survives at $k_3\tau_0$ while one fails in $((i-1)\tau_0, i\tau_0]$, $1 \leq i \leq k_3$. Then

$$P_{ik_3} = 2\int_{(i-1)\tau_0}^{i\tau_0} f(x)\overline{F}(x)K_{k_3\tau_0}(x)\mathrm{d}x \qquad (8)$$

where $K_{k_3\tau_0}$ is given as in equation (7).

Let $p_{k,k} = q_k$, denote $p_k^{(2)}$ be the probability that the system is renewed at $k\tau_0$ which yields

$$\begin{aligned} p_k^{(2)} &= \sum_{i=1}^{k} p_{i,k}, k = 1, 2, \cdots, k_3 - 1; \\ p_{k_3}^{(2)} &= 1 - \sum_{i=1}^{k_3-1} p_k^{(2)}. \end{aligned}$$

3.2.3. Average long-run cost per unit time

The cost rate in the long run $C^{(2)}(k_3)$ under maintenance policy 2 is therefore:

$$C^{(2)}(k_3) = \frac{1}{\sum_{k=1}^{k_3} p_k^{(2)} k\tau_0}(c_r + \frac{c_2}{2}\sum_{k=1}^{k_3}\sum_{i=1}^{k} p_{i,k}(\hat{n}_{k-1}+\hat{n}_{i-1})) \tag{9}$$
$$+\frac{1}{\sum_{k=1}^{k_3} p_k^{(2)} k\tau_0}\frac{c_2}{2}\sum_{i=1}^{k_3} P_{i,k_3}(\hat{n}_{i-1}+\hat{n}_{k_3-1})$$
$$+\frac{1}{\sum_{k=1}^{k_3} p_k^{(2)} k\tau_0}(c_2\overline{F}^2(k_3\tau_0)\hat{n}_{k_3-1} + c_p\sum_{k=1}^{k_3}\sum_{i=1}^{k} p_{i,k})$$

3.3. Average cost evaluation under policy 3

3.3.1. The failure rate

Under this policy, the imperfect repair is carried out at time $n\tau$, $n = 1, 2, \cdots$ if both the two components are functional. Moreover, we assume that if one component fails in $](i-1)\tau_0, i\tau_0]$, then the survival component will be repaired imperfectly at $i\tau_0$, $i\tau_0 + j\tau^{'}$, $j = 1, 2, \cdots$, $\tau^{'} = k_2^{'}\tau_0 < \tau$. Other conditions are similar as in policy 2. The failure rate of the long-lived component under policy 3 given that $min(X_1, X_2) = x$ can be represented as

$$r_3(t) = \begin{cases} r_1(t) & t \le x \\ 2la_1 t^{a_1-1} - \lambda la(n_x\tau)^{a-1} & x < t \le M_x\tau_0 \\ 2la_1(t^{a_1-1} - \lambda(M_x\tau_0 + i\tau^{'})^{a_1-1}) & M_x\tau_0 + i\tau^{'} < t \le M_x\tau_0 + (i+1)\tau^{'} \end{cases}$$

for $i = 0, 1, \cdots$, where $M_x = \lceil \frac{x}{\tau_0} \rceil$

3.3.2. The system replacement and failure probability

Define $p_{i,k}^{(3)}$ be the probability that the component failures occur in $((i-1)\tau_0, i\tau_0]$, $((k-1)\tau_0, k\tau_0]$ respectively under policy 3, $P_{i,k_3}^{(3)}$ be the probability that one component survives at $k_3\tau_0$ while one fails in $((i-1)\tau_0, i\tau_0]$, $1 \le i \le k_3$ under policy 3. One can deduce

$$p_{i,k}^{(3)} = 2\mathbb{P}((i-1)\tau_0 < X_1 \le i\tau_0, (k-1)\tau_0 < X_2 \le k\tau_0) \tag{10}$$
$$= 2\int_{(i-1)\tau_0}^{i\tau_0} f(x)\overline{F}(x))(K_{(k-1)\tau_0}^{(3)}(x) - K_{k\tau_0}^{(3)}(x))\mathrm{d}x$$

