Language elements helping to see enlargement as a multiplicative situation

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This paper presents an analysis of Grade 5 pupils' work with similarity, based on an adaptation of Brousseau's classical puzzle. My main interest is in investigating the pupils' language repertoire and mathematical strategies for expressing enlargement in a situation involving similar shapes. A crucial point is to identify elements of the language use that contribute to seeing enlargement to preserve similarity as a multiplicative situation.

Keywords: Enlargement, similarity, multiplicative structures, Theory of Didactical Situations

Introduction

A large part of the mathematics taught in compulsory school in Norway and elsewhere can be related to multiplicative structures (Vergnaud, 1983). The span reaches from the early work with multiplication, usually seen as repeated addition, in the lower grades to more advanced topics such as proportionality, similarity, combinatorics and growth rate in the higher grades. In the project Language Use and Development in the Mathematics Classroom (LaUDiM) — an intervention study carried out in collaboration between researchers at the Norwegian University of Science and Technology and two local primary schools, we have acknowledged the central role of multiplicative structures by making this a recurring theme over several years, from basic models for multiplication and division, to combinatorial problems, and, as in this paper, where the pupils in Grade 5 work with proportional growth in a task which can be seen as an introduction to similarity.

It is widely acknowledged that an important achievement in the development of pupils' numerical thinking is the transition from additive to multiplicative (proportional) thinking and that humans acquire additive reasoning (AR) before proportional reasoning (PR) (Gläser & Riegler, 2015). This may lead to an application of AR in situations where PR is appropriate but there is also evidence to show that when pupils have become familiar with PR, they tend to apply PR in situations where this is *not* appropriate (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2012; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). These studies seem to indicate that language use may play an important role for pupils' success when choosing the correct type of reasoning.

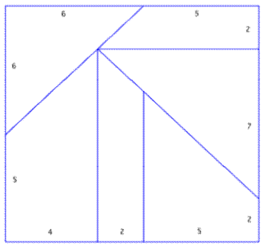
By zooming in on discussions between pupils, and pupils and teacher, I will identify language elements that play a role when pupils development their reasoning from additive to multiplicative (proportional).

The task given to the pupils

The task, presented in Figure 1, is an adaptation of the puzzle described in (Brousseau, 1997, p. 177).

The pieces in the puzzle to the right shall be **enlarged** to a puzzle having the same shape as the one shown to the right. This should be done in such a way that the length which is **4 cm** in the original puzzle becomes **6 cm** in the enlarged puzzle.

1) Each member of the group shall enlarge two pieces.
2) Put the pieces together to an enlarged puzzle.



In Brousseau's example, the given enlargement is from 4 cm to 7 cm. We chose the measures 4 cm and 6 cm in order to make calculations simpler. A similar situation is reported in Erath (2019), with the enlargement factor 7/4 but with slightly different, and fewer pieces.

Multiplicative structures

Greer (1992) presented a classification of situations modelled by multiplication and division. A shortened version of his table is given below (Greer, 1992, p. 280):

- Equal groups and equal measures
- Rate
- Measure conversion
- Multiplicative comparison and multiplicative change
- Part/whole
- Cartesian products
- Rectangular area

The situation in the task in Figure 1, fits with Greer's class multiplicative change, as in his example: "A piece of elastic 4.2 meters long can be stretched to 13.9 meters. By what factor is it lengthened?" (Greer, 1992, p. 280). The crucial point in the task in Figure 1 is to identify the enlargement factor, 1.5, although it is not explicitly said that one is looking for an enlargement *factor*.

Vergnaud sees multiplication, division, fraction, ratio, proportion, similarity, linear functions, and other concepts as belonging to one *conceptual field* and presents three classes of multiplicative structures (Vergnaud, 1983, 2009): *Isomorphy of measures*, *Product of measures*, and *Multiple proportions*. Isomorphy of measures is described as a direct proportion between two measure spaces, M_1 and M_2, as illustrated in Table 1.

