

UNIQUENESS AND PROPERTIES OF DISTRIBUTIONAL SOLUTIONS OF NONLOCAL EQUATIONS OF POROUS MEDIUM TYPE

FÉLIX DEL TESO, JØRGEN ENDAL, AND ESPEN R. JAKOBSEN

ABSTRACT. We study the uniqueness, existence, and properties of bounded distributional solutions of the initial value problem for the anomalous diffusion equation $\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0$. Here \mathcal{L}^μ can be any nonlocal symmetric degenerate elliptic operator including the fractional Laplacian and numerical discretizations of this operator. The function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is only assumed to be continuous and nondecreasing. The class of equations include nonlocal (generalized) porous medium equations, fast diffusion equations, and Stefan problems. In addition to very general uniqueness and existence results, we obtain stability, L^1 -contraction, and a priori estimates. We also study local limits, continuous dependence, and properties and convergence of a numerical approximation of our equations.

1. INTRODUCTION

In this paper, we obtain uniqueness, existence, and various other properties for bounded distributional solutions of a class of possibly degenerate nonlinear anomalous diffusion equations of the form:

$$(1.1) \quad \partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 \quad \text{in } Q_T := \mathbb{R}^N \times (0, T)$$

$$(1.2) \quad u(x, 0) = u_0(x) \quad \text{on } \mathbb{R}^N$$

where $u = u(x, t)$ is the solution and $T > 0$. The nonlinearity φ is an arbitrary continuous nondecreasing function, while the anomalous or nonlocal diffusion operator \mathcal{L}^μ is defined for any $\psi \in C_c^\infty(\mathbb{R}^N)$ as

$$(1.3) \quad \mathcal{L}^\mu[\psi](x) = \int_{\mathbb{R}^N \setminus \{0\}} \left(\psi(x+z) - \psi(x) - z \cdot D\psi(x) \mathbf{1}_{|z| \leq 1} \right) d\mu(z),$$

where D is the gradient, $\mathbf{1}_{|z| \leq 1}$ a characteristic function, and μ a nonnegative symmetric possibly singular measure satisfying the Lévy condition $\int |z|^2 \wedge 1 d\mu(z) < \infty$. For the precise assumptions, we refer to Section 2.

The class of nonlocal diffusion operators we consider coincide with the generators of the symmetric pure-jump Lévy processes [9, 7, 39] like e.g. compound Poisson processes, CGMY processes in Finance, and symmetric s -stable processes. Included are the well-known fractional Laplacians $-(-\Delta)^{\frac{s}{2}}$ for $s \in (0, 2)$ (where $d\mu(z) = c_{N,s} \frac{dz}{|z|^{N+s}}$ for some $c_{N,s} > 0$ [24, 7]), along with degenerate operators, and surprisingly, numerical discretizations of these operators!

In the language of [48], equation (1.1) is a generalized porous medium equation. On one hand, since φ is only assumed to be continuous, the full range of porous medium and fast diffusion nonlinearities are included: $\varphi(r) = r|r|^{m-1}$ for $m >$

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0. This is somehow optimal for power nonlinearities since if $m < 0$ (ultra fast diffusion), then not only uniqueness, but also existence may fail [12]. On the other hand, since φ is only assumed to be nondecreasing, it can be constant on sets of positive measure and then equation (1.1) is strongly degenerate. This case include Stefan type of problems, like e.g. when $c_1, c_2, T > 0$ and

$$\varphi(r) = \begin{cases} c_2 r, & r < 0, \\ c_1 (r - T)^+, & r \geq 0. \end{cases}$$

Many physical problems can be modelled by equations like (1.1). We mention flow in a porous medium of e.g. oil, gas, and groundwater, nonlinear heat transfer, and population dynamics. For more information and examples, we refer to Chapter 2 and 21 in [48] for local problems, and to [49, 34, 7, 46, 47] for nonlocal problems.

A key result in this paper is the uniqueness result for bounded distributional solutions of (1.1) and (1.2). Almost half of the paper is devoted to the proof of this result. Once we have it, we prove a general stability result, and then we obtain other properties like existence, L^1 -contraction, and many a priori estimates from more regular problems via approximation and compactness arguments. As straightforward applications of all of these estimates, we then obtain the following results: (i) Convergence as $s \rightarrow 2^-$ of distributional solutions of

$$(1.4) \quad \partial_t u + (-\Delta)^{\frac{s}{2}} \varphi(u) = 0 \quad \text{in} \quad Q_T,$$

to distributional solutions of the local equation

$$(1.5) \quad \partial_t u - \Delta \varphi(u) = 0 \quad \text{in} \quad Q_T;$$

(ii) continuous dependence in $(m, s) \in (0, \infty) \times (0, 2]$ for the porous medium equation of [37],

$$(1.6) \quad \partial_t u + (-\Delta)^{\frac{s}{2}} u |u|^{m-1} = 0 \quad \text{in} \quad Q_T,$$

including for the first time also the fast diffusion range; and (iii) convergence of semi-discrete numerical approximations of a class of equations including (1.1) (cf. (2.7) and (2.8) in Section 2.2).

The uniqueness result is hard to prove because of our very general assumptions on the initial value problem combined with a very weak solution concept – merely bounded distributional solutions. This combination means that many classical techniques do not work: Fourier techniques are hard to apply because the problem is nonlinear and the Fourier symbol of \mathcal{L}^μ could be merely a bounded function, energy estimates do not imply uniqueness because φ is not strictly increasing, and L^1 -contraction arguments do not apply since we do not assume additional entropy conditions (cf. e.g. [5] for the local case), or equivalently, additional regularity in time as in [37] (see the uniqueness result for so-called strong solutions). The (weighted) L^1 -contraction argument for ordered solutions given in [15] avoids these additional conditions, but it cannot be adapted here since it strongly depends on the equation being like (1.6) with $0 < m < 1$ and $s \in (0, 2)$. Finally, since our solutions are not assumed to have finite energy, the classical uniqueness argument of Oleinik [32] cannot be adapted either. We refer to [32, 48] for the local case, and the uniqueness argument for so-called weak solutions in [37] for results in the nonlocal case.

For the local equation (1.5), uniqueness for bounded distributional solution was proven by Brezis and Crandall in [18] under similar assumptions on φ and u_0 . Their argument is quite indirect and rely on a clever idea using resolvents and their integral representations (fundamental solutions). In this paper, we adapt such an approach to our nonlocal setting. But because of the generality of our diffusion operators, we cannot rely on explicit fundamental solutions for our proofs. Instead,

we have to develop this part of the theory from scratch, using the equation and the regularity that comes with our solutions concept to obtain the necessary estimates. To do this, a key tool is to approximate the possibly singular integral operator \mathcal{L}^μ by a bounded integral operator and then carefully pass to the limit. This procedure, and hence also the proof, is truly nonlocal – there is no similar approximation by local operators. The proof necessarily becomes much more involved than in [18], and includes a number of approximations, a priori estimates, L^1 -contraction estimates, comparison principles, compactness and regularity arguments. It also includes new Stroock-Varoupolous inequalities and a new Liouville type of result for nonlocal operators. Both our approach and intermediate results should be of independent interest.

Let us give the main references for the well-posedness of the Cauchy problems for (1.1) and (1.5). We start with the local case (1.5). In the linear case, when $\varphi(u) = u$, it is the classical heat equation, cf. e.g. [26]. When $\varphi(u) = u^m$, it is a porous medium equation, and a very complete theory can be found in [48]. In the general case, (1.5) is a generalized porous medium equation (or filtration equation). We refer again to [48]. Uniqueness of distributional solutions of this equation was proven in [18] for bounded initial data and continuous, nondecreasing φ , and in [28] for locally integrable initial data, $\varphi(r) = r^m$ for $0 < m < 1$, and with regularity assumptions on $\partial_t u$. Some nonuniqueness results can be found in e.g. [44, 45]. In the presence of convection, or if general L^1 -contraction results are sought, then so-called entropy solutions are a useful tool to obtain well-posedness [31, 20]. A very general well-posedness result which cover the case of merely continuous φ can then be found in [5].

In the nonlocal case, one linear special case of (1.1) is the fractional heat equation $\partial_t u + (-\Delta)^{\frac{s}{2}} u = 0$ for $s \in (0, 2)$. As in the local case, the initial value problem has a classical solution $u(x, t) = (K_s(\cdot, t) * u(\cdot, 0))(x)$ for $\mathcal{F}(K_s(\cdot, t))(\xi) = e^{-|\xi|^s t}$. It is well-posed even for measure data and solutions growing at infinity [8, 14]. The fractional porous medium equations (1.6) are examples of nonlinear equations of the form (1.1). In [36, 37], existence, uniqueness and a priori estimates for (1.6) are proven for so-called weak L^1 -energy solutions – possibly unbounded solutions with finite energy. In [15] there are existence and uniqueness results for minimal distributional solutions of (1.6) with $0 < m < 1$ in weighted L^1 -spaces (solutions can grow at infinity). We also mention that logarithmic diffusion ($\varphi(u) = \log(1 + u)$) is considered in [38], singular or ultra fast diffusions in [12], weighted equations with measure data in [27], and problems on bounded domains in [13, 16, 17]. Energy solutions of equations with a larger class of nonlinearities φ and nonlocal operators \mathcal{L}^μ are studied in the recent paper [35]. The authors obtain results on well-posedness, continuity/regularity, and long time asymptotics. The setting, solution concept, and techniques are different from ours. Their operators \mathcal{L}^μ can have some x -dependence, but the (singular part) must be comparable to a fractional Laplacian (i.e. be nondegenerate). Initial data in $L^\infty \cap L^1$ is assumed for uniqueness. In the x -independent case their assumptions are less general than ours, especially those for \mathcal{L}^μ and the regularity of the solutions. Other types of equations of the form (1.1) can be found in [6]. These equations involve bounded diffusion operators that can be represented by nonsingular integral operators of the form (1.3). Because of this, at least the well-posedness is easier to handle in this case.

It should be clear from the previous discussion that even if our uniqueness result is very general, it is usually not strictly comparable to the other results. E.g. a price to pay to work with general φ and a very weak solution concept, is that solutions u have to be bounded. Our method of proof also requires that $u - u_0 \in L^1(Q_T)$. For particular choices of φ , these assumptions may not be optimal. E.g. if you change

the solution concept and assume finite energy, then there are uniqueness results for unbounded solutions of (1.6) in L^1 in [36, 37]. There are even uniqueness results in weighted L^1 -spaces, see [15]. Here the solutions are allowed to grow at infinity, but the uniqueness result is weaker in the sense that it only holds for *minimal* distributional solutions.

There are other ways to generalize the porous medium equation to a nonlocal setting. In [11, 19, 40, 10, 41] the authors consider a so-called porous medium equations with fractional pressure. These equations are in a divergence form, and no uniqueness is known except when $N = 1$. Finally, we mention that in the presence of (nonlinear) convection, additional entropy conditions are needed to have uniqueness as in the local case. Nonuniqueness of distributional solutions is proven in [2], and several well-posedness results for entropy solutions are given in [1, 22, 25]. These latter results requires φ to be linear or locally Lipschitz and hence do not apply to our case where φ is merely continuous.

Outline. In Section 2 we state the assumptions and present and discuss our main results. The proof of the uniqueness result is given in Section 3. This proof requires a number of results and estimates for a resolvent equation – an auxiliary elliptic equation – and these are proven in Section 6. In Section 4, we prove the main stability and existence result, along with a number of a priori estimates. We then apply these results to prove the convergence to the local case, continuous dependence, and the properties and convergence of the numerical scheme in Section 5. Finally, after Section 6, there is an appendix with the proofs of some technical results.

Notation. For $x \in \mathbb{R}$, $x^+ := \max\{x, 0\}$, $x^- := (-x)^+$, and $\text{sign}^+(x)$ is $+1$ for $x > 0$ and 0 for $x \leq 0$. We let $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$, $\mathbf{1}_A(x)$ be 1 for $x \in A \subset \mathbb{R}^N$ and 0 otherwise, and $\text{supp } \psi$ be the support of a function ψ . Derivatives are denoted by $'$, $\frac{d}{dt}$, ∂_t , ∂_{x_i} , and $D\psi$ and $D^2\psi$ denote the x -gradient and Hessian matrix of ψ . Convolution is defined as $f * g(x) = [f * g](x) = \int_{\mathbb{R}^N} f(x - y)g(y) dy$, and $(f, g) = \int_{\mathbb{R}^N} fg dx$ whenever the integral is well-defined. If $f, g \in L^2(\mathbb{R}^N)$, we write $(f, g)_{L^2(\mathbb{R}^N)}$. The L^2 -adjoint of an operator \mathcal{T} is denoted by \mathcal{T}^* , and the reader may check that $(\mathcal{L}^\mu)^* = \mathcal{L}^{\mu^*}$ (see below for the definition of μ^*). A modulus of continuity is a nonnegative function $\lambda(\varepsilon)$ which is continuous in ε with $\lambda(0) = 0$. By a classical solution, we mean a solution such that the equation holds pointwise everywhere.

Function spaces: C_0 , C_b , C_b^∞ and C_c^∞ are spaces of continuous functions that are vanishing at infinity; bounded; bounded with bounded derivatives of all orders; and smooth functions with compact support respectively. $C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N))$ is the space of measurable functions $\psi : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$ such that (i) $\psi(\cdot, t) \in L_{\text{loc}}^1(\mathbb{R}^N)$ for every $t \in [0, T]$; (ii) for all compact $K \subset \mathbb{R}^N$, $\int_K |\psi(x, t) - \psi(x, s)| dx \rightarrow 0$ when $t \rightarrow s \in [0, T]$; and (iii) $\|\psi\|_{C([0, T]; L^1(K))} := \text{ess sup}_{t \in [0, T]} \int_K |\psi(x, t)| dx < \infty$.

Measures: $\delta_a(x)$ denotes the delta measure centered at $a \in \mathbb{R}^N$. Let $X \subset \mathbb{R}^N$ be open and μ a Borel measure on X . For $x \in X$ and $\Omega \subset X$ Borel, we denote $\mu_x(\Omega) = \mu(\Omega + x)$ where $\Omega + x = \{y + x : y \in \Omega\}$. Moreover, μ^* is defined as $\mu^*(B) = \mu(-B)$ for all Borel sets B , and we say that μ is symmetric if $\mu^* = \mu$. The support of a Borel measure μ on is

$$\text{supp } \mu = \{x \in X : \mu(B(x, r) \cap X) > 0 \text{ for all } r > 0\}.$$

The Lebesgue measure of \mathbb{R}^N is denoted by dw if w is a generic variable on \mathbb{R}^N . Moreover, the tensor product $d\mu(z) dw$ is a well-defined nonnegative Radon measure since μ is σ -finite (for more details, consult [3, Section 2.1.2].)

For the rest of the paper, we fix two families of mollifiers $\omega_\delta, \rho_\delta$ defined by

$$(1.7) \quad \omega_\delta(\sigma) := \frac{1}{\delta^N} \omega\left(\frac{\sigma}{\delta}\right)$$

for fixed $0 \leq \omega \in C_c^\infty(\mathbb{R}^N)$ satisfying $\text{supp } \omega \subseteq \overline{B}(0, 1)$, $\omega(\sigma) = \omega(-\sigma)$, $\int \omega = 1$; and

$$(1.8) \quad \rho_\delta(\tau) := \frac{1}{\delta} \rho\left(\frac{\tau}{\delta}\right)$$

for fixed $0 \leq \rho \in C_c^\infty([0, T])$, $\text{supp } \rho \subseteq [-1, 1]$, $\rho(\tau) = \rho(-\tau)$, $\int \rho = 1$.

2. THE MAIN RESULTS

In this section, we present the main results: first of all uniqueness, and then stability, existence and a number of estimates for the solutions of (1.1) and (1.2). As an application of our main results, we give compactness and continuous dependence estimates. We introduce a semi-discrete numerical scheme for even more general equations and show that convergence and other properties easily follow from our previous results. Finally, we establish a new existence result that also cover local diffusion equations.

Throughout the paper we assume that

(A $_\varphi$) $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and nondecreasing;

(A $_{u_0}$) $u_0 \in L^\infty(\mathbb{R}^N)$;

(A $_\mu$) μ is a nonnegative symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z| \leq 1} |z|^2 d\mu(z) + \int_{|z| > 1} 1 d\mu(z) < \infty.$$

Remark 2.1. (a) Without loss of generality, we can assume $\varphi(0) = 0$ (by adding a constant to φ).

(b) A nonlocal operator defined by (1.3) is a nonpositive operator (see Lemma 3.7).

We use the following definition of distributional solutions of (1.1) and (1.2).

Definition 2.2. Let $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $u \in L^1_{\text{loc}}(Q_T)$. Then

(a) u is a distributional solution of equation (1.1) if

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 \quad \text{in } \mathcal{D}'(Q_T),$$

(b) u is a distributional solution of the initial condition (1.2) if

$$\text{ess lim}_{t \rightarrow 0^+} \int_{\mathbb{R}^N} u(x, t) \psi(x, t) dx = \int_{\mathbb{R}^N} u_0(x) \psi(x, 0) dx \quad \forall \psi \in C_c^\infty(\mathbb{R}^N \times [0, T]).$$

The equation in part (a) is well-defined when e.g. (A $_\varphi$) and (A $_\mu$) hold and $u \in L^\infty(Q_T)$. Note as well that the initial condition u_0 is assumed in the distributional sense (u_0 is a weak initial trace). See Lemma 2.21 below for an equivalent definition.

We state the main result of this paper.

Theorem 2.3. Assume (A $_\varphi$) and (A $_\mu$). Let $u(x, t)$ and $\hat{u}(x, t)$ satisfy

$$(2.1) \quad u, \hat{u} \in L^\infty(Q_T),$$

$$(2.2) \quad u - \hat{u} \in L^1(Q_T),$$

$$(2.3) \quad \partial_t u - \mathcal{L}^\mu[\varphi(u)] = \partial_t \hat{u} - \mathcal{L}^\mu[\varphi(\hat{u})] \quad \text{in } \mathcal{D}'(Q_T)$$

$$(2.4) \quad \text{ess lim}_{t \rightarrow 0^+} \int_{\mathbb{R}^N} (u(x, t) - \hat{u}(x, t)) \psi(x, t) dx = 0 \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N \times [0, T]).$$

Then $u = \hat{u}$ a.e. in Q_T .

Sections 3 and 6 are devoted to the (long) proof of this result.

Corollary 2.4 (Uniqueness). *Assume (A_φ) , (A_{u_0}) and (A_μ) . Then there is at most one distributional solution u of (1.1) and (1.2) such that $u \in L^\infty(Q_T)$ and $u - u_0 \in L^1(Q_T)$.*

Proof. Assume there are two solutions u and \hat{u} . Then all assumptions of Theorem 2.3 obviously hold ($\|u - \hat{u}\|_{L^1} \leq \|u - u_0\|_{L^1} + \|\hat{u} - u_0\|_{L^1} < \infty$), and $u = \hat{u}$ a.e. \square

Remark 2.5. Uniqueness holds for $u_0 \notin L^1$, for example $u_0(x) = c + \phi(x)$ for $c \in \mathbb{R}$ and $\phi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. However, periodic u_0 are not included. In Section 2.3 below we discuss some extensions of the uniqueness result.

Next, we study under which assumptions solutions of

$$(2.5) \quad \partial_t u_n - \mathcal{L}^{\mu_n}[\varphi_n(u_n)] = 0 \quad \text{in } Q_T,$$

converge to solutions of

$$(2.6) \quad \partial_t u - \mathcal{L}[\varphi(u)] = 0 \quad \text{in } Q_T.$$

Theorem 2.6 (Stability). *Assume $\mathcal{L} : C_c^\infty(Q_T) \rightarrow L^1(Q_T)$, μ_n satisfies (A_μ) , φ_n and φ satisfy (A_φ) , and $u_n, u \in L^\infty(Q_T)$ for every $n \in \mathbb{N}$. Then if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence of distributional solutions of (2.5), $\sup_n \|u_n\|_{L^\infty(Q_T)} < \infty$, and*

(i) $\mathcal{L}^{\mu_n}[\psi] \rightarrow \mathcal{L}[\psi]$ in $L^1(\mathbb{R}^N)$ for all $\psi \in C_c^\infty(\mathbb{R}^N)$;

(ii) $\varphi_n \rightarrow \varphi$ locally uniformly;

(iii) $u_n \rightarrow u$ pointwise a.e. in Q_T ;

then u is a distributional solution of (2.6).

This result is proven in Section 4.

Remark 2.7. The limit operator \mathcal{L} need not satisfy (A_μ) , we can recover any operator of the form $\mathcal{L}[\psi] = \text{tr}[\sigma\sigma^T D^2\psi] + \mathcal{L}^\mu[\psi]$: the general form of the generator of a symmetric Lévy process [7]. See sections 2.2 and 5.2 for more details and examples. An extension of this result will be discussed in Section 2.3 below.

The stability result will be used along with approximation and compactness arguments to obtain the following existence result and a priori estimates.

Theorem 2.8 (Existence and uniqueness). *Assume (A_φ) , (A_μ) , and $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Then there exists a unique distributional solution u of (1.1) and (1.2) satisfying*

$$u \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)).$$

Remark 2.9. Existence results for merely bounded (and more general) initial data can be found in Theorem 3.1 in [15] in the setting of the fractional porous medium equation (1.6) with $0 < m < 1$.

Theorem 2.10 (A priori estimates). *Assume (A_φ) , (A_μ) , $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Let u, \hat{u} be the distributional solutions of (1.1) with initial data u_0, \hat{u}_0 in the sense of Definition 2.2 (b), respectively. Then*

(a) (L^1 -contraction) $\int_{\mathbb{R}^N} (u(x, t) - \hat{u}(x, t))^+ dx \leq \int_{\mathbb{R}^N} (u_0(x) - \hat{u}_0(x))^+ dx$, $t \in [0, T]$;

(b) (Comparison principle) If $u_0 \leq \hat{u}_0$ a.e. in \mathbb{R}^N , then $u \leq \hat{u}$ a.e. in Q_T ;

(c) (L^1 -bound) $\|u(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)}$, $t \in [0, T]$;

(d) (L^∞ -bound) $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}$, $t \in [0, T]$;

(e) (Time regularity) For every $t, s \in [0, T]$ and compact set $K \subset \mathbb{R}^N$,

$$\|u(\cdot, t) - u(\cdot, s)\|_{L^1(K)} \leq \lambda_{u_0} \left(|t - s|^{\frac{1}{3}} \right) + C_{K, \varphi, u_0, \mu} \left(|t - s|^{\frac{1}{3}} + |t - s| \right)$$

where $\lambda_{u_0}(\delta) = \max_{|\sigma| \leq \delta} \|u_0 - u_0(\cdot + \sigma)\|_{L^1(\mathbb{R}^N)}$, $|K|$ is the Lebesgue measure of K , and for some constant C independent of K , φ , u_0 , and μ ,

$$C_{K, \varphi, u_0, \mu} = C|K| \left(\sup_{|r| \leq \|u_0\|_{L^\infty}} |\varphi(r)| + 1 \right) \int_{|z| > 0} \min\{|z|^2, 1\} d\mu(z).$$

(f) (Mass conservation) If, in addition, there exists $L, \delta > 0$ such that $|\varphi(r)| \leq L|r|$ for $|r| \leq \delta$, then

$$\int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx, \quad t \in [0, T].$$

These results are proven in Section 4.

Remark 2.11. The condition $|\varphi(r)| \leq L|r|$ in Theorem 2.10 (f) is sharp in the following sense: If $\varphi(r) = r^m$ for any $m < 1$, then there is $\mathcal{L}^\mu = -(-\Delta)^{\frac{s}{2}}$ such that positive solutions u of (1.1) and (1.2) has extinction in finite time and hence $\int u \neq \int u_0$. Simply take $N \in \mathbb{N}$ and $s \in (0, 2)$ such that $m \leq \frac{(N-s)^+}{N}$: see [37] for the details.

We now present several applications of the previous results.

2.1. Application 1: Compactness, local limits, continuous dependence.

We start by a compactness and convergence result for very general approximations of (1.1) and (1.2).

Theorem 2.12 (Compactness and convergence). *Assume $\mathcal{L} : C_c^\infty(Q_T) \rightarrow L^1(Q_T)$, μ_n satisfies (A_μ) , φ_n and φ satisfy (A_φ) , and $u_{0,n} \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ for every $n \in \mathbb{N}$. Then if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence of distributional solutions of (2.5) with initial data $\{u_{0,n}\}_{n \in \mathbb{N}}$ in the sense of Definition 2.2 (b), and*

(i) $\sup_n \int_{|z| > 0} \min\{|z|^2, 1\} d\mu_n(z) < \infty$;

(ii) $\sup_n \|u_{0,n}\|_{L^\infty(\mathbb{R}^N)} < \infty$;

(iii) $\mathcal{L}^{\mu_n}[\psi] \rightarrow \mathcal{L}[\psi]$ in $L^1(\mathbb{R}^N)$ for all $\psi \in C_c^\infty(\mathbb{R}^N)$;

(iv) $\varphi_n \rightarrow \varphi$ locally uniformly;

(v) $u_{0,n} \rightarrow u_0$ in $L_{\text{loc}}^1(\mathbb{R}^N)$.

