A displacement-free formulation for the Timoshenko beam problem and a corresponding isogeometric collocation approach

J. Kiendl[1,*], F. Auricchio[2,3], A. Reali[2,3,4]

[1]Department of Marine Technology, Norwegian University of Science and Technology, Trondheim, Norway
[2]Department of Civil Engineering and Architecture, University of Pavia, Italy
[3]Institute for Applied Mathematics and Information Technology (IMATI), National Research Council (CNR), Pavia, Italy
[4]Institute for Advanced Study (IAS), Technical University of Munich, Garching, Germany

[*]corresponding author

Abstract

We present a reformulation of the classical Timoshenko beam problem, resulting in a single differential equation with the rotation as the only primal variable. We show that this formulation is equivalent to the standard formulation and the same types of boundary conditions apply. Moreover, we develop an isogeometric collocation scheme to solve the problem numerically. The formulation is completely locking-free and involves only half the degrees of freedom compared to a standard formulation. Numerical tests are presented to confirm the performance of the proposed approach.

1 Introduction

Isogeometric collocation [2] is a novel development in computational mechanics and has been introduced as an alternative to Galerkin-based isogeometric analysis (IGA) [14, 9]. The main idea behind isogeometric collocation (IGA-C) is to discretize the geometry and the unknowns by isogeometric functions (such as B-splines, NURBS, or T-splines [8]), and to solve the differential equations in the strong form by collocating them on a set of suitable collocation points [25] . With this approach, the computation of integrals is avoided, resulting in a greatly reduced computational effort compared to Galerkin-based IGA. In the latter, numerical quadrature constitutes an issue in terms of computational costs, since the standard quadrature rules, adopted from finite element analysis (FEA), become inefficient with increasing continuity of the shape functions. In IGA-C, instead, only one collocation point per unknown is necessary, which roughly corresponds to one point per element, independently of polynomial degree. A profound study and comparison of the computational costs in classical FEA, IGA, and IGA-C can be found in [27]. Isogeometric collocation has been employed successfully to solve linear and nonlinear mechanical problems including elastostatics and explicit dynamics [3], structural mechanics of beams [7, 16, 5], spatial rods [4, 21, 28, 22], plates [15, 24], and shells [17], large deformation elasticity [18], contact [10, 18], phase-field modeling [13], and fracture [26]. The convergence and stability properties of IGA-C [19] depend crucially on the chosen collocation points and different approaches have been proposed, see [2, 12, 1, 23].

In the context of shear deformable structures, it was shown that standard displacement-based collocation formulations can exhibit locking problems, similar to what is known from Galerkin methods, if the discretization spaces of the primal variables are not well balanced. Henceforth, locking-free mixed collocation formulations were proposed for straight Timoshenko beams [7], spatial Timoshenko rods [4, 28, 22], and Reissner-Mindlin plates [15]. A different approach to obtain locking-free formulations for Timoshenko beams was presented in [16], based on a reformulation of the differential equations in a single equation with only one unknown variable. For this reformulation, a new variable, the so-called bending displacement, was introduced and then considered as the primal variable to be solved for. Both Galerkin and collocation formulations based on this approach were presented in [16] and were shown to be locking-free ab initio.

In the present contribution, we pursue an approach which has similarities with [16], but, instead of introducing an artificial variable like the bending displacement, we consider the rotation as the primal variable. We first show how the standard Timoshenko beam differential equations, consisting of two equations in terms of displacement and rotation, can be reduced to a single equation in terms of the rotation only, and then develop a corresponding isogeometric collocation scheme. Numerical tests confirm the good performance and the locking-free characteristic of the method. The paper is organized as follows. In Section 2, we first recall the governing equations of the Timoshenko beam problem as well as the standard form of the resulting differential equations. Then, we reformulate the problem with the rotation as the only unknown, resulting in a single differential equation of third order. This formulation is used to develop a corresponding collocation approach in Section 3, which is tested numerically in Section 4. In Section 5, finally, we draw conclusions.

2 Formulation

We consider linear static analysis of straight beams subjected to a distributed transverse load f. Cross sections are assumed to remain straight during deformation, but not necessarily perpendicular to the beam axis due to shear deformability, as depicted in Figure 1, where w is the vertical deflection of the beam axis, φ is the total rotation of the cross section and γ is the transverse shear strain. All variables are functions of the coordinate x and a prime symbol $(\cdot)'$ indicates a derivative with respect to x, i.e., $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}x$.

The beam's material and cross-sectional parameters are defined by the Young's modulus E and second moment of inertia I for bending, and by the shear modulus G and cross-sectional area A for shear deformation. For a compact notation, we introduce the beam's bending stiffness as $K_b = EI$ and the shear stiffness as $K_s = \alpha GA$, with α as the shear correction factor. Within this paper we assume the stiffness parameters to be constant throughout the beam's length.

In the following, we first review the governing equations of the classical Timoshenko beam theory, consisting of kinematic, constitutive, and equilibrium equations as well as the standard formulation of the problem with displacements and rotations as unknown variables. Then, we present a new formulation, which satisfies the same governing equations, but with the rotation as the only unknown variable.

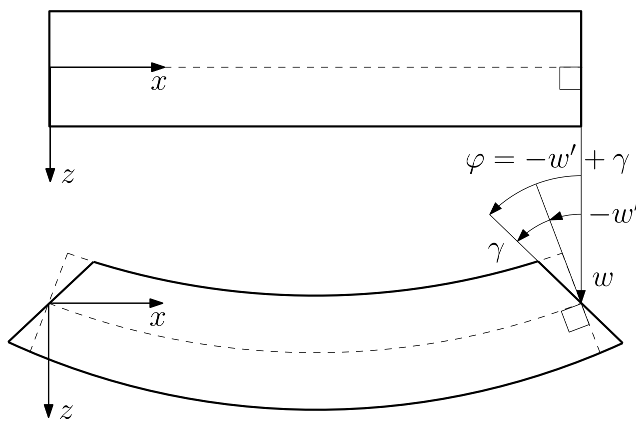

Figure 1: Beam model and kinematic variables.

2.1 Governing equations of the Timoshenko beam

The kinematic equations of a Timoshenko beam are given by

$$w' = -\varphi + \gamma, \tag{1}$$
$$\varphi' = \kappa, \tag{2}$$

with κ as the curvature. The bending moment M and the shear force Q are then obtained as

$$M = K_b \kappa, \tag{3}$$
$$Q = K_s \gamma, \tag{4}$$

and the equilibrium equations are given by

$$M' = Q, \tag{5}$$
$$Q' = -f, \tag{6}$$

where equation (5) represents moment equilibrium and (6) transverse force equilibrium.

2.2 Standard and previous alternative formulations

Substituting the constitutive equations into the equilibrium equations and expressing the shear deformation in terms of displacement and rotation, we obtain the standard form of the differential equations for the Timoshenko beam

$$K_b\varphi'' - K_s(\varphi + w') = 0, \tag{7}$$
$$K_s(\varphi' + w'') = -f, \tag{8}$$

which is a system of two equations in the two unknowns φ and w, both appearing in the second derivatives. Accordingly, four boundary conditions are necessary to complete the boundary value problem. Boundary conditions can be either of kinematic type, prescribing w or φ, or of static type, prescribing M or Q. In the following, we discuss some possible alternative formulations based on different choices of primal variables. However, from the modeling point of view it is important that boundary conditions are always to be imposed on the classical variables w, φ, M, and Q.

An alternative formulation could be obtained by considering w and γ as primal variables, as it has been done recently in the context of plates and shells in [6, 20, 11]. The advantage of such a formulation is that it is completely free of shear locking, since $\lim_{t\to 0}\gamma = 0$ can be represented independently of the chosen discretization spaces. Nevertheless, the problem is still governed by two equations and two primal unknowns.

In [16], a single-variable form of the Timoshenko beam problem was presented. To this end, the displacement was split into two parts, namely bending and shear displacement, and it was shown that the differential equations (7)-(8) could be reduced to a single equation in terms of the bending displacement w_b

$$K_b w_b'''' = f. \tag{9}$$

In this form, the Timoshenko beam problem is described by a fourth order differential equation and four boundary conditions are again necessary to complete the boundary value problem.

2.3 New formulation in terms of the rotation

Similar to what was shown in [16], we use the rotational equilibrium equation (5) together with equations (2)-(4) in order to obtain a direct relation between the rotation φ and the shear deformation γ

$$K_b\varphi'' = K_s\gamma. \tag{10}$$

Therefore, we can express γ as a functions of φ as follows

$$\gamma = \frac{K_b}{K_s}\varphi''. \tag{11}$$

Accordingly, also w can be expressed in terms of φ by substituting (11) into (1) and subsequent integration

$$w = \int_0^x \left(-\varphi + \frac{K_b}{K_s}\varphi''\right) \mathrm{d}x + w_0, \tag{12}$$

where the integration constant represents the displacement at the left end, i.e., $w_0 = w(0)$. An alternative form can be obtained by taking the second term out of the integral

$$w = -\int_0^x \varphi \,\mathrm{d}x + \frac{K_b}{K_s}\varphi' + c. \tag{13}$$

It should be noted that in this case, the integration constant can no longer be interpreted as the displacement at the left end, i.e., $c \neq w(0)$. Bending moment and shear force are simply obtained by derivatives of φ

$$M = K_b\varphi' \tag{14}$$

$$Q = K_b\varphi'' \tag{15}$$

and the differential equation finally reads as

$$K_b\varphi''' = -f. \tag{16}$$

As can be seen, the differential equation is of third order. Furthermore, we have the integration constant for w as an additional unknown, such that the boundary value problem is again completed by four boundary conditions. The boundaries of the beam are denoted by $\Gamma = \{0\} \cup \{l\}$, and Γ_w, Γ_φ, Γ_M, Γ_Q indicate the boundaries with prescribed w, φ, M, and Q, respectively. The boundary conditions can then be formulated as follows, with barred symbols indicating the prescribed boundary values

$$\int_0^x \left(-\varphi + \frac{K_b}{K_s}\varphi''\right) \mathrm{d}x + w_0 = \bar{w} \quad \text{on } \Gamma_w, \tag{17}$$

$$\varphi = \bar{\varphi} \quad \text{on } \Gamma_\varphi, \tag{18}$$

$$K_b\,\varphi' = \pm\bar{M} \quad \text{on } \Gamma_M, \tag{19}$$

$$K_b\,\varphi'' = \pm\bar{Q} \quad \text{on } \Gamma_Q. \tag{20}$$

The integral in (17) disappears for the left boundary, $x = 0$, such that the equation reduces to $w_0 = \bar{w}$. In (19)-(20), the minus sign refers to $x = 0$ while the plus sign refers to $x = l$. At each boundary, we have to prescribe two boundary conditions, one on either w or Q and one on either φ or M.

With equations (16)-(20) the strong form of the Timoshenko beam problem can be stated as follows: Given a distributed load $f(x)$ and prescribed boundary values $\bar{w}, \bar{\varphi}, \bar{M}, \bar{Q}$, find $\varphi(x)$ and w_0 such that (16) is satisfied on $]0, l[$ and (17)-(20) are satisfied on the boundaries.

The fact that w depends on the integral of the primal variable φ makes this formulation not a suitable basis for establishing Galerkin-type numerical formulations. Nevertheless, the strong form equations are well suited for a collocation approach as will be shown in the following section.

3 Isogeometric collocation

In this section we develop an isogeometric collocation approach for solving the strong form equations presented above. The geometry and the unknown rotation field are discretized by B-splines, and the Greville abscissae [2] are used as collocation points.

B-splines are piecewise polynomials defined over a so-called knot vector $\{\xi_1, ..., \xi_{n+p+1}\}$, where p is the polynomial degree and n is the number of B-spline functions, and can be computed with the following recursion formula

For $p = 0$ (piecewise constants)

$$N_{i,0}(\xi) = \begin{cases} 1 \text{ if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 \text{ otherwise}, \end{cases} \tag{21}$$

for $p \geq 1$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi). \tag{22}$$

The Greville abscissae are defined as knot averages by the following formula

$$\overline{\xi}_i = \frac{\xi_{i+1} + \xi_{i+2} + \ldots + \xi_{i+p}}{p}. \tag{23}$$

Furthermore, we can define Greville abscissae related to the k-th derivative space as

$$\overline{\xi}_i^{(k)} = \frac{\xi_{i+1+k} + \xi_{i+2+k} + \ldots + \xi_{i+p}}{p - k} \tag{24}$$

According to equation (23), the number of Greville abscissae is always equal to the number of basis functions n, while from equation (24) we obtain $n-k$ Greville abscissae related to the k-th derivative space.

The primal variable φ is approximated by φ^h

$$\varphi^h(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\,\hat{\varphi}_i \tag{25}$$

where $\hat{\varphi}_i$ are the control variables. Equation (25) can also be written more compact in matrix form

$$\varphi^h = \mathbf{N}\,\hat{\boldsymbol{\varphi}} \tag{26}$$

where $\hat{\boldsymbol{\varphi}}$ is the vector of control variables $\hat{\boldsymbol{\varphi}} = [\hat{\varphi}_1\,\hat{\varphi}_2\ldots]^{\mathrm{T}}$, and $\mathbf{N}$ is the row vector of shape functions $\mathbf{N} = [N_{1,p}\,N_{2,p}\ldots]$. In the following, derivatives of φ^h with respect to x are necessary which requires consideration of the isoparametric mapping $x(\xi)$. Since we only consider straight beams with a linear parametrization, we can assume $x = \xi$ and, therefore, $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}x = \mathrm{d}(\cdot)/\mathrm{d}\xi$.

Corresponding to the differential equation (16) being of third order, we use the Greville abscissae related to the third derivative space (24) as collocation points, and obtain the following set of equations

$$K_b\,\mathbf{N}'''(\bar{\xi}_i^{(3)})\,\hat{\boldsymbol{\varphi}} \;=\; -f(\bar{\xi}_i^{(3)}) \qquad \text{for } i = 1,\ldots,n-3, \tag{27}$$

which is completed by four boundary conditions, which are obtained as the discrete version of equations (17)-(20)

$$\int_0^x \left(-\mathbf{N} + \frac{K_b}{K_s}\mathbf{N}''\right)\mathrm{d}x\,\hat{\boldsymbol{\varphi}} + w_0 = \bar{w} \qquad \text{on } \Gamma_w \tag{28}$$

$$\mathbf{N}\,\hat{\boldsymbol{\varphi}} = \bar{\varphi} \qquad \text{on } \Gamma_\varphi \tag{29}$$

$$K_b\,\mathbf{N}'\,\hat{\boldsymbol{\varphi}} = \pm\bar{M} \quad \text{on } \Gamma_M \tag{30}$$

$$K_b\,\mathbf{N}''\,\hat{\boldsymbol{\varphi}} = \pm\bar{Q} \quad \text{on } \Gamma_Q \tag{31}$$

It is a general feature of IGA-C that boundary conditions are imposed by simply collocating the respective equations on the boundaries. This makes it quite easy to impose also unusual equations like (28). The integral in equation (28) is computed numerically by Gauss quadrature, and it is important to note that this has no significant effect on the computational cost of the method, since this integral has to be computed only for one equation at most, namely for the boundary equation $w(l) = \bar{w}$. If the displacement at $x = l$ is free, no integral at all needs to be computed for assembling the equation system (however, numerical integration may be needed in post-processing for recovering the results of w). It is also worth noting that due to the interpolating property of NURBS at the boundaries, equation (29) reduces to $\hat{\varphi}_j = \bar{\varphi}$ ($j \in \{1, n\}$), which means that the prescribed rotation values can be directly assigned to the boundary degrees of freedom, as classically done in Galerkin methods.

The equation system obtained with (27)-(31) has in total $n+1$ unknowns, namely the n coefficients of $\hat{\boldsymbol{\varphi}}$ and the constant w_0. With the proposed collocation scheme, using $\overline{\xi}_i^{(3)}$ as collocation points, we naturally obtain $n+1$ equations, resulting in a square system of equations. As can be seen in equation (24), the Greville abscissae related to the k-th derivative space are defined for $p \geq k+1$. Accordingly, polynomial degrees $p \geq 4$ can be used for the presented formulation, while $p \geq 5$ had to be used for the collocation approach in terms of the bending displacement in [16].

A common problem of numerical methods based on the standard form of the Timoshenko beam problem (7)-(8) is shear locking, which can appear if the discrete spaces of the two primal variables are not compatible. In [7, 4] locking-free mixed collocation formulations have been proposed for Timoshenko beams and spatial rods. In such formulations, a third variable, typically a force variable like shear stress, is considered as an additional unknown, resulting in a larger system of equations to be solved. Instead, the formulation proposed in the present paper is locking-free by construction since there is only one variable to be discretized. Furthermore, having only one unknown variable significantly reduces the size of the discrete system to be solved. For the same "mesh", i.e., knot vector and polynomial degree, we have only half the number of degrees of freedom compared to a standard formulation and only one third compared to a mixed formulation.

4 Numerical tests

In this section, we test the proposed method on different benchmark problems with analytic solutions. We first demonstrate that the approach can handle arbitrary boundary conditions and then study the error convergence for the various variables of interest. For the convergence study we compare two cases, one of a thick beam and one of very thin beam in order to demonstrate that the formulation is locking-free.

4.1 Study on different boundary conditions

In order to test the imposition of boundary conditions, we consider different cases, namely a clamped beam ($w(0) = \varphi(0) = w(l) = \varphi(l) = 0$), a simply supported beam ($w(0) = M(0) = w(l) = M(l) = 0$), and a cantilever beam ($w(0) = \varphi(0) = M(l) = Q(l) = 0$). The geometrical and material parameters are as follows: Length $l = 1$, rectangular cross-section of width $b = 0.1$ and thickness $t = 0.01$, Young's modulus

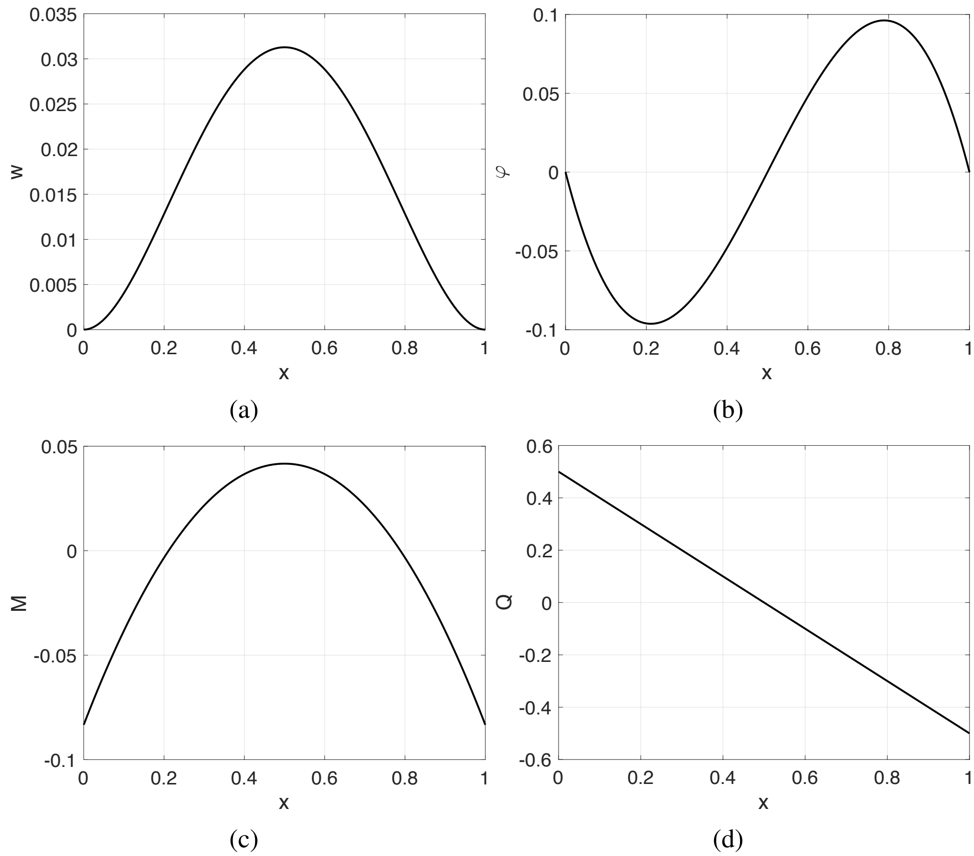

Figure 2: Clamped beam with constant load. Results for (a) displacement w, (b) rotation φ, (c) bending moment M, and (d) shear force Q.

$E = 10^7$, Poisson's ratio $\nu = 0.2$, and shear correction factor $\alpha = 5/6$. A constant load $f = 1$ is applied, such that the analytical solution in all cases is of fourth order for the displacement and of third order for the rotation.

For analysis, we use only one element with $p = 4$, which is the minimum degree according to equation (24). Accordingly, the numerical solutions are expected to exactly represent the analytical solutions. Figures 2-4 show the obtained results for the different variables of interest (w, φ, M, Q) for the three different cases. In all cases, the analytical solutions are obtained exactly up to machine precision.

4.2 Convergence study

In this study, we consider a sinusoidal load function and investigate the error convergence for the various variables of interest. The beam is supported by a slider support at the left end and clamped at the right end, which means that the integral in equation (28) is not vanishing. The load function is given by $f(x) = 16\,\pi^4 \cos(2\pi x)$,

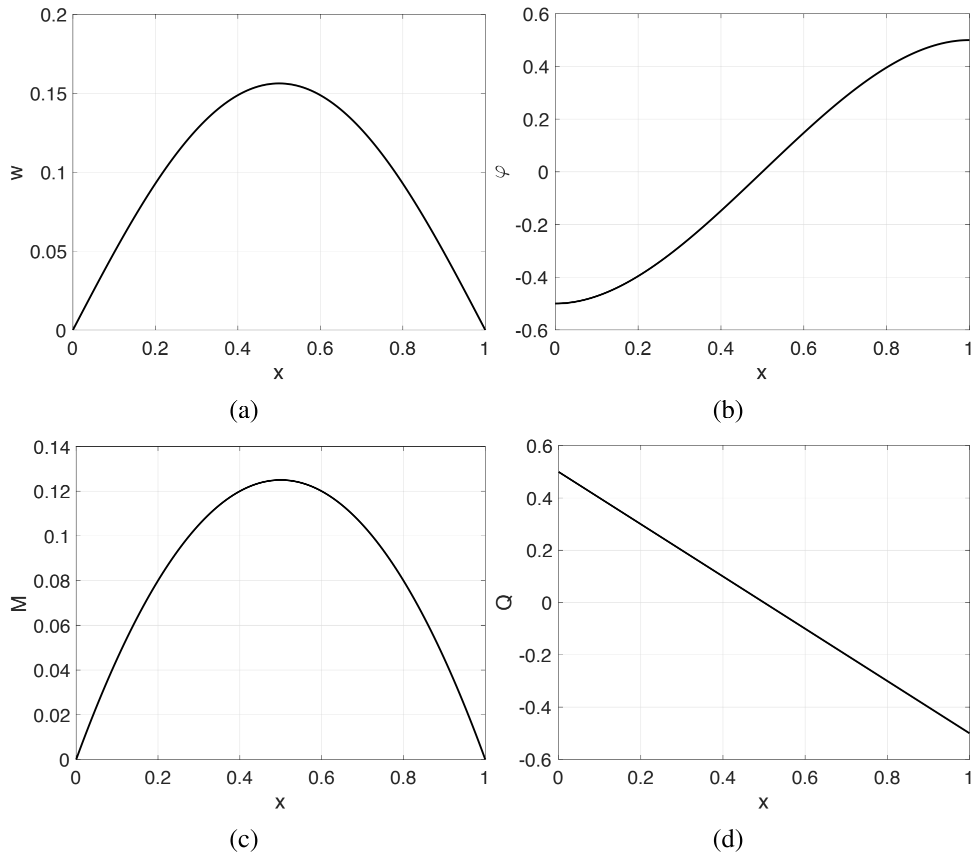

Figure 3: Simply supported beam with constant load. Results for (a) displacement w, (b) rotation φ, (c) bending moment M, and (d) shear force Q.

and the problem setup is shown in Figure 5. The analytical solutions for displacement, rotation, bending moment, and shear force are given by

$$w(x) = \frac{1}{K_b}(\cos(2\pi x) - 1) + \frac{1}{K_s}\left(4\pi^2\cos(2\pi x) - 4\pi^2\right) \tag{32}$$

$$\varphi(x) = \frac{1}{K_b}2\pi\sin(2\pi x) \tag{33}$$

$$M(x) = 4\pi^2\cos(2\pi x) \tag{34}$$

$$Q(x) = -8\pi^3\sin(2\pi x) \tag{35}$$

The geometrical and material parameters are the same as in the previous study, apart from the thickness, where we consider two different values, namely $t = 10^{-1}$ and $t = 10^{-4}$, corresponding to a thick and a very thin beam. We perform convergence studies for polynomial degrees $p = 4, 5, 6, 7, 8$ and evaluate the errors in the L^2-norm for all variables. The results are shown in Figure 6 for the thick beam and in Figure 7 for the thin beam. Very good convergence for all variables can be observed in both cases, with dashed reference lines indicating the convergence order. In

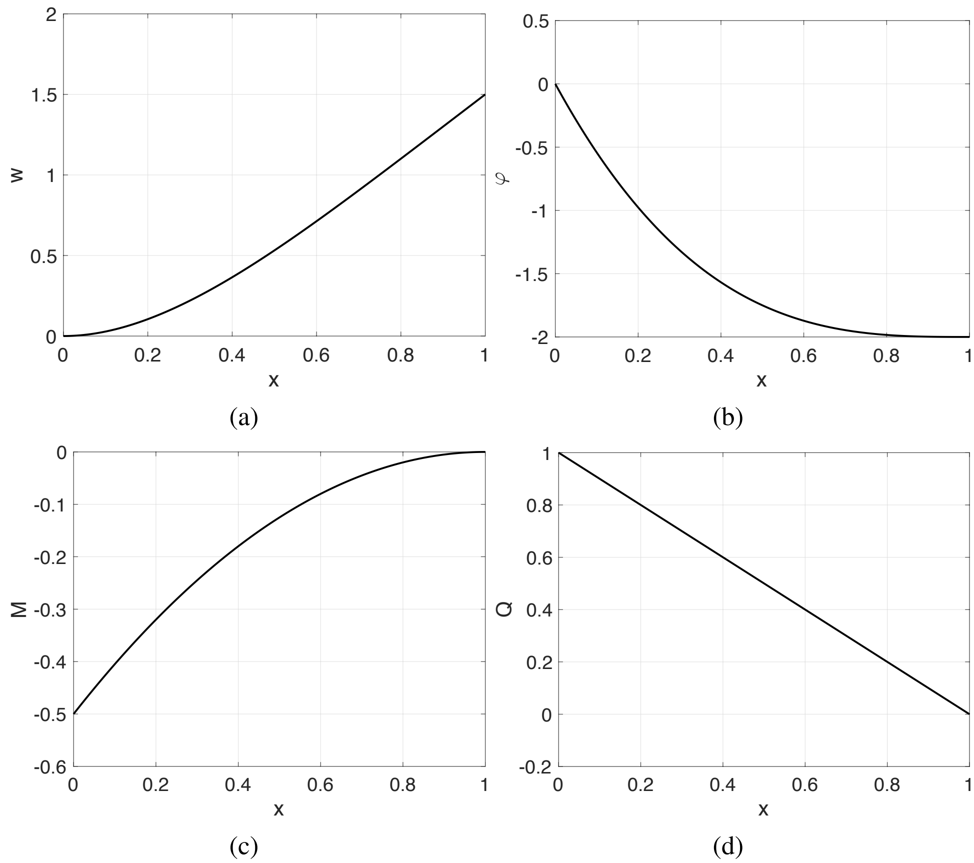

Figure 4: Cantilever beam with constant load. Results for (a) displacement w, (b) rotation φ, (c) bending moment M, and (d) shear force Q.

particular, it can be seen that the results are practically identical for the thick and thin beams, confirming that this formulation is fully locking-free. Moreover, the errors are almost identical for all different variables. The same characteristics have been observed also for collocation of the single-variable formulation based on the bending displacement [16]. However, the results for a given polynomial degree are found to be superior in the present formulation than in [16], in terms of both the convergence rate and the absolute values of the error. In particular, it can be observed that the results obtained with the present formulation for a polynomial degree p match closely the results for $p + 1$ with the collocation approach in [16].

5 Conclusions

We have presented a reformulation of the Timoshenko beam problem with the rotation as primal variable. The resulting differential equation is of third order with an additional unknown constant, which is the integration constant for the displace-

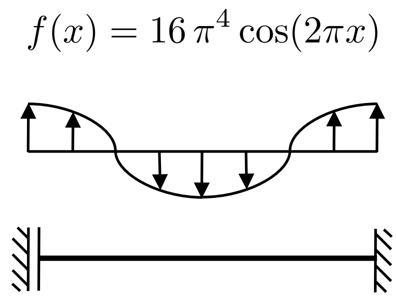

Figure 5: Beam with sinusoidal load. Slider support on the left boundary, clamped on the right boundary.

ment. With the classical boundary conditions of Timoshenko beams, the problem is complete and well posed. Moreover, we have derived an isogeometric collocation scheme to solve this problem numerically. The formulation is completely locking-free and, at the same time, involves only half the number of unknowns compared to a standard formulation and only one third compared to a mixed formulation. Numerical tests confirmed the performance and the locking-free behavior of the method. As future work, we plan the extension to frame structures and to vibration problems.

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7 Compliance with Ethical Standards

The authors declare that they have no conflict of interest.

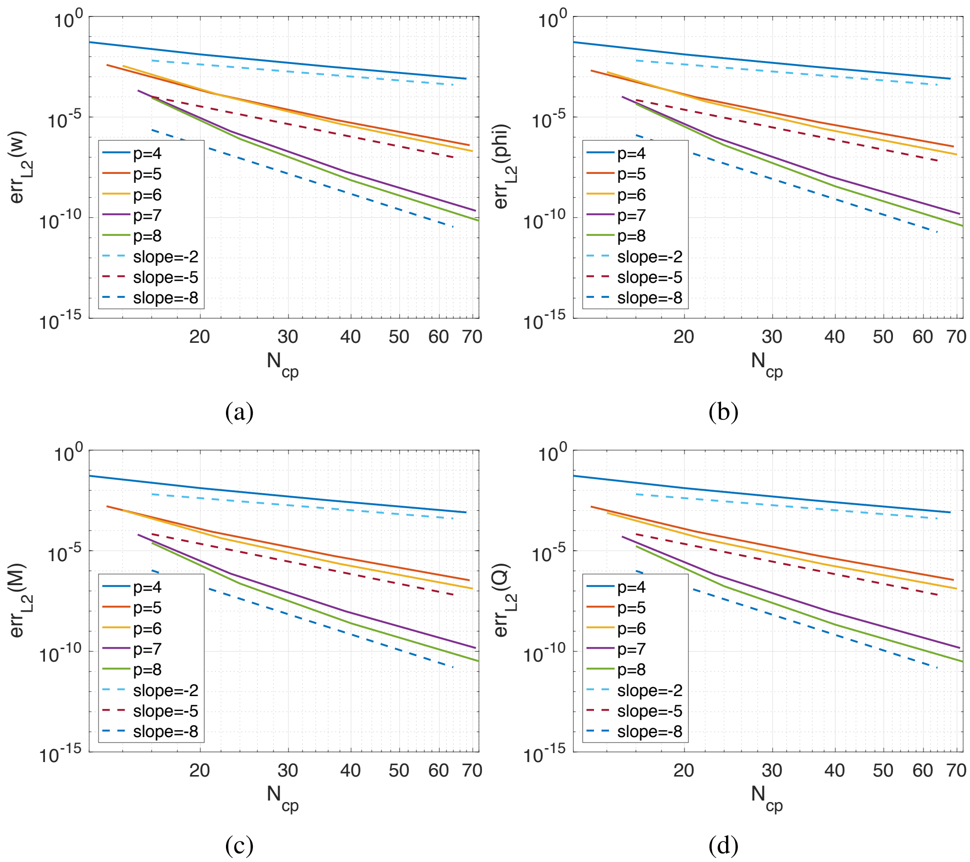

Figure 6: Convergence studies for a *thick* beam, $t = 10^{-1}$. Error in the L^2-norm for (a) displacement w, (b) rotation φ, (c) bending moment M, and (d) shear force Q.

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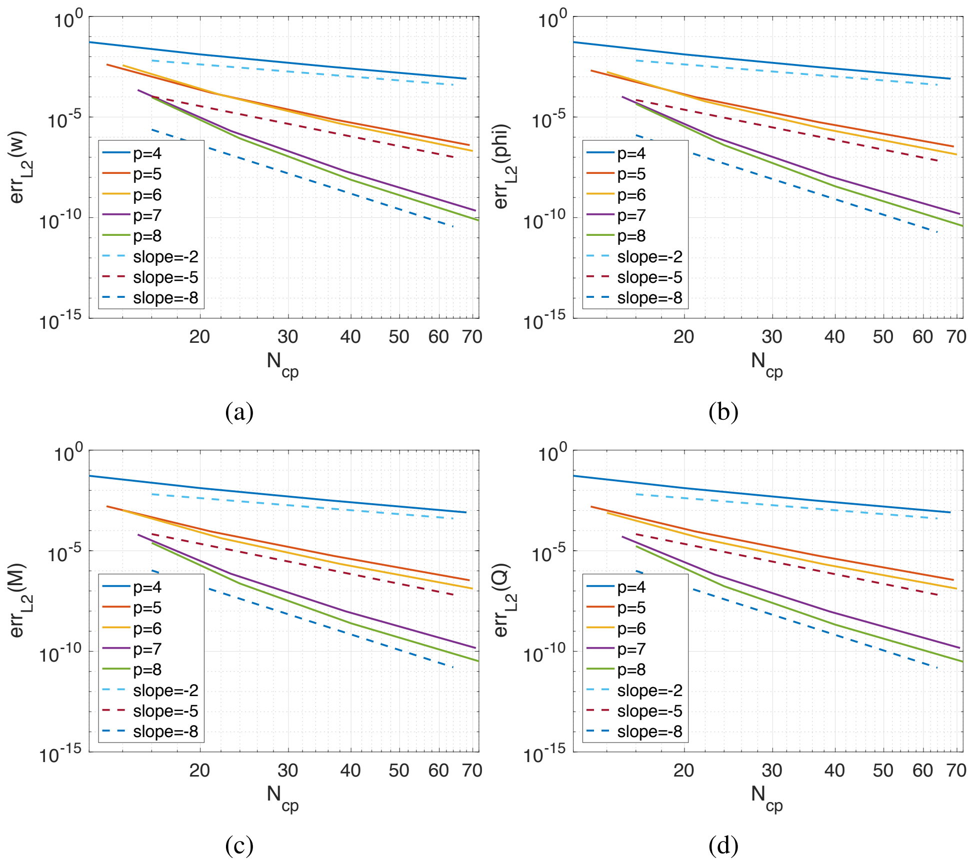

Figure 7: Convergence studies for a *thin* beam, $t = 10^{-4}$. Error in the L^2-norm for (a) displacement w, (b) rotation φ, (c) bending moment M, and (d) shear force Q.

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