# THE CATEGORY OF EXTENSIONS AND A CHARACTERISATION OF $n$-EXANGULATED FUNCTORS

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Abstract. Additive categories play a fundamental role in mathematics and related disciplines. Given an additive category equipped with a biadditive functor, one can construct its category of extensions, which encodes important structural information. We study how functors between categories of extensions relate to those at the level of the original categories. When the additive categories in question are $n$-exangulated, this leads to a characterisation of $n$-exangulated functors.

Our approach enables us to study $n$-exangulated categories from a 2-categorical perspective. We introduce $n$-exangulated natural transformations and characterise them using categories of extensions. Our characterisations allow us to establish a 2-functor between the 2-categories of small $n$-exangulated categories and small exact categories. A similar result with no smallness assumption is also proved.

We employ our theory to produce various examples of $n$-exangulated functors and natural transformations. Although the motivation for this article stems from representation theory and the study of $n$-exangulated categories, our results are widely applicable: several require only an additive category equipped with a biadditive functor with no extra assumptions; others can be applied by endowing an additive category with its split $n$-exangulated structure.

## Contents



## 1. Introduction

*Additive categories* appear in various branches of mathematics and related disciplines. For the more algebraically inclined mathematician, perhaps the category of abelian groups is the prototypical example; for the more analytical, perhaps the category of Banach spaces over the real numbers; and the geometer may opt for some category of sheaves. The unsuspecting theoretical physicist might uncover that certain additive categories control possibilities in

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string theory or, more broadly, particle physics. Phenomena of this last kind demonstrate the power of category theory and the importance of its study.

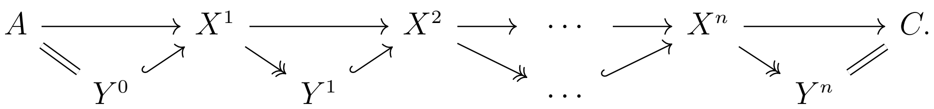
The motivation for this article stems from algebra, yet many of the results are widely applicable. Indeed, as we build up our theory, we ask only for an additive category $\mathcal{C}$ equipped with a biadditive functor $\mathbb{E}(-,-)\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to \mathsf{Ab}$, where $\mathsf{Ab}$ denotes the category of abelian groups. As an example, for any additive category $\mathcal{C}$, one can already take $\mathbb{E}(C,A)$ to be the abelian group $\mathcal{C}(C,A)$ of morphisms $C\to A$; see Example 3.3. More interesting choices of $\mathbb{E}$ can be made depending on the structure of $\mathcal{C}$; see Section 5.

For now, let us focus on two examples from classical homological algebra, namely *abelian categories* and *triangulated categories*. If $\mathcal{C}$ is a skeletally small abelian (or exact) category, then a possible choice for $\mathbb{E}$ is the functor $\mathrm{Ext}^1_{\mathcal{C}}$. In this case, the abelian group $\mathbb{E}(C,A)=\mathrm{Ext}^1_{\mathcal{C}}(C,A)$ consists of equivalence classes of short exact sequences in $\mathcal{C}$ of the form $0\longrightarrow A\longrightarrow - \longrightarrow C\longrightarrow 0$. If $\mathcal{C}$ is a triangulated category with suspension functor $\Sigma$, then one could set $\mathbb{E}(C,A)=\mathcal{C}(C,\Sigma A)$, which is in bijection with equivalence classes of distinguished triangles of the form $A\longrightarrow - \longrightarrow C\longrightarrow \Sigma A$. In both these examples, we see that the bifunctor $\mathbb{E}$ encodes the basic building blocks of the homological structure of $\mathcal{C}$.

Nakaoka–Palu [57] recently used the observations above to establish the theory of *extriangulated categories*, giving a simultaneous generalisation of exact and triangulated categories. An extriangulated category consists of a triplet $(\mathcal{C},\mathbb{E},\mathfrak{s})$, where $\mathcal{C}$ is an additive category, $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathsf{Ab}$ is a biadditive functor and $\mathfrak{s}$ is a *realisation* (see Definition 2.3), satisfying certain axioms. This new framework has proven significance: aside from permitting the unification and extension of many known results (see e.g. [23, 24, 52, 55]), it has led to novel insights and explained previously mysterious connections (see e.g. [41, 42, 60]). This again underlines the benefits of abstraction in mathematics.

The realisation $\mathfrak{s}$ of an extriangulated category $(\mathcal{C},\mathbb{E},\mathfrak{s})$ encapsulates a core idea from classical homological algebra, namely that one *realises* each $\delta\in\mathbb{E}(C,A)$ by an equivalence class $\mathfrak{s}(\delta)=[\ A\longrightarrow B\longrightarrow C\ ]$ of a pair of composable morphisms. Hence, the realisation allows us to visualise the structure encoded by $\mathbb{E}$ via 3-term sequences. For an integer $n\geqslant 1$, Iyama [32] discovered that $n$*-cluster tilting* subcategories of module categories exhibit structures reminiscent of those from classical homological algebra, but now involving longer sequences. It was demonstrated that $(n+2)$-term sequences could be used to study a higher-dimensional analogue of Auslander–Reiten theory in such settings. Jasso [36] formalised this idea by introducing $n$-exact and $n$-abelian categories. Similar observations in the triangulated setting (see e.g. [32], [33], [35]) motivated the axiomatisation of $(n+2)$*-angulated categories* by Geiss–Keller–Oppermann [21]. These formal frameworks constitute what has become known as *higher homological algebra*, where the case $n=1$ recovers classical theory of exact, abelian and triangulated categories. Higher homological algebra is linked to modern developments in various branches of mathematics, ranging from representation theory, cluster theory, commutative algebra, algebraic geometry, homological mirror symmetry and symplectic geometry to string theory, conformal field theory and combinatorics (see e.g. [1, 14, 16, 21, 25, 28, 34, 39, 59, 66]).

A central idea in the higher setup is that a suitable $n$-cluster tilting subcategory $\mathcal{T}$ of an abelian (resp. triangulated) category $\mathcal{A}$ is $n$-abelian [36, Thm. 3.16] (resp. $(n+2)$-angulated [21, Thm. 1]). Each admissible $n$-exact sequence $A \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow C$ in the $n$-abelian category $\mathcal{T}$ is obtained by splicing together $n$ short exact sequences $Y^{i-1} \hookrightarrow X^i \twoheadrightarrow Y^i$ from $\mathcal{A}$ as indicated in the following diagram


$(*)$

In this way, the higher structure of $\mathcal{T}$ is compatible with the classical structure of $\mathcal{A}$.

Of course, now one asks: What does "compatible" *formally* mean? The answer is work in preparation by the authors (see [5, 4]), but is inspired by the results and methodology from the present article. The approach taken involves the higher analogue of extriangulated categories, namely *$n$-exangulated categories* in the sense of Herschend–Liu–Nakaoka [29] (see Definition 2.5). As for extriangulated categories, an $n$-exangulated category consists of a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ where $\mathcal{C}$ is an additive category and $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ a biadditive functor. For each pair of objects $A, C \in \mathcal{C}$ and to each *extension* $\delta \in \mathbb{E}(C, A)$, the realisation associates an equivalence class $\mathfrak{s}(\delta) = [X^\bullet] = [\ A \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow C\ ]$ of an $(n+2)$-term sequence. Each $(n+2)$-angulated and $n$-exact category is $n$-exangulated (see Examples 5.3 and 5.4), and a category is extriangulated if and only if it is 1-exangulated (see Example 5.1).

Structure-preserving functors between $n$-exangulated categories have been formalised recently in [6]. Given $n$-exangulated categories $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$, an *$n$-exangulated functor* $(\mathscr{F}, \Gamma)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is a pair consisting of an additive functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\Gamma\colon \mathbb{E}(-,-) \Rightarrow \mathbb{E}'(\mathscr{F}-, \mathscr{F}-)$ satisfying a certain condition (see Definition 3.15). Compatibility of structures can be naturally expressed by means of $n$-exangulated functors in the case where the domain and codomain categories are both $n$-exangulated for the same $n$.

Let us return to the example of the inclusion functor $\mathcal{T} \hookrightarrow \mathcal{A}$ of an $n$-cluster tilting subcategory $\mathcal{T}$ into an ambient abelian category $\mathcal{A}$. As soon as $n > 1$, we have that $\mathcal{T}$ and $\mathcal{A}$ are higher abelian categories of differing "dimension", and hence the established notion of an $n$-exangulated functor does not apply. This demonstrates the need for terminology that allows one to describe compatibility of structures also in this more general setup. A naive attempt to fill this gap might be to define an *$(n,1)$-exangulated functor* from an $n$-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ to a 1-exangulated category $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ as a pair $(\mathscr{F}, \Gamma)$, where $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor as before, but where $\Gamma$ is now a natural transformation from $\mathbb{E}$ to an $n$-fold product arising from $\mathbb{E}'$ satisfying some compatibility conditions. For instance, in the situation of $(*)$ above, one would want $\Gamma$ to take one equivalence class $\delta = [\ A \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow C\ ]$ of an admissible $n$-exact sequence to an $n$-tuple $(\rho_n, \ldots, \rho_1)$ of equivalence classes of short exact sequences with $\rho_i = [\ Y^{i-1} \hookrightarrow X^i \twoheadrightarrow Y^i\ ]$. However, the careful reader quickly spots that for $n > 1$ there cannot be a natural transformation $\Gamma$ of this kind, as the domain and codomain of $\Gamma$ are functors with different domains. Furthermore, note that although we have focused on the $(n,1)$-case above for expository purposes, we more generally aim to study $(n,q)$-exangulated functors for $q \geqslant 1$.

Instead of expecting to describe compatibility of $n$-exangulated and $q$-exangulated structures by use of a natural transformation $\Gamma$ as above, our main result of Section 3 (see Theorem A) opens another avenue of approach. In this result, we characterise $n$-exangulated functors via the corresponding *categories of extensions*. Given an additive category $\mathcal{C}$ with a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$, there is an associated category denoted by $\mathbb{E}\text{-Ext}(\mathcal{C})$, which has as its objects extensions $\delta \in \mathbb{E}(C, A)$ as $A, C$ vary over objects in $\mathcal{C}$. For the unexplained terminology used in Theorem A, see Definitions 3.6 and 3.16.

**Theorem A** (See Theorem 3.17)**.** *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ be $n$-exangulated categories. Then there is a one-to-one correspondence*

$$\left\{ \begin{array}{c} n\text{-exangulated functors} \\ (\mathscr{F},\Gamma)\colon (\mathcal{C},\mathbb{E},\mathfrak{s}) \to (\mathcal{C}',\mathbb{E}',\mathfrak{s}') \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{pairs } (\mathscr{F},\mathscr{E}) \text{ of additive functors } \mathscr{F}: \mathcal{C} \to \mathcal{C}' \\ \text{and } \mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}'), \text{ where } \mathscr{E} \text{ respects} \\ \text{morphisms and distinguished } n\text{-exangles over } \mathscr{F} \end{array} \right\}.$$

The category $\mathbb{E}\text{-Ext}(\mathcal{C})$ comes equipped with an exact structure $\mathcal{X}_{\mathbb{E}}$ determined by the sections and retractions in $\mathcal{C}$; see Proposition 3.2 and Remark 3.4. Furthermore, if $\mathscr{F}: \mathcal{C} \to \mathcal{C}'$ is an additive functor, then any functor $\mathscr{E}: \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}')$ which respects morphisms over $\mathscr{F}$ satisfies $\mathscr{E}(\mathcal{X}_{\mathbb{E}}) \subseteq \mathcal{X}_{\mathbb{E}'}$, i.e. $\mathscr{E}$ is exact; see Proposition 3.11.

In addition to permitting a new perspective on the problem of defining structure-preserving functors between higher exangulated categories of possibly different dimensions, the one-to-one correspondence above is interesting in its own right. From Theorem A we deduce Corollary B, which provides a characterisation of what it means for an additive functor between $n$-exangulated categories to be $n$-exangulated. This is a useful tool for detecting $n$-exangulated functors, because it is often easier to observe that distinguished $n$-exangles are sent to distinguished $n$-exangles in a functorial way, than to check that the corresponding natural transformation is indeed natural; see Examples 5.9 and 5.11.

**Corollary B.** *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ be $n$-exangulated categories. For an additive functor $\mathscr{F}: \mathcal{C} \to \mathcal{C}'$, the following statements are equivalent.*

(i) *There exists a natural transformation $\Gamma\colon \mathbb{E}(-,-) \Rightarrow \mathbb{E}'(\mathscr{F}-, \mathscr{F}-)$ such that the pair $(\mathscr{F}, \Gamma)$ is an $n$-exangulated functor.*

(ii) *There exists an additive functor $\mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}')$ which respects both morphisms and distinguished $n$-exangles over $\mathscr{F}$.*

In Section 4 we study $n$-exangulated categories in a 2-category-theoretic setting by considering morphisms of $n$-exangulated functors. We introduce a higher version of natural transformations of extriangulated functors as defined by Nakaoka–Ogawa–Sakai [56, Def. 2.11(3)], which we call *$n$-exangulated natural transformations*; see Definition 4.1. Applying Theorem A and using the notation $\mathscr{E}_{(\mathscr{F},\Gamma)}$ for the exact functor $(\mathbb{E}\text{-Ext}(\mathcal{C}), \mathcal{X}_{\mathbb{E}}) \to (\mathbb{E}'\text{-Ext}(\mathcal{C}'), \mathcal{X}_{\mathbb{E}'})$ arising from an $n$-exangulated functor $(\mathscr{F}, \Gamma)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$, we give a characterisation of $n$-exangulated natural transformations; see Theorem C. In the following we use the Hebrew letter ב (beth). See Definition 4.18 for the meaning of *balanced*.

**Theorem C** (See Theorem 4.19). *Suppose* $(\mathscr{F},\Gamma),(\mathscr{G},\Lambda)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ *are $n$-exangulated functors. Then there is a one-to-one correspondence between $n$-exangulated natural transformations* $(\mathscr{F},\Gamma)\overset{\beth}{\Longrightarrow}(\mathscr{G},\Lambda)$ *and balanced natural transformations* $\mathscr{E}_{(\mathscr{F},\Gamma)}\overset{\langle\beth\rangle}{\Longrightarrow}\mathscr{E}_{(\mathscr{G},\Lambda)}$.

For $n\geqslant 1$, we consider the category $n$-$\mathsf{Exang}$ of *all* $n$-exangulated categories, which has properties just like a 2-category (see e.g. Proposition 4.12). However, due to the set-theoretic issue outlined in Remark 4.13, we cannot formally call $n$-$\mathsf{Exang}$ a 2-category. If we consider instead *small* categories, then we avoid such problems, and may talk of the 2-category $n$-$\mathsf{exang}$ of small $n$-exangulated categories. We use the correspondences of Theorems A and C to construct a 2-functor from $n$-$\mathsf{exang}$ to the 2-category $\mathsf{exact}$ of small exact categories.

**Theorem D** (See Corollary 4.25). *There is* 2*-functor* $n$-$\mathsf{exang}\to\mathsf{exact}$*, which sends a* 0*-cell* $(\mathcal{C},\mathbb{E},\mathfrak{s})$ *to* $(\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C}),\mathcal{X}_{\mathbb{E}})$*, a* 1*-cell* $(\mathscr{F},\Gamma)$ *to* $\mathscr{E}_{(\mathscr{F},\Gamma)}$ *and a* 2*-cell* $\beth$ *to* $\langle\beth\rangle$*.*

Theorem D is a consequence of Theorem 4.22. This latter result is more general, in that one can construct a functor from $n$-$\mathsf{Exang}$ to the category $\mathsf{Exact}$ of all exact categories which behaves just like the 2-functor described in Theorem D. Ignoring Remark 4.13, one should interpret Theorem 4.22 as establishing a 2-functor $n$-$\mathsf{Exang}\to\mathsf{Exact}$.

In Section 5 we provide several examples of $n$-exangulated categories, functors and natural transformations. Some of these examples also produce $n$-exangulated subcategories in the sense of [24, Def. 3.7].

**Conventions.** We write $A\in\mathcal{C}$ to denote that an object $A$ lies in a category $\mathcal{C}$. For $A,B\in\mathcal{C}$, we write $\mathcal{C}(A,B)$ for the collection of morphisms $A\to B$ in $\mathcal{C}$. Unless stated otherwise: our subcategories are always assumed to be full; and our functors are always assumed to be covariant. We write $\mathsf{Ab}$ for the category of abelian groups. Throughout this paper, let $n\geqslant 1$ denote a positive integer.

## 2. Preliminaries on $n$-exangulated categories

We follow [29, Sec. 2] in briefly recalling the definition of an $n$-exangulated category, which is a higher analogue of an extriangulated category as introduced in [57]. See also [30].

**Setup 2.1.** Throughout this section, we assume that $\mathcal{C}$ is an additive category and that $\mathbb{E}\colon\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathsf{Ab}$ is a *biadditive* functor. The latter means that for all $A,C\in\mathcal{C}$, the functors $\mathbb{E}(C,-)\colon\mathcal{C}\to\mathsf{Ab}$ and $\mathbb{E}(-,A)\colon\mathcal{C}^{\mathrm{op}}\to\mathsf{Ab}$ are both additive.

Let $A,C\in\mathcal{C}$ be arbitrary. The identity element of the abelian group $\mathbb{E}(C,A)$ is denoted by ${}_{A}0_{C}$. An element $\delta$ of $\mathbb{E}(C,A)$ is called an $\mathbb{E}$-*extension*, and we set

$$x_{\mathbb{E}}\delta:=\mathbb{E}(C,x)(\delta)\in\mathbb{E}(C,X)\qquad\text{and}\qquad z^{\mathbb{E}}\delta:=\mathbb{E}(z,A)(\delta)\in\mathbb{E}(Z,A)$$

for morphisms $x\colon A\to X$ and $z\colon Z\to C$ in $\mathcal{C}$. It follows that $z^{\mathbb{E}}x_{\mathbb{E}}\delta=\mathbb{E}(z,x)(\delta)=x_{\mathbb{E}}z^{\mathbb{E}}\delta$. Given $\delta\in\mathbb{E}(C,A)$ and $\rho\in\mathbb{E}(D,B)$, a *morphism of* $\mathbb{E}$-*extensions* from $\delta$ to $\rho$ is a pair $(a,c)$ of morphisms $a\colon A\to B$ and $c\colon C\to D$ in $\mathcal{C}$ such that

$$a_{\mathbb{E}}\delta=c^{\mathbb{E}}\rho. \tag{2.1}$$

If there is no confusion about the biadditive functor involved, $\mathbb{E}$-extensions and morphisms of $\mathbb{E}$-extensions are simply called *extensions* and *morphisms of extensions*, respectively. The Yoneda Lemma yields two natural transformations denoted and defined by

$$\begin{array}{rl} {}_{\mathbb{E}}\delta\colon \mathcal{C}(A,-) \Longrightarrow \mathbb{E}(C,-) \\ {}_{\mathbb{E}}\delta_X\colon \qquad x \longmapsto x_{\mathbb{E}}\delta \end{array} \quad \text{and} \quad \begin{array}{rl} {}^{\mathbb{E}}\delta\colon \mathcal{C}(-,C) \Longrightarrow \mathbb{E}(-,A) \\ {}^{\mathbb{E}}\delta_Z\colon \qquad z \longmapsto z^{\mathbb{E}}\delta. \end{array}$$

In order to explain how to associate a homotopy class of a complex to each extension, we recall some terminology and notation. We denote by $\mathsf{C}^n_{\mathcal{C}}$ the subcategory of the category of complexes $\mathsf{C}_{\mathcal{C}}$ in $\mathcal{C}$ consisting of complexes concentrated in degrees $0, 1, \dots, n, n+1$. That is, if $X^\bullet \in \mathsf{C}^n_{\mathcal{C}}$, then $X^i$ is zero if $i < 0$ or $i > n+1$. We write such a complex as

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1}.$$

**Definition 2.2.** (See [29, Def. 2.13].) Suppose $X^\bullet \in \mathsf{C}^n_{\mathcal{C}}$ and $\delta \in \mathbb{E}(X^{n+1}, X^0)$. If

$$\mathcal{C}(-,X^0) \xRightarrow{\mathcal{C}(-,d_X^0)} \mathcal{C}(-,X^1) \xRightarrow{\mathcal{C}(-,d_X^1)} \cdots \xRightarrow{\mathcal{C}(-,d_X^n)} \mathcal{C}(-,X^{n+1}) \xRightarrow{{}^{\mathbb{E}}\delta} \mathbb{E}(-,X^0)$$

and

$$\mathcal{C}(X^{n+1},-) \xRightarrow{\mathcal{C}(d_X^n,-)} \mathcal{C}(X^n,-) \xRightarrow{\mathcal{C}(d_X^{n-1},-)} \cdots \xRightarrow{\mathcal{C}(d_X^0,-)} \mathcal{C}(X^0,-) \xRightarrow{{}_{\mathbb{E}}\delta} \mathbb{E}(X^{n+1},-)$$

are both exact sequences of functors, then we call the pair $\langle X^\bullet, \delta\rangle$ an *$n$-exangle.*

For $A, C \in \mathcal{C}$ the not-necessarily-full subcategory $\mathsf{C}^n_{(A,C)}$ of $\mathsf{C}^n_{\mathcal{C}}$ is defined as follows. Objects of $\mathsf{C}^n_{(A,C)}$ are complexes $X^\bullet \in \mathsf{C}^n_{\mathcal{C}}$ with $X^0 = A$ and $X^{n+1} = C$. Given $X^\bullet, Y^\bullet \in \mathsf{C}^n_{(A,C)}$, set

$$\mathsf{C}^n_{(A,C)}(X^\bullet, Y^\bullet) \coloneqq \{\, f^\bullet = (f^0, \dots, f^{n+1}) \in \mathsf{C}^n_{\mathcal{C}}(X^\bullet, Y^\bullet) \mid f^0 = \mathrm{id}_A \text{ and } f^{n+1} = \mathrm{id}_C \,\}.$$

The usual notion of a *homotopy* between morphisms of complexes restricts to give an equivalence relation $\sim$ on $\mathsf{C}^n_{(A,C)}(X^\bullet, Y^\bullet)$. This gives rise to a new category $\mathsf{K}^n_{(A,C)}$ with the same objects as $\mathsf{C}^n_{(A,C)}$ and with $\mathsf{K}^n_{(A,C)}(X^\bullet, Y^\bullet) \coloneqq \mathsf{C}^n_{(A,C)}(X^\bullet, Y^\bullet)/\!\sim$. If the image of a morphism $f^\bullet \in \mathsf{C}^n_{(A,C)}(X^\bullet, Y^\bullet)$ in $\mathsf{K}^n_{(A,C)}(X^\bullet, Y^\bullet)$ is an isomorphism, then $f^\bullet$ is called a *homotopy equivalence* and we say that $X^\bullet$ and $Y^\bullet$ are *homotopy equivalent.* We denote the isomorphism class in $\mathsf{K}^n_{(A,C)}$ of an object $X^\bullet$ by $[X^\bullet]$.

**Definition 2.3.** (See [29, Def. 2.22].) Let $\mathfrak{s}$ be a correspondence that, for each $A, C \in \mathcal{C}$, associates to an extension $\delta \in \mathbb{E}(C, A)$ an isomorphism class $\mathfrak{s}(\delta) = [X^\bullet]$ in $\mathsf{K}^n_{(A,C)}$. Such an $\mathfrak{s}$ is said to be an *exact realisation of* $\mathbb{E}$ if the following conditions are satisfied.

(R0) Let $\delta \in \mathbb{E}(C, A)$ and $\rho \in \mathbb{E}(D, B)$ be extensions with $\mathfrak{s}(\delta) = [X^\bullet]$ and $\mathfrak{s}(\rho) = [Y^\bullet]$. For any morphism of extensions $(a, c)\colon \delta \to \rho$, there exists $f^\bullet \in \mathsf{C}^n_{\mathcal{C}}(X^\bullet, Y^\bullet)$ such that $f^0 = a$ and $f^{n+1} = c$. In this setting, we say that $X^\bullet$ *realises* $\delta$ and $f^\bullet$ *realises* $(a, c)$.

(R1) The pair $\langle X^\bullet, \delta\rangle$ is an $n$-exangle whenever $\mathfrak{s}(\delta) = [X^\bullet]$.