Then

(a) there exist a subsequence $\{u_{n_j}\}_{j \in \mathbb{N}}$ and a $u \in C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N))$ such that

$$u_{n_j} \rightarrow u \quad \text{in} \quad C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N)) \quad \text{as} \quad j \rightarrow \infty;$$

(b) the limit u from part (a) is a distributional solution of (2.6) and (1.2).

The proof can be found in Section 5.1. Using this result, we study the case $\mathcal{L}^\mu = -(-\Delta)^{\frac{s}{2}}$, $s \in (0, 2)$. As expected, we find that solutions of the fractional equation (1.4) converge as $s \rightarrow 2^-$ to the solution of the local equation (1.5). Then we obtain a new result about continuous dependence on (m, s) for the porous medium equation of [37], that is, equation (1.6).

Corollary 2.13. *Assume (A_φ) and $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.*

(a) *The distributional solution u_s of (1.4) and (1.2), converges in $C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N))$ as $s \rightarrow 2^-$ to a function u , and u is a distributional solution of (1.5) and (1.2).*

(b) Let u_n and \bar{u} be distributional solutions of (1.6) and (1.2) with $(m, s) = (m_n, s_n)$ and $(m, s) = (\bar{m}, \bar{s})$ respectively. If

$$(0, \infty) \times (0, 2) \ni (m_n, s_n) \longrightarrow (\bar{m}, \bar{s}) \in (0, \infty) \times (0, 2],$$

then $u_n \rightarrow \bar{u}$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$.

The proof of this result can also be found in section 5.1.

Remark 2.14. When $u_0 \in L^1(\mathbb{R}^N)$, the authors of [37] show continuous dependence in $C([0, T]; L^1(\mathbb{R}^N))$ for (1.6) and (1.2) for $(m, s) \in \left(\frac{(N-s)^+}{N}, \infty\right) \times (0, 2]$. When $m \leq \frac{(N-s)^+}{N}$, we are in the fast diffusion range and Corollary 2.13 (b) provides the first continuous dependence result for this case.

2.2. Application 2: Numerical approximation, convergence, existence.

Surprisingly, our class of operators \mathcal{L}^μ is so wide that it contains a lot of its own numerical discretizations! It even contains common discretizations of local operators as well. We illustrate this by giving one such discretization, a basic and very natural one, and then analyzing the resulting semidiscrete numerical method for (1.1), or rather (2.7). We prove that it satisfies many properties including convergence, and conclude a second and more general existence result. Consider

$$(2.7) \quad \partial_t u - (L^\sigma + \mathcal{L}^\mu)[\varphi(u)] = 0 \quad \text{in} \quad Q_T,$$

where \mathcal{L}^μ is defined as before and L^σ is a possibly degenerate local operator

$$L^\sigma[\psi](x) := \text{tr}[\sigma \sigma^T D^2 \psi(x)]$$

where $\sigma = (\sigma_1, \dots, \sigma_P) \in \mathbb{R}^{N \times P}$, $P \in \mathbb{N}$, and $\sigma_i \in \mathbb{R}^N$. Note that $L^\sigma + \mathcal{L}^\mu$ is the generator of a *symmetric* Lévy process, and conversely, any *symmetric* Lévy process has a generator like $L^\sigma + \mathcal{L}^\mu$ (cf. [7]). Moreover, equation (1.1) and (1.5) are special cases of (2.7) since σ and μ may be degenerate or even zero.

For any $h > 0$, we approximate (2.7) in the following way,

$$(2.8) \quad \partial_t u_h - (L_h^\sigma + \mathcal{L}_h^\mu)[\varphi(u_h)] = 0 \quad \text{in} \quad Q_T.$$

where

$$(2.9) \quad L_h^\sigma[\psi](x) := \sum_{i=1}^P \frac{\psi(x + \sigma_i h) + \psi(x - \sigma_i h) - 2\psi(x)}{h^2},$$

$$(2.10) \quad \mathcal{L}_h^\mu[\psi](x) := \sum_{\alpha \neq 0} (\psi(x + z_\alpha) - \psi(x)) \mu(z_\alpha + R_h),$$

and $z_\alpha = h\alpha$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Z}^N$, $R_h = \frac{h}{2}[-1, 1]^N$. This is a finite difference approximation of L^σ and quadrature approximation of \mathcal{L}^μ .

Remark 2.15. (a) When $\sigma = e_i$, a standard basis vector of \mathbb{R}^N , then $L^{e_i} = \frac{\partial^2}{\partial x_i^2}$ and

$$L_h^{e_i} \psi(x) = \frac{\psi(x + h e_i) - 2\psi(x) + \psi(x - h e_i)}{h^2}: \text{ a classical finite difference approximation.}$$

(b) Both L_h^σ and \mathcal{L}_h^μ are in form (1.3) and satisfy (A_μ) : cf. Lemma 5.2 and 5.3.

(c) $L^\sigma \psi(x) = \sum_{i=1}^P \sigma_i^T D^2 \psi(x) \sigma_i = \sum_{i=1}^P (\sigma_i^T D)^2 \psi(x) \approx L_h^\sigma \psi(x)$.

(d) $\mathcal{L}^\mu[\psi](x) = \sum_{\alpha \in \mathbb{Z}^N} \int_{z_\alpha + R_h} \psi(x + z) - \psi(x) d\mu(z) \approx \mathcal{L}_h^\mu[\psi](x)$.

(e) To avoid $\mu(R_h)$ which may be infinite, we do not sum over $\alpha = 0$ in \mathcal{L}_h^μ .

We now show that the scheme has many good properties, including convergence.

Proposition 2.16 (Properties of approximation). *Assume (A_φ) , (A_μ) , $\sigma \in \mathbb{R}^{N \times P}$, $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, and $h > 0$.*

(a) (Existence and uniqueness) *There exists a unique distributional solution $u_h \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ of (2.8) and (1.2).*

(b) (L^p -stable) $\|u_h(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}^{\frac{p-1}{p}} \|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}}, \quad p \in [1, \infty], t \in [0, T].$

(c) (L^1 -consistent) *For all $\psi \in C_c^\infty(\mathbb{R}^N)$*

$$\|(L_h^\sigma + \mathcal{L}_h^\mu)[\psi] - (L^\sigma + \mathcal{L}^\mu)[\psi]\|_{L^1(\mathbb{R}^N)} \rightarrow 0 \quad \text{as} \quad h \rightarrow 0^+.$$

(d) (Monotone) *If $u_0 \leq \hat{u}_0$ a.e. in \mathbb{R}^N , then $u_h \leq \hat{u}_h$ a.e. in Q_T .*

(e) (Conservative) *If in addition, there exists $\delta, L > 0$ such that $|\varphi(r)| \leq L|r|$ for $|r| \leq \delta$, then for all $t \in [0, T]$*

$$\int_{\mathbb{R}^N} u_h(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx.$$

Proposition 2.17 (Compactness of approximation). *Assume (A_φ) , (A_μ) , $\sigma \in \mathbb{R}^{N \times P}$, $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, and $h > 0$. Then there is subsequence of distributional solutions u_h of (2.8) and (1.2) that converges in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ as $h \rightarrow 0^+$ to some function u . Moreover, $u \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ and u is a distributional solution of (2.7) and (1.2).*

Note that Proposition 2.17 also provide a new existence result:

Corollary 2.18 (Existence for (2.7)). *Under the assumptions of Proposition 2.17, there exists a distributional solution $u \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ of (2.7) and (1.2).*

In many cases we can combine the compactness result with uniqueness results for the limit equations, and hence obtain convergence for the approximation.

Theorem 2.19 (Convergence of approximation). *Under the assumptions of Proposition 2.17, and if in addition either $\sigma \equiv 0$ or $\mu \equiv 0$ and $\sigma = I$ (the identity matrix), then the distributional solutions u_h of (2.8) and (1.2) converges in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ as $h \rightarrow 0^+$ to the unique distributional solution $u \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ of (2.7) and (1.2).*

The proofs will be given in Section 5.2.

Remark 2.20. (a) Our approximation is well-defined and converge for *any* problem of the type (2.7), including strongly degenerate Stefan problems and fast diffusion equations. The scheme and convergence result thus cover cases that have not been considered before in the literature. For nonlocal problems of this type, there are very few results, and only for locally Lipschitz φ [43, 23, 42].

(b) To obtain a fully discrete numerical method, it remains to (i) restrict the method to some spacial grid and (ii) discretize also in time. Time discretization is easier and leads to a problem that no longer has the form (1.1); we will discuss it in a future work. Restriction to a spacial grid can always be done after a change of coordinate system: see Section 2.3 below.

(c) The existence result is a result where existence for problems involving nonlocal operators \mathcal{L}^μ are exported to problems involving the “closure” of this class of operators – namely, operators of the form $L^\sigma + \mathcal{L}^\mu$. The proof is completely different from proofs based on nonlinear semigroup theory; see e.g. Chp. 10 in [48], and [37].

2.3. Remarks and extensions.

Alternative definition of distributional solutions.

(1) A more compact form that we will use in the proofs is the following:

Lemma 2.21. *Assume (A_φ) , (A_{u_0}) , (A_μ) and $u \in L^\infty(Q_T)$. Then u is a distributional solution of (1.1) and (1.2) if and only if*

$$\int_0^T \int_{\mathbb{R}^N} \left(u(x, t) \partial_t \psi(x, t) + \varphi(u(x, t)) \mathcal{L}^\mu[\psi(\cdot, t)](x) \right) dx dt + \int_{\mathbb{R}^N} u_0(x) \psi(x, 0) dx = 0$$

for all $\psi \in C_c^\infty(\mathbb{R}^N \times [0, T])$.

The easy and standard proof is omitted.

About the initial conditions.

(2) The solutions provided by Theorem 2.8 belong to $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ and hence satisfy the initial condition in the strong L^1_{loc} -sense: For all compact $K \subset \mathbb{R}^N$,

$$\int_K |u(x, t) - u_0(x)| dx \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

(3) If the initial conditions are satisfied in the strong L^1_{loc} -sense, then they are of course also satisfied in the distributional sense of Definition 2.2.

Extensions of the uniqueness result Corollary 2.4.

(4) With the same proof, we also get uniqueness for the initial value problem for the inhomogenous equation

$$\partial_t u + \mathcal{L}^\mu[\varphi(u)] = g(x, t).$$

(5) A close inspection of the proof reveals that we can replace continuity of φ in (A_φ) by continuity at zero, Borel measurability, and $\varphi(u) \in L^\infty(Q_T)$ (cf. [18]).

Extensions of the stability result Theorem 2.6.

(6) When φ_n is independent of n , we only need weak convergence of \mathcal{L}^{μ_n} in (i):

$$\mathcal{L}^{\mu_n}[\psi] \rightarrow \mathcal{L}[\psi] \quad \text{weakly in } L^1(\mathbb{R}^N) \text{ for all } \psi \in C_c^\infty(Q_T).$$

Moreover, by considering subsequences we can replace (iii) by $u_n \rightarrow u$ in $L^1_{\text{loc}}(Q_T)$. These observations follow by slight changes in the proof of Theorem 2.6 in Section 4.

(7) A general condition for L^1 -weak convergence of \mathcal{L}^{μ_n} [21]: There exist $\sigma \in \mathbb{R}^{N \times P}$ and a nonnegative Radon measure μ such that for all $A \in \mathbb{R}^{N \times N}$

$$(i) \quad \sup_n \int_{|z|>0} \min\{|z|^2, 1\} d\mu_n(z) < \infty;$$

$$(ii) \quad \int_{|z|\leq 1} z A z^T d\mu_n(z) \rightarrow \text{tr}(\sigma \sigma^T A) + \int_{|z|\leq 1} z A z^T d\mu(z);$$

$$(iii) \quad \int_{|z|>1} d\mu_n(z) \rightarrow \int_{|z|>1} d\mu(z).$$

Here $\mathcal{L} = \text{tr}[\sigma \sigma^T D^2] + \mathcal{L}^\mu$: see [21] for a general discussion and more examples.

Defining the scheme (2.8) on a grid.

(8) By a coordinate transformation $x = Ay$, $L^\sigma + \mathcal{L}^\mu$ can be transformed into

$$L^{I_0} + \mathcal{L}^{\tilde{\mu}} \quad \text{where} \quad I_0 := \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array} \right] \in \mathbb{R}^{N \times N},$$

I is an identity matrix, and $d\tilde{\mu}(z) = d\mu(A^{-1}z)$ satisfies (A_μ) . Up to permutations of the components of y , $A = QJ$ where $Q \in \mathbb{R}^{N \times N}$ is orthonormal, $Q\sigma\sigma^T Q^T = \text{diag}(\lambda_i)$ for $\lambda_i \geq 0$, and $J = \text{diag}(\sqrt{c_i})$ where $c_i = 1$ if $\lambda_i = 0$ and $c_i = \frac{1}{\lambda_i}$ if $\lambda_i > 0$ for $i = 1, \dots, N$.

- (9) For the new operator $L^{I_0} + \mathcal{L}^{\tilde{\mu}}$, our approximations produce an operator $L_h^{I_0} + \mathcal{L}_h^{\tilde{\mu}}$ that can be restricted to the (y) -grid $\mathcal{G}_h := h\mathbb{Z}^N$ ($h > 0$), that is $L_h^{I_0} + \mathcal{L}_h^{\tilde{\mu}} : \mathbb{R}^{\mathcal{G}_h} \rightarrow \mathbb{R}^{\mathcal{G}_h}$ is well-defined.

3. THE PROOF OF UNIQUENESS

3.1. Preliminary results. A crucial part in the proof is played by the following linear elliptic equation

$$(3.1) \quad \varepsilon v_\varepsilon(x) - \mathcal{L}^\mu[v_\varepsilon](x) = g(x) \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$ and \mathcal{L}^μ defined by (1.3). Its solutions will be denoted by

$$B_\varepsilon^\mu[g](x) := v_\varepsilon(x).$$

Formally, $B_\varepsilon^\mu = (\varepsilon I - \mathcal{L}^\mu)^{-1}$ is the resolvent of \mathcal{L}^μ . Note that \mathcal{L}^μ may be very degenerate and therefore Fourier techniques do not easily apply (cf. Example 3.1 and Remark 3.8 (a) below). The main results about equation (3.1) are given below, while most of the proofs will be given in Section 6. Note that in [18] such results are easy in view of an explicit representation formula for B_ε^μ . Here, on the other hand, they are not easy and we have to work quite a lot to prove these estimates. The method of proof is different, more nonlocal, and requires less of the operator.

Theorem 3.1 (Classical and distributional solutions). *Assume (A_μ) and $\varepsilon > 0$.*

- (a) *If $g \in C_b^\infty(\mathbb{R}^N)$, then there exists a unique classical solution $B_\varepsilon^\mu[g] \in C_b^\infty(\mathbb{R}^N)$ of (3.1). Moreover, for each multiindex $\alpha \in \mathbb{N}^N$,*

$$\varepsilon \|D^\alpha B_\varepsilon^\mu[g]\|_{L^\infty} \leq \|D^\alpha g\|_{L^\infty}.$$

- (b) *If $g \in L^1(\mathbb{R}^N)$, then there exists a unique distributional solution $B_\varepsilon^\mu[g] \in L^1(\mathbb{R}^N)$ of (3.1). Moreover,*

$$\varepsilon \|B_\varepsilon^\mu[g]\|_{L^1(\mathbb{R}^N)} \leq \|g\|_{L^1(\mathbb{R}^N)}.$$

- (c) *If $g \in L^\infty(\mathbb{R}^N)$, then there exists a unique distributional solution $B_\varepsilon^\mu[g] \in L^\infty(\mathbb{R}^N)$ of (3.1). Moreover,*

$$\varepsilon \|B_\varepsilon^\mu[g]\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}.$$

Remark 3.2. If $g \in L^1 \cap L^\infty$, then $\varepsilon \|B_\varepsilon^\mu[g]\|_{L^p} \leq \|g\|_{L^\infty}^{\frac{p-1}{p}} \|g\|_{L^1}^{\frac{1}{p}}$ for any $p \in (1, \infty)$.

When a smooth g depends also on time, then $B_\varepsilon^\mu[g]$ will be smooth in time and space.

Corollary 3.3. *Assume (A_μ) , $\varepsilon > 0$, and $\gamma \in C_c^\infty(\mathbb{R}^N \times [0, T])$. Then*

- (a) $B_\varepsilon^\mu[\gamma] \in C_b^\infty(\mathbb{R}^N \times [0, T])$.
(b) $B_\varepsilon^\mu[\gamma](x, \cdot)$ is compactly supported in $[0, T]$.
(c) $\partial_t(B_\varepsilon^\mu[\gamma]) = B_\varepsilon^\mu[\partial_t \gamma]$ and $B_\varepsilon^\mu[\gamma], B_\varepsilon^\mu[\partial_t \gamma], \mathcal{L}^\mu[B_\varepsilon^\mu[\gamma]] \in L^1(Q_T)$.

Proof. (a) A standard argument using difference quotients, linearity and uniqueness of the problem, the L^∞ -bound of Theorem 3.1 (a), and induction on n , gives that

$$(3.2) \quad \partial_t^n D^\alpha B_\varepsilon^\mu[\gamma] = B_\varepsilon^\mu[\partial_t^n D^\alpha \gamma] \quad \text{in } Q_T$$

for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^N$. This argument is almost exactly the same as the one given in the proof of Proposition 6.8 (d) below. Then by Theorem 3.1 (a),

$$\varepsilon \|\partial_t^n D^\alpha B_\varepsilon^\mu[\gamma]\|_{L^\infty(Q_T)} \leq \|\partial_t^n D^\alpha \gamma\|_{L^\infty(Q_T)}.$$

- (b) Holds since B_ε^μ is an operator in the spatial variable x and $B_\varepsilon^\mu[0] = 0$.

(c) Note that $\partial_t B_\varepsilon^\mu[\gamma] = B_\varepsilon^\mu[\partial_t \gamma]$ by (3.2), and by Theorem 3.1 (b) and the time continuity of γ and $B_\varepsilon^\mu[\gamma]$,

$$\varepsilon \|B_\varepsilon^\mu[\gamma]\|_{L^1(Q_T)} \leq \|\gamma\|_{L^1(Q_T)},$$

which is finite because $\gamma \in C_c^\infty(Q_T)$. Hence it follows that

$$\varepsilon \|\partial_t(B_\varepsilon^\mu[\gamma])\|_{L^1(Q_T)} = \varepsilon \|B_\varepsilon^\mu[\partial_t \gamma]\|_{L^1(Q_T)} \leq \|\partial_t \gamma\|_{L^1(Q_T)},$$

By equation (3.1), $\mathcal{L}^\mu[B_\varepsilon^\mu[\gamma]] = \varepsilon B_\varepsilon^\mu[\gamma] - \gamma$ for all $(x, t) \in Q_T$. Since both $B_\varepsilon^\mu[\gamma]$ and γ are in $L^1(Q_T)$, it follows that also $\mathcal{L}^\mu[B_\varepsilon^\mu[\gamma]] \in L^1(Q_T)$. \square

The operator B_ε^μ is self-adjoint in the following sense:

Lemma 3.4. *Assume (A_μ) , $g \in L^\infty(\mathbb{R}^N)$, $f \in L^1(\mathbb{R}^N)$, and $\varepsilon > 0$. Then*

$$\int_{\mathbb{R}^N} B_\varepsilon^\mu[g](x) f(x) \, dx = \int_{\mathbb{R}^N} g(x) B_\varepsilon^\mu[f](x) \, dx.$$

The proof is given in section 6. To prove these and other results in this paper, we will need some properties of the nonlocal operator \mathcal{L}^μ that are given below.

Lemma 3.5. *Assume (A_μ) .*

(a) *If $\psi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then*

$$|\mathcal{L}^\mu[\psi](x)| \leq \frac{1}{2} \max_{|z| \leq 1} |D^2 \psi(x+z)| \int_{|z| \leq 1} |z|^2 \, d\mu(z) + 2\|\psi\|_{L^\infty(\mathbb{R}^N)} \int_{|z| > 1} d\mu(z).$$

(b) *Let $p \in \{1, \infty\}$ be fixed. If $\psi \in W^{2,p}(\mathbb{R}^N)$, then*

$$\|\mathcal{L}^\mu[\psi]\|_{L^p(\mathbb{R}^N)} \leq \frac{1}{2} \|D^2 \psi\|_{L^p(\mathbb{R}^N)} \int_{|z| \leq 1} |z|^2 \, d\mu(z) + 2\|\psi\|_{L^p(\mathbb{R}^N)} \int_{|z| > 1} d\mu(z).$$

(c) *If $\psi_1 \in W^{2,1}(\mathbb{R}^N)$ and $\psi_2 \in W^{2,\infty}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \psi_1 \mathcal{L}^\mu[\psi_2] \, dx = \int_{\mathbb{R}^N} \mathcal{L}^\mu[\psi_1] \psi_2 \, dx.$$

Remark 3.6. (a) If $\psi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then $\mathcal{L}^\mu[\psi](x)$ is well-defined by (a).

(b) If $\mu(\mathbb{R}^N) < \infty$, a density argument and the symmetry of μ reveals that

$$\mathcal{L}^\mu[\phi](x) = \int_{|z| > 0} (\phi(x+z) - \phi(x)) \, d\mu(z),$$

and the assumptions of Lemma 3.4 can be relaxed to $g \in L^\infty(\mathbb{R}^N)$, $f \in L^p(\mathbb{R}^N)$ for $p \in \{1, \infty\}$, and $\psi_1 \in L^1(\mathbb{R}^N)$ and $\psi_2 \in L^\infty(\mathbb{R}^N)$ respectively in (a), (b), and (c). The second derivative part of the estimates in (a) and (b) then have to be dropped and the remaining term modified accordingly.

A proof of Lemma 3.5 can be found e.g. in Sections 1 and 4 in [3].

Lemma 3.7. *Assume (A_μ) and $\psi \in C_c^\infty(\mathbb{R}^N)$. Then*

$$\mathcal{F}(\mathcal{L}^\mu[\psi])(\xi) = -\sigma_{\mathcal{L}^\mu}(\xi) \mathcal{F}(\psi)(\xi),$$

where

$$\sigma_{\mathcal{L}^\mu}(\xi) := \int_{|z| > 0} 1 - \cos(z \cdot \xi) \, d\mu(z).$$

Moreover, $\sigma_{\mathcal{L}^\mu}(\xi) \geq 0$ and

$$\left(\psi, \mathcal{L}^\mu[\psi] \right)_{L^2(\mathbb{R}^N)} = - \left\| (\mathcal{L}^\mu)^{\frac{1}{2}}[\psi] \right\|_{L^2(\mathbb{R}^N)}^2.$$

Remark 3.8. (a) $\sigma_{\mathcal{L}^\mu}$ is the Fourier symbol of \mathcal{L}^μ . In our generality it may not be invertible or have any smoothing properties. An extreme example is $\mu = \delta_{z_0}$ for $z_0 \neq 0$, where $\sigma_{\mathcal{L}^\mu}(\xi) = 1 - \cos z_0 \cdot \xi$; this is a bounded function with infinitely many zeros.

(b) If $\psi, \mathcal{L}^\mu[\psi] \in L^2(\mathbb{R}^N)$, then a density argument shows that the Fourier symbol exists and the conclusions of Lemma 3.7 still hold.

- (c) The notation $(\mathcal{L}^\mu)^{\frac{1}{2}}$ is used to denote the square root of the operator \mathcal{L}^μ in the Fourier transform sense.

Proof. By the definition of \mathcal{L}^μ , Fubini's theorem, and the symmetry of μ ,

$$\begin{aligned} \mathcal{F}(\mathcal{L}^\mu[\psi])(\xi) &= (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-ix \cdot \xi} \int_{|z|>0} \\ &\quad \left(\psi(x+z) - \psi(x) - z \cdot D\psi(x) \mathbf{1}_{|z|\leq 1} \right) d\mu(z) dx \\ &= \int_{|z|>0} \left(e^{iz \cdot \xi} \mathcal{F}(\psi)(\xi) - \mathcal{F}(\psi)(\xi) - iz \cdot \xi \mathbf{1}_{|z|\leq 1} \mathcal{F}(\psi)(\xi) \right) d\mu(z) \\ &= \mathcal{F}(\psi)(\xi) \int_{|z|>0} \left(\cos(z \cdot \xi) - 1 \right) d\mu(z). \end{aligned}$$

To show the second part of the lemma, note that $\sigma_{\mathcal{L}^\mu} \geq 0$ and $\psi, \mathcal{L}^\mu[\psi] \in L^2(\mathbb{R}^N)$ (cf. Lemma 3.5 (b)). It follows that $\mathcal{F}(\psi), \sigma_{\mathcal{L}^\mu} \mathcal{F}(\psi) \in L^2(\mathbb{R}^N)$, and then by the inequality $2ab \leq a^2 + b^2$, $(\sigma_{\mathcal{L}^\mu})^{\frac{1}{2}} \mathcal{F}(\psi) \in L^2(\mathbb{R}^N)$. By Plancherel's theorem,

$$\begin{aligned} \left(\psi, \mathcal{L}^\mu[\psi] \right)_{L^2(\mathbb{R}^N)} &= \left(\mathcal{F}(\psi), \mathcal{F}(\mathcal{L}^\mu[\psi]) \right)_{L^2(\mathbb{R}^N)} = \left(\mathcal{F}(\psi), -\sigma_{\mathcal{L}^\mu} \mathcal{F}(\psi) \right)_{L^2(\mathbb{R}^N)} \\ &= - \left((\sigma_{\mathcal{L}^\mu})^{\frac{1}{2}} \mathcal{F}(\psi), (\sigma_{\mathcal{L}^\mu})^{\frac{1}{2}} \mathcal{F}(\psi) \right)_{L^2(\mathbb{R}^N)} = - \left\| (\mathcal{L}^\mu)^{\frac{1}{2}}[\psi] \right\|_{L^2(\mathbb{R}^N)}^2, \end{aligned}$$

which completes the proof. \square

The following theorem is a key technical tool in our uniqueness argument.

