CYCLICITY IN THE DRURY-ARVESON SPACE AND OTHER WEIGHTED BESOV SPACES

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Abstract. Let $\mathcal{H}$ be a space of analytic functions on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ with multiplier algebra $\mathrm{Mult}(\mathcal{H})$. A function $f \in \mathcal{H}$ is called cyclic if the set $[f]$, the closure of $\{\varphi f : \varphi \in \mathrm{Mult}(\mathcal{H})\}$, equals $\mathcal{H}$. For multipliers we also consider a weakened form of the cyclicity concept. Namely for $n \in \mathbb{N}_0$ we consider the classes

$$\mathcal{C}_n(\mathcal{H}) = \{\varphi \in \mathrm{Mult}(\mathcal{H}) : \varphi \neq 0, [\varphi^n] = [\varphi^{n+1}]\}.$$

Many of our results hold for N:th order radially weighted Besov spaces on $\mathbb{B}_d$, $\mathcal{H} = B^N_\omega$, but we describe our results only for the Drury-Arveson space H^2_d here.

Letting $\mathbb{C}_{\mathrm{stable}}[z]$ denote the stable polynomials for $\mathbb{B}_d$, i.e. the d-variable complex polynomials without zeros in $\mathbb{B}_d$, we show that

$$\text{if } d \text{ is odd, then } \mathbb{C}_{\mathrm{stable}}[z] \subseteq \mathcal{C}_{\frac{d-1}{2}}(H^2_d), \text{ and}$$
$$\text{if } d \text{ is even, then } \mathbb{C}_{\mathrm{stable}}[z] \subseteq \mathcal{C}_{\frac{d}{2}-1}(H^2_d).$$

For $d = 2$ and $d = 4$ these inclusions are the best possible, but in general we can only show that if $0 \leq n \leq \frac{d}{4} - 1$, then $\mathbb{C}_{\mathrm{stable}}[z] \nsubseteq \mathcal{C}_n(H^2_d)$.

For functions other than polynomials we show that if $f, g \in H^2_d$ such that $f/g \in H^\infty$ and f is cyclic, then g is cyclic. We use this to prove that if f, g extend to be analytic in a neighborhood of $\overline{\mathbb{B}_d}$, have no zeros in $\mathbb{B}_d$, and the same zero sets on the boundary, then f is cyclic in $\in H^2_d$ if and only if g is. Furthermore, if the boundary zero set of $f \in H^2_d \cap C(\overline{\mathbb{B}_d})$ embeds a cube of real dimension ≥ 3, then f is not cyclic in the Drury-Arveson space.

1. Introduction

Investigations about cyclic vectors of spaces of single variable analytic functions are classical. Beurling [13] showed that the cyclic vectors of the Hardy space H^2 are the outer functions, that is, those $f \in H^2$

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such that

$$-\infty < \log|f(0)| = \int_0^{2\pi} \log|f(e^{it})| \frac{dt}{2\pi}.$$

Korenblum [32, 33] established complete results for the topological algebra $A^{-\infty} = \{f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| = O((1-|z|)^{-n}) \text{ for some } n \in \mathbb{N}\}$, and for the weighted Dirichlet spaces $\{f \in \mathrm{Hol}(\mathbb{D}) : f^{(n)} \in H^2\}$. Complete characterizations of the cyclic functions in the Bergman space $L^2_a = \{f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f|^2 dA < \infty\}$ and Dirichlet space $D = \{f \in \mathrm{Hol}(\mathbb{D}) : f' \in L^2_a\}$ are lacking, but the area is rich with deep results that have clarified the function theory for these spaces, see for example [21, 23, 28].

