Task design using a realization tree: The case of the derivative in the context of chemistry

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We have recently developed a realization tree for the derivative at a point based on its five main realizations (See Haghjoo et al., 2023). In this paper, we discuss how this realization tree can be used for task design, particularly when the focus of teaching is helping students realize the applications of the derivative in the real world and learning more about the physical and numerical realizations of the derivative. To achieve this, after discussing the main realizations of the derivative and what a realization tree is, we briefly present our realization tree. Then we discuss the task we have designed in the context of chemistry and how such a rich task addresses different realizations of the derivative.

The derivative and its main five realizations

The derivative is one of the core topics in calculus with applications in various disciplines (e.g., engineering and medicine) (Hass et al., 2018). Previous studies reported that many students struggle to learn the derivative due to the complexity of its definition and various representations (in the commognition term, *realizations*) (e.g., Biza, 2021). Five main different realizations have been discussed in the literature for the derivative: symbolic (the formal definition of the derivative, the limit of a difference quotient), graphical (the slope of the tangent line), numerical ($\frac{f(x_2)-f(x_1)}{x_2-x_1}$, when Δx is very small but does not approach zero as in the formal procedure), verbal (the instantaneous rate of change), and physical (the measuring procedure prior to calculating the derivative using numerical approaches). Furthermore, for each main realization, three layers of the derivative (i.e., ratio, limit, and function) map a function into its derivative (See Roundy et al., 2015; Zandieh, 2000). Past research pointed out many students struggle to make meaningful connections between these realizations (e.g., Biza, 2021; Zandieh, 2000).

The realization tree: A tool from commognition theory

In the commognition theory, mathematics is considered a type of discourse with unique objects and ways of doing and saying (Sfard, 2008). It is distinguishable from other discourses by its four interrelated characteristics: *Word use* (e.g., differentiable function), *visual mediators* (e.g., a derivative function drawn in Desmos), *routines* (e.g., how to find the absolute extrema of a continuous function on a finite closed interval), and *endorsed narratives* (e.g., the extreme value theorem) (Sfard, 2008). In the commognition theory, the term *realization* has been used instead of the well-known term *representation*. In addition, a visual mediator in the form of a connected graph has been introduced and named *realization tree*. It is defined as a “hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations, and so forth” (Sfard, 2008, p. 301).

A realization tree for the derivative

In our realization tree for the derivative at a point (see Haghjoo et al., 2023), we have 17 roots: two roots for numerical (e.g., $N1$: $f'(x) \approx \frac{f(x_0+h)-f(x_0-h)}{2h}$), three roots for symbolic (e.g., $S3$: $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$), ten roots for graphical, and one root for each verbal and physical realization. Furthermore, average and instantaneous rates of change have many applications in the real world. Consequently, we identified 26 verbal realizations across eight disciplines (e.g., chemistry and biology) for the derivative by exploring a few calculus textbooks (e.g., Hass et al., 2018), such as reaction rate in chemistry.

The task: the reaction between calcium carbonate and hydrochloric acid

Task designers, when focusing on the verbal realizations of the derivative, have many contexts to choose from. However, we recommend selecting a context relevant to students' majors and future careers. Furthermore, we recommend selecting a context where measuring the dependent and independent variables is easy, not taking too much time, and, if possible, can be done by the lecturer or students in the lecture room. That makes the task very close to what is called in the literature as *tasks with authentic context* (Vos, 2020). Using such contextual tasks could make teaching more interesting for many students, especially those interested in learning mathematics because of its *use-value*. Furthermore, it gives meaning to the mathematical concepts discussed in the task, could increase the task's accessibility and may help students develop their mathematical understanding using their out-of-school/university knowledge (Vos, 2020). In this paper, we focus on the reaction rate in chemistry for calcium carbonate and hydrochloric acid because such an experiment could be done in the classroom/lecture by taking some safety measures, and the materials needed for doing this experiment are not expensive. The chemical equation for this experiment is $CaCO_3\,(s) + 2HCl\,(aq) \rightarrow CaCl_2\,(aq) + CO_2\,(g) + H_2O(l)$. When calcium carbonate is added to hydrochloric acid (we used a 70% solution of hydrochloric acid), calcium carbonate will dissolve. Furthermore, gas bubbles will appear at the top of the solution due to the formation of carbon dioxide gas. So, the lecturer or students could calculate the reaction rate by measuring the weight of the solution over some chosen intervals. Doing so shows how much carbon dioxide gas has been released through the solution. With the help of a digital scale and a timer, we recorded the weight of the solution, and consequently, according to the law of conservation of mass, we were able to record how much carbon dioxide was released (Table 1). We measured the weight of the solution in six-second intervals according to the sensitivity of the scale. The experiment ended after 36 seconds. The measuring process discussed above could help students to have a better realization of the P17 (physical realization). It also provides the necessary information for students to engage with numerical realizations of the derivative. In chemistry, it is impossible to convert the mass of one element directly (for example, in grams) to the mass of another. Instead, mass-to-mole conversion should be used. This is also true for calculating the rate of change of reactions. One mole of carbon dioxide is approximately 44 grams ($\mathrm{CO_2}: 12 + 2 \times 16$). Therefore, to calculate the average rate of the reaction, the measured mass in grams should be converted to mol first by dividing them by 44. Afterwards, students can calculate the average rate of change in the six-second intervals by finding the corresponding difference quotients (e.g., $\overline{\mathrm{R}}(\mathrm{CO_2}) = \frac{5.5\times 10^{-3}-0}{6-0} = 0.92 \times 10^{-3}$) (Table 2).

T (second)	0	6	12	18	24	30	36
Weight of solution (g)	16.07	15.83	15.69	15.60	15.55	15.51	15.47
Weight of carbon dioxide (g)	0	0.24	0.38	0.47	0.52	0.56	0.60
Weight of carbon dioxide (mol$\times 10^3$)	0	5.5	8.6	10.7	11.8	12.7	13.6

Table 1: The weight of solution and carbon dioxide in our experiment

Intervals	1st: 0-6	2nd: 6-12	3rd: 12-18	4th: 18-24	5th: 24-30	6th: 30-36
$\overline{R}(CO_2)$ $(\frac{mol}{Second})$	0.92×10^{-3}	0.52×10^{-3}	0.35×10^{-3}	0.18×10^{-3}	0.15×10^{-3}	0.15×10^{-3}

Table 2: The average rate of change of the reaction over six-second intervals

The next step is to focus on approximating the instantaneous rate of change of the reaction. We chose to focus on $t = 18s$. We can use $N1$ and approximate the instantaneous rate of change of the reaction at $t = 18s$ using the average rate of change of the reaction in the third and fourth intervals (i.e., $(0.35 \times 10^{-3} + 0.18 \times 10^{-3})/2 = 0.265 \times 10^{-3}$). The next part could focus on the graphical realizations. We can start by putting the values of carbon dioxide we measured on an $x - y$ plane on Desmos or other suitable platforms. Then ask students what would be a suitable function to fit these points. Students can manipulate themselves on the platform and try different types of functions. One of the elementary functions accessible to students and provides a reasonable approximation here is the square root functions in the form of $y = a\sqrt{x}$ (see The reaction between calcium carbonate and hydrochloric acid (desmos.com); in this link, for convenience, the data on the y-axis is multiplied by 10^4). Then, the lecturer can suggest a few functions used more often in chemistry that fit the obtained data from the reaction much better. Two examples are natural logarithms and homographic functions. However, if students zoom in on the graphs, they would realize that these functions are not the best fit for the obtained data. This could provide an opportunity for the lecturer to discuss with the students that in more advanced courses (e.g., introductory numerical analysis course), they would learn how to approximate functions based on a given set of points. Here in the Desmos link, we also included the Lagrange polynomial interpolation method. Then the lecturer could continue with a homographic or a natural logarithm function, as the polynomial obtained by the Lagrange method is complex for first-year calculus students (in our opinion).

A reasonable fit for the obtained data from the reaction is $y = 53\ln(t + 3) - 57\ ;\ \ 0 \leq t \leq 36$. Its graph could be used to discuss some of the graphical realizations of the derivative with the help of Desmos, such as Zooming in on the function to discuss $G6$ and drawing secant lines to discuss $G7$-10. We can also discuss some of the symbolic realizations of the derivative with this function. For instance, the instantaneous rate of change could be estimated: $R(CO_2) = \lim_{\Delta t \to 0} \frac{\Delta n(Co_2)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{53}{t+3}$. Substituting $t = 18$ gives us another estimation for the derivative at this point which is $0.252 \times 10^{-3} \frac{m}{s}$; that is close to our other estimation.

Concluding words

While our suggested task is related to chemistry, such a design could be used for designing tasks in other contexts. To conclude, we suggest task designers start with the physical realization, which provides the necessary information for engaging students with the numerical realizations as suggested in the literature for teaching the derivative (e.g., Diaz Eaton et al., 2019; Roundy et al., 2015). Then, the graphical, symbolic, and verbal realizations could be the focus of teaching. In our search among several textbooks used for teaching calculus (e.g., Hass et al., 2018; Hughes-Hallett et al., 2017), the physical realization was not in focus when introducing the derivative. In Hughes-Hallett et al. (2017), the derivative is first introduced in a physical context (i.e., velocity) by focusing on the numerical realizations, whereas in Hass et al. (2018), it is first introduced by graphical and symbolic realizations. We hope this work also inspires textbook developers to consider the physical realization in calculus textbooks due to its importance, as highlighted in the literature: Mathematics could be taught "like the sciences as a laboratory discipline" (Diaz Eaton et al., 2019, p. 807). This could, among other things, help students feel "more agency to readily engage with the conversation on models and modeling" (Diaz Eaton et al., 2019, p. 807).

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