Theorem 3.9. *Assume (A_μ) and $\text{supp } \mu \neq \emptyset$. If $v \in C_0(\mathbb{R}^N)$ solves*

$$\mathcal{L}^\mu[v] = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N),$$

then $v \equiv 0$ for all $x \in \mathbb{R}^N$.

We give the proof of Theorem 3.9 in Appendix A. In the local case [18] such a result follows for example from the Liouville theorem for the Laplacian. On one hand, our result is much weaker since we need to ask for some kind of decay at infinity. On the other hand, Theorem 3.9 covers very degenerate operators \mathcal{L}^μ which do not satisfy any sort of Liouville theorem.

Example 3.1. Let $\mu = \delta_{2\pi} + \delta_{-2\pi}$. Note that (A_μ) holds and that for smooth functions v ,

$$\mathcal{L}^\mu[v](x) = v(x+2\pi) - 2v(x) + v(x-2\pi).$$

The function $v = \cos \in C_b^\infty(\mathbb{R})$ is an example of a nonconstant function that satisfies $\mathcal{L}^\mu[v](x) = 0$ in \mathbb{R} , and hence the Liouville theorem does not hold for \mathcal{L}^μ .

3.2. The proof of Theorem 2.3. We define

$$U(x, t) := u(x, t) - \hat{u}(x, t) \quad \text{and} \quad \Phi(x, t) := \varphi(u(x, t)) - \varphi(\hat{u}(x, t)).$$

By the assumptions (2.1), (2.2), and (A_φ) ,

$$U \in L^1(Q_T) \cap L^\infty(Q_T), \quad \Phi \in L^\infty(Q_T),$$

and by (2.3), (2.4), and Lemma 2.21

$$(3.3) \quad \int_0^T \int_{\mathbb{R}^N} \left(U \partial_t \psi + \Phi \mathcal{L}^\mu[\psi] \right) dx dt = 0 \quad \text{for all} \quad \psi \in C_c^\infty(\mathbb{R}^N \times [0, T]).$$

We emphasize that this equation also incorporates a zero initial condition for U .

We now define the function $h_\varepsilon(t)$ which will play the main role in the proof:

$$(3.4) \quad h_\varepsilon(t) := (B_\varepsilon^\mu[U](\cdot, t), U(\cdot, t)) = \int_{\mathbb{R}^N} B_\varepsilon^\mu[U(\cdot, t)](x) U(x, t) dx.$$

Note that $h_\varepsilon \in L^1((0, T))$ since $\|h_\varepsilon\|_{L^1((0, T))} \leq \frac{1}{\varepsilon} \|U\|_{L^\infty(Q_T)} \|U\|_{L^1(Q_T)}$ by Theorem 3.1 (b). For the proof of Theorem 2.3, we will now show that there is a sequence $\varepsilon_n \rightarrow 0^+$ such that $\lim_{\varepsilon_n \rightarrow 0^+} h_{\varepsilon_n}(t) = 0$. To do that we start by the following lemma:

Lemma 3.10. *Assume (A_μ) , $U \in L^1(Q_T) \cap L^\infty(Q_T)$, $\Phi \in L^\infty(Q_T)$, and (3.3) holds. Then*

$$(a) \iint_{Q_T} \left(B_\varepsilon^\mu[U] \partial_t \psi + (\varepsilon B_\varepsilon^\mu[\Phi] - \Phi) \psi \right) dx dt = 0 \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N \times [0, T]).$$

$$(b) B_\varepsilon^\mu[U(\cdot, t)](x) = \int_0^t \left(\varepsilon B_\varepsilon^\mu[\Phi(\cdot, s)](x) - \Phi(x, s) \right) ds \quad \text{a.e. } (x, t) \in \mathbb{R}^N \times (0, T).$$

$$(c) \text{ For a.e. } t \in (0, T), \|B_\varepsilon^\mu[U](\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq 2t \|\Phi\|_{L^\infty(Q_T)}.$$

Proof. (a) We fix $\gamma \in C_c^\infty(\mathbb{R}^N \times [0, T])$ and take $\psi = B_\varepsilon^\mu[\gamma]$ as a test function in (3.3). Note that ψ is an admissible test function by a density argument using Corollary 3.3 (a)–(c) and $U, \Phi \in L^\infty(Q_T)$. Then by (3.1) and Corollary 3.3 (c),

$$\begin{aligned} 0 &= \iint_{Q_T} \left(U \partial_t (B_\varepsilon^\mu[\gamma]) + \Phi \mathcal{L}^\mu [B_\varepsilon^\mu[\gamma]] \right) dx dt \\ &= \iint_{Q_T} \left(U B_\varepsilon^\mu[\partial_t \gamma] + \Phi (\varepsilon B_\varepsilon^\mu[\gamma] - \gamma) \right) dx dt. \end{aligned}$$

Finally, the self-adjointness of B_ε^μ (cf. Lemma 3.4) yields

$$\int_0^T \int_{\mathbb{R}^N} \left(B_\varepsilon^\mu[U] \partial_t \gamma + (\varepsilon B_\varepsilon^\mu[\Phi] - \Phi) \gamma \right) dx dt = 0,$$

which completes the proof.

(b) This result follows from (a) and a special choice of test function. For $0 < s < T$, $a > 0$, and $0 < \delta < T - a$, we define

$$\theta_a(t) = \begin{cases} 1 & t \leq s - a \\ 1 - \frac{1}{a}(t - s + a) & s - a < t < s \\ 0 & t \geq s \end{cases} \quad \text{and} \quad \theta_{a,\delta}(t) = \theta_a * \rho_\delta(t),$$

where the mollifier ρ_δ is defined in (1.8). Then $\theta_{a,\delta} \in C_b^\infty((0, T)) \cap L^1((0, T))$ and $\text{supp}\{\theta_{a,\delta}\} \subset [-\infty, T)$. Let $\gamma \in C_c^\infty(\mathbb{R}^N)$ and take $\psi(x, t) = \theta_{a,\delta}(t) \gamma(x) \in C_c^\infty(\mathbb{R}^N \times [0, T])$ as a test function in part (a). Then we use properties of mollifiers and Lebesgue's dominated convergence theorem to send $\delta \rightarrow 0^+$ and get

$$\iint_{Q_T} \left(B_\varepsilon^\mu[U] \theta'_a + (\varepsilon B_\varepsilon^\mu[\Phi] - \Phi) \theta_a \right) \gamma dx dt = 0.$$

By Fubini's theorem and since $\theta'_a(t) = -\frac{1}{a} \mathbf{1}_{s-a < t < s}$ and $\text{supp}\{\theta_a\} = [0, s]$, we find that

$$\int_{\mathbb{R}^N} \left(\frac{1}{a} \int_{s-a}^s B_\varepsilon^\mu[U] dt + \int_0^s (\varepsilon B_\varepsilon^\mu[\Phi] - \Phi) \theta_a dt \right) \gamma dx = 0.$$

We now send $a \rightarrow 0^+$. Since $\int_{\mathbb{R}^N} B_\varepsilon^\mu[U(\cdot, t)](x) \gamma(x) dx \in L^1(0, T)$ by Fubini's theorem,

$$\frac{1}{a} \int_{s-a}^s \int_{\mathbb{R}^N} B_\varepsilon^\mu[U(\cdot, t)](x) \gamma(x) dx dt \rightarrow \int_{\mathbb{R}^N} B_\varepsilon^\mu[U(\cdot, s)](x) \gamma(x) dx \quad \text{as } a \rightarrow 0^+$$

for a.e. s by Lebesgue's differentiation theorem. For the other term, we may use Lebesgue's dominated convergence theorem to pass to the limit. Since $\theta_a \rightarrow \mathbf{1}_{[0, s]}$ pointwise, we find that for a.e. $s \in [0, T]$,

$$\int_{\mathbb{R}^N} \left(B_\varepsilon^\mu[U(\cdot, s)](x) + \int_0^s \left(\varepsilon B_\varepsilon^\mu[\Phi(\cdot, t)](x) - \Phi(x, t) \right) dt \right) \gamma(x) dx = 0.$$

Since $\gamma \in C_c^\infty(\mathbb{R}^N)$ is arbitrary, part (b) follows.

(c) By part (b) and Theorem 3.1 (c), $\|B_\varepsilon^\mu[U](\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq 2t\|\Phi\|_{L^\infty(Q_T)}$ a.e. \square

Proposition 3.11. *Assume (A_μ) , $U \in L^1(Q_T) \cap L^\infty(Q_T)$, $\Phi \in L^\infty(Q_T)$, and (3.3) holds. Then $h_\varepsilon(t)$ defined by (3.4) is absolutely continuous and*

$$h'_\varepsilon(t) = 2(\varepsilon B_\varepsilon^\mu[\Phi](\cdot, t) - \Phi(\cdot, t), U(\cdot, t)) \quad \text{in } \mathcal{D}'((0, T)).$$

The proof below is an adaptation of the proof in [18, pp. 157–158].

Proof. Let the mollifier $\rho_\delta = \rho_\delta(t)$ be defined in (1.8), the extension \bar{U} be U on Q_T and zero outside Q_T , and

$$\bar{U}_\delta(x, t) := \bar{U}(x, \cdot) * \rho_\delta(t) = \int_{\mathbb{R}} \bar{U}(x, s) \rho_\delta(t - s) ds.$$

By Young's inequality, $\|\bar{U}_\delta\|_{L^\infty(Q_T)} \leq \|U\|_{L^\infty(Q_T)}$ and $\|\bar{U}_\delta\|_{L^1(Q_T)} \leq \|U\|_{L^1(Q_T)}$. Moreover, the time continuity of \bar{U}_δ , Corollary 3.3 (c), and Lemma 3.4 yields

$$(3.5) \quad \frac{d}{dt} \int_{\mathbb{R}^N} B_\varepsilon^\mu[\bar{U}_\delta] \bar{U}_\delta dx = 2 \int_{\mathbb{R}^N} \partial_t (B_\varepsilon^\mu[\bar{U}_\delta]) \bar{U}_\delta dx = 2 \int_{\mathbb{R}^N} \partial_t (\bar{U}_\delta) B_\varepsilon^\mu[\bar{U}_\delta] dx$$

for $t \in \mathbb{R}$.

Let us show that

$$(3.6) \quad B_\varepsilon^\mu[\bar{U}_\delta(\cdot, t)](x) = \int_{\mathbb{R}} B_\varepsilon^\mu[\bar{U}(\cdot, s)](x) \rho_\delta(t - s) ds \quad \text{in } Q_T.$$

First assume that $\bar{U} \in C_b^\infty(Q_T) \cap L^1(Q_T)$. Then $B_\varepsilon^\mu[\bar{U}(\cdot, t)] \in C_b^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ for $t \in [0, T]$, and thus, it solves (3.1) pointwise in \mathbb{R}^N . Multiply this equation by $\rho_\delta(s - t)$, integrate over \mathbb{R} , and use Fubini's theorem and the uniqueness in Theorem 3.1 (b) and (c) to find that (3.6) holds. A density/mollification argument using uniqueness and $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ estimates from Theorem 3.1 then shows that (3.6) also holds (a.e.!) for $\bar{U} \in L^1(Q_T) \cap L^\infty(Q_T)$.

Let the extension $\bar{\Phi}$ be Φ on Q_T and zero outside Q_T . Using Lemma 3.10 (a) with test functions $\psi \in C_c^\infty(\mathbb{R}^N \times (\delta, T - \delta))$ we get that

$$\partial_t B_\varepsilon^\mu[\bar{U}_\delta(\cdot, t)](x) = \left((\varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi})(x, \cdot) * \rho_\delta \right)(t) \quad \text{a.e. in } \mathbb{R}^N \times (\delta, T - \delta).$$

For any $\Theta \in C_c^\infty((0, T))$ and sufficiently small δ , we then conclude from (3.5) that

$$- \int_0^T (B_\varepsilon^\mu[\bar{U}_\delta](\cdot, t), \bar{U}_\delta(\cdot, t)) \Theta'(t) dt = 2 \int_0^T ((\varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi}) * \rho_\delta(t), \bar{U}_\delta(\cdot, t)) \Theta(s) ds.$$

By properties of mollifiers and Theorem 3.1 (b) and (c),

$$\begin{aligned} \bar{U}_\delta &\rightarrow U \quad \text{in } L^1(Q_T), \\ (\varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi}) * \rho_\delta &\rightarrow \varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi} \quad \text{a.e. in } Q_T, \\ \varepsilon \|B_\varepsilon^\mu[\bar{U}_\delta]\|_{L^\infty(Q_T)} &\leq \|U\|_{L^\infty(Q_T)}, \\ |(\varepsilon B_\varepsilon^\mu[\bar{\Phi}] - \bar{\Phi}) * \rho_\delta| &\leq 2\|\Phi\|_{L^\infty(Q_T)}. \end{aligned}$$

Now we send $\delta \rightarrow 0^+$ using Lebesgue's dominated convergence theorem, and then by the definition of h_ε , we find that

$$- \int_0^T h_\varepsilon(t) \Theta'(t) dt = 2 \int_0^T (\varepsilon B_\varepsilon^\mu[\Phi](\cdot, t) - \Phi(\cdot, t), U(\cdot, t)) \Theta(t) dt.$$

That is, h_ε is weakly differentiable and the weak derivative is

$$h'_\varepsilon(t) = 2(\varepsilon B_\varepsilon^\mu[\Phi](\cdot, t) - \Phi(\cdot, t), U(\cdot, t)).$$

Moreover, $h'_\varepsilon \in L^1((0, T))$ since by Theorem 3.1 (c),

$$\int_0^T |h'_\varepsilon(t)| dt \leq 4\|\Phi\|_{L^\infty(Q_T)}\|U\|_{L^1(Q_T)}.$$

Hence, $h_\varepsilon(t)$ is absolutely continuous, and the proof is complete. \square

Proposition 3.12. *Assume (A_φ) , (A_μ) , $U \in L^1(Q_T) \cap L^\infty(Q_T)$, $\Phi \in L^\infty(Q_T)$ and (3.3) holds. Then*

(a) *For a.e. $t \in [0, T]$*

$$h_\varepsilon(t) = \varepsilon\|B_\varepsilon^\mu[U](\cdot, t)\|_{L^2}^2 + \|(\mathcal{L}^\mu)^{\frac{1}{2}}[B_\varepsilon^\mu[U]](\cdot, t)\|_{L^2}^2.$$

(b) *If a sequence $\varepsilon_n B_{\varepsilon_n}^\mu[U] \rightarrow 0$ a.e. in Q_T as $\varepsilon_n \rightarrow 0^+$, then for a.e. $t \in [0, T]$,*

$$\lim_{\varepsilon_n \rightarrow 0^+} h_{\varepsilon_n}(t) = 0.$$

We need a technical lemma (cf. [18]).

Lemma 3.13. *Assume (A_φ) and (2.2). Then the Lebesgue measure of the set*

$$S^\xi := \{(x, t) \in Q_T : |\varphi(u(x, t)) - \varphi(\hat{u}(x, t))| > \xi\},$$

is finite for all $\xi > 0$.

Proof. Define the set

$$S_u^\delta = \{(x, t) \in Q_T : |u(x, t) - \hat{u}(x, t)| > \delta\}.$$

If $(x, t) \in S^\xi$, then by the continuity of φ there exists a $\delta > 0$ such that $|u(x, t) - \hat{u}(x, t)| > \delta$, that is, $S^\xi \subset S_u^\delta$. By (2.2),

$$\delta|S_u^\delta| < \iint_{Q_T} |u(x, t) - \hat{u}(x, t)| dx dt < \infty,$$

and thus, S^ξ also has finite Lebesgue measure. \square

Proof of Proposition 3.12. (a) By the assumptions, Theorem 3.1 (b) and (c), interpolation between $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$, and Fubini's theorem, we have for a.e. $t \in [0, T]$ that $U, B_\varepsilon^\mu[U] \in L^2(\mathbb{R}^N)$ and

$$(3.7) \quad \varepsilon B_\varepsilon^\mu[U] - \mathcal{L}^\mu[B_\varepsilon^\mu[U]] = U \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N).$$

Hence it follows that $\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]] \in L^2(\mathbb{R}^N)$, where $\overline{\mathcal{L}}^\mu$ is defined through the relation

$$\int_{\mathbb{R}^N} \overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]]\psi dx dt = \int_{\mathbb{R}^N} B_\varepsilon^\mu[U]\mathcal{L}^\mu[\psi] dx dt \quad \text{for all} \quad \psi \in C_c^\infty(\mathbb{R}^N).$$

Using Plancherel's theorem and Lemma 3.7, we then find that for any $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} \int_{\mathbb{R}^N} \mathcal{F}(\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]])\mathcal{F}(\psi) d\xi &= \int_{\mathbb{R}^N} \mathcal{F}(B_\varepsilon^\mu[U])\mathcal{F}(\mathcal{L}^\mu[\psi]) d\xi \\ &= - \int_{\mathbb{R}^N} \mathcal{F}(B_\varepsilon^\mu[U])\sigma_{\mathcal{L}^\mu}(\xi)\mathcal{F}(\psi) d\xi, \end{aligned}$$

and hence

$$\int_{\mathbb{R}^N} \mathcal{F}(\psi)(\xi) \left(\mathcal{F}(\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]])(\xi) + \sigma_{\mathcal{L}^\mu}(\xi)\mathcal{F}(B_\varepsilon^\mu[U])(\xi) \right) d\xi = 0.$$

Then by a density argument, we conclude that

$$\mathcal{F}(\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]])(\xi) = -\sigma_{\mathcal{L}^\mu}(\xi)\mathcal{F}(B_\varepsilon^\mu[U])(\xi) \quad \text{in} \quad L^2(\mathbb{R}^N),$$

and thus, for a.e. $t \in [0, T]$, we have $\overline{\mathcal{L}}^\mu[B_\varepsilon^\mu[U]] = \mathcal{L}^\mu[B_\varepsilon^\mu[U]]$ in $L^2(\mathbb{R}^N)$.

Since $U, B_\varepsilon^\mu[U], \mathcal{L}^\mu[B_\varepsilon^\mu[U]] \in L^2(\mathbb{R}^N)$, equation (3.7) holds in $L^2(\mathbb{R}^N)$. By Lemma 3.7, Remark 3.8 (b), and the definition of h_ε (see (3.4)), we have for a.e. $t \in [0, T]$ that

$$\begin{aligned} h_\varepsilon(t) &= (B_\varepsilon^\mu[U](\cdot, t), U(\cdot, t))_{L^2(\mathbb{R}^N)} \\ &= (B_\varepsilon^\mu[U](\cdot, t), \varepsilon B_\varepsilon^\mu[U](\cdot, t) - \mathcal{L}^\mu[B_\varepsilon^\mu[U]](\cdot, t))_{L^2(\mathbb{R}^N)} \\ &= \varepsilon \|B_\varepsilon^\mu[U](\cdot, t)\|_{L^2(\mathbb{R}^N)}^2 - (B_\varepsilon^\mu[U](\cdot, t), \mathcal{L}^\mu[B_\varepsilon^\mu[U]](\cdot, t))_{L^2(\mathbb{R}^N)} \\ &= \varepsilon \|B_\varepsilon^\mu[U](\cdot, t)\|_{L^2(\mathbb{R}^N)}^2 + \|(\mathcal{L}^\mu)^{\frac{1}{2}}[B_\varepsilon^\mu[U]]\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

(b) By part (a), Proposition 3.11, and $U\Phi = (u - \hat{u})(\varphi(u) - \varphi(\hat{u})) \geq 0$,

$$\begin{aligned} (3.8) \quad 0 \leq h_\varepsilon(t) &= h_\varepsilon(0+) + \int_0^t h_\varepsilon'(s) \, ds \\ &\leq h_\varepsilon(0+) + 2 \int_0^t (\varepsilon B_\varepsilon^\mu[\Phi](\cdot, s), U(\cdot, s)) \, ds. \end{aligned}$$

By the (absolute) continuity of h_ε , Hölder's inequality, Lemma 3.10 (c), and Lebesgue's dominated convergence theorem (valid since $U \in L^1(Q_T)$),

$$\begin{aligned} h_\varepsilon(0+) &= \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t h_\varepsilon(s) \, ds \leq \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \|B_\varepsilon^\mu[U](\cdot, s)\|_{L^\infty(\mathbb{R}^N)} \|U(\cdot, s)\|_{L^1(\mathbb{R}^N)} \, ds \\ &\leq 2\|\Phi\|_{L^\infty(Q_T)} \lim_{t \rightarrow 0^+} \int_0^t \|U(\cdot, s)\|_{L^1(\mathbb{R}^N)} \mathbf{1}_{(0,t)}(s) \, ds = 0. \end{aligned}$$

Let $\xi > 0$. By the self-adjointness of B_ε^μ (cf. Lemma 3.4) and Theorem 3.1 (b), we get for a.e. $t \in [0, T]$

$$\begin{aligned} (\varepsilon B_\varepsilon^\mu[\Phi](\cdot, t), U(\cdot, t)) &= \int_{\mathbb{R}^N} \Phi(x, t) \varepsilon B_\varepsilon^\mu[U(\cdot, t)](x) \, dx \\ &\leq \|\Phi\|_{L^\infty} \int_{\{|\Phi(x,t)| > \xi\}} |\varepsilon B_\varepsilon^\mu[U]| \, dx + \xi \int_{\{|\Phi(x,t)| \leq \xi\}} |\varepsilon B_\varepsilon^\mu[U]| \, dx \\ &\leq \|\Phi\|_{L^\infty} \int_{\mathbb{R}^N} |\varepsilon B_\varepsilon^\mu[U(\cdot, t)]| \mathbf{1}_{|\Phi(x,t)| > \xi} \, dx + \xi \|U(\cdot, t)\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

Let t be a point where this inequality holds and $\varepsilon_n B_{\varepsilon_n}^\mu[U(\cdot, t)] \rightarrow 0$ a.e. x and $|\varepsilon B_\varepsilon^\mu[U(\cdot, t)](x)| \leq \|U\|_{L^\infty(Q_T)}$ a.e. x (using Theorem 3.1 (c)). For any $\eta > 0$, take ξ such that $\xi \|U(\cdot, t)\|_{L^1} < \frac{1}{2}\eta$. Then note that $|\varepsilon B_\varepsilon^\mu[U]| \mathbf{1}_{|\Phi(x,t)| > \xi}$ is dominated by $\|U\|_{L^\infty} \mathbf{1}_{|\Phi(x,t)| > \xi}$ which is integrable by Lemma 3.13. By Lebesgue's dominated convergence theorem it then follows that $\int_{\mathbb{R}^N} |\varepsilon_n B_{\varepsilon_n}^\mu[U(\cdot, t)]| \mathbf{1}_{|\Phi(x,t)| > \xi} \, dx < \frac{1}{2}\eta$ when ε_n is small enough. Since this holds for a.e. $t \in [0, T]$, we have proven that

$$\lim_{\varepsilon_n \rightarrow 0^+} (\varepsilon_n B_{\varepsilon_n}^\mu[\Phi](\cdot, t), U(\cdot, t)) \leq 0 \quad \text{for a.e. } t \in [0, T].$$

We conclude the proof using Lebesgue's dominated convergence theorem to send $\varepsilon_n \rightarrow 0^+$ in (3.8) (the integrand is dominated by $\|\Phi\|_{L^\infty(Q_T)} \|U(\cdot, t)\|_{L^1(\mathbb{R}^N)} \in L^1((0, T))$ since $U \in L^1(Q_T) \cap L^\infty(Q_T)$). \square

Proposition 3.14. *Assume (A_μ) , $\text{supp } \mu \neq \emptyset$, and $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there exists a sequence such that $\varepsilon_n B_{\varepsilon_n}^\mu[g] \rightarrow 0$ a.e. in \mathbb{R}^N as $\varepsilon_n \rightarrow 0^+$.*

This proposition will be proven later in this section. We are now ready to prove our main result.

Proof of Theorem 2.3. In the case that $\text{supp } \mu = \emptyset$, $\mu \equiv 0$ and $\mathcal{L}^\mu \equiv 0$. Then equation (1.1) becomes the ODE $u_t = 0$, and uniqueness follows by standard arguments (e.g. one can easily deduce that $\int_{\mathbb{R}^N} |u(x, t) - \hat{u}(x, t)| \, dx \leq \int_{\mathbb{R}^N} |u(x, 0) - \hat{u}(x, 0)| \, dx$).