M_1	M_2
1	a
b	x

Table 1: Isomorphy of measures

The terms *scalar operator* and *function operator* are used to denote the numbers b and a, respectively. The scalar operator operates *within* one measure space, whereas the function operator operates *between* two measure spaces. This model contains both equal groups, multiplicative comparison and change, as well as rate in Greer's list. Problems fitting into Vergnaud's model *Isomorphy of measures* are of a type referred to by Behr, Harel, Post and Lesh as "the rule of three problems" (1992, p. 297). One can construct problems that on the surface look like rule-of-three-problems, but that not are multiplicative. Several authors have studied and compared pupils' reasoning in such problems

(Fernández et al., 2012; Van Dooren et al., 2005, 2010). Van Dooren et al. (2010) report that there is an increase in the use of PR when AR is appropriate as pupils get older but for young pupils the inappropriate use of AR is dominating. This seems to indicate that when the ability for PR is established it tends to be used also in situations where it is not appropriate. An example of a problem requiring AR, which has the structure of a rule-of-three-problem, is the following:

> Ellen and Kim are running around a track. **They run equally fast but Ellen started later.** When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run? (Van Dooren et al., 2010, p. 368, my emphasis)

In the study by Van Dooren et al. (2010) the percentage of pupils that solved additive problems of this type using PR increased from 17 % in Grade 3 to 67.9 % in Grade 6. The language cue indicating that this is an AR task is "[t]hey run equally fast but Ellen started later". In the task in Figure 1, the cue indicating PR is that the new puzzle should have the same shape as the original one. In addition, the physical fitting together of the pieces was intended to support the choice of PR. As will be seen, this did not suffice, and only after additional language support, elements of PR started to develop. My interest in this paper is to investigate the nature of these elements of language support.

Method

Throughout the whole LaUDiM project, classroom sessions have been designed based on principles from the *Theory of Didactical Situations (TDS)* (Brousseau, 1997). A central feature of TDS is to create an *adidactical situation*, a situation in which the pupils take a mathematical problem as their own and try to solve it without the teacher's guidance and without trying to interpret the teacher's intention with it. For an adidactical situation to be successful, it is important that the pupils get relevant feedback from the *milieu* (Brousseau, 1997) so that they have a chance to know when the task is solved correctly without being told so by the teacher. For this to happen, the task must contain an inner logic, guiding the pupils to the correct solution without the teacher's intervention. To create tasks with such an inner logic has proved to be very challenging (Rønning & Strømskag, 2017). In the task discussed here, the feedback from the milieu is that if the pieces are not enlarged in the correct way, they will not fit together as a puzzle. It is then expected that this feedback will strengthen the connection between the language cue "same shape" and proportional reasoning.

The task was provided for the pupils to work with in groups of three-four, after a brief introduction by the teacher. After some time, the teacher interrupted and gave some additional information, and the pupils continued their work. In the next mathematics lesson, three days later, the lesson started with the teacher giving an introduction where she drew on what she had observed on the first day. During this whole class session, the correct solution of the task was established, with active participation from the pupils. All whole class situations are videotaped, as well as the group work of three groups on day 1 and five groups on day 2. For this paper I follow one group of three pupils, Frances, George and Mary, throughout their work on day 1. In addition, I base my analysis on the teacher's intervention in whole class situations as well as on written work collected from the pupils. In the whole class session on day 2 it became evident that she had observed something interesting in the group with Frank, Roger, Nora and Brenda, and to look closely at this, I also use part of their discussion as data for the analysis.

Analysis of the task

The task given in Figure 1 can be seen as a rule-of-three-problem where the measure spaces M_1 and M_2 are the original and the enlarged puzzle, respectively. Since it is not given what length in the enlarged puzzle corresponds to the length 1 in the original puzzle, a slightly modified version of the Vergnaud table is therefore appropriate for this task (Table 2).

M_1	M_2
4	6
b	x

Table 2: The Vergnaud table for the task

Here b can be any of the other numbers in the original puzzle and the task is to find the corresponding x. The givens in Table 2 will make it possible to compute the enlargement factor, 6/4 = 1.5, which in Vergnaud's (1983) terms is the *function operator*.

The task requires that the pieces should be *enlarged*, and to secure a multiplicative and not an additive enlargement the cue is given that the enlarged puzzle should have *the same shape* as the original puzzle. The task is therefore to produce a new puzzle where each piece, and therefore the whole puzzle, is *similar* to the original one. This can be seen as what Hölzl (2018) describes as a *dynamic* approach to similarity, where enlargement is seen as a dynamic process, something should be done to the original figure to get the new one. This can be contrasted with at *static* view, where only the ratio between corresponding distances is considered.

