Realizing LTI models by identifying characteristic parameters using least squares optimization*

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***Abstract*— This paper considers the realization of discrete-time linear time-invariant dynamical systems using input-output data. Starting from a generalized state-space representation, a state-independent system representation is derived using the Cayley-Hamilton theorem and characteristic parameters are introduced to describe the system dynamics in an alternative way. Given input-output data, we present two formulations to account for model deviations and to identify characteristic parameters by minimizing considered error terms in a least squares sense. The applicability of the proposed subspace identification method is demonstrated with physical data of the identification database DaISy.**

I. INTRODUCTION

The analysis of process characteristics and the relationships between variables using input-output data holds prime importance for the innovation, design, monitoring, control and prediction of process systems. System identification is concerned with estimating models of dynamical systems through observed/experimental data (see [1], [2], [3] and references therein for extensive information). The branch of subspace identification (SID) addresses the identification of discrete-time linear time-invariant (LTI) state-space models offering the benefit of non-iterative, numerical efficient and general parameterizations [4]. Classical related SID methods like *Canonical Variate Analysis* (CVA) [5], *Multivariable Output Error State Space* (MOESP) [6] and *Numerical Algorithms for Subspace State Space System Identification* (N4SID) [7] provide asymptotic convergence guarantees for the learned system model [8]. To address closed-loop identification, the approaches of *innovation estimation* [9], ARX modeling (SSARX) [10] or the idea of constructing a state predictor [11] were introduced and applied in *Predictor-Based Subspace Identification* (PBSID) [12]. A survey of various SID methods is provided in [13].

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In contrast to the conventional asymptotic methods, there has been an increasing interest in finite sample size complexity and non-asymptotic analysis from the machine learning community leading to non-asymptotic SID methods like the single-trajectory method of Oymak and Ozay [14] or the multiple-trajectory method of Zheng and Li [15], which derive error-bounds for their respective least squares estimators. Least squares optimization is also taken up by De Moor in [16] demonstrating that the realization of autonomous single-output LTI systems corresponds to the solution of a multiparameter eigenvalue problem. A main concept therein is the elimination of the state vector using the Cayley-Hamilton theorem. In this paper, we adopt the idea of state vector elimination as described in [17] for multiple-input-multiple-output LTI models and write the input-to-output relation of the state-space representation as an equivalent autoregressive model. By introducing characteristic parameters as system descriptive quantities, we propose an ordinary least squares (OLS) formulation and a total least squares (TLS) formulation for identifying them. A suggested state-space realization comprising characteristic parameters completes *characteristic parameters identification* (CPI) as a standalone SID algorithm.

This paper is structured as follows: Section II provides a description of the problem statement which essentially involves a generalized subspace identification problem. In Section III, we derive an alternative, equivalent input-output expression of the system, introduce characteristic parameters, and provide a method for realizing the state-space representation. Subsequently, Section IV covers two least squares minimization approaches for identifying characteristic parameters. In Section V, CPI is benchmarked via cross-validation against well-known and recent SID methods using physical data sets of the identification database DaISy. Conclusions are given in Section VI.

II. PROBLEM FORMULATION

We consider the following n-dimensional state-space representation of a deterministic discrete-time LTI system

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{e} \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) + \mathbf{f} \end{aligned} \tag{1}$$

with the system matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times m}$, the state vector $\mathbf{x} \in \mathbb{R}^n$, the input vector $\mathbf{u} \in \mathbb{R}^m$, the output vector $\mathbf{y} \in \mathbb{R}^p$, constant vectors $\mathbf{e} \in \mathbb{R}^n$, $\mathbf{f} \in \mathbb{R}^p$ and the time index $k \in \mathbb{N}_0$. By extending the standard state-space representation with the vectors $\mathbf{e}$ and $\mathbf{f}$, a static state offset and output offset can be explicitly

considered at the cost of a small increase in the number of model parameters. Our objective is to identify a model of this structure from given input-output data $\{\mathbf{u}(k), \mathbf{y}(k)\}$.

III. CHARACTERISTIC PARAMETERS

The goal of this section is to derive an input-output relation equivalent to that of the system in (1) by introducing characteristic parameters. For this purpose we use the Cayley-Hamilton theorem and eliminate the influence of the state vector $\mathbf{x}$ in a similar way as in [17]. A state-space realization of the derived relation including initial state determination is provided.