Now consider the case $\text{supp } \mu \neq \emptyset$. By Proposition 3.14 and 3.12 (a) and (b), there is a sequence such that for a.e. $t \in [0, T]$,

$$(3.9) \quad \varepsilon_n \|B_{\varepsilon_n}^\mu[U](\cdot, t)\|_{L^2}^2 + \|(\mathcal{L}^\mu)^{\frac{1}{2}}[B_{\varepsilon_n}^\mu[U]](\cdot, t)\|_{L^2}^2 \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0^+.$$

Let $\psi \in C_c^\infty(\mathbb{R}^N)$. By Plancherel's theorem, Lemma 3.7, and Cauchy-Schwarz' inequality, and finally, by (3.9), we get for a.e. $t \in [0, T]$ that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} B_{\varepsilon_n}^\mu[U] \mathcal{L}^\mu[\psi] \, dx \right| &= \left| - \int_{\mathbb{R}^N} (\mathcal{L}^\mu)^{\frac{1}{2}}[B_{\varepsilon_n}^\mu[U]] (\mathcal{L}^\mu)^{\frac{1}{2}}[\psi] \, dx \right| \\ &\leq \|(\mathcal{L}^\mu)^{\frac{1}{2}}[B_{\varepsilon_n}^\mu[U]]\|_{L^2(\mathbb{R}^N)} \|(\mathcal{L}^\mu)^{\frac{1}{2}}[\psi]\|_{L^2(\mathbb{R}^N)} \rightarrow 0 \end{aligned}$$

as $\varepsilon_n \rightarrow 0^+$. Moreover, by Cauchy-Schwarz' inequality and (3.9), we have for a.e. $t \in [0, T]$

$$\left| \int_{\mathbb{R}^N} \varepsilon_n B_{\varepsilon_n}^\mu[U] \psi \, dx \right| \leq \|\varepsilon_n B_{\varepsilon_n}^\mu[U]\|_{L^2(\mathbb{R}^N)} \|\psi\|_{L^2(\mathbb{R}^N)} \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0^+.$$

Hence we conclude that as $\varepsilon_n \rightarrow 0^+$,

$$U = \varepsilon_n B_{\varepsilon_n}^\mu[U] - \mathcal{L}^\mu[B_{\varepsilon_n}^\mu[U]] \rightarrow 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

for a.e. $t \in [0, T]$. That is,

$$u - \hat{u} = U \equiv 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

for a.e. $t \in [0, T]$, and then a.e. in Q_T by du Bois-Reymond's lemma. \square

In the rest of this section, we prove Proposition 3.14. For $\gamma \in C_c^\infty(\mathbb{R}^N)$, we let $v_\varepsilon := \varepsilon B_\varepsilon^\mu[\gamma]$ be the unique smooth classical solution (see Theorem 3.1 (a) and Corollary 3.3 (a)) of

$$(3.10) \quad \varepsilon v_\varepsilon(x) - \mathcal{L}^\mu[v_\varepsilon](x) = \varepsilon \gamma(x) \quad \text{for all } x \in \mathbb{R}^N.$$

We want to prove that there exists a sequence such that $v_{\varepsilon_n} = \varepsilon_n B_{\varepsilon_n}^\mu[\gamma] \rightarrow 0$ as $\varepsilon_n \rightarrow 0^+$ for every $x \in \mathbb{R}^N$ and every $\gamma \in C_c^\infty(\mathbb{R}^N)$.

Lemma 3.15. *Assume (A $_\mu$) and $\gamma \in C_c^\infty(\mathbb{R}^N)$. Then there exists a sequence $\{\varepsilon_n B_{\varepsilon_n}^\mu[\gamma]\}_{n \in \mathbb{N}}$ that converges locally uniformly in \mathbb{R}^N as $\varepsilon_n \rightarrow 0^+$. Moreover, the corresponding limit v is uniformly continuous, $\lim_{|x| \rightarrow \infty} v = 0$ and satisfies*

$$\mathcal{L}^\mu[v](x) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Lemma 3.16 (Barbálat). *If $\psi \in L^1(\mathbb{R}^N)$ is uniformly continuous, then $\lim_{|x| \rightarrow \infty} \psi(x) = 0$.*

For a proof, see e.g. Lemma 5.2 in [30] (take $G = \mathbb{R}^N$ and $B = \mathbb{R}$).

Proof of Lemma 3.15. We recall that $v_\varepsilon := \varepsilon B_\varepsilon^\mu[\gamma]$. By Theorem 3.1 (a),

$$\|D^\alpha v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq \|D^\alpha \gamma\|_{L^\infty(\mathbb{R}^N)}$$

for each multiindex $\alpha \in \mathbb{N}^N$. So, then any sequence $\{v_{\varepsilon_n}\}_{n \in \mathbb{N}}$ is equibounded and equilipschitz. By Arzelà-Ascoli's theorem, there exists a subsequence such that $v_{\varepsilon_n} \rightarrow v$ locally uniformly as $n \rightarrow \infty$. Since v_{ε_n} is uniformly continuous (the derivative of v_{ε_n} exists and is bounded) and by the local uniform convergence, for every $\eta > 0$ and $R > 0$ we can find some $n > 0$ such that $\max\{|v(x) - v_{\varepsilon_n}(x)| : |x| \leq R\} < \eta$. Thus, we have the following estimate for every $R > 0$ and $|x|, |y| \leq R$,

$$\begin{aligned} |v(x) - v(y)| &\leq |v(x) - v_{\varepsilon_n}(x)| + |v_{\varepsilon_n}(x) - v_{\varepsilon_n}(y)| + |v_{\varepsilon_n}(y) - v(y)| \\ &\leq 2\eta + \|D\gamma\|_{L^\infty(\mathbb{R}^N)} |x - y| \end{aligned}$$

As R is arbitrary, v is Lipschitz continuous with Lipschitz constant $\|D\gamma\|_{L^\infty(\mathbb{R}^N)}$, and thus, uniformly continuous. Furthermore, Fatou's lemma and Theorem 3.1 (b)

give that $\|v\|_{L^1} \leq \liminf_{n \rightarrow \infty} \|v_{\varepsilon_n}\|_{L^1} \leq \|\gamma\|_{L^1}$. By Lemma 3.16, $\lim_{|x| \rightarrow \infty} v(x) = 0$.

Multiplying (3.10) by a test function, integrating over \mathbb{R}^N , and using self-adjointness (cf. Lemma 3.5) of \mathcal{L}^μ we get

$$\varepsilon_n \int_{\mathbb{R}^N} v_{\varepsilon_n} \psi \, dx - \int_{\mathbb{R}^N} v_{\varepsilon_n} \mathcal{L}^\mu[\psi] \, dx = \varepsilon_n \int_{\mathbb{R}^N} \gamma \psi \, dx \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N).$$

Since $\|v_{\varepsilon_n}\|_{L^\infty} \leq \|\gamma\|_{L^\infty}$ by Theorem 3.1 (c), we use Lebesgue's dominated convergence theorem to take the limit as $\varepsilon_n \rightarrow 0^+$, to find that

$$0 = \lim_{\varepsilon_n \rightarrow 0^+} \int_{\mathbb{R}^N} v_{\varepsilon_n} \mathcal{L}^\mu[\psi] \, dx = \int_{\mathbb{R}^N} v \mathcal{L}^\mu[\psi] \, dx \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N),$$

which completes the proof. \square

Lemma 3.17. *Assume (A_μ) and $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there exists a sequence $\{\varepsilon_n B_{\varepsilon_n}^\mu[g]\}_{n \in \mathbb{N}}$ that converges in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon_n \rightarrow 0^+$.*

Proof. Note that $u_\varepsilon := \varepsilon B_\varepsilon^\mu[g]$ is the unique distributional solution (see Theorem 3.1 (b) and (c)) of the following elliptic problem

$$\varepsilon u_\varepsilon(x) - \mathcal{L}^\mu[u_\varepsilon](x) = \varepsilon g(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

By Theorem 3.1 (b) and (c) and the linearity of the above equation, for any $h \in \mathbb{R}^N$,

$$\|u_\varepsilon\|_{L^\infty} \leq \|g\|_{L^\infty}, \quad \|u_\varepsilon\|_{L^1} \leq \|g\|_{L^1} \quad \text{and} \quad \|u_\varepsilon(\cdot + h) - u_\varepsilon\|_{L^1} \leq \|g(\cdot + h) - g\|_{L^1}.$$

Now let $K \subset \mathbb{R}^N$ be any compact set, and define $w_\varepsilon^K(x) = u_\varepsilon(x) \mathbf{1}_K(x)$. The uniform in ε bound ensures that the family $M := \{w_\varepsilon^K\}_{\varepsilon > 0} \subset L^1(\mathbb{R}^N)$ is uniformly bounded in $L^1(\mathbb{R}^N)$. Moreover, by continuity of the L^1 -translation, Theorem 3.1 (b) and (c), and Lebesgue's dominated convergence theorem,

$$\begin{aligned} & \|w_\varepsilon^K(\cdot + h) - w_\varepsilon^K\|_{L^1} \\ & \leq \|(u_\varepsilon(\cdot + h) - u_\varepsilon) \mathbf{1}_K(\cdot + h)\|_{L^1} + \|u_\varepsilon(\mathbf{1}_K(\cdot + h) - \mathbf{1}_K)\|_{L^1} \\ & \leq \|g(\cdot + h) - g\|_{L^1} + \|g\|_{L^\infty} \int_{\mathbb{R}^N} |\mathbf{1}_K(x + h) - \mathbf{1}_K(x)| \, dx \rightarrow 0 \quad \text{as } |h| \rightarrow 0. \end{aligned}$$

Combining the above results, we see that M is relatively compact by Kolmogorov's compactness theorem (see e.g. [29, Theorem A.5]). Hence, there is a convergent subsequence in $L^1(K)$.

Now, cover \mathbb{R}^N by a countable number of balls B_n . Then the above argument holds for $K := \overline{B_n}$ for every $n \in \mathbb{N}$. A diagonal argument then allows us to pick a subsequence which converges in $L^1(\overline{B_n})$ for each n , and thus in $L^1_{\text{loc}}(\mathbb{R}^N)$. \square

Remark 3.18. By Theorem 3.1 (a) and Arzelà-Ascoli, we can have $D^\alpha v_\varepsilon \rightarrow w_\alpha$ locally uniformly in \mathbb{R}^N as $\varepsilon \rightarrow 0^+$ for all multiindex $\alpha \in \mathbb{N}^N$. However, because of the lack of uniqueness in $\mathcal{L}^\mu[v](x) = 0$, we do not know if $D^\alpha v = w_\alpha$. Hence, we are forced to work with distributional solutions of $\mathcal{L}^\mu[v](x) = 0$.

Lemma 3.19. *Assume (A_μ) , $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and $\{\varepsilon_n B_{\varepsilon_n}^\mu[g]\}_{n \in \mathbb{N}}$ converges in $L^1_{\text{loc}}(\mathbb{R}^N)$. If $\varepsilon_n B_{\varepsilon_n}^\mu[\gamma](x) \rightarrow 0$ as $\varepsilon_n \rightarrow 0^+$ for every $x \in \mathbb{R}^N$ and every $\gamma \in C_c^\infty(\mathbb{R}^N)$, then $\varepsilon_n B_{\varepsilon_n}^\mu[g] \rightarrow 0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon_n \rightarrow 0^+$.*

Proof. By the self-adjointness given in Lemma 3.4, and the definitions $u_{\varepsilon_n} := \varepsilon_n B_{\varepsilon_n}^\mu[g]$, $v_{\varepsilon_n} := \varepsilon_n B_{\varepsilon_n}^\mu[\gamma]$, we have

$$\int_{\mathbb{R}^N} u_{\varepsilon_n}(x) \gamma(x) \, dx = \int_{\mathbb{R}^N} g(x) v_{\varepsilon_n}(x) \, dx.$$

Since $\|v_{\varepsilon_n}\|_{L^\infty} \leq \|\gamma\|_{L^\infty}$ by Theorem 3.1 (c), $|g(x)v_{\varepsilon_n}(x)| \leq |g(x)|\|\gamma\|_{L^\infty}$. Then by the assumption and Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon_n \rightarrow 0^+} \int_{\mathbb{R}^N} u_{\varepsilon_n}(x)\gamma(x) dx = 0 \quad \text{for all } \gamma \in C_c^\infty(\mathbb{R}^N),$$

Hence $u_{\varepsilon_n} \rightarrow 0$ in $\mathcal{D}'(\mathbb{R}^N)$, and since the distributional and L^1_{loc} limits coincide (by uniqueness), it follows that $u_{\varepsilon_n} \rightarrow 0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon_n \rightarrow 0^+$. \square

Proof of Proposition 3.14. Let $\gamma \in C_c^\infty(\mathbb{R}^N)$ be arbitrary, and recall the definitions $\varepsilon B_\varepsilon^\mu[\gamma] = v_\varepsilon$ and $\varepsilon B_\varepsilon^\mu[g] = u_\varepsilon$. Lemma 3.15 yields a subsequence such that $v_{\varepsilon_n} \rightarrow v$ locally uniformly as $\varepsilon_n \rightarrow 0^+$ with $v \in C_0(\mathbb{R}^N)$ and $\mathcal{L}^\mu[v](x) = 0$ in $D'(\mathbb{R}^N)$. Then, Theorem 3.9 ensures that $v(x) = 0$ for every $x \in \mathbb{R}^N$.

Hence, Lemma 3.17 and 3.19 give that $u_{\varepsilon_n} \rightarrow 0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon_n \rightarrow 0^+$. Finally, take a further subsequence (still denoted by ε_n) such that $u_{\varepsilon_n} \rightarrow 0$ a.e. in \mathbb{R}^N as $\varepsilon_n \rightarrow 0^+$. \square

4. STABILITY, EXISTENCE AND A PRIORI RESULTS

In this section, we will start by showing the stability result stated in Section 2, and then we continue by showing existence and a priori results for (1.1). The latter part will follow by regularization and compactness from results in [23] for the case $\varphi \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

Proof of Theorem 2.6. Since u_n are distributional solutions of (1.1), we will take the limit as $n \rightarrow \infty$ to see that so are also u .

Assumption (iii) and the uniform boundedness of $\|u_n\|_{L^\infty(Q_T)}$ gives for all $\psi \in C_c^\infty(Q_T)$ that

$$\int_0^T \int_{\mathbb{R}^N} u_n \partial_t \psi dx dt \rightarrow \int_0^T \int_{\mathbb{R}^N} u \partial_t \psi dx dt \quad \text{as } n \rightarrow \infty.$$

To prove convergence of the \mathcal{L}^{μ_n} -term in the distributional formulation we proceed as follows

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} (\varphi_n(u_n) \mathcal{L}^{\mu_n}[\psi] - \varphi(u) \mathcal{L}[\psi]) dx dt \\ &= \int_0^T \int_{\mathbb{R}^N} \varphi_n(u_n) (\mathcal{L}^{\mu_n}[\psi] - \mathcal{L}[\psi]) dx dt + \int_0^T \int_{\mathbb{R}^N} (\varphi_n(u_n) - \varphi(u_n)) \mathcal{L}[\psi] dx dt \\ & \quad + \int_0^T \int_{\mathbb{R}^N} (\varphi(u_n) - \varphi(u)) \mathcal{L}[\psi] dx dt. \end{aligned}$$

Since $\|u_n\|_{L^\infty(Q_T)}$ is uniformly bounded, $\varphi_n \rightarrow \varphi$ locally uniformly in \mathbb{R} by assumption (ii), and $|\varphi_n(u_n)| \leq |\varphi_n(u_n) - \varphi(u_n)| + |\varphi(u_n)|$, we obtain for n sufficiently large

$$(4.1) \quad \|\varphi_n(u_n)\|_{L^\infty(Q_T)} \leq \sup\{|\varphi(r)| : |r| \leq C\} + 1 =: C_\varphi.$$

Then, using assumption (i), we get

$$\left| \int_0^T \int_{\mathbb{R}^N} \varphi_n(u_n) (\mathcal{L}^{\mu_n}[\psi] - \mathcal{L}[\psi]) dx dt \right| \leq C_\varphi \int_0^T \int_{\mathbb{R}^N} |\mathcal{L}^{\mu_n}[\psi] - \mathcal{L}[\psi]| dx dt \rightarrow 0$$

as $n \rightarrow \infty$. By the uniform boundedness of $\|u_n\|_{L^\infty(Q_T)}$, and since $\varphi_n \rightarrow \varphi$ locally uniformly in \mathbb{R} by assumption (ii),

$$\|\varphi_n(u_n) - \varphi(u_n)\|_{L^\infty(Q_T)} \leq \sup\{|\varphi_n(r) - \varphi(r)| : |r| \leq C\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since we assume that $\mathcal{L}[\psi] \in L^1(Q_T)$,

$$\left| \int_0^T \int_{\mathbb{R}^N} (\varphi_n(u_n) - \varphi(u_n)) \mathcal{L}[\psi] \, dx \, dt \right| \leq \|\varphi_n(u_n) - \varphi(u_n)\|_{L^\infty} \|\mathcal{L}[\psi]\|_{L^1} \rightarrow 0$$

as $n \rightarrow \infty$. By assumption (iii) and (A_φ) , $|\varphi(u_n) - \varphi(u)| \rightarrow 0$ a.e. in Q_T as $n \rightarrow \infty$, and $\|\varphi(u_n)\|_{L^\infty(Q_T)} \leq C$ for some C independent of n . Hence, $|\varphi(u_n) - \varphi(u)|$ is bounded by $2C$. Moreover, since $\mathcal{L}[\psi] \in L^1(Q_T)$, Lebesgue's dominated convergence theorem yields

$$\left| \int_0^T \int_{\mathbb{R}^N} (\varphi(u_n) - \varphi(u)) \mathcal{L}[\psi] \, dx \, dt \right| \leq \int_0^T \int_{\mathbb{R}^N} |\varphi(u_n) - \varphi(u)| |\mathcal{L}[\psi]| \, dx \, dt \rightarrow 0$$

as $n \rightarrow \infty$. The proof is complete. \square

Let us turn our attention to proving the other main results in this section.

Theorem 4.1. *Assume (A_φ) , (A_μ) , $\varphi \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$, $\varphi(0) = 0$, and $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.*

- (a) *There exists a unique entropy solution $u \in L^\infty(Q_T) \cap C([0, T]; L^1(\mathbb{R}^N))$ of (1.1).*
 (b) *If u, \hat{u} are entropy solutions of (1.1) with initial data u_0, \hat{u}_0 respectively, then for all $t \in [0, T]$*

$$\|u(\cdot, t) - \hat{u}(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0 - \hat{u}_0\|_{L^1(\mathbb{R}^N)}.$$

- (c) *If u is a entropy solution of (1.1) with initial data u_0 , then for all $t \in [0, T]$*

$$\|u(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)} \quad \text{and} \quad \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

Entropy solutions are defined in Definition 2.1 in [23], and the result holds by Theorem 5.5 in [23] and Theorem 5.2 in [22].

In what follows, we let $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and define

$$(4.2) \quad \varphi_\eta(x) := \varphi * \omega_\eta(x) - \varphi * \omega_\eta(0) \quad \text{where} \quad \omega_\eta \quad \text{is given by (1.7) with } N = 1.$$

Hence $\varphi_\eta \in W_{\text{loc}}^{1,\infty}(\mathbb{R}) \subset C(\mathbb{R})$, it is nondecreasing by (A_φ) , $\varphi_\eta(0) = 0$, and $\varphi_\eta \rightarrow \varphi$ locally uniformly in \mathbb{R} . Let u_η be the entropy solution of (1.1) with φ_η replacing φ . Since entropy solutions are distributional solutions (cf. Theorem 2.5 ii) and Section 5 in [22]),

$$(4.3) \quad \int_0^T \int_{\mathbb{R}^N} \left(u_\eta \partial_t \psi + \varphi_\eta(u_\eta) \mathcal{L}^\mu[\psi] \right) \, dx \, dt + \int_{\mathbb{R}^N} u_0 \psi|_{t=0} \, dx = 0 \quad \forall \psi \in C_c^\infty(\mathbb{R}^N \times [0, T]).$$

Going to the limit as $\eta \rightarrow 0^+$ in (4.3), we will prove the existence and the a priori results given in Theorems 2.8 and 2.10.

Remark 4.2. We will prove that the L^1 -contraction holds for limits of the functions $\{u_\eta\}_{\eta>0}$. As a consequence of uniqueness (Corollary 2.4), this result then holds for all $L^\infty \cap L^1$ -distributional solutions of (1.1).

Before these results can be proven, we need an auxiliary lemma.

Lemma 4.3. *Assume (A_μ) , $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, φ_η satisfy (A_φ) for all $\eta > 0$, and $\varphi_\eta \rightarrow \varphi$ locally uniformly as $\eta \rightarrow 0^+$. If u_η solves (4.3) and satisfies Theorem 4.1 (b) and (c), then there exists a subsequence $\{u_{\eta_n}\}_{n \in \mathbb{N}}$ and a $u \in C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N))$ such that as $\eta_n \rightarrow 0^+$*

$$u_{\eta_n} \rightarrow u \quad \text{in} \quad C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N)).$$

Moreover, for all $t \in [0, T]$

$$\|u(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)} \quad \text{and} \quad \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

Proof. We will use Kolmogorov's compactness theorem in the form of Theorem A.8 in [29]. Let $K \subset \mathbb{R}^N$ be any compact set.

Step 1: u_η is bounded independently of η in Q_T by Theorem 4.1 (c).

Step 2: Since (1.1) is translation invariant, $v(x, t) = u_\eta(x + h, t)$ solves (4.3) with initial data $v_0(x) = u_0(x + h)$ for every $h \in \mathbb{R}^N$. Let $\gamma \in \mathbb{R}^N$. By Theorem 4.1 (b) and since translations are continuous in L^1 ,

$$\begin{aligned} \sup_{|h| \leq |\gamma|} \int_K |u_\eta(x + h, t) - u_\eta(x, t)| dx &\leq \sup_{|h| \leq |\gamma|} \int_{\mathbb{R}^N} |u_\eta(x + h, t) - u_\eta(x, t)| dx \\ &\leq \sup_{|h| \leq |\gamma|} \int_{\mathbb{R}^N} |u_0(x + h) - u_0(x)| dx \leq \max_{|h| \leq |\gamma|} \tilde{\lambda}_{u_0}(|h|) =: \lambda_{u_0}(|\gamma|) \end{aligned}$$

for some moduli of continuity $\tilde{\lambda}_{u_0}, \lambda_{u_0}$.

Step 3: Let ω_δ be defined by (1.7) and let $\Theta \in C_c^\infty((0, T))$. For any $x \in \mathbb{R}^N$ take $\psi(y, t) = \Theta(t)\omega_\delta(x - y)$ as a test function in (4.3) to find that

$$\begin{aligned} (4.4) \quad 0 &= \int_0^T \int_{\mathbb{R}^N} \left(u_\eta(y, t)\omega_\delta(x - y)\Theta'(t) + \varphi_\eta(u_\eta(y, t))\mathcal{L}^\mu[\omega_\delta](x - y)\Theta(t) \right) dy dt \\ &= \int_0^T \left((u_\eta(\cdot, t) * \omega_\delta)(x)\Theta'(t) + (\varphi_\eta(u_\eta(\cdot, t)) * \mathcal{L}^\mu[\omega_\delta])(x)\Theta(t) \right) dt. \end{aligned}$$

For $\rho_\delta(t)$ defined by (1.8), we choose

$$\Theta(t) := \Theta_{\tilde{\delta}}(t) = \int_{-\infty}^t \left(\rho_{\tilde{\delta}}(\tau - t_1) - \rho_{\tilde{\delta}}(\tau - t_2) \right) d\tau,$$

where $0 < t_1 < t_2 < T$. For $\tilde{\delta} > 0$ small enough, $\Theta_{\tilde{\delta}}(t)$ is supported in $[0, T]$ and is a smooth approximation to a square pulse which is one in $[t_1, t_2]$ and zero otherwise. By (4.4),

$$\begin{aligned} \int_0^T \rho_{\tilde{\delta}}(t - t_2)(u_\eta(\cdot, t) * \omega_\delta)(x) dt &= \int_0^T \rho_{\tilde{\delta}}(t - t_1)(u_\eta(\cdot, t) * \omega_\delta)(x) dt \\ &\quad + \int_0^T \Theta_{\tilde{\delta}}(t)(\varphi_\eta(u_\eta(\cdot, t)) * \mathcal{L}^\mu[\omega_\delta])(x) dt. \end{aligned}$$

Let $u_\eta^\delta(x, t) := u_\eta(\cdot, t) * \omega_\delta(x)$. By Theorem 4.1 (c) and the properties of mollifiers, we send $\tilde{\delta} \rightarrow 0^+$ in the previous equality to obtain the following pointwise identity,

$$(4.5) \quad u_\eta^\delta(x, t_2) - u_\eta^\delta(x, t_1) = \int_{t_1}^{t_2} (\varphi_\eta(u_\eta(\cdot, t)) * \mathcal{L}^\mu[\omega_\delta])(x) dt.$$

Now, we need to estimate the integral involving the mollified version of u_η . Let $t, s \in [0, T]$ and take $\delta < \min\{t, s\}$. Use (4.5) to find that

$$\begin{aligned} \int_K |u_\eta^\delta(x, t) - u_\eta^\delta(x, s)| dx &\leq \int_K \int_s^t |(\varphi_\eta(u_\eta(\cdot, \tau)) * \mathcal{L}^\mu[\omega_\delta])(x)| d\tau dx \\ &= \int_s^t \int_K \int_{\mathbb{R}^N} |\varphi_\eta(u_\eta(x - y, \tau))| |\mathcal{L}^\mu[\omega_\delta](y)| dy dx d\tau \\ &\leq \|\varphi_\eta(u_\eta)\|_{L^\infty(Q_T)} \|\mathcal{L}^\mu[\omega_\delta]\|_{L^1(\mathbb{R}^N)} |K| |t - s|, \end{aligned}$$

where $|K|$ denotes the Lebesgue measure of the compact set K . As in the proof of Theorem 2.6 (see (4.1)), we obtain for η sufficiently small

$$\|\varphi_\eta(u_\eta)\|_{L^\infty(Q_T)} \leq \sup\{|\varphi(r)| : |r| \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}\} + 1.$$

Moreover, Lemma 3.5 (b) and the properties of mollifiers yield

$$\|\mathcal{L}^\mu[\omega_\delta]\|_{L^1(\mathbb{R}^N)} \leq (\delta^{-2}\|D^2\omega\|_{L^1(\mathbb{R}^N)} + 1) \int_{|z|>0} \min\{|z|^2, 1\} d\mu(z).$$

Hence, taking $\delta^2 := |t - s|^{\frac{2}{3}}$ we see that

$$(4.6) \quad \int_K |u_\eta^\delta(x, t) - u_\eta^\delta(x, s)| dx \leq \tilde{C}_{K, \varphi, u_0, \mu} \left(|t - s|^{\frac{1}{3}} + |t - s| \right),$$

where

$$\begin{aligned} & \tilde{C}_{K, \varphi, u_0, \mu} \\ &= |K| \left(\|D^2\omega\|_{L^1(\mathbb{R}^N)} + 1 \right) \left(\sup_{|r| \leq \|u_0\|_{L^\infty}} |\varphi(r)| + 1 \right) \int_{|z|>0} \min\{|z|^2, 1\} d\mu(z). \end{aligned}$$

By the triangle inequality and Theorem 4.1 (b),

$$\begin{aligned} & \int_K |u_\eta(x, t) - u_\eta(x, s)| dx \\ & \leq \int_K |u_\eta(x, t) - u_\eta^\delta(x, t)| dx + \int_K |u_\eta^\delta(x, t) - u_\eta^\delta(x, s)| dx \\ & \quad + \int_K |u_\eta^\delta(x, s) - u_\eta(x, s)| dx \\ & \leq \sup_{|\sigma| \leq \delta} \|u_\eta(\cdot, t) - u_\eta(\cdot + \sigma, t)\|_{L^1(\mathbb{R}^N)} + \int_K |u_\eta^\delta(x, t) - u_\eta^\delta(x, s)| dx \\ & \quad + \sup_{|\sigma| \leq \delta} \|u_\eta(\cdot, s) - u_\eta(\cdot + \sigma, s)\|_{L^1(\mathbb{R}^N)} \\ & \leq 2 \sup_{|\sigma| \leq \delta} \|u_0 - u_0(\cdot + \sigma)\|_{L^1(\mathbb{R}^N)} + \int_K |u_\eta^\delta(x, t) - u_\eta^\delta(x, s)| dx \\ & \leq 2 \max_{|\sigma| \leq \delta} \tilde{\lambda}_{u_0}(\delta) + \int_K |u_\eta^\delta(x, t) - u_\eta^\delta(x, s)| dx, \end{aligned}$$

where $\tilde{\lambda}_{u_0}$ is defined in Step 2 . Hence, by (4.6)

$$\begin{aligned} \int_K |u_\eta(x, t) - u_\eta(x, s)| dx & \leq \lambda_{u_0} \left(|t - s|^{\frac{1}{3}} \right) + \tilde{C}_{K, \varphi, u_0, \mu} \left(|t - s|^{\frac{1}{3}} + |t - s| \right) \\ & =: \Lambda_{K, \varphi, u_0, \mu}(|t - s|) \end{aligned}$$

for some moduli of continuity λ_{u_0} and $\Lambda_{K, \varphi, u_0, \mu}$.