(R2) For all $A \in \mathcal{C}$, we have $\mathfrak{s}({}_A0_0) = [\ A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\ ]$ and $\mathfrak{s}({}_00_A) = [\ 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\mathrm{id}_A} A\ ]$.

If $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$ and $\mathfrak{s}(\delta) = [X^\bullet] = [\ X^0 \xrightarrow{d_X^0} X^1 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1}\ ]$, then $d_X^0$ is known as an $\mathfrak{s}$-*inflation* and $d_X^n$ as an $\mathfrak{s}$-*deflation*.

Before stating the main definition of this section, we recall the notion of a mapping cone.

**Definition 2.4.** (See [29, Def. 2.27].) Suppose $f^\bullet \in \mathsf{C}^n_{\mathcal{C}}(X^\bullet, Y^\bullet)$ with $f^0 = \mathrm{id}_A$ for some $A = X^0 = Y^0$. The *mapping cone* $M^\bullet \coloneqq \mathrm{MC}(f)^\bullet$ of $f^\bullet$ is the complex

$$X^1 \xrightarrow{d_M^0} X^2 \oplus Y^1 \xrightarrow{d_M^1} X^3 \oplus Y^2 \xrightarrow{d_M^2} \cdots \xrightarrow{d_M^{n-1}} X^{n+1} \oplus Y^n \xrightarrow{d_M^n} Y^{n+1}$$

in $\mathsf{C}^n_{\mathcal{C}}$, where $d_M^0 \coloneqq \begin{pmatrix} -d_X^1 \\ f^1 \end{pmatrix}$, $d_M^n \coloneqq (\, f^{n+1}\ \ d_Y^n\,)$, and $d_M^i \coloneqq \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix}$ when $0 < i < n$.

**Definition 2.5.** (See [29, Def. 2.32].) Let $\mathcal{C}$ be an additive category, $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ a biadditive functor and $\mathfrak{s}$ an exact realisation of $\mathbb{E}$. The triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called an *$n$-exangulated category* if the following conditions are satisfied.

(EA1) The composition of any two $\mathfrak{s}$-inflations is again an $\mathfrak{s}$-inflation, and the composition of any two $\mathfrak{s}$-deflations is again an $\mathfrak{s}$-deflation.

(EA2) For any $\delta \in \mathbb{E}(D, A)$ and any $c \in \mathcal{C}(C, D)$ with $\mathfrak{s}(c^{\mathbb{E}}\delta) = [X^\bullet]$ and $\mathfrak{s}(\delta) = [Y^\bullet]$, there exists a morphism $f^\bullet\colon X^\bullet \to Y^\bullet$ realising $(\mathrm{id}_A, c)$ such that $\mathfrak{s}((d_X^0)_{\mathbb{E}}\delta) = [\mathrm{MC}(f)^\bullet]$.

(EA2$^{\mathrm{op}}$) Dual of (EA2).

If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an $n$-exangulated category and $\mathfrak{s}(\delta) = [X^\bullet]$ for an extension $\delta \in \mathbb{E}(C, A)$, then we call $\langle X^\bullet, \delta \rangle$ a *distinguished* $n$-exangle.

## 3. The category of extensions and $n$-exangulated functors

Our main result in Section 3 is Theorem 3.17, which is Theorem A from Section 1. In Section 3.1 we recall the definition of the category of extensions associated to an additive category equipped with a biadditive functor. In Subsection 3.2 we characterise natural transformations of a certain form; see Setup 3.5 and Proposition 3.14. In Subsection 3.3 we use this characterisation to prove Theorem 3.17.

3.1. **The category of extensions.** For this subsection, assume that $\mathcal{C}$ is an additive category and that $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ is a biadditive functor (see Setup 2.1). The *category* $\mathbb{E}\text{-Ext}(\mathcal{C})$ *of* $\mathbb{E}$*-extensions* was considered in [57, Def. 2.3]. The authors thank Thomas Brüstle for informing them that similar ideas already appeared in the literature before; see Remark 3.4.

The objects and morphisms of $\mathbb{E}\text{-Ext}(\mathcal{C})$ are given by $\mathbb{E}$-extensions and morphisms of $\mathbb{E}$-extensions, respectively, as defined in Section 2. Recall that, for $\delta \in \mathbb{E}(C, A)$ and $\rho \in \mathbb{E}(D, B)$, a morphism $\delta \to \rho$ of $\mathbb{E}$-extensions is a pair $(a, c)$ of morphisms $a\colon A \to B$ and $c\colon C \to D$ in $\mathcal{C}$ such that $a_{\mathbb{E}}\delta = c^{\mathbb{E}}\rho$; see (2.1). If $\delta$ is an extension in $\mathbb{E}(C, A)$, then the identity morphism $\mathrm{id}_\delta$ of $\delta$ is given by the pair $(\mathrm{id}_A, \mathrm{id}_C)$. The composition of morphisms $(a, c)\colon \delta \to \rho$ and $(b, d)\colon \rho \to \eta$ in $\mathbb{E}\text{-Ext}(\mathcal{C})$ is the pair $(ba, dc)$. It is straightforward to check that $(ba, dc)$ is again a morphism of extensions and that $\mathbb{E}\text{-Ext}(\mathcal{C})$ is a category under this composition rule.

As one might expect, the category of $\mathbb{E}$-extensions is an additive category. Moreover, motivated by [18, Sec. 9.1, Exam. 5], we show that $\mathbb{E}\text{-Ext}(\mathcal{C})$ can be equipped with an exact structure $\mathcal{X}_{\mathbb{E}}$ that is not necessarily the split exact structure. Suppose $\delta \in \mathbb{E}(C,A)$, $\rho' \in \mathbb{E}(D,B)$ and $\eta \in \mathbb{E}(C',A')$. We declare that a sequence

$$\delta \xrightarrow{(a,c)} \rho' \xrightarrow{(b,d)} \eta \tag{3.1}$$

of morphisms in $\mathbb{E}$-Ext$(\mathcal{C})$ lies in $\mathcal{X}_{\mathbb{E}}$ if and only if $a$ and $c$ are both sections with $b = \operatorname{coker} a$ and $d = \operatorname{coker} c$. It follows from [63, Prop. 2.7] that (3.1) belongs to $\mathcal{X}_{\mathbb{E}}$ if and only if $b$ and $d$ are both retractions with $a = \ker b$ and $c = \ker d$. This is again equivalent to the underlying sequences $A \xrightarrow{a} B \xrightarrow{b} A'$ and $C \xrightarrow{c} D \xrightarrow{d} C'$ being split exact in $\mathcal{C}$. We call a sequence (3.1) belonging to $\mathcal{X}_{\mathbb{E}}$ a *conflation*, and in this case the morphism $(a,c)$ an *inflation* and $(b,d)$ a *deflation*. Notice that $\mathcal{X}_{\mathbb{E}}$ is closed under isomorphisms.

We use column and row notation $\iota_X = \left(\begin{smallmatrix} \mathrm{id}_X \\ 0 \end{smallmatrix}\right) \colon X \to X \oplus Y$ and $\pi_X = \left(\begin{smallmatrix} \mathrm{id}_X & 0 \end{smallmatrix}\right) \colon X \oplus Y \to X$ for the canonical inclusion and projection, respectively, associated to the biproduct of two objects $X$ and $Y$ in the additive category $\mathcal{C}$. Then, given a conflation (3.1), there are isomorphisms $h \colon B \to A \oplus A'$ and $g \colon D \to C \oplus C'$ in $\mathcal{C}$ such that (3.1) is isomorphic to

$$\delta \xrightarrow{(\iota_A, \iota_C)} \rho \xrightarrow{(\pi_{A'}, \pi_{C'})} \eta, \tag{3.2}$$

where $\rho = (g^{-1})^{\mathbb{E}} h_{\mathbb{E}} \rho'$. Moreover, the sequence (3.2) also lies in $\mathcal{X}_{\mathbb{E}}$. If $(a,c)$ is a morphism in $\mathbb{E}$-Ext$(\mathcal{C})$ consisting of a pair of sections both admitting cokernels, it is not a priori clear that it is an inflation. The following lemma verifies this.

**Lemma 3.1.** *Let $(a,c) \colon \delta \to \rho'$ be a morphism in $\mathbb{E}$-Ext$(\mathcal{C})$ for $\delta \in \mathbb{E}(C,A)$ and $\rho' \in \mathbb{E}(D,B)$. Suppose $a$ and $c$ are sections with cokernels $b = \operatorname{coker} a$ and $d = \operatorname{coker} c$. Then $(a,c)$ completes to a kernel-cokernel pair* (3.1)*, which is in $\mathcal{X}_{\mathbb{E}}$.*

*Proof.* By our remarks above, we may assume $B = A \oplus A'$, $D = C \oplus C'$ and that $(a,c)$ is of the form $(\iota_A, \iota_C) \colon \delta \to \rho$. Consider the sequence (3.2) with $\eta \coloneqq (\iota_{C'})^{\mathbb{E}} (\pi_{A'})_{\mathbb{E}} \rho$. Using that $(\iota_A, \iota_C)$ is a morphism in $\mathbb{E}$-Ext$(\mathcal{C})$ and writing $\mathrm{id}_D$ as $\iota_C \pi_C + \iota_{C'} \pi_{C'}$, it is straightforward to check that $(\pi_{A'}, \pi_{C'})$ is a morphism of extensions $\rho \to \eta$. It follows that (3.2) is in $\mathcal{X}_{\mathbb{E}}$.

To show that (3.2) is a kernel-cokernel pair, we first observe that $(\pi_{A'}, \pi_{C'})(\iota_A, \iota_C) = 0$. Let $\alpha \in \mathbb{E}(Z,X)$ be an extension and consider a morphism $(\left(\begin{smallmatrix} x \\ x' \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ z' \end{smallmatrix}\right)) \colon \alpha \to \rho$ in $\mathbb{E}$-Ext$(\mathcal{C})$ with $(\pi_{A'}, \pi_{C'}) (\left(\begin{smallmatrix} x \\ x' \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ z' \end{smallmatrix}\right)) = (0,0)$. This implies that $x'$ and $z'$ are zero. We claim that $(x,z)$ is a morphism of extensions $\alpha \to \delta$. Indeed, we have

$$(\iota_A)_{\mathbb{E}} x_{\mathbb{E}} \alpha = \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)_{\mathbb{E}} \alpha = \left(\begin{smallmatrix} x \\ x' \end{smallmatrix}\right)_{\mathbb{E}} \alpha = \left(\begin{smallmatrix} z \\ z' \end{smallmatrix}\right)^{\mathbb{E}} \rho = \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)^{\mathbb{E}} \rho = z^{\mathbb{E}} (\iota_C)^{\mathbb{E}} \rho = (\iota_A)_{\mathbb{E}} z^{\mathbb{E}} \delta,$$

where the last equality follows from $(\iota_A, \iota_C)$ being a morphism of extensions. As $(\iota_A)_{\mathbb{E}}$ is monic since $\iota_A$ is a section, this yields $x_{\mathbb{E}} \alpha = z^{\mathbb{E}} \delta$. Moreover, we conclude that $(x,z)$ is the unique morphism $\alpha \to \delta$ satisfying $(\left(\begin{smallmatrix} x \\ x' \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ z' \end{smallmatrix}\right)) = (\iota_A, \iota_C)(x,z)$, so $(\iota_A, \iota_C)$ is a kernel of $(\pi_{A'}, \pi_{C'})$. Similarly, one can verify that $(\pi_{A'}, \pi_{C'})$ is a cokernel of $(\iota_A, \iota_C)$. ■

We are now ready to show that $\mathcal{X}_{\mathbb{E}}$ is an exact structure on $\mathbb{E}$-Ext$(\mathcal{C})$.

**Proposition 3.2.** *The pair $(\mathbb{E}\text{-Ext}(\mathcal{C}), \mathcal{X}_{\mathbb{E}})$ is an exact category.*

*Proof.* We first verify that $\mathbb{E}$-Ext$(\mathcal{C})$ is an additive category. Let $\delta \in \mathbb{E}(C,A)$ and $\rho \in \mathbb{E}(D,B)$ be extensions. The collection of morphisms of extensions $\delta \to \rho$ in $\mathbb{E}$-Ext$(\mathcal{C})$ is a set, since

$\mathcal{C}(A,B)$ and $\mathcal{C}(C,D)$ are both groups and hence sets. The addition of morphisms $(a,c)\colon \delta\to\rho$ and $(a',c')\colon \delta\to\rho$ is defined by the pair $(a+a',c+c')$, which is a morphism $\delta\to\rho$ of extensions as $\mathbb{E}$ is biadditive. This establishes that $\mathbb{E}\text{-Ext}(\mathcal{C})$ is preadditive. By the biadditivity of $\mathbb{E}$, we have a natural isomorphism $\mathbb{E}(C\oplus D, A\oplus B)\cong \mathbb{E}(C,A)\oplus\mathbb{E}(C,B)\oplus\mathbb{E}(D,A)\oplus\mathbb{E}(D,B)$. As in [29, Def. 2.6], we let $\delta\oplus\rho\in\mathbb{E}(C\oplus D, A\oplus B)$ denote the element corresponding to $(\delta,0,0,\rho)$ via this isomorphism. It is straightforward to check that this gives a biproduct of extensions making $\mathbb{E}\text{-Ext}(\mathcal{C})$ an additive category; see, for instance, Liu–Tan [64, Rem. 2]. In particular, note that the biproduct inclusion and projection morphisms are of the form $\iota_\delta=(\iota_A,\iota_C)\colon \delta\to\delta\oplus\rho$ and $\pi_\delta=(\pi_A,\pi_C)\colon\delta\oplus\rho\to\delta$.

Next we show that $\mathcal{X}_\mathbb{E}$ is an exact structure on $\mathbb{E}\text{-Ext}(\mathcal{C})$. It follows from Lemma 3.1 that $\mathcal{X}_\mathbb{E}$ consists of kernel-cokernel pairs, using that (co)kernels are unique up to isomorphism. To check the axioms as in Bühler [10, Def. 2.1] of an exact category, it suffices to consider sequences of the form (3.2) as $\mathcal{X}_\mathbb{E}$ is closed under isomorphisms. The identity morphism of $\delta\in\mathbb{E}(C,A)$ is $\mathrm{id}_\delta=(\mathrm{id}_A,\mathrm{id}_C)$, which is a pair of sections admitting cokernels, so (E0) holds. The collection of morphisms of extensions that consist of pairs of sections that admit cokernels is closed under composition, so (E1) follows from Lemma 3.1. We prove (E2) below, and note that axioms ($\mathrm{E0^{op}}$), ($\mathrm{E1^{op}}$) and ($\mathrm{E2^{op}}$) can be shown dually.

For (E2), suppose we have a conflation (3.2). Let $(u,w)\colon\delta\to\beta$ be an arbitrary morphism where $\beta\in\mathbb{E}(W,U)$. By the universal property of the product in $\mathbb{E}\text{-Ext}(\mathcal{C})$, there exists a unique morphism $(e,f)\colon\delta\to\beta\oplus\rho$ for which $\pi_\beta(e,f)=(-u,-w)$ and $\pi_\rho(e,f)=(\iota_A,\iota_C)$. It follows that $(e,f)=\left(\left(\begin{smallmatrix}-u\\ \mathrm{id}_A\\ 0\end{smallmatrix}\right),\left(\begin{smallmatrix}-w\\ \mathrm{id}_C\\ 0\end{smallmatrix}\right)\right)$ is a pair of sections with $l:=\operatorname{coker} e=\left(\begin{smallmatrix}\mathrm{id}_U & u & 0\\ 0 & 0 & \mathrm{id}_{A'}\end{smallmatrix}\right)$ and $m:=\operatorname{coker} f=\left(\begin{smallmatrix}\mathrm{id}_W & w & 0\\ 0 & 0 & \mathrm{id}_{C'}\end{smallmatrix}\right)$. By Lemma 3.1, this implies that $(e,f)$ fits into a conflation $\delta\xrightarrow{(e,f)}\beta\oplus\rho\xrightarrow{(l,m)}\gamma$ with $\gamma\in\mathbb{E}(W\oplus C',U\oplus A')$. It is straightforward to check that $\gamma$ equipped with $(l,m)\iota_\beta\colon\beta\to\gamma$ and $(l,m)\iota_\rho\colon\rho\to\gamma$ is a pushout of $(\iota_A,\iota_C)$ along $(u,w)$. Lastly, we note that $(l,m)\iota_\beta=\left(\left(\begin{smallmatrix}\mathrm{id}_U\\ 0\end{smallmatrix}\right),\left(\begin{smallmatrix}\mathrm{id}_W\\ 0\end{smallmatrix}\right)\right)$ is a pair of sections admitting cokernels, and hence an inflation by Lemma 3.1. ■

In Example 3.3 we consider the category of extensions given by the Hom-bifunctor.

**Example 3.3.** Consider the biadditive functor $\mathbb{E}(-,-):=\mathcal{C}(-,-)\colon\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathsf{Ab}$. With this choice, the objects in $\mathbb{E}\text{-Ext}(\mathcal{C})$ coincide with morphisms in $\mathcal{C}$. For $\delta\in\mathbb{E}(C,A)$ and $\rho\in\mathbb{E}(D,B)$, a morphism $\delta\to\rho$ in $\mathbb{E}\text{-Ext}(\mathcal{C})$ is given by a pair $(a,c)$ with $a\in\mathcal{C}(A,B)$ and $c\in\mathcal{C}(C,D)$ such that $a_\mathbb{E}\delta=c^\mathbb{E}\rho$, i.e. such that the square

$$\begin{array}{ccc} C & \xrightarrow{\delta} & A\\ \big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle a}\\ D & \xrightarrow{\rho} & B\end{array}$$

commutes in $\mathcal{C}$. It follows that $\mathbb{E}\text{-Ext}(\mathcal{C})$ is the *arrow category of* $\mathcal{C}$. Furthermore, note that given $\delta\in\mathbb{E}(C,A)$ and $\rho\in\mathbb{E}(D,B)$, the biproduct $\delta\oplus\rho$ in the additive category $\mathbb{E}\text{-Ext}(\mathcal{C})$ is the morphism $\left(\begin{smallmatrix}\delta & 0\\ 0 & \rho\end{smallmatrix}\right)\colon C\oplus D\to A\oplus B$.

A sequence (3.2) in $\mathcal{X}_\mathbb{E}$ corresponds to a morphism $\left(\begin{smallmatrix}\delta & \alpha\\ 0 & \eta\end{smallmatrix}\right)\colon C\oplus C'\to A\oplus A'$, where $\alpha\colon C'\to A$ can be taken to be arbitrary. Such a conflation is trivial if and only if we have $\alpha=\delta\gamma-\beta\eta$

for some $\gamma\colon C' \to C$ and some $\beta\colon A' \to A$. For an example of a non-trivial conflation, one may hence take $\mathcal{C} = \mathsf{Ab}$, $\alpha = \mathrm{id}_{\mathbb{Z}}$ and $\delta = \eta$ to be the endomorphism of $\mathbb{Z}$ given by $d \mapsto 2d$.

*Remark* 3.4. The authors are grateful to Thomas Brüstle for pointing out that variants of the category $\mathbb{E}$-Ext$(\mathcal{C})$ have been studied before. Gabriel–Nazarova–Roiter–Sergeichuk–Vossieck [17, Sec. 1] considered a category of *$M$-spaces* for a functor $M$ from an *aggregate* (that is, a skeletally small, Hom-finite, Krull–Schmidt category) to a category of vector spaces. Gabriel–Roiter [18, p. 88, Exam. 5] looked more generally at a category defined by a bifunctor on a pair of aggregates, and this context was generalised further by Dräxler–Reiten–Smalø–Solberg [13, p. 670]. These examples also have analogues, where one restricts the focus to extensions of the form $\delta \in \mathbb{E}(C, A)$ with $A = C$; see, for example, Crawley-Boevey [11], Tiefenbrunner [65], Geiß [20] and Brüstle–Hille [9].

3.2. **Functors between categories of extensions.** In this subsection we discuss how functors between categories of extensions relate to functors on the underlying categories. This culminates in Proposition 3.14, from which Theorem 3.17 in the next subsection will follow.

**Setup 3.5.** For the remainder of this subsection, let $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ be a functor between additive categories $\mathcal{C}$ and $\mathcal{C}'$. Suppose also that $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ and $\mathbb{E}'\colon (\mathcal{C}')^{\mathrm{op}} \times \mathcal{C}' \to \mathsf{Ab}$ are biadditive functors.

In Setup 3.5 we do not assume the functor $\mathscr{F}$ to be additive. We explicitly impose this requirement whenever needed in the results that follow. Associated to $\mathscr{F}$ is the *opposite functor* $\mathscr{F}^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}')^{\mathrm{op}}$, and we usually abuse notation by writing $\mathscr{F}$ instead of $\mathscr{F}^{\mathrm{op}}$.

**Definition 3.6.** We say that a functor $\mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}')$ *respects morphisms over* $\mathscr{F}$ if, for every morphism $(a, c)\colon \delta \to \rho$ in $\mathbb{E}$-Ext$(\mathcal{C})$, the morphism $\mathscr{E}(a, c)\colon \mathscr{E}(\delta) \to \mathscr{E}(\rho)$ in $\mathbb{E}'$-Ext$(\mathcal{C}')$ is given by the pair $(\mathscr{F}a, \mathscr{F}c)$.

Building on Example 3.3, the following shows that a functor $\mathscr{F}$ between additive categories always induces a functor between categories of extensions that respects morphisms over $\mathscr{F}$.

**Example 3.7.** Recall that if we put $\mathbb{E}(-,-) = \mathcal{C}(-,-)$ and $\mathbb{E}'(-,-) = \mathcal{C}'(-,-)$, then $\mathbb{E}$-Ext$(\mathcal{C})$ and $\mathbb{E}'$-Ext$(\mathcal{C}')$ coincide with the arrow categories of $\mathcal{C}$ and $\mathcal{C}'$, respectively; see Example 3.3. Since functors preserve commutative squares, any functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ induces a functor $\mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}')$. This functor is defined by $\mathscr{E}(\delta) = \mathscr{F}\delta$ for $\delta \in \mathbb{E}(C, A)$, and by $\mathscr{E}(a, c) = (\mathscr{F}a, \mathscr{F}c)\colon \mathscr{E}(\delta) \to \mathscr{E}(\rho)$ for each morphism $(a, c)\colon \delta \to \rho$ in $\mathbb{E}$-Ext$(\mathcal{C})$. Note that $\mathscr{E}$ respects morphisms over $\mathscr{F}$ by construction.

Note that Lemma 3.10 below holds trivially in the setup of this example. In particular, the equations in Lemma 3.10(ii) say that $\mathscr{F}$ respects composition of morphisms. It is clear that if $\mathscr{F}$ is additive, then $\mathscr{E}$ is additive. The converse also holds, but involves a trick; see the proof of Proposition 3.11. In this case, we have $\mathscr{E}(\delta_1 + \delta_2) = \mathscr{E}(\delta_1) + \mathscr{E}(\delta_2)$ for $\delta_1, \delta_2 \in \mathbb{E}(C, A)$. These statements hold more generally; see Proposition 3.11 and Proposition 3.13.

*Remark* 3.8. Note that even though a functor $\mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}')$ that respects morphisms over $\mathscr{F}$ sends a pair $(a, c)$ of morphisms of $\mathcal{C}$ to the pair $(\mathscr{F}a, \mathscr{F}c)$ of morphisms of $\mathcal{C}'$, this does *not* mean that $\mathscr{E}$ is determined on all morphisms of $\mathbb{E}$-Ext$(\mathcal{C})$. For instance, in Example 3.7 one could also consider a functor $\widetilde{\mathscr{E}}$ that respects morphisms over $\mathscr{F}$, but is defined by $\widetilde{\mathscr{E}}(\delta) = -\mathscr{F}\delta$ on objects. Despite $\mathscr{E}(a,c)$ and $\widetilde{\mathscr{E}}(a,c)$ both being equal to $(\mathscr{F}a, \mathscr{F}c)$ as pairs of morphisms of $\mathcal{C}'$, we might have $\mathscr{E}(a,c) \neq \widetilde{\mathscr{E}}(a,c)$ as morphisms in $\mathbb{E}'$-Ext$(\mathcal{C}')$, since the domains or the codomains may not agree. Indeed, if $\mathscr{F}\delta \neq -\mathscr{F}\delta$ in $\mathbb{E}'(\mathscr{F}C, \mathscr{F}A)$, then the morphisms $\mathscr{E}(\mathrm{id}_\delta) = \mathrm{id}_{\mathscr{F}\delta}$ and $\widetilde{\mathscr{E}}(\mathrm{id}_\delta) = \mathrm{id}_{-\mathscr{F}\delta}$ are not the same identity morphisms in $\mathbb{E}'$-Ext$(\mathcal{C}')$, even though they are both given by the pair $(\mathrm{id}_{\mathscr{F}A}, \mathrm{id}_{\mathscr{F}C})$. Hence, the notion of respecting morphisms is not as strict as it may seem. In particular, a functor that is determined on all morphisms is automatically also determined on all objects, but this is not necessarily the case for functors respecting morphisms.

