A maritime inventory routing problem: Discrete time formulations and valid inequalities

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Abstract

A single product maritime inventory routing problem (MIRP) in which the production and consumption rates vary over the planning horizon is studied. The problem includes a heterogeneous fleet and multiple production and consumption ports with limited storage capacity.

Two discrete time formulations are developed. An original model is reformulated and appear as a fixed charge network flow model. Mixed integer sets arising from the decomposition of the formulations are identified. Several lot-sizing relaxations are derived for the formulations and used to establish valid inequalities to strengthen the proposed formulations.

So far, the derivation of models and valid inequalities for MIRPs has mainly been inspired by the developments in the routing community. Here, we have developed a new model and new valid inequalities and generalized existing ones for MIRPs based on recent advances from the lot-sizing literature.

Considering a set of instances based on real data, a computational study is conducted to test the formulations and the effectiveness of the inclusion of valid inequalities. By using a branch and bound scheme based on the strengthened fixed charge network formulation most of the instances with up to sixty time periods are solved to optimality.

Keywords: Inventory routing, maritime transportation, mixed integer linear formulation, lot-sizing relaxations.

1 Introduction

Maritime transportation is a major mode of transportation covering more than 80% of the world trade by volume, UNCTAD [31]. Large quantities are transported over long distances, and often

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inventories exist at the loading or discharge ports of the sailing legs. When one actor or cooperating actors in the maritime supply chain have the responsibility of both the transportation of goods and the inventories at the ports, the underlying planning problem is a maritime inventory routing problem (MIRP). Such problems are very complex, but a modest improvement in the fleet utilization and loading/discharge quantities can translate into large increase in profit due to a capital intensive industry. This means that there is a great potential and need for research in the area of MIRPs.

The problem analyzed in this paper is a single product MIRP. The product is produced at loading (production) ports and consumed at discharge (consumption) ports. It is possible to store the product in inventories with time dependent capacities at both types of ports. The production and consumption rates are deterministic but may vary over the planning horizon. There are berth capacities at the ports, limiting the number of ships that can load or discharge at the same time. A heterogeneous fleet of ships is used to transport the product. Each ship has a given capacity, speed, and loading/discharge rate. The ships can wait outside a port before entering for a loading or discharge operation. A ship can both load and discharge at multiple ports in succession. The initial position and load on board each ship is known at the beginning of the planning horizon. The sailing costs, waiting costs and port costs are all ship dependent. The planning problem is to design routes and schedules for the fleet that minimize the transportation and port costs and determine the load or discharge quantity at each port visit without exceeding the storage capacities. Depending on the segment the fleet is operating in, the typical planning period spans from one week up to several months.

Maritime inventory routing problems have achieved increasing attention in the literature the last decade; see the surveys on MIRPs in Andersson et al. [3] and Christiansen and Fagerholt [5] and the general reviews on ship routing and scheduling by Christiansen et al. [7] and Christiansen et al. [8]. Most of the published contributions are based on real cases from the industry due to a demand for support when taking complex routing and inventory management decisions. Similar to the problem analyzed here, many of the studies describe single product MIRPs; see for instance Christiansen [4] and Flatberg et al. [12] considering ammonia supply chains, Furman et al. [13] focusing on the transportation of oil products and Grønhaug et al. [16] discussing liquefied natural gas (LNG) distribution. However, several cases are described in the literature where multiple products need to be taken into account; see for instance Al-Khayyal and Hwang [1], Christiansen et al. [6], Rakke et al. [23], Ronen [24], and Siswanto et al. [28].

As discussed in both Andersson et al. [3] and Song and Furman [29], most combined maritime routing and inventory management problems described in the literature are a particular version of the MIRP and tailor-made methods are developed to solve the problem. These methods are often based on heuristics or decomposition techniques. The choice of these solution approaches might be explained by the high complexity of real MIRPs and the opportunity to utilize the special structure of the problem. However, constant hardware developments combined with the theoretical advances in optimization techniques have produced optimization solvers capable of handling increasingly larger instances. Currently, it is possible to obtain optimal or near optimal solutions to small real instances occurring in maritime transportation problems using commercial solvers, see Agra et al. [2], and Sherali and Al-Yakoob [26, 27]. It is well-known that the choice of mathematical formulation for mixed integer programming problems is of crucial importance to efficiently solve a problem, Nemhauser and Wolsey [18]. This makes the study of the mathematical formulation a key issue to solve larger MIRPs.

The study of valid inequalities for mixed integer sets and the derivation of extended formulations is currently receiving large attention both in solving routing and lot-sizing problems. However, relatively little work has been done on applying these techniques to maritime transportation problems. Sherali et al. [25] include valid inequalities in order to strengthen the formulations of an oil products transportation problem, and Persson and Göthe-Lundgren [19] develope valid inequalities within a column generation approach for a combined MIRP and production scheduling problem. Also, Grønhaug et al. [16] include valid inequalities to improve the path flow formulation presented for an LNG inventory routing problem. Agra et al. [2] develop strong mixed integer formulations for a short sea fuel oil distribution problem. Finally, Song and Furman [29] present valid inequalities for MIRPs including several practical constraints for solving problems in different shipping segments. Even though Savelsbergh and Song [30] do not handle a MIRP, their inventory routing problem has many parallels to the MIRP studied in this paper and is relevant due to the formulation and valid inequalities presented.

The objective of this research has been to study a general MIRP with time varying production and consumption rates and to develop tight mixed integer linear programming formulations for the problem. Therefore the paper starts with a formulation called the original formulation for the MIRP studied, and then a stronger formulation which is a fixed charge network flow formulation (FCNF) is presented. In addition, valid inequalities for the problem are developed that are based on known families of valid inequalities from the lot-sizing literature. Several of these valid inequalities can potentially be used for other inventory routing problems and in tailor-made solution approaches such as column generation to solve even larger instances than those presented here. Research on models and valid inequalities for inventory routing problems has mainly come from the routing community. Now, we develop a new model and valid inequalities for the MIRP from the lot-sizing theory.

The remainder of the paper is organized as follows. Section 2 presents the two alternative mixed integer linear formulations for the MIRP. In Sections 3 and 4 several mixed integer relaxations are derived for the formulations. These relaxations are used to develop valid inequalities to strengthen the proposed formulations. Section 5 presents the computational study. Some concluding remarks follow in Section 6.

2 Problem formulations

To formulate the problem as a mixed integer linear program, a number of modeling decisions have been made. The first consideration is whether to work with continuous or discrete time periods. Continuous time models can be found in the literature for the MIRP when the production and/or consumption rates are considered given and fixed during the planning horizon; see for instance Christiansen [4], Al-Khayyal and Hwang [1], and Siswanto et al. [28]. In Ronen [24], Grønhaug and Christiansen [15], Grønhaug et al. [16], Engineer et al. [11], Furman et al. [13], Rakke et al. [23], and Song and Furman [29] discrete time models are developed to overcome the complicating factors with variable production and consumption rates. Since both production and consumption rates may vary over the planning horizon in the problem described in this paper, discrete time formulations are proposed. It is therefore assumed that the waiting time, the time for loading and discharge and the sailing times can be expressed as an integer multiple of a basic time period. The length of the time period depends on the actual shipping segment.

In each time period, a ship can either be waiting, operating in port (loading or discharging), or sailing. In the following, we will use the terms *operating in port* or just *operating* for loading and discharging. It does not include any waiting or sailing. Two assumptions are made: i) a ship does not visit a port without carrying out a loading/discharge operation, and ii) waiting always takes place on arrival at a port before any port operations start. The first is natural while the second can in certain not very likely circumstances result in a worse optimal solution. We discuss how the models can be adapted if these assumptions are dropped at the end of Section 2. The assumptions imply that if a ship operates (loads or discharges) in a port in one time period, it can either continue to operate in that port or sail to another port in the next time period. It cannot wait in a port and sail to another port immediately after. This also means that if a ship waits outside a port in one time period, it can either continue waiting or start operating in the port in the next time period, but it cannot sail to another port before it has operated. When a ship has started operating in a port, it continues until it starts to sail. This means that it is not possible to wait for one or several time periods in a port after the loading/discharging has started.

The movement of a ship is illustrated in the time expanded network in Figure 1. The ship starts at its initial position O and sails to Port 1. At Port 1 the ship operates for two periods

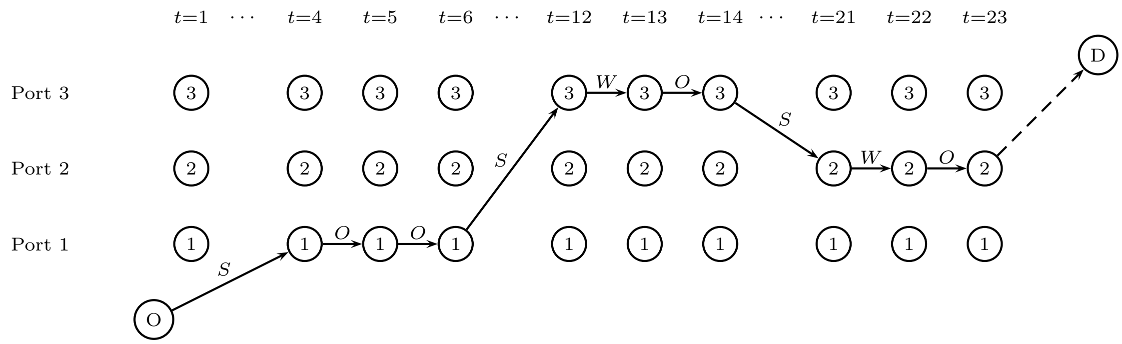

Figure 1: Example of the movement of a ship in a time expanded network. The arc labels are O for operating, W for waiting and S for sailing.

(periods 4 and 5) before sailing to Port 3 where it waits for one period before operating. The ship then sails to Port 2 where it waits and operates before it ends its schedule. For modeling purposes it is assumed that the ship then sails to an artificial end node D. The sailing to this node is marked with a dashed line in Figure 1. Each path through the network defines a *schedule* for the ship. A schedule consists of a geographical route, i.e. a sequence of ports, and the time periods when the ship operates at the ports.

In Section 2.1 a mixed integer linear formulation of the problem is given. This formulation has some similarities in the definitions of arc, quantity and load variables to other MIRP formulations. However, here the port operations are modeled in more detail than can be found in several other published discrete time MIRP models where the loading and discharge are assumed to take one time period or a given number of time periods independently of the quantity loaded/discharged; see for instance Song and Furman [29] and Grønhaug and Christiansen [15]. This means that the proposed models in this paper fit short sea shipping instances with long loading and discharge times relatively to the sailing times. The first model is called the original formulation. This formulation is then reformulated as a fixed charge network flow (FCNF) model in Section 2.2. The main difference between the models can be found in the precision of how the load on each ship is modeled. Some advantages with a FCNF formulation are that it leads to a tighter linear programming relaxation, and the formulation comes from an established literature with known families of valid inequalities.

2.1 Original formulation

To model the problem as a mixed integer linear program, the following notation is introduced

Sets

N^P	set of production ports with indices i and j,
N^D	set of consumption ports with indices i and j,
N	set of production and consumption ports with indices i and j, $N = N^P \cup N^D$,
T	set of time periods with index t,
V	set of ships with index v.