Evaluating the state equation and the output equation of (1) for different time steps, the output of the model at time index $k+i$ for $i \in \mathbb{N}_0$ may be written as

$$\begin{aligned}\mathbf{y}(k+i) = \mathbf{C}\mathbf{A}^i\mathbf{x}(k) + \sum_{j=0}^{i-1} \mathbf{C}\mathbf{A}^{i-j-1}\big(\mathbf{B}\mathbf{u}(k+j) + \mathbf{e}\big) \\ + \mathbf{D}\mathbf{u}(k+i) + \mathbf{f}.\end{aligned} \tag{2}$$

For ease of notation, the Markov parameters

$$\mathbf{M}(i) = \begin{cases}\mathbf{D}, & \text{if } i = 0 \\ \mathbf{C}\mathbf{A}^{i-1}\mathbf{B}, & \text{if } i \geq 1\end{cases}$$

and the constants

$$\mathbf{c}(i) = \begin{cases}\mathbf{f}, & \text{if } i = 0 \\ \mathbf{C}\mathbf{A}^{i-1}\mathbf{e}, & \text{if } i \geq 1\end{cases}$$

are introduced, so that the expression in (2) can be rewritten as

$$\mathbf{y}(k+i) = \mathbf{C}\mathbf{A}^i\mathbf{x}(k) + \sum_{j=0}^{i} \left(\mathbf{M}(i-j)\mathbf{u}(k+j) + \mathbf{c}(i-j)\right). \tag{3}$$

From the Cayley-Hamilton theorem it is obtained that

$$p_{\mathbf{A}}(\mathbf{A}) = \alpha_n\mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}_n = \mathbf{O} \tag{4}$$

holds, where $\alpha_n = 1$ and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$ are the coefficients of the characteristic polynomial $p_{\mathbf{A}}$. Combining (3) and (4) yields

$$\sum_{i=0}^{n} \alpha_i \mathbf{y}(k+i) = \mathbf{v} + \sum_{i=0}^{n} \mathbf{G}_i \mathbf{u}(k+i) \tag{5}$$

with

$$\mathbf{v} = \sum_{i=0}^{n}\sum_{j=0}^{i} \alpha_i \mathbf{c}(j) \quad \text{and} \quad \mathbf{G}_i = \sum_{j=0}^{n-i} \alpha_{i+j}\mathbf{M}(j). \tag{6}$$

Remark 1: Any similarity transformation $\bar{\mathbf{x}} = \mathbf{T}\mathbf{x} + \mathbf{b}$ with a nonsingular matrix $\mathbf{T} \in \mathbb{R}^{n\times n}$ and a constant vector $\mathbf{b} \in \mathbb{R}^n$, such that

$$\begin{aligned}\bar{\mathbf{x}}(k+1) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(k) + \bar{\mathbf{B}}\mathbf{u}(k) + \bar{\mathbf{e}} \\ \mathbf{y}(k) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(k) + \bar{\mathbf{D}}\mathbf{u}(k) + \bar{\mathbf{f}}\end{aligned}$$

with

$$\begin{aligned}&\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \quad \bar{\mathbf{B}} = \mathbf{T}\mathbf{B}, \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}, \quad \bar{\mathbf{D}} = \mathbf{D}, \\ &\bar{\mathbf{e}} = \mathbf{T}\mathbf{e} + \left(\mathbf{I} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)\mathbf{b}, \quad \bar{\mathbf{f}} = \mathbf{f} - \mathbf{C}\mathbf{T}^{-1}\mathbf{b}\end{aligned}$$

preserves $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, the matrices $\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_n$ and the vector $\mathbf{v}$. Hence, α_i, $\mathbf{G}_i$ and $\mathbf{v}$ are independent of the realization and referred to as characteristic parameters.

Remark 2: Given the characteristic parameters $\mathbf{v}$, α_i, $\mathbf{G}_i$ for $i \in \{0, 1, \ldots, n\}$, equation (6) can be used to find a realization of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{I} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{O} \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{I} \\ -\alpha_0\mathbf{I} & \cdots & -\alpha_{n-3}\mathbf{I} & -\alpha_{n-2}\mathbf{I} & -\alpha_{n-1}\mathbf{I} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix}\mathbf{M}(1) \\ \mathbf{M}(2) \\ \vdots \\ \mathbf{M}(n)\end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix}\mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{v}\end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix}\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O}\end{bmatrix}, \quad \mathbf{D} = \mathbf{M}(0), \quad \mathbf{f} = \mathbf{0},$$

where $\mathbf{O}, \mathbf{I} \in \mathbb{R}^{p\times p}$. However, for $p > 1$ this observable realization may not be of minimal dimension [18].