As the point made in Remark 3.8 is subtle, we now spell out explicitly what it means for two morphisms in the category $\mathbb{E}$-Ext$(\mathcal{C})$ to be equal.

*Remark* 3.9. Let $\delta \in \mathbb{E}(C,A)$, $\delta' \in \mathbb{E}(C',A')$, $\rho \in \mathbb{E}(D,B)$ and $\rho' \in \mathbb{E}(D',B')$ be extensions. Suppose one fixes morphisms $(a,c)\colon \delta \to \rho$ and $(a',c')\colon \delta' \to \rho'$ in the category $\mathbb{E}$-Ext$(\mathcal{C})$. These morphisms are equal in $\mathbb{E}$-Ext$(\mathcal{C})$ if and only if we have the equalities:

(i) $A = A'$, $B = B'$, $C = C'$ and $D = D'$ as objects in $\mathcal{C}$;

(ii) $\delta = \delta'$ as elements in $\mathbb{E}(C,A)$ and $\rho = \rho'$ as elements in $\mathbb{E}(D,B)$; and

(iii) $a = a'$ and $c = c'$ as morphisms in $\mathcal{C}$.

When checking that two morphisms $(a,c)\colon \delta \to \rho$ and $(a',c')\colon \delta' \to \rho'$ as above are equal, it is usually straightforward—but nonetheless essential—to verify requirements (i) and (ii). The verification of (iii) is typically less straightforward and often involves Definition 3.6.

Our next lemma, which we use to prove Propositions 3.13 and 3.14, describes natural compatibility properties for a functor that respects morphisms over $\mathscr{F}$.

**Lemma 3.10.** *Let $\mathscr{E}\colon \mathbb{E}$-Ext$(\mathcal{C}) \to \mathbb{E}'$-Ext$(\mathcal{C}')$ be a functor that respects morphisms over $\mathscr{F}$. The following statements hold for $\delta \in \mathbb{E}(C,A)$.*

(i) *The extension $\mathscr{E}(\delta)$ lies in $\mathbb{E}'(\mathscr{F}C, \mathscr{F}A)$.*

(ii) *If $x\colon A \to X$ and $z\colon Z \to C$ are morphisms in $\mathcal{C}$, then*

$$\mathscr{E}(x_{\mathbb{E}}\delta) = (\mathscr{F}x)_{\mathbb{E}'}\mathscr{E}(\delta) \qquad \text{and} \qquad \mathscr{E}(z^{\mathbb{E}}\delta) = (\mathscr{F}z)^{\mathbb{E}'}\mathscr{E}(\delta).$$

*Proof.* (i) As $\mathscr{E}$ is a functor, we have $\mathrm{id}_{\mathscr{E}(\delta)} = \mathscr{E}(\mathrm{id}_A, \mathrm{id}_C)$. Since $\mathscr{E}$ respects morphisms over $\mathscr{F}$, this equals $(\mathscr{F}\mathrm{id}_A, \mathscr{F}\mathrm{id}_C) = (\mathrm{id}_{\mathscr{F}A}, \mathrm{id}_{\mathscr{F}C})$, so $\mathscr{E}(\delta) \in \mathbb{E}'(\mathscr{F}C, \mathscr{F}A)$.

(ii) We only demonstrate the first identity, as the second is dual. By [29, Rem. 2.4], the pair $(x, \mathrm{id}_C)\colon \delta \to x_{\mathbb{E}}\delta$ is a morphism in $\mathbb{E}$-Ext$(\mathcal{C})$. Since $\mathscr{E}$ is a functor and respects morphisms over $\mathscr{F}$, this implies that $\mathscr{E}(x, \mathrm{id}_C) = (\mathscr{F}x, \mathrm{id}_{\mathscr{F}C})$ is a morphism $\mathscr{E}(\delta) \to \mathscr{E}(x_{\mathbb{E}}\delta)$ in $\mathbb{E}'$-Ext$(\mathcal{C}')$. Consequently, we have $(\mathscr{F}x)_{\mathbb{E}'}\mathscr{E}(\delta) = (\mathrm{id}_{\mathscr{F}C})^{\mathbb{E}'}\mathscr{E}(x_{\mathbb{E}}\delta) = \mathscr{E}(x_{\mathbb{E}}\delta)$. ■

The following proposition establishes the connection between the additivity of a functor $\mathscr{F}$ and that of, or the exactness of, a functor which respects morphisms over $\mathscr{F}$.

**Proposition 3.11.** *Let $\mathscr{E}\colon \mathbb{E}$-Ext$(\mathcal{C}) \to \mathbb{E}'$-Ext$(\mathcal{C}')$ be a functor that respects morphisms over $\mathscr{F}$. The following statements are equivalent.*

(i) *The functor* $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ *is additive.*

(ii) *The functor* $\mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}')$ *is additive.*

(iii) *The functor* $\mathscr{E}\colon (\mathbb{E}\text{-Ext}(\mathcal{C}), \mathcal{X}_{\mathbb{E}}) \to (\mathbb{E}'\text{-Ext}(\mathcal{C}'), \mathcal{X}_{\mathbb{E}'})$ *is exact.*

*Proof.* (i) $\Rightarrow$ (iii) For a conflation (3.1) in $\mathcal{X}_{\mathbb{E}}$, the underlying sequences $A \longrightarrow B \longrightarrow A'$ and $C \longrightarrow D \longrightarrow C'$ are split exact in $\mathcal{C}$. As $\mathscr{F}$ is additive, their images $\mathscr{F}A \longrightarrow \mathscr{F}B \longrightarrow \mathscr{F}A'$ and $\mathscr{F}C \longrightarrow \mathscr{F}D \longrightarrow \mathscr{F}C'$ under $\mathscr{F}$ are split exact in $\mathcal{C}'$. Since $\mathscr{E}$ respects morphisms over $\mathscr{F}$, the image $\mathscr{E}(\delta) \to \mathscr{E}(\rho') \to \mathscr{E}(\eta)$ of (3.1) under $\mathscr{E}$ is a sequence in $\mathcal{X}_{\mathbb{E}'}$.

(iii) $\Rightarrow$ (ii) An exact functor preserves finite biproducts and is hence additive.

(ii) $\Rightarrow$ (i) Recall that for any morphism $a\colon A \to B$ in $\mathcal{C}$ and for the unique element $\delta \in \mathbb{E}(0, A)$, there is a morphism $(a, \mathrm{id}_0)\colon \delta \to a_{\mathbb{E}}\delta$ in $\mathbb{E}\text{-Ext}(\mathcal{C})$. Let $a, b\colon A \to B$ be morphisms in $\mathcal{C}$. Note that the morphisms $(a, \mathrm{id}_0)\colon \delta \to a_{\mathbb{E}}\delta$ and $(b, \mathrm{id}_0)\colon \delta \to b_{\mathbb{E}}\delta$ in $\mathbb{E}\text{-Ext}(\mathcal{C})$ have the same codomain, as $a_{\mathbb{E}}\delta = b_{\mathbb{E}}\delta$ in the trivial abelian group $\mathbb{E}(0, B)$. Consequently, we can add $(a, \mathrm{id}_0)$ and $(b, \mathrm{id}_0)$ as morphisms $\delta \to a_{\mathbb{E}}\delta$, and so the sum $\mathscr{E}(a, \mathrm{id}_0) + \mathscr{E}(b, \mathrm{id}_0)$ also makes sense. We then have

$$
\begin{aligned}
(\mathscr{F}(a+b), \mathscr{F}\mathrm{id}_0) &= \mathscr{E}(a+b, \mathrm{id}_0) && \text{as } \mathscr{E} \text{ respects morphisms over } \mathscr{F} \\
&= \mathscr{E}(a+b, \mathrm{id}_0 + \mathrm{id}_0) && \text{as } \mathcal{C}(0,0) = \{\, \mathrm{id}_0 \,\} \text{ is trivial} \\
&= \mathscr{E}(a, \mathrm{id}_0) + \mathscr{E}(b, \mathrm{id}_0) && \text{as } \mathscr{E} \text{ is additive by assumption} \\
&= (\mathscr{F}a, \mathscr{F}\mathrm{id}_0) + (\mathscr{F}b, \mathscr{F}\mathrm{id}_0) && \text{as } \mathscr{E} \text{ respects morphisms over } \mathscr{F} \\
&= (\mathscr{F}a + \mathscr{F}b, \mathscr{F}\mathrm{id}_0 + \mathscr{F}\mathrm{id}_0).
\end{aligned}
$$

The computation yields $\mathscr{F}(a+b) = \mathscr{F}a + \mathscr{F}b$, and so $\mathscr{F}$ is additive. ∎

In the proof of Proposition 3.14, we use that functors which respect morphisms over additive functors preserve the additivity of extensions. This is shown in Proposition 3.13 below, for which we first recall some notation.

**Notation 3.12.** (See [29, Def. 2.6].) Given an object $X$ in an additive category, we use the notation $\iota_{X,1}$ and $\iota_{X,2}$ for the canonical inclusion morphisms $X \to X \oplus X$ into the first and second summand of the biproduct, respectively. Similarly, we write $\pi_{X,1}$ and $\pi_{X,2}$ for the canonical projection morphisms $X \oplus X \to X$. Let $\Delta_X\colon X \to X \oplus X$ (resp. $\nabla_X\colon X \oplus X \to X$) denote the diagonal (resp. codiagonal) morphism of $X$, i.e. the unique morphism such that $\pi_{X,i} \circ \Delta_X = \mathrm{id}_X$ (resp. $\nabla_X \circ \iota_{X,i} = \mathrm{id}_X$) for $i = 1, 2$. Note that for any $A, C \in \mathcal{C}$ and any extensions $\delta_1, \delta_2 \in \mathbb{E}(C, A)$, the addition in the abelian group $\mathbb{E}(C, A)$ relates to the biproduct in $\mathbb{E}\text{-Ext}(\mathcal{C})$ via the equation

$$\delta_1 + \delta_2 = \mathbb{E}(\Delta_C, \nabla_A)(\delta_1 \oplus \delta_2). \tag{3.3}$$

**Proposition 3.13.** *Suppose that* $\mathscr{F}$ *is additive and let* $\mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}')$ *be a functor that respects morphisms over* $\mathscr{F}$*. For all* $A, C \in \mathcal{C}$ *and for all extensions* $\delta_1, \delta_2 \in \mathbb{E}(C, A)$*, we have that* $\mathscr{E}(\delta_1 + \delta_2) = \mathscr{E}(\delta_1) + \mathscr{E}(\delta_2)$*.*

*Proof.* Since $\mathscr{F}$ is additive, there exists an isomorphism $f_X : \mathscr{F}(X \oplus X) \to \mathscr{F}X \oplus \mathscr{F}X$ for each $X \in \mathcal{C}$, such that for each $i = 1, 2$ the diagram

$$\begin{array}{ccccc} & & \mathscr{F}(X \oplus X) & & \\ & \overset{\mathscr{F}\iota_{X,i}}{\nearrow} & & \overset{\mathscr{F}\pi_{X,i}}{\searrow} & \\ \mathscr{F}X & & f_X \downarrow \cong & & \mathscr{F}X \\ & \underset{\iota_{\mathscr{F}X,i}}{\searrow} & & \underset{\pi_{\mathscr{F}X,i}}{\nearrow} & \\ & & \mathscr{F}X \oplus \mathscr{F}X & & \end{array} \tag{3.4}$$

in $\mathcal{C}'$ commutes. For $1 \leqslant i, j \leqslant 2$, it follows that

$$\mathbb{E}'(\iota_{\mathscr{F}C,i}, \pi_{\mathscr{F}A,j}) = \mathbb{E}'(\mathscr{F}\iota_{C,i}, \mathscr{F}\pi_{A,j}) \circ \mathbb{E}'(f_C, f_A^{-1}). \tag{3.5}$$

Recall from [29, Def. 2.6] that $\delta_1 \oplus \delta_2$ is the unique element in $\mathbb{E}(C \oplus C, A \oplus A)$ satisfying

$$\mathbb{E}(\iota_{C,i}, \pi_{A,j})(\delta_1 \oplus \delta_2) = \begin{cases} \delta_i & \text{if } i = j \\ {}_A0_C & \text{if } i \neq j, \end{cases} \tag{3.6}$$

and that $\mathscr{E}(\delta_1) \oplus \mathscr{E}(\delta_2)$ is the unique element in $\mathbb{E}'(\mathscr{F}C \oplus \mathscr{F}C, \mathscr{F}A \oplus \mathscr{F}A)$ satisfying

$$\mathbb{E}'(\iota_{\mathscr{F}C,i}, \pi_{\mathscr{F}A,j})(\mathscr{E}(\delta_1) \oplus \mathscr{E}(\delta_2)) = \begin{cases} \mathscr{E}(\delta_i) & \text{if } i = j \\ {}_{\mathscr{F}A}0_{\mathscr{F}C} & \text{if } i \neq j. \end{cases} \tag{3.7}$$

Consider the element $\eta \coloneqq \mathbb{E}'(f_C, f_A^{-1})^{-1}(\mathscr{E}(\delta_1 \oplus \delta_2))$ in $\mathbb{E}'(\mathscr{F}C \oplus \mathscr{F}C, \mathscr{F}A \oplus \mathscr{F}A)$. We claim that $\eta$ and $\mathscr{E}(\delta_1) \oplus \mathscr{E}(\delta_2)$ are equal. Note that it suffices to show that $\eta$ satisfies the defining equations (3.7) of $\mathscr{E}(\delta_1) \oplus \mathscr{E}(\delta_2)$. In order to verify this, let us first consider the zero morphism $0_A \colon A \to A$. The induced homomorphism $(0_A)_{\mathbb{E}} = \mathbb{E}(C, 0_A) \colon \mathbb{E}(C, A) \to \mathbb{E}(C, A)$ is trivial, and hence $(0_A)_{\mathbb{E}}({}_A0_C) = {}_A0_C$. Similarly, the homomorphism $(0_{\mathscr{F}A})_{\mathbb{E}'}$ is the zero map, so $(0_{\mathscr{F}A})_{\mathbb{E}'}(\mathscr{E}({}_A0_C)) = {}_{\mathscr{F}A}0_{\mathscr{F}C}$. Using these observations, we have that

$$\begin{aligned} \mathscr{E}({}_A0_C) &= \mathscr{E}((0_A)_{\mathbb{E}}({}_A0_C)) & \\ &= (\mathscr{F}0_A)_{\mathbb{E}'}(\mathscr{E}({}_A0_C)) & \text{by Lemma 3.10(ii)} \\ &= (0_{\mathscr{F}A})_{\mathbb{E}'}(\mathscr{E}({}_A0_C)) & \text{since } \mathscr{F} \text{ is additive} \\ &= {}_{\mathscr{F}A}0_{\mathscr{F}C}. & \end{aligned}$$

This gives

$$\mathbb{E}'(\mathscr{F}\iota_{C,i}, \mathscr{F}\pi_{A,j})(\mathscr{E}(\delta_1 \oplus \delta_2)) = \mathscr{E}(\mathbb{E}(\iota_{C,i}, \pi_{A,j})(\delta_1 \oplus \delta_2)) = \begin{cases} \mathscr{E}(\delta_i) & \text{if } i = j \\ {}_{\mathscr{F}A}0_{\mathscr{F}C} & \text{if } i \neq j, \end{cases} \tag{3.8}$$

where the first equality is by Lemma 3.10(ii) and the second holds by (3.6) and the observations above. We next see that

$$\begin{aligned} \mathbb{E}'(\iota_{\mathscr{F}C,i}, \pi_{\mathscr{F}A,j})(\eta) &= \mathbb{E}'(\mathscr{F}\iota_{C,i}, \mathscr{F}\pi_{A,j}) \circ \mathbb{E}'(f_C, f_A^{-1})(\eta) & \text{using (3.5)} \\ &= \mathbb{E}'(\mathscr{F}\iota_{C,i}, \mathscr{F}\pi_{A,j})(\mathscr{E}(\delta_1 \oplus \delta_2)) & \text{using the definition of } \eta \\ &= \begin{cases} \mathscr{E}(\delta_i) & \text{if } i = j \\ {}_{\mathscr{F}A}0_{\mathscr{F}C} & \text{if } i \neq j \end{cases} & \text{by (3.8).} \end{aligned}$$

Uniqueness hence yields $\mathbb{E}'(f_C, f_A^{-1})^{-1}(\mathscr{E}(\delta_1 \oplus \delta_2)) = \eta = \mathscr{E}(\delta_1) \oplus \mathscr{E}(\delta_2)$, which implies that

$$\mathscr{E}(\delta_1 \oplus \delta_2) = \mathbb{E}'(f_C, f_A^{-1})(\mathscr{E}(\delta_1) \oplus \mathscr{E}(\delta_2)). \tag{3.9}$$

The commutativity of (3.4) combined with uniqueness statements from the universal properties of the diagonal $\Delta_{\mathscr{F}C}$ and the codiagonal $\nabla_{\mathscr{F}A}$ gives

$$\Delta_{\mathscr{F}C} = f_C \circ \mathscr{F}\Delta_C \qquad \text{and} \qquad \nabla_{\mathscr{F}A} = (\mathscr{F}\nabla_A) \circ f_A^{-1}. \tag{3.10}$$

Altogether, we conclude that

$$\begin{aligned}
\mathscr{E}(\mathbb{E}(\Delta_C, \nabla_A)(\delta_1 \oplus \delta_2)) &= \mathbb{E}'(\mathscr{F}\Delta_C, \mathscr{F}\nabla_A)(\mathscr{E}(\delta_1 \oplus \delta_2)) && \text{by Lemma 3.10(ii)} \\
&= \mathbb{E}'(\mathscr{F}\Delta_C, \mathscr{F}\nabla_A)\mathbb{E}'(f_C, f_A^{-1})(\mathscr{E}(\delta_1) \oplus \mathscr{E}(\delta_2)) && \text{by (3.9)} \\
&= \mathbb{E}'(\Delta_{\mathscr{F}C}, \nabla_{\mathscr{F}A})(\mathscr{E}(\delta_1) \oplus \mathscr{E}(\delta_2)) && \text{by (3.10).}
\end{aligned}$$

Thus, using the description in (3.3), we see that $\mathscr{E}(\delta_1 + \delta_2) = \mathscr{E}(\delta_1) + \mathscr{E}(\delta_2)$, as required. ■

We are now ready to prove the main result of this subsection, characterising the existence of an additive functor between categories of extensions that respects morphisms. This is the key ingredient in the proof of Theorem 3.17. We remark that, by Proposition 3.11, the functors $\mathscr{E}$ appearing in the right-hand side of the statement of Proposition 3.14 are exact functors of the form $(\mathbb{E}\text{-Ext}(\mathcal{C}), \mathcal{X}_{\mathbb{E}}) \to (\mathbb{E}'\text{-Ext}(\mathcal{C}'), \mathcal{X}_{\mathbb{E}'})$. Moreover, we note that the assignment from left to right in Proposition 3.14 has been proven independently by Børve–Trygsland; see [8, Lem. 4.2]. We include the argument for completeness.

**Proposition 3.14.** *For an additive functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$, there is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{natural transformations} \\ \Gamma\colon \mathbb{E}(-,-) \Rightarrow \mathbb{E}'(\mathscr{F}-, \mathscr{F}-) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{additive functors } \mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}') \\ \textit{that respect morphisms over } \mathscr{F} \end{array} \right\}$$

$$\begin{aligned} \Gamma &\longmapsto \mathscr{E}_{(\mathscr{F},\Gamma)} \\ \Gamma_{(\mathscr{F},\mathscr{E})} &\longleftarrow \mathscr{E}, \end{aligned}$$

*where $\mathscr{E}_{(\mathscr{F},\Gamma)}(\delta) \coloneqq \Gamma_{(C,A)}(\delta)$ and $(\Gamma_{(\mathscr{F},\mathscr{E})})_{(C,A)}(\delta) \coloneqq \mathscr{E}(\delta)$ for each $\delta \in \mathbb{E}(C, A)$.*

*Proof.* Suppose that $\Gamma\colon \mathbb{E}(-,-) \Rightarrow \mathbb{E}'(\mathscr{F}-, \mathscr{F}-)$ is a natural transformation. We define $\mathscr{E}_{(\mathscr{F},\Gamma)}$ by setting $\mathscr{E}_{(\mathscr{F},\Gamma)}(\delta) \coloneqq \Gamma_{(C,A)}(\delta)$ for objects $\delta \in \mathbb{E}(C, A)$ and $\mathscr{E}_{(\mathscr{F},\Gamma)}(a, c) \coloneqq (\mathscr{F}a, \mathscr{F}c)$ for morphisms $(a, c)\colon \delta \to \rho$ in $\mathbb{E}\text{-Ext}(\mathcal{C})$. We need to show that this gives an additive functor $\mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathcal{C}')$ which respects morphisms over $\mathscr{F}$.

Let us first check that $\mathscr{E}_{(\mathscr{F},\Gamma)}(a, c) = (\mathscr{F}a, \mathscr{F}c)$ is a morphism from $\mathscr{E}_{(\mathscr{F},\Gamma)}(\delta) = \Gamma_{(C,A)}(\delta)$ to $\mathscr{E}_{(\mathscr{F},\Gamma)}(\rho) = \Gamma_{(D,B)}(\rho)$ in $\mathbb{E}'\text{-Ext}(\mathcal{C}')$ whenever $(a, c)$ is a morphism from $\delta \in \mathbb{E}(C, A)$ to $\rho \in \mathbb{E}(D, B)$ in $\mathbb{E}\text{-Ext}(\mathcal{C})$. Note that by the naturality of $\Gamma$, for any pair of morphisms $a\colon A \to B$ and $c\colon C \to D$ of $\mathcal{C}$, the diagram

$$\begin{array}{ccccc}
\mathbb{E}(C, A) & \xrightarrow{\mathbb{E}(C, a)} & \mathbb{E}(C, B) & \xleftarrow{\mathbb{E}(c, B)} & \mathbb{E}(D, B) \\
\downarrow{\scriptstyle \Gamma_{(C,A)}} & & \downarrow{\scriptstyle \Gamma_{(C,B)}} & & \downarrow{\scriptstyle \Gamma_{(D,B)}} \\
\mathbb{E}'(\mathscr{F}C, \mathscr{F}A) & \xrightarrow{\mathbb{E}'(\mathscr{F}C, \mathscr{F}a)} & \mathbb{E}'(\mathscr{F}C, \mathscr{F}B) & \xleftarrow{\mathbb{E}'(\mathscr{F}c, \mathscr{F}B)} & \mathbb{E}'(\mathscr{F}D, \mathscr{F}B)
\end{array}$$

commutes. Suppose furthermore that the pair $(a,c)$ defines a morphism $\delta \to \rho$ in $\mathbb{E}$-Ext$(\mathcal{C})$, where $\delta \in \mathbb{E}(C,A)$ and $\rho \in \mathbb{E}(D,B)$. Assuming additionally $(a,c)\colon \delta \to \rho$ is a morphism in $\mathbb{E}$-Ext$(\mathcal{C})$, we have $\mathbb{E}(C,a)(\delta) = \mathbb{E}(c,B)(\rho)$. Applying $\Gamma_{(C,B)}$ to this equality, the commutativity above gives

$$\mathbb{E}'(\mathscr{F}C,\mathscr{F}a)(\Gamma_{(C,A)}(\delta)) = \mathbb{E}'(\mathscr{F}c,\mathscr{F}B)(\Gamma_{(D,B)}(\rho)),$$

so $(\mathscr{F}a,\mathscr{F}c)\colon \Gamma_{(C,A)}(\delta) \to \Gamma_{(D,B)}(\rho)$ is a morphism in $\mathbb{E}'$-Ext$(\mathcal{C}')$. It is clear that the assignment $\mathscr{E}_{(\mathscr{F},\Gamma)}$ respects identity morphisms and composition, and thus defines a functor $\mathbb{E}$-Ext$(\mathcal{C}) \to \mathbb{E}'$-Ext$(\mathcal{C}')$. Notice that $\mathscr{E}_{(\mathscr{F},\Gamma)}$ respects morphisms over $\mathscr{F}$ by construction. The additivity of $\mathscr{E}_{(\mathscr{F},\Gamma)}$ follows from the additivity of $\mathscr{F}$ by Proposition 3.11.