Parameters

B_{it}	berth capacity in number of ships at port i in time period t,
C^T_{ijv}	sailing cost from port i to port j with ship v,
C^W_v	waiting cost for ship v per time period,
C^P_{iv}	port cost at port i for ship v per time period,
D_{it}	consumption at port i in period t,
P_{it}	production at port i in period t,
K_v	capacity of ship v,
L^0_v	initial load on board ship v,
Q_v	upper bound on the amount ship v loads/discharges per time period,
$\overline{S}_{it}$	upper bound on the inventory level at port i at the end of time period t,
$\underline{S}_{it}$	lower bound on the inventory level at port i at the end of time period t,
S^0_i	inventory level in port i at the beginning of the planning horizon,
$o(v)$	initial position for ship v,
$d(v)$	artificial end node for ship v,
T_{ijv}	sailing time from port i to port j for ship v.

Variables

o_{ivt}	1 if ship v operates(loads/discharge) in port i in time period t, 0 otherwise,
x_{ijvt}	1 if ship v sails from port i to port j, starting in time period t, 0 otherwise,
w_{ivt}	1 if ship v is waiting outside port i in time period t, 0 otherwise,
l_{vt}	load on board ship v at the end of time period t,
q_{ivt}	quantity loaded/discharged in time period t at port i by ship v,
s_{it}	inventory level in port i at the end of time period t.

Only variables associated with relevant nodes and arcs are defined, and the network construction is done implicitly within the model. The problem can now be formulated as follows

$$\min \sum_{v\in V} \sum_{i\in N\cup\{o(v)\}} \sum_{j\in N\cup\{d(v)\}} \sum_{t\in T} C^T_{ijv} x_{ijvt} + \sum_{v\in V}\sum_{i\in N}\sum_{t\in T} C^P_{iv} o_{ivt} + \sum_{v\in V}\sum_{i\in N}\sum_{t\in T} C^W_v w_{ivt}, \quad (1)$$

subject to:

$$\sum_{j\in N\cup\{d(v)\}}\sum_{t\in T} x_{o(v)jvt} = 1, \qquad \forall v \in V, \tag{2}$$

$$\sum_{i\in N\cup\{o(v)\}}\sum_{t\in T} x_{id(v)vt} = 1, \qquad \forall v \in V, \tag{3}$$

$$\sum_{j\in N\cup\{o(v)\}} x_{jiv,t-T_{jiv}} + w_{iv,t-1} + o_{iv,t-1} = \sum_{j\in N\cup\{d(v)\}} x_{ijvt} + w_{ivt} + o_{ivt}, \qquad \forall v \in V, i \in N, t \in T, \tag{4}$$

$$o_{iv,t-1} \leq \sum_{j\in N\cup\{d(v)\}} x_{ijvt} + o_{ivt}, \qquad \forall v \in V, i \in N, t \in T, \tag{5}$$

$$o_{iv,t-1} \geq \sum_{j\in N\cup\{d(v)\}} x_{ijvt}, \qquad \forall v \in V, i \in N, t \in T, \tag{6}$$

$$\sum_{v\in V} o_{ivt} \leq B_{it}, \qquad \forall i \in N, t \in T, \tag{7}$$

$$0 \leq q_{ivt} \leq Q_v o_{ivt}, \qquad \forall v \in V, i \in N, t \in T, \tag{8}$$

$$s_{i,t-1} + \sum_{v\in V} q_{ivt} = D_{it} + s_{it}, \qquad \forall i \in N^D, t \in T, \tag{9}$$

$$s_{i,t-1} + P_{it} = \sum_{v\in V} q_{ivt} + s_{it}, \qquad \forall i \in N^P, t \in T, \tag{10}$$

$$\underline{S}_{it} \leq s_{it} \leq \overline{S}_{it}, \qquad \forall i \in N, t \in T, \tag{11}$$

$$s_{i0} = S_i^0, \qquad \forall i \in N, \tag{12}$$

$$l_{v,t-1} + \sum_{i\in N^P} q_{ivt} - \sum_{i\in N^D} q_{ivt} - l_{vt} = 0, \qquad \forall v \in V, t \in T, \tag{13}$$

$$0 \leq l_{vt} \leq K_v, \qquad \forall v \in V, t \in T, \tag{14}$$

$$l_{v0} = L_v^0, \qquad \forall v \in V, \tag{15}$$

$$x_{ijvt} \in \{0,1\}, \qquad \forall v \in V, i \in N \cup \{o(v)\}, \; j \in N \cup \{d(v)\}, t \in T, \tag{16}$$

$$o_{ivt}, w_{ivt} \in \{0,1\}, \qquad \forall v \in V, i \in N, t \in T. \tag{17}$$

The objective function (1) is the sum of all sailing costs, operating costs and waiting costs. Constraints (2) and (3) ensure that each ship starts and finishes a schedule. Note that a ship can be idle the whole planning horizon by sailing directly from the initial node to the artificial end node. Constraints (4) are the ship flow conservation constraints at each port in each period. Constraints (5) prevent a ship from waiting at a port after an operation, while constraints (6) make sure that a ship can only sail after operating. The berth capacities are stated in constraints (7). Constraints (8) ensure that a ship cannot load/discharge if it is not in operating mode and defines the upper bound on the quantity loaded/discharged. The inventory balances for consumption and production ports are expressed in constraints (9) and (10) respectively. Constraints (11) and (12) define the upper and lower inventory limits and the initial inventory. Constraints (13), (14), and (15) guarantee the equilibrium of the quantity on board the ship. All binary variable restrictions are stated in constraints (16) and (17).

The connections between the variables in the formulation are shown in Figure 2. Ship v arrives at port i at the start of time period 2 and waits for one period before starting to discharge. The figure shows the movement of the ship through the x, w, and o variables, the quantity discharged from the ship, the q variables, the external demand, the D parameters, and the inventory levels at the discharge port, the s variables. Note that q_{ivt} only can be positive if the ship is in loading/discharge mode, i.e $o_{ivt} = 1$, q_{iv1} and q_{iv2} are therefore zero and not marked with bold arrows.

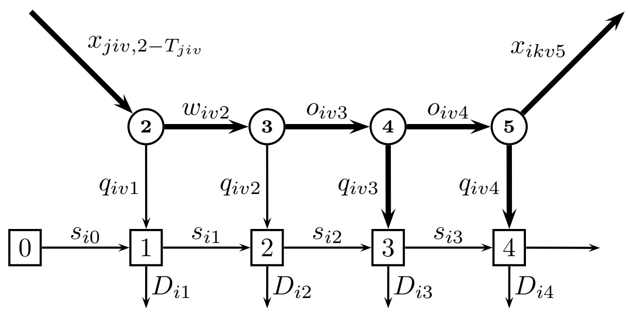

Figure 2: Discharge operation at port i for ship v.

2.2 Fixed charge network flow formulation

As the linear programming bounds provided by formulation (1) – (17) are weak, it is natural to try to strengthen the formulation. One way to do this is provided by the observation that the problem can be viewed as a single commodity fixed charge network flow (FCNF) problem in which the commodity is supplied externally at loading ports, flows along the arcs corresponding to the ships' routes before being deposited at the discharge ports where it can satisfy the external demands.

This cannot be modeled in the network similar to that presented in Figure 2 since there it is possible for a ship to wait throughout its visit to a port and not operate at all. Thus we have chosen to model the problem as a single commodity FCNF problem. This allows us to take advantage of known inequalities for such problems. To keep the FCNF structure, an extended network is needed in which each arc representing either waiting or operating is split into one arc representing waiting and another arc representing operating. In addition each node in the upper layer in Figure 2 is split into one node in which a ship can enter the port and one node in which it can depart from the port.

Since a ship only can depart from a port after an operation, it is also necessary to distinguish between the first time the ship operates during each call to a port and the following operating periods. Thus new nodes and arcs are introduced along with the corresponding binary arc and flow variables in order to model the operations of the ship: o^A_{ivt} indicates whether ship v starts to operate at port i in period t and o^B_{ivt} indicates the succeeding operations at that port. Keep in mind that a ship has to load or discharge continuously in port when the ship has first started the port operation.

Figure 3 illustrates the extended network corresponding to the situations shown in Figure 2. The ship has arrived at port i at the beginning of period 2, waits in period 2, starts operating (unloading) in period 3, continues operating in period 4 and then leaves for port k in period 5. The ship can only depart from the second layer, so it is forced to operate at least once.

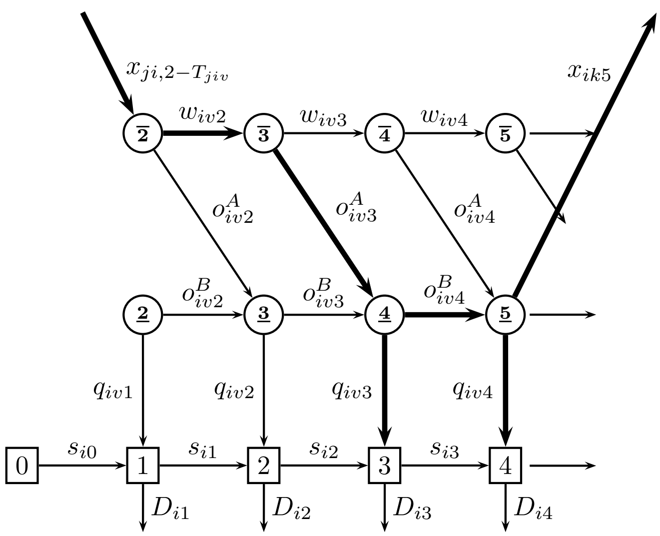

Figure 3: Discharge operation at port i for ship v in the extended network.

With the new o^A_{ivt}, o^B_{ivt} variables, the ship flow conservation constraints (4) now can be formulated as

$$\sum_{j \in N \cup \{o(v)\}} x_{jiv,t-T_{jiv}} + w_{iv,t-1} = w_{ivt} + o^A_{ivt} \qquad \forall v \in V, i \in N, t \in T, \tag{18}$$

$$o^A_{iv,t-1} + o^B_{iv,t-1} = o^B_{ivt} + \sum_{j \in N \cup \{d(v)\}} x_{ijvt}, \qquad \forall v \in V, i \in N, t \in T, \tag{19}$$

$$o^A_{ivt}, o^B_{ivt} \in \{0, 1\}, \qquad \forall v \in V, i \in N, t \in T, \tag{20}$$

and together with constraints (2), (3), (16), and (17) they describe the movement of the ships through the extended network given in Figure 3.

The coordination between the path of the ships and the loading or discharge of the product in a port is provided by the constraints

$$o_{ivt}^A + o_{ivt}^B = o_{ivt}, \qquad \forall v \in V, i \in N, t \in T, \tag{21}$$

which also provide the link between the old and the new operating variables.

To complete the fixed charge network flow formulation, the variable l_{vt} and the constraints (13) – (15) describing the quantity on board the ships are replaced by flow variables and flow conservation constraints. A flow variable is defined for each arc in the extended network.

- f_{ijvt}^X load on board ship v when traveling from port i to port j, leaving at time period t,
- f_{ivt}^{OA} load on board ship v when starting to operate at port i in time period t when it has not operated in time period $t-1$,
- f_{ivt}^{OB} load on board ship v before continuing to operate at port i in time period t after an operation in time period $t-1$,
- f_{ivt}^W load on board ship v while waiting during time period t at port i.