where

$$K_t^{(3)}(x) \begin{cases} e^{\left[-2l\left(t^{a_1}-x^{a_1}\right)+\lambda la(n_x\tau)^{a-1}(t-x)\right]}, & n(t) < 0 \\ e^{\left[-2l\left(t^{a_1}-x^{a_1}\right)+\lambda la(n_x\tau)^{a-1}(i\tau_0-x)+2l\lambda\sum_{j=0}^{n(t)}\delta_j a_1(i\tau_0+j\tau^{'})^{a_1-1}\right]}, & \text{otherwise} \end{cases} \tag{11}$$

where $n_x = \lfloor \frac{x}{\tau} \rfloor$, $n(t) = \lfloor \frac{t - i\tau_0}{\tau'} \rfloor$, $\delta_j = (t - i\tau_0 - n(t)\tau')$ when $j = n(t)$ and τ' otherwise

Utilizing the same method we have

$$P_{i,k_3}^{(3)} = 2 \int_{(i-1)\tau_0}^{i\tau_0} f(x)\overline{F}(x) K_{k_3\tau_0}^{(3)}(x)\mathrm{d}x \tag{12}$$

where $K_{k_3\tau_0}^{(3)}$ is defined in equation (11).

Similarly, let $p_{k,k}^{(3)} = q_k$, denote by $p_k^{(3)}$ the probability that the system is renewed at $k\tau_0$, then

$$p_k^{(3)} = \sum_{i=1}^{k} p_{i,k}^{(3)}, k = 1, 2, \cdots, k_3 - 1;$$
$$p_{k_3}^{(3)} = 1 - \sum_{i=1}^{k_3-1} p_k^{(3)}.$$

3.3.3. Average long-run cost per unit time

The cost rate in the long run $C^{(3)}(k_3)$ under maintenance policy 3 is:

$$\begin{aligned} C^{(3)}(k_3) = \frac{1}{\sum_{k=1}^{k_3} p_k^{(3)} k\tau_0} (c_r &+ \frac{c_2}{2} \sum_{k=1}^{k_3} \left(\sum_{i=1}^{k-1} p_{i,k}^{(3)} (\tilde{n}_{k-1-i} + 1 \ + 2\hat{n}_{i-1}) + 2q_k \hat{n}_{k-1} \right) \\ &+ \frac{c_2}{2} \left(\sum_{i=1}^{k_3-1} P_{i,k_3} (2\hat{n}_{i-1} + \tilde{n}_{k_3-1-i} + 1) + 2P_{k_3,k_3} \hat{n}_{k_3-1} \right) \\ &+ c_2 \overline{F}^2(k_3\tau_0) \hat{n}_{k_3-1} + c_p \sum_{k=1}^{k_3} \sum_{i=1}^{k} p_{i,k}^{(3)}) \end{aligned} \tag{13}$$

4. The existence of minimal average cost

In order to minimize the average cost rate under each maintenance policy, the following theorems are presented.

Theorem 1: The optimal cost rate $C(k_3)$ under policy 1 exists.

Proof. As mentioned in [35], the failure rate of the short-lived component satisfies

$$r_1(t) \leq (1 - \lambda) a t^{a-1} l$$

which yields

$$\lim_{k\to\infty} p_k \leq \lim_{k\to\infty} \exp(-(1-\lambda)((k-1)\tau_0)^a l) = 0$$

Since $0 \leq q_k \leq p_k$, so p_k and q_k go to 0 when k goes to infinity.