Conversely, suppose we are given an additive functor $\mathscr{E}\colon \mathbb{E}$-Ext$(\mathcal{C}) \to \mathbb{E}'$-Ext$(\mathcal{C}')$ that respects morphisms over $\mathscr{F}$. Note that by Lemma 3.10(i), we have $\mathscr{E}(\delta) \in \mathbb{E}'(\mathscr{F}C,\mathscr{F}A)$ whenever $\delta \in \mathbb{E}(C,A)$. For each pair $A, C \in \mathcal{C}$, we can hence write $(\Gamma_{(\mathscr{F},\mathscr{E})})_{(C,A)}(\delta) \coloneqq \mathscr{E}(\delta)$ to define a function $(\Gamma_{(\mathscr{F},\mathscr{E})})_{(C,A)}\colon \mathbb{E}(C,A) \to \mathbb{E}'(\mathscr{F}C,\mathscr{F}A)$. It follows from Proposition 3.13 that the functions $(\Gamma_{(\mathscr{F},\mathscr{E})})_{(C,A)}$ are group homomorphisms. The diagram

$$\begin{array}{ccccc}
\mathbb{E}(Z,A) & \xleftarrow{\mathbb{E}(z,A)} & \mathbb{E}(C,A) & \xrightarrow{\mathbb{E}(C,x)} & \mathbb{E}(C,X) \\
\downarrow{\scriptstyle (\Gamma_{(\mathscr{F},\mathscr{E})})_{(Z,A)}} & & \downarrow{\scriptstyle (\Gamma_{(\mathscr{F},\mathscr{E})})_{(C,A)}} & & \downarrow{\scriptstyle (\Gamma_{(\mathscr{F},\mathscr{E})})_{(C,X)}} \\
\mathbb{E}'(\mathscr{F}Z,\mathscr{F}A) & \xleftarrow{\mathbb{E}'(\mathscr{F}z,\mathscr{F}A)} & \mathbb{E}'(\mathscr{F}C,\mathscr{F}A) & \xrightarrow{\mathbb{E}'(\mathscr{F}C,\mathscr{F}x)} & \mathbb{E}'(\mathscr{F}C,\mathscr{F}X)
\end{array}$$

commutes for any pair of morphisms $x\colon A \to X$ and $z\colon Z \to C$ of $\mathcal{C}$ by Lemma 3.10(ii). Commutativity of diagrams of the above form imply the naturality of the transformation $\Gamma_{(\mathscr{F},\mathscr{E})} = \{(\Gamma_{(\mathscr{F},\mathscr{E})})_{(C,A)}\}_{(C,A)\in\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}\colon \mathbb{E}(-,-) \Longrightarrow \mathbb{E}'(\mathscr{F}-,\mathscr{F}-)$.

By the arguments above, we see that the assignments $\Gamma \mapsto \mathscr{E}_{(\mathscr{F},\Gamma)}$ and $\mathscr{E} \mapsto \Gamma_{(\mathscr{F},\mathscr{E})}$ from the statement of the proposition are well-defined. It is straightforward to check that they are mutually inverse, and hence define a one-to-one correspondence. ■

3.3. **A characterisation of $n$-exangulated functors.** In this subsection we first recall the definition of an $n$-exangulated functor as introduced in [6]. This notion captures what it means for a functor between $n$-exangulated categories to be structure-preserving. We then prove the main result of Section 3, namely Theorem 3.17, which gives a characterisation of $n$-exangulated functors in terms of functors on the associated categories of extensions. We conclude with a lemma pertaining to the composition of these kinds of functors, which will allow us to take a 2-categorical perspective on $n$-exangulated categories in Section 4.

Recall that given an additive functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ between additive categories, there is an induced functor $\mathscr{F}_{\mathsf{C}}\colon \mathsf{C}_{\mathcal{C}} \to \mathsf{C}_{\mathcal{C}'}$ between the corresponding categories of complexes. For $X^\bullet \in \mathsf{C}_{\mathcal{C}}$, the object $\mathscr{F}_{\mathsf{C}}X^\bullet \in \mathsf{C}_{\mathcal{C}'}$ has $(\mathscr{F}_{\mathsf{C}}X^\bullet)^i = \mathscr{F}(X^i)$ in degree $i \in \mathbb{Z}$. The differential of $\mathscr{F}_{\mathsf{C}}X^\bullet$ is given by $d^i_{\mathscr{F}_{\mathsf{C}}X} = \mathscr{F}(d^i_X)$, where $d_X$ denotes the differential of $X^\bullet$. For a morphism $f^\bullet$ in $\mathsf{C}_{\mathcal{C}}$, we have $\mathscr{F}_{\mathsf{C}}f^\bullet = (\ldots,\mathscr{F}(f^{i-1}),\mathscr{F}(f^i),\mathscr{F}(f^{i+1}),\ldots)$. For the remainder of this section, let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ and $(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ be $n$-exangulated categories.

**Definition 3.15.** (See [6, Def. 2.32].) Let $\mathscr{F}\colon \mathcal{C}\to\mathcal{C}'$ be an additive functor and let

$$\Gamma = \{\Gamma_{(C,A)}\}_{(C,A)\in\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}\colon \mathbb{E}(-,-)\Longrightarrow \mathbb{E}'(\mathscr{F}^{\mathrm{op}}-,\mathscr{F}-)$$

be a natural transformation. We call the pair $(\mathscr{F},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ an *$n$-exangulated functor* if, for all $X^0,X^{n+1}\in\mathcal{C}$ and each $\delta\in\mathbb{E}(X^{n+1},X^0)$, we have that $\mathfrak{s}(\delta)=[X^\bullet]$ implies $\mathfrak{s}'(\Gamma_{(X^{n+1},X^0)}(\delta))=[\mathscr{F}_{\mathsf{C}}X^\bullet]$.

A similar structure-preservation condition exists for functors on categories of extensions.

**Definition 3.16.** Suppose $\mathscr{F}\colon\mathcal{C}\to\mathcal{C}'$ is an additive functor. We say that a functor $\mathscr{E}\colon\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C})\to\mathbb{E}'\text{-}\mathrm{Ext}(\mathcal{C}')$ *respects distinguished $n$-exangles over $\mathscr{F}$* if $\mathfrak{s}(\delta)=[X^\bullet]$ implies $\mathfrak{s}'(\mathscr{E}(\delta))=[\mathscr{F}_{\mathsf{C}}X^\bullet]$.

We are now ready to prove Theorem A from Section 1. Again, by Proposition 3.11, each $\mathscr{E}$ in the statement below is an exact functor $(\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C}),\mathcal{X}_{\mathbb{E}})\to(\mathbb{E}'\text{-}\mathrm{Ext}(\mathcal{C}'),\mathcal{X}_{\mathbb{E}'})$.

**Theorem 3.17.** *There is a one-to-one correspondence*

$$\left\{\begin{array}{c} n\text{-exangulated functors}\\ (\mathscr{F},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')\end{array}\right\}\longleftrightarrow\left\{\begin{array}{c}\text{pairs }(\mathscr{F},\mathscr{E})\text{ of additive functors }\mathscr{F}\colon\mathcal{C}\to\mathcal{C}'\\ \text{and }\mathscr{E}\colon\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C})\to\mathbb{E}'\text{-}\mathrm{Ext}(\mathcal{C}'),\text{ where }\mathscr{E}\text{ respects}\\ \text{morphisms and distinguished }n\text{-exangles over }\mathscr{F}\end{array}\right\}$$

$$\begin{array}{rcl}(\mathscr{F},\Gamma) & \longmapsto & (\mathscr{F},\mathscr{E}_{(\mathscr{F},\Gamma)})\\ (\mathscr{F},\Gamma_{(\mathscr{F},\mathscr{E})}) & \longleftarrow & (\mathscr{F},\mathscr{E}),\end{array}$$

*where $\mathscr{E}_{(\mathscr{F},\Gamma)}(\delta)\coloneqq\Gamma_{(C,A)}(\delta)$ and $(\Gamma_{(\mathscr{F},\mathscr{E})})_{(C,A)}(\delta)\coloneqq\mathscr{E}(\delta)$ for $\delta\in\mathbb{E}(C,A)$.*

*Proof.* Suppose first that $(\mathscr{F},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ is an $n$-exangulated functor. The functor $\mathscr{F}\colon\mathcal{C}\to\mathcal{C}'$ is hence additive, and Proposition 3.14 yields that $\mathscr{E}_{(\mathscr{F},\Gamma)}$ as defined above is an additive functor $\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C})\to\mathbb{E}'\text{-}\mathrm{Ext}(\mathcal{C}')$ that respects morphisms over $\mathscr{F}$. As $(\mathscr{F},\Gamma)$ is $n$-exangulated, we have that $\mathfrak{s}(\delta)=[X^\bullet]$ implies $\mathfrak{s}'(\mathscr{E}_{(\mathscr{F},\Gamma)}(\delta))=\mathfrak{s}'(\Gamma_{(X^{n+1},X^0)}(\delta))=[\mathscr{F}_{\mathsf{C}}X^\bullet]$, so $\mathscr{E}_{(\mathscr{F},\Gamma)}$ respects distinguished $n$-exangles over $\mathscr{F}$.

On the other hand, consider a pair $(\mathscr{F},\mathscr{E})$ from the right-hand side of the claimed correspondence. Proposition 3.14 yields that $\Gamma_{(\mathscr{F},\mathscr{E})}$ as defined above is a natural transformation $\mathbb{E}(-,-)\Rightarrow\mathbb{E}'(\mathscr{F}-,\mathscr{F}-)$. If $\mathfrak{s}(\delta)=[X^\bullet]$, then we have $\mathfrak{s}'((\Gamma_{(\mathscr{F},\mathscr{E})})_{(X^{n+1},X^0)}(\delta))=[\mathscr{F}_{\mathsf{C}}X^\bullet]$ by assumption, so $(\mathscr{F},\Gamma_{(\mathscr{F},\mathscr{E})})\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ is an $n$-exangulated functor.

Consequently, the assignments $(\mathscr{F},\Gamma)\mapsto(\mathscr{F},\mathscr{E}_{(\mathscr{F},\Gamma)})$ and $(\mathscr{F},\mathscr{E})\mapsto(\mathscr{F},\Gamma_{(\mathscr{F},\mathscr{E})})$ from the statement of theorem are well-defined. It is straightforward to check that these two assignments are mutually inverse, and hence define a one-to-one correspondence. ∎

Recall that Corollary B of Section 1 interprets, in terms of functors between categories of extensions, what it means for an additive functor between $n$-exangulated categories to be structure-preserving. This corollary is an immediate consequence of Theorem 3.17.

For the remainder of this section, suppose also that $(\mathcal{C}'',\mathbb{E}'',\mathfrak{s}'')$ is an $n$-exangulated category and that $(\mathscr{F},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ and $(\mathscr{L},\Phi)\colon(\mathcal{C}',\mathbb{E}',\mathfrak{s}')\to(\mathcal{C}'',\mathbb{E}'',\mathfrak{s}'')$ are $n$-exangulated functors. There is then a natural transformation

$$\Phi_{\mathscr{F}\times\mathscr{F}}=\{\Phi_{(\mathscr{F}C,\mathscr{F}A)}\}_{(C,A)\in\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}\colon\mathbb{E}'(\mathscr{F}-,\mathscr{F}-)\Longrightarrow\mathbb{E}''(\mathscr{L}\mathscr{F}-,\mathscr{L}\mathscr{F}-).$$

This is known as the *whiskering of* $\mathscr{F} \times \mathscr{F}$ *and* $\Phi$. We also use whiskerings in Section 4; see Notation 4.5 and onward.

Whiskerings enable us to define the composition of $n$-exangulated functors. This is a higher analogue of the composition of extriangulated functors from [56, Def. 2.11(2)].

**Definition 3.18.** (i) The *identity $n$-exangulated functor* of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is the pair $(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathbb{E}})$.

(ii) The *composite* of $(\mathscr{F}, \Gamma)$ and $(\mathscr{L}, \Phi)$ is $(\mathscr{L}, \Phi) \circ (\mathscr{F}, \Gamma) \coloneqq (\mathscr{L} \circ \mathscr{F}, \Phi_{\mathscr{F} \times \mathscr{F}} \circ \Gamma)$.

We conclude the section by justifying our terminology in Definition 3.18 and showing that the left-to-right assignment in Theorem 3.17 is compatible with identity and composition of $n$-exangulated functors.

**Lemma 3.19.** *The following statements hold.*

(i) *The pair* $(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathbb{E}})$ *is an $n$-exangulated functor* $(\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

(ii) *The composite* $(\mathscr{L}, \Phi) \circ (\mathscr{F}, \Gamma) \colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}'', \mathbb{E}'', \mathfrak{s}'')$ *is $n$-exangulated. This composition is associative and unital with respect to identity $n$-exangulated functors.*

(iii) *There are equalities* $\mathscr{E}_{(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathbb{E}})} = \mathrm{id}_{\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C})}$ *and* $\mathscr{E}_{(\mathscr{L}, \Phi) \circ (\mathscr{F}, \Gamma)} = \mathscr{E}_{(\mathscr{L}, \Phi)} \circ \mathscr{E}_{(\mathscr{F}, \Gamma)}$.

*Proof.* Checking (i), (ii) and the first part of (iii) is straightforward. For the second claim of (iii), note that $\mathscr{E}_{(\mathscr{L}, \Phi) \circ (\mathscr{F}, \Gamma)} = \mathscr{E}_{(\mathscr{L}\mathscr{F}, \Phi_{\mathscr{F} \times \mathscr{F}} \Gamma)}$ and $\mathscr{E}_{(\mathscr{L}, \Phi)} \circ \mathscr{E}_{(\mathscr{F}, \Gamma)}$ agree on objects of $\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C})$. By Theorem 3.17, the functors $\mathscr{E}_{(\mathscr{F}, \Gamma)}$, $\mathscr{E}_{(\mathscr{L}, \Phi)}$ and $\mathscr{E}_{(\mathscr{L}\mathscr{F}, \Phi_{\mathscr{F} \times \mathscr{F}} \Gamma)}$ respect morphisms over $\mathscr{F}$, $\mathscr{L}$ and $\mathscr{L}\mathscr{F}$, respectively. Since $\mathscr{E}_{(\mathscr{L}, \Phi)}(\mathscr{E}_{(\mathscr{F}, \Gamma)}(a, c)) = (\mathscr{L}\mathscr{F}a, \mathscr{L}\mathscr{F}c) = \mathscr{E}_{(\mathscr{L}\mathscr{F}, \Phi_{\mathscr{F} \times \mathscr{F}} \Gamma)}(a, c)$ for each morphism $(a, c)$ in $\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C})$, it follows from Remark 3.9 that $\mathscr{E}_{(\mathscr{L}\mathscr{F}, \Phi_{\mathscr{F} \times \mathscr{F}} \Gamma)}$ and $\mathscr{E}_{(\mathscr{L}, \Phi)} \circ \mathscr{E}_{(\mathscr{F}, \Gamma)}$ also agree on morphisms of $\mathbb{E}\text{-}\mathrm{Ext}(\mathcal{C})$, and hence are equal as functors. ■

## 4. A 2-categorical perspective on $n$-exangulated categories

The authors are very grateful to Hiroyuki Nakaoka for encouraging them to think about morphisms between $n$-exangulated functors, which prompted the results in this section. In particular, we introduce the notion of $n$-exangulated natural transformations, which recovers [56, Def. 2.11(3)] in the case $n = 1$. This enables us to make considerations that are 2-category-theoretic in the sense of [54, Sec. XII.3]. Some definitions have been developed independently in He–He–Zhou [26] and in Enomoto–Saito [15]. The 2-category of small abelian categories has been studied before; see, for example, work of Prest–Rajani [62].

A 2-*category* consists of 0-cells, 1-cells and 2-cells, which should be thought of as objects, morphisms between objects, and morphisms between morphisms, respectively, satisfying some axioms; see e.g. [54, p. 273]. We show that the category $n\text{-}\mathsf{exang}$ of small $n$-exangulated categories is a 2-category, with $n$-exangulated functors as 1-cells and $n$-exangulated natural transformations as 2-cells; see Corollary 4.15. Recall that a category is said to be *small* if the class of objects and the class of morphisms are sets. More generally, we prove that similar properties hold for the category $n\text{-}\mathsf{Exang}$ of all $n$-exangulated categories; see Proposition 4.12.

We start this section by giving the definition of $n$-exangulated natural transformations, before considering their compositions in Subsection 4.1. Having established a notion of morphisms between $n$-exangulated functors, we will be in position to introduce and study

$n$-exangulated adjoints and equivalences in Subsection 4.2. In Subsection 4.3 we continue our 2-categorical approach, leading to the construction of a 2-functor ✡: $n$ - exang → exact to the 2-category of small exact categories; see Corollary 4.25, which yields Theorem D from Section 1. The proof of this statement goes via a more general result, namely Theorem 4.22, where we establish a functor ✡: $n$ - Exang → Exact with similar properties, but without any smallness assumptions. If one ignores the set-theoretic issue described in Remark 4.13, one can interpret our work in Subsection 4.3 as constructing a 2-functor ✡ from the category $n$ - Exang to the category Exact of all exact categories. A fundamental step in defining ✡: $n$ - Exang → Exact is the characterisation of $n$-exangulated natural transformations given in Theorem 4.19. A full definition of ✡ is given in Definition 4.20.

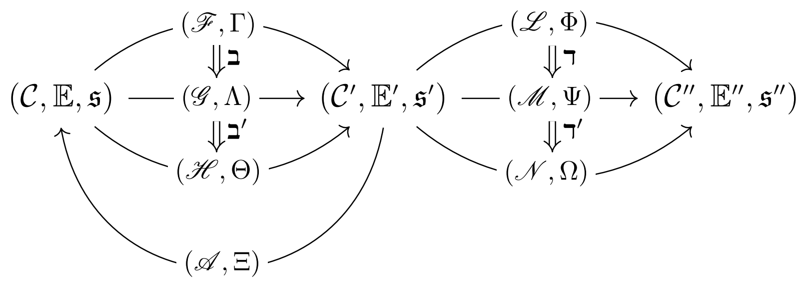
For $n$-exangulated natural transformations we use Hebrew letters $\beth$ (beth) and $\daleth$ (daleth).

**Definition 4.1.** Let $(\mathscr{F},\Gamma),(\mathscr{G},\Lambda)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ be $n$-exangulated functors and $\beth\colon\mathscr{F}\Rightarrow\mathscr{G}$ a natural transformation. We call $\beth\colon(\mathscr{F},\Gamma)\Rightarrow(\mathscr{G},\Lambda)$ an *$n$-exangulated natural transformation* if, for all $A,C\in\mathcal{C}$ and each $\delta\in\mathbb{E}(C,A)$, the pair $(\beth_A,\beth_C)$ is a morphism $\Gamma_{(C,A)}(\delta)\to\Lambda_{(C,A)}(\delta)$ in $\mathbb{E}'$-Ext$(\mathcal{C}')$, i.e.

$$(\beth_A)_{\mathbb{E}'}\Gamma_{(C,A)}(\delta)=(\beth_C)^{\mathbb{E}'}\Lambda_{(C,A)}(\delta). \tag{4.1}$$

See Examples 5.1, 5.3 and 5.4 for discussions on the notion of an $n$-exangulated natural transformation in some familiar settings.

**Setup 4.2.** For the remainder of this section, we use the standing assumptions and notation as indicated in the diagram below. Fix $n\geqslant 1$. We assume that the categories $(\mathcal{C},\mathbb{E},\mathfrak{s})$, $(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ and $(\mathcal{C}'',\mathbb{E}'',\mathfrak{s}'')$ are $n$-exangulated. The seven functors between these categories, drawn horizontally, and the four natural transformations, draw vertically, are assumed to be $n$-exangulated functors and $n$-exangulated natural transformations, respectively.




4.1. **Composing $n$-exangulated natural transformations.** In a 2-category, one should be able to compose 2-cells in two ways that are associative and unital [54, p. 273]. Our aim in this section is hence to consider two notions of composition of $n$-exangulated natural transformations. These are defined by using the classical notions of vertical and horizontal compositions, which apply to natural transformations in general [54, pp. 40, 42].

**Definition 4.3.** (i) We define the *identity* $\mathrm{id}_{(\mathscr{F},\Gamma)}$ of $(\mathscr{F},\Gamma)$ to be $\mathrm{id}_{\mathscr{F}}\colon\mathscr{F}\Rightarrow\mathscr{F}$.

(ii) The *vertical composition* $\beth'\circ_v\beth$ is given by $(\beth'\circ_v\beth)_X:=\beth'_X\beth_X$ for each $X\in\mathcal{C}$.

(iii) The $n$-exangulated natural transformation $\beth$ is said to be an *$n$-exangulated natural isomorphism* if it has an $n$-exangulated inverse under vertical composition. Note that this is equivalent to $\beth_X$ being an isomorphism for every $X \in \mathcal{C}$.

(iv) The *horizontal composition* $\daleth \circ_h \beth$ is given by $(\daleth \circ_h \beth)_X := \daleth_{\mathscr{G}X} \circ (\mathscr{L}\beth_X)$ for each $X \in \mathcal{C}$.

It follows from classical theory that the vertical and horizontal compositions of $n$-exangulated natural transformations are again natural [54, pp. 40, 42–43]. However, it is not clear that these compositions are $n$-exangulated. This is checked in Proposition 4.8.

First, we verify associativity and unitality, as well as a useful commutativity property known as the *interchange law* (or *middle-four exchange*).

**Lemma 4.4.** *The following statements hold.*

(i) *The identity* $\mathrm{id}_{(\mathscr{F},\Gamma)} \colon (\mathscr{F},\Gamma) \Rightarrow (\mathscr{F},\Gamma)$ *of* $(\mathscr{F},\Gamma)$ *is $n$-exangulated.*

(ii) *Both* $\circ_v$ *and* $\circ_h$ *are associative and unital on $n$-exangulated natural transformations.*

(iii) *The interchange law* $(\daleth' \circ_h \beth') \circ_v (\daleth \circ_h \beth) = (\daleth' \circ_v \daleth) \circ_h (\beth' \circ_v \beth)$ *holds.*

*Proof.* Equation (4.1) is trivially satisfied when $\Lambda = \Gamma$ and $\beth = \mathrm{id}_{\mathscr{F}}$, which yields (i). The claims of (ii) and (iii) hold for natural transformations in general; see [54, pp. 40, 43]. ■

To show that compositions of $n$-exangulated natural transformations are again $n$-exangulated, we use whiskering [54, p. 275], which is a special case of horizontal composition.

**Notation 4.5.** (i) The *whiskering* $\daleth_{\mathscr{G}} \colon \mathscr{L}\mathscr{G} \Rightarrow \mathscr{M}\mathscr{G}$ *of* $\mathscr{G}$ *and* $\daleth$ is the natural transformation given by $(\daleth_{\mathscr{G}})_X := \daleth_{\mathscr{G}X}$ for $X \in \mathcal{C}$.

(ii) The *whiskering* $\mathscr{L}\beth \colon \mathscr{L}\mathscr{F} \Rightarrow \mathscr{L}\mathscr{G}$ *of* $\beth$ *and* $\mathscr{L}$ is the natural transformation given by $(\mathscr{L}\beth)_X := \mathscr{L}\beth_X$ for $X \in \mathcal{C}$.