The Norwegian word used in the task is 'å forstørre', which has the same meaning as 'to enlarge', namely 'to make something larger'. To make something larger can be done by adding a positive number, or by multiplying with a number greater than 1. In the task, the enlargement should lead to new shapes *similar* to the original shapes, meaning that the ratio between corresponding lengths should be the same. Hence, the dynamic enlargement process should lead to a static situation with a constant ratio. Both Brousseau (1997) and Erath (2019), experienced that the first attempt of the pupils was to add a constant number to each length, and indeed, this also happened in our case. Van Dooren et al. (2010) investigated pupils from Grades 3-6 and found that the tendency to use AR for PR was decreasing as pupils got older, whereas the tendency to use PR when AR was appropriate was increasing. One would therefore expect that the pupils in my study (Grade 5) would have developed PR some extent.

The research question that is addressed in this paper is: *What language cues can be identified in the classroom sessions that support a transition from additive reasoning to multiplicative reasoning?*

Analysis of the teaching sequence

Group work, day 1, Frances, George and Mary

The pupils are given the pieces of the original puzzle cut out in blue cardboard with the correct measures. The task is printed on a piece of paper and the pupils are given a large piece of yellow cardboard on which they are supposed to draw the enlarged pieces. Each pupil makes two pieces. When they have drawn their pieces, they are supposed to cut them out and place them together. Almost immediately, when looking at the task, Mary says "we can take plus two. Since this was plus two, we can take all the others plus two". There seems to be agreement in the group that this is the thing to do and based on this, they make their pieces. However, already when cutting out the new shapes they start to suspect that something is wrong. Mary says that "the shapes are not the same". It seems that when enlarging the right-angled triangles, they have added 2 cm to all three sides, and then the new triangles are no longer right-angled. Obviously, the pieces then don't fit together. They think this is because they have not been accurate enough and they try to make new versions of some of the shapes. Although they see that the pieces don't fit, they don't question the "plus 2 strategy". Discussing with the teacher, they realise that when adding 2 to all pieces, the new total shape will no longer be a square. George observes that by adding 2 to each length, the sides composed of three parts will be longer than the sides composed of two parts. Therefore, they decide to concentrate on the frame and make this into a square of 13 x 13 cm (instead of 11 x 11 cm) and adjust the interior parts to fit into the frame. They don't finish this strategy before the teacher intervenes.

Teacher intervention in whole class

The teacher has observed that all groups have used the strategy of adding 2. She decides to help them further by giving the additional information is that the sides which are 5 should become 7.5.

Group work, day 1, Frances, George and Mary, continued

The teacher's new information leads to a new hypothesis in the group, that the even numbers get +2 and the odd numbers get +2.5. This is consistent with the information that 4 → 6 and 5 → 7.5. They use this procedure to compute the new lengths, in combination with trying to get a square. They realise that this procedure will give 15.5 cm for two of the edges and 17.5 cm for the other two, hence not a square. The discussion gives rise to two different hypotheses to solve this problem. One comes from George who suggests that if the length that is 6 becomes 10 and not 8, they will get a 17.5 x 17.5 square. The second hypothesis is to let the lengths that are 2 remain as 2. Then they think that they get a 15.5 x 15.5 square. Here they overlook one of the lengths of 2. They don't get time to finish testing their hypotheses but they seem to be prepared to construct rather elaborate additive structures to solve the problem, where the operator takes on a different form depending on the values of the side lengths.

Teacher in whole class, day 2

The teacher starts by presenting the methods she has observed in the groups on day 1, first the "+2 strategy", and then variations of this, mainly based on distinguishing between even and odd numbers. In the group with Frank, Roger, Nora and Brenda, the teacher has observed something different, and she asks Nora to present the strategy from her group.

Nora: We thought first that four should be six. So then we added two. Then we found that on the other we should add two point five. So we added one half more on the next. Then we tried to do that upwards on the other numbers.

Teacher: What we add, should be larger and larger the longer the edges are.

Nora's utterance "added one half more on the other" gives an indication that they are moving away from adding a constant to adding something which varies, which could be a first step towards PR.

Looking into the video from this particular group on day 1, I can see that these pupils also started with adding 2 for even numbers and 2.5 for odd numbers and when they realised that this did not work they were stuck. They sit for some time, not knowing what to do. Then the teacher comes and asks how they are doing. Frank says “not good”. He has written 4 = 6 and 5 = 7.5. The teacher points to this and the following conversation takes place.

Teacher: Is there something else with these apart from that here we add two and here we add two point five? Is there another pattern we can think of in addition to this?