Remark 3: Assuming $(\mathbf{A}, \mathbf{C})$ is observable and input-output data $\{\mathbf{u}(k), \mathbf{y}(k)\}$ is available for $k \in \{0, 1, \ldots, q-1\}$ with $q \geq n$, expression (3) can be used to determine the initial state $\mathbf{x}(0)$, since

$$y_q = \mathcal{O}_q\mathbf{x}(0) + \mathcal{M}_q u_q + \mathcal{Z}_q \mathbf{1}_q$$

with

$$y_q = \begin{bmatrix}\mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(q-1)\end{bmatrix}, \quad \mathcal{O}_q = \begin{bmatrix}\mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{q-1}\end{bmatrix},$$

$$\mathcal{M}_q = \begin{bmatrix} \mathbf{M}(0) & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{M}(1) & \mathbf{M}(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{O} \\ \mathbf{M}(q-1) & \cdots & \mathbf{M}(1) & \mathbf{M}(0) \end{bmatrix},$$

$$u_q = \begin{bmatrix}\mathbf{u}(0) \\ \mathbf{u}(1) \\ \vdots \\ \mathbf{u}(q-1)\end{bmatrix},$$

$$\mathcal{Z}_q = \begin{bmatrix} \mathbf{c}(0) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{c}(1) & \mathbf{c}(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{c}(q-1) & \cdots & \mathbf{c}(1) & \mathbf{c}(0) \end{bmatrix}$$

and therefore

$$\mathbf{x}(0) = \mathcal{O}_q^{\dagger}\left(y_q - \mathcal{M}_q u_q - \mathcal{Z}_q \mathbf{1}_q\right),$$

where $\mathbf{1}_q \in \mathbb{R}^q$ indicates the vector of ones and $\mathcal{O}_q^{\dagger}$ denotes the Moore–Penrose inverse of $\mathcal{O}_q$ which can be calculated by

$$\mathcal{O}_q^{\dagger} = \left(\mathcal{O}_q^{\top}\mathcal{O}_q\right)^{-1}\mathcal{O}_q^{\top}.$$

IV. IDENTIFYING CHARACTERISTIC PARAMETERS

There are several ways to identify the values of the characteristic parameters based on input-output data. For noisy data, the obtained values may strongly depend on how the noise is modeled. For the purpose of estimating the characteristic parameters, two least squares optimization formulations inspired by [19] are derived.

A. Ordinary Least Squares Formulation

A least squares estimate of the characteristic parameters can be obtained by minimizing the error related to (5) defined by

$$\boldsymbol{\varepsilon}(k) = -\mathbf{v} + \sum_{i=0}^{n} \left(\alpha_i \mathbf{y}(k+i) - \mathbf{G}_i \mathbf{u}(k+i)\right). \tag{7}$$

It is important to note that $\boldsymbol{\varepsilon}(k) = \mathbf{0}$ if and only if (5) is satisfied. This equation can be written in the form of an autoregressive exogenous (ARX) model as

$$\mathbf{y}(k+n) = \mathbf{v} - \sum_{i=0}^{n-1} \alpha_i \mathbf{y}(k+i) + \sum_{j=0}^{n} \mathbf{G}_j \mathbf{u}(k+j) + \boldsymbol{\varepsilon}(k).$$

Furthermore, all model parameters can be grouped such that

$$\mathbf{y}(k+n) = \mathbf{R}(k)\mathbf{p} + \boldsymbol{\varepsilon}(k)$$

with parameter vector

$$\mathbf{p} = \begin{bmatrix} \alpha_0 & \cdots & \alpha_{n-1} & \mathbf{v}^\top & \mathrm{vec}(\mathbf{G}_0)^\top & \cdots & \mathrm{vec}(\mathbf{G}_n)^\top \end{bmatrix}^\top$$

and regressor

$$\mathbf{R}(k) = \begin{bmatrix} -\mathbf{Y}(k) & \mathbf{U}(k) \end{bmatrix},$$

comprising

$$\begin{aligned} \mathbf{Y}(k) &= \begin{bmatrix} \mathbf{y}(k) & \cdots & \mathbf{y}(k+n-1) \end{bmatrix}, \\ \mathbf{U}(k) &= \begin{bmatrix} \mathbf{I} & \mathbf{u}(k)^\top \otimes \mathbf{I} & \cdots & \mathbf{u}(k+n)^\top \otimes \mathbf{I} \end{bmatrix}. \end{aligned} \tag{8}$$