Much less is known in the setting of spaces of functions of several complex variables. Norm closed ideals in the ball and polydisc algebras have been investigated by Hedenmalm, [29, 30], and there are results for the analogous questions in the Drury-Arveson context by Clouâtre and Davidson, [18]. Cyclicity of polynomials in weighted Dirichlet spaces of the bidisc has been investigated by Bénéteau-Condori-Liaw-Seco-Sola [10], Bénéteau-Knese-Kosiński-Liaw-Seco-Sola [11], and Knese-Kosiński-Ransford-Sola [31]. The paper [25] by Guo-Zhou contains results specifically for the Hardy space of the bidisc, while the paper [12] by Bergqvist treats the polydisc. In [44] Sola extended known bidisc results to the unit ball of two complex variables. Some cyclicity results for the Drury-Arveson space H^2_d can be found in [41, Theorems 1.4 and 1.5]. We refer the reader to [26] for a general introduction to the Drury-Arveson space H^2_d.

The purpose of this paper is to study the properties of both cyclic and non-cyclic functions $f \in H^2_d$ which have no zeros in $\mathbb{B}_d$. Since H^2_d is central to multivariable operator theory connected with $\mathbb{B}_d$, we expect our results to have significance in that context. The Drury-Arveson space is known to be an example of two types of important classes of function spaces: it is a Hilbert function space with a complete Pick kernel, and it is a radially weighted Besov space on the unit ball of $\mathbb{C}^d$. Many of our methods apply in this generality. Thus, our presentation will be very general. In the special case of the Dirichlet space of the unit disc our approach provides a new proof of Corollary 5.5 of [39].

It is known that for complete Pick spaces there is a 1-1 correspondence between multiplier invariant subspaces of $\mathcal{H}$ and weak* closed ideals of $\mathrm{Mult}(\mathcal{H})$, see [20]. As our main interest is in H^2_d we have stated our results as results about invariant subspaces of B^N_ω, even though many of them are also results about weak* closed ideals in $\mathrm{Mult}(B^N_\omega)$.

Before stating and discussing our main results in Section 3, we will present necessary background material on radially weighted Besov spaces and their multipliers in Section 2. In this preliminary section we also introduce the classes $\mathcal{C}_n(\mathcal{H})$ of pseudo-cyclic multipliers. In Section 4 we illustrate our results through a number of examples. In Section 5 we prove the theorems which apply for general radially weighted Besov spaces, while we treat those spaces which are also complete Pick spaces in Section 6. Finally, we consider some natural open questions in Section 7.

After completion of this paper we were made aware of the paper [34] in which the authors determine in the case of d=2 for which α the stable polynomials are cyclic in $D_\alpha(\mathbb{B}_d)$ (definitions below). In particular, Theorem 6 of [34] also implies the case of d=2 of (3.1)

2. Preliminaries

2.1. **Weighted Besov spaces.** Let X be a set. A Hilbert function space $\mathcal{H}$ on X is a Hilbert space of complex valued functions on X such that for each $z \in X$ the evaluation functional $f \mapsto f(z)$ is continuous on $\mathcal{H}$.

Let $d \in \mathbb{N}$ and let $\mathbb{B}_d$ denote the open unit ball of $\mathbb{C}^d$. We will use $\mathrm{Hol}(\mathbb{B}_d)$ to denote the analytic functions on $\mathbb{B}_d$, and we will write $\mathbb{D} = \mathbb{B}_1$, when we want to emphasize that $d = 1$. The main focus of this paper will be on radially weighted Besov spaces

$$B^N_\omega = \{f \in \mathrm{Hol}(\mathbb{B}_d) : R^N f \in L^2(\omega)\}.$$

Here N is a non-negative integer, $R = \sum_{j=1}^d z_j \frac{\partial}{\partial z_j}$ is the radial derivative operator, and ω is an *admissible radial measure* on $\overline{\mathbb{B}_d}$. That is, ω is of the type $d\omega(z) = d\mu(r)d\sigma(w)$, where $z = rw$, σ is the normalized rotationally invariant measure on $\partial\mathbb{B}_d$, and μ is a Borel measure on $[0,1]$ with $\mu((r,1]) > 0$ for each real r with $0 < r < 1$. If μ has a point mass at 1, then the $L^2(\omega)$-norm of an analytic function f is to be understood by $\|f\|^2_{L^2(\omega)} = \int_{\mathbb{B}_d} |f|^2 d\omega + \mu(\{1\})\|f\|^2_{H^2(\partial\mathbb{B}_d)}$. Recall that the Hardy space $H^2(\partial\mathbb{B}_d)$ consists of analytic functions in $\partial\mathbb{B}_d$ with