Step 4: The assumptions of Theorem A.8 in [29] hold by Steps 1–3, so we conclude that there is a subsequence $\{u_{\eta_n}\}_{n \in \mathbb{N}}$ such that

$$u_{\eta_n} \rightarrow u \quad \text{in} \quad C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$$

as $\eta_n \rightarrow 0^+$. Finally, u inherits the properties of u_η given in Theorem 4.1 (c) by Fatou's lemma, and the fact that the limit of a uniformly bounded sequence which converges a.e. is also bounded. \square

Remark 4.4. If \mathcal{L}^μ was not fixed in the above result, but rather $\mu = \mu_n$ (with μ_n satisfying (A_μ)), then the result still holds and the proof is the same provided we also assume that for some $a > 0$ there exists a function $f \in L^\infty_{\text{loc}}((0, a))$ such that

$$\|\mathcal{L}^{\mu_n}[\omega_\delta]\|_{L^1(\mathbb{R}^N)} \leq f(\delta) \quad \text{for every} \quad \delta \in (0, a),$$

where ω_δ is defined by (1.7). Observe that the above inequality follows from the assumption $\sup_n \int_{|z|>0} \min\{|z|^2, 1\} d\mu_n(z) < \infty$ in Theorem 2.12.

Now, the proofs of the existence and the a priori results follow.

Proof of Theorem 2.8. Let u_{η_n} be the solutions of (4.2) (cf. Theorem 4.1), $u \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ the function provided by Lemma 4.3, and define $\mathcal{L}^{\mu_n} := \mathcal{L}^\mu$ (that is, $\mathcal{L}^{\mu_n} := \mathcal{L}^\mu = \mathcal{L}$), $\varphi_n := \varphi_{\eta_n}$, and $u_n := u_{\eta_n}$. Then assumptions (i), (ii), and (iii) in Theorem 2.6 are satisfied by the n -independence of \mathcal{L}^μ , (4.2), and Lemma 4.3. Moreover, $\sup_n \|u_n\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} < \infty$ by Theorem 4.1 (c). Hence, by Theorem 2.6, u satisfies (1.1) in the sense of distributions: cf. Lemma 2.21 and Definition 2.2. Moreover, we have that $u - u_0 \in L^1(Q_T)$. So, u is in fact a distributional solution of (1.1) in the sense of Definition 2.2, and it is unique by Corollary 2.4.

Thus, any subsequence has the same limit, and hence, the whole sequence $\{u_\eta\}_{\eta>0}$ converges since it is bounded by Theorem 4.1 (c). \square

Proof of Theorem 2.10. (a) Let u_η be the entropy solution of (4.2) (cf. Theorem 4.1). Using the semi entropy-entropy flux pairs

$$(u_\eta - k)^\pm \quad \text{and} \quad \pm \text{sign}^\pm(u_\eta - k)(f(u_\eta) - f(k)) \quad \text{for all } k \in \mathbb{R},$$

and the corresponding definitions for entropy solutions in stead of the Kruřkov entropy-entropy flux pairs in [22], we obtain

$$\int_{\mathbb{R}^N} (u_\eta(x, t) - \hat{u}_\eta(x, t))^+ dx \leq \int_{\mathbb{R}^N} (u_0(x) - \hat{u}_0(x))^+ dx$$

for $u_\eta, \hat{u}_\eta \in L^\infty(Q_T) \cap L^1(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ with initial data $u_0, \hat{u}_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. See [25] for the result and a proof.

By Lemma 4.3, we can take subsequences such that $u_{\eta_n}, \hat{u}_{\eta_n} \rightarrow u, \hat{u}$ a.e. in Q_T as $\eta_n \rightarrow 0^+$. Thus, Fatou's lemma yield the result.

(b) By the contraction estimate obtained in part (a) and $u_0 \leq \hat{u}_0$ a.e. in \mathbb{R}^N , for all $t \in (0, T)$, $\int_{\mathbb{R}^N} (u(x, t) - \hat{u}(x, t))^+ dx \leq 0$. Hence, $(u - \hat{u})^+ = 0$ and $u \leq \hat{u}$ a.e. in Q_T .

(c) Follows by Lemma 4.3.

(d) Follows by Lemma 4.3.

(e) Using the triangle inequality, and taking u, u_{η_n} as in Lemma 4.3, we obtain by Step 3 in the proof of that lemma that for all $t, s \in [0, T]$ and any compact set $K \subset \mathbb{R}^N$

$$\begin{aligned} & \|u(\cdot, t) - u(\cdot, s)\|_{L^1(K)} \\ & \leq \|u(\cdot, t) - u_{\eta_n}(\cdot, t)\|_{L^1(K)} + \|u_{\eta_n}(\cdot, t) - u_{\eta_n}(\cdot, s)\|_{L^1(K)} + \|u_{\eta_n}(\cdot, s) - u(\cdot, s)\|_{L^1(K)} \\ & \leq 2\|u(\cdot, t) - u_{\eta_n}(\cdot, t)\|_{C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))} + \Lambda_{K, \varphi, u_0, \mu}(|t - s|) \end{aligned}$$

for the modulus of continuity $\Lambda_{K, \varphi, u_0, \mu}$ (see the above mentioned proof). Since $u_{\eta_n} \rightarrow u$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ by Lemma 4.3, the proof is complete.

(f) Consider a standard cut-off function $0 \leq \mathcal{X} \in C_c^\infty(\mathbb{R}^N)$ such that $\mathcal{X}(x) = 1$ for $|x| \leq 1$ and $\mathcal{X}(x) = 0$ for $|x| \geq 2$. We will write $\mathcal{X}_R(x) = \mathcal{X}(\frac{x}{R})$ for $R > 0$. Following the proof of Lemma 3.10 (b), with θ_a as defined there, we can take $\psi(x, t) = \mathcal{X}_R(x)\theta_a(t)$ for any $R > 0$ as a test function in Definition 2.2 (cf. Lemma 2.21). Hence

$$\begin{aligned} \frac{1}{a} \int_{s-a}^s \int_{\mathbb{R}^N} u(x, t) \mathcal{X}_R(x) dx dt &= \int_0^s \theta_a(t) \int_{\mathbb{R}^N} \varphi(u(x, t)) \mathcal{L}^\mu[\mathcal{X}_R](x) dx dt \\ &+ \int_{\mathbb{R}^N} u_0(x) \mathcal{X}_R(x) dx. \end{aligned}$$

Since \mathcal{X}_R is compactly supported and $u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$, we can pass to the limit as $a \rightarrow 0^+$ in the first integral to get $\int_{\mathbb{R}^N} u(x, s) \mathcal{X}_R(x) dx$. For the second integral, we know that $\varphi(u) \in L^\infty(Q_T)$, $\mathcal{L}^\mu[\mathcal{X}_R] \in L^1(\mathbb{R}^N)$ and $\theta_a \rightarrow \mathbf{1}_{[0, s]}$ pointwise a.e. as $a \rightarrow 0^+$, and thus, it converges as $a \rightarrow 0^+$ to $\int_0^s \int_{\mathbb{R}^N} \varphi(u(x, t)) \mathcal{L}^\mu[\mathcal{X}_R](x) dx dt$ by Lebesgue's dominated convergence theorem. In this way, we get

$$\int_{\mathbb{R}^N} u(x, s) \mathcal{X}_R(x) dx = \int_0^s \int_{\mathbb{R}^N} \varphi(u(x, t)) \mathcal{L}^\mu[\mathcal{X}_R](x) dx dt + \int_{\mathbb{R}^N} u_0(x) \mathcal{X}_R(x) dx.$$

The function \mathcal{X}_R converges pointwise as $R \rightarrow \infty$ to 1, and it is also bounded by 1. Then, since $u(\cdot, s), u_0 \in L^1(\mathbb{R}^N)$, Lebesgue's dominated convergence theorem allows us to pass to the limit as $R \rightarrow \infty$ in the first and the last integrals to get $\int_{\mathbb{R}^N} u(x, s) dx$ and $\int_{\mathbb{R}^N} u_0(x) dx$, respectively, for all $s \in (0, T)$. Consider the nonsingular part of the Lévy operator, i.e., $\int_{|z|>1} \mathcal{X}_R(x+z) - \mathcal{X}_R(x) d\mu(z)$ which is bounded by $2\mu(\{z \in \mathbb{R}^N : |z| > 1\})$ for every $x \in \mathbb{R}^N$. Since $\mathcal{X}_R(y) \rightarrow 1$ pointwise as $R \rightarrow \infty$ for all $y \in \mathbb{R}^N$, Lebesgue's dominated convergence theorem shows the pointwise convergence to 0 of the nonsingular part. For the singular part, Lemma 3.5 (b) gives

$$\left| \int_{0 < |z| \leq 1} \mathcal{X}_R(x+z) - \mathcal{X}_R(x) d\mu(z) \right| \leq \frac{1}{R^2} \|D^2 \mathcal{X}\|_{L^\infty(\mathbb{R}^N)} \int_{|z| \leq 1} |z|^2 d\mu(z)$$

which also goes to 0 as $R \rightarrow \infty$. Moreover, by the assumption $|\varphi(r)| \leq L_\delta |r|$ for $|r| \leq \delta$,

$$\|\varphi(u(x, t))\|_{L^1(Q_T)} \leq \int_0^T \int_{|u| \leq \delta} L_\delta |u(x, t)| dx dt + \|\varphi(u)\|_{L^\infty(Q_T)} \int_0^T \int_{|u| > \delta} dx dt.$$

Since $u \in L^1(Q_T)$, both terms on the right-hand side of the estimate above are finite. Then by Lebesgue's dominated convergence theorem,

$$\left| \int_0^s \int_{\mathbb{R}^N} \varphi(u(x, t)) \mathcal{L}^\mu[\mathcal{X}_R](x) dx dt \right| \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty.$$

The proof is complete. \square

5. APPLICATIONS OF STABILITY

This section focuses on proving the results stated in Sections 2.1 and 2.2.

5.1. Compactness, local limits and continuous dependence.

Proof of Theorem 2.12. (a) Note that the sequence of solutions $\{u_n\}_{n \in \mathbb{N}}$ satisfy the hypothesis of Theorem 2.10. By the assumptions, Remark 4.4, and Lemma 4.3, the result follows.

(b) This is a consequence of the stability given in Theorem 2.6. For the initial condition, note that by the assumption $\sup_n \|u_{0,n}\|_{L^\infty(\mathbb{R}^N)} < \infty$ and Fatou's lemma, $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, and the convergence of $\int_{\mathbb{R}^N} u_{0,n}(x) \psi(x, 0) dx$ follows by the L^1_{loc} -convergence of $\{u_{0,n}\}_{n \in \mathbb{N}}$. \square

Lemma 5.1. *Assume (A_μ) , $s \in (0, 2)$, $\mathcal{L}^\mu = -(-\Delta)^{\frac{s}{2}}$, and $\psi \in C_c^\infty(\mathbb{R}^N)$. Then*

$$\lim_{s \rightarrow 2^-} \left\| -(-\Delta)^{\frac{s}{2}} \psi - \Delta \psi \right\|_{L^1(\mathbb{R}^N)} = 0.$$

Proof. The fractional Laplacian has a representation in the form (1.3) with measure

$$d\mu = c_{N,s} \frac{dz}{|z|^{N+s}} \quad \text{for} \quad c_{N,s} = \left(N \int_{\mathbb{R}^N} \frac{1 - \cos(z_1)}{|z|^{N+s}} dz \right)^{-1},$$

where

$$(5.1) \quad \lim_{s \rightarrow 2^-} c_{N,s} = 0,$$

see e.g. Proposition 4.1 in [33]. Hence

$$\begin{aligned} -(-\Delta)^{\frac{s}{2}}\psi(x) &= c_{N,s} \int_{|z| \leq 1} \frac{\psi(x+z) - \psi(x) - z \cdot D\psi(x)}{|z|^{N+s}} dz \\ &\quad + c_{N,s} \int_{|z| > 1} \frac{\psi(x+z) - \psi(x)}{|z|^{N+s}} dz, \end{aligned}$$

where the last term goes to zero in $L^1(\mathbb{R}^N)$ as $s \rightarrow 2^-$ since it is bounded in $L^1(\mathbb{R}^N)$ by $c_{N,s} 2 \|\psi\|_{L^1(\mathbb{R}^N)} \int_{|z| > 1} |z|^{-N-1} dz$ for $s \geq 1$.

The explicit form of $c_{N,s}$ given in (5.1) yields

$$\Delta\psi(x) = \Delta\psi(x) c_{N,s} N \int_{|z| \leq 1} \frac{1 - \cos(z_1)}{|z|^{N+s}} dz + \Delta\psi(x) c_{N,s} N \int_{|z| > 1} \frac{1 - \cos(z_1)}{|z|^{N+s}} dz.$$

Again, the last term goes to zero in $L^1(\mathbb{R}^N)$ as $s \rightarrow 2^-$ since $|1 - \cos(z_1)| \leq 2$ and then it is bounded in $L^1(\mathbb{R}^N)$ by $c_{N,s} 2N \int_{|z| > 1} |z|^{-N-1} dz$ for $s \geq 1$. Using Taylor's theorem, we see that $1 - \cos(z_1) = \frac{1}{2}z_1^2 - \frac{1}{24}z_1^4 \cos(\xi)$ for some $\xi \in [0, z_1]$. Hence,

$$(5.2) \quad \int_{|z| \leq 1} \frac{1 - \cos(z_1)}{|z|^{N+s}} dz = \frac{1}{2} \int_{|z| \leq 1} \frac{z_1^2}{|z|^{N+s}} dz - \frac{1}{24} \cos(\xi) \int_{|z| \leq 1} \frac{z_1^4}{|z|^{N+s}} dz,$$

and the following estimate holds:

$$\left\| \frac{N}{24} \Delta\psi(x) c_{N,s} \int_{|z| \leq 1} \frac{\cos(\xi) z_1^4}{|z|^{N+s}} dz \right\|_{L^1(\mathbb{R}^N)} \leq \frac{N}{24} c_{N,s} \|\Delta\psi\|_{L^1(\mathbb{R}^N)} \int_{|z| \leq 1} \frac{1}{|z|^{N-2}} dz$$

which goes to zero since $\int_{|z| \leq 1} \frac{1}{|z|^{N-2}} dz < \infty$ and (5.1) hold.

To estimate the remaining term in (5.2), note that for all $r > 0$,

$$N \Delta\psi(x) \int_{|z| \leq r} z_1^2 dz = \Delta\psi(x) \int_{|z| \leq r} |z|^2 dz = \int_{|z| \leq r} D^2\psi(x) z \cdot z dz,$$

and then

$$\frac{1}{2} c_{N,s} N \Delta\psi(x) \int_{|z| \leq 1} \frac{z_1^2}{|z|^{N+s}} dz = c_{N,s} \int_{|z| \leq 1} \frac{\frac{1}{2} D^2\psi(x) z \cdot z}{|z|^{N+s}} dz.$$

We combine the all above estimates to get

$$\begin{aligned} &\lim_{s \rightarrow 2^-} \| -(-\Delta)^{\frac{s}{2}}\psi(x) - \Delta\psi(x) \|_{L^1(\mathbb{R}^N)} \\ &= \lim_{s \rightarrow 2^-} c_{N,s} \left\| \int_{|z| \leq 1} \frac{\psi(x+z) - \psi(x) - z \cdot D\psi(x) - \frac{1}{2} D^2\psi(x) z \cdot z}{|z|^{N+s}} dz \right\|_{L^1(\mathbb{R}^N)} + 0 \\ &\leq \lim_{s \rightarrow 2^-} c_{N,s} \frac{1}{6} \|D^3\psi\|_{L^1(\mathbb{R}^N)} \int_{|z| \leq 1} \frac{|z|^3}{|z|^{N+s}} dz, \end{aligned}$$

where the last inequality follows from Taylor's and Fubini's theorems. Since the z -integral is bounded by $\int_{|z| \leq 1} \frac{1}{|z|^{N-1}} dz < \infty$ and (5.1) hold, the limit is zero and the proof is complete. \square

Proof of Corollary 2.13. (a) We will use Theorem 2.12 and Remark 4.4 to prove the result, and now we verify the assumptions. By Lemma 5.1, $-(-\Delta)^{\frac{s}{2}}\psi \rightarrow \Delta\psi$ in $L^1(\mathbb{R}^N)$ as $s \rightarrow 2^-$ for all $\psi \in C_c^\infty(\mathbb{R}^N)$. Moreover, by Lemma 3.5 (b),

properties of mollifiers, $\lim_{s \rightarrow 2^-} \int_{|z| \leq 1} |z|^2 \frac{c_{N,s} dz}{|z|^{N+s}} = 1$, and $\lim_{s \rightarrow 2^-} \int_{|z| > 1} \frac{c_{N,s} dz}{|z|^{N+s}} = 0$ (see previous proof),

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}} \omega_\delta\|_{L^1} &= \frac{1}{2} \|D^2 \omega_\delta\|_{L^1} \int_{|z| \leq 1} |z|^2 \frac{c_{N,s} dz}{|z|^{N+s}} + 2 \|\omega_\delta\|_{L^1} \int_{|z| > 1} \frac{c_{N,s} dz}{|z|^{N+s}} \\ &\leq C \left(1 + \frac{1}{\delta^2} \|D^2 \omega\|_{L^1} \right) \end{aligned}$$

for s close to 2. Hence, since also φ is fixed (independently of s), we may use Theorem 2.12 and Remark 4.4 to get a subsequence $\{u_{s_j}\}_{j \in \mathbb{N}}$ and a $u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ such that $u_{s_j} \rightarrow u$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ as $j \rightarrow \infty$. Finally, the uniqueness for the limit (equation) [18], and the boundedness of the sequence $\{u_s\}_{s \in (0, 2)}$ (Theorem 2.10 (d)), ensures that the whole sequence converges.

(b) Since $(-\Delta)^{\frac{s_n}{2}} \psi \rightarrow (-\Delta)^{\frac{s}{2}} \psi$ in $L^1(\mathbb{R}^N)$ as $n \rightarrow \infty$ (a similar argument as in Lemma 5.1), $\varphi_{m_n}(r) = r^{m_n} \rightarrow \varphi_m(r) = r^m$ locally uniformly as $n \rightarrow \infty$, and $\|(-\Delta)^{\frac{s_n}{2}} \omega_\delta\|_{L^1} \leq C(1 + \delta^{-2})$ by the proof of part (a), convergence for a subsequence follows by Theorem 2.12. Moreover, the convergence of the whole sequence follows from uniqueness of the limit (Corollary 2.4) and boundedness of the sequence (Theorem 2.10 (d)). \square

5.2. Numerical approximation, convergence and existence. We start by showing that a standard finite difference approximations of the Laplacian can be written in the form (1.3) and that convergence of the resulting scheme then follows from our theory.

Example 5.1. Let $e_i \in \mathbb{R}^n$ for $i = 1, \dots, N$ be points with i -th component 1 and the other components 0. Using δ -measures and $h > 0$, we define

$$\mu_h = \sum_{i=1}^N \frac{\delta_{he_i} + \delta_{-he_i}}{h^2}.$$

It is clear that μ_h is a measure satisfying (A_μ) for every $h > 0$. Moreover,

$$\mathcal{L}^{\mu_h}[v](x) := \int_{\mathbb{R}^N} v(x+z) - v(x) d\mu_h(z) = \sum_{i=1}^N \frac{v(x+he_i) + v(x-he_i) - 2v(x)}{h^2}.$$

With $\mu = \mu_h$, problem (2.5) can be reformulated as

$$(5.3) \quad \partial_t u_h(x, t) - \sum_{i=1}^N \frac{\varphi(u_h(x+he_i, t)) + \varphi(u_h(x-he_i, t)) - 2\varphi(u_h(x, t))}{h^2} = 0$$

in $\mathcal{D}'(Q_T)$.

For $\psi \in C_c^\infty(\mathbb{R}^N)$, an application of Taylor's theorem reveals that there is a $C > 0$ such that

$$\int_{\mathbb{R}^N} |\mathcal{L}^{\mu_h}[\psi](x) - \Delta \psi(x)| dx \leq h^2 C \|D^4 \psi\|_{L^1(\mathbb{R}^N)} \rightarrow 0 \quad \text{as } h \rightarrow 0^+.$$

Moreover, for h small enough,

$$\sup_h \int_{|z| > 0} \min\{|z|^2, 1\} d\mu_h(z) = \sup_h \sum_{i=1}^N \frac{|he_i|^2 + |-he_i|^2}{h^2} = 2N$$

Then by Theorem 2.12, there exists a subsequence $\{u_{h_j}\}_{j \in \mathbb{N}}$ of solutions of (5.3), and a $u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ such that $u_{h_j} \rightarrow u$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ as $j \rightarrow \infty$. Moreover, the limit u satisfies equation (1.5):

$$\partial_t u - \Delta \varphi(u) = 0 \quad \text{in } \mathcal{D}'(Q_T).$$

In fact, as in the proof of Corollary 2.13, the whole sequence $\{u_h\}_{h > 0}$ converges.

We can proceed as in this example to get convergence for a more general class of second order local operators.