*Remark* 4.6. One can view horizontal composition as the vertical composition of some whiskerings [54, p. 43]. Note that $\daleth_{\mathscr{G}} = \daleth \circ_h \mathrm{id}_{\mathscr{G}}$ and $\mathscr{L}\beth = \mathrm{id}_{\mathscr{L}} \circ_h \beth$. Hence, unitality of vertical composition combined with the interchange law (see (ii) and (iii) of Lemma 4.4) yields

$$\daleth \circ_h \beth = (\daleth \circ_v \mathrm{id}_{\mathscr{L}}) \circ_h (\mathrm{id}_{\mathscr{G}} \circ_v \beth) = (\daleth \circ_h \mathrm{id}_{\mathscr{G}}) \circ_v (\mathrm{id}_{\mathscr{L}} \circ_h \beth) = \daleth_{\mathscr{G}} \circ_v \mathscr{L}\beth. \tag{4.2}$$

Recall from Theorem 3.17 the exact functor $\mathscr{E}_{(\mathscr{F},\Gamma)} \colon (\mathbb{E}\text{-Ext}(\mathcal{C}), \mathcal{X}_{\mathbb{E}}) \to (\mathbb{E}'\text{-Ext}(\mathcal{C}'), \mathcal{X}_{\mathbb{E}'})$ given by $\mathscr{E}_{(\mathscr{F},\Gamma)}(\delta) = \Gamma_{(C,A)}(\delta)$ on objects $\delta \in \mathbb{E}(C,A)$, and recall also that $\mathscr{E}_{(\mathscr{F},\Gamma)}$ respects morphisms and distinguished $n$-exangles over $\mathscr{F}$.

**Proposition 4.7.** *The following statements hold.*

(i) *The whiskering* $\daleth_{\mathscr{G}} \colon (\mathscr{L},\Phi) \circ (\mathscr{G},\Lambda) \Rightarrow (\mathscr{M},\Psi) \circ (\mathscr{G},\Lambda)$ *is $n$-exangulated.*

(ii) *The whiskering* $\mathscr{L}\beth \colon (\mathscr{L},\Phi) \circ (\mathscr{F},\Gamma) \Rightarrow (\mathscr{L},\Phi) \circ (\mathscr{G},\Lambda)$ *is $n$-exangulated.*

*Proof.* (i) By Lemma 3.19(ii), we have that the composites $(\mathscr{L},\Phi) \circ (\mathscr{G},\Lambda) = (\mathscr{L}\mathscr{G}, \Phi_{\mathscr{G}\times\mathscr{G}}\Lambda)$ and $(\mathscr{M},\Psi) \circ (\mathscr{G},\Lambda) = (\mathscr{M}\mathscr{G}, \Psi_{\mathscr{G}\times\mathscr{G}}\Lambda)$ are $n$-exangulated functors from $(\mathcal{C},\mathbb{E},\mathfrak{s})$ to $(\mathcal{C}'',\mathbb{E}'',\mathfrak{s}'')$. Given an extension $\delta \in \mathbb{E}(C,A)$, we have $\Lambda_{(C,A)}(\delta) \in \mathbb{E}'(\mathscr{G}C, \mathscr{G}A)$. As $\daleth \colon (\mathscr{L},\Phi) \Rightarrow (\mathscr{M},\Psi)$ is an $n$-exangulated natural transformation, we thus obtain

$$(\daleth_{\mathscr{G}A})_{\mathbb{E}''}\Phi_{(\mathscr{G}C,\mathscr{G}A)}(\Lambda_{(C,A)}(\delta)) = (\daleth_{\mathscr{G}C})^{\mathbb{E}''}\Psi_{(\mathscr{G}C,\mathscr{G}A)}(\Lambda_{(C,A)}(\delta)),$$

which verifies that $\daleth_{\mathscr{G}}$ is an $n$-exangulated natural transformation.

(ii) The pairs $(\mathscr{L},\Phi)\circ(\mathscr{F},\Gamma) = (\mathscr{L}\mathscr{F},\Phi_{\mathscr{F}\times\mathscr{F}}\Gamma)$ and $(\mathscr{L},\Phi)\circ(\mathscr{G},\Lambda) = (\mathscr{L}\mathscr{G},\Phi_{\mathscr{G}\times\mathscr{G}}\Lambda)$ are $n$-exangulated functors from $(\mathcal{C},\mathbb{E},\mathfrak{s})$ to $(\mathcal{C}'',\mathbb{E}'',\mathfrak{s}'')$ by Lemma 3.19(ii). Consider the functor $\mathscr{E}_{(\mathscr{L},\Phi)} : \mathbb{E}'\text{-Ext}(\mathcal{C}') \to \mathbb{E}''\text{-Ext}(\mathcal{C}'')$ from Theorem 3.17. For $\delta \in \mathbb{E}(C,A)$, we have

$$\begin{aligned}
(\mathscr{L}\beth_A)_{\mathbb{E}''}\Phi_{(\mathscr{F}C,\mathscr{F}A)}(\Gamma_{(C,A)}(\delta)) &= (\mathscr{L}\beth_A)_{\mathbb{E}''}\mathscr{E}_{(\mathscr{L},\Phi)}(\Gamma_{(C,A)}(\delta)) && \text{using the definition of } \mathscr{E}_{(\mathscr{L},\Phi)}\\
&= \mathscr{E}_{(\mathscr{L},\Phi)}((\beth_A)_{\mathbb{E}'}\Gamma_{(C,A)}(\delta)) && \text{by Lemma 3.10(ii)}\\
&= \mathscr{E}_{(\mathscr{L},\Phi)}((\beth_C)^{\mathbb{E}'}\Lambda_{(C,A)}(\delta)) && \text{as } \beth \text{ is } n\text{-exangulated}\\
&= (\mathscr{L}\beth_C)^{\mathbb{E}''}\mathscr{E}_{(\mathscr{L},\Phi)}(\Lambda_{(C,A)}(\delta)) && \text{by Lemma 3.10(ii)}\\
&= (\mathscr{L}\beth_C)^{\mathbb{E}''}\Phi_{(\mathscr{G}C,\mathscr{G}A)}(\Lambda_{(C,A)}(\delta)) && \text{using the definition of } \mathscr{E}_{(\mathscr{L},\Phi)}.
\end{aligned}$$

This verifies that $\mathscr{L}\beth$ is an $n$-exangulated natural transformation. ■

We are now in position to show that the collection of $n$-exangulated natural transformations is closed under vertical and horizontal composition.

**Proposition 4.8.** *The following statements hold.*

(i) *The vertical composition* $\beth'\circ_v\beth\colon(\mathscr{F},\Gamma)\Rightarrow(\mathscr{H},\Theta)$ *is $n$-exangulated.*

(ii) *The horizontal composition* $\daleth\circ_h\beth\colon(\mathscr{L},\Phi)\circ(\mathscr{F},\Gamma)\Rightarrow(\mathscr{M},\Psi)\circ(\mathscr{G},\Lambda)$ *is $n$-exangulated.*

*Proof.* (i) Let $\delta\in\mathbb{E}(C,A)$ be arbitrary. We have

$$\begin{aligned}
((\beth'\circ_v\beth)_A)_{\mathbb{E}'}\Gamma_{(C,A)}(\delta) &= (\beth'_A)_{\mathbb{E}'}(\beth_A)_{\mathbb{E}'}\Gamma_{(C,A)}(\delta) && \text{using the definition of } \circ_v\\
&= (\beth'_A)_{\mathbb{E}'}(\beth_C)^{\mathbb{E}'}\Lambda_{(C,A)}(\delta) && \text{as } \beth \text{ is } n\text{-exangulated}\\
&= (\beth_C)^{\mathbb{E}'}(\beth'_C)^{\mathbb{E}'}\Theta_{(C,A)}(\delta) && \text{as } \beth' \text{ is } n\text{-exangulated}\\
&= ((\beth'\circ_v\beth)_C)^{\mathbb{E}'}\Theta_{(C,A)}(\delta) && \text{using the definition of } \circ_v,
\end{aligned}$$

which verifies that $\beth'\circ_v\beth$ is $n$-exangulated.

(ii) This follows from (4.2), Proposition 4.7 and part (i) above. ■

4.2. ***n*-exangulated adjoints and equivalences.** In this subsection we discuss adjunctions and equivalences between $n$-exangulated categories. An important property of the functor $✡\colon n\text{-}\mathsf{Exang}\to\mathsf{Exact}$, which will be defined in Definition 4.20, is that it preserves adjoint pairs and equivalences; see Corollary 4.23.

Recall from Setup 4.2 that we consider $n$-exangulated functors $(\mathscr{F},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ and $(\mathscr{A},\Xi)\colon(\mathcal{C}',\mathbb{E}',\mathfrak{s}')\to(\mathcal{C},\mathbb{E},\mathfrak{s})$. In the case $n=1$, part (ii) in Definition 4.9 recovers the notion of an equivalence of extriangulated categories as defined in [56, Prop. 2.13]. In the following we use the Hebrew letters צ (tsadi) and מ (mem).

**Definition 4.9.** (i) We call $((\mathscr{F},\Gamma),(\mathscr{A},\Xi))$ an *$n$-exangulated adjoint pair* if $(\mathscr{F},\mathscr{A})$ is an adjoint pair for which the unit צ and counit מ give $n$-exangulated natural transformations צ$\colon(\mathrm{id}_{\mathcal{C}},\mathrm{id}_{\mathbb{E}})\Rightarrow(\mathscr{A},\Xi)\circ(\mathscr{F},\Gamma)$ and מ$\colon(\mathscr{F},\Gamma)\circ(\mathscr{A},\Xi)\Rightarrow(\mathrm{id}_{\mathcal{C}'},\mathrm{id}_{\mathbb{E}'})$.

(ii) We call $(\mathscr{F},\Gamma)$ an *$n$-exangulated equivalence* if there is an $n$-exangulated adjoint pair $((\mathscr{F},\Gamma),(\mathscr{A},\Xi))$ whose unit and counit give $n$-exangulated natural isomorphisms.

Recall that if $(\mathscr{F},\mathscr{A})$ is an adjoint pair with unit $\text{צ}\colon \mathrm{id}_{\mathcal{C}} \Rightarrow \mathscr{A}\mathscr{F}$ and counit $\text{מ}\colon \mathscr{F}\mathscr{A} \Rightarrow \mathrm{id}_{\mathcal{C}'}$, then one has the *triangle identities* (or *counit-unit equations*)

$$\text{מ}_{\mathscr{F}X}(\mathscr{F}\text{צ}_X) = \mathrm{id}_{\mathscr{F}X} \qquad \text{and} \qquad (\mathscr{A}\text{מ}_Y)\text{צ}_{\mathscr{A}Y} = \mathrm{id}_{\mathscr{A}Y} \tag{4.3}$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$; see e.g. [54, Thm. IV.1.1(ii)]. In terms of vertical and horizontal compositions of ($n$-exangulated) natural transformations, the equations in (4.3) give

$$(\text{מ} \circ_h \mathrm{id}_{(\mathscr{F},\Gamma)}) \circ_v (\mathrm{id}_{(\mathscr{F},\Gamma)} \circ_h \text{צ}) = \mathrm{id}_{(\mathscr{F},\Gamma)} \quad \text{and} \quad (\mathrm{id}_{(\mathscr{A},\Xi)} \circ_h \text{מ}) \circ_v (\text{צ} \circ_h \mathrm{id}_{(\mathscr{A},\Xi)}) = \mathrm{id}_{(\mathscr{A},\Xi)}. \tag{4.4}$$

If $(\mathscr{F},\mathscr{A})$ is an adjoint equivalence, we also have $\text{מ}^{-1}_{\mathscr{F}X} = \mathscr{F}\text{צ}_X$ in $\mathcal{C}'$ and $\text{צ}^{-1}_{\mathscr{A}Y} = \mathscr{A}\text{מ}_Y$ in $\mathcal{C}$.

We use the following lemma to characterise $n$-exangulated equivalences. Notice the similarity between the equations in the statement below and the triangle identities above.

**Lemma 4.10.** *Let $((\mathscr{F},\Gamma),(\mathscr{A},\Xi))$ be an $n$-exangulated adjoint pair with unit* $\text{צ}$ *and counit* $\text{מ}$. *Then for all $A, C \in \mathcal{C}$ and $B, D \in \mathcal{C}'$, we have:*

(i) $(\text{מ}_{\mathscr{F}A})_{\mathbb{E}'}(\mathscr{F}\text{צ}_C)^{\mathbb{E}'}(\Gamma_{\mathscr{A}\times\mathscr{A}}\Xi)_{(\mathscr{F}C,\mathscr{F}A)} = \mathrm{id}_{\mathbb{E}'(\mathscr{F}C,\mathscr{F}A)}$; *and*

(ii) $(\mathscr{A}\text{מ}_B)_{\mathbb{E}}(\text{צ}_{\mathscr{A}D})^{\mathbb{E}}(\Xi_{\mathscr{F}\times\mathscr{F}}\Gamma)_{(\mathscr{A}D,\mathscr{A}B)} = \mathrm{id}_{\mathbb{E}(\mathscr{A}D,\mathscr{A}B)}$.

*Proof.* We just show (i), as the proof of (ii) is similar. Let $\delta' \in \mathbb{E}'(\mathscr{F}C,\mathscr{F}A)$ be arbitrary. Since $\text{מ}\colon (\mathscr{F}\mathscr{A},\Gamma_{\mathscr{A}\times\mathscr{A}}\Xi) \Rightarrow (\mathrm{id}_{\mathcal{C}'},\mathrm{id}_{\mathbb{E}'})$ is an $n$-exangulated natural transformation, we get

$$(\mathscr{F}\text{צ}_C)^{\mathbb{E}'}(\text{מ}_{\mathscr{F}A})_{\mathbb{E}'}(\Gamma_{\mathscr{A}\times\mathscr{A}}\Xi)_{(\mathscr{F}C,\mathscr{F}A)}(\delta') = (\mathscr{F}\text{צ}_C)^{\mathbb{E}'}(\text{מ}_{\mathscr{F}C})^{\mathbb{E}'}(\delta') = \delta',$$

where the first equality follows from (4.1) and the second from (4.3). ■

Proposition 4.11 below is a higher analogue of [56, Prop. 2.13], giving a characterisation of when an $n$-exangulated functor is an $n$-exangulated equivalence. One direction in the proof is provided in [56] for the extriangulated case and easily translates to the $n$-exangulated setting, so we omit this here. We provide a proof for the other implication. The following statement has also appeared independently; see [26, Prop. 2.14].

**Proposition 4.11.** *The pair $(\mathscr{F},\Gamma)$ is an $n$-exangulated equivalence if and only if $\mathscr{F}$ is an equivalence of categories and $\Gamma$ is a natural isomorphism.*

*Proof.* ($\Rightarrow$) Suppose that we are given an $n$-exangulated adjoint pair $((\mathscr{F},\Gamma),(\mathscr{A},\Xi))$ with corresponding $n$-exangulated natural isomorphisms $\text{צ}\colon (\mathrm{id}_{\mathcal{C}},\mathrm{id}_{\mathbb{E}}) \Rightarrow (\mathscr{A},\Xi)\circ(\mathscr{F},\Gamma)$ and $\text{מ}\colon (\mathscr{F},\Gamma)\circ(\mathscr{A},\Xi) \Rightarrow (\mathrm{id}_{\mathcal{C}'},\mathrm{id}_{\mathbb{E}'})$ defined by the unit and counit, respectively. It follows from classical theory that $\mathscr{F}$ is an equivalence. Thus, it remains to show that for all $A, C \in \mathcal{C}$, the induced homomorphism $\Gamma_{(C,A)}\colon \mathbb{E}(C,A) \to \mathbb{E}'(\mathscr{F}C,\mathscr{F}A)$ is invertible. We claim that the composition

$$(\text{צ}^{-1}_A)_{\mathbb{E}}(\text{צ}_C)^{\mathbb{E}}\Xi_{(\mathscr{F}C,\mathscr{F}A)}\colon \mathbb{E}'(\mathscr{F}C,\mathscr{F}A) \to \mathbb{E}(C,A)$$

is a two-sided inverse of $\Gamma_{(C,A)}$. First, for each $\delta \in \mathbb{E}(C,A)$, notice that

$$(\text{צ}^{-1}_A)_{\mathbb{E}}(\text{צ}_C)^{\mathbb{E}}\Xi_{(\mathscr{F}C,\mathscr{F}A)}\Gamma_{(C,A)}(\delta) = (\text{צ}^{-1}_A)_{\mathbb{E}}(\text{צ}_A)_{\mathbb{E}}\mathrm{id}_{\mathbb{E}}(\delta) = \delta,$$

where the first equality follows from (4.1) for $\text{צ}$. It remains to check that $(\text{צ}_A^{-1})_{\mathbb{E}}(\text{צ}_C)^{\mathbb{E}}\Xi_{(C,A)}$ is a right inverse of $\Gamma_{(C,A)}$. For $\delta' \in \mathbb{E}'(\mathscr{F}C, \mathscr{F}A)$, we have that

$$\begin{aligned}\Gamma_{(C,A)}(\text{צ}_A^{-1})_{\mathbb{E}}(\text{צ}_C)^{\mathbb{E}}\Xi_{(\mathscr{F}C,\mathscr{F}A)}(\delta') &= (\mathscr{F}\text{צ}_A^{-1})_{\mathbb{E}'}(\mathscr{F}\text{צ}_C)^{\mathbb{E}'}\Gamma_{(\mathscr{A}\mathscr{F}C,\mathscr{A}\mathscr{F}A)}\Xi_{(\mathscr{F}C,\mathscr{F}A)}(\delta') \\ &= (\text{מ}_{\mathscr{F}A})_{\mathbb{E}'}(\mathscr{F}\text{צ}_C)^{\mathbb{E}'}\Gamma_{(\mathscr{A}\mathscr{F}C,\mathscr{A}\mathscr{F}A)}\Xi_{(\mathscr{F}C,\mathscr{F}A)}(\delta'),\end{aligned}$$

where the first equality is by the naturality of $\Gamma$ and the second follows from (4.3). This last term is equal to $\delta'$ by Lemma 4.10(i), as required. ∎

4.3. **A 2-categorical viewpoint.** We start this subsection by using what we have shown so far to prove that $n$-$\mathsf{exang}$ is a 2-category. More generally, we establish a Hom-category for each pair of $n$-exangulated categories, as explained in the proposition below.

**Proposition 4.12.** *For each pair $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ of $n$-exangulated categories, there is a category $\mathscr{N} := n\text{-}\mathsf{Exang}((\mathcal{C}, \mathbb{E}, \mathfrak{s}), (\mathcal{C}', \mathbb{E}', \mathfrak{s}'))$ whose objects are $n$-exangulated functors $(\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ and whose morphisms are $n$-exangulated natural transformations.*

*Proof.* Define composition of morphisms in $\mathscr{N}$ to be vertical composition $\circ_v$ of natural transformations, which is well-defined by Proposition 4.8(i). Lemma 4.4(i) and (ii) imply that $\mathscr{N}$ is a category. ∎

*Remark* 4.13. Note that in this article, and in particular in Proposition 4.12, the term 'category' does not require *smallness*, i.e. the collections of objects and morphisms associated to a category are not assumed to form sets. In spite of this, we have so far usually considered additive categories. These are necessarily *locally small*, which means that the collection of morphisms between any two objects is a set, since a group is a set.

Due to a set-theoretic issue, care must be taken when referring to the established notion of a 2-*category*. In order to ensure that the collection of 2-cells between a pair of fixed 1-cells forms a set instead of a proper class, it is common from a 2-categorical viewpoint to only consider small categories; see e.g. [54, p. 43]. Consequently, we do not refer to the categories $n$-$\mathsf{Exang}$ and $\mathsf{Exact}$ as 2-categories, since collections of 2-cells between pairs of 1-cells need not form sets. Yet, it is still natural to adopt a 2-categorical perspective on these categories. Nevertheless, for accuracy and in the interest of not abusing established terminology, we avoid the terms '2-category' and '2-functor' in Proposition 4.12 and Theorem 4.22.

So far, we have used the notation $n$-$\mathsf{Exang}$ (resp. $\mathsf{Exact}$) for the 1-category of all $n$-exangulated (resp. exact) categories. Similarly, we have used $n$-$\mathsf{exang}$ and $\mathsf{exact}$ to denote the subcategories obtained by restricting to small categories. In order to formally place $n$-$\mathsf{exang}$ and $\mathsf{exact}$ in a 2-categorical framework, we make our terminology more precise by indicating below the 0-cells, 1-cells and 2-cells of these structures.

**Notation 4.14.** We write $n$-$\mathsf{Exang}$ and $\mathsf{Exact}$ for the collections of 0-cells, 1-cells and 2-cells described in the table below.

| | $n$ - Exang | Exact |
|---|---|---|
| 0-cells | $n$-exangulated categories | exact categories |
| 1-cells | $n$-exangulated functors | exact functors |
| 2-cells | $n$-exangulated natural transformations | natural transformations |

The $i$-cells above also induce $i$-cells for $n$ - exang and exact, where the only difference is that for 0-cells we restrict to small categories.

It is well-known that the composition of two exact functors is an exact functor, and hence exact is a 2-category; see e.g. [50, 1.4(a)]. From the theory developed in Subsection 4.1 and in this subsection so far, one can deduce that $n$ - Exang has the characteristics of a 2-category. When restricting to small categories, the set-theoretic issue described in Remark 4.13 is avoided and we obtain the following immediate corollary.

**Corollary 4.15.** *The category $n$ - exang is a 2-category.*

The final aim of this section is to provide a 2-categorical understanding of how $n$-exangulated concepts relate to notions on the level of associated categories of extensions, bringing together our work in Sections 3 and 4. We do this by constructing a functor $✡\colon n\text{-}\mathsf{Exang} \to \mathsf{Exact}$ that respects the 2-categorical properties of the categories involved. In particular, this induces a 2-*functor* $✡\colon n\text{-}\mathsf{exang} \to \mathsf{exact}$ in the sense of [54, p. 278].

In order to have a 2-functor, one must give an assignment of $i$-cells in the domain 2-category to $i$-cells in the codomain 2-category for $i \in \{0, 1, 2\}$, satisfying certain compatibility conditions. Based on the theory developed in Section 3, we can define the functor ✡ on 0-cells by sending an $n$-exangulated category to its category of extensions, and on 1-cells by sending an $n$-exangulated functor $(\mathscr{F}, \Gamma)$ to $\mathscr{E}_{(\mathscr{F},\Gamma)}$ as described in Theorem 3.17. It remains to consider how ✡ acts on 2-cells, that is, on $n$-exangulated natural transformations. The next lemma constitutes a first step in this direction. We use the Hebrew letter $\aleph$ (aleph).

**Lemma 4.16.** *Any natural transformation $\aleph\colon \mathscr{E}_{(\mathscr{F},\Gamma)} \Rightarrow \mathscr{E}_{(\mathscr{G},\Lambda)}$ gives rise to natural transformations $\aleph^{\ell}\colon \mathscr{F} \Rightarrow \mathscr{G}$ and $\aleph^{r}\colon \mathscr{F} \Rightarrow \mathscr{G}$ such that $\aleph_{\delta} = (\aleph^{\ell}_{A}, \aleph^{r}_{C})$ for each $\delta \in \mathbb{E}(C, A)$.*

*Proof.* Since $\aleph\colon \mathscr{E}_{(\mathscr{F},\Gamma)} \Rightarrow \mathscr{E}_{(\mathscr{G},\Lambda)}$ is a natural transformation, for each $\delta \in \mathbb{E}(C, A)$, there is a morphism $\aleph_{\delta}\colon \Gamma_{(C,A)}(\delta) \to \Lambda_{(C,A)}(\delta)$ in $\mathbb{E}'$-Ext$(\mathcal{C}')$. This implies that there are morphisms $\ell_{\delta}\colon \mathscr{F}A \to \mathscr{G}A$ and $r_{\delta}\colon \mathscr{F}C \to \mathscr{G}C$ in $\mathcal{C}'$ such that $\aleph_{\delta} = (\ell_{\delta}, r_{\delta})$. We claim that $\ell_{\delta}$ depends only on the object $A$. To see this, recall that for each $Z \in \mathcal{C}$ we write ${}_{Z}0_{0}$ for the trivial element of the abelian group $\mathbb{E}(0, Z)$. Consider the morphism $(\mathrm{id}_{A}, 0)\colon {}_{A}0_{0} \to \delta$ for a fixed extension $\delta \in \mathbb{E}(C, A)$. By Remark 3.9, we have the equalities

$$\begin{aligned}
((\mathscr{G}\mathrm{id}_{A})\ell_{{}_{A}0_{0}}, 0) &= \mathscr{E}_{(\mathscr{G},\Lambda)}(\mathrm{id}_{A}, 0) \circ \aleph_{{}_{A}0_{0}} && \text{as } \mathscr{E}_{(\mathscr{G},\Lambda)} \text{ respects morphisms over } \mathscr{G} \\
&= \aleph_{\delta} \circ \mathscr{E}_{(\mathscr{F},\Gamma)}(\mathrm{id}_{A}, 0) && \text{as } \aleph \text{ is natural} \\
&= (\ell_{\delta}(\mathscr{F}\mathrm{id}_{A}), 0) && \text{as } \mathscr{E}_{(\mathscr{F},\Gamma)} \text{ respects morphisms over } \mathscr{F},
\end{aligned}$$

as morphisms $\Gamma_{(0,A)}({}_{A}0_{0}) \to \Lambda_{(C,A)}(\delta)$ in $\mathbb{E}'$-Ext$(\mathcal{C}')$. Consequently, the morphism $\ell_{\delta} = \ell_{{}_{A}0_{0}}$ depends only on $A$, and we write $\aleph^{\ell}_{A} \coloneqq \ell_{\delta}\colon \mathscr{F}A \to \mathscr{G}A$. Similarly, the morphism $r_{\delta}$ depends only on $C$, and we write $\aleph^{r}_{C} \coloneqq r_{\delta}\colon \mathscr{F}C \to \mathscr{G}C$.

It remains to show that $\aleph^\ell := \{\, \aleph^\ell_A \,\}_{A\in\mathcal{C}}$ and $\aleph^r := \{\, \aleph^r_C \,\}_{C\in\mathcal{C}}$ define natural transformations $\mathscr{F} \Rightarrow \mathscr{G}$. Fix a morphism $x\colon X \to Y$ in $\mathcal{C}$. The pair $(x,0)\colon {}_X0_0 \to {}_Y0_0$ is a morphism in $\mathbb{E}$-Ext$(\mathcal{C})$, so $\aleph_{{}_Y0_0}\mathscr{E}_{(\mathscr{F},\Gamma)}(x,0) = \mathscr{E}_{(\mathscr{G},\Lambda)}(x,0)\aleph_{{}_X0_0}$ as $\aleph$ is natural. This yields $\aleph^\ell_Y\mathscr{F}x = (\mathscr{G}x)\aleph^\ell_X$, so $\aleph^\ell$ is a natural transformation. The proof that $\aleph^r$ is natural is similar. ■

Note that it does not necessarily follow that $\aleph^\ell$ and $\aleph^r$ coincide in the lemma above, as demonstrated by the following example.

**Example 4.17.** Let $\mathcal{C}$ be a non-zero additive category. Equip $\mathcal{C}$ with its split $n$-exangulated structure $(\mathcal{C},\mathbb{E},\mathfrak{s})$, which is induced by the split $n$-exact structure as explained in Example 5.5. Consider the identity $n$-exangulated functor $(\mathrm{id}_\mathcal{C},\mathrm{id}_\mathbb{E})\colon (\mathcal{C},\mathbb{E},\mathfrak{s}) \to (\mathcal{C},\mathbb{E},\mathfrak{s})$. By Theorem 3.17 and Lemma 3.19(iii), we obtain the additive functor $\mathscr{E}_{(\mathrm{id}_\mathcal{C},\mathrm{id}_\mathbb{E})} = \mathrm{id}_{\mathbb{E}\text{-Ext}(\mathcal{C})}$. For $\delta \in \mathbb{E}(C,A)$, define $\aleph_\delta\colon \delta = \mathscr{E}_{(\mathrm{id}_\mathcal{C},\mathrm{id}_\mathbb{E})}(\delta) \to \mathscr{E}_{(\mathrm{id}_\mathcal{C},\mathrm{id}_\mathbb{E})}(\delta) = \delta$ by $\aleph_\delta = (\mathrm{id}_A, 0)$. Note that $\aleph_\delta$ is a morphism $\delta \to \delta$ in $\mathbb{E}$-Ext$(\mathcal{C})$ since $\mathbb{E}(C,A)$ is trivial. It is straightforward to check that $\aleph = \{\aleph_\delta\}_{\delta\in\mathbb{E}\text{-Ext}(\mathcal{C})}$ defines a natural transformation $\mathscr{E}_{(\mathrm{id}_\mathcal{C},\mathrm{id}_\mathbb{E})} \Rightarrow \mathscr{E}_{(\mathrm{id}_\mathcal{C},\mathrm{id}_\mathbb{E})}$. Since $\aleph^\ell_A = \mathrm{id}_A \neq 0 = \aleph^r_A$ for any non-zero $A \in \mathcal{C}$, we have that $\aleph$ is not *balanced* in the sense of Definition 4.18 below.

**Definition 4.18.** Let $\aleph\colon \mathscr{E}_{(\mathscr{F},\Gamma)} \Rightarrow \mathscr{E}_{(\mathscr{G},\Lambda)}$ be a natural transformation. In the notation of Lemma 4.16, we call $\aleph$ *balanced* provided that $\aleph^\ell = \aleph^r$.

We can now prove Theorem C from Section 1.

**Theorem 4.19.** *There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} n\text{-exangulated natural transformations} \\ \beth\colon (\mathscr{F},\Gamma) \Longrightarrow (\mathscr{G},\Lambda) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{balanced natural transformations} \\ \aleph\colon \mathscr{E}_{(\mathscr{F},\Gamma)} \Longrightarrow \mathscr{E}_{(\mathscr{G},\Lambda)} \end{array} \right\}$$

$$\begin{array}{rcl} \beth & \longmapsto & \langle\beth\rangle \\ \aleph^\ell = \aleph^r & \longleftarrow & \aleph, \end{array}$$

*where* $\langle\beth\rangle_\delta = (\beth_A, \beth_C)$ *for all* $A, C \in \mathcal{C}$ *and each* $\delta \in \mathbb{E}(C,A)$.

*Proof.* Suppose first that $\beth\colon (\mathscr{F},\Gamma) \Rightarrow (\mathscr{G},\Lambda)$ is an $n$-exangulated natural transformation, and consider the collection $\langle\beth\rangle = \{\, \langle\beth\rangle_\delta \,\}_{\delta\in\mathbb{E}\text{-Ext}(\mathcal{C})}$ where $\langle\beth\rangle_\delta$ is as defined in the statement of the theorem. Since $\beth$ is $n$-exangulated, each pair $\langle\beth\rangle_\delta = (\beth_A, \beth_C)$ is a morphism from $\mathscr{E}_{(\mathscr{F},\Gamma)}(\delta) = \Gamma_{(C,A)}(\delta)$ to $\Lambda_{(C,A)}(\delta) = \mathscr{E}_{(\mathscr{G},\Lambda)}(\delta)$ in $\mathbb{E}'$-Ext$(\mathcal{C}')$. Let $(a,c)\colon \delta \to \rho$ be a morphism in $\mathbb{E}$-Ext$(\mathcal{C})$ with $\delta \in \mathbb{E}(C,A)$ and $\rho \in \mathbb{E}(D,B)$. We must show that the square

$$\begin{array}{ccc} \mathscr{E}_{(\mathscr{F},\Gamma)}(\delta) & \xrightarrow{\langle\beth\rangle_\delta} & \mathscr{E}_{(\mathscr{G},\Lambda)}(\delta) \\ {\scriptstyle \mathscr{E}_{(\mathscr{F},\Gamma)}(a,c)}\downarrow & & \downarrow{\scriptstyle \mathscr{E}_{(\mathscr{G},\Lambda)}(a,c)} \\ \mathscr{E}_{(\mathscr{F},\Gamma)}(\rho) & \xrightarrow{\langle\beth\rangle_\rho} & \mathscr{E}_{(\mathscr{G},\Lambda)}(\rho) \end{array}$$

commutes in $\mathbb{E}'\text{-Ext}(\mathcal{C}')$. By Remark 3.9, it is enough to observe that

$$\begin{aligned}
\mathscr{E}_{(\mathscr{G},\Lambda)}(a,c)\langle\beth\rangle_{\delta} &= (\mathscr{G}a,\mathscr{G}c)(\beth_A,\beth_C) && \text{as } \mathscr{E}_{(\mathscr{G},\Lambda)} \text{ respects morphisms over } \mathscr{G}\\
&= (\beth_B\mathscr{F}a,\beth_D\mathscr{F}c) && \text{since } \beth \text{ is natural}\\
&= \langle\beth\rangle_{\rho}\mathscr{E}_{(\mathscr{F},\Gamma)}(a,c) && \text{as } \mathscr{E}_{(\mathscr{F},\Gamma)} \text{ respects morphisms over } \mathscr{F}.
\end{aligned}$$

This means that $\langle\beth\rangle\colon \mathscr{E}_{(\mathscr{F},\Gamma)} \Rightarrow \mathscr{E}_{(\mathscr{G},\Lambda)}$ is natural, and it is balanced by construction.

On the other hand, suppose $\aleph\colon \mathscr{E}_{(\mathscr{F},\Gamma)} \Rightarrow \mathscr{E}_{(\mathscr{G},\Lambda)}$ is balanced. Write $\aleph^{\ell} = \aleph^{r}\colon \mathscr{F} \Rightarrow \mathscr{G}$ for the natural transformation from Lemma 4.16 satisfying $\aleph_{\delta} = (\aleph^{\ell}_A,\aleph^{\ell}_C)$ for each $\delta\in\mathbb{E}(C,A)$. The pair $\aleph_{\delta} = (\aleph^{\ell}_A,\aleph^{\ell}_C)$ is a morphism from $\Gamma_{(C,A)}(\delta) = \mathscr{E}_{(\mathscr{F},\Gamma)}(\delta)$ to $\mathscr{E}_{(\mathscr{G},\Lambda)}(\delta) = \Lambda_{(C,A)}(\delta)$, so $\aleph^{\ell}$ is an $n$-exangulated natural transformation.

We can thus conclude that the two assignments $\beth \mapsto \langle\beth\rangle$ and $\aleph \mapsto \aleph^{\ell} = \aleph^{r}$ from the statement of the theorem are well-defined. It is straightforward to check that they are mutually inverse, and hence define a one-to-one correspondence. ∎

Just as Theorem 3.17 allowed us to define the functor $\davidsstar\colon n\text{-}\mathsf{Exang} \to \mathsf{Exact}$ on 1-cells, the characterisation in Theorem 4.19 enables us to define $\davidsstar$ on 2-cells by sending an $n$-exangulated natural transformation $\beth$ to the balanced natural transformation $\langle\beth\rangle$. We can hence complete the definition of $\davidsstar$. For $i\in\{0,1,2\}$, denote by $n\text{-}\mathsf{Exang}_i$ and $\mathsf{Exact}_i$ the collection of $i$-cells of $n\text{-}\mathsf{Exang}$ and $\mathsf{Exact}$, respectively.

**Definition 4.20.** Let $\davidsstar = (\davidsstar_0,\davidsstar_1,\davidsstar_2)\colon n\text{-}\mathsf{Exang} \to \mathsf{Exact}$ be defined by the assignments $\davidsstar_i\colon n\text{-}\mathsf{Exang}_i \to \mathsf{Exact}_i$, where:

$$\begin{aligned}
\davidsstar_0(\mathcal{C},\mathbb{E},\mathfrak{s}) &\coloneqq (\mathbb{E}\text{-Ext}(\mathcal{C}),\mathcal{X}_{\mathbb{E}}),\\
\davidsstar_1(\mathscr{F},\Gamma) &\coloneqq \mathscr{E}_{(\mathscr{F},\Gamma)},\\
\davidsstar_2(\beth) &\coloneqq \langle\beth\rangle.
\end{aligned}$$

*Remark* 4.21. We discuss Definition 4.20 with a view towards explaining Theorem 4.22.

(i) The assignments $\davidsstar_i$ are well-defined: by Proposition 3.2, the assignment $\davidsstar_0$ takes an object of $n\text{-}\mathsf{Exang}$ to an object of $\mathsf{Exact}$; by Theorem 3.17 and Proposition 3.11, $\davidsstar_1$ associates an exact functor $\davidsstar_1(\mathscr{F},\Gamma) = \mathscr{E}_{(\mathscr{F},\Gamma)}$ from $\davidsstar_0(\mathcal{C},\mathbb{E},\mathfrak{s})$ to $\davidsstar_0(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ to each $n$-exangulated functor $(\mathscr{F},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s}) \to (\mathcal{C}',\mathbb{E}',\mathfrak{s}')$; and $\davidsstar_2$ takes an $n$-exangulated natural transformation $\beth\colon(\mathscr{F},\Gamma)\Rightarrow(\mathscr{G},\Lambda)$ to a natural transformation $\davidsstar_2(\beth) = \langle\beth\rangle\colon \davidsstar_1(\mathscr{F},\Gamma) \to \davidsstar_1(\mathscr{G},\Lambda)$ by Theorem 4.19.

(ii) In Theorem 4.22 below, given $n$-exangulated categories $(\mathcal{C},\mathbb{E},\mathfrak{s})$ and $(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$, we denote by $\mathcal{A} \coloneqq \mathsf{Exact}((\mathbb{E}\text{-Ext}(\mathcal{C}),\mathcal{X}_{\mathbb{E}}),(\mathbb{E}'\text{-Ext}(\mathcal{C}'),\mathcal{X}_{\mathbb{E}'}))$ the category whose objects are exact functors from $(\mathbb{E}\text{-Ext}(\mathcal{C}),\mathcal{X}_{\mathbb{E}})$ to $(\mathbb{E}'\text{-Ext}(\mathcal{C}'),\mathcal{X}_{\mathbb{E}'})$ and whose morphisms are natural transformations. Composition of morphisms in $\mathcal{A}$ is vertical composition of natural transformations. It follows from [54, pp. 40, 43] that $\mathcal{A}$ is a category.

(iii) If one ignores the set-theoretic issue from Remark 4.13, then Theorem 4.22 below can be interpreted as showing that the triplet $\davidsstar$ satisfies the properties of a 2-functor $n\text{-}\mathsf{Exang} \to \mathsf{Exact}$; see [38, Prop. 4.1.8].

**Theorem 4.22.** *The following statements hold for the assignments $\davidsstar_0$, $\davidsstar_1$ and $\davidsstar_2$.*

(i) *The pair* $(\davidsstar_0, \davidsstar_1)$ *defines a functor* $n$-$\mathsf{Exang} \to \mathsf{Exact}$.

(ii) *Given $n$-exangulated categories* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *and* $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$*, the pair* $(\davidsstar_1, \davidsstar_2)$ *defines a functor* $\mathcal{N} \to \mathcal{A}$ *in the notation of* Proposition 4.12 *and* Remark 4.21(ii).

(iii) *The assignment* $\davidsstar_2$ *preserves horizontal composition.*

*Proof.* (i) This part follows from Remark 4.21(i) and Lemma 3.19(iii).

(ii) The second statement holds by Remark 4.21(i) combined with a straightforward check to see that $\davidsstar_2$ preserves identity morphisms (i.e. identity $n$-exangulated natural transformations) and (vertical) composition.

(iii) We must show that the natural transformations $\davidsstar_2(\daleth \circ_h \beth) \colon \mathscr{E}_{(\mathscr{L},\Phi)\circ(\mathscr{F},\Gamma)} \Rightarrow \mathscr{E}_{(\mathscr{M},\Psi)\circ(\mathscr{G},\Lambda)}$ and $(\davidsstar_2 \daleth) \circ_h (\davidsstar_2 \beth) \colon \mathscr{E}_{(\mathscr{L},\Phi)} \circ \mathscr{E}_{(\mathscr{F},\Gamma)} \Rightarrow \mathscr{E}_{(\mathscr{M},\Psi)} \circ \mathscr{E}_{(\mathscr{G},\Lambda)}$ are equal. Notice first that their domains (resp. codomains) agree by Lemma 3.19(iii). Consequently, for $\delta \in \mathbb{E}(C, A)$, the morphisms $(\davidsstar_2(\daleth \circ_h \beth))_\delta$ and $((\davidsstar_2 \daleth) \circ_h (\davidsstar_2 \beth))_\delta$ have the same domain (resp. codomain). Thus, by Remark 3.9, it is enough to note that

$$
\begin{aligned}
(\davidsstar_2(\daleth \circ_h \beth))_\delta &= \langle \daleth \circ_h \beth \rangle_\delta && \text{by the definition of } \davidsstar_2 \\
&= ((\daleth \circ_h \beth)_A, (\daleth \circ_h \beth)_C) && \text{see Theorem 4.19} \\
&= (\daleth_{\mathscr{G}A}(\mathscr{L}\beth_A), \daleth_{\mathscr{G}C}(\mathscr{L}\beth_C)) && \text{see Definition 4.3} \\
&= (\daleth_{\mathscr{G}A}, \daleth_{\mathscr{G}C})(\mathscr{L}\beth_A, \mathscr{L}\beth_C) && \text{by the definition of composition in } \mathbb{E}''\text{-Ext}(\mathcal{C}'') \\
&= \langle \daleth \rangle_{\mathscr{E}_{(\mathscr{G},\Lambda)}(\delta)} \mathscr{E}_{(\mathscr{L},\Phi)}(\langle \beth \rangle_\delta) && \text{as } \mathscr{E}_{(\mathscr{L},\Phi)} \text{ respects morphisms over } \mathscr{L} \\
&= (\langle \daleth \rangle \circ_h \langle \beth \rangle)_\delta && \text{using the definition of } \circ_h \\
&= ((\davidsstar_2 \daleth) \circ_h (\davidsstar_2 \beth))_\delta && \text{by the definition of } \davidsstar_2. \qquad \blacksquare
\end{aligned}
$$

As Theorem 4.22 establishes that the functor $\davidsstar \colon n$-$\mathsf{Exang} \to \mathsf{Exact}$ behaves just like a 2-functor, it enjoys similar properties. For example, it is known that 2-functors preserve adjunctions; see e.g. [38, Prop. 6.1.7]. Thus, we deduce the following result, which is readily shown by applying $\davidsstar_2$ to the triangle identities (4.4), and using that $\davidsstar_2$ preserves vertical and horizontal composition and identities of 2-cells.

**Corollary 4.23.** *If* $((\mathscr{F}, \Gamma), (\mathscr{A}, \Xi))$ *is an $n$-exangulated adjoint pair, then* $(\mathscr{E}_{(\mathscr{F},\Gamma)}, \mathscr{E}_{(\mathscr{A},\Xi)})$ *is an adjoint pair of exact functors.*

Corollary 4.25 below yields Theorem D from Section 1. It follows immediately from Theorem 4.22 that the restriction of $\davidsstar = (\davidsstar_0, \davidsstar_1, \davidsstar_2)$ to $n$-$\mathsf{exang}$ is a 2-functor. One checks that $\davidsstar$ is faithful on 1-cells by using that any morphism $x \colon X \to Y$ in $\mathcal{C}$ induces a morphism $(x, 0) \colon {}_X0_0 \to {}_Y0_0$ of extensions. For an example showing that $\davidsstar$ is not full on 1-cells, see Example 4.24 below. The 2-functor is faithful but not full on 2-cells by Theorem 4.19 and Example 4.17, respectively.

**Example 4.24.** Let $\mathcal{C}$ be a non-zero additive category. Equip $\mathcal{C}$ with the split $n$-exangulated structure $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, which is induced by the split $n$-exact structure as explained in Example 5.5. Hence, the category $\mathbb{E}$-Ext$(\mathcal{C})$ consists of objects of the form ${}_A0_C$, one for each pair $A, C \in \mathcal{C}$. Any conflation in $(\mathbb{E}\text{-Ext}(\mathcal{C}), \mathcal{X}_{\mathbb{E}})$ is isomorphic to one of the form ${}_{A}0_{C} \longrightarrow {}_{A\oplus A'}0_{C\oplus C'} \longrightarrow {}_{A'}0_{C'}$ given by $(\iota_A, \iota_C)$ and $(\pi_{A'}, \pi_{C'})$; see the discussion around (3.2).

Define $\mathscr{E}\colon \mathbb{E}\text{-Ext}(\mathcal{C}) \to \mathbb{E}\text{-Ext}(\mathcal{C})$ by ${}_{A}0_{C} \mapsto {}_{C}0_{A}$ and $(a,c) \mapsto (c,a)$. It is straightforward to check that $\mathscr{E}$ is a functor by noting that any pair of morphisms $a\colon A \to B$ and $c\colon C \to D$ in $\mathcal{C}$ defines a morphism $(a,c)\colon {}_{A}0_{C} \to {}_{B}0_{D}$ in $\mathbb{E}\text{-Ext}(\mathcal{C})$. Furthermore, a conflation of the form described above is sent under $\mathscr{E}$ to the sequence ${}_{C}0_{A} \longrightarrow {}_{C\oplus C'}0_{A\oplus A'} \longrightarrow {}_{C'}0_{A'}$ given by the morphisms $(\iota_C, \iota_A)$ and $(\pi_{C'}, \pi_{A'})$. This is an element of $\mathcal{X}_{\mathbb{E}}$, so $\mathscr{E}$ is exact.

We claim that $\mathscr{E} \neq \mathscr{E}_{(\mathscr{F},\Gamma)}$ for every $n$-exangulated functor $(\mathscr{F},\Gamma)\colon (\mathcal{C},\mathbb{E},\mathfrak{s}) \to (\mathcal{C},\mathbb{E},\mathfrak{s})$. To see this, assume there exists some additive functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}$ such that $\mathscr{E}$ respects morphisms over $\mathscr{F}$. Then it must be the case that $\mathscr{F}X = X$ for all $X \in \mathcal{C}$, as

$$(\mathrm{id}_X, \mathrm{id}_X) = \mathscr{E}(\mathrm{id}_X, \mathrm{id}_X) = (\mathscr{F}\mathrm{id}_X, \mathscr{F}\mathrm{id}_X) = (\mathrm{id}_{\mathscr{F}X}, \mathrm{id}_{\mathscr{F}X}).$$

Now consider $A, C \in \mathcal{C}$ with $A \ncong C$. The identity $(\mathrm{id}_A, \mathrm{id}_C)\colon {}_{A}0_{C} \to {}_{A}0_{C}$ is equal to $(\mathrm{id}_{\mathscr{F}A}, \mathrm{id}_{\mathscr{F}C}) = (\mathscr{F}\mathrm{id}_A, \mathscr{F}\mathrm{id}_C) = \mathscr{E}(\mathrm{id}_A, \mathrm{id}_C) = (\mathrm{id}_C, \mathrm{id}_A)$. This implies that $\mathrm{id}_A = \mathrm{id}_C$, which is a contradiction.

**Corollary 4.25.** *Restriction of the assignments* $✡_i$ *from* Definition 4.20 *yields a* 2*-functor* $✡ = (✡_0, ✡_1, ✡_2)\colon n\text{-}\mathsf{exang} \to \mathsf{exact}$. *This* 2*-functor is faithful on* 1*-cells and* 2*-cells, but is full on neither* 1*-cells nor* 2*-cells.*

## 5. Examples of $n$-exangulated categories, functors and natural transformations

Let $n \geqslant 1$ be an integer. We begin this section by recalling some known classes of $n$-exangulated categories arising from extriangulated, $(n+2)$-angulated and $n$-exact settings; see Examples 5.1, 5.3 and 5.4, respectively. In each of these cases, we discuss what it means for functors and natural transformations to respect the $n$-exangulated structure. We then show that any additive category admits a “smallest” $n$-exangulated structure in Example 5.5.

We move on to considering $n$-exangulated functors for which the type of structure of the domain category *differs* from that of the codomain category, providing examples of $n$-exangulated functors which are neither $(n+2)$-angulated nor $n$-exact in general. Our characterisation in Theorem 3.17 is applied to establish many of these examples. In Examples 5.6 and 5.8, we study structure-preserving functors from $n$-exact to $(n+2)$-angulated categories, before the canonical functor from a Frobenius $n$-exangulated category to its $(n+2)$-angulated stable category is shown to be $n$-exangulated in Example 5.9. Additionally, we demonstrate how the relative theory of $n$-exangulated categories can be used to equip a triangulated category with its pure-exact extriangulated structure; see Examples 5.10 and 5.11. In doing so, we show that the restricted Yoneda embedding gives an example of an extriangulated functor which is neither exact nor triangulated in general.

Our first three examples each consists of two parts. In part (i) we focus on $n$-exangulated functors. In part (ii) we discuss what it means for natural transformations to be $n$-exangulated.

**Example 5.1.** *Extriangulated* categories were introduced by Nakaoka–Palu in [57] as a simultaneous generalisation of triangulated and exact categories. Examples, which are neither triangulated nor exact in general, include extension-closed subcategories [57, Rem. 2.18] and certain ideal quotients [57, Prop. 3.30] of triangulated categories. A category is extriangulated if and only if it is 1-exangulated [29, Prop. 4.3]. Thus, we obtain a plethora of categories with interesting $n$-exangulated structures for $n = 1$.

(i) A 1-exangulated functor is also called *extriangulated* [6, Def. 2.32]. In Examples 5.6 and 5.11, we exhibit extriangulated functors from work of Keller [44] and Krause [48], respectively.

(ii) Following (i), by an *extriangulated natural transformation* we refer to the case $n = 1$ in Definition 4.1. *Morphisms of extriangulated functors* were introduced by Nakaoka–Ogawa–Sakai [56, Def. 2.11(3)]. The equation [56, (2.2)] defining such morphisms is precisely (4.1) in the case $n = 1$, and so this notion coincides with that of an extriangulated natural transformation. In Example 5.6 we exhibit extriangulated natural transformations which arise in [44].

In Examples 5.3 and 5.4, we use the following notation.

**Notation 5.2.** Suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ are $n$-exangulated categories, and let $\beth\colon \mathscr{F} \Rightarrow \mathscr{G}$ be a natural transformation of additive functors $\mathcal{C} \to \mathcal{C}'$. For $\delta \in \mathbb{E}(C, A)$ with $\mathfrak{s}(\delta) = [X^{\bullet}]$, we note that setting $\beth^{i}_{X} := \beth_{X^{i}}$ for each $i$ defines a morphism of complexes $\beth^{\bullet}_{X} \colon \mathscr{F}_{\mathsf{C}} X^{\bullet} \to \mathscr{G}_{\mathsf{C}} X^{\bullet}$ by the naturality of $\beth$. In particular, we have $\beth^{0}_{X} = \beth_{A}$ and $\beth^{n+1}_{X} = \beth_{C}$.

**Example 5.3.** An $(n+2)$-*angulated* category (see [21, Def. 2.1]) is the higher homological analogue of a triangulated category. Any $(n+2)$-angulated category $(\mathcal{C}, \Sigma_{n}, \pentagon)$ has the structure of an $n$-exangulated category $(\mathcal{C}, \mathbb{E}_{\pentagon}, \mathfrak{s}_{\pentagon})$; see [29, Sec. 4.2]. In this case, we have $\mathbb{E}_{\pentagon}(C, A) := \mathcal{C}(C, \Sigma_{n} A)$ and $\mathfrak{s}_{\pentagon}(\delta) := [X^{\bullet}]$ whenever $\delta \in \mathbb{E}_{\pentagon}(X^{n+1}, X^{0}) = \mathcal{C}(X^{n+1}, \Sigma_{n} X^{0})$ completes to a distinguished $(n+2)$-*angle*

$$X^{0} \xrightarrow{d^{0}_{X}} X^{1} \longrightarrow \cdots \longrightarrow X^{n} \xrightarrow{d^{n}_{X}} X^{n+1} \xrightarrow{\delta} \Sigma_{n} X^{0}. \tag{5.1}$$

(i) For $(n+2)$-angulated categories $(\mathcal{C}, \Sigma_{n}, \pentagon)$ and $(\mathcal{C}', \Sigma'_{n}, \pentagon')$, the notion of an $(n+2)$-angulated functor $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ was introduced by Bergh–Thaule [7, Sec. 4]. The functor $\mathscr{F}$ is $(n+2)$-*angulated* if it is additive and comes equipped with a natural isomorphism $\Theta\colon \mathscr{F}\Sigma_{n} \Rightarrow \Sigma'_{n}\mathscr{F}$ such that for any distinguished $(n+2)$-angle of the form (5.1) one obtains a distinguished $(n+2)$-angle

$$\mathscr{F}X^{0} \xrightarrow{\mathscr{F}d^{0}_{X}} \mathscr{F}X^{1} \longrightarrow \cdots \longrightarrow \mathscr{F}X^{n} \xrightarrow{\mathscr{F}d^{n}_{X}} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{X_0} \circ \mathscr{F}\delta} \Sigma'_{n}\mathscr{F}X^{0}.$$

For $\mathscr{F}$ and $\Theta$ as above, setting $\Gamma_{(X^{n+1},X^{0})}(\delta) = \Theta_{X_0} \circ \mathscr{F}\delta$ defines a natural transformation $\Gamma\colon \mathcal{C}(-, \Sigma_{n}-) \Rightarrow \mathcal{C}'(\mathscr{F}-, \Sigma'_{n}\mathscr{F}-)$. In [6, Thm. 2.33], it is shown that the existence of an $(n+2)$-angulated functor $\mathscr{F}$ with natural isomorphism $\Theta$ is equivalent to the existence of an $n$-exangulated functor $(\mathscr{F}, \Gamma)$ from $(\mathcal{C}, \mathbb{E}_{\pentagon}, \mathfrak{s}_{\pentagon})$ to $(\mathcal{C}', \mathbb{E}_{\pentagon'}, \mathfrak{s}_{\pentagon'})$.

(ii) As noted in [56, Rem. 2.12], the notion of an extriangulated natural transformation is equivalent to the definition of a *morphism of triangulated functors* in the sense of

Kashiwara–Schapira [43, Def. 10.1.9(ii)], whenever the extriangulated categories involved correspond to triangulated categories.

Keeping the notation from above, suppose that $(\mathscr{F},\Gamma)$ and $(\mathscr{G},\Lambda)$ are $n$-exangulated functors $(\mathcal{C},\mathbb{E}_{\pentagon},\mathfrak{s}_{\pentagon}) \to (\mathcal{C}',\mathbb{E}_{\pentagon'},\mathfrak{s}_{\pentagon'})$ corresponding to $(n+2)$-angulated functors. Note that a natural transformation $\beth\colon \mathscr{F} \Rightarrow \mathscr{G}$ satisfies (4.1) if and only if each square

$$\begin{array}{ccc} \mathscr{F}X^{n+1} & \xrightarrow{\Gamma_{(X^{n+1},X^0)}(\delta)} & \Sigma'_n\mathscr{F}X^0 \\ {\scriptstyle \beth_{X^{n+1}}}\downarrow & & \downarrow{\scriptstyle \Sigma'_n\beth_{X^0}} \\ \mathscr{G}X^{n+1} & \xrightarrow[\Lambda_{(X^{n+1},X^0)}(\delta)]{} & \Sigma'_n\mathscr{G}X^0 \end{array}$$

commutes, in which case the sequence $(\beth^1_X,\dots,\beth^n_X)$ defines a *morphism of* $(n+2)$-$\Sigma'_n$-*sequences* in $\mathcal{C}'$ in the sense of [21, Def. 2.1].

Parallel to the $(n+2)$-angulated story is the $n$-exact one.

**Example 5.4.** Higher versions of abelian and exact categories were introduced in [36]. Analogously to the classical theory, every $n$-*abelian* category carries an $n$-*exact* structure [36, Thm. 4.4]. Any skeletally small $n$-exact category $(\mathcal{C},\mathcal{X})$, where $\mathcal{X}$ is the collection of *admissible* $n$-exact sequences, gives rise to an $n$-exangulated category $(\mathcal{C},\mathbb{E}_{\mathcal{X}},\mathfrak{s}_{\mathcal{X}})$; see [29, Sec. 4.3]. In this case, we have $\mathbb{E}_{\mathcal{X}}(X^{n+1},X^0) \coloneqq \{\, [X^\bullet] \mid X^\bullet \in \mathcal{X} \,\}$, where $[X^\bullet]$ is the equivalence class with respect to the homotopy relation $\sim$ described in Section 2.

(i) For $n$-exact categories $(\mathcal{C},\mathcal{X})$ and $(\mathcal{C}',\mathcal{X}')$, an additive functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ is called $n$-*exact* provided $\mathscr{F}_{\mathsf{C}}X^\bullet \in \mathcal{X}'$ whenever $X^\bullet \in \mathcal{X}$ [6, Def. 2.18]. Assuming that the categories involved are skeletally small and considering the associated $n$-exangulated structures, the existence of an $n$-exact functor $\mathscr{F}$ is equivalent to the existence of an $n$-exangulated functor $(\mathscr{F},\Gamma)$ from $(\mathcal{C},\mathbb{E}_{\mathcal{X}},\mathfrak{s}_{\mathcal{X}})$ to $(\mathcal{C}',\mathbb{E}_{\mathcal{X}'},\mathfrak{s}_{\mathcal{X}'})$; see [6, Thm. 2.34]. In this case, the natural transformation $\Gamma$ is uniquely defined by $\Gamma_{(X^{n+1},X^0)}([X^\bullet]) = [\mathscr{F}_{\mathsf{C}}X^\bullet]$.

(ii) Keeping the notation from above, suppose that $\mathscr{F}$ and $\mathscr{G}$ are $n$-exact functors $\mathcal{C} \to \mathcal{C}'$ and consider a natural transformation $\beth\colon \mathscr{F} \Rightarrow \mathscr{G}$. As noted in [56, Rem. 2.12], equation (4.1) is automatically satisfied for the corresponding $n$-exangulated functors in the case $n=1$. We now show that this holds for any integer $n \geqslant 1$.

Consider the equivalence class $\delta = [X^\bullet] \in \mathbb{E}_{\mathcal{X}}(C,A)$ of an admissible $n$-exact sequence $X^\bullet \in \mathcal{X}$. As $\mathscr{F}$ and $\mathscr{G}$ are $n$-exact, we have $\mathscr{F}_{\mathsf{C}}X^\bullet, \mathscr{G}_{\mathsf{C}}X^\bullet \in \mathcal{X}'$, so $\beth^\bullet_X\colon \mathscr{F}_{\mathsf{C}}X^\bullet \to \mathscr{G}_{\mathsf{C}}X^\bullet$ is a morphism of admissible $n$-exact sequences. By the existence of $n$-pushout diagrams in the $n$-exact category $(\mathcal{C}',\mathcal{X}')$, we obtain a morphism $p^\bullet\colon \mathscr{F}_{\mathsf{C}}X^\bullet \to (\beth_A)_{\mathbb{E}_{\mathcal{X}'}}\mathscr{F}_{\mathsf{C}}X^\bullet$ with $p^0 = \beth_A$ and $p^{n+1} = \mathrm{id}_{\mathscr{F}C}$; see [36, Def. 4.2(E2) and Prop. 4.8]. Dually, there is also a morphism $q^\bullet\colon (\beth_C)^{\mathbb{E}_{\mathcal{X}'}}\mathscr{G}_{\mathsf{C}}X^\bullet \to \mathscr{G}_{\mathsf{C}}X^\bullet$ with $q^0 = \mathrm{id}_{\mathscr{G}A}$ and $q^{n+1} = \beth_C$. By [36, Prop. 4.9], there exists a morphism $l^\bullet\colon (\beth_A)_{\mathbb{E}_{\mathcal{X}'}}\mathscr{F}_{\mathsf{C}}X^\bullet \to \mathscr{G}_{\mathsf{C}}X^\bullet$ satisfying $l^\bullet p^\bullet \sim \beth^\bullet_X$, $l^0 = \mathrm{id}_{\mathscr{G}A}$ and $l^{n+1} = \beth_C$. On the other hand, the dual of [36, Prop. 4.9] yields a morphism $m^\bullet\colon (\beth_A)_{\mathbb{E}_{\mathcal{X}'}}\mathscr{F}_{\mathsf{C}}X^\bullet \to (\beth_C)^{\mathbb{E}_{\mathcal{X}'}}\mathscr{G}_{\mathsf{C}}X^\bullet$ with $q^\bullet m^\bullet \sim l^\bullet$, $m^0 = l^0 = \mathrm{id}_{\mathscr{G}A}$ and $m^{n+1} = \mathrm{id}_{\mathscr{F}C}$. Note that $m^\bullet$ is an *equivalence* of $n$-exact sequences in the sense of [36, Def. 2.9]. This implies that $(\beth_A)_{\mathbb{E}_{\mathcal{X}'}}[\mathscr{F}_{\mathsf{C}}X^\bullet] = [(\beth_A)_{\mathbb{E}_{\mathcal{X}'}}\mathscr{F}_{\mathsf{C}}X^\bullet] = [(\beth_C)^{\mathbb{E}_{\mathcal{X}'}}\mathscr{G}_{\mathsf{C}}X^\bullet] = (\beth_C)^{\mathbb{E}_{\mathcal{X}'}}[\mathscr{G}_{\mathsf{C}}X^\bullet]$ by [36, Prop. 4.10], which verifies (4.1).

Our next example shows that any additive category $\mathcal{C}$ carries a smallest $n$-exangulated structure $(\mathcal{C}, \mathbb{E}_{n\text{-split}}, \mathfrak{s}_{n\text{-split}})$ arising from an $n$-exact structure for each integer $n \geqslant 1$. In particular, we have that $(\mathcal{C}, \mathbb{E}_{n\text{-split}}, \mathfrak{s}_{n\text{-split}})$ is an $n$-exangulated subcategory, in the sense of [24, Def. 3.7], of any $n$-exangulated structure $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ we impose on $\mathcal{C}$. Recall that an *$n$-exangulated subcategory* of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an isomorphism-closed subcategory $\mathcal{A}$ of $\mathcal{C}$ with an $n$-exangulated structure $(\mathcal{A}, \mathbb{E}', \mathfrak{s}')$ for which the inclusion $\mathscr{F}$ of $\mathcal{A}$ in $\mathcal{C}$ gives an $n$-exangulated functor $(\mathscr{F}, \Gamma)\colon (\mathcal{A}, \mathbb{E}', \mathfrak{s}') \to (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ where each $\Gamma_{(C,A)}$ is an inclusion of abelian groups.

**Example 5.5.** By [36, Rem. 4.7], any additive category $\mathcal{C}$ admits a smallest $n$-exact structure $(\mathcal{C}, \mathcal{X}_{n\text{-split}})$, where $\mathcal{X}_{n\text{-split}}$ denotes the class of all contractible $n$-exact sequences. Recall that a sequence is *contractible* if it is homotopy equivalent to the zero complex. We call $\mathcal{X}_{n\text{-split}}$ the *split $n$-exact structure* of $\mathcal{C}$. Notice that for all $A, C \in \mathcal{C}$, any contractible $n$-exact sequence starting in $A$ and ending in $C$ is homotopic to

$$A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow C \xrightarrow{\mathrm{id}_C} C. \tag{5.2}$$

Note that in the case $n = 1$, the sequence (5.2) is of the form $A \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_A \\ 0 \end{smallmatrix}\right)} A \oplus C \xrightarrow{\left(\begin{smallmatrix} 0 & \mathrm{id}_C \end{smallmatrix}\right)} C$. As seen in Example 5.4, the $n$-exact category $(\mathcal{C}, \mathcal{X}_{n\text{-split}})$ yields an $n$-exangulated structure $(\mathcal{C}, \mathbb{E}_{n\text{-split}}, \mathfrak{s}_{n\text{-split}})$, which we call the *split $n$-exangulated structure* of $\mathcal{C}$. Note that no set-theoretic issues arise as $\mathbb{E}_{n\text{-split}}(C, A) = \{{}_A0_C\}$ is the trivial abelian group for all $A, C \in \mathcal{C}$.

We claim that this is the smallest $n$-exangulated structure on $\mathcal{C}$. To see this, suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an $n$-exangulated category. Consider the identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ and the natural transformation $\Gamma\colon \mathbb{E}_{n\text{-split}} \Rightarrow \mathbb{E}$ given by $\Gamma_{(C,A)}({}_A0_C) = {}_A0_C$. Note that, by (R2) and [29, Prop. 3.3], we have

$$\mathfrak{s}_{n\text{-split}}({}_A0_C) = \mathfrak{s}({}_A0_C) = [\ A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow C \xrightarrow{\mathrm{id}_C} C\ ].$$

Consequently, the pair $(\mathrm{id}_{\mathcal{C}}, \Gamma)\colon (\mathcal{C}, \mathbb{E}_{n\text{-split}}, \mathfrak{s}_{n\text{-split}}) \to (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an $n$-exangulated functor. As $\Gamma_{(C,A)}$ is an inclusion for all $A, C \in \mathcal{C}$, this implies that $(\mathcal{C}, \mathbb{E}_{n\text{-split}}, \mathfrak{s}_{n\text{-split}})$ is an $n$-exangulated subcategory of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. In particular, the split $n$-exangulated structure is the smallest $n$-exangulated structure we can impose on $\mathcal{C}$.

In Example 5.6 we describe extriangulated functors from an exact or abelian category to a triangulated category. Some of these examples are classical and others have been of very recent interest. The authors would like to thank Peter Jørgensen for pointing out Linckelmann [51, Rem. 6.8], which is used in (ii) below.

**Example 5.6.** Structure-preserving functors from exact to triangulated categories, or more generally from exact to *suspended* categories in the sense of Keller–Vossieck [46], have been considered in the literature previously. Such functors are called *$\delta$-functors* in [44, pp. 701–702]. Let $\mathcal{A}$ be an exact category for which $\mathrm{Ext}^1_{\mathcal{A}}(C, A)$ is a set for all $A, C \in \mathcal{A}$. Consider a triangulated category $\mathcal{C}$ with suspension functor $\Sigma$. Suppose that $\mathscr{F}\colon \mathcal{A} \to \mathcal{C}$ is an additive functor and that $\Gamma\colon \mathrm{Ext}^1_{\mathcal{A}}(-,-) \Rightarrow \mathcal{C}(\mathscr{F}-, \Sigma\mathscr{F}-)$ is a natural transformation. The pair

$(\mathscr{F},\Gamma)$ is called a *$\delta$-functor* if each conflation

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{5.3}$$

in $\mathcal{A}$ is sent to a distinguished triangle in $\mathcal{C}$ of the form

$$\mathscr{F}A \xrightarrow{\mathscr{F}f} \mathscr{F}B \xrightarrow{\mathscr{F}g} \mathscr{F}C \xrightarrow{\Gamma_{(C,A)}(\rho)} \Sigma\mathscr{F}A, \tag{5.4}$$

where $\rho$ denotes the equivalence class of the conflation (5.3). Before translating the language of $\delta$-functors to that of extriangulated functors, we recall some examples.

(i) Let $\mathcal{A}$ be a skeletally small exact category and $\mathcal{C}$ its *derived* category with suspension functor $\Sigma$; see Keller [45, p. 692]. If $\mathscr{F}\colon \mathcal{A} \to \mathcal{C}$ denotes the canonical inclusion, then there is a natural transformation $\Gamma\colon \operatorname{Ext}^1_{\mathcal{A}}(-,-) \Rightarrow \mathcal{C}(\mathscr{F}-,\Sigma\mathscr{F}-)$ such that the equivalence class of a conflation (5.3) is sent to a distinguished triangle (5.4); see [44, pp. 701–702]. The pair $(\mathscr{F},\Gamma)$ is a $\delta$-functor $\mathcal{A}\to\mathcal{C}$.

(ii) Suppose $\mathcal{C}$ is a triangulated category with suspension functor $\Sigma$. Let $\mathscr{F}\colon \mathcal{A}\to\mathcal{C}$ denote the inclusion of an isomorphism-closed additive subcategory $\mathcal{A}\subseteq\mathcal{C}$. Assume that $\mathcal{A}$ is a *distinguished abelian subcategory* in the sense of [51, Def. 1.1]. This means that $\mathcal{A}$ is abelian and, given a short exact sequence (5.3) in $\mathcal{A}$, there exists a morphism $\Gamma_{(C,A)}(\rho)\colon \mathscr{F}C \to \Sigma\mathscr{F}A$ in $\mathcal{C}$ such that (5.4) is a distinguished triangle in $\mathcal{C}$. A morphism of short exact sequences in the abelian category $\mathcal{A}$ uniquely determines a morphism of distinguished triangles in the triangulated category $\mathcal{C}$; see [51, Rem. 6.8]. This implies that $\Gamma = \{\Gamma_{(C,A)}\}_{(C,A)\in\mathcal{A}^{\mathrm{op}}\times\mathcal{A}}\colon \operatorname{Ext}^1_{\mathcal{A}}(-,-)\Rightarrow \mathcal{C}(\mathscr{F}-,\Sigma\mathscr{F}-)$ is a natural transformation. The pair $(\mathscr{F},\Gamma)$ is hence a $\delta$-functor $\mathcal{A}\to\mathcal{C}$. Similarly, the inclusions of *proper abelian subcategories* (see Jørgensen [40, Def. 1.2]) and *admissible abelian subcategories* (see Beĭlinson–Bernstein–Deligne [2, Def. 1.2.5]) into their ambient triangulated categories give rise to $\delta$-functors; see [40, Rem. 1.3].

Consider the extriangulated categories that arise from $\mathcal{A}$ and $\mathcal{C}$ being exact and triangulated, respectively. Following Example 5.3 and Example 5.4 with $n=1$, we obtain extriangulated categories $(\mathcal{A},\mathbb{E},\mathfrak{s})$ and $(\mathcal{C},\mathbb{E}',\mathfrak{s}')$ with $\mathbb{E} := \operatorname{Ext}^1_{\mathcal{A}}(-,-)$ and $\mathbb{E}' := \mathcal{C}(-,\Sigma-)$. The pair $(\mathscr{F},\Gamma)$ is a $\delta$-functor $\mathcal{A}\to\mathcal{C}$ if and only if it is an extriangulated functor $(\mathcal{A},\mathbb{E},\mathfrak{s})\to(\mathcal{C},\mathbb{E}',\mathfrak{s}')$.

Furthermore, *morphisms of $\delta$-functors* are also defined and studied in [44, p. 702]. If $(\mathscr{F},\Gamma)$ and $(\mathscr{G},\Lambda)$ are $\delta$-functors $\mathcal{A}\to\mathcal{C}$, then a morphism $(\mathscr{F},\Gamma)\Rightarrow(\mathscr{G},\Lambda)$ is given by a natural transformation $\beth\colon \mathscr{F}\Rightarrow\mathscr{G}$ satisfying $(\Sigma\beth_A)\circ\Gamma_{(C,A)}(\rho) = \Lambda_{(C,A)}(\rho)\circ\beth_C$ for each extension $\rho\in\mathbb{E}(C,A)$. This defining condition is precisely (4.1) in the context of this example. Thus, the notion of a morphism of $\delta$-functors coincides with that of an extriangulated natural transformation.

*Remark* 5.7. The prototypical example of a $\delta$-functor is the canonical embedding of an exact category into its derived category; see Example 5.6(i). Morphisms of $\delta$-functors were studied in [44] in the search for some universal property of the derived category. Thus, interesting examples of $n$-exangulated natural transformations with $n>1$ may arise from a generalisation of this to a higher-dimensional setting. This would require the construction of a derived category of an $n$-exact category—a problem for which there seems so far to be no obvious solution; see Jasso–Külshammer [37].

Although $\delta$-functors provide formal language to express what it means for functors to send conflations to distinguished triangles in a functorial way, it has two apparent limitations. First, in its current form this notion cannot be used in higher homological algebra. Second, a $\delta$-functor must go from an exact category to a triangulated (or suspended) category, but not vice versa. The language of $n$-exangulated functors addresses both these limitations, as we will see across Examples 5.8, 5.9 and 5.11.

In the next example, we show how recent work of Klapproth [47] (see also [27, 68]) produces examples of $n$-exangulated functors from an $n$-exact category into an ambient $(n+2)$-angulated category. This also gives more examples of $n$-exangulated subcategories.