Here, $\mathrm{vec}(\mathbf{G}_i)$ is the vectorization of the matrix $\mathbf{G}_i$ and $\otimes$ denotes the Kronecker product. It is supposed that input-output data is available for $k \in \{0, 1, 2, \ldots, N-1\}$ for some constant $N \in \mathbb{N}$. If N is sufficiently large and the provided data is sufficiently rich, the characteristic parameter vector $\mathbf{p}$ may be identified by minimizing the sum of squares error vector of $\boldsymbol{\varepsilon}(k)$ for $k \in \{0, 1, \ldots, N-n-1\}$. This can be done by computing the values of $\mathbf{p}$ that correspond to the solution of the optimization problem

$$\min_{\mathbf{p}, \bar{\boldsymbol{\varepsilon}}} \|\bar{\boldsymbol{\varepsilon}}\|^2, \quad \text{s.t.} \quad \bar{\mathbf{y}} = \bar{\mathbf{R}}\mathbf{p} + \bar{\boldsymbol{\varepsilon}} \tag{9}$$

with

$$\bar{\mathbf{y}} = \begin{bmatrix} \mathbf{y}(n) \\ \mathbf{y}(n+1) \\ \vdots \\ \mathbf{y}(N-1) \end{bmatrix}, \quad \bar{\mathbf{R}} = \begin{bmatrix} \mathbf{R}(0) \\ \mathbf{R}(1) \\ \vdots \\ \mathbf{R}(N-n-1) \end{bmatrix},$$

$$\bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} \boldsymbol{\varepsilon}(0) \\ \boldsymbol{\varepsilon}(1) \\ \vdots \\ \boldsymbol{\varepsilon}(N-n-1) \end{bmatrix},$$

where $\|\cdot\|$ denotes the Euclidean norm. This is an ordinary least squares problem with solution

$$\begin{aligned} \mathbf{p} &= \bar{\mathbf{R}}^\dagger \bar{\mathbf{y}} = \left(\bar{\mathbf{R}}^\top \bar{\mathbf{R}}\right)^{-1} \bar{\mathbf{R}}^\top \bar{\mathbf{y}} \qquad \text{and} \\ \bar{\boldsymbol{\varepsilon}} &= \left(\mathbf{I} - \bar{\mathbf{R}}\bar{\mathbf{R}}^\dagger\right) \bar{\mathbf{y}} = \left(\mathbf{I} - \bar{\mathbf{R}} \left(\bar{\mathbf{R}}^\top \bar{\mathbf{R}}\right)^{-1} \bar{\mathbf{R}}^\top\right) \bar{\mathbf{y}}, \end{aligned}$$

assuming that

$$\mathrm{rank}(\bar{\mathbf{R}}) = (n+1)mp + n + p, \tag{10}$$

so that the solution of the optimization problem is unique and, therefore, the values of the identified characteristic parameters are unique.

Remark 4: To satisfy the rank condition (10), the number of data points must fulfill

$$N \geq \left\lceil (n+1)(m+1) + \frac{n}{p} \right\rceil,$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

B. Total Least Squares Formulation

Instead of defining error vectors as in (7), an error can be associated with each measured output $\mathbf{y}(k)$, according to

$$\sum_{i=0}^{n} \alpha_i \left(\mathbf{y}(k+i) - \boldsymbol{\epsilon}_i(k)\right) = \mathbf{v} + \sum_{i=0}^{n} \mathbf{G}_i \mathbf{u}(k+i),$$

where, for any given k and $i \in \{0, 1, \ldots, n\}$, $\boldsymbol{\epsilon}_i(k)$ is the error vector associated with $\mathbf{y}(k+i)$. This equation can be written as

$$\mathbf{E}(k)\boldsymbol{\alpha} = \tilde{\mathbf{Y}}(k)\boldsymbol{\alpha} - \mathbf{U}(k)\mathbf{g},$$

with

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} \mathbf{v} \\ \mathrm{vec}(\mathbf{G}_0) \\ \mathrm{vec}(\mathbf{G}_1) \\ \vdots \\ \mathrm{vec}(\mathbf{G}_n) \end{bmatrix},$$

$$\begin{aligned} \mathbf{E}(k) &= \begin{bmatrix} \boldsymbol{\epsilon}_0(k) & \boldsymbol{\epsilon}_1(k) & \ldots & \boldsymbol{\epsilon}_n(k) \end{bmatrix}, \\ \tilde{\mathbf{Y}}(k) &= \begin{bmatrix} \mathbf{y}(k) & \mathbf{y}(k+1) & \ldots & \mathbf{y}(k+n) \end{bmatrix} \end{aligned}$$