Lemma 5.2. *Assume $h > 0$, $P \in \mathbb{N}$, $\sigma = (\sigma_1, \dots, \sigma_P)$, $\sigma_i \in \mathbb{R}^N$ for $i = 1, \dots, P$, L_h^σ is defined by (2.9), and $\psi \in C_c^\infty(\mathbb{R}^N)$. Then*

$$L_h^\sigma[\psi](x) = \int_{|z|>0} \left(\psi(x+z) - \psi(x) \right) d\mu_h(z) =: \mathcal{L}^{\mu_{h,\sigma}}[\psi](x),$$

where the measure $\mu_{h,\sigma} = \frac{1}{h^2} \sum_{i=1}^P (\delta_{h\sigma_i} + \delta_{-h\sigma_i})$. Moreover, $\mu_{h,\sigma}$ satisfies (A_μ) ,

$$\sup_h \int_{|z|>0} \min\{|z|^2, 1\} d\mu_{h,\sigma}(z) < \infty,$$

and

$$\|\mathcal{L}^{\mu_{h,\sigma}}[\psi] - \text{tr}[\sigma\sigma^T D^2\psi]\|_{L^1} \rightarrow 0 \quad \text{as} \quad h \rightarrow 0^+.$$

Proof. By an elementary identity and Talyor's theorem,

$$\begin{aligned} \text{tr}[\sigma\sigma^T D^2\psi(x)] &= \sum_{i=1}^P (\sigma_i \cdot D)^2 \psi(x) \\ &= \sum_{i=1}^P \frac{v(x+h\sigma_i) + v(x-h\sigma_i) - 2v(x)}{h^2} + h^2 \sum_{i=1}^P \sum_{|\beta|=4} \frac{1}{\beta!} \sigma_i^\beta D^\beta \psi(\xi_i) \end{aligned}$$

Here we use standard multiindex notation, with multiindex $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^N$, to account for the 4-th order derivatives. Since the first term of the last line is $L_h^\sigma[\psi](x)$, the rest of the proof follows along the arguments of Example 5.1. \square

We aim to consider the general operator \mathcal{L}^μ defined in (1.3). In order to use our stability result, we would like to prove that the operator \mathcal{L}_h^μ defined in (2.10) is a particular case of the operators studied in this paper. The following result ensures this fact.

Lemma 5.3. *Assume (A_μ) , $h > 0$, \mathcal{L}_h^μ is defined in (2.10), and $\psi \in C_c^\infty(\mathbb{R}^N)$. Then*

$$\mathcal{L}_h^\mu[\psi](x) = \int_{|z|>0} \left(\psi(x+z) - \psi(x) \right) d\nu_h(z) =: \mathcal{L}^{\nu_h}[\psi](x)$$

where the measure $\nu_h = \sum_{\alpha \neq 0} \mu(z_\alpha + R_h) \delta_{z_\alpha}$. Moreover, ν_h satisfies (A_μ) and

$$\sup_h \int_{|z|>0} \min\{|z|^2, 1\} d\nu_h(z) < \infty.$$

Proof. By the definition of δ_{z_α} , it immetiati follows that $\mathcal{L}_h^\mu = \mathcal{L}^{\nu_h}$. It remains to show that ν_h satisfies (A_μ) . For $h < 1/\sqrt{N}$,

$$\int_{|z|>1} d\nu_h(z) = \sum_{|z_\alpha|>1} \mu(z_\alpha + R_h) \leq \mu \left(\left\{ |z| > 1 - \sqrt{N} \frac{h}{2} \right\} \right) \leq \mu \left(\left\{ |z| > \frac{1}{2} \right\} \right),$$

which is finite since μ satisfies (A_μ) . Moreover, for $h > 0$ small enough,

$$\begin{aligned} &\int_{|z| \leq 1} |z|^2 d\nu_h(z) \\ &\leq \sum_{0 < |z_\alpha| \leq 1} \int_{z_\alpha + R_h} |z_\alpha|^2 d\mu(z) \leq \sum_{0 < |z_\alpha| \leq 1} \int_{z_\alpha + R_h} \left(|z| + \sqrt{N} \frac{h}{2} \right)^2 d\mu(z) \\ &\leq \int_{h/2 \leq |z| \leq 1 + \sqrt{N} \frac{h}{2}} \left(|z| + \sqrt{N} \frac{h}{2} \right)^2 d\mu(z) \leq (1 + \sqrt{N})^2 \int_{|z| \leq 2} |z|^2 d\mu(z), \end{aligned}$$

which is also finite since μ satisfies (A_μ) . The proof is complete. \square

Lemma 5.4. *Assume (A_μ) , \mathcal{L}^μ and \mathcal{L}_h^μ are defined in (1.3) and (2.10) respectively, and $\psi \in C_c^\infty(\mathbb{R}^N)$. Then*

$$\|\mathcal{L}_h^\mu[\psi] - \mathcal{L}^\mu[\psi]\|_{L^1} \rightarrow 0 \quad \text{as } h \rightarrow 0^+.$$

Proof. The following inequality is just a use of the definitions,

$$\begin{aligned} & \int_{\mathbb{R}^N} |\mathcal{L}_h^\mu[\psi](x) - \mathcal{L}^\mu[\psi](x)| \, dx \\ &= \int_{\mathbb{R}^N} \left| \sum_{\alpha \neq 0} (\psi(x + z_\alpha) - \psi(x)) \int_{z_\alpha + R_h} d\mu(z) \right. \\ & \quad \left. - \sum_{\alpha \in \mathbb{Z}^N} \int_{z_\alpha + R_h} (\psi(x + z) - \psi(x)) \, d\mu(z) \right| dx \\ &\leq \int_{\mathbb{R}^N} \left(\left| \int_{R_h} (\psi(x + z) - \psi(x)) \, d\mu(z) \right| \right. \\ & \quad \left. + \left| \sum_{\alpha \neq 0} \int_{z_\alpha + R_h} (\psi(x + z_\alpha) - \psi(x + z)) \, d\mu(z) \right| \right) dx. \end{aligned}$$

We will show that both terms go to zero with h . Indeed, for $|z| \leq 1$ we have that $|z|^2 \mathbf{1}_{R_h}(z) \rightarrow 0$ pointwise as $h \rightarrow 0^+$. Then, by Lebesgue's dominated convergence theorem, (A_μ) , and Lemma 3.5 (b), we have as $h \rightarrow 0^+$

$$\int_{\mathbb{R}^N} \left| \int_{R_h} (\psi(x + z) - \psi(x)) \, d\mu(z) \right| dx \leq \frac{1}{2} \|D^2\psi\|_{L^1} \int_{|z| \leq 1} |z|^2 \mathbf{1}_{R_h}(z) \, d\mu(z) \rightarrow 0.$$

For the second term, we need to consider separately the cases when when we are close or far from the origin. First note that for any $z \in z_\alpha + R_h$ we have that $|z_\alpha - z| \leq \sqrt{N} \frac{h}{2}$. Since μ satisfies (A_μ) and $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} I_{\text{ext}} &:= \int_{\mathbb{R}^N} \left| \sum_{|\alpha|/h > 1} \int_{z_\alpha + R_h} (\psi(x + z_\alpha) - \psi(x + z)) \, d\mu(z) \right| dx \\ &\leq \|D\psi\|_{L^1(\mathbb{R}^N)} \sum_{|\alpha|/h > 1} \int_{z_\alpha + R_h} |z_\alpha - z| \, d\mu(z) \\ &\leq h \frac{\sqrt{N}}{2} \|D\psi\|_{L^1(\mathbb{R}^N)} \int_{|z| > 1/2} d\mu(z) \rightarrow 0 \quad \text{as } h \rightarrow 0^+. \end{aligned}$$

On the other hand, by the symmetry of μ and also of the term in the sum, we have that

$$\begin{aligned} I_{\text{int}} &:= \int_{\mathbb{R}^N} \left| \sum_{0 < |\alpha|/h \leq 1} \int_{z_\alpha + R_h} (\psi(x + z_\alpha) - \psi(x + z)) \, d\mu(z) \right| dx \\ &\leq \int_{\mathbb{R}^N} \left| \sum_{0 < |\alpha|/h \leq 1} \int_{z_\alpha + R_h} (\psi(x + z_\alpha) - \psi(x + z) - (z_\alpha - z) \cdot D\psi(x)) \, d\mu(z) \right| dx. \end{aligned}$$

We make use of the Taylor expansions

$$\begin{aligned} \psi(x + z_\alpha) &= \psi(x + z) + D\psi(x + z) \cdot (z_\alpha - z) + G_1(x, z, z_\alpha)(z_\alpha - z) \cdot (z_\alpha - z) \\ D\psi(x + z) &= D\psi(x) + G_2(x, z) \cdot z \end{aligned}$$

where $\|G_1\|_{L^1(\mathbb{R}^N, dx)} + \|G_2\|_{L^1(\mathbb{R}^N, dx)} \leq C\|D^2\psi\|_{L^1(\mathbb{R}^N)}$ for some constant $C > 0$. In this way,

$$\begin{aligned} I_{\text{int}} &\leq \int_{\mathbb{R}^N} \left| \sum_{0 < |\alpha| h \leq 1} \int_{z_\alpha + R_h} \left(G_2 z \cdot (z_\alpha - z) + G_1 (z_\alpha - z) \cdot (z_\alpha - z) \right) d\mu(z) \right| dx \\ &= C\|D^2\psi\|_{L^1(\mathbb{R}^N)} \sum_{0 < |\alpha| h \leq 1} \int_{z_\alpha + R_h} \left(|z| |z_\alpha - z| + |z_\alpha - z|^2 \right) d\mu(z) \\ &\leq C\|D^2\psi\|_{L^1(\mathbb{R}^N)} \int_{\frac{h}{2} < |z| \leq 1 + \sqrt{N} \frac{h}{2}} \frac{h}{2} \left(|z| + \frac{h}{2} \right)^2 d\mu(z) \rightarrow 0 \quad \text{as } h \rightarrow 0^+. \end{aligned}$$

Since the integrand is dominated by $2|z|^2$ which is an integrable function with respect to the measure μ on the set $\{z \in \mathbb{R}^N : |z| \leq 1\}$ by (A_μ) , the last term goes to zero by Lebesgue's dominated convergence theorem. \square

Proof of Proposition 2.16. Note that by Lemmas 5.2 and 5.3, L_h^σ and \mathcal{L}_h^μ are in the class of operators defined by (1.3) and (A_μ) .

(a) Existence, uniqueness and regularity follow from Theorem 2.8.

(b) Follows from Theorem 2.10 (c) and (d) and interpolation.

(c) Lemmas 5.2 and 5.4 ensure the L^1 -consistency.

(d) Follows from Theorem 2.10 (b).

(e) Follows from Theorem 2.10 (f). \square

Proof of Proposition 2.17. By Lemmas 5.2 and 5.3 and Proposition 2.16 (c),

$$\sup_h \int_{|z| > 0} \min\{|z|^2, 1\} d(\mu_{h,\sigma} + \nu_h)(z) < \infty,$$

and

$$\|(L_h^\sigma + \mathcal{L}_h^\mu)[\psi] - (L^\sigma + \mathcal{L}^\mu)[\psi]\|_{L^1} \rightarrow 0 \quad \text{as } h \rightarrow 0^+.$$

Since also φ and u_0 are fixed (that is, independent of h), by Theorem 2.12 there is a subsequence $\{u_{h_n}\}_{n \in \mathbb{N}}$ of solutions of (2.5), that converge in $C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N))$ to a function u . Moreover, this function u is a distributional solution of (2.6). Finally, u also belongs to $L^\infty(Q_T) \cap L^1(Q_T)$ by Proposition 2.16 (b) and Fatou's lemma. \square

Proof of Corollary 2.18. Any limit point u from Proposition 2.17 is a distributional solution of (2.7) and (1.2). \square

Proof of Theorem 2.19. By Proposition 2.17 there is a converging subsequence with a limit u which has the right regularity and is a distributional solution of (2.6). Assume there is a subsequence that converge to another limit v . Then by Proposition 2.17 again, there is a subsubsequence that converge to a limit which is a distributional solution. By uniqueness of the limit, v is a distributional solution. But then $v = u$ by the uniqueness given in Corollary 2.4 for the case $\sigma \equiv 0$ or the local result in [18]. Hence all subsequence limits are equal to u and since the sequence itself is bounded (Proposition 2.16 (b)), the whole sequence converges to u . \square

6. AUXILIARY ELLIPTIC EQUATION

In this section we study the elliptic equation (3.1) introduced in Section 3 with the ultimate goal to prove Theorem 3.1 and Lemma 3.4. We will also need the following approximation of (3.1) where the measure μ is replaced by $\mu_r := \mathbf{1}_{|z|>r}\mu$:

$$(6.1) \quad \varepsilon v_{\varepsilon,r}(x) - \mathcal{L}^{\mu_r}[v_{\varepsilon,r}](x) = g(x) \quad \text{in } \mathbb{R}^N,$$

with $\varepsilon > 0$,

$$\mathcal{L}^{\mu_r}[\psi](x) = \int_{|z|>0} (\psi(x+z) - \psi(x)) \, d\mu_r(z).$$

Note that for any $r > 0$, the operator $\mathcal{L}^{\mu_r}[\psi]$ is well-defined for merely bounded ψ , and that Lemma 3.5 also holds for \mathcal{L}^{μ_r} : see Remark 3.6 (b). Also recall the notation $B_\varepsilon^\mu = (\varepsilon I - \mathcal{L}^\mu)^{-1}$ and define $B_\varepsilon^{\mu_r} := (\varepsilon I - \mathcal{L}^{\mu_r})^{-1}$.

Remark 6.1. (a) Since (3.1) and (6.1) are linear equations, we have formally, for any multiindex $\alpha \in \mathbb{N}^N$, that $D^\alpha v$ is a solution of (3.1) or (6.1) with right hand side $D^\alpha g$ if v is a solution of the same equation with right hand side g .

(b) Let $\psi \in C_b^2(\mathbb{R}^N)$, and let $p \in \{1, \infty\}$. Since

$$(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[\psi](x) = \int_{|z|\leq r} (\psi(x+z) - \psi(x) - z \cdot D\psi(x)) \, d\mu(z),$$

we have that $\mathcal{L}^{\mu_r}[\psi] \rightarrow \mathcal{L}^\mu[\psi]$ in $L^p(\mathbb{R}^N)$ as $r \rightarrow 0^+$ by Lemma 3.5 (b) and Lebesgue's dominated convergence theorem.

6.1. Preliminary results. We will state and prove a very general Stroock-Varopoulos type of inequality which is of independent interest. First we consider the bounded operators \mathcal{L}^{μ_r} .

Lemma 6.2. *Assume (A_μ) , $\psi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $\zeta \in C(\mathbb{R})$ is nondecreasing. Then for any $r > 0$ we have,*

$$\begin{aligned} I_r &:= \int_{\mathbb{R}^N} \zeta(\psi(x)) \mathcal{L}^{\mu_r}[\psi](x) \, dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^N} \int_{|z|>0} (\zeta(\psi(x+z)) - \zeta(\psi(x))) (\psi(x+z) - \psi(x)) \, d\mu_r(z) \, dx, \end{aligned}$$

and in particular, $I_r \leq 0$.

Remark 6.3. More generally, the above lemma holds as long as the integral I_r is well-defined for ψ and $\zeta(\psi)$.

In the proof we need a technical lemma which will be proven in Appendix A.

Lemma 6.4. *Assume ν is a nonnegative, symmetric and locally finite Borel measure on \mathbb{R}^N . Let A, B be Borel sets on \mathbb{R}^N , and let*

$$M_1(A, B) = \int_A \left(\int_{B-z} d\nu(x) \right) dz = \int_A \nu(B-z) dz.$$

$$M_2(A, B) = \int_B \left(\int_{A-z} d\nu(x) \right) dz = \int_B \nu(A-z) dz.$$

Then $M_1(A, B) = M_2(A, B)$.

Proof of Lemma 6.2. Observe that $\zeta(\psi) \in L^\infty(\mathbb{R}^N)$, and since $\int_{\mathbb{R}^N} |\mathcal{L}^{\mu_r}[\psi]| \, dx \leq 2\|\psi\|_{L^1} \int_{|z|>r} d\mu(z)$, $\mathcal{L}^{\mu_r}[\psi] \in L^1(\mathbb{R}^N)$. Hence I_r is well-defined.

By the symmetry of μ , the gradient term in the nonlocal operator vanishes. Fubini's theorem and a relabelling of the variables gives

$$\begin{aligned} I_r &= \int_{\mathbb{R}^N} \zeta(\psi(x)) \int_{|z|>0} (\psi(x+z) - \psi(x)) d\mu_r(z) dx \\ &= \int_{\mathbb{R}^N} \int_{|z-x|>0} \zeta(\psi(x)) (\psi(z) - \psi(x)) \mathbf{1}_{|z-x|>r} d\mu_{-x}(z) dx \\ &= \int_{\mathbb{R}^N} \int_{|x-z|>0} \zeta(\psi(z)) (\psi(x) - \psi(z)) \mathbf{1}_{|x-z|>r} d\mu_{-z}(x) dz. \end{aligned}$$

Since μ is a nonnegative, symmetric and finite Radon measure on \mathbb{R}^N (and hence a Borel measures), we can use Lemma 6.4 to see that

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_{|x-z|>0} \zeta(\psi(z)) (\psi(x) - \psi(z)) \mathbf{1}_{|x-z|>r} d\mu_{-z}(x) dz \\ &= \int_{\mathbb{R}^N} \int_{|x-z|>0} \zeta(\psi(z)) (\psi(x) - \psi(z)) \mathbf{1}_{|x-z|>r} d\mu_{-x}(z) dx. \end{aligned}$$

It then follows that

$$\begin{aligned} 2I_r &= - \int_{\mathbb{R}^N} \int_{|z-x|>0} \zeta(\psi(x)) (\psi(x) - \psi(z)) \mathbf{1}_{|z-x|>r} d\mu_{-x}(z) dx \\ &\quad + \int_{\mathbb{R}^N} \int_{|x-z|>0} \zeta(\psi(z)) (\psi(x) - \psi(z)) \mathbf{1}_{|x-z|>r} d\mu_{-x}(z) dx \\ &= - \int_{\mathbb{R}^N} \int_{|z-x|>0} (\zeta(\psi(x)) - \zeta(\psi(z))) (\psi(x) - \psi(z)) \mathbf{1}_{|z-x|>r} d\mu_{-x}(z) dx. \end{aligned}$$

Since $(\zeta(\psi(x)) - \zeta(\psi(z))) (\psi(x) - \psi(z)) \geq 0$ for all $x, z \in \mathbb{R}^N$, $I_r \leq 0$. \square

Now we give the general result, considering the general nonlocal operator \mathcal{L}^μ .

Corollary 6.5 (General Stroock-Varopoulos). *Assume (A_μ) , and $\zeta \in C^1(\mathbb{R})$ such that $\zeta' \geq 0$.*

(a) *Let $\psi \in W^{1,\infty}(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$. Then*

$$\begin{aligned} I &:= \int_{\mathbb{R}^N} \zeta(\psi(x)) \mathcal{L}^\mu[\psi](x) dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^N} \int_{|z-x|>0} (\zeta(\psi(z)) - \zeta(\psi(x))) (\psi(z) - \psi(x)) d\mu(z) dx \\ &\leq 0. \end{aligned}$$

(b) *Let $\psi \in W^{2,\infty}(\mathbb{R}^N) \cap W^{2,1}(\mathbb{R}^N)$. If $Z \in C^2(\mathbb{R})$ is such that $Z(0) = 0$ and $(Z')^2 = \zeta'$, then*

$$\int_{\mathbb{R}^N} \zeta(\psi(x)) \mathcal{L}^\mu[\psi](x) dx \leq -\frac{1}{2} \int_{\mathbb{R}^N} \int_{|z-x|>0} (Z(\psi(z)) - Z(\psi(x)))^2 d\mu_{-x}(z) dx.$$

Moreover,

$$\begin{aligned} \left(Z(\psi), \mathcal{L}^\mu[Z(\psi)] \right)_{L^2(\mathbb{R}^N)} &= -\frac{1}{2} \int_{\mathbb{R}^N} \int_{|z-x|>0} (Z(\psi(z)) - Z(\psi(x)))^2 d\mu_{-x}(z) dx \\ &= - \left\| (\mathcal{L}^\mu)^{\frac{1}{2}} [Z(\psi)] \right\|_{L^2}^2. \end{aligned}$$

Remark 6.6. The (energy) norm in part (b) is much studied when $\mathcal{L}^\mu = -(-\Delta)^{\frac{s}{2}}$, $s \in (0, 2)$, and $Z = I$ (see [7, 33]). In this case

$$\left(\psi, (-\Delta)^{\frac{s}{2}} \psi \right)_{L^2(\mathbb{R}^N)} = \frac{1}{2} \int_{\mathbb{R}^N} \int_{|z-x|>0} \frac{(\psi(z) - \psi(x))^2}{|z-x|^{N+s}} dz dx = \left\| (-\Delta)^{\frac{s}{4}} \psi \right\|_{L^2(\mathbb{R}^N)}^2.$$

This is called the Gagliardo (semi)norm of ψ and is denoted by $[\psi]_{W^{\frac{\mu}{2},2}(\mathbb{R}^N)}$.

Proof. (a) By Remark 6.1 (b),

$$\left| \int_{\mathbb{R}^N} \zeta(\psi)(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[\psi] \, dx \right| \leq \|\zeta(\psi)\|_{L^\infty} \|(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[\psi]\|_{L^1} \rightarrow 0 \quad \text{as } r \rightarrow 0^+,$$

and we may send $r \rightarrow 0^+$ in Lemma 6.2 to get

$$\begin{aligned} & \int_{\mathbb{R}^N} \zeta(\psi(x)) \mathcal{L}^\mu[\psi](x) \, dx \\ &= -\frac{1}{2} \lim_{r \rightarrow 0^+} \int_{\mathbb{R}^N} \int_{|z-x|>r} (\zeta(\psi(z)) - \zeta(\psi(x))) (\psi(z) - \psi(x)) \, d\mu_{-x}(z) \, dx. \end{aligned}$$

By the assumptions on ζ, ψ and (A_μ) , $(\zeta(\psi(z)) - \zeta(\psi(x))) (\psi(z) - \psi(x)) \geq 0$ is integrable with respect to $d\mu_{-x}(z) \, dx$ on $\mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}$ since

$$\begin{aligned} & \int_{\mathbb{R}^N} \left| \int_{|z-x|>0} (\zeta(\psi(z)) - \zeta(\psi(x))) (\psi(z) - \psi(x)) \, d\mu_{-x}(z) \right| \, dx \\ (6.2) \quad & \leq \|\zeta'(\psi)\|_{L^\infty} \|D\psi\|_{L^\infty} \|D\psi\|_{L^1} \int_{|z|\leq 1} |z|^2 \, d\mu(z) \\ & \quad + 4\|\zeta(\psi)\|_{L^\infty} \|\psi\|_{L^1} \int_{|z|>1} d\mu(z). \end{aligned}$$

Thus, Lebesgue's dominated convergence theorem gives the desired result.

(b) For $a, b \in \mathbb{R}$, the Fundamental Theorem of Calculus and Jensen's inequality gives the following pointwise inequality:

$$\begin{aligned} (6.3) \quad (Z(b) - Z(a))^2 &= \left(\int_a^b Z'(t) \, dt \right)^2 \leq (b-a) \int_a^b (Z'(t))^2 \, dt \\ &= (b-a) \int_a^b \zeta'(t) \, dt = (b-a)(\zeta(b) - \zeta(a)). \end{aligned}$$

By the assumptions, we can easily check that $Z(\psi) \in W^{2,\infty}(\mathbb{R}^N) \cap W^{2,1}(\mathbb{R}^N)$. So the integral

$$-\frac{1}{2} \int_{\mathbb{R}^N} \int_{|z-x|>0} (Z(\psi(z)) - Z(\psi(x)))^2 \, d\mu_{-x}(z) \, dx$$

is well-defined using a similar argument as in (6.2). Then, part (a) and (6.3) gives the first result of part (b).

Next, part (a) yields

$$\left(Z(\psi), \mathcal{L}^\mu[Z(\psi)] \right)_{L^2(\mathbb{R}^N)} = -\frac{1}{2} \int_{\mathbb{R}^N} \int_{|z-x|>0} (Z(\psi(z)) - Z(\psi(x)))^2 \, d\mu_{-x}(z) \, dx.$$

Moreover, since $Z(\psi) \in W^{2,\infty}(\mathbb{R}^N) \cap W^{2,1}(\mathbb{R}^N)$, then by Lemma 3.5 (b) and interpolation, both $Z(\psi)$ and $\mathcal{L}^\mu[Z(\psi)]$ are in $L^2(\mathbb{R}^N)$. We then conclude the proof by application of Lemma 3.7 and Remark 3.8 (b). \square

6.2. Results for the approximate elliptic equation (6.1). We will now focus on proving some a priori, uniqueness, existence, and stability results for (6.1).

Proposition 6.7. *Assume (A_μ) .*

(a) *If $g \in L^\infty(\mathbb{R}^N)$ and $v_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$ solves $\varepsilon v_{\varepsilon,r} - \mathcal{L}^{\mu_r}[v_{\varepsilon,r}] \leq g$ a.e., then*

$$\varepsilon \|(v_{\varepsilon,r})^+\|_{L^\infty(\mathbb{R}^N)} \leq \|(g)^+\|_{L^\infty(\mathbb{R}^N)}.$$

(b) If $g \in L^\infty(\mathbb{R}^N)$ and $v_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$ is an a.e. solution of (6.1), then

$$\varepsilon \|v_{\varepsilon,r}\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}.$$

(c) Let $g, \hat{g}, v_{\varepsilon,r}, \hat{v}_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$, $\varepsilon v_{\varepsilon,r} - \mathcal{L}^{\mu_r}[v_{\varepsilon,r}] \leq g$ a.e. and $\varepsilon \hat{v}_{\varepsilon,r} - \mathcal{L}^{\mu_r}[\hat{v}_{\varepsilon,r}] \geq \hat{g}$ a.e. If $g \leq \hat{g}$ a.e., then $v_{\varepsilon,r} \leq \hat{v}_{\varepsilon,r}$ a.e.