Let be $C(N)$ and $C(N+1)$ the long-run cost rate when the preventive replacement is carried out at $N\tau_0$ and $(N+1)\tau_0$ respectively. Denoted by p_i, p'_j, q_i, q'_j the probabilities that the system is renewed at $i\tau_0$ and $j\tau_0$, that both the two components fail in $](i-1)\tau_0, i\tau_0]$, $[(j-1)\tau_0, j\tau_0]$ respectively, $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, N+1$. Let $A_k = \lfloor \frac{k}{k_2} \rfloor \; c_2 + c_r$, $x = c_r \sum_{i=1}^{N} q_i + \sum_{i=1}^{N} p_i A_i$, $y = \sum_{i=1}^{N} p_i i\tau_0$. Then

$$C(N+1) - C(N) = \frac{x + c_p q'_{N+1} + p'_{N+1}(A_{N+1} - A_N)}{y + p'_{N+1}\tau_0} - \frac{x}{y}$$

As p'_{N+1} and q'_{N+1} go to 0 when N goes to infinity and $(A_{N+1} - A_N) \leq c_2$ is bounded, we know that $C(N)$ is convergent when N goes to ∞. Let $N_0\tau_0$ be the convergence time, the optimal long run cost C^* can be given as

$$C^* = \begin{cases} C(n\tau_0), & n = minC(N), n < N_0 \\ C(N_0\tau_0), & N_0 = minC(N) \end{cases}$$

$\diamond$

Theorem 2: The optimal cost rate $C^{(2)}(k_3)$ under policy 2 exists.

Proof. Similarly, denoted $C^{(2)}(N)$ and $C^{(2)}(N+1)$ the long-run cost rate when the preventive replacement is carried out at $N\tau_0$ and $(N+1)\tau_0$ respectively under policy 2. Let $x^{(2)}$ and $y^{(2)}$ be the numerator and denominator of $C^{(2)}(k_3)$ respectively in equation (9). When the system preventive replacement are carried out at $(N+1)\tau_0$, let p'_{N+1} be the probability that the system are replaced at $(N+1)\tau_0$ under policy 1, $p^{(2)'}_{N+1}$ be the probability that the system is replaced at $(N+1)\tau_0$ under policy 2, $p'_{i,N+1}$ be the probability that the two components fail in $((i-1)\tau_0, i\tau_0]$ and $((N\tau_0, (N+1)\tau_0$ under policy 2 respectively, $B_N = c_2(\hat{n}_N - \hat{n}_{N-1})$ Then we have

$$C^{(2)}(N+1) - C^{(2)}(N) = \frac{x^{(2)} + B_N \sum_{i=1}^{N} P_{iN} + 2c_2 p'_{N+1}\hat{n}_{N-1} + c_p \sum_{i=1}^{N+1} p'_{i,N+1}}{y^{(2)} + p^{(2)'}_{N+1}\tau_0} - \frac{x^{(2)}}{y^{(2)}}$$

where $\hat{n}_N = \lfloor \frac{N}{k_2} \rfloor$. It can be seen that

$$\lim_{N\to\infty} p_{N+1}^{(2)'} \leq \lim_{N\to\infty} \overline{F}(N\tau_0) \leq \lim_{N\to\infty} \exp(-(1-\lambda)l(N\tau_0)^a) = 0$$

$$\lim_{N\to\infty} p'_{N+1}\hat{n}_{N-1} \leq \lim_{N\to\infty} \frac{N}{\overline{F}^2(N\tau_0)} \leq \lim_{N\to\infty} \frac{N}{\exp(2(1-\lambda)l(N\tau_0)^a)} = 0$$

$$\lim_{N\to\infty} \sum_{i=1}^{N} P_{iN} \leq \lim_{N\to\infty} \overline{F}((N-1)\tau_0) = 0$$

Therefore $C^{(2)}(N)$ is convergent when N goes to infinity and so the existence of its optimal value. $\diamond$

Theorem 3: The optimal cost rate $C^{(3)}(k_3)$ under policy 3 exists.

Proof. . Under policy 3, denoted $C^{(3)}(N)$ and $C^{(3)}(N+1)$ the long-run cost rate when the preventive replacement are carried out at $N\tau_0$ and $(N+1)\tau_0$ respectively . Let $x^{(3)}$ and $y^{(3)}$ be the numerator and denominator of $C^{(3)}(k_3)$ respectively in equation (13). Then

$$C^{(3)}(N+1) - C^{(3)}(N) =$$
$$\frac{x^{(3)} + \sum_{i=1}^{N-1} P_{iN}\tilde{B}(N,i) + 2c_2\overline{F}^2(N\tau_0)B_N + c_p \sum_{i=1}^{N+1} p_{i,N+1}^{'(3)}}{y^{(3)} + p_{N+1}^{'(3)}\tau_0} - \frac{x^{(3)}}{y^{(3)}}$$