Hence, f_{ijvt}^X, f_{ivt}^{OA}, f_{ivt}^{OB} and f_{ivt}^W represent the flow on the arcs defined by the binary variables x_{ijvt}, o_{ivt}^A, o_{ivt}^B and w_{ivt}, respectively. This leads to the flow conservation constraints:

$$\sum_{j \in N \cup \{o(v)\}} f_{jiv,t-T_{jiv}}^X + f_{iv,t-1}^W = f_{ivt}^W + f_{ivt}^{OA} \qquad \forall v \in V, i \in N, t \in T, \tag{22}$$

$$f_{iv,t-1}^{OA} + f_{iv,t-1}^{OB} + q_{iv,t-1} = f_{ivt}^{OB} + \sum_{j \in N \cup \{d(v)\}} f_{ijvt}^X, \quad \forall v \in V, i \in N^P \cup \{o(v)\}, t \in T, \tag{23}$$

$$f_{iv,t-1}^{OA} + f_{iv,t-1}^{OB} - q_{iv,t-1} = f_{ivt}^{OB} + \sum_{j \in N \cup \{d(v)\}} f_{ijvt}^X, \quad \forall v \in V, i \in N^D \cup \{o(v)\}, t \in T, \tag{24}$$

$$f_{o(v)jvt}^X = L_v^0 x_{o(v)jvt}, \qquad \forall v \in V, j \in N \cup \{d(v)\}, t \in T, \tag{25}$$

and the variable upper bound and nonnegativity constraints:

$$0 \le f_{ijvt}^X \le K_v x_{ijvt} \qquad \forall v \in V, i \in N \cup \{o(v)\}, j \in N \cup \{d(v)\}, t \in T, \tag{26}$$

$$0 \le f_{ivt}^{OA} \le K_v o_{ivt}^A \qquad \forall v \in V, i \in N, t \in T, \tag{27}$$

$$0 \le f_{ivt}^{OB} \le K_v o_{ivt}^B \qquad \forall v \in V, i \in N, t \in T, \tag{28}$$

$$0 \le q_{ivt} \le Q_v o_{ivt} \qquad \forall v \in V, i \in N, t \in T, \tag{29}$$

$$0 \le f_{ivt}^W \le K_v w_{ivt} \qquad \forall v \in V, i \in N, t \in T. \tag{30}$$

The FCNF formulation is defined by (1) – (3), (7) – (12), (16), (17), and (18) – (30). We denote by $\mathbb{X}^{FCNF}$the set of feasible solutions of the FCNF formulation.

The original formulation can be related to the FCNF formulation as follows: (4) – (6) are replaced by (18) – (21) and (13) – (15) are replaced by (22) – (30). It can also be shown that constraints (4) – (6) and (13) – (15) are valid for the FCNF formulation, so that the FCNF formulation is stronger than the original formulation.

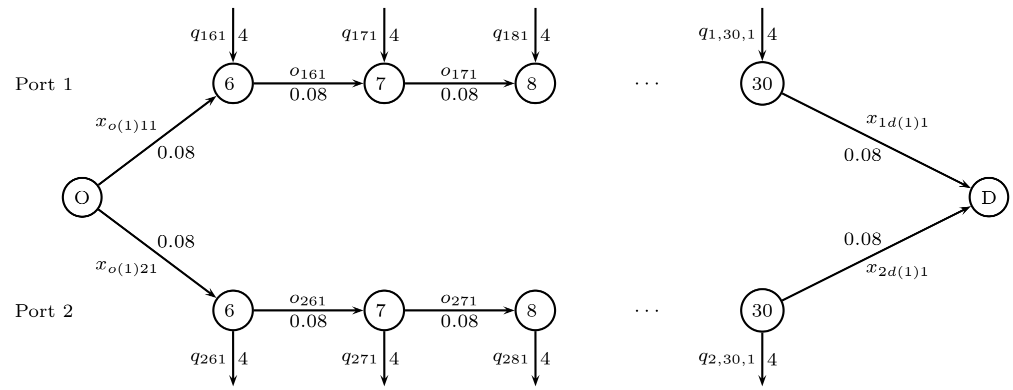

Figure 4: Solution of the linear relaxation using the original formulation. Next to each arc the variable and its value is represented.

Example 2.1 *Consider the following instance for* $T = \{1, \ldots, 30\}$, *with one loading port* $N^P = \{1\}$, *one discharge port* $N^D = \{2\}$, *and a single ship. Thus, we omit the index* v *from variables and parameters. Assume the initial position* $o(v)$ *coincides with Port* 1. *Let* $B_{it} = 1, \forall t \in T$, $C^T_{12} = C^T_{21} = 50$, $C^W = 1$, $C^P_1 = C^P_2 = 2$, $D_{2t} = 10, \forall t \in T$, $P_{1t} = 10, \forall t \in T$, $K = Q = 50$, $L^0 = 0$, $\underline{S}_{1t} = \underline{S}_{2t} = 0$ $\overline{S}_{1t} = \overline{S}_{2t} = 200$ $S^0_1 = 0, S^0_2 = 200$ $T_{12} = T_{21} = 5$.

The optimal solution has a total cost of 162. An optimal route is given by $x_{o(v),1,1} = x_{1,2,6} = x_{1,2,18} = x_{2,1,12} = x_{2,d(v),24} = 1$. *There are two loading operations, in periods 5 and 17, and two unloading operations, in periods 11 and 23, all of them at the maximum load/unload level of 50.*

Using the original formulation, the value of the linear relaxation is 12. This cost mainly comes from the port operations. The transportation costs are very low because the routing variables are $x_{o(v),1,6} = 0.08, x_{o(v),2,1} = 0.08$. *From period 6 to period 30, the ship simultaneously loads 4 units at port 1 and discharges 4 units at port 2. All the sailing variables between the two ports are zero. This happens because constraints* (13) *only ensure the equilibrium onboard the ship. There are no flow constraints linking each unit loaded at Port 1 to the same unit discharged at Port 2, see Figure 4.*

The linear relaxation of the FCNF formulation has value 160. In this case, there are many fractional routing variables (that for brevity we omit their values here) that ensure the connection between the two ports since the load flow constraints force any unit discharged at Port 2 to have been loaded at Port 1.

In Figure 5 the graph corresponding to loading port i and ship v is depicted. The two top layers model the ship operations while the third layer is for the port inventory. If there is more than one ship, then the two top layers must be replicated with one such network for each ship.

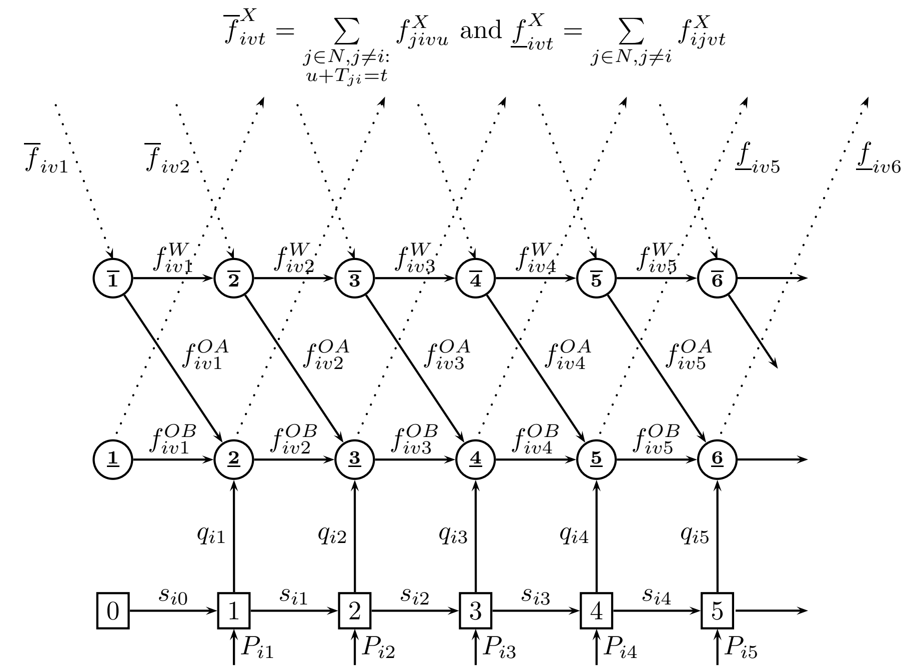

Figure 5: Example of the fixed charge network flow model at loading port i, only ship v is shown.

The aggregate arriving and departing flows $\overline{f}^X_{ivt}$ and $\underline{f}^X_{ivt}$ are introduced to ease the presentation.

Remark 2.2 *If the initial model assumptions are dropped, i.e. obliging a ship to operate at least once during a visit to a port and imposing that a ship only waits before arrival at a port, it suffices to replace the equality (21) by the inequality* $o_{ivt} \leq o^A_{ivt} + o^B_{ivt}$*. Now periods in which* $o^A_{ivt} + o^B_{ivt} = 1$ *and* $o_{ivt} = 0$ *are waiting periods, so the cost term* $C^W_v(o^A_{ivt} + o^B_{ivt} - o_{ivt})$ *must be added to the objective function. If a ship is forced to operate at least once, the constraint* $q_{ivt} \geq \underline{Q} o_{ivt}$ *is added where* $\underline{Q} > 0$ *is an appropriate minimum load/unload amount can be added.*

3 Strengthening the fixed charge network flow formulation

The FCNF formulation can be tightened by adding inequalities that are valid inequalities for mixed integer sets derived as relaxations of the FCNF formulation. In this section several such relaxations are identified while Section 5 shows how the addition of valid inequalities for these relaxations can be very important in solving the test instances. The relaxations can be grouped into two major types: mixed integer relaxations resulting from single row relaxations along with simple or variable bound constraints, such as knapsack sets or single row mixed integer sets, and lot-sizing relaxations which can be regarded as sets with more structure. The sets of valid inequalities for different relaxations may overlap. In Pochet and Wolsey [22] a comprehensive study of valid inequalities and reformulations for the mixed integer sets used in this paper is given. Some inequalities that are discussed in this section, such as knapsack inequalities, are also valid or can be easily adapted for the standard formulation.

3.1 Mixed integer relaxations

For each port, simple mixed integer relaxations are obtained from bounding the flow across cut-sets separating the given port from the remaining ports in the FCNF network.

Loading ports

The idea here is to look at the flow in and out of loading port i over a given time interval. Define the time interval $\mathtt{T} = [k, \ell] \subseteq T$. For each ship v, define a set $\mathtt{T}_v \subseteq \mathtt{T}$ representing a subset of the time periods in $\mathtt{T}$ in which ship v is assumed to operate at port i. Also define $\mathtt{T}_v^+ = \{t \in \mathtt{T}_v : t+1 \notin \mathtt{T}_v\}$ as the time periods in $\mathtt{T}$ followed immediately by the departure of ship v from port i and $\mathtt{T}_v^- = \{t \in \mathtt{T}_v : t-1 \notin \mathtt{T}_v\}$ as the time periods in $\mathtt{T}$ in which ship v starts operating at i.

Summing the flow conservation constraints (24) for loading port i over all ships $v \in V$ and time periods $t-1 \in \mathtt{T}_v$, gives

$$\sum_{v \in V} \sum_{t \in \mathtt{T}_v} q_{ivt} = \sum_{v \in V} \sum_{t \in \mathtt{T}_v} (f^{OB}_{iv,t+1} - f^{OB}_{ivt}) + \sum_{v \in V} \sum_{j \in N \cup \{d(v)\}} \sum_{t \in \mathtt{T}_v} f^{X}_{ijv,t+1} - \sum_{v \in V} \sum_{t \in \mathtt{T}_v} f^{OA}_{ivt}.$$

Using

$$\sum_{v \in V} \sum_{t \in \mathtt{T}_v} (f^{OB}_{iv,t+1} - f^{OB}_{ivt}) = \sum_{v \in V} \sum_{t \in \mathtt{T}_v^+} f^{OB}_{iv,t+1} - \sum_{v \in V} \sum_{t \in \mathtt{T}_v^-} f^{OB}_{ivt}$$

and nonnegativity of f^{OA}_{ivt} and f^{OB}_{ivt} gives

$$\sum_{v \in V} \sum_{t \in \mathtt{T}} q_{ivt} \le \sum_{v \in V} \left(\sum_{t \in \mathtt{T}_v^+} f^{OB}_{iv,t+1} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in \mathtt{T}_v} f^{X}_{ijv,t+1} + \sum_{t \in \mathtt{T} \setminus \mathtt{T}_v} q_{ivt} \right). \tag{31}$$

Summing the inventory constraints (10) over $\mathtt{T}$, and taking $\underline{S}_{it}$ as an under estimator of s_{it}, i.e. $s_{it} \ge \underline{S}_{it}$, it follows from (31) that

$$s_{ik} + \sum_{v \in V} \left(\sum_{t \in \mathtt{T}_v^+} f^{OB}_{iv,t+1} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in \mathtt{T}_v} f^{X}_{ijv,t+1} + \sum_{t \in \mathtt{T} \setminus \mathtt{T}_v} q_{ivt} \right) \ge \sum_{t \in \mathtt{T}} P_{it} + \underline{S}_{i,\ell-1}. \tag{32}$$

Using the variable upper bound constraints (26) – (30), inequalities (32) imply:

$$s_{ik} + \sum_{v \in V} \left(\sum_{t \in \mathtt{T}_v^+} K_v o^{B}_{iv,t+1} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in \mathtt{T}_v} K_v x_{ijv,t+1} + \sum_{t \in \mathtt{T} \setminus \mathtt{T}_v} Q_v o_{ivt} \right) \ge \sum_{t \in \mathtt{T}} P_{it} + \underline{S}_{i,\ell-1}. \tag{33}$$

Inequality (33) can be viewed as a continuous binary knapsack set $\{(s, y) \in \mathbb{R}^1_+ \times \{0,1\}^n : \sum_{j=1}^n a_j y_j \le b + s\}$, see Pochet and Wolsey [22].

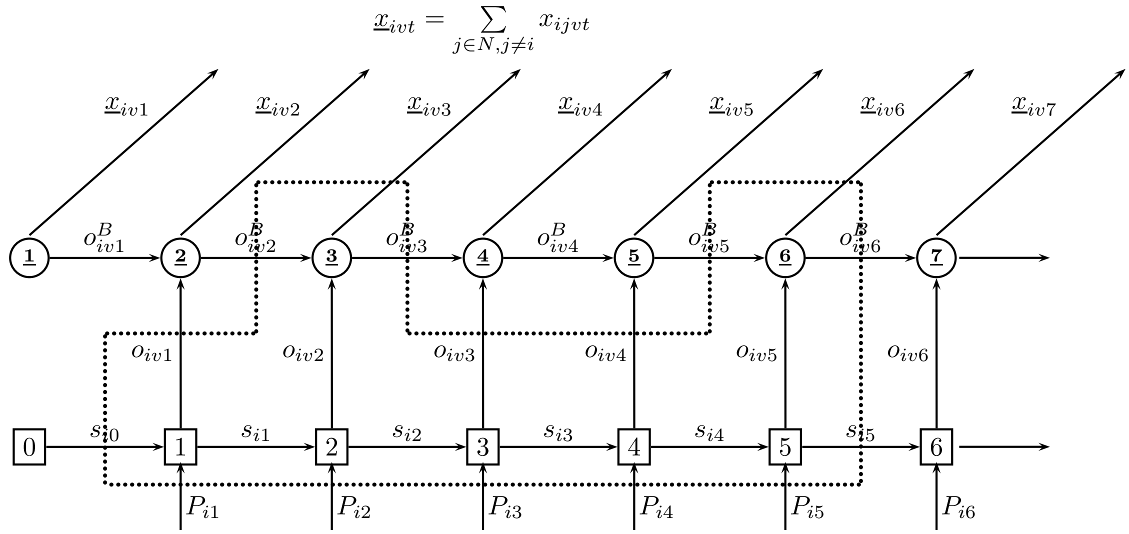

Figure 6: Mixed integer knapsack relaxation for ship v at loading port i.

Replacing s_{ik} by its upper bound $\overline{S}_{ik}$ gives knapsack sets. Valid inequalities for these knapsack sets are valid for $\mathbb{X}^{FCNF}$. Thus for arbitrary $Q > 0$, the following Chvátal-Gomory inequalities are valid for $\mathbb{X}^{FCNF}$:

$$\sum_{v\in V}\left(\sum_{t\in \mathtt{T}_v^+}\left\lceil\frac{K_v}{Q}\right\rceil o^B_{iv,t+1} + \sum_{j\in N\cup\{d(v)\}}\sum_{t\in \mathtt{T}_v}\left\lceil\frac{K_v}{Q}\right\rceil x_{ijv,t+1} + \sum_{t\in \mathtt{T}\setminus \mathtt{T}_v}\left\lceil\frac{Q_v}{Q}\right\rceil o_{ivt}\right) \geq \left\lceil\frac{\sum_{t\in\mathtt{T}} P_{it} + \underline{S}_{i,\ell-1} - \overline{S}_{ik}}{Q}\right\rceil. \quad (34)$$

In Section 5.1 appropriate values for parameter Q are considered.

Example 3.1 *Inequality* (34) *is derived for the situation illustrated in Figure 6. A loading port i, a single ship v, and a time interval $\mathtt{T} = [1, 5]$ are given. Taking $\mathtt{T}_v = \{2, 5\}$, implying that the ship can leave the port in time periods 3 or 6, one has by definition $\mathtt{T}_v^+ = \{2, 5\}$ and $\mathtt{T}_v^- = \{2, 5\}$. The ship has capacity $K_v = 110$ and loading rate $Q_v = 60$. Assume $S_i^0 = 90$, $\overline{S}_{i5} = 120$ and $P_{it} = 20$ for all $t \in T$. Choosing $Q = Q_v = 60$, inequality* (34) *gives*

$$2o^B_{iv3} + 2o^B_{iv6} + 2\underline{x}_{iv3} + 2\underline{x}_{iv6} + o_{iv1} + o_{iv3} + o_{iv4} \geq \left\lceil\frac{100 + 90 - 120}{60}\right\rceil = 2$$

where $\underline{x}_{ivt} = \sum_{j\in N, j\neq i} x_{ijvt}$.

Two special cases of inequalities (34) lead to simpler inequalities. First, taking $\mathtt{T}_v = \mathtt{T}$ implies $\mathtt{T}_v^+ = k$ and $\mathtt{T}\setminus\mathtt{T}_v = \emptyset$. Second, taking $\mathtt{T}_v = \emptyset$ implies $\mathtt{T}_v^+ = \emptyset$ and $\mathtt{T}\setminus\mathtt{T}_v = \mathtt{T}$. With $\overline{K} = \max\{K_v :$

$v \in V\}$ and $\overline{Q} = \max\{Q_v : v \in V\}$, the corresponding knapsack inequalities are:

$$\sum_{v\in V}\left(o^B_{iv,k+1} + \sum_{t\in \mathtt{T}}\sum_{j\in N\cup\{d(v)\}} x_{ijv,t+1}\right) \geq \left\lceil \frac{\sum_{t\in\mathtt{T}} P_{it} + \underline{S}_{i,\ell-1} - \overline{S}_{ik}}{\overline{K}} \right\rceil, \tag{35}$$

$$\sum_{v\in V}\sum_{t\in\mathtt{T}} o_{ivt} \geq \left\lceil \frac{\sum_{t\in\mathtt{T}} P_{it} + \underline{S}_{i,\ell-1} - \overline{S}_{ik}}{\overline{Q}} \right\rceil, \tag{36}$$

Note that all variables have binary coefficients in inequalities (35) and (36). Inequalities (35) impose a minimum number of ship departures and inequalities (36) impose a minimum number of loading periods at port i. These inequalities can also be generalized by aggregating over any nonempty subset of loading ports. Similar inequalities to (35) and (36) have been used for related problems, see Grønhaug et al. [16], Song and Furman [29], and Savelsbergh and Song [30].

Other inequalities can also be derived for the continuous binary knapsack set. Dividing (33) by $Q > 0$, one obtains:

$$\frac{s_{ik}}{Q} + \sum_{v\in V}\left(\sum_{t\in\mathtt{T}^+_v}\frac{K_v}{Q} o^B_{iv,t+1} + \sum_{j\in N\cup\{d(v)\}}\sum_{t\in\mathtt{T}_v}\frac{K_v}{Q} x_{ijv,t+1} + \sum_{t\in\mathtt{T}\setminus\mathtt{T}_v}\frac{Q_v}{Q} o_{ivt}\right) \geq \left(\sum_{t\in\mathtt{T}} P_{it} + \underline{S}_{i,\ell-1}\right)/Q.$$

Setting $y = \sum_{v\in V}\left(\sum_{t\in\mathtt{T}^+_v}\left\lceil\frac{K_v}{Q}\right\rceil o^B_{iv,t+1} + \sum_{j\in N\cup\{d(v)\}}\sum_{t\in\mathtt{T}_v}\left\lceil\frac{K_v}{Q}\right\rceil x_{ijv,t+1} + \sum_{t\in\mathtt{T}\setminus\mathtt{T}_v}\left\lceil\frac{Q_v}{Q}\right\rceil o_{ivt}\right)$, $s = s_{ik}/Q$ and $b = \left(\sum_{t\in\mathtt{T}} P_{it} + \underline{S}_{i,\ell-1}\right)/Q$, this becomes a basic-mip set of the form: $\{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : s + y \geq b\}$ for which the simple mixed integer rounding inequality is derived:

$$s + fy \geq f\lceil b\rceil \tag{37}$$

where $f = b - \lfloor b\rfloor$. For a given $Q > 0$ and $\mathtt{T}$, the separation problem for the inequalities (34) and (37) can be solved in polynomial time by finding a minimum capacity cut in a simple network similar to the one depicted in Figure 6, see Nemhauser and Wolsey [18] for more details.

Discharge ports

The simple mixed integer relaxations used to derive valid inequalities for loading ports, see Section 3.1, are based on ship arcs leaving a subgraph. For discharge ports the network structure is a little more complex since ship arcs entering a subgraph are used. This means that the subgraph includes all three layers of the network, see Figure 7, and the corresponding incident arcs.

Define the time interval $\mathtt{T} = [\ell, k] \subseteq T$ as before. To construct the subgraph for ship v, $\mathtt{T}$ is partitioned into three disjoint sets R^0_v, R^1_v, and R^2_v. For all $t \in \mathtt{T}$; if $t \in R^0_v$, node t from the lowest layer is included in the subgraph, if $t \in R^1_v$, node t and node $\underline{t+1}$ are included in the

$$\overline{x}_{ivt} = \sum_{j \in N \cup o(v), j \neq i : u - T_{ji} = t} x_{jivu}$$

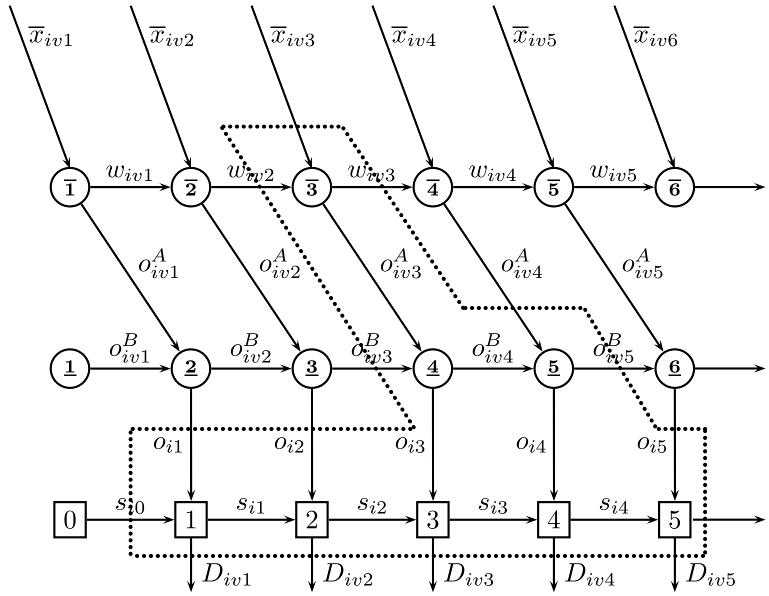

Figure 7: Cut for the fixed charge network flow formulation at discharge port i for ship v.

subgraph and if $t \in R_v^2$, node t, node $\underline{t+1}$, and node $\overline{t}$ are included in the subgraph. Also define $\mathtt{T}_v^2 = \{t \in R_v^2 : t-1 \notin R_v^2\}$ and $\mathtt{T}_v^1 = \{t \in R_v^1 \cup R_v^2 : t-1 \notin (R_v^1 \cup R_v^2)\}$. In Figure 7 this gives $R_v^0 = \{1, 2, 5\}$, $R_v^1 = \{4\}$, $R_v^2 = \{3\}$, and $\mathtt{T}_v^1 = \mathtt{T}_v^2 = \{3\}$.

Summing the inventory balance constraints (9) over $\mathtt{T}$ and using $\underline{S}_{ik}$ as the lower bound on s_{ik} gives the following inequalities written in terms of the partition R_v^0, R_v^1, and R_v^2

$$s_{i,\ell-1} + \sum_{v \in V} \left(\sum_{t \in R_v^0} q_{ivt} + \sum_{t \in R_v^1} q_{ivt} + \sum_{t \in R_v^2} q_{ivt} \right) \geq \sum_{t \in \mathtt{T}} D_{it} + \underline{S}_{ik}. \tag{38}$$

Summing the flow conservation constraints (22) and (24) over R_v^2 and $R_v^2 \cup R_v^1$ respectively gives

$$\sum_{t \in R_v^2} f_{ivt}^{OA} = \sum_{t \in R_v^2} \left(\sum_{j \in N \cup \{o(v)\}} f_{jiv,t-T_{jiv}}^X + f_{iv,t-1}^W - f_{ivt}^W \right), \tag{39}$$

$$\sum_{t \in R_v^2 \cup R_v^1} \left(f_{iv,t-1}^{OA} + f_{iv,t-1}^{OB} - q_{iv,t-1} \right) = \sum_{t \in R_v^2 \cup R_v^1} \left(f_{ivt}^{OB} + \sum_{j \in N \cup \{d(v)\}} f_{ijvt}^X \right). \tag{40}$$

Simplifying equations (39) by canceling out variables f_{ivt}^W, and equations (40) by canceling out variables f_{ivt}^{OB}, and using the nonnegativity of f_{ivt}^W, f_{ijvt}^X and f_{ivt}^{OB}, we obtain from (38),

$$s_{i,\ell-1} + \sum_{v \in V} \left(\sum_{t \in R_v^0} q_{ivt} + \sum_{t \in R_v^1} f_{ivt}^{OA} + \sum_{t \in R_v^2} \sum_{j \in N \cup \{o(v)\}} f_{jiv,t-T_{ji}}^X + \sum_{t \in \mathtt{T}_v^2} f_{iv,t-1}^W + \sum_{t \in \mathtt{T}_v^1} f_{ivt}^{OB} \right) \geq \sum_{t \in \mathtt{T}} D_{it} + \underline{S}_{ik}. \tag{41}$$

Using the variable upper bound constraints (26) – (30), inequality (41) can be relaxed as follows:

$$s_{i,\ell-1} + \sum_{v\in V}\left(\sum_{t\in R_v^0} Q_v o_{ivt} + \sum_{t\in R_v^1} K_v o_{ivt}^A + \sum_{j\in N\cup\{o(v)\}}\sum_{t\in R_v^2} K_v x_{jiv,t-T_{jiv}} \right.$$
$$\left. + \sum_{t\in \mathtt{T}_v^2} K_v w_{iv,t-1} + \sum_{t\in \mathtt{T}_v^1} K_v o_{ivt}^B\right) \geq \sum_{t\in\mathtt{T}} D_{it} + \underline{S}_{ik}. \quad (42)$$

Constraints (42) have the same structure as constraints (33) and are thus the defining constraints of continuous binary knapsack sets. Setting $s_{i,\ell-1}$ to its upper bound ($\overline{S}_{i,\ell-1}$ if $\ell > 1$ and S_i^0 if $\ell = 1$) gives an integer knapsack constraint. Using Chvátal-Gomory rounding, we obtain for arbitrary $Q > 0$

$$\sum_{v\in V}\left(\sum_{t\in R_v^0}\left\lceil\frac{Q_v}{Q}\right\rceil o_{ivt} + \sum_{t\in R_v^1}\left\lceil\frac{K_v}{Q}\right\rceil o_{ivt}^A + \sum_{j\in N\cup\{o(v)\}}\sum_{t\in R_v^2}\left\lceil\frac{K_v}{Q}\right\rceil x_{jiv,t-T_{jiv}}\right.$$
$$\left. + \sum_{t\in\mathtt{T}_v^2}\left\lceil\frac{K_v}{Q}\right\rceil w_{iv,t-1} + \sum_{t\in\mathtt{T}_v^1}\left\lceil\frac{K_v}{Q}\right\rceil o_{ivt}^B\right) \geq \left\lceil\frac{\sum_{t\in\mathtt{T}} D_{it} - \overline{S}_{i,\ell-1} + \underline{S}_{ik}}{Q}\right\rceil. \quad (43)$$

Three special cases of these inequalities are obtained by setting $R_v^2 = \mathtt{T}$, $R_v^1 = \mathtt{T}$, and $R_v^0 = \mathtt{T}$ respectively. Choosing $\overline{K} = \max\{K_v : v \in V\}$ and $\overline{Q} = \max\{Q_v : v\in V\}$, one obtains:

$$\sum_{v\in V}\left(\sum_{j\in N\cup\{o(v)\}}\sum_{t\in\mathtt{T}} x_{jiv,t-T_{jiv}} + w_{iv,\ell-1} + o_{iv\ell}^B\right) \geq \left\lceil\frac{\sum_{t\in\mathtt{T}} D_{it} - \overline{S}_{i,\ell-1} + \underline{S}_{ik}}{\overline{K}}\right\rceil, \quad (44)$$

$$\sum_{v\in V}\left(\sum_{t\in\mathtt{T}} o_{ivt}^A + o_{iv\ell}^B\right) \geq \left\lceil\frac{\sum_{t\in\mathtt{T}} D_{it} - \overline{S}_{i,\ell-1} + \underline{S}_{ik}}{\overline{K}}\right\rceil, \quad (45)$$

$$\sum_{v\in V}\sum_{t\in\mathtt{T}} o_{ivt} \geq \left\lceil\frac{\sum_{t\in\mathtt{T}} D_{it} - \overline{S}_{i,\ell-1} + \underline{S}_{ik}}{\overline{Q}}\right\rceil. \quad (46)$$

Inequalities (44) establish the minimum number of arrivals at port i, (45) establish the minimum number of times a ship must start operating, and inequalities (46) impose a minimum number of operations. Inequalities (44) – (46) can be generalized for any nonempty subset of discharge ports.

Example 3.2 *Inequality* (43) *is derived for the situation illustrated in Figure 7 based on the entering arcs crossing the cut-set shown. A discharge port i, a single ship v, and a time interval* $\mathtt{T} = [1,5]$ *are given.* $\mathtt{T}$ *is partitioned into* $R_v^2 = \{3\}$, $R_v^1 = \{4\}$, *and* $R_v^0 = \{1,2,5\}$. *The ship has capacity* $K_v = 110$ *and its discharge rate is* $Q_v = 60$. *Assume* $S_i^0 = 40, \underline{S}_{i5} = 10$ *and* $D_{it} = 20, \forall t \in T$. *Choosing* $Q = Q_v = 60$ *then gives*

$$o_{iv1} + o_{iv2} + 2o_{iv4}^A + o_{iv5} + 2\overline{x}_{iv3} + 2w_{iv2} + 2o_{iv3}^B \geq \left\lceil\frac{100-40+10}{60}\right\rceil = 2$$

where $\overline{x}_{ivt} = \sum_{j\in N\cup o(v), j\neq i: u-T_{ji}=t} x_{jivu}$.

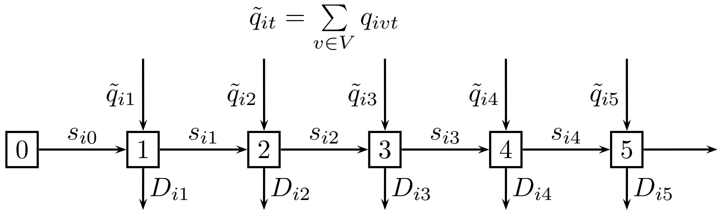

Figure 8: A simple lot-sizing structure at discharge port i.

4 Lot-sizing relaxations

In this section, several possible single item lot-sizing sets, see Pochet and Wolsey [22], that arise from relaxations and decompositions of $\mathbb{X}^{FCNF}$are presented. Single item lot-sizing is concerned with the production of a single product for which there is a demand D_t in each time period. To model the problem as a mixed integer program, the production q_t in each period, the stock of the product s_t at the end of the period and a binary set-up variable taking the value $o_t = 1$ if there is production in the period ($q_t > 0$) are defined. Additional aspects involve upper and lower bounds on the stock, production capacity implying an upper bound Q_t on the amount produced per period and start-up variables $o_t^A = 1$ if period t is the first period of an interval of set-ups ($o_t = 1$ and $o_{t-1} = 0$). The variable o_t can also be viewed as an integer variable, in which case it represents the number of batches of maximum size Q that is required to produce q_t. As we will show below, this corresponds very closely to the situation in a discharge port, but it is also explained how to adapt it for loading ports.

4.1 Constant capacitated lot-sizing relaxations

The first lot-sizing relaxations that we derive correspond to one level of the fixed charge network at discharge port i, see Figure 8, by taking into account a constant upper bound on the q_{ivt} variables. The constraints (7) – (9), (11), (12), (17) lead to the relaxation

$$s_{i,t-1} + \sum_{v \in V} q_{ivt} = D_{it} + s_{it}, \qquad \forall t \in T, \tag{47}$$

$$0 \leq q_{ivt} \leq Q_v o_{ivt}, \qquad \forall v \in V, t \in T, \tag{48}$$

$$\sum_{v \in V} o_{ivt} \leq B_{it}, \qquad \forall t \in T, \tag{49}$$

$$\underline{S}_{it} \leq s_{it} \leq \overline{S}_{it}, \qquad \forall t \in T, \tag{50}$$

$$s_{i0} = S_i^0, \tag{51}$$

$$o_{ivt} \in \{0, 1\}, \qquad \forall v \in V, t \in T. \tag{52}$$

For the constant capacity lot-sizing set, the upper bounds on the inventory level variables s are relaxed. The lower bounds on the same variables can then be eliminated. To do so, one first updates the bounds as follows:

$$\underline{S}_{it}^{M} = S_i^0 \text{ if } t = 0, \text{ and } \underline{S}_{it}^{M} = \max\{\underline{S}_{it}, \underline{S}_{i,t-1} - D_{it}\} \text{ if } \ t \in T \tag{53}$$

where $\underline{S}_{it}^{M}$ is the modified lower bound and $\underline{S}_{i0} = S_i^0$. Now, a new net inventory level variable $\tilde{s}_{it}$ and demand $\tilde{D}_{it}$ can be defined as:

$$\tilde{s}_{it} = 0 \text{ if } t = 0, \text{ and } \tilde{s}_{it} = s_{it} - \underline{S}_{it}^{M} \text{ if } t \in T, \text{ and } \tilde{D}_{it} = D_{it} - \underline{S}_{i,t-1}^{M} + \underline{S}_{it}^{M}. \tag{54}$$

Based on the subset (47) – (52) and setting $\tilde{q}_{it} = \sum_{v \in V} q_{ivt}$, $\tilde{o}_{it} = \sum_{v \in V} o_{ivt}$, and $\overline{Q} = \max\{Q_v : v \in V\}$, one obtains the constant capacitated lot-sizing set, LSCC, for discharge port i:

$$\tilde{s}_{i,t-1} + \tilde{q}_{it} = \tilde{D}_{it} + \tilde{s}_{it}, \qquad \forall t \in T, \tag{55}$$

$$\tilde{q}_{it} \leq \overline{Q}\tilde{o}_{it}, \qquad \forall t \in T, \tag{56}$$

$$\tilde{q}_{it}, \tilde{s}_{it} \geq 0, \qquad \forall t \in T, \tag{57}$$

$$\tilde{o}_{it} \in Z_+^1 \qquad \forall t \in T. \tag{58}$$

If it is assumed that the berth capacity $B_{it} = 1$, then (58) becomes $\tilde{o}_{it} \in \{0, 1\}$. Several valid inequalities for LSCC are known, see Pochet and Wolsey [20]. For discharge port i, a relaxation of (55)-(58), known as the Wagner-Whitin relaxation WWCC, can now be given:

$$\tilde{s}_{i,k-1} + \overline{Q}\sum_{u=k}^{t} \tilde{o}_{iu} \geq \sum_{u=k}^{t} \tilde{D}_{iu}, \qquad \forall k \in T, t \in T, k \leq t,$$

$$\tilde{s}_{it} \geq 0, \tilde{o}_{it} \in Z_+^1, \qquad \forall t \in T.$$

A complete polyhedral description of the convex hull of solutions of WWCC is known, as well as a polynomial size extended formulation, see Pochet and Wolsey [21]. For a comprehensive survey on the valid inequalities for LSCC and WWCC, see Pochet and Wolsey [22]. Valid inequalities for LSCC and WWCC can be converted back into valid inequalities for $\mathbb{X}^{FCNF}$using the linear transformations:

$$\tilde{s}_{it} = s_{it} - \underline{S}_{it}^{M}, \quad \tilde{q}_{it} = \sum_{v \in V} q_{ivt}, \quad \tilde{o}_{it} = \sum_{v \in V} o_{ivt}.$$

Example 4.1 *Consider an instance based on Figure 8 over five time periods $T = \{1, 5\}$ with demands $D_i = (3, 2, 3, 4, 2)$, lower bounds on the inventory levels $\underline{S}_i = (2, 2, 2, 2, 2)$, initial inventory $S_i^0 = 6$ and the capacity of the largest ship $\overline{Q} = 5$.*

Calculating the modified lower bounds on the inventory levels according to (53) and the demand according to (54) gives $\underline{S}_{it}^{M} = (6, 3, 2, 2, 2, 2)$ and $\tilde{D}_{it} = (0, 1, 3, 4, 2)$. For the corresponding WWCC relaxation, a valid inequality is:

$$\tilde{s}_{i2} \geq 3(1 - \tilde{o}_{i3}) + 1(2 - \tilde{o}_{i3} - \tilde{o}_{i4} - \tilde{o}_{i5})$$

Transforming back to the original variables $\tilde{s}_{i2} = s_{i2} - \underline{S}^M_{i2}$, $\tilde{o}_{it} = \sum_{v \in V} o_{ivt}$ *and collecting the terms one obtains the valid inequality for* $\mathbb{X}^{FCNF}$*:*

$$s_{i2} \geq 7 - 4\sum_{v \in V} o_{iv3} - \sum_{v \in V} o_{iv4} - \sum_{v \in V} o_{iv5}.$$

For a loading port i, the relaxation is defined by constraints (7), (8), (10), (11), (12), and (17)

$$s_{i,t-1} - \sum_{v \in V} q_{ivt} = -P_{it} + s_{it}, \qquad \forall t \in T,$$

$$0 \leq q_{ivt} \leq Q_v o_{ivt}, \qquad \forall v \in V, \forall t \in T,$$

$$\sum_{v \in V} o_{ivt} \leq B_{it}, \qquad \forall t \in T,$$

$$\underline{S}_{it} \leq s_{it} \leq \overline{S}_{it}, \qquad \forall t \in T,$$

$$s_{i0} = S^0_i,$$

$$o_{ivt} \in \{0, 1\}, \qquad \forall v \in V, t \in T.$$

To formulate this problem as a lot-sizing problem, new variables $\hat{s}_{it} = \overline{S}_i - s_{it}$ are introduced that measure the spare stock capacity available at period t in port i, where $\overline{S}_i = \max\{S^0_i, \max_{t \in T} \overline{S}_{it}\}$. This leads to the following equivalent formulation

$$\hat{s}_{i,t-1} + \sum_{v \in V} q_{ivt} = P_{it} + \hat{s}_{it}, \qquad \forall t \in T, \tag{59}$$

$$0 \leq q_{ivt} \leq Q_v o_{ivt} \qquad \forall v \in V, t \in T, \tag{60}$$

$$\sum_{v \in V} o_{ivt} \leq B_{it}, \qquad \forall t \in T, \tag{61}$$

$$\overline{S}_i - \underline{S}_{it} \geq \hat{s}_{it} \geq \overline{S}_i - \overline{S}_{it}, \qquad \forall t \in T, \tag{62}$$

$$\hat{s}_{i0} = \overline{S}_i - S^0_i, \tag{63}$$

$$o_{ivt} \in \{0, 1\}, \qquad \forall v \in V, t \in T. \tag{64}$$

This formulation can now be used to derive the same relaxations as for the discharge ports.

4.2 Two level lot-sizing relaxations

The two level relaxations are derived from two levels of the fixed charge network, see Figure 9. In multi-level lot-sizing problems it is useful to aggregate the levels which also makes it natural to consider aggregated stocks, known as echelon stocks. For a discharge port i such as the one depicted in Figure 9, the appropriate echelon stock in period t is shown below to be $s_{it} + f^{OB}_{i,t+1}$.

Extending the lot-sizing relaxations defined in Section 4.1, the two level lot-sizing set (2LLS)

$$\tilde{q}_{it} = \sum_{v\in V} q_{ivt},\ \tilde{f}^{OA}_{it} = \sum_{v\in V} f^{OA}_{ivt},\ \tilde{f}^{OB}_{it} = \sum_{v\in V} f^{OB}_{ivt},\ \tilde{f}^{X}_{it} = \sum_{v\in V} f^{X}_{ivt}$$

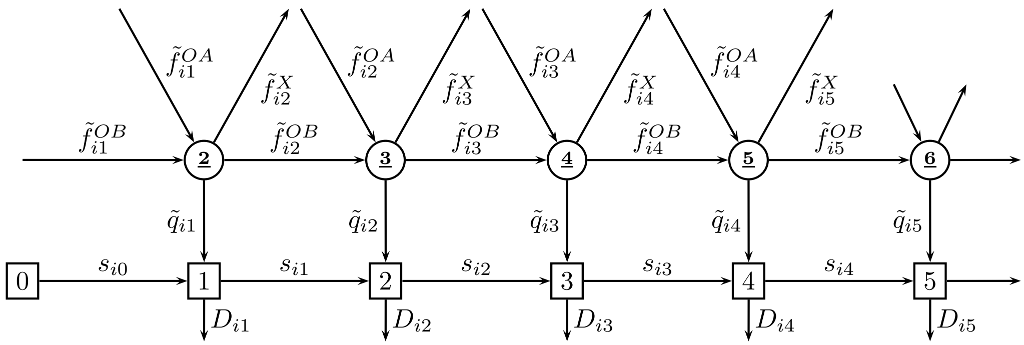

Figure 9: A two level lot-sizing structure at discharge port i.

for discharge port i can be defined as

$$\tilde{f}^{OA}_{it} + \tilde{f}^{OB}_{it} = \tilde{q}_{it} + \tilde{f}^{OB}_{i,t+1} + \tilde{f}^{X}_{i,t+1}, \qquad \forall t \in T, \tag{65}$$

$$\tilde{f}^{OA}_{it} \leq \overline{K}\tilde{o}^{A}_{it}, \qquad \forall t \in T, \tag{66}$$

$$\tilde{f}^{OA}_{it}, \tilde{f}^{OB}_{it}, \tilde{f}^{X}_{it} \geq 0, \qquad \forall t \in T, \tag{67}$$

$$\tilde{o}^{A}_{it} \in \mathbb{Z}^{1}_{+}, \qquad \forall t \in T, \tag{68}$$

$$\text{and } (55) - (58) \tag{69}$$

where

$$\tilde{f}^{OA}_{it} = \sum_{v\in V} f^{OA}_{ivt}, \tilde{f}^{OB}_{it} = \sum_{v\in V} f^{OB}_{ivt}, \tilde{f}^{X}_{it} = \sum_{v\in V}\sum_{j\in N\cup\{d(v)\}} f^{X}_{ivt}, \tilde{q}_{it} = \sum_{v\in V} q_{ivt}, \tilde{o}^{A}_{it} = \sum_{v\in V} o^{A}_{ivt}$$

and $\overline{K} = \max\{K_v : v \in V\}$. Constraints (65) are the flow balance constraints (24) summed over v.

Summing constraints (65) and (55) and introducing the echelon stock variable $e_{it} = \tilde{f}^{OB}_{i,t+1} + \tilde{s}_{it}$ gives the relaxation:

$$e_{i,t-1} + \tilde{f}^{OA}_{it} = \tilde{D}_{it} + e_{it} + \tilde{f}^{X}_{i,t+1}, \qquad \forall t \in T,$$

$$\tilde{f}^{OA}_{it} \leq \bar{K}\tilde{o}^{A}_{it}, \qquad \forall t \in T,$$

$$e_{it}, \tilde{f}^{OA}_{it} \geq 0, \tilde{o}^{A}_{it} \in Z^{1}_{+}, \qquad \forall t \in T.$$

From this we again obtain a Wagner-Whitin constant capacity relaxation:

$$e_{i,k-1} + \overline{K}\sum_{u=k}^{t} \tilde{o}^{A}_{iu} \geq \sum_{u=k}^{t} \tilde{D}_{iu}, \qquad \forall k \in T, t \in T, k \leq t,$$

$$e_t \geq 0, \tilde{o}^{A}_{it} \in \{0,1\}, \qquad \forall t \in T.$$

Valid inequalities for this relaxation, denoted 2LWWCC, can be derived, and then converted into valid inequalities for $\mathbb{X}^{FCNF}$.

In order to derive similar inequalities for loading port i, new variables, $\overline{f}^{OA}_{ivt}$, $\overline{f}^{OB}_{ivt}$, and $\overline{f}^{X}_{ijvt}$ are defined. These variables indicate the unused capacity of each ship operating at that port, i.e.

$$\overline{f}^{OA}_{ivt} = K_v o^A_{ivt} - f^{OA}_{ivt}, \overline{f}^{OB}_{ivt} = K_v o^B_{ivt} - f^{OB}_{ivt}, \overline{f}^{X}_{ijvt} = K_v x_{ijvt} - f^{X}_{ijvt}.$$

Using these new variables, a two level lot-sizing set similar to (65) – (69) for loading ports can be formulated. Thus two level lot-sizing relaxations can be derived at loading ports.

4.3 Lot-sizing with start-up relaxations

An important extension of the lot-sizing problem is to include start-up costs, i.e. a cost associated with the first period of an interval of set-ups, and several lot-sizing relaxations of it have been studied in the literature. The first period a ship operates in a port can be seen as a start-up and thus these relaxations can be used to derive valid inequalities. When deriving lot-sizing relaxations from the standard formulation, as in Section 4.1, it is not possible to handle start-ups since the variable o_{ivt} does not give information on whether ship v operated at port i at time period $t-1$ or not. In the fixed charge network flow problem, o^A_{ivt} can be interpreted as a start-up variable and can be used to derive valid inequalities. Necessarily in the binary case, the start-up variable $o^A_{ivt} = 1$ if $o_{ivt} = 1$ and $o_{iv,t-1} = 0$.

This can be expressed by

$$o^A_{ivt} \leq o_{ivt}, \qquad \forall t \in T, \tag{70}$$

$$o^A_{ivt} \geq o_{ivt} - o_{iv,t-1}, \qquad \forall t \in T, \tag{71}$$

$$o_{ivt}, o^A_{ivt} \in \{0,1\}, \qquad \forall t \in T. \tag{72}$$

Constraints (70) ensure that ship v starts operating if there is a start-up, while constraints (71) force a start-up if the ship operates in the current time period and did not operate in the previous time period.

Several lot-sizing relaxations with start-ups can be derived by adding (70) – (72) to an existing lot-sizing set. In particular, valid inequalities can be derived for the capacitated lot-sizing set with start-ups, see Constantino [9], and then used to derive valid inequalities for $\mathbb{X}^{FCNF}$.

Here we derive a discrete constant capacity lot-sizing with start-ups relaxation (DLSCCS), for which valid inequalities have been proposed by van Eijl and van Hoesel [32].

Constraints (70) – (72) are aggregated by summing over v. This gives

$$\tilde{o}^A_{it} \leq \tilde{o}_{it}, \qquad \forall t \in T, \tag{73}$$

$$\tilde{o}^A_{it} \geq \tilde{o}_{it} - \tilde{o}_{i,t-1}, \qquad \forall t \in T, \tag{74}$$

$$\tilde{o}_{it}, \tilde{o}^A_{it} \in \{0, 1, \cdots, B_{it}\}, \qquad \forall t \in T, \tag{75}$$

where $\tilde{o}^A_{it} = \sum_{v\in V} o^A_{ivt}$ and $\tilde{o}_{it} = \sum_{v\in V} o_{ivt}$. Here in the integer case $\tilde{o}^A_{ivt}$ is the increase of $\tilde{o}_{ivt}$ from period $t-1$ to t. However if it is assumed that the berth capacity $B_{it} = 1$, the aggregated variables are still binary.

Now let $\tilde{O}_{it} = \left\lceil \frac{\sum_{u=1}^{t} \tilde{D}_{iu}}{\overline{Q}} \right\rceil$, where $\tilde{D}_{it}$ is the modified demand from (54) and $\overline{Q} = \max\{Q_v : v \in V\}$ is the largest ship capacity. Also set $\delta_{it} = \tilde{O}_{it} - \tilde{O}_{i,t-1}$. Note that $\tilde{O}_{it}$ is a lower bound on the number of operating periods needed during the first t periods. The set DLSCCS is obtained by adding the constraints

$$\sum_{u=1}^{t} \tilde{o}_{iu} \geq \tilde{O}_{it}, \qquad \forall t \in T, \tag{76}$$

to constraints (73) – (75).

The following set of inequalities was proved to be valid for DLSCCS by van Eijl and van Hoesel [32] in the case where δ_{it} and B_{it} are binary, and thus $\tilde{o}_{it}$ and $\tilde{o}^A_{it}$ are binary variables.

Proposition 4.2 *Consider a time interval $[k, \ell] \subseteq T$ with $\delta_{i\ell} = 1$. Let $\sum_{t=k}^{\ell} \delta_{it} = p > 0$ and let $t_1 < t_2 < \cdots < t_p = \ell$ be the periods in $[k, \ell]$ in which $\delta_{it} = 1$. The inequality*

$$\sigma_{i,k-1} + \sum_{j=1}^{p} (\tilde{o}_{i,k+j-1} + \tilde{o}^A_{i,k+j} + \cdots + \tilde{o}^A_{it_j}) \geq p \tag{77}$$

is valid for DLSCCS, where $\sigma_{it} = \sum_{u=1}^{t} \tilde{o}_{iu} - \tilde{O}_{it} \geq 0$ and $\sigma_{i0} = 0$.

Example 4.3 *Consider the data from Example 4.1. Since $\tilde{D}_{it} = (0, 1, 3, 4, 2)$ and $\overline{Q} = 5$ it follows that $\delta_{it} = (0, 1, 0, 1, 0)$. Let $[k, \ell] = [1, 4]$. This gives $t_1 = 2$, $t_2 = 4$ and $p = \sum_{t=k}^{\ell} \tilde{O}_{it} = 2$. A valid inequality derived from (77) is then*

$$\sigma_{i0} + (\tilde{o}_{i1} + \tilde{o}^A_{i2}) + (\tilde{o}_{i2} + \tilde{o}^A_{i3} + \tilde{o}^A_{i4}) \geq 2 \Rightarrow \tilde{o}_{i1} + \tilde{o}_{i2} + \tilde{o}^A_{i2} + \tilde{o}^A_{i3} + \tilde{o}^A_{i4} \geq 2$$

Hence the following inequality is valid for $\mathbb{X}^{FCNF}$:

$$\sum_{v\in V} o_{iv1} + \sum_{v\in V} o_{iv2} + \sum_{v\in V} o^A_{iv2} + \sum_{v\in V} o^A_{iv3} + \sum_{v\in V} o^A_{iv4} \geq 2.$$

5 Computational results

This section presents some of the computational experiments carried out to test different strategies for the solution of instances of the maritime inventory routing problem. The strategies tested include the comparison of the two mathematical formulations presented in Section 2, the effectiveness of the inclusion of the valid inequalities discussed in Sections 3 and 4 and the use of branching priorities.

First the original and FCNF formulations with and without the inclusion of valid inequalities are compared. This initial study leads to the selection of some relevant inequalities. Then, taking the two formulations tightened with the selected inequalities, different branching priorities are tested. Thus for each formulation, several different combinations of valid inequalities and branching priorities are tested. Finally, the scalability of the approaches are tested by changing the discretization of the time periods and the length of the time horizon.

The instances used were generated from seven instances based on real data. They come from the short sea segment with long loading and discharge times relative to the sailing times. The number of ports and ships of each instance is given in the second column of Table 1. The time horizon is 30 days. Traveling, operating and waiting costs are time invariant.

All tests were run on a computer with processor Intel Core 2 Duo, CPU 2.2GHz, with 4GB of RAM using the optimization software Xpress Optimizer Version 21.01.00 with Xpress Mosel Version 3.2.0. Unless stated otherwise, all inequalities used to tighten the formulations were added a priori to the MIP model which was then fed to the MIP solver.

In the last six columns of Table 1 we provide summary information for the two formulations considered. Columns "Rows" and "Columns" indicate the total number of constraints and variables, respectively. The column "Int. Var." indicates the number of integer variables. In parentheses we provide the corresponding values after the preprocessing phase.

		Original Formulation			FCNF Formulation		
Inst.	$(\mid N \mid, \mid V \mid)$	Rows	Columns	Int. Var.	Rows	Columns	Int. Var.
A	(3,2)	982 (575)	952 (662)	694 (459)	1682 (1042)	1873 (1320)	875 (552)
B	(4,2)	1308 (757)	1128 (795)	838 (569)	2050 (1252)	2235 (1466)	1078 (668)
C	(4,2)	1724 (1197)	1756 (1542)	1364 (1200)	3100 (2225)	3574 (2677)	1726 (1243)
D	(5,2)	2138 (1445)	2355 (1863)	1882 (1461)	4016 (2928)	4793 (3670)	2320 (1714)
E	(5,2)	2138 (1446)	2367 (1878)	1894 (1476)	4028 (2949)	4817 (3699)	2332 (1731)
F	(4,3)	2466 (1696)	2548 (2237)	2023 (1775)	4502 (3249)	5249 (3961)	2564 (1847)
G	(6,5)	5836 (3150)	5652 (4165)	4692 (3346)	9731 (6350)	11537 (8089)	5678 (3837)

Table 1: Summary statistics for the base instances using the two models (with and without preprocessing).

5.1 Formulations, valid inequalities and reformulations

The original formulation consists of (1) – (17), while the FCNF formulation is defined by (1) – (3), (7) – (12), (16), (17), and (18) – (30).

The following valid inequalities and reformulations have been tested:

Knapsack inequalities. These inequalities refer to (34) for the loading ports and (43) for discharge ports. In both cases $\mathtt{T}$ includes either the first period or the last period, that is, $\mathtt{T} = 1, \ldots, t$ or $\mathtt{T} = t, \ldots, \mid T \mid, t \in T$. Inequalities (34) are generated for case $\mathtt{T}_v = \mathtt{T}$, for all $Q \in \bigcup_{v \in V}\{K_v\}$, and for case $\mathtt{T}_v = \emptyset$, for all $Q \in \bigcup_{v \in V}\{Q_v\}$. Inequalities (43) are generated for the cases $R_v^2 = \mathtt{T}$, $R_v^1 = \mathtt{T}$, and $R_v^0 = \mathtt{T}$ for $Q \in \bigcup_{v \in V}\{K_v\}$ in the first two cases and for $Q \in \bigcup_{v \in V}\{Q_v\}$ in the last one. These inequalities will henceforth be denoted K.

Mixed integer rounding inequalities. These inequalities are stated in (37) and are generated for all $Q \in \bigcup_{v \in V}\{Q_v\}$. They are added dynamically as cuts (valid inequalities that cut off the current factional solution), and will henceforth be denoted M.

Wagner-Whitin constant capacitated lot-sizing reformulations. These reformulations are given in Pochet and Wolsey [22] (denoted by XFormWWCC for the constant capacitated case and XFormWWU for the uncapacitated case) for the WWCC relaxation described in Section 4.1 and the two-level relaxation 2LWWCC described in Section 4.2. These reformulations are denoted W.

Inequalities for lot-sizing with start-up relaxations. These inequalities are stated in (77) and will henceforth be denoted D. These inequalities consider every subset $[k, \ell]$ of T.

Table 2 gives some characteristics of each instance and provides information on the lower bounds obtained with the original formulation. The first column specifies the instance, the second column contains the optimal value, and the third column gives the linear relaxation bound, denoted L, of the original formulation. The last four columns present the percentage of the gap closed with the inclusion of additional valid inequalities. X gives the gap reduced in the root node after the inclusion of cuts from Xpress. K means that the valid inequalities K are added to the formulation, K, W means that valid inequalities K and W are added.

Table 3 shows the results obtained with some of the more interesting and/or effective combinations of valid inequalities and reformulations for the FCNF formulation. Again column L gives the linear relaxation bound of the FCNF formulation. The last eight columns give the gap reduced with the inclusion of inequalities. To ease the presentation, Ω is introduced to denote the inclusion of all valid inequalities, i.e. $\Omega = K, M, W, D$ and $\Omega - \Delta$ denotes the inclusion of all valid inequalities except Δ, where $\Delta \in \{K, M, W, D\}$.