Proof. (a) Assume first that $g, v_{\varepsilon,r} \in C_b(\mathbb{R}^N)$. Then for all $\delta > 0$ there exists a $x_\delta \in \mathbb{R}^N$ such that

$$v_{\varepsilon,r}(x_\delta) + \delta > \sup\{v_{\varepsilon,r}\}.$$

Then, since $v_{\varepsilon,r}$ is an a.e. solution,

$$\begin{aligned} \varepsilon v_{\varepsilon,r}(x_\delta) &\leq g(x_\delta) + \int_{|z|>0} (v_{\varepsilon,r}(x_\delta + z) - v_{\varepsilon,r}(x_\delta)) \, d\mu_r(z) \\ &\leq \|g\|_{L^\infty(\mathbb{R}^N)} + \int_{|z|>r} (\sup\{v_{\varepsilon,r}\} - v_{\varepsilon,r}(x_\delta)) \, d\mu(z) \\ &\leq \|g\|_{L^\infty(\mathbb{R}^N)} + \delta \mu(\{z \in \mathbb{R}^N : |z| > r\}). \end{aligned}$$

Hence,

$$\varepsilon \sup\{v_{\varepsilon,r}\} < \varepsilon v_{\varepsilon,r}(x) + \varepsilon \delta \leq \|g\|_{L^\infty(\mathbb{R}^N)} + \delta(\varepsilon + \mu(\{z \in \mathbb{R}^N : |z| > r\})),$$

and we pass to the limit as $\delta \rightarrow 0^+$ to get

$$\varepsilon \sup\{v_{\varepsilon,r}\} \leq \|g\|_{L^\infty}^+.$$

In the general case, when $g, v_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$, we need a regularization argument. Let $v_{\varepsilon,r}^\delta := \omega_\delta * v_{\varepsilon,r}$ and mollify the inequality to see that

$$\varepsilon v_{\varepsilon,r}^\delta - \mathcal{L}^{\mu_r}[v_{\varepsilon,r}^\delta] \leq g_\delta \quad \text{in } \mathbb{R}^N.$$

By the first part of the proof and the properties of mollifiers,

$$v_{\varepsilon,r}(x) \leq |v_{\varepsilon,r}^\delta(x) - v_{\varepsilon,r}(x)| + v_{\varepsilon,r}^\delta(x) \leq o(1) + \frac{1}{\varepsilon} \|g\|_{L^\infty(\mathbb{R}^N)}^+ \quad \text{as } \delta \rightarrow 0^+$$

for a.e. $x \in \mathbb{R}^N$. Part (a) follows.

(b) In a similar way as in (a), we find that $\varepsilon \sup\{-v_{\varepsilon,r}\} \leq \|g\|_{L^\infty(\mathbb{R}^N)}^-$ and combine with (a) to conclude that $\varepsilon \|v_{\varepsilon,r}\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}$.

(c) Since $w = v_{\varepsilon,r} - \hat{v}_{\varepsilon,r}$ solves $\varepsilon w - \mathcal{L}^{\mu_r}[w] \leq g - \hat{g}$, by (a) and the assumptions, it follows that $\varepsilon \sup\{w\} \leq \|(g - \hat{g})^+\|_{L^\infty(\mathbb{R}^N)} = 0$. \square

Proposition 6.8 (Existence and uniqueness). *Assume (A_μ) .*

(a) If $g \in C_b(\mathbb{R}^N)$, then there exists a unique classical solution $v_{\varepsilon,r} \in C_b(\mathbb{R}^N)$ of (6.1).

(b) If $g \in L^\infty(\mathbb{R}^N)$, then there exists a unique a.e. solution $v_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$ of (6.1).

(c) If $g \in L^1(\mathbb{R}^N)$, then there exists a unique a.e. solution $v_{\varepsilon,r} \in L^1(\mathbb{R}^N)$ of (6.1).

(d) If $g \in C_b^\infty(\mathbb{R}^N)$, then there exists a unique classical solution $v_{\varepsilon,r} \in C_b^\infty(\mathbb{R}^N)$ of (6.1). Moreover,

$$\varepsilon \|D^\alpha v_{\varepsilon,r}\|_{L^\infty} \leq \|D^\alpha g\|_{L^\infty}$$

for each multiindex $\alpha \in \mathbb{N}^N$.

Proof. The proofs of (a), (b), and (c) follow from standard arguments using Banach's fixed point theorem. Let X denote any of one of the spaces $C_b(\mathbb{R}^N)$, $L^\infty(\mathbb{R}^N)$, and $L^1(\mathbb{R}^N)$, and note that X is a Banach space. Let the operator T be such that (6.1) is equivalent to the fixed point equation $T[u] = u$:

$$T[v_\varepsilon](x) := \frac{1}{\varepsilon + \int_{|z|>r} d\mu(z)} \left(\int_{|z|>r} v_\varepsilon(x+z) \, d\mu(z) + g(x) \right).$$

It is easy to check that T is a bounded linear operator on X , and straightforward computations also shows it is a contraction:

$$\|T[v_\varepsilon] - T[\hat{v}_\varepsilon]\|_X \leq \alpha \|v_\varepsilon - \hat{v}_\varepsilon\|_X \quad \text{for} \quad \alpha = \frac{\int_{|z|>r} d\mu(z)}{\varepsilon + \int_{|z|>r} d\mu(z)} < 1.$$

Hence by Banach's fixed point theorem there exists a unique $v_\varepsilon \in X$ such that $v_\varepsilon = T[v_\varepsilon]$ in X and then also a.e. (everywhere if $X = C_b$).

(d) Let $v_{\varepsilon,r} = B_\varepsilon^{\mu_r}[g]$ and define $\delta_{i,h}\psi$ by

$$\delta_{i,h}\psi(x) = \frac{\psi(x + he_i) - \psi(x)}{h}.$$

By part (a), we have uniqueness for $C_b(\mathbb{R}^N)$ solutions of (6.1). Hence, $v_{\varepsilon,r}(x + he_i) = B_\varepsilon^{\mu_r}[g(\cdot + he_i)](x)$, and then by uniqueness and linearity $\delta_{i,h}v = B_\varepsilon^{\mu_r}[\delta_{i,h}g]$. In addition, there exists a unique $w_{i,\varepsilon,r} \in C_b(\mathbb{R}^N)$ such that $w_{i,\varepsilon,r} = B_\varepsilon^{\mu_r}[\partial_{x_i}g]$.

Using linearity and Proposition 6.7 (b), we get

$$\varepsilon \|w_{i,\varepsilon,r} - \delta_{i,h}v_{\varepsilon,r}\|_{L^\infty} \leq \|\partial_{x_i}g - \delta_{i,h}g\|_{L^\infty}.$$

When $h \rightarrow 0^+$, $\delta_{i,h}g \rightarrow \partial_{x_i}g$ uniformly on \mathbb{R}^N , and hence $\delta_{i,h}v_{\varepsilon,r} \rightarrow w_{i,\varepsilon,r}$ in L^∞ . This implies that $\partial_{x_i}v_{\varepsilon,r} = w_{i,\varepsilon,r}$. Moreover, by Proposition 6.7 (b),

$$\|\partial_{x_i}v_{\varepsilon,r}\|_{L^\infty} = \|w_{i,\varepsilon,r}\|_{L^\infty} \leq \|\partial_{x_i}g\|_{L^\infty}.$$

A similar argument shows that for each multiindex $\alpha \in \mathbb{N}^N$, $D^\alpha v_{\varepsilon,r} = B_\varepsilon^{\mu_r}[D^\alpha g]$, and hence belongs to $C_b(\mathbb{R}^N)$. \square

Corollary 6.9. *Assume (A_μ) and $g \in C_b(\mathbb{R}^N)$. If $(g)^+ \in L^1(\mathbb{R}^N)$, then $(B_\varepsilon^{\mu_r}[g])^+ \in L^1(\mathbb{R}^N)$.*

Proof. Note that $g \in C_b(\mathbb{R}^N)$ implies that $(g)^+ \in C_b(\mathbb{R}^N)$. By Proposition 6.8 (a) and (c), and the assumption on $(g)^+$, we have that $B_\varepsilon^{\mu_r}[g] \in C_b(\mathbb{R}^N)$ and $B_\varepsilon^{\mu_r}[(g)^+] \in L^1(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$ are the unique classical solutions of (6.1) with right-hand sides $g, (g)^+$, respectively. Proposition 6.7 (c) ensures that $B_\varepsilon^{\mu_r}[(g)^+] \geq 0$ since $(g)^+ \geq 0$. In the same way, we get $B_\varepsilon^{\mu_r}[-(g)^-] \in C_b(\mathbb{R}^N)$ and $B_\varepsilon^{\mu_r}[-(g)^-] \leq 0$.

Adding the equations for $B_\varepsilon^{\mu_r}[(g)^+]$ and $B_\varepsilon^{\mu_r}[-(g)^-]$, and noting that $(g)^+ - (g)^- = g \in C_b(\mathbb{R}^N)$, we get

$$\varepsilon \left(B_\varepsilon^{\mu_r}[(g)^+] - B_\varepsilon^{\mu_r}[(g)^-] \right) - \mathcal{L}^{\mu_r} \left[B_\varepsilon^{\mu_r}[(g)^+] - B_\varepsilon^{\mu_r}[(g)^-] \right] = g.$$

It follows that $B_\varepsilon^{\mu_r}[g] = B_\varepsilon^{\mu_r}[(g)^+] - B_\varepsilon^{\mu_r}[(g)^-]$ by uniqueness. We conclude that $0 \leq (B_\varepsilon^{\mu_r}[g])^+ \leq B_\varepsilon^{\mu_r}[(g)^+]$, and thus, $(B_\varepsilon^{\mu_r}[g])^+ \in L^1(\mathbb{R}^N)$. \square

6.3. Results for the elliptic equation (3.1). Now, we state and prove comparison, uniqueness and existence results for classical solutions of (3.1). These results will be obtained from the corresponding results for (6.1) and limit procedures.

Lemma 6.10 (Comparison). *Assume (A_μ) , $g, \hat{g} \in L^\infty(\mathbb{R}^N)$, and $v_\varepsilon, \hat{v}_\varepsilon \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ are solutions of (3.1) with right-hand sides g, \hat{g} respectively. If $g \leq \hat{g}$ a.e., then $v_\varepsilon \leq \hat{v}_\varepsilon$ in \mathbb{R}^N .*

Proof. Note that $w = v_\varepsilon - \hat{v}_\varepsilon$ solves $\varepsilon w - \mathcal{L}^\mu[w] \leq 0$, and hence, also

$$\varepsilon w - \mathcal{L}^{\mu_r}[w] \leq \|(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[w]\|_{L^\infty(\mathbb{R}^N)}.$$

By Proposition 6.7 (a), it then follows that

$$\varepsilon \|w\|_{L^\infty(\mathbb{R}^N)} \leq \|(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[w]\|_{L^\infty(\mathbb{R}^N)}.$$

Assume for moment that $w \in C_b^2(\mathbb{R}^N)$. Then by Remark 6.1 (b),

$$\|(\mathcal{L}^\mu - \mathcal{L}^{\mu_r})[w]\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0 \quad \text{as} \quad r \rightarrow 0^+,$$

and we conclude that $w \leq 0$.

The general case follows by mollification: $w_\delta = w * \omega_\delta$ (cf. (1.7)) satisfies $\varepsilon w_\delta - \mathcal{L}^\mu[w_\delta] \leq 0$ and hence by the first part of the proof and properties of mollifiers,

$$w(x) \leq w_\delta(x) + |w(x) - w_\delta(x)| \leq 0 + o(1) \quad \text{as } \delta \rightarrow 0^+$$

for every $x \in \mathbb{R}^N$. The proof is complete. \square

Corollary 6.11 (Uniqueness). *Assume (A_μ) , and $g \in L^\infty(\mathbb{R}^N)$. Then there is at most one classical solution $v_\varepsilon \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ of (3.1).*

Proof. If $g = \hat{g}$ a.e., then Lemma 6.10 gives $v_\varepsilon = \hat{v}_\varepsilon$ in \mathbb{R}^N . \square

Proposition 6.12 (Existence and Stability). *Assume (A_μ) , $g \in C_b^\infty(\mathbb{R}^N)$, $\varepsilon > 0$.*

(a) *There exists a unique classical solution $B_\varepsilon^\mu[g] = v_\varepsilon \in C_b^2$ of (3.1).*

(b) *Any sequence $\{v_{\varepsilon, r_n}\}_{n \in \mathbb{N}}$ of solutions of (6.1) converges locally uniformly to $v_\varepsilon = B_\varepsilon^\mu[g]$ of part (a) as $r_n \rightarrow 0^+$.*

Proof. (a) Let $0 < r_n \rightarrow 0^+$ as $n \rightarrow \infty$, and let $v_n := v_{\varepsilon, r_n} \in C_b^\infty(\mathbb{R}^N)$ be the unique solution of (6.1) given by Proposition 6.8 (d). Moreover, for all $n > 0$,

$$\begin{aligned} \varepsilon \|v_n\|_{L^\infty} &\leq \|g\|_{L^\infty}, & \varepsilon \|Dv_n\|_{L^\infty} &\leq \|Dg\|_{L^\infty}, \\ \varepsilon \|D^2v_n\|_{L^\infty} &\leq \|D^2g\|_{L^\infty}, & \varepsilon \|D^3v_n\|_{L^\infty} &\leq \|D^3g\|_{L^\infty}. \end{aligned}$$

The sequences $\{v_n\}_{n>0}$, $\{Dv_n\}_{n>0}$ and $\{D^2v_n\}_{n>0}$ are thus equibounded and equicontinuous. By Arzelà-Ascoli's theorem there exists a subsequence (still denoted by v_n , Dv_n and D^2v_n) such that (v_n, Dv_n, D^2v_n) converges locally uniformly (and hence a.e.) as $n \rightarrow \infty$ to a limit $(\bar{v}, \overline{Dv}, \overline{D^2v})$ which is bounded and continuous.

We check that $D\bar{v} = \overline{Dv}$ and $D^2\bar{v} = \overline{D^2v}$. Let $\alpha \in \mathbb{N}^N$ denote a multiindex. By Taylor's theorem

$$\begin{aligned} (6.4) \quad v_n(y) &= v_n(x) + Dv_n(x) \cdot (y-x) + \frac{1}{2} D^2v_n(x)(y-x) \cdot (y-x) \\ &+ \sum_{|\alpha|=3} \frac{3}{\alpha!} (y-x)^\alpha \int_0^1 (1-t)^2 D^\alpha v_n(x+t(y-x)) dt. \end{aligned}$$

Since

$$\left| \sum_{|\alpha|=3} \frac{3}{\alpha!} (y-x)^\alpha \int_0^1 (1-t)^2 D^\alpha v_n(x+t(y-x)) dt \right| \leq \frac{1}{\varepsilon} \|D^3g\|_{L^\infty} \sum_{|\alpha|=3} \frac{|y-x|^\alpha}{\alpha!},$$

we can take the locally uniform limit in (6.4) as $n \rightarrow \infty$ to obtain that

$$\bar{v}(y) = \bar{v}(x) + \overline{Dv}(x) \cdot (y-x) + \frac{1}{2} \overline{D^2v}(x)(y-x) \cdot (y-x) + o(|y-x|^2) \quad \text{as } y \rightarrow x.$$

By definition, it then follows that $D\bar{v} = \overline{Dv}$ and $D^2\bar{v} = \overline{D^2v}$.

We now go to the limit in (6.1) as $r_n \rightarrow 0^+$, and we may assume that $r_n < 1$. In order to show the convergence, the nonlocal operator in (6.1) will be written as

$$\mathcal{L}^{\mu_{r_n}}[v_n](x) = \mathcal{L}_1^{\mu_{r_n}}[v_n](x) + \int_{|z|>1} (v_n(x+z) - v_n(x)) d\mu(z),$$

with

$$\mathcal{L}_1^{\mu_{r_n}}[v_n](x) := \int_{|z|\leq 1} (v_n(x+z) - v_n(x) - z \cdot Dv_n(x)) d\mu_{r_n}(z).$$

By the triangle inequality and Lemma 3.5 (a),

$$\begin{aligned} & |\mathcal{L}_1^{\mu_{r_n}}[v_n](x) - \mathcal{L}_1^\mu[\bar{v}](x)| \\ & \leq |\mathcal{L}_1^{\mu_{r_n}}[v_n - \bar{v}](x)| + |(\mathcal{L}_1^{\mu_{r_n}} - \mathcal{L}_1^\mu)[\bar{v}](x)| \\ & \leq \frac{1}{2} \max_{|z| \leq 1} |D^2 v_n(x+z) - D^2 \bar{v}(x+z)| \int_{|z| \leq 1} |z|^2 d\mu(z) \\ & \quad + \frac{1}{2} \max_{|z| \leq 1} |D^2 \bar{v}(x+z)| \int_{|z| \leq 1} |z|^2 \mathbf{1}_{|z| \leq r_n} d\mu(z). \end{aligned}$$

So, the local uniform convergence and Lebesgue's dominated convergence theorem ensures that $|\mathcal{L}_1^{\mu_{r_n}}[v_n](x) - \mathcal{L}_1^\mu[\bar{v}](x)| \rightarrow 0$ as $r_n \rightarrow 0^+$ for all $x \in \mathbb{R}^N$. The remaining term in the nonlocal operator also converges by Lebesgue's dominated convergence theorem:

$$\int_{|z| > 1} (v_n(x+z) - v_n(x)) d\mu(z) \rightarrow \int_{|z| > 1} (\bar{v}(x+z) - \bar{v}(x)) d\mu(z) \quad \text{as } r_n \rightarrow 0^+.$$

Sending $r_n \rightarrow 0^+$ in (6.1) then shows that \bar{v} solves (3.1). Moreover, the limit is unique by Corollary 6.11.

(b) In fact, part (a) shows that all limit points of the sequences $\{v_{\varepsilon, r_n}\}_{n \in \mathbb{N}}$ coincide by uniqueness (see Corollary 6.11). By Proposition 6.8 (d), every sequence is bounded, and hence, the whole sequence converge locally uniformly to the solution of (3.1) as $r_n \rightarrow 0^+$. \square

Proposition 6.13. *Assume (A_μ) , $g \in C_b^\infty(\mathbb{R}^N)$, $\varepsilon > 0$, and $v_\varepsilon = B_\varepsilon^\mu[g]$. If $(g)^+ \in L^1(\mathbb{R}^N)$, then*

$$\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^+ dx \leq \int_{\mathbb{R}^N} (g)^+ dx.$$

Proof. By Proposition 6.8 (d), for any $r > 0$, there exists a unique function $v_{\varepsilon, r} \in C_b^\infty(\mathbb{R}^N)$ such that

$$\varepsilon v_{\varepsilon, r}(x) - \mathcal{L}^{\mu_r}[v_{\varepsilon, r}](x) = g(x) \quad \text{in } \mathbb{R}^N.$$

Consider $\mathcal{X} \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \mathcal{X}$,

$$\mathcal{X}(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 2 \end{cases},$$

and define $\mathcal{X}_R(x) = \mathcal{X}(\frac{x}{R})$ for $R > 0$. Then for every $r > 0$, by Proposition 6.8 (d), there exists a function $u_R \in C_b^\infty(\mathbb{R}^N)$ such that

$$(6.5) \quad \varepsilon u_R(x) - \mathcal{L}^{\mu_r}[u_R](x) = g(x)\mathcal{X}_R(x) \quad \text{for all } x \in \mathbb{R}^N.$$

Let $\zeta_\delta : \mathbb{R} \rightarrow \mathbb{R}_+$ be a smooth approximation of the sign^+ function. More precisely, $\zeta_\delta(x) = 0$ for $x \leq 0$, $\zeta_\delta'(x) \geq 0$ and $0 < \zeta_\delta(x) \leq 1$ for $x > 0$. Since $0 \leq (g\mathcal{X}_R)^+ \leq (g)^+ \in L^1(\mathbb{R}^N)$, $(u_R)^+ \in L^1(\mathbb{R}^N)$ by Corollary 6.9, and

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u_R \zeta_\delta(u_R) dx \right| & \leq \|(u_R)^+\|_{L^1} \|\zeta_\delta(u_R)\|_{L^\infty} \\ \left| \int_{\mathbb{R}^N} g\mathcal{X}_R \zeta_\delta(u_R) dx \right| & \leq \|g\|_{L^\infty} \int_{|x| \leq 2R} dx. \end{aligned}$$

Then by (6.5), $|\int_{\mathbb{R}^N} \mathcal{L}^{\mu_r}[u_R] \zeta_\delta(u_R) dx| < \infty$, and we may multiply (6.5) by ζ_δ and integrate over \mathbb{R}^N to find that

$$\varepsilon \int_{\mathbb{R}^N} u_R \zeta_\delta(u_R) dx = \underbrace{\int_{\mathbb{R}^N} \mathcal{L}^{\mu_r}[u_R] \zeta_\delta(u_R) dx}_{I_r} + \int_{\mathbb{R}^N} g\mathcal{X}_R \zeta_\delta(u_R) dx.$$

So, Lemma 6.2 and Remark 6.3 gives that $I_r \leq 0$ and hence

$$\varepsilon \int_{\mathbb{R}^N} u_R \zeta_\delta(u_R) \, dx \leq \int_{\mathbb{R}^N} g \mathcal{X}_R \zeta_\delta(u_R) \, dx \leq \int_{\mathbb{R}^N} (g)^+ \, dx.$$

Letting $\zeta_\delta(u_R) \rightarrow \text{sign}^+(u_R)$ as $\delta \rightarrow 0^+$ in the above inequality (using Fatou's lemma on the left-hand side since $u_R \zeta_\delta(u_R) \geq 0$) yields

$$(6.6) \quad \varepsilon \int_{\mathbb{R}^N} (u_R)^+ \, dx \leq \int_{\mathbb{R}^N} (g)^+ \, dx.$$

We note that the sequence $\{u_R\}_{R>0}$ is equibounded and equilipschitz since Proposition 6.8 (d) gives

$$\begin{aligned} \|u_R\|_{L^\infty} &\leq \frac{1}{\varepsilon} \|g\|_{L^\infty} \\ \|Du_R\|_{L^\infty} &\leq \frac{1}{\varepsilon} \|D(g\mathcal{X}_R)\|_{L^\infty} \leq \frac{1}{\varepsilon} \|Dg\|_{L^\infty} + \frac{1}{\varepsilon} \mathcal{O}\left(\frac{1}{R}\right). \end{aligned}$$

Hence, by Arzelà-Ascoli, $u_R \rightarrow u$ as $R \rightarrow \infty$ locally uniformly in \mathbb{R}^N (and thus a.e. in \mathbb{R}^N). Sending $R \rightarrow \infty$ in (6.5) shows that $u = v_{\varepsilon,r}$, that is, the unique solution of (3.1) given by Proposition 6.8 (d). Furthermore, we can send $R \rightarrow \infty$ in (6.6) (again using Fatou's lemma) to obtain

$$\varepsilon \int_{\mathbb{R}^N} (v_{\varepsilon,r})^+ \, dx \leq \int_{\mathbb{R}^N} (g)^+ \, dx.$$

By Fatou's lemma and Proposition 6.12 (b), we can let $r_n \rightarrow 0^+$ in the above estimate to get

$$\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^+ \, dx \leq \int_{\mathbb{R}^N} (g)^+ \, dx,$$

where v_ε is the classical solution of (3.1). □

Corollary 6.14. *Assume (A_μ) , $g \in C_b^\infty(\mathbb{R}^N)$, $\varepsilon > 0$, and $v_\varepsilon = B_\varepsilon^\mu[g]$.*

(a) *If $(g)^- \in L^1(\mathbb{R}^N)$, then $\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^- \, dx \leq \int_{\mathbb{R}^N} (g)^- \, dx$.*

(b) *If $g \in L^1(\mathbb{R}^N)$, then $\varepsilon \int_{\mathbb{R}^N} |v_\varepsilon| \, dx \leq \int_{\mathbb{R}^N} |g| \, dx$*

Proof. (a) Note that $(g)^- \in L^1(\mathbb{R}^N)$ implies that $(-g)^+ \in L^1(\mathbb{R}^N)$. Since $B_\varepsilon^\mu[-g] = -B_\varepsilon^\mu[g]$, we have by Proposition 6.13 that

$$\varepsilon \int_{\mathbb{R}^N} (B_\varepsilon^\mu[g])^- \, dx = \varepsilon \int_{\mathbb{R}^N} (B_\varepsilon^\mu[-g])^+ \, dx \leq \int_{\mathbb{R}^N} (-g)^+ = \int_{\mathbb{R}^N} (g)^- \, dx.$$

(b) Follows by noting that $(v_\varepsilon)^+ + (v_\varepsilon)^- = |v_\varepsilon|$. □

Below, we collect the main results for (3.1).