$$\|f\|^2_{H^2(\partial\mathbb{B}_d)} = \sup_{r\in[0,1)} \int_{\partial\partial\mathbb{B}_d} |f(rz)|^2 d\sigma(z) = \lim_{r\to 1^-} \int_{\partial\partial\mathbb{B}_d} |f(rz)|^2 d\sigma(z) < \infty.$$

We define a norm on B^N_ω by

$$\|f\|^2_{B^N_\omega} = \begin{cases} \|f\|^2_{L^2(\omega)}, & \text{if } N = 0, \\ \omega(\mathbb{B}_d)|f(0)|^2 + \|R^N f\|^2_{L^2(\omega)}, & \text{if } N > 0, \end{cases} \tag{2.1}$$

and we note that the hypothesis on μ implies that each B^N_ω is a Hilbert function space on $\mathbb{B}_d$. For later reference we also note that

$$\|f\|^2_{B^N_\omega} = \omega(\mathbb{B}_d)|f(0)|^2 + \|Rf\|^2_{B^{N-1}_\omega} \tag{2.2}$$

holds for all $N > 0$.

An important example is the Drury-Arveson space H^2_d. It can be defined as the space of analytic functions f in $\mathbb{B}_d$ such that

$$\|f\|^2_{H^2_d} = \sum_{\alpha\in\mathbb{N}_0^d} \frac{\alpha!}{|\alpha|!}|\hat{f}(\alpha)|^2 < \infty,$$

where f is given by the power series $f(z) = \sum_{\alpha\in\mathbb{N}_0^d} \hat{f}(\alpha)z^\alpha$ in multinomial notation. One calculates $\|f_n\|^2_{H^2(\partial\mathbb{B}_d)} = \frac{n!(d-1)!}{(n+d-1)!}\|f_n\|^2_{H^2_d}$, whenever f_n is a homogeneous polynomial of degree n, see e.g. [41, Section 2]. For an arbitrary radially weighted Besov space we therefore have that

$$\|f\|^2_{B^N_\omega} = \omega(\mathbb{B}_d)|f(0)|^2 + \sum_{n=1}^\infty n^{2N}\omega_n\|f_n\|^2_{H^2_d}, \tag{2.3}$$

where $f = \sum_{n=0}^\infty f_n$ is the representation of f as a sum of homogeneous polynomials of degree n, and $\omega_n = \frac{n!(d-1)!}{(n+d-1)!}\int_{[0,1]} r^{2n}d\mu(r)$. In particular, the Drury-Arveson space is itself a radially weighted Besov space, up to norm equivalence. In fact,

$$H^2_d = \begin{cases} B^{(d-1)/2}_\omega & \text{if } d \text{ is odd and } \omega = \sigma \\ B^{d/2}_\omega & \text{if } d \text{ is even and } \omega = V. \end{cases} \tag{2.4}$$

Here V denotes normalized Lebesgue measure on $\mathbb{B}_d$.