and $\mathbf{U}(k)$ as in (8). Similar to (9), the optimal values of $\mathbf{g}$ and $\boldsymbol{\alpha}$ can be determined by minimizing all errors in a least squares sense. Then, the optimization problem is given by

$$\min_{\boldsymbol{\alpha}, \mathbf{g}, \bar{\mathbf{E}}} \|\bar{\mathbf{E}}\|_{\mathrm{F}}^2, \quad \text{s.t.} \quad \bar{\mathbf{E}}\boldsymbol{\alpha} = \bar{\mathbf{Y}}\boldsymbol{\alpha} - \bar{\mathbf{U}}\mathbf{g}, \tag{11}$$

with

$$\bar{\mathbf{E}} = \begin{bmatrix} \mathbf{E}(0) \\ \mathbf{E}(1) \\ \vdots \\ \mathbf{E}(N-n-1) \end{bmatrix}, \quad \bar{\mathbf{Y}} = \begin{bmatrix} \tilde{\mathbf{Y}}(0) \\ \tilde{\mathbf{Y}}(1) \\ \vdots \\ \tilde{\mathbf{Y}}(N-n-1) \end{bmatrix},$$

$$\bar{\mathbf{U}} = \begin{bmatrix} \mathbf{U}(0) \\ \mathbf{U}(1) \\ \vdots \\ \mathbf{U}(N-n-1) \end{bmatrix}.$$

Here, $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. It is to note that

$$\|\bar{\mathbf{E}}\|_{\mathrm{F}}^2 = \left\|\bar{\mathbf{E}}\left(\mathbf{I} - \frac{\boldsymbol{\alpha}\boldsymbol{\alpha}^\top}{\|\boldsymbol{\alpha}\|^2}\right)\right\|_{\mathrm{F}}^2 + \left\|\bar{\mathbf{E}}\frac{\boldsymbol{\alpha}\boldsymbol{\alpha}^\top}{\|\boldsymbol{\alpha}\|^2}\right\|_{\mathrm{F}}^2.$$

Therefore, minimizing the optimization problem in (11) with respect to $\bar{\mathbf{E}}$ yields

$$\min_{\boldsymbol{\alpha},\mathbf{g}} \left\|\left(\bar{\mathbf{Y}}\boldsymbol{\alpha} - \bar{\mathbf{U}}\mathbf{g}\right)\frac{\boldsymbol{\alpha}^\top}{\|\boldsymbol{\alpha}\|^2}\right\|_{\mathrm{F}}^2$$

with minimizer

$$\bar{\mathbf{E}} = \left(\bar{\mathbf{Y}}\boldsymbol{\alpha} - \bar{\mathbf{U}}\mathbf{g}\right)\frac{\boldsymbol{\alpha}^\top}{\|\boldsymbol{\alpha}\|^2}.$$

Subsequently minimizing with respect to $\mathbf{g}$ leads to

$$\begin{aligned}&\min_{\boldsymbol{\alpha}} \left\|\left(\mathbf{I} - \bar{\mathbf{U}}\bar{\mathbf{U}}^\dagger\right)\bar{\mathbf{Y}}\frac{\boldsymbol{\alpha}\boldsymbol{\alpha}^\top}{\|\boldsymbol{\alpha}\|^2}\right\|_{\mathrm{F}}^2 \\ &= \min_{\boldsymbol{\alpha}} \left\|\left(\mathbf{I} - \bar{\mathbf{U}}\left(\bar{\mathbf{U}}^\top\bar{\mathbf{U}}\right)^{-1}\bar{\mathbf{U}}^\top\right)\bar{\mathbf{Y}}\frac{\boldsymbol{\alpha}\boldsymbol{\alpha}^\top}{\|\boldsymbol{\alpha}\|^2}\right\|_{\mathrm{F}}^2,\end{aligned} \tag{12}$$

assuming that

$$\mathrm{rank}(\bar{\mathbf{U}}) = (n+1)mp + p.$$

The value of $\mathbf{g}$ that minimizes (12) is then given by

$$\mathbf{g} = \bar{\mathbf{U}}^\dagger\bar{\mathbf{Y}}\boldsymbol{\alpha} = \left(\bar{\mathbf{U}}^\top\bar{\mathbf{U}}\right)^{-1}\bar{\mathbf{U}}^\top\bar{\mathbf{Y}}\boldsymbol{\alpha}. \tag{13}$$

Changing the optimization variables from $\boldsymbol{\alpha}$ to

$$\boldsymbol{\beta} = \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|},$$

the optimization problem in (12) may be rewritten as

$$\min_{\boldsymbol{\beta}} \left\|\left(\mathbf{I} - \bar{\mathbf{U}}\left(\bar{\mathbf{U}}^\top\bar{\mathbf{U}}\right)^{-1}\bar{\mathbf{U}}^\top\right)\bar{\mathbf{Y}}\boldsymbol{\beta}\right\|^2, \text{ s.t. } \|\boldsymbol{\beta}\| = 1. \tag{14}$$

This optimization problem can be solved by considering the singular value decomposition of $\left(\mathbf{I} - \bar{\mathbf{U}}\left(\bar{\mathbf{U}}^\top\bar{\mathbf{U}}\right)^{-1}\bar{\mathbf{U}}^\top\right)\bar{\mathbf{Y}}$ denoted by

$$\left(\mathbf{I} - \bar{\mathbf{U}}\left(\bar{\mathbf{U}}^\top\bar{\mathbf{U}}\right)^{-1}\bar{\mathbf{U}}^\top\right)\bar{\mathbf{Y}} = \mathbf{U}_{\boldsymbol{\beta}}\boldsymbol{\Sigma}_{\boldsymbol{\beta}}\mathbf{V}_{\boldsymbol{\beta}}^\top,$$

where $\mathbf{U}_{\boldsymbol{\beta}}$ and $\mathbf{V}_{\boldsymbol{\beta}}$ are orthonormal matrices and $\boldsymbol{\Sigma}_{\boldsymbol{\beta}}$ is a diagonal matrix containing the singular values in descending order. Consequently, the vector $\boldsymbol{\beta}$ that minimizes (14) is equal to the last column of $\mathbf{V}_{\boldsymbol{\beta}}$, which corresponds to the smallest singular value [19]. To satisfy $\alpha_n = 1$, see (4), the elements of $\boldsymbol{\beta}$ are numbered from zero to n (i.e. $\boldsymbol{\beta} = [\beta_0, \beta_1, \ldots, \beta_n]^\top$). Then, the optimal values of $\boldsymbol{\alpha}$ are given by

$$\alpha_i = \frac{\beta_i}{\beta_n} \tag{15}$$

for $i \in \{0, 1, \ldots, n\}$, where it is assumed that β_n is nonzero. Hence, all optimal values of the characteristic parameters can be obtained using (13) and (15).

Remark 5: As with the OLS formulation, the number of data points must fulfill

$$N \geq \left\lceil (n+1)(m+1) + \frac{n}{p} \right\rceil$$

to ensure that the optimization problem (11) of the TLS formulation has a unique solution.

V. EXPERIMENTAL COMPARISON

In this section the two formulations of CPI are applied to different types of systems, such as a heating system, a glass furnace, a tank reactor and an industrial winding process, as listed in Table I. Consistent with the previous sections, N denotes the number of data points, m the number of inputs and p the number of outputs. The first data set originates from

TABLE I
EXAMPLES FROM DAISY

Data set	N	m	p	Ref.
Heating system	801	1	1	[20]
Glass furnace (Philips)	1247	3	6	[8]
Stirred tank reactor	7500	1	2	[21]
Industrial winding process	2500	5	2	[22]

a single-input-single-output heating system. The input signal triggers a halogen lamp mounted a few centimeters above a thin steel plate. The output is a thermocouple measurement taken from the back of the plate. Secondly, the data of a glass furnace is used, where the inputs are the heating and cooling inlets. The signals of six temperature sensors in a cross section of the furnace comprise the output. The third data set emerges from a continuously stirred tank reactor wherein an exothermic reaction takes place. The concentration is controlled by regulating the coolant flow. The temperature and the concentration are monitored. Lastly, the data of a test setup of an industrial winding process is considered. The main part of the plant consists of a plastic web that is unwinded from the first reel, passes over the traction reel and is finally rewinded on the rewinding reel. Reel one and three are coupled with a DC-motor that is controlled with two input currents. The angular speed of each reel are considered as further inputs, where the tensions in the web between reel one and two are measured by tension meters. Input-output data of all these systems is provided by the DaISy database [23] which is built for verifying and comparing identification algorithms. The four selected datasets are appropriate for testing SID algorithms as [24] indicates.

To benchmark CPI against well-known SID methods, Matlab® implementations of MOESP, CVA, SSARX, PBSID and the modern non-asymptotic method of Oymak and Ozay [14] are also applied to the four sample data sets. Source code of the PBSID algorithm is available in the Predictor-Based Subspace Identification Toolbox [25] and the asymptotic methods MOESP and CVA as well as the SSARX algorithm can be called via the *n4sid* command. It is worth noting that the *n4sid* function of the Matlab® System Identification Toolbox [26] can indicate a suitable dimension n of the state-space model to be parameterized. Furthermore, it should be noted that the method of Oymak and Ozay does not represent a general SID method, since it has additional requirements to provide non-asymptotic identification results, such as a zero initial state and the assumption that the data originates from LTI systems. These prerequisites may be difficult to fulfill if physical data is used. Table II describes all considered SID algorithms. To assess the methods via *cross-validation*, the

TABLE II

SUBSPACE IDENTIFICATION METHODS

Name	Description	Ref.
CPI OLS	ordinary least squares formulation of CPI	this paper
CPI TLS	total least squares formulation of CPI	this paper
MOESP	*n4sid* with 'N4Weight' set to 'MOESP'	[6], [26]
CVA	*n4sid* with 'N4Weight' set to 'CVA'	[5], [26]
SSARX	*n4sid* with 'N4Weight' set to 'SSARX'	[10], [26]
PBSID	predictor-based subspace identification	[12], [25]
Oymak	non-asymptotic subspace identification	[14]

data is split into identification and verification parts. The first 70% of the total number of samples is used for estimation, while the final 30% is used for evaluating the model quality. The validation criteria utilized here are the normalized root-mean-square error (NRMSE) fitness value

$$v_{\text{fit}} = \text{fit}\left(\mathbf{y}, \hat{\mathbf{y}}\right) = \max\left(1 - \frac{\|\mathbf{y} - \hat{\mathbf{y}}\|}{\|\mathbf{y} - \bar{\mathbf{y}}\|}, 0\right) \cdot 100\%$$

and the variance accounted for (VAF)

$$v_{\text{vaf}} = \text{vaf}\left(\mathbf{y}, \hat{\mathbf{y}}\right) = \max\left(1 - \frac{\text{var}(\mathbf{y} - \hat{\mathbf{y}})}{\text{var}(\mathbf{y})}, 0\right) \cdot 100\%,$$

indicating how well the model output sequence $\hat{\mathbf{y}}$ matches the original data sequence $\mathbf{y}$, where $\bar{\mathbf{y}}$ denotes the arithmetic mean and $\text{var}(\mathbf{y})$ the variance of $\mathbf{y}$. The higher v_{fit} and v_{vaf}, the lower the prediction error and the better the model. The average execution time over ten identification runs is recorded as an additional performance criterion. The outputs of the identified state-space models are visualized in Fig. 1 - 4 and evaluated in Table III - VI. By comparing the

TABLE III

QUALITY OF HEATING SYSTEM MODELS ($n = 2$)

Method	v_{fit}	v_{vaf}	execution time
CPI OLS	48.00%	92.97%	21ms
CPI TLS	41.26%	91.25%	25ms
MOESP	49.34%	80.25%	157ms
CVA	49.17%	80.05%	143ms
SSARX	13.51%	33.97%	146ms
PBSID	57.87%	86.57%	2ms
Oymak	35.48%	94.03%	78ms

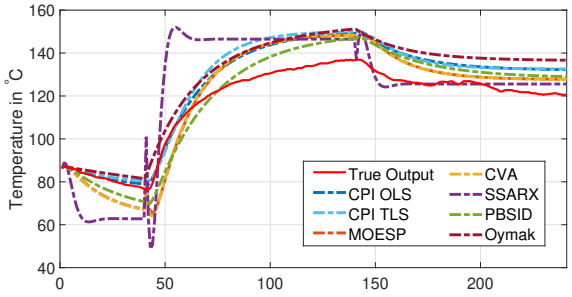

Fig. 1. True output and model outputs of a heating system

outputs of the identified two-dimensional state-space models with the real output of the heating system in Fig. 1 and by computing the quality of the models (see Table III), it can be seen that all SID methods provide consistent identification results, although the values of v_{fit} and v_{vaf} of the SSARX method are noticeably lower. This demonstrates the general functionality of the SID methods compared here.

TABLE IV

QUALITY OF GLASS FURNACE MODELS ($n = 7$)

Method	v_{fit}	v_{vaf}	execution time
CPI OLS	47.36%	70.67%	176ms
CPI TLS	45.63%	68.69%	546ms
MOESP	52.24%	75.16%	653ms
CVA	49.90%	72.58%	682ms
SSARX	46.22%	67.52%	684ms
PBSID	26.80%	61.84%	23ms

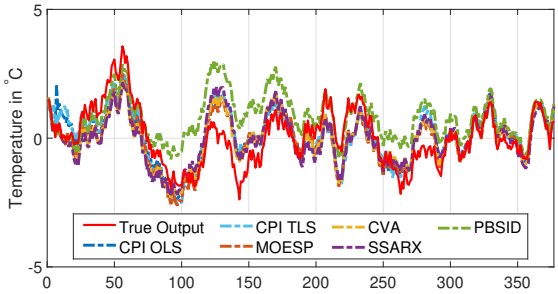

Fig. 2. True output and model outputs of a glass furnace

TABLE V

QUALITY OF TANK REACTOR MODELS ($n = 7$)

Method	v_{fit}	v_{vaf}	execution time
CPI OLS	87.99%	98.41%	726ms
CPI TLS	86.02%	97.97%	1714ms
MOESP	59.39%	92.15%	1119ms
CVA	37.50%	47.41%	1047ms
SSARX	14.71%	25.97%	1048ms
PBSID	87.98%	98.42%	47ms

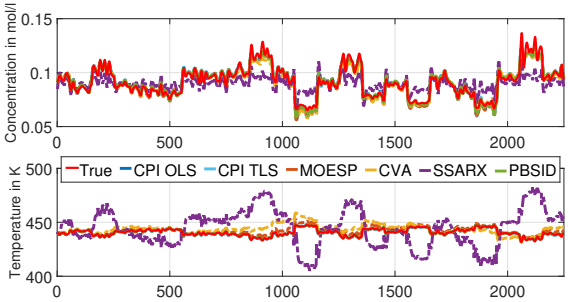

Fig. 3. True output and model outputs of a tank reactor

Applying the considered SID methods to the data of a glass furnace, a tank reactor and a winding process leads to models whose outputs are illustrated in Fig. 2 - 4. For a clear visualization, Fig. 2 shows the representative curve of one of the six temperature sensors inside the furnace.

TABLE VI

QUALITY OF WINDING PROCESS MODELS ($n = 5$)

Method	v_{fit}	v_{vaf}	execution time
CPI OLS	74.62%	92.76%	224ms
CPI TLS	62.85%	84.60%	332ms
MOESP	74.76%	93.03%	443ms
CVA	75.60%	93.45%	411ms
SSARX	75.56%	93.42%	459ms
PBSID	73.07%	92.05%	12ms

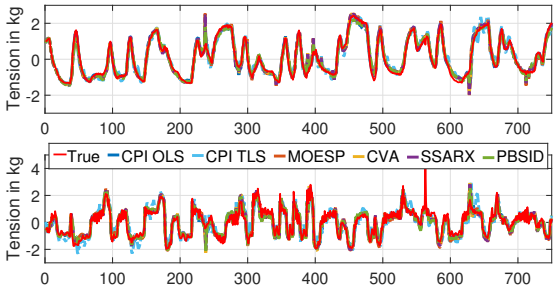

Fig. 4. True output and model outputs of a winding process

The quality characteristics are specified in Table IV - VI. Since the identified model outputs provided by the method of Oymak and Ozay and the true outputs differ significantly, the respective curves are not included in Fig. 2, Fig. 3 and Fig. 4. This performance may be caused by unmet prerequisites, which come into effect when using data generated in practical experiments. The six remaining techniques model the data more accurately, and the low execution time of PBSID and CPI OLS, coupled with decent identification results, is worth emphasizing.

Among the considered subspace methods the overall best fit of the four datasets is achieved by the CPI OLS formulation, which furthermore requires less execution time compared to the related TLS formulation. The identification results of MOESP and CVA are similar to each other. PBSID is appropriate for time-critical identification tasks, as this technique exhibits minimum execution times and achieves nearly as accurate results as the CPI OLS method.

VI. CONCLUSION

In this paper, we have derived a new SID method based on least squares optimization. By considering additional offsets within the state-space representation, the presented approach places comparatively lower demands on the data of the system to be identified. For describing the system dynamics, we have introduced characteristic parameters which form the basis of the state-space realization and derived two approaches of identifying them in a least squares sense. Practical experiments with the DaISy database using Matlab® indicate high performance with comparably low computation time for the OLS as well as for the TLS formulation. For future work, it would be interesting to extend the idea of characteristic parameters identification to closed-loop data, and investigate whether alternative formulations regarding Kalman filtering or optimization are appropriate.

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