Theorem 6.15. *Assume (A_μ) , $g \in L_{loc}^1(\mathbb{R}^N)$, and $v_\varepsilon \in L_{loc}^1(\mathbb{R}^N)$ is a distributional solution of (3.1).*

(a) *If $(g)^+ \in L^1(\mathbb{R}^N)$, then*

$$\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^+ \, dx \leq \int_{\mathbb{R}^N} (g)^+ \, dx.$$

(b) *If $g \geq 0$ a.e. on \mathbb{R}^N , then $v_\varepsilon \geq 0$ a.e. on \mathbb{R}^N .*

Proof. (a) Let $\omega_\delta \in C_c^\infty(\mathbb{R}^N)$ be defined in (1.7), and let $v_{\varepsilon,\delta} = v_\varepsilon * \omega_\delta \in C_b^\infty(\mathbb{R}^N)$. By assumption,

$$\varepsilon \int_{\mathbb{R}^N} v_\varepsilon \psi \, dy - \int_{\mathbb{R}^N} v_\varepsilon \mathcal{L}^\mu[\psi] \, dy = \int_{\mathbb{R}^N} g \psi \, dy$$

for all $\psi \in C_c^\infty(\mathbb{R}^N)$. Taking $\psi(y) = \omega_\delta(x - y)$ for $x \in \mathbb{R}^N$, we get the pointwise equation

$$\varepsilon v_{\varepsilon, \delta} - \mathcal{L}^\mu[v_{\varepsilon, \delta}] = g * \omega_\delta \quad \text{in } \mathbb{R}^N.$$

Note that $0 \leq (g * \omega_\delta)^+ \leq (g)^+ * \omega_\delta \in L^1(\mathbb{R}^N)$ (see e.g. Lemma 5.1 in [25]), so Proposition 6.13 gives

$$\varepsilon \int_{\mathbb{R}^N} (v_{\varepsilon, \delta})^+ dx \leq \int_{\mathbb{R}^N} (g * \omega_\delta)^+ dx.$$

Then by Fatou's lemma

$$\varepsilon \int_{\mathbb{R}^N} \liminf_{\delta \rightarrow 0^+} (v_{\varepsilon, \delta})^+ dx \leq \liminf_{\delta \rightarrow 0^+} \varepsilon \int_{\mathbb{R}^N} (v_{\varepsilon, \delta})^+ dx \leq \liminf_{\delta \rightarrow 0^+} \int_{\mathbb{R}^N} (g)^+ * \omega_\delta dx.$$

Since $(\cdot)^+$ is continuous, $(g)^+ \in L^1(\mathbb{R}^N)$, and $v_{\varepsilon, \delta} \in L^1_{\text{loc}}(\mathbb{R}^N)$, the properties of mollifiers yields

$$\varepsilon \int_{\mathbb{R}^N} (v_\varepsilon)^+ dx \leq \int_{\mathbb{R}^N} (g)^+ dx.$$

(b) Note that $-v_\varepsilon$ solves (3.1) with right-hand side $-g$. If $-g \leq 0$ a.e. on \mathbb{R}^N , then $(-g)^+ = 0 \in L^1(\mathbb{R}^N)$. By part (a), we deduce that $\varepsilon \int_{\mathbb{R}^N} (-v_\varepsilon)^+ dx \leq 0$, and hence that $-v_\varepsilon \leq 0$ a.e. on \mathbb{R}^N . \square

We are now ready to prove our main theorem for the elliptic equation (3.1).

Proof of Theorem 3.1. (a) By the assumptions and Proposition 6.8 (d), for every $r > 0$, there exists a unique classical solution $v_{\varepsilon, r} \in C_b^\infty(\mathbb{R}^N)$ of (6.1) satisfying

$$\varepsilon \|D^\alpha v_{\varepsilon, r}\|_{L^\infty} \leq \|D^\alpha g\|_{L^\infty} \quad \text{for all } \alpha \in \mathbb{N}^N.$$

An Arzelà-Ascoli argument as in the proof of Proposition 6.12 (in this case combined with a diagonal extraction argument), shows the existence of classical solutions $v_\varepsilon \in C_b^\infty(\mathbb{R}^N)$ of (3.1) satisfying

$$\varepsilon \|D^\alpha v_\varepsilon\|_{L^\infty} \leq \|D^\alpha g\|_{L^\infty} \quad \text{for all } \alpha \in \mathbb{N}^N.$$

Moreover, Corollary 6.11 ensures that the classical solutions v_ε are unique.

(b) *Existence of L^1 -solutions:* Let $\delta > 0$, $g_\delta = g * \omega_\delta \in C_b^\infty(\mathbb{R}^N)$, where ω_δ is defined by (1.7), and $v_{\varepsilon, \delta} \in C_b^\infty(\mathbb{R}^N)$ be the solution of (3.1) with g_δ as right hand side. By Remark 6.1 (a), a difference of solutions is also a solution, and then by Corollary 6.14 (b),

$$\varepsilon \|v_{\varepsilon, \delta_1} - v_{\varepsilon, \delta_2}\|_{L^1} \leq \|g_{\delta_1} - g_{\delta_2}\|_{L^1} \quad \text{for every } \delta_1, \delta_2 > 0.$$

Hence, $\{v_{\varepsilon, \delta}\}_{\delta > 0}$ is Cauchy and there exists $v_\varepsilon \in L^1(\mathbb{R}^N)$ such that $\|v_{\varepsilon, \delta} - v_\varepsilon\|_{L^1} \rightarrow 0$ as $\delta \rightarrow 0^+$.

Since $v_{\varepsilon, \delta}$ satisfies (3.1) with right-hand side g_δ ,

$$\varepsilon \int_{\mathbb{R}^N} v_{\varepsilon, \delta} \psi dx - \int_{\mathbb{R}^N} v_{\varepsilon, \delta} \mathcal{L}^\mu[\psi] dx = \int_{\mathbb{R}^N} g_\delta \psi dx \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N),$$

and since $v_{\varepsilon, \delta}, g_\delta \rightarrow v_\varepsilon, g$ in $L^1(\mathbb{R}^N)$ as $\delta \rightarrow 0^+$, we send $\delta \rightarrow 0^+$ and find that v_ε is an L^1 -distributional solution of (3.1).

Uniqueness: Note that $L^1 \subset L^1_{\text{loc}}$. Consider two distributional solutions $v_\varepsilon, \hat{v}_\varepsilon$ of (3.1) with right-hand sides $g, \hat{g} \in L^1(\mathbb{R}^N)$. If $g - \hat{g} = 0$ a.e., then $v_\varepsilon - \hat{v}_\varepsilon = B_\varepsilon^\mu[g - \hat{g}] = 0$ by Theorem 6.15 (b).

L^1 -estimate: By the assumptions, we can take $v_\varepsilon \in L^1(\mathbb{R}^N) \subset L^1_{\text{loc}}(\mathbb{R}^N)$ and $g \in L^1(\mathbb{R}^N)$. Then Theorem 6.15 (a) gives

$$\varepsilon \|(v_\varepsilon)^+\|_{L^1} \leq \|(g)^+\|_{L^1}.$$

A similar argument as in the proof of Corollary 6.14 concludes the proof.

(c) *Existence of L^∞ -solutions:* Proposition 6.8 (b) ensures that there exists a unique a.e. solution $v_{\varepsilon,r} \in L^\infty(\mathbb{R}^N)$ of

$$\varepsilon v_{\varepsilon,r} - \mathcal{L}^{\mu r}[v_{\varepsilon,r}] = g,$$

and $\varepsilon \|v_{\varepsilon,r}\|_{L^\infty} \leq \|g\|_{L^\infty}$. Then, by Alaoglu's theorem there exists $\bar{v}_\varepsilon \in L^\infty(\mathbb{R}^N)$ such that, up to a subsequence, $v_{\varepsilon,r_n} \xrightarrow{*} \bar{v}_\varepsilon$ in $L^\infty(\mathbb{R}^N)$ as $r_n \rightarrow 0^+$. That is,

$$\lim_{r_n \rightarrow 0^+} \int_{\mathbb{R}^N} v_{\varepsilon,r_n} \psi \, dx = \int_{\mathbb{R}^N} \bar{v}_\varepsilon \psi \, dx \quad \text{for all } \psi \in L^1(\mathbb{R}^N).$$

To finish the existence proof, we need to show that \bar{v}_ε is in fact a distributional solution of (3.1). Consider a function $\gamma \in C_c^\infty(\mathbb{R}^N)$, then $\gamma, \mathcal{L}^{\mu r_n}[\gamma], \mathcal{L}^\mu[\gamma] \in L^1(\mathbb{R}^N)$ (see Lemma 3.5 (b)). Since v_{ε,r_n} is a pointwise a.e. solution and $v_{\varepsilon,r_n}, \mathcal{L}^{\mu r_n}[v_{\varepsilon,r_n}] \in L^\infty(\mathbb{R}^N)$, we have by integration and self-adjointness of $\mathcal{L}^{\mu r_n}$ (cf. Lemma 3.5 and Remark 3.6 (b)) that

$$\varepsilon \int_{\mathbb{R}^N} v_{\varepsilon,r_n} \gamma \, dx - \int_{\mathbb{R}^N} v_{\varepsilon,r_n} \mathcal{L}^{\mu r_n}[\gamma] \, dx = \int_{\mathbb{R}^N} g \gamma \, dx \quad \text{for all } \gamma \in C_c^\infty(\mathbb{R}^N).$$

The weak* L^∞ -convergence ensures that

$$\lim_{r_n \rightarrow 0^+} \int_{\mathbb{R}^N} v_{\varepsilon,r_n} \gamma \, dx = \int_{\mathbb{R}^N} \bar{v}_\varepsilon \gamma \, dx \quad \text{for all } \gamma \in C_c^\infty(\mathbb{R}^N).$$

By Remark 6.1 (b), we have that, for any $\gamma \in C_c^\infty(\mathbb{R}^N)$, $\mathcal{L}^{\mu r_n}[\gamma] \rightarrow \mathcal{L}^\mu[\gamma]$ in $L^1(\mathbb{R}^N)$ as $r_n \rightarrow 0^+$. Then, since $\|v_{\varepsilon,r_n}\|_{L^\infty} \leq \frac{1}{\varepsilon} \|g\|_{L^\infty}$, we get as $r_n \rightarrow 0^+$

$$\begin{aligned} & \int_{\mathbb{R}^N} v_{\varepsilon,r_n} \mathcal{L}^{\mu r_n}[\gamma] \, dx \\ &= \int_{\mathbb{R}^N} v_{\varepsilon,r_n} \mathcal{L}^\mu[\gamma] \, dx + \int_{\mathbb{R}^N} v_{\varepsilon,r_n} (\mathcal{L}^{\mu r_n} - \mathcal{L}^\mu)[\gamma] \, dx \rightarrow \int_{\mathbb{R}^N} \bar{v}_\varepsilon \mathcal{L}^\mu[\gamma] \, dx, \end{aligned}$$

for all $\gamma \in C_c^\infty(\mathbb{R}^N)$. This shows that

$$\varepsilon \int_{\mathbb{R}^N} \bar{v}_\varepsilon \gamma \, dx - \int_{\mathbb{R}^N} \bar{v}_\varepsilon \mathcal{L}^\mu[\gamma] \, dx = \int_{\mathbb{R}^N} g \gamma \, dx \quad \text{for all } \gamma \in C_c^\infty(\mathbb{R}^N),$$

that is, \bar{v}_ε is an L^∞ -distributional solution of (3.1).

Uniqueness: Note that $L^\infty \subset L_{\text{loc}}^1$. Consider two distributional solutions $v_\varepsilon, \hat{v}_\varepsilon$ of (3.1) with right-hand sides $g, \hat{g} \in L^\infty(\mathbb{R}^N)$. If $g - \hat{g} = 0$ on \mathbb{R}^N , then $v_\varepsilon - \hat{v}_\varepsilon = 0$ by Theorem 6.15 (b).

L^∞ -estimate: Observe that $\pm \frac{1}{\varepsilon} \|g\|_{L^\infty} \in L^\infty(\mathbb{R}^N) \subset L_{\text{loc}}^1(\mathbb{R}^N)$ are distributional solutions of (3.1) with $\pm \|g\|_{L^\infty}$ as right-hand sides. Moreover, $-\|g\|_{L^\infty} \leq g \leq \|g\|_{L^\infty}$. Then Theorem 6.15 (b) gives $|v_\varepsilon| \leq \frac{1}{\varepsilon} \|g\|_{L^\infty}$. \square

This section is concluded by a proof of the self-adjointness of B_ε^μ .

Proof of Lemma 3.4. Let $f_\delta = f * \omega_\delta$ and $g_\delta = g * \omega_\delta$ where ω_δ is defined by (1.7). Then $f_\delta \in C_b^\infty(\mathbb{R}^N) \cap W^{2,1}(\mathbb{R}^N)$ and $g \in C_b^\infty(\mathbb{R}^N)$, and then by Theorem 3.1 (a)–(c), $B_\varepsilon^\mu[f_\delta] \in C_b^\infty(\mathbb{R}^N) \cap W^{2,1}(\mathbb{R}^N)$, $B_\varepsilon^\mu[g_\delta] \in C_b^\infty(\mathbb{R}^N)$, and

$$\begin{aligned} \varepsilon B_\varepsilon^\mu[f_\delta] - \mathcal{L}^\mu[B_\varepsilon^\mu[f_\delta]] &= f_\delta(x) & \text{in } \mathbb{R}^N, \\ \varepsilon B_\varepsilon^\mu[g_\delta] - \mathcal{L}^\mu[B_\varepsilon^\mu[g_\delta]] &= g_\delta(x) & \text{in } \mathbb{R}^N. \end{aligned}$$

By the regularity and integrability of the terms of the equations (cf. Lemma 3.5), we may multiply the first equation by $B_\varepsilon^\mu[g_\delta]$ and the second by $B_\varepsilon^\mu[f_\delta]$, and then

integrate both equations in x over \mathbb{R}^N . By self-adjointness of \mathcal{L}^μ (Lemma 3.5 (c)), we then find that

$$\begin{aligned}
(6.7) \quad \int_{\mathbb{R}^N} f_\delta B_\varepsilon^\mu[g_\delta] \, dx &= \int_{\mathbb{R}^N} (\varepsilon B_\varepsilon^\mu[f_\delta] - \mathcal{L}^\mu[B_\varepsilon^\mu[f_\delta]]) B_\varepsilon^\mu[g_\delta] \, dx \\
&= \int_{\mathbb{R}^N} (\varepsilon B_\varepsilon^\mu[g_\delta] - \mathcal{L}^\mu[B_\varepsilon^\mu[g_\delta]]) B_\varepsilon^\mu[f_\delta] \, dx \\
&= \int_{\mathbb{R}^N} g_\delta B_\varepsilon^\mu[f_\delta] \, dx
\end{aligned}$$

To pass to the limit as $\delta \rightarrow 0^+$, we first subtract equations to find that

$$\varepsilon B_\varepsilon^\mu[f] - \varepsilon B_\varepsilon^\mu[f_\delta] - \mathcal{L}^\mu[B_\varepsilon^\mu[f] - B_\varepsilon^\mu[f_\delta]] = f - f_\delta \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N),$$

and hence by Theorem 3.1 (b), linearity, and properties of mollifiers,

$$\varepsilon \|B_\varepsilon^\mu[f] - B_\varepsilon^\mu[f_\delta]\|_{L^1} = \varepsilon \|B_\varepsilon^\mu[f - f_\delta]\|_{L^1} \leq \|f - f_\delta\|_{L^1} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0^+.$$

On the other hand, by Theorem 3.1 (b) and (c), and properties of the mollifiers,

$$\varepsilon \|B_\varepsilon^\mu[f_\delta]\|_{L^1} \leq \|f\|_{L^1}, \quad \varepsilon \|B_\varepsilon^\mu[f_\delta]\|_{L^\infty} \leq \|f\|_{L^\infty}, \quad \varepsilon \|B_\varepsilon^\mu[g_\delta]\|_{L^\infty} \leq \|g\|_{L^\infty},$$

and $g_\delta \rightarrow g$ a.e. Using L^1 -convergence for the f -terms and the dominated convergence theorem for the g -terms, we may send $\delta \rightarrow 0^+$ in (6.7) to get the result. \square

APPENDIX A. TECHNICAL RESULTS

A.1. Proof of Liouville type of theorem.

Proof of Theorem 3.9. By the definition of distributional solutions,

$$\int_{\mathbb{R}^N} v(y) \mathcal{L}^\mu[\psi](y) \, dy = 0 \quad \text{for all} \quad \psi \in C_c^\infty(\mathbb{R}^N).$$

Let $x \in \mathbb{R}^N$, take $\psi(y) = \omega_\delta(x - y)$, where ω_δ is defined in (1.7), and let $v_\delta = v * \omega_\delta \in C_0(\mathbb{R}^N) \cap C_b^\infty(\mathbb{R}^N)$. By Lemma 3.5 (b), $\mathcal{L}^\mu[\psi] \in L^1$, and we may use Fubini's theorem to see that

$$(A.1) \quad \mathcal{L}^\mu[v_\delta](x) = 0 \quad \text{for every} \quad x \in \mathbb{R}^N.$$

Assume that there exists an $\tilde{x} \in \mathbb{R}^N$ such that $v_\delta(\tilde{x}) \neq 0$. We only consider the case $v_\delta(\tilde{x}) > 0$; the proof in the other case is similar. Then $M := \sup_{x \in \mathbb{R}^N} v_\delta > 0$, and since $v_\delta \in C_0(\mathbb{R}^N)$ there exists an x_0 such that

$$0 < M = \max_{x \in \mathbb{R}^N} v_\delta = v_\delta(x_0).$$

By equation (A.1) and Lemma 3.5 (b), we then find that

$$\begin{aligned}
0 = \mathcal{L}^\mu[v_\delta](x_0) &= \int_{|z| \leq \kappa} (v_\delta(x_0 + z) - v_\delta(x_0) - z \cdot Dv_\delta(x_0)) \, d\mu(z) \\
&\quad + \int_{|z| > \kappa} (v_\delta(x_0 + z) - v_\delta(x_0)) \, d\mu(z) \\
&\leq \|D^2 v_\delta\|_{L^\infty(\overline{B}(x_0, \kappa))} \int_{|z| \leq \kappa} |z|^2 \, d\mu(z) \\
&\quad + \int_{|z| > \kappa} (v_\delta(x_0 + z) - M) \, d\mu(z).
\end{aligned}$$

Take any $z_0 \in \text{supp } \mu$. By definition, $z_0 \neq 0$ and $\mu(B(z_0, r)) > 0$ for all $r > 0$. Hence we can take $r, \kappa \in (0, 1)$ small enough such that

$$B(z_0, r) \cap \{z \in \mathbb{R}^N : |z| \leq \kappa\} = \emptyset.$$

Since $\kappa < 1$, $v_\delta(x_0 + z) - M \leq 0$, and $B(z_0, r) \subset \{z \in \mathbb{R}^N : |z| > \kappa\}$, the above inequality yields that

$$\int_{B(z_0, r)} (v_\delta(x_0 + z) - M) d\mu(z) \geq -\|D^2 v_\delta\|_{L^\infty(\overline{B}(x_0, 1))} \int_{|z| \leq 1} |z|^2 \mathbf{1}_{|z| \leq \kappa} d\mu(z).$$

Taking the limit as $\kappa \rightarrow 0^+$ using Lebesgue's dominated convergence theorem (the integrand is dominated by $|z|^2$ which is integrable by (A_μ)) gives

$$\int_{B(z_0, r)} (v_\delta(x_0 + z) d\mu(z) - M\mu(B(z_0, r))) d\mu(z) \geq 0.$$

Then by continuity, $v_\delta(x_0 + z) = v_\delta(x_0 + z_0) + \lambda(|z - z_0|)$ in $B(z_0, r)$ for some modulus of continuity λ , and we find that

$$v_\delta(x_0 + z_0) + \lambda(r) \geq \frac{1}{\mu(B(z_0, r))} \int_{B(z_0, r)} v_\delta(x_0 + z) d\mu(z) \geq M.$$

Hence, we may send $r \rightarrow 0^+$ and get that $v_\delta(x_0 + z_0) \geq M$. It follows that $v_\delta(x_0 + z_0) = M$ since M is the maximum of v_δ .

Repeating the above argument, we find that $v_\delta(x_0 + nz_0) = M$ for every $n \in \mathbb{N}$, and thus

$$\limsup_{n \rightarrow \infty} v_\delta(x_0 + nz_0) \geq M > 0.$$

This is a contradiction since $\lim_{|x| \rightarrow \infty} v_\delta(x) = 0$. So, we conclude that $v_\delta(x) = 0$ for every $x \in \mathbb{R}^N$.

By the properties of mollifiers, $v_\delta \rightarrow v$ locally uniformly in \mathbb{R}^N as $\delta \rightarrow 0^+$, and hence it follows that also $v(x) = 0$ for every $x \in \mathbb{R}^N$. \square

A.2. Proof of a measure theory result.

Proof of Lemma 6.4. Remember that we defined

$$M_1(A, B) = \int_A \left(\int_{B-z} d\nu(x) \right) dz = \int_A \nu(B-z) dz,$$

$$M_2(A, B) = \int_B \left(\int_{A-z} d\nu(x) \right) dz = \int_B \nu(A-z) dz,$$

and that we want to show that $M_1(A, B) = M_2(A, B)$.

Consider the set $C \subset \mathbb{R}^{2N}$ defined as

$$C = \{(x, z) \in \mathbb{R}^{2N} : z \in A, x \in B - z\}.$$

Furthermore, define the sets

$$S = \{x = x_B - x_A : x_A \in A, x_B \in B\} = \bigcup_{x_A \in A} (B - x_A),$$

$$G_x = \{z \in A : x \in B - z\} = \{z \in A : z \in B - x\} = A \cap (B - x).$$

Note that C can also be expressed as

$$C = \{(x, z) \in \mathbb{R}^{2N} : x \in S, z \in G_x\}.$$

Then

(A.2)

$$\begin{aligned} M_1(A, B) &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathbf{1}_A(z) \mathbf{1}_{B-z}(x) d\nu(x) \right) dz = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathbf{1}_C(x, z) d\nu(x) \right) dz \\ &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathbf{1}_C(x, z) dz \right) d\nu(x) = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathbf{1}_S(x) \mathbf{1}_{G_x}(z) dz \right) d\nu(x) \\ &= \int_S \left(\int_{G_x} dz \right) d\nu(x) = \int_S |G_x| d\nu(x), \end{aligned}$$

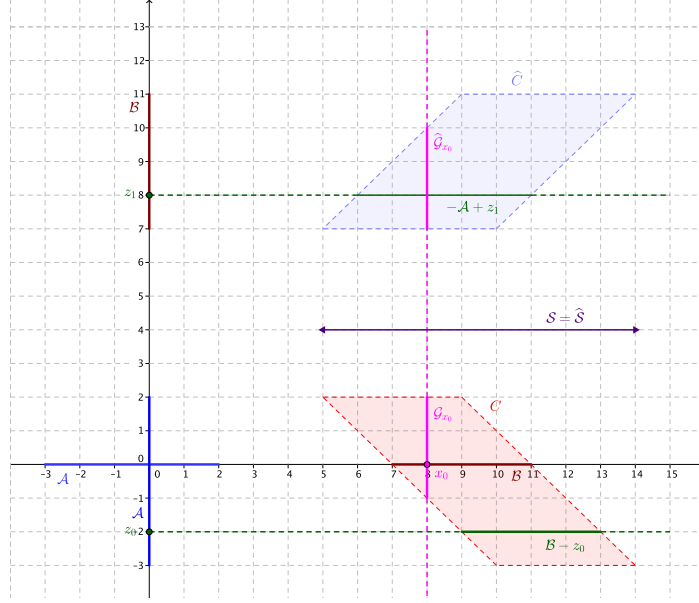


FIGURE 1.

where the third equality follows by Tonelli's theorem (the tensor measure is a non-negative Radon measure), and $|\cdot|$ denotes the Lebesgue measure on \mathbb{R}^N .

We can proceed in the same way to change the order of integration in the expression for $M_2(A, B)$, but first we make use of the symmetry of ν

$$M_2(A, B) = \int_B \nu(A - z) dz = \int_B \nu(-A + z) dz = \int_B \left(\int_{z-A} d\nu(x) \right) dz.$$

Using the same technique we consider the sets,

$$\widehat{C} = \{(x, z) \in \mathbb{R}^{2N} : z \in B, x \in -A + z\},$$

$$\widehat{S} = \{x = x_B + x_A : x_A \in -A, x_B \in B\} = \bigcup_{x_B \in B} (x_B - A),$$

$$\widehat{G}_x = \{z \in B : x \in -A + z\} = \{z \in B : z \in A + x\} = B \cap (A + x).$$

Second, we follow (A.2) to get

$$(A.3) \quad M_2(A, B) = \int_{\widehat{S}} |\widehat{G}_x| d\nu(x).$$

Now, note that $S = \widehat{S}$. Moreover, $G_x = A \cap (B - x)$ is just a translation of $\widehat{G}_x = B \cap (A + x)$. Then $|G_x| = |\widehat{G}_x|$, since the Lebesgue measure is invariant under translations. (Consult Figure 1 for a visual overview of the sets.)

Finally, we can conclude by (A.2) and (A.3) that

$$M_1(A, B) = \int_S |G_x| d\nu(x) = \int_{\widehat{S}} |\widehat{G}_x| d\nu(x) = M_2(A, B),$$

which completes the proof. \square

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(F. del Teso) BASQUE CENTER FOR APPLIED MATHEMATICS (BCAM), BILBAO, SPAIN

E-mail address: felix.delteso@uam.es

URL: <http://www.bcamath.org/es/people/fdelteso>

(J. Endal) DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (NTNU), N-7491 TRONDHEIM, NORWAY

E-mail address: jorgen.endal@math.ntnu.no

URL: <http://www.math.ntnu.no/~jorgeen/>

(E. R. Jakobsen) DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (NTNU), N-7491 TRONDHEIM, NORWAY

E-mail address: erj@math.ntnu.no

URL: <http://www.math.ntnu.no/~erj/>