The Drury-Arveson space is part of a one-parameter scale of spaces of analytic functions. For $\alpha \in \mathbb{R}$ and $f \in \mathrm{Hol}(\mathbb{B}_d)$ define

$$\|f\|^2_{D_\alpha(\mathbb{B}_d)} = \sum_{n=0}^\infty (n+1)^\alpha\|f_n\|^2_{H^2_d} \approx \sum_{n=0}^\infty (n+1)^{\alpha+d-1}\int_{\partial\mathbb{B}_d}|f_n(z)|^2 d\sigma(z),$$

where $f = \sum_{n=0}^\infty f_n$ is the expansion of f into homogeneous polynomials of degree n, and let $D_\alpha(\mathbb{B}_d) = \{f \in \mathrm{Hol}(\mathbb{B}_d) : \|f\|^2_{D_\alpha(\mathbb{B}_d)} < \infty\}$. Then $D_0(\mathbb{B}_d) = H^2_d$, $D_{-d+1}(\mathbb{B}_d) = H^2(\partial\mathbb{B}_d)$ is the Hardy space, $D_{-d}(\mathbb{B}_d) = L^2_a(\mathbb{B}_d)$ is the Bergman space, and $D = D_1(\mathbb{D})$ is the Dirichlet space. Here, and throughout, the equality of spaces is to be understood to include the possibility that the norms are not equal, but equivalent.

More generally, for $\alpha < -d+1$ we have

$$(n+1)^{\alpha+d-1} \approx \int_0^1 r^n(1-r)^{-(\alpha+d)}dr.$$

Thus, by setting

$$d\omega_\alpha(w) = \begin{cases} d\sigma(w) & \text{if } \alpha = -d+1 \\ (1-r)^{-(\alpha+d)} dr d\sigma(z) & \text{if } w = rz \text{ and } \alpha < -d+1 \end{cases}$$

we see that $B^0_{\omega_\alpha} = D_\alpha(\mathbb{B}_d)$, whenever $\alpha \le -d+1$. This implies that $D_\alpha(\mathbb{B}_d) = B^N_{\omega_{\alpha-2N}}$, whenever $N \in \mathbb{N}_0$ with $N \ge \frac{\alpha+d-1}{2}$, cf. (2.3). If $\alpha > 1$, then a simple argument with the Cauchy-Schwarz inequality implies that the spaces $D_\alpha(\mathbb{B}_d)$ are contained in the ball algebra with $\|f\|_\infty \le C\|f\|_{D_\alpha(\mathbb{B}_d)}$, see [43] for the case $d = 1$. Furthermore, since $f \in D_\alpha(\mathbb{B}_d) \Leftrightarrow Rf \in D_{\alpha-2}(\mathbb{B}_d)$, we conclude that evaluation of $f, Rf, \dots, R^{N-1}f$ at points $z \in \partial\mathbb{B}_d$ defines bounded linear functionals on $D_\alpha(\mathbb{B}_d)$ whenever $\alpha > 2N-1$. For more information about these spaces and their multipliers see e.g. [4, 7, 14, 17, 19, 37, 38, 41], and Section 14 of [1].

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert function spaces, then

$$\mathrm{Mult}(\mathcal{H},\mathcal{K}) = \{\varphi : \varphi f \in \mathcal{K} \text{ for all } f \in \mathcal{H}\}$$

are the multipliers from $\mathcal{H}$ to $\mathcal{K}$. $\mathrm{Mult}(\mathcal{H},\mathcal{K})$ is a Banach space with norm $\|\varphi\|_{\mathrm{Mult}(\mathcal{H},\mathcal{K})} = \sup\{\|\varphi f\|_{\mathcal{K}} : f \in \mathcal{H}, \|f\|_{\mathcal{H}} \le 1\}$. We will write $\mathrm{Mult}(\mathcal{H}) = \mathrm{Mult}(\mathcal{H},\mathcal{H})$, and we note that

$$\|\varphi\|_\infty \le \|\varphi\|_{\mathrm{Mult}(B^N_\omega)}$$

with equality whenever $N = 0$. As is well known, equality does not hold in general when $N > 0$. Standard examples are the weighted Dirichlet spaces on the unit disc, or the Drury-Arveson space.

Furthermore, one checks that if a function f extends to be analytic in a neighborhood of $\overline{\mathbb{B}_d}$, then $f \in \mathrm{Mult}(B^N_\omega)$ for all admissible radial measures ω and all $N \in \mathbb{N}_0$.