As expected, the FCNF formulation provides better bounds. It can also be observed that best bounds when only one type of inequalities is tested were obtained with the inclusion of K

			Gap closed (%)			
Inst.	Opt.	L	X	K	W	K, W
A	137.4	22.3	80.1	100.0	24.2	100.0
B	370.6	32.0	21.8	78.6	48.4	91.4
C	413.5	44.7	16.0	79.6	51.0	89.2
D	357.9	53.6	43.4	75.5	46.3	85.1
E	355.5	52.3	25.8	74.5	43.9	85.4
F	504.9	105.2	11.3	81.3	23.5	79.2
G	747.9	213.6	19.1	92.0	43.0	71.6

Table 2: Lower bounds based on the original formulation.

			Gap closed (%)							
Inst.	Opt.	L	X	W	D	K, M	$\Omega - K, M$	$\Omega - W$	$\Omega - D$	Ω
A	137.4	69.6	53.0	100	100	100	100	100	100	100
B	370.6	263.4	72.2	44.5	16.3	42.4	45.9	43.2	44.6	46.0
C	413.5	235.9	54.5	56.6	10.6	64.0	56.5	64.0	64.3	64.3
D	357.9	204.1	52.6	52.9	10.9	58.8	53.4	61.8	61.3	62.1
E	355.5	206.2	54.8	52.2	9.2	56.0	52.8	57.6	58.5	59.0
F	504.9	350.3	64.3	58.8	7.1	60.5	59.0	58.0	58.0	58.7
G	747.9	618.2	80.0	66.0	16.7	79.3	66.1	79.6	79.7	80.1

Table 3: Lower bounds based on the FCNF formulation.

inequalities and M cuts. K and M are considered in the same type of inequality since K can be generated from M. On the other hand, dropping reformulations W or dropping inequalities D leads to a slight worsening of the bound. This suggests that a good formulation should be based on some inequalities K and M. However, extended testing (not reported in Table 3) showed that it is necessary to add many M cuts to get significant improvements on the lower bounds. That experience also showed that most of the gap closed by K and M can be closed by K. So to achieve a similar bound, many more M cuts would have to be added compared to K inequalities.

Table 4 gives the average integrality gap over the seven instances, where $gap = \frac{Opt - LB}{Opt} \times 100$ and LB is the value of the lower bound provided by the corresponding relaxation. The X indicates the use of Xpress cuts. When X is present, the gap reported is the gap at the root node after the inclusion of cuts from Xpress. For example, Ω, X under FCNF formulation means that the gap is measured at the root node when the FCNF formulation is used with the addition of all valid inequalities (or reformulations) and Xpress cuts are added.

Original formulation				FCNF formulation			
L	X	Ω	Ω, X	L	X	Ω	Ω, X
83.6	57.5	14.4	10.1	36.3	8.8	11.4	6.4

Table 4: Average integrality gaps for both formulations.

The valid inequalities added to the original formulation are much stronger than the general cuts added by Xpress, while the general cuts by Xpress gives a stronger FCNF formulation. Xpress recognizes the FCNF structure of the problem, and exploits it in the generation of cuts. Combining valid inequalities and cuts from Xpress further reduces the gap of both formulations.

5.2 Branching strategies

It is well known that branching decisions within a Branch and Bound algorithm may have great influence on the performance of the algorithm. Usually, solvers, as Xpress Optimiser, allow the user to define his own branching scheme. One possible branching strategy is to establish different branching priorities on variables. Here we followed this approach by considering new variables (resulting from aggregation of the original variables) providing information related to the total number of visits each ship makes to each port.

Based on the results in Table 2 and 3 and related runs it was decided to use the following strategies for further tests:

Sx - set highest priority to variables $Sx_{iv} = \sum_{t\in T} \sum_{j\in N\cup\{o(v)\}} x_{jivt}$ that represent the number of times ship v visits port i;

So^A - set highest priority to variables $So^A_{iv} = \sum_{t\in T} o^A_{ivt}$ that represent the number of start-ups of ship v at port i.

For the original formulation only strategy Sx can be used. We tested the use of this strategy combined with the inclusion of inequalities K. For the FCNF formulation both strategies have been tested. These strategies were combined with the inclusion of inequalities K and D. The choice of D inequalities was motivated by the possibility of combining valid inequalities involving the start-up variables o^A with the branching strategy based on the same set of variables. Inequalities K are included a priori in the formulation while inequalities D are added to the cut pool. Since slightly better results were obtained with So^A for the harder instances, only results for So^A are provided. Tables 5 and 6 show the results for the original and FCNF formulations, respectively.

Each pair (V, B) in the header row of the tables indicates the combination of valid inequalities (V) and branching priority (B) used. The symbol – means that no inequality or branching priority is added. For each such pair, the time T in seconds and the number of branch and bound nodes N is given. A $*$ means that the optimal solution could not be found within a three hours limit.

	(–,–)		$(K$,–)		$(K$,$Sx)$	
Inst.	T	N	T	N	T	N
A	1.3	53	0.2	1	0.2	1
B	11	4349	6	1320	3	323
C	1700	550676	310	57590	105	17734
D	117	21195	5	35	5	17
E	268	55715	10	253	4	59
F	$*$	$*$	$*$	$*$	1754	156262
G	$*$	$*$	$*$	$*$	3236	24278

Table 5: Branching priorities for branch and bound with the original formulation.

	(–,–)		$(K$,–)		$(K+D$,–)		$(K$,$So^A)$		$(K+D$,$So^A)$	
Inst.	T	N	T	N	T	N	T	N	T	N
A	7	221	0.5	1	0.5	1	0.5	1	0.5	1
B	6	425	5	33	5	33	4	23	4	11
C	$*$	$*$	145	7520	147	8342	28	745	19	197
D	94	3789	12	3	9	3	11	7	10	7
E	136	5810	26	99	21	33	11	23	16	17
F	$*$	$*$	635	11825	317	6386	71	711	53	307
G	$*$	$*$	152	297	111	119	188	393	54	35

Table 6: Branching priorities for branch and bound with the FCNF formulation.

Tables 5 and 6 show that the use of branching priorities is essential to solve the instances tested. An efficient approach is the use of the FCNF formulation with the combination of inequalities K and D with a branching priority on the o^A variables.

In order to further test this strategy more computational experiments were conducted. Five new instances for each base instance were created by randomly generating the initial inventory, using a uniform distribution on $[\underline{S}_{i1}, \overline{S}_{i1}]$ in each port $i \in N$. We choose the first five feasible instances generated. The average solution times and average number of nodes over the five instances for each base instance are given in Table 7. The random instances based on initial instance G turned out to be much harder than G. Only three of them were solved within the limit of three hours, and the final gaps of the two other instances were 12.3% and 9.3%. Thus, these instances are not presented in the table. The ** in column (K,–) indicates that some of the corresponding five instances were not solved within three hours (three instances were solved to optimality and the two other instances were stopped with gaps of 13.3% and 15.8%).

	Original formulation				FCNF formulation			
	(K,–)		(K,Sx)		($K+D$,–)		($K+D$,So^A)	
Inst.	T	N	T	N	T	N	T	N
A	0.5	6.6	0.4	26.2	0.7	3.4	0.9	4.2
B	5.1	1150.8	3.5	390.4	4.4	197.8	2.5	24.6
C	37.5	5031.8	9.4	857.6	27	971.6	8	31.2
D	80.9	6757.4	17.4	853.8	56.3	1188.6	18.1	93
E	35	4129	11.5	591.2	80.4	2215.2	29.5	296
F	**	**	1650.2	113951.2	1066.5	17285.8	182.1	1907

Table 7: Branching priorities and D inequalities for random instances.

Using the FCNF formulation with inequalities K and D (using o^A variables), and using branching priority So^A performs well on most instances. The good performance of this approach based on the "start-ups" is also reinforced with the results given in Section 5.3.

5.3 Scalability study

Seven larger instances were constructed to test the time discretization. Each day is split into two periods, doubling the number of periods. The demand/production of the first new period is set to zero and the demand of the second is set to the demand/production of the day. The same settings as in Table 7 have been used. Table 8 gives the results for these instances. Again it can be seen that the use of branching priorities is essential, and the best results are obtained when inequalities K and D are added. The FCNF formulation with the addition of inequalities K and D, and with the use of the branching priority So^A is particularly successful for large test instances.

Finally, different time horizons were tested. In order to extend the time horizon it was necessary to change the port consumption rates D_{it} and production rates P_{it} for the instances. The results using the FCNF formulation with inequalities K added a priori and inequalities D added to the cut pool, and the branching priority So^A are given in Table 9. A $*$ means that the optimal solution could not be found within a three hours limit. For the case of instance G with 45 days, the integrality gap after three hours is about 25%, and for 60 days no feasible solution was found within the running time limit.

		Original formulation				FCNF formulation			
		$(K,–)$		(K,Sx)		$(K+D,–)$		$(K+D,So^A)$	
Inst.	Opt.	T	N	T	N	T	N	T	N
A	132.7	1	1	1.6	1	3.2	1	3.2	1
B	367	573	11458	23	1633	270	12293	50	1049
C	407	*	*	*	*	*	*	1874	36891
D	352.3	545	11970	25	109	504	2162	83	35
E	350.2	612	17884	97	2063	658	2613	159	85
F	502.5	*	*	*	*	*	*	2571	7410
G	747.9	*	*	*	*	*	*	8473	3006

Table 8: Results for the two periods per day case.

	30 days		45 days		60 days	
Inst.	T	N	T	N	T	N
A	1	1	1	3	80	28
B	13	3	1	5	20285	1159
C	17	6	53	31	7084	1907
D	1	5	69	52	29356	10853
E	3	10	1257	233	6999	4098
F	17	14	3537	1420	315	1051
G	725	249	*	*	*	*

Table 9: Results for the 30, 45 and 60 days using the FCNF formulation.

6 Concluding remarks

A maritime inventory routing problem with varying production and consumption rates is studied in this paper. Two discrete time formulations are introduced, an original formulation and a fixed charge network flow (FCNF) formulation that models the ship sequence of actions as a path on a given network. These formulations are strengthened using valid inequalities from (mixed) integer sets that arise as relaxations of these two formulations. In particular, several lot-sizing relaxations are derived for the FCNF formulation. It has been observed in studying lot-sizing problems that valid inequalities linking stocks and binary set-up variables indicating whether a period is a production period can often be strengthened by using additional binary variables indicating a start-up period at the beginning of one or more production periods. Taking production periods to correspond to loading/discharge periods and ship arrivals to correspond to the start-up variables mean that such strengthening is also possible here. In addition a branching strategy based on these start-up variables turns out to be better than a similar strategy based on the set-up variables.

The FCNF formulation tightened with valid inequalities and using a branching strategy based on the start-up variables instances including up to 60 periods could be solved to optimality.

In general the FCNF formulation provides better bounds than the original formulation. The general cuts generated by the optimizer Xpress gives a much stronger FCNF formulation compared with the original formulation. Xpress recognizes the FCNF structure of the problem, and exploits it in the generation of cuts for the FCNF formulation. Therefore the valid inequalities added to the original formulation are particularly useful and are much stronger than the general cuts added by Xpress. Combining valid inequalities and cuts from Xpress further reduces the gap of both formulations.

As future research, it would be interesting to investigate heuristics that could provide feasible solutions quickly, since there are instances with few feasible solutions for which it is hard to get good upper bounds early in the search. Combining such heuristics with a branch and cut approach might be fruitful. Another direction is to investigate other valid inequalities for different lot-sizing models as well as valid inequalities for the ship routing aspect of the problem. Investigation of the possibility of using the valid inequalities presented here together with column generation in a branch and price and cut framework is another interesting path for further research.

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