Frank: Perhaps that when there is a larger number, then we add one half more.

Teacher: OK, so what do you think if it had been six here [points to the number 5]?

Frank: [writes 6 = 9 below 5 = 7.5]

Teacher: You think it will be nine. What about seven?

Frank continues writing and finally he produces the table shown in Figure 2. The teacher turns to the other pupils and encourages them to try out this method.

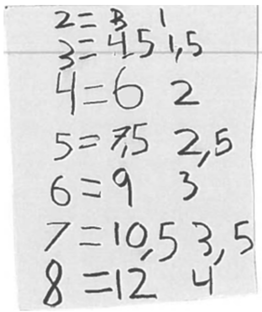


Figure 2: From the notes by Frank

In the whole class session on day 2, the teacher presents the table that Frank had made and says:

Teacher: Do you see any connection between the lengths in the original puzzle and how much we should add?

Frances: I think we have to add a half each time.

Teacher: What half then, do you think? What is it half of?

Mary: Perhaps when it is eight, we have to add half of eight.

When Frances says that “we have to add a half each time” it is reasonable to think that she means that she thinks about the increase in the increment and not the increase itself, since her utterance comes after the teacher has pointed to the table in Figure 2. A language cue to bring the reasoning forwards may be when the teacher then asks “what is it half of?” Then the word ‘half’ changes from being an additive constant to being a multiplicative operator. It is no longer the number ‘one half’ but it is ‘one half of something’. There are also traces of the operator aspect in what Frank said in the group, “when there is a larger number, then we add one half more”. When the teacher has emphasized “what is it half *of*”, Mary is able to express the general function $x \rightarrow x + \frac{1}{2} x$, using ‘eight’ as a generic example, “when it is eight, we have to add half of eight”. Finally, the teacher produces the table shown in Figure 3 on the board.

1	2	3	4	5	6	7	8	9	11
1.5	3	4.5	6	7.5	9	10.5	12	13.5	16.5
+0.5	+1	+1.5	+2	+2.5	+3	+3.5	+4	+4.5	+5.5

Figure 3: The table produced by the teacher

Here the teacher has indicated that what is to be added is half of the original value of the lengths, so in effect the teacher has made a representation of the function $x \rightarrow f(x)$ shown in Figure 4.

$$f(x) = x + \frac{1}{2}x$$

Figure 4: The enlargement function

Summary

All the groups started by adding 2 cm to each length in the puzzle, an indication that the word *enlarged* made them think of adding a constant. During their work, they got feedback from the milieu as they observed that the new pieces did not have the same shape as the original pieces and also that they did not fit together without gaps. However, they explained this by inaccurate measuring and inaccurate cutting. They did not question the procedure. This is completely in line with the experiences made by Brousseau (1997, p. 177): "It is not the model, it is the realization that is put into question." When the teacher gave the information that 5 → 7.5, in addition to 4 → 6 they realised that they should not add the same number each time. This information led to new procedures, which, in the effort to preserve the shape, turned out to be rather complicated.

Already in grade 3, the pupils worked with situations of multiplicative comparison, which is closely related to multiplicative change. These situations were of the type "the short walls of a room are 3 meters long, and the long walls are five times longer. How long are the long walls?" In Vergnaud's model, this means that the measure spaces M_1 and M_2 are the short and the long walls, respectively, and "five times longer" is the function operator. Hence, in $f: M_1 \rightarrow M_2: a \xrightarrow{k\cdot} b$, a and k are given and b is unknown. In the current situation, a and b are given and k is unknown.

In line with previous research (Ahl, 2019; Erath, 2019; Fernández et al., 2012; Gläser & Riegler, 2015; Van Dooren et al., 2005, 2010) done with students in different age groups, the present study shows that using AR instead of PR is firmly established in problems of the kind studied here. Seen from a language perspective, an interesting question is, what elements of language use may support a transition from additive to multiplicative thinking. A turning point towards multiplicative reasoning takes place when Frank suggests that "when there is a larger number, then we add one half more". Since only integers are considered, I interpret the statement "when there is a larger number, then we add one half more" to mean that when a length in the original puzzle is increased by 1, the increment increases by a constant, ½. This is extended by Mary to say that "when it is eight, we have to add half of eight". Although the multiplicative factor is not clearly identified and the function in Figure 4 is a mixed additive/multiplicative model, it seems that the pupils are beginning to develop a language involving a multiplicative structure.

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