We are interested in multiplier invariant subspaces of $\mathcal{H}$, i.e. those subspaces $\mathcal{M} \subseteq \mathcal{H}$ that satisfy $\varphi f \in \mathcal{M}$, whenever $f \in \mathcal{M}$ and $\varphi \in \mathrm{Mult}(\mathcal{H})$. If $f \in \mathcal{H}$, then $[f] = \mathrm{clos}_{\mathcal{H}}\{\varphi f : \varphi \in \mathrm{Mult}(\mathcal{H})\}$ denotes the invariant subspace generated by f. A function $f \in \mathcal{H}$ is called cyclic in $\mathcal{H}$ if $[f] = \mathcal{H}$. If $\mathcal{H}$ is a radially weighted Besov space, then all polynomials are multipliers, and they are densely contained in $\mathcal{H}$. Thus, in this case, f is cyclic if and only if $1 \in [f]$.

If $\mathcal{H} = H^2(\mathbb{D})$, then the cyclic functions are the classical outer functions. If $f \in \mathcal{H} = H^2_d$, then each slice function $f_z(\lambda) = f(\lambda z)$, $z \in \partial\mathbb{B}_d$, is in $H^2(\mathbb{D})$ with $\|f_z\|_{H^2(\mathbb{D})} \le \|f\|_{H^2_d}$; see Section 6.2 for a generalization to other spaces. It follows that if f is cyclic in H^2_d, then each slice function f_z must be outer in $H^2(\mathbb{D})$. However, we will see in Section 4 that for $d \ge 2$, there are noncyclic functions $0 \ne f \in H^2_d$ such that every slice is outer.

As a tool to investigate the cyclic behaviour of functions in $\mathcal{H}$, we define the following sets of multipliers, each indexed by an integer $n \geq 0$:

$$\mathcal{C}_n(\mathcal{H}) = \{\varphi \in \mathrm{Mult}(\mathcal{H}) : \varphi \neq 0 \text{ and } [\varphi^n] = [\varphi^{n+1}]\}.$$

We also define

$$\mathcal{C}_\infty(\mathcal{H}) = \{\varphi \in \mathrm{Mult}(\mathcal{H}) : \bigcap_{n=1}^{\infty} [\varphi^n] \neq (0)\}.$$

We consider membership in $\mathcal{C}_n(\mathcal{H})$ to be a weakened form of cyclicity. Indeed, if the multipliers are dense in $\mathcal{H}$, then $\mathcal{C}_0(\mathcal{H})$ consists of the cyclic multipliers and it is easy to prove that

$$\mathcal{C}_0(\mathcal{H}) \subseteq \mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{C}_2(\mathcal{H}) \cdots \subseteq \mathcal{C}_\infty(\mathcal{H}). \tag{2.5}$$

In the case $\mathcal{H} = H^2(\mathbb{D})$ one has equality throughout, each set equaling the outer functions in $H^\infty(\mathbb{D})$, as can be seen from the inner-outer factorization. Similarly, if $\alpha \geq 0$, then we show in Section 6.2 that if $f \in \mathcal{C}_n(D_\alpha(\mathbb{B}_d))$ for some $n \in \mathbb{N}$, then for each $z \in \partial\mathbb{B}_d$, the slice function f_z is an outer function in $H^\infty(\mathbb{D})$. One easily checks that for $d = 1$ the same conclusion holds for $f \in \mathcal{C}_\infty(D_\alpha(\mathbb{B}_d))$. However, in Proposition 6.8 we will present an example to show that if $d \geq 2$, then functions in $\mathcal{C}_\infty(H^2_d)$ may have slices with non-trivial singular inner factors. It follows that $\bigcup_{n=0}^{\infty} \mathcal{C}_n(H^2_d) \subsetneqq \mathcal{C}_\infty(H^2_d)$.