where $\tilde{B}(N,i) = c_2(\tilde{n}_{N-i} - \tilde{n}_{N-1-i})$, $B_N = c_2(\hat{n}_N - \hat{n}_{N-1})$ $p_{N+1}^{'(3)}$ is the probability that the system is replaced at $(N+1)\tau_0$ under policy 3, $p_{i,N+1}^{'(3)}$ is the probability that the two components fail in $((i-1)\tau_0, i\tau_0]$ and $((N\tau_0, (N+1)\tau_0$ under policy 3 respectively. By the similar method as in policy 2, it can be proved that

$$\lim_{N\to\infty} \sum_{i=1}^{N} p_{iN}^{'(3)} = 0$$

$$\lim_{N\to\infty} p_N^{'(3)} = 0$$

$$\lim_{N\to\infty} \overline{F}^2(N\tau_0)B_N = 0$$

Therefore $C^{(3)}(N)$ is convergent when N goes to infinity and so the existence of its optimal value. $\diamond$

5. Parameter estimation

There are many load sharing parallel systems in industry and engineering field. For instance, one can enumerate the sensors which take the workload together in a distributed computer system; the pumps sharing the workload in a hydraulic system; the welded joints in a bridge support [36]; the cables in a suspension bridge system [37] and the hoisting ropes in the mining system. Hoisting rope plays a significant role in mining system as its tensile strength and lifetime affect directly the system reliability and the system operation state. According to the literature [38, 39], the break of ropes is relevant to the fretting wear, mechanical damage, operating environment like temperature, corrosive gas, distortion, etc. Therefore, regular inspections, lubrication and overhaul are necessary for the enterprises to increase the effective operation of systems and decrease the probability of failure. Our model can be applied to a mining hoist system with two hoisting rope. The two ropes share the system load uniformly. Whenever one fails,

- the survival component bear the whole system load.
- the sudden component failure can be regarded as a shock which increases the failure rate of the survival one.

Different maintenance policies are provided to slow down the rope deterioration and to maintain the rope a good condition. Furthermore, for the safety, the two components can be replaced together when the age of the system arrives at a predetermined time limit. To develop maintenance policies to equilibrate the owner costs and the system safety, the primary issue is to obtain efficient evaluation of the system failure-related properties which implies the estimation of a, l and a_1 respectively in this study.

We propose a two-step method to estimate the system parameters a and a_1. Suppose that the test number is n and the failure times of the short-lived component (resp. the long-lived component) are t_i (resp. $\hat{t}_i$), $i = 1, 2, \cdots, n$ where $t_i < \hat{t}_i$ and $\{t_i, i = 1, 2, \cdots, n\}$ (resp. $\{\hat{t}_i, i = 1, 2, \cdots, n\}$) are independent. The system load is known and equal to $2l$. Therefore, according to the failure rates, the likelihood function of $\{t_i, i = 1, 2, \cdots, n\}$ is given as follows:

$$f(t_1, t_2, \cdots, t_n; a) = (al)^n \Pi_{i=1}^n {t_i}^{a-1} \exp(-l \sum_{i=1}^n {t_i}^a)$$

By calculating respectively the first derivative of the log-likelihood function with respect to a and l the maximum-likelihood estimate $\hat{a}$ and $\hat{l}$ can be obtained from

$$\frac{n}{\hat{a}} + \sum_{i=1}^{n} \log t_i - \hat{l}\hat{a} \sum_{i=1}^{n} t_i^{\hat{a}-1} = 0 \tag{14}$$

$$\hat{l} = \frac{n}{\sum_{i=1}^{n} t_i^{\hat{a}}} \tag{15}$$

Besides, it is reasonable to assume that $\hat{t}_i$ is independent of $\hat{t}_j$ $(i \neq j)$ given the value of t_i for any $i, j \in \{1, 2, \cdots, n\}$. Therefore, the conditional likelihood function of $\{\hat{t}_i, i = 1, 2, \cdots, n\}$ is as follows:

$$f(\hat{t}_1, \hat{t}_2, \cdots \hat{t}_n \mid t_1, t_2, \cdots, t_n; a_1) = (2\hat{l}a_1)^n \Pi_{i=1}^{n} {t_i}^{a_1-1} \exp(-2\hat{l} \sum_{i=1}^{n} {t_i}^{a_1} + 2\hat{l} \sum_{i=1}^{n} {t_i}^{a_1})$$

Similarly, the estimate $\hat{a}_1$ can be obtained from

$$\frac{n}{\hat{a}_1} + \log \sum_{i=1}^{n} \hat{t}_i - 2\hat{l}\hat{a}_1 \sum_{i=1}^{n} ({t_i}^{\hat{a}_1-1} - {t_i}^{\hat{a}_1-1}) = 0 \tag{16}$$

6. Numerical analysis

In the following, we present a numerical example to illustrate the system behavior. Afterward, for different maintenance policies, we analyse the impact of different parameters on the long run average total cost.

6.1. Optimization Algorithm

Since the existence of the minimal cost is proved the maintenance optimization can be carried out with different methods.

For the numerical minimization the following algorithm is implemented.

1. Set parameters $\tau, \tau_0, \tau', a, a_1, l, \lambda, c_2, c_r, c_p$, and the accuracy ε
2. Set $k_3 = 1$.
3. for an arbitrary k_3 presented in Section 3 calculate $C(k_3)$ and $C(k_3+1)$
   - if $|C(k_3+1) - C(k_3)| < \varepsilon$ and $|C(k_3+2) - C(k_3+1)| < \varepsilon$ go to 4
   - else $k_3 = k_3 + 1$ repeat 3
4. Compare the value from $C(1)$ to $C(k_3+1)$ and let the optimal value be $C^* = min\{C(i)\}$, $i = 1, 2, \cdots, k_3 + 1$.

optimal	$l = 0.03$	0.04	0.06	λ =0.4	0.5	0.6
N^*	20	16	12	16	16	24
C^*	19.8975	24.0127	32.1823	25.1940	24.0127	22.6421

Table 1: The optimal cost rate with different load l and maintenance effect λ under policy 1.

6.2. Sensitivity analysis of the long run average maintenance cost

Numerical examples are given to describe the optimal average long run cost with various parameters.

6.2.1. Policy 1: component based policy

Let $\tau = 4$, $\tau_0 = 1$, $c_2 = 25$, $c_r = 100$, $c_p = 220$, $l = 0.04$, $\lambda = 0.5$, $a = 1.3$, $a_1 = 2$, $\varepsilon = 0.0001$. Denote by C^* and $N^*\tau_0$ the optimal long run average total cost and the optimal time of preventive replacement under policy 1 correspondingly. Tables 1-4 show the variation of the optimal long run cost rate under policy 1 with one parameter while other system parameters are unchanged.

optimal	a =1.3	1.4	1.5	a_1 =1.8	2	2.1
N^*	16	12	12	20	16	16
C^*	24.0127	26.6954	29.3361	21.6656	24.0127	25.1899

Table 2: The optimal cost rate with different deterioration parameters a and a_1 under policy 1.

optimal	$c_2 = 20$	25	30	c_r =90	100	120
N^*	16	16	12	16	16	20
C^*	23.1970	24.0127	24.8285	22.7558	24.0127	26.4438

Table 3: The optimal cost rate with different maintenance cost units c_2 and c_r under policy 1.

optimal	c_p =150	220	250
N^*	24	16	16
C^*	21.5265	24.0127	25.0171

Table 4: The optimal cost rate with different penalty cost unit c_p under policy 1.

The numerical results in Tables 1, 2, 3 and 4 indicate the following:

- Table 1 shows that the total maintenance cost and they maintenance frequency increase with the load. Moreover, the closer is the imperfect maintenance to the perfect maintenance the cheaper is the total maintenance cost. As λ increases, the system replacement become less frequent.

- It can be noticed in Table 2, long run average total maintenance cost increases as the components deteriorate faster. More replacements are required for fast deteriorations.

- When a, a_1, λ, l are high, the system deteriorates faster, it is then sensible to carry out the preventive maintenance more often. The increasing parameter a of Weibull distribution, which is positively correlated with the failure rate of the system, impacts lightly the optimal average costs, while it is more sensitive to the variation of l.

- The optimal average costs are quiet robust in the sense that they don't vary significantly with the degradation parameters.

- Unsurprisingly, the optimal cost rate C^* is increasing with imperfect maintenance cost c_2, system renewal cost c_r and failure penalty cost c_p.

6.2.2. Policy 2: system based policy

Under policy 2, we take the same parameters as in policy 2. The optimal average long-run cost $C^{(2)*}$ under different parameter settings are illustrated in Tables 5, 6, 7, 8. The following features can be pointed out.

- Similarly to policy 1, it can be noticed that the $C^{(2)*}$ decreases with the system load l and shows an decreasing tendency with λ . Because the larger λ is, the better is the repair. Therefore, the system is more robust and is liable to survive. The higher is the system load, the larger is the system failure rate. Thus the system fails more frequently which causes an increase of the maintenance cost.

- Table 6 considers the variation of $C^{(2)*}$ under different deterioration parameters a and a_1. In our example, the long-run average total cost is not very sensitive to the variations of a_1. It is more sensitive to small values of a.

- In Table 7-8 the impact of maintenance costs units variations on $C^{(2)*}$ are presented. Obviously $C^{(2)*}$ increases with respect to maintenance costs units. It is shown in Table 8 that the system is replaced earlier when the penalty is high. Therefore, the system owner are recommended to consider the potential risk he or she should undertake when the consequence of system failure is serious.

optimal	$l = 0.03$	0.04	0.06	λ =0.4	0.5	0.6
$N^{(2)*}$	24	20	16	12	20	28
$C^{(2)*}$	28.6937	34.6901	45.9031	36.6135	34.6901	32.2203

Table 5: The optimal cost rate with different load l and maintenance effect λ under policy 2.

optimal	a =1.3	1.4	1.5	a_1 =1.8	2	2.1
$N^{(2)*}$	20	8	4	8	20	20
$C^{(2)*}$	34.6901	38.6017	39.9732	32.3945	34.6901	35.3785

Table 6: The optimal cost rate with different deterioration parameters a and a_1 under policy 2.

optimal	$c_2 = 20$	25	30	c_r =90	100	120
$N^{(2)*}$	20	20	16	16	20	24
$C^{(2)*}$	33.8679	34.6901	35.5110	33.6349	34.6901	36.7052

Table 7: The optimal cost rate with different maintenance costs units c_2 and c_r under policy 2.

optimal	c_p =150	220	250
$N^{(2)*}$	28	20	16
$C^{(2)*}$	28.1650	34.6901	37.4398

Table 8: The optimal cost rate with different c_p under policy 2.

6.2.3. Policy 3: component based policy, variant

Under policy 3, the system optimal long-run average maintenance cost and the optimal preventive maintenance time are denoted respectively by $C^{(3)*}$ and $N^{(3)*}\tau_0$. By adopting the parameters setting as in the cost analysis under policy 1 and assuming that $k_2^{'} = 3$, Tables 9, 10, 11 and 12 elucidate the similar average cost variation as in policy 2.

optimal	l =0.03	0.04	0.06	λ =0.4	0.5	0.6
$N^{(3)*}$	20	16	16	8	16	28
$C^{(3)*}$	28.6769	34.6156	45.2158	36.3411	34.6156	32.0843

Table 9: The optimal cost rate with different load l and maintenance efficiency λ under policy 3.

optimal	a =1.3	1.4	1.5	a_1 =1.8	2	2.1
$N^{(3)*}$	16	4	4	8	16	20
$C^{(3)*}$	34.6156	37.2893	38.4369	31.5371	34.6156	35.4033

Table 10: The optimal cost rate with different deterioration parameters a and a_1 under policy 3.

optimal	$c_2 = 20$	25	30	c_r =90	100	120
$N^{(3)*}$	20	16	16	8	16	24
$C^{(3)*}$	33.7100	34.6156	35.4971	33.4508	34.6156	36.6353

Table 11: The optimal cost rate with different maintenance unit costs c_2 and c_r under policy 3.

optimal	c_p =150	220	250
$N^{(3)*}$	28	16	8
$C^{(3)*}$	28.2491	34.6156	37.2236

Table 12: The optimal cost rate with different penalty c_p under policy 3.

It is pointed out that in our example policy 1 is the most economical one comparing to policy 2 and 3. In most cases, policy 3 is a second-best choice which indicates that it is necessary to consider the period to carry out the imperfect maintenance when only one component is functional in the system.

In all policies, the maintenance costs are very sensitive to the system load. In sum, under high loads, low quality system and non efficient maintenance operations, the maintenance policies are very costly. The penalty cost may have influence on the policy preference. For example when $c_p = 150$ or 220, the cost difference between policy 2 and policy 3 is tiny while policy 3 is more favored when the failure consequence is serious. To distinguish the policies when similar optimal maintenance costs are revealed, by considering the system safety we introduce the concept of dangerousness rate in the following.

6.2.4. Dangerousness rate of the maintenance policy

Here we introduce the dangerousness rate to measure the accident risk of the system. The dangerousness rate is defined as the probability that both of the two components failure occur before the system replacement time $N\tau_0$, $N \in \mathbb{N}^*$.

Denote by $dr(i, N)$ the system dangerousness rate under policy i when the system preventive renewal time is $N\tau_0$. Then

$$
\begin{aligned}
dr(1, N) &= \sum_{k=1}^{N} q_k \\
dr(2, N) &= \sum_{k=1}^{N} q_k + \sum_{k=1}^{N} \sum_{i=1}^{k-1} p_{i,k} \\
dr(3, N) &= \sum_{k=1}^{N} q_k + \sum_{k=1}^{N} \sum_{i=1}^{k-1} p_{i,k}^{(3)}
\end{aligned}
$$

where q_k, $p_{i,k}$ and $p_{i,k}^{(3)}$ are given in equation (4), equation (6) and equation (10) respectively.

By taking the same parameter settings as in policy 1-3 in the above, Figure 6 demonstrates the dangerousness rate under the three policies. It can be noticed that policy 1 is the safest maintenance policy and policy 2 which is the the most dangerous policy. Decision-makers are recommended to consider the two aspects and find an equilibrium considering both the maintenance cost and the system reliability.

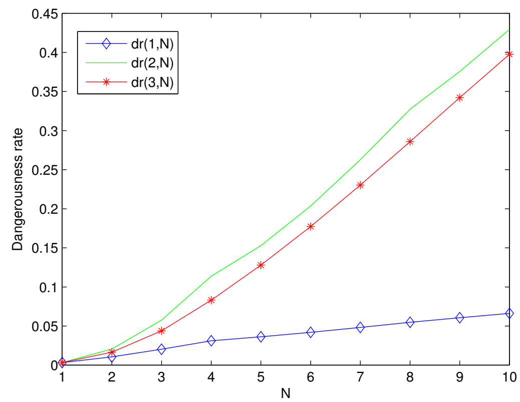

Figure 5: the dangerous rate under different policies

7. Conclusions

In this study, different maintenance policies for a two-component load-sharing system are proposed. To avoid system failure, imperfect preventive maintenance and preventive system replacement are applied. The long-run average cost of the system under different maintenance policies are obtained. Numerical examples are illustrated and it is shown that policy 1 is the most cost saving policy. The system dangerousness rate criteria is proposed to describe the system accident risk. It is recommended to the decision maker to consider an equilibrium between the average maintenance cost and the system reliability. This work can be generalized by considering a dynamic load of a multi-component system and analyse its induced cost complexity. The next step is to apply real deterioration data of the hoisting ropes in the mining system and implement statistical inference to estimate different model parameters. Sensitivity analysis of the maintenance policy performances can be presented.

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