**Example 5.8.** Let $(\mathcal{C}, \Sigma_n, \pentagon)$ be a Krull–Schmidt $(n+2)$-angulated category and consider the $n$-exangulated structure $(\mathcal{C}, \mathbb{E}_{\pentagon}, \mathfrak{s}_{\pentagon})$ described in Example 5.3. Let $\mathcal{A}$ be a subcategory of $\mathcal{C}$ which is closed under direct sums and summands, and *closed under $n$-extensions*, meaning that for all $A, C \in \mathcal{A}$ and each $\delta \in \mathcal{A}(C, \Sigma_n A)$ there is a distinguished $(n+2)$-angle

$$A \xrightarrow{d_X^0} X^1 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{d_X^n} C \xrightarrow{\delta} \Sigma_n A \tag{5.5}$$

in $\mathcal{C}$ with $X^i \in \mathcal{A}$ for $i = 1, \dots, n$; see [47, Def. 1.1]. This also implies $\mathcal{A}$ is an *extension closed* subcategory of $(\mathcal{C}, \mathbb{E}_{\pentagon}, \mathfrak{s}_{\pentagon})$ in the sense of [30, Def. 4.1].

Suppose, moreover, that $\mathcal{A}(\Sigma_n C, A) = 0$ for any $A, C \in \mathcal{A}$. Following [47, Sec. 3], an $\mathcal{A}$-*conflation* is a complex $X^\bullet\colon\ A \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow C$ that forms part of a distinguished $(n+2)$-angle (5.5). Let $\mathbb{E}_{\mathcal{A}} \coloneqq \mathcal{A}(-, \Sigma_n -)$ be the restriction of $\mathbb{E}_{\pentagon}$ to $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$. Restricting $\mathfrak{s}_{\pentagon}$ defines an exact realisation $\mathfrak{s}_{\mathcal{A}}$ of $\mathbb{E}_{\mathcal{A}}$ by [30, Prop. 4.2(1)]. In an $\mathcal{A}$-conflation $X^\bullet$ as above, the morphism $d_X^0$ is called an $\mathcal{A}$-*inflation* and $d_X^{n+1}$ is called an $\mathcal{A}$-*deflation.* By [47, Lem. 3.8] and its dual, we have that both $\mathcal{A}$-inflations and $\mathcal{A}$-deflations are closed under composition. This implies that $(\mathcal{A}, \mathbb{E}_{\mathcal{A}}, \mathfrak{s}_{\mathcal{A}})$ is an $n$-exangulated category by [30, Prop. 4.2(2)]. Furthermore, it is an $n$-exangulated subcategory of $(\mathcal{C}, \mathbb{E}_{\pentagon}, \mathfrak{s}_{\pentagon})$; see [24, Exam. 3.8(1)].

We denote the collection of all $\mathcal{A}$-conflations by $\mathcal{X}_{\mathcal{A}}$, each member of which is an $n$-exact sequence by [47, Lem. 3.3]. It follows from [47, Thm. I(1)] that the pair $(\mathcal{A}, \mathcal{X}_{\mathcal{A}})$ is an $n$-exact category. For $A, C \in \mathcal{A}$, there is an abelian group $\mathrm{YExt}^n_{(\mathcal{A},\mathcal{X}_{\mathcal{A}})}(C, A)$ of equivalence classes $[X^\bullet]$ of $\mathcal{A}$-conflations $X^\bullet \in \mathcal{X}_{\mathcal{A}}$ with $A = X^0$ and $C = X^{n+1}$. Note that no set-theoretic problems arise here, since the size of $\mathrm{YExt}^n_{(\mathcal{A},\mathcal{X}_{\mathcal{A}})}(C, A)$ is bounded by the size of the set $\mathcal{A}(C, \Sigma_n A)$. Hence, this gives a biadditive functor $\mathbb{E} \coloneqq \mathrm{YExt}^n_{(\mathcal{A},\mathcal{X}_{\mathcal{A}})}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$ and an $n$-exangulated category $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ as in Example 5.4. By [47, Thm. I(2)], there is a natural isomorphism $\Gamma\colon \mathbb{E} \Rightarrow \mathbb{E}_{\mathcal{A}}$ given by $\Gamma([X^\bullet]) = \delta$, where $X^\bullet$ is part of (5.5). Thus, the pair $(\mathrm{id}_{\mathcal{A}}, \Gamma)\colon (\mathcal{A}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{A}, \mathbb{E}_{\mathcal{A}}, \mathfrak{s}_{\mathcal{A}})$ is an $n$-exangulated equivalence by Proposition 4.11.

*Frobenius exact categories* are studied in Happel [22, Sec. I.2], and their higher analogues were introduced in [36, Sec. 5]. In these setups, the quotient functor from a Frobenius exact (resp. $n$-exact) category to its stable category sends admissible exact (resp. $n$-exact) sequences to distinguished triangles (resp. $(n+2)$-angles). In Example 5.9 we follow the terminology introduced by Liu–Zhou [53, Def. 3.2], and we show that these aforementioned quotient functors are instances of extriangulated (resp. $n$-exangulated) functors.

**Example 5.9.** Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an $n$-exangulated category. An object $I\in\mathcal{C}$ is called $\mathbb{E}$-*injective* if for each distinguished $n$-exangle $\langle X^{\bullet},\delta\rangle$, which is depicted as

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \overset{\delta}{\dashrightarrow}, \tag{5.6}$$

and for each $x\in\mathcal{C}(X^0,I)$, there exists $y\in\mathcal{C}(X^1,I)$ such that $yd_X^0=x$. The category $\mathcal{C}$ is said to have *enough* $\mathbb{E}$-*injectives* if for any $X^0\in\mathcal{C}$, there is a distinguished $n$-exangle

$$X^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow Z \overset{\delta_{X^0}}{\dashrightarrow}, \tag{5.7}$$

where $I^i$ is $\mathbb{E}$-injective for $1\leqslant i\leqslant n$. Dually, one defines what it means for an object of $\mathcal{C}$ to be $\mathbb{E}$-*projective* and for the category $\mathcal{C}$ to *have enough* $\mathbb{E}$-*projectives*. If $\mathcal{C}$ has enough $\mathbb{E}$-projectives and enough $\mathbb{E}$-injectives, and if an object in $\mathcal{C}$ is $\mathbb{E}$-projective if and only if it is $\mathbb{E}$-injective, then we say that $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is a *Frobenius* $n$-exangulated category.

Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be Frobenius $n$-exangulated. Denote by $\overline{\mathcal{C}}$ the stable category $\mathcal{C}/\mathcal{I}$ in the sense of [53, p. 169], where $\mathcal{I}$ is the subcategory of $\mathbb{E}$-projective-injectives. Consider the canonical quotient functor $\mathscr{Q}\colon\mathcal{C}\to\overline{\mathcal{C}}$. Note that $\mathscr{Q}(X)=X$ in $\overline{\mathcal{C}}$ for each $X\in\mathcal{C}$. Since $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is Frobenius, setting $SX^0\coloneqq Z$ in (5.7) yields a well-defined autoequivalence of $\overline{\mathcal{C}}$; see [53, Prop. 3.7]. Given any distinguished $n$-exangle (5.6), there is a morphism

$$\begin{array}{ccccccccccc} X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \overset{\delta}{\dashrightarrow} \\ \| & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle d_X^{n+1}} & \\ X^0 & \longrightarrow & I^1 & \longrightarrow & \cdots & \longrightarrow & I^n & \longrightarrow & SX^0 & \overset{\delta_{X^0}}{\dashrightarrow} \end{array} \tag{5.8}$$

of distinguished $n$-exangles in $\mathcal{C}$, using that $I^1$ is $\mathbb{E}$-injective and [29, Prop. 3.6(1)]. Furthermore, there is a natural isomorphism $\mathbb{E}(-,-)\cong\overline{\mathcal{C}}(\mathscr{Q}-,S\mathscr{Q}-)$ given by $\delta\mapsto\mathscr{Q}(d_X^{n+1})$; see [53, Lem. 3.11]. It is shown in [53, Thm. 3.13] (see also Zheng–Wei [67, Prop. 3.17]) that there is an $(n+2)$-angulation of $(\overline{\mathcal{C}},S)$ consisting of $(n+2)$-angles of the form

$$X^0 \xrightarrow{\mathscr{Q}(d_X^0)} X^1 \xrightarrow{\mathscr{Q}(d_X^1)} \cdots \xrightarrow{\mathscr{Q}(d_X^{n-1})} X^n \xrightarrow{\mathscr{Q}(d_X^n)} X^{n+1} \xrightarrow{\mathscr{Q}(d_X^{n+1})} SX^0. \tag{5.9}$$

This gives an $n$-exangulated category $(\overline{\mathcal{C}},\overline{\mathbb{E}},\overline{\mathfrak{s}})$, where $\overline{\mathbb{E}}(X^{n+1},X^0)=\overline{\mathcal{C}}(X^{n+1},SX^0)$ and

$$\overline{\mathfrak{s}}(\mathscr{Q}(d_X^{n+1}))=[\ X^0 \xrightarrow{\mathscr{Q}(d_X^0)} X^1 \xrightarrow{\mathscr{Q}(d_X^1)} \cdots \xrightarrow{\mathscr{Q}(d_X^{n-1})} X^n \xrightarrow{\mathscr{Q}(d_X^n)} X^{n+1}\ ]$$

whenever the morphism $\mathscr{Q}(d_X^{n+1})\in\overline{\mathcal{C}}(X^{n+1},SX^0)$ fits into an $(n+2)$-angle (5.9).

Using Theorem 3.17, it is straightforward to check that there exists a natural transformation $\Gamma\colon\mathbb{E}(-,-)\Longrightarrow\overline{\mathbb{E}}(\mathscr{Q}-,\mathscr{Q}-)$, such that $(\mathscr{Q},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\overline{\mathcal{C}},\overline{\mathbb{E}},\overline{\mathfrak{s}})$ is an $n$-exangulated functor. Indeed, in the notation above, consider the functor $\mathscr{E}\colon\mathbb{E}\text{-Ext}(\mathcal{C})\to\overline{\mathbb{E}}\text{-Ext}(\overline{\mathcal{C}})$ given by $\mathscr{E}(\delta)=\mathscr{Q}(d_X^{n+1})$ on objects and by $\mathscr{E}(a,c)=(\mathscr{Q}a,\mathscr{Q}c)$ on morphisms. Note that $(\mathscr{Q}(f^0),\mathscr{Q}(f^n))$ is a morphism in $\overline{\mathbb{E}}\text{-Ext}(\overline{\mathcal{C}})$ by [53, Lem. 3.11]. It is clear that $\mathscr{E}$ respects morphisms and distinguished $n$-exangles over $\mathscr{Q}$. By Proposition 3.11, we see that $\mathscr{E}$ is additive, and thus Theorem 3.17 applies. Without the theorem, proving the naturality of $\Gamma$ requires a non-trivial amount of extra work.

In Examples 5.10 and 5.11, we use *relative* $n$-exangulated structures. We provide some key definitions here, but refer the reader to [29, Sec. 3.2] for details. Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an $n$-exangulated category and $\mathcal{I}$ be a subcategory of $\mathcal{C}$. Then the assignment $\mathbb{E}_{\mathcal{I}}\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathsf{Ab}$ given by $\mathbb{E}_{\mathcal{I}}(C,A) := \{\, \delta\in\mathbb{E}(C,A) \mid {}^{\mathbb{E}}\delta_X = 0 \text{ for all } X\in\mathcal{I}\,\}$ defines a subfunctor of $\mathbb{E}$. The restriction $\mathfrak{s}_{\mathcal{I}}$ of $\mathfrak{s}$ to the extensions $\delta\in\mathbb{E}_{\mathcal{I}}(C,A)$ for $A,C\in\mathcal{C}$ is an exact realisation of $\mathbb{E}_{\mathcal{I}}$. Moreover, it follows from [29, Props. 3.16, 3.19] that $(\mathcal{C},\mathbb{E}_{\mathcal{I}},\mathfrak{s}_{\mathcal{I}})$ is an $n$-exangulated category, and by [41, Thm. 2.12] that it is an $n$-exangulated subcategory of $(\mathcal{C},\mathbb{E},\mathfrak{s})$.

The next two examples concern compactly generated triangulated categories. We recall relevant definitions in Example 5.10; for more details, see work of the first author [3], Garkusha–Prest [19], Krause [48], Neeman [58], and Prest [61]. In this example we recall how equipping a compactly generated triangulated category with its class of pure-exact triangles results in an extriangulated substructure of the triangulated structure. This was first noted in Hu–Zhang–Zhou [31, Rem. 3.3] from a different perspective. In Example 5.11 we show that a certain restricted Yoneda functor is extriangulated, and preserves and reflects injective objects.

**Example 5.10.** Let $\mathcal{C}$ be a triangulated category with suspension functor $\Sigma$. Then $\mathcal{C}$ has the structure of an extriangulated category, which we denote by $(\mathcal{C},\mathbb{E},\mathfrak{s})$; see Example 5.3. Suppose $\mathcal{C}$ has all set-indexed coproducts. Following Neeman [58, pp. 210–211], an object $X\in\mathcal{C}$ is called *compact* if the functor $\mathcal{C}(X,-)$ commutes with all coproducts, and we denote by $\mathcal{C}^{\mathrm{c}}$ the subcategory of $\mathcal{C}$ consisting of compact objects. The category $\mathcal{C}$ is *compactly generated* provided there is a set $\mathcal{S}$ of objects in $\mathcal{C}^{\mathrm{c}}$ such that, for each $A\in\mathcal{C}$, if $\mathcal{C}(X,A)=0$ for all $X\in\mathcal{S}$, then $A$ must be the zero object. Suppose that $\mathcal{C}$ is compactly generated.

As defined in [48, Def. 1.1(3)], a distinguished triangle

$$A \longrightarrow B \longrightarrow C \overset{\delta}{\longrightarrow} \Sigma A \tag{5.10}$$

in $\mathcal{C}$ is called *pure-exact* if, for any object $X\in\mathcal{C}^{\mathrm{c}}$, there is an induced short exact sequence $0\longrightarrow\mathcal{C}(X,A)\longrightarrow\mathcal{C}(X,B)\longrightarrow\mathcal{C}(X,C)\longrightarrow 0$ of abelian groups. A morphism $\delta\colon C\to\Sigma A$ in $\mathcal{C}$ is *phantom* if $\mathcal{C}(X,\delta)$ is the zero morphism $\mathcal{C}(X,C)\to\mathcal{C}(X,\Sigma A)$ for all $X\in\mathcal{C}^{\mathrm{c}}$ (see [48, p. 104]). For a morphism $\delta\colon C\to\Sigma A$ that fits into a distinguished triangle (5.10), there is a natural transformation ${}^{\mathbb{E}}\delta_{\_} = \mathcal{C}(-,\delta)\colon\mathcal{C}(-,C)\Rightarrow\mathcal{C}(-,\Sigma A)$ (see Section 2). Consequently, the morphism $\delta$ is phantom if and only if ${}^{\mathbb{E}}\delta_X = 0$ for all $X\in\mathcal{C}^{\mathrm{c}}$. By Krause [49, Lem. 1.3], one has that the distinguished triangle (5.10) is pure-exact if and only if $\delta\colon C\to\Sigma A$ is phantom. This implies that $(\mathcal{C},\mathbb{E}_{\mathcal{C}^{\mathrm{c}}},\mathfrak{s}_{\mathcal{C}^{\mathrm{c}}})$ is an extriangulated category in which $\langle\, A\overset{f}{\longrightarrow}B\overset{g}{\longrightarrow}C\,,\delta\rangle$ is a distinguished extriangle if and only if $A\overset{f}{\longrightarrow}B\overset{g}{\longrightarrow}C\overset{\delta}{\longrightarrow}\Sigma A$ is a pure-exact triangle in $\mathcal{C}$. By our discussion above this example, we have that $(\mathcal{C},\mathbb{E}_{\mathcal{C}^{\mathrm{c}}},\mathfrak{s}_{\mathcal{C}^{\mathrm{c}}})$ is an *extriangulated* (i.e. 1-exangulated) subcategory of the (ex)triangulated category $(\mathcal{C},\mathbb{E},\mathfrak{s})$. Furthermore, if $\mathcal{C}$ contains a non-zero object, then $(\mathcal{C},\mathbb{E}_{\mathcal{C}^{\mathrm{c}}},\mathfrak{s}_{\mathcal{C}^{\mathrm{c}}})$ is not triangulated; and if there is a non-split pure-exact distinguished triangle, then it is not exact (cf. [31, Rem. 3.3]).

Building on Example 5.10, the next example shows that a certain restricted Yoneda functor $\mathscr{Y}\colon \mathcal{C} \to \mathsf{Mod}-\mathcal{C}^{\mathrm{c}}$ preserves extriangles. Like in Example 5.9, the characterisation of $n$-exangulated functors of in Theorem 3.17 makes light work of this. We also show that $\mathscr{Y}$ both preserves and reflects injective objects. Relevant definitions are provided as needed.

**Example 5.11.** Suppose that $\mathcal{C}$ is a triangulated category with suspension functor $\Sigma$. Assume that $\mathcal{C}$ has all set-indexed coproducts and is compactly generated. Denote by $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ the extriangulated category arising from the triangulated structure on $\mathcal{C}$. In Example 5.10 we showed that the relative structure induced by the subcategory $\mathcal{C}^{\mathrm{c}}$ of compact objects in $\mathcal{C}$ gives an extriangulated subcategory $(\mathcal{C}, \mathbb{E}_{\mathcal{C}^{\mathrm{c}}}, \mathfrak{s}_{\mathcal{C}^{\mathrm{c}}})$. Recall that the distinguished extriangles of $(\mathcal{C}, \mathbb{E}_{\mathcal{C}^{\mathrm{c}}}, \mathfrak{s}_{\mathcal{C}^{\mathrm{c}}})$ correspond to the pure-exact triangles in $\mathcal{C}$.

Assume that $\mathcal{C}^{\mathrm{c}}$ is skeletally small. Let $\mathsf{Mod}-\mathcal{C}^{\mathrm{c}}$ be the category of additive functors $(\mathcal{C}^{\mathrm{c}})^{\mathrm{op}} \to \mathsf{Ab}$, which is a Grothendieck abelian category; see [61, Thm. 10.1.3]. In particular, it has enough injectives (see [61, Thm. E.1.8]), so for all $L, N \in \mathsf{Mod}-\mathcal{C}^{\mathrm{c}}$, the collection of equivalence classes of short exact sequences of the form $0 \longrightarrow L \longrightarrow - \longrightarrow N \longrightarrow 0$ is a set. Thus, we have that the Ext-bifunctor $\mathbb{E}' \coloneqq \operatorname{Ext}^1_{\mathsf{Mod}-\mathcal{C}^{\mathrm{c}}}\colon (\mathsf{Mod}-\mathcal{C}^{\mathrm{c}})^{\mathrm{op}} \times \mathsf{Mod}-\mathcal{C}^{\mathrm{c}} \to \mathsf{Ab}$ is well-defined. Consequently, equipping $\mathsf{Mod}-\mathcal{C}^{\mathrm{c}}$ with $\mathbb{E}'$ and the canonical realisation $\mathfrak{s}'$ yields an extriangulated category $(\mathsf{Mod}-\mathcal{C}^{\mathrm{c}}, \mathbb{E}', \mathfrak{s}')$.

Write $\mathscr{Y}\colon \mathcal{C} \to \mathsf{Mod}-\mathcal{C}^{\mathrm{c}}$ for the *restricted Yoneda functor*, which is defined on objects by $\mathscr{Y}(Z) = \mathcal{C}(-, Z)|_{\mathcal{C}^{\mathrm{c}}}$ (see [48, p. 105]). In the rest of this example, we show the following two statements.

(i) There is an extriangulated functor $(\mathscr{Y}, \Gamma)\colon (\mathcal{C}, \mathbb{E}_{\mathcal{C}^{\mathrm{c}}}, \mathfrak{s}_{\mathcal{C}^{\mathrm{c}}}) \to (\mathsf{Mod}-\mathcal{C}^{\mathrm{c}}, \mathbb{E}', \mathfrak{s}')$.

(ii) The functor $\mathscr{Y}$ both preserves and reflects injective objects.

Let us first prove (i). By Theorem 3.17, it is sufficient to define an additive functor

$$\mathscr{E}\colon \mathbb{E}_{\mathcal{C}^{\mathrm{c}}}\text{-Ext}(\mathcal{C}) \to \mathbb{E}'\text{-Ext}(\mathsf{Mod}-\mathcal{C}^{\mathrm{c}})$$

which respects both morphisms and distinguished extriangles over $\mathscr{Y}$. Given $\delta \in \mathbb{E}_{\mathcal{C}^{\mathrm{c}}}(C, A)$, there is a pure-exact distinguished triangle $A \longrightarrow B \longrightarrow C \overset{\delta}{\longrightarrow} \Sigma A$ that is unique up to isomorphism. Thus, we define $\mathscr{E}$ on objects by setting

$$\mathscr{E}(\delta) \coloneqq [\ 0 \longrightarrow \mathscr{Y}(A) \longrightarrow \mathscr{Y}(B) \longrightarrow \mathscr{Y}(C) \longrightarrow 0\ ].$$

A morphism $(a, c)\colon \delta \to \delta'$ in $\mathbb{E}_{\mathcal{C}^{\mathrm{c}}}\text{-Ext}(\mathcal{C})$ extends to a morphism of triangles between the pure-exact triangles associated to $\delta$ and $\delta'$. Hence, the pair $(\mathscr{Y}a, \mathscr{Y}c)\colon \mathscr{E}(\delta) \to \mathscr{E}(\delta')$ is a morphism in $\mathbb{E}'\text{-Ext}(\mathsf{Mod}-\mathcal{C}^{\mathrm{c}})$, and we can define $\mathscr{E}(a, c) \coloneqq (\mathscr{Y}a, \mathscr{Y}c)$. It is straightforward to check that $\mathscr{E}$ is a functor, and it respects morphisms and distinguished extriangles over $\mathscr{Y}$ by construction. Since $\mathscr{Y}$ is additive, so is $\mathscr{E}$ by Proposition 3.11. This verifies (i). Note that if $\mathcal{C}$ contains a non-zero object and at least one non-split pure exact triangle, then $(\mathscr{Y}, \Gamma_{(\mathscr{Y}, \mathscr{E})})$ is neither an exact functor nor a triangulated functor.

Now we show (ii). A morphism $f\colon A \to B$ in the triangulated category $\mathcal{C}$ is a *pure monomorphism* if the morphism $\mathscr{Y}(f)_X = \mathcal{C}(X, f)\colon \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is injective for each compact object $X \in \mathcal{C}^{\mathrm{c}}$ (see [48, Def. 1.1(1)]). An object $C \in \mathcal{C}$ is *pure-injective* provided any pure monomorphism with domain $C$ splits (see [48, Def. 1.1(2)]). Following the dual of

[57, Def. 3.23, Prop. 3.24], an object $I$ in an extriangulated category $(\mathcal{D}, \mathbb{F}, \mathfrak{t})$ is $\mathbb{F}$-*injective* if and only if $\mathbb{F}(D, I) = 0$ for all $D \in \mathcal{D}$. Thus, it follows from [48, Lem. 1.4] that an object in the extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{C}^c}, \mathfrak{s}_{\mathcal{C}^c})$ is $\mathbb{E}_{\mathcal{C}^c}$-injective if and only if it is pure-injective in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Furthermore, by [48, Thm. 1.8], we have that $A \in \mathcal{C}$ is pure-injective in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $\mathscr{Y}A$ is $\mathbb{E}'$-injective in $(\mathsf{Mod}\!-\!\mathcal{C}^c, \mathbb{E}', \mathfrak{s}')$. That is, the extriangulated functor $(\mathscr{Y}, \Gamma)$ from $(\mathcal{C}, \mathbb{E}_{\mathcal{C}^c}, \mathfrak{s}_{\mathcal{C}^c})$ to $(\mathsf{Mod}\!-\!\mathcal{C}^c, \mathbb{E}', \mathfrak{s}')$ preserves and reflects injective objects.

*Remark* 5.12. Suppose that in Examples 5.10 and 5.11 we replace the compactly generated triangulated category with a *finitely accessible* category, and also that we swap compact objects with *finitely presented* objects. With this exchange, one can make analogous observations about the restricted Yoneda functor to those made in Example 5.11 using results of Crawley-Boevey [12]. We omit this example, however, since the restricted Yoneda functor in this case is in fact an exact functor.

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