For the Dirichlet space $D = D_1(\mathbb{D})$ it was shown in [40, Theorem 4.3] that the class $\mathcal{C}_1(D)$ equals the outer functions in $\mathrm{Mult}(D)$. On the other hand, it is known [14] that there are non-cyclic outer functions in $\mathrm{Mult}(D)$, and therefore $\mathcal{C}_0(D) \neq \mathcal{C}_1(D)$. In this particular case we have that $\mathcal{C}_1(D) = \mathcal{C}_\infty(D)$, but nothing like this is true in general.

Example 2.1. If $\mathcal{H} = D_4(\mathbb{D}) = B^2_\omega$, $d\omega = d\delta_1 \frac{|dz|}{2\pi}$, then $\varphi(z) = 1 - z$ is an example of an outer function in $\mathcal{C}_2(B^2_\omega) \setminus \mathcal{C}_1(B^2_\omega)$.

This holds by observing that the evaluation of functions and their derivatives at $z = 1$ define bounded linear functionals on $D_4(\mathbb{D})$. Hence $1 - z \notin [(1 - z)^2]$. It is elementary to show that $1 - z \in \mathcal{C}_2(\mathcal{H})$, cf. Theorem 3.5.

We now list and prove a few more elementary properties of the sets $\mathcal{C}_\infty(\mathcal{H})$.

Lemma 2.2. *Assume that* $\mathrm{Mult}(\mathcal{H}) \subseteq \mathcal{H}$. *Then*

(a) If $\varphi \in \mathcal{C}_\infty(\mathcal{H})$, *then* $\varphi(z) \neq 0$ *for all* $z \in \mathbb{B}_d$.

(b) If $n, m \in \mathbb{N}_0$ *and if* $\psi, \varphi \in \mathrm{Mult}(\mathcal{H})$ *such that* $[\varphi^n] = [\varphi^{n+1}]$ *and* $[\psi^m] = [\psi^{m+1}]$, *then* $[\varphi^n\psi^m] = [\varphi^{n+1}\psi^{m+1}]$.

Proof. (a) Let $\varphi \in \mathrm{Mult}(\mathcal{H})$ and $z_0 \in \mathbb{B}_d$ such that $\varphi(z_0) = 0$. We need to show that $\bigcap_{n=1}^\infty [\varphi^n] = (0)$, so let $f \in [\varphi^n]$ for each $n \in \mathbb{N}$. Let $r > 0$ such that the closure of $B = \{z : |z - z_0| < r\}$ is contained in $\mathbb{B}_d$. It will be sufficient to show that $f(z) = 0$ for each $z \in B$. It is clear that $f(z) = 0$ at every point z where $\varphi(z) = 0$. Let $z \in B$ with $\varphi(z) \neq 0$. For $\lambda \in \mathbb{C}$ set $\psi_z(\lambda) = z_0 + \lambda(z - z_0)$. Then $\varphi_z = \varphi \circ \psi_z$ and $f_z = f \circ \psi_z$ are analytic in a neighborhood of the unit disc. Since $f(z) = f_z(1)$ it suffices to show that $f_z = 0$.

We have $\varphi(z_0) = 0$, but $\varphi(z) \neq 0$. Thus, the function φ_z satisfies $\varphi_z(0) = 0$, but φ_z is not indentically 0. Hence there is $k \geq 1$ and an analytic function h such that $h(0) \neq 0$ and $\varphi_z(\lambda) = \lambda^k h(\lambda)$. Now fix $n \in \mathbb{N}$. Then $f \in [\varphi^n]$ implies that there are multipliers g_j such that $g_j \varphi^n \to f$ in $\mathcal{H}$. Then $g_j \circ \psi_z \varphi_z^n \to f_z$ uniformly on the closed unit disc. Since each $g_j \varphi^n$ has a zero of multiplicity nk at 0, we conclude that f_z has a zero of multiplicity nk at 0. Since $k \geq 1$ and n is arbitrary, this implies that $f_z = 0$.

(b) Suppose that n, m, φ, ψ are as in the hypothesis. Since $\varphi^{n+1}\psi^{m+1} = (\varphi\psi)\varphi^n\psi^m \in [\varphi^n\psi^m]$ we only have to show that $\varphi^n\psi^m \in [\varphi^{n+1}\psi^{m+1}]$. The hypothesis for φ implies that there is a sequence of multipliers u_j such that $u_j\varphi^{n+1} \to \varphi^n$ in $\mathcal{H}$. Then $u_j\varphi^{n+1}\psi^{m+1} \to \varphi^n\psi^{m+1}$ in $\mathcal{H}$. This implies $\varphi^n\psi^{m+1} \in [\varphi^{n+1}\psi^{m+1}]$. Similarly, there is a sequence of multipliers v_j such that $v_j\psi^{m+1} \to \psi^m$ in $\mathcal{H}$. Then $\varphi^n v_j \psi^{m+1} \to \varphi^n\psi^m$ and hence $\varphi^n\psi^m \in [\varphi^n\psi^{m+1}] \subseteq [\varphi^{n+1}\psi^{m+1}]$. ∎

In particular, if $\mathrm{Mult}(\mathcal{H})$ is densely contained in $\mathcal{H}$, then multiplication by cyclic functions preserves each of the classes $\mathcal{C}_n(\mathcal{H})$.

2.2. **Background about complete Pick spaces.** Recall that each Hilbert function space $\mathcal{H}$ has a reproducing kernel $k : X \times X \to \mathbb{C}$. Writing $k_w(z) = k(z, w)$ it satisfies $f(w) = \langle f, k_w \rangle$ for all $f \in \mathcal{H}$, $w \in X$. A reproducing kernel k on X is called a normalized complete Pick kernel, if there is $w_0 \in X$ and a function b from X into some auxiliary Hilbert space $\mathcal{K}$ such that $b(w_0) = 0$ and

$$k_w(z) = \frac{1}{1 - \langle b(z), b(w) \rangle_{\mathcal{K}}}.$$

In the interesting case where $\mathcal{H}$ is a Hilbert space of analytic functions one easily shows that $\mathcal{H}$ is separable, and then one may assume that $\mathcal{K}$ is separable also.

We say that a Hilbert function space $\mathcal{H}$ on X is a complete Pick space, if there is a Hilbert space norm on $\mathcal{H}$ that is equivalent to the original norm, and such that the reproducing kernel that $\mathcal{H}$ has with respect to the new norm is a normalized complete Pick kernel. The

Hardy space H^2 of the unit disc is the easiest example of a complete Pick space. The spaces $D_\alpha(\mathbb{B}_d)$ are complete Pick spaces for all $\alpha \geq 0$. This includes the Dirichlet space D and the Drury-Arveson space H^2_d, which has reproducing kernel $k_w(z) = \frac{1}{1-\langle z,w\rangle}$.

In [4] general conditions were given on ω that imply that B^N_ω is a complete Pick space. For example, if $\alpha > -1$ and if $d\omega(z) = w(z)dV(z)$ where $\frac{w(z)}{(1-|z|^2)^\alpha}$ is non-decreasing as $|z| \to 1$, then for $N \geq \frac{\alpha+d}{2}$ the space B^N_ω is a complete Pick space, see Theorem 1.4 of [4].

We now recall some important basic properties of complete Pick spaces.

Lemma 2.3. *If k is a normalized complete Pick kernel and if $w \in X$, then $k_w \in \mathrm{Mult}(\mathcal{H})$ with $\|k_w\|_{\mathrm{Mult}(\mathcal{H})} \leq 2\|k_w\|^2_{\mathcal{H}}$.*

For spaces of analytic functions this was proved in [24]. For the general version see [42, Proposition 4.4] or [1, Lemma 7.2].

Corollary 2.4. *If $\mathcal{H}$ is a complete Pick space, then $\mathrm{Mult}(\mathcal{H})$ is dense in $\mathcal{H}$. In particular, $f \in \mathcal{H}$ is cyclic if and only if $1 \in [f]$.*

We shall need another useful estimate in spaces with a normalized complete Pick kernel. For spaces of analytic functions this was also proved in [24], while the general version given here follows from Corollary 3.3 in [3].

Lemma 2.5. *If k is a normalized complete Pick kernel, $f \in \mathcal{H}$, and $w \in X$, then*

$$|f(w)|^2 \leq 2\mathrm{Re}\langle f, k_w f\rangle_{\mathcal{H}} - \|f\|^2_{\mathcal{H}}.$$

Finally we recall a special case of [3, Theorem 1.1 (i)].

Theorem 2.6. *Let k be a normalized complete Pick kernel, $k_{w_0} = 1$. For $f : X \to \mathbb{C}$, the following are equivalent:*

(i) *$f \in \mathcal{H}$ and $\|f\| \leq 1$;*
(ii) *there are multipliers $\varphi, \psi \in \mathrm{Mult}(\mathcal{H})$ such that*
  (a) *$f = \frac{\varphi}{1-\psi}$*
  (b) *$\psi(w_0) = 0$, and*
  (c) *$\|\psi h\|^2 + \|\varphi h\|^2 \leq \|h\|^2$ for every $h \in \mathcal{H}$.*

3. Statements of the main results

In this section we present our main results. Throughout we will suppose that $N \in \mathbb{N}$ and let ω denote an admissible radial measure.

Theorem 3.1. *If $\varphi, \psi \in \mathrm{Mult}(B^N_\omega)$ with $\varphi/\psi \in H^\infty(\mathbb{B}_d)$, then for each $k \in \mathbb{N}$ we have $\varphi^{N+k-1} \in [\psi^k] \subseteq [\psi]$.*

Consequently, if $N = 1$ or $\varphi \in \mathcal{C}_1(B^N_\omega)$, then $|\varphi(z)| \leq |\psi(z)|$ implies $\varphi \in [\psi]$. In particular, the cyclicity of φ implies cyclicity of ψ. A main technical step in the proof of Theorem 3.1 is interesting in its own right:

Theorem 3.2. *If $\varphi, \psi \in \mathrm{Mult}(B^N_\omega)$ with $\varphi/\psi \in H^\infty(\mathbb{B}_d)$, then $\varphi^{N+1}/\psi \in \mathrm{Mult}(B^N_\omega)$.*

Of course, if $N = 0$, then $\mathcal{H}$ is a weighted Bergman space or the Hardy space and $\mathrm{Mult}(B^N_\omega) = H^\infty(\mathbb{B}_d)$. Thus, in this case the Theorem is trivial. However, if $N > 0$, then it may happen that $\mathrm{Mult}(B^N_\omega) \subsetneq H^\infty(\mathbb{B}_d)$, and hence one may have to choose $n > 1$ in order for φ^n/ψ to be a multiplier.

By applying Theorem 3.2 with $\varphi = 1$ we recover the following theorem from [35].

Corollary 3.3. *If $\psi \in \mathrm{Mult}(B^N_\omega)$ with $|\psi(z)| \geq 1$ for all $z \in \mathbb{B}_d$, then $1/\psi \in \mathrm{Mult}(B^N_\omega)$. Consequently, for all $\varphi \in \mathrm{Mult}(B^N_\omega)$ we have $\sigma(M_\varphi) = \overline{\varphi(\mathbb{B}_d)}$.*

In other words, the corollary asserts that the "one function Corona Theorem" holds for $\mathrm{Mult}(B^N_\omega)$.

Theorem 3.2 is perhaps reminiscent of Wolff's Ideal Theorem for H^∞, [45]. And indeed, as in [9], Theorem 3.2 does imply a simple condition for membership in radical ideals generated by principal ideals in $\mathrm{Mult}(B^N_\omega)$. For $\psi \in \mathrm{Mult}(B^N_\omega)$ let

$$\mathrm{Rad}(\psi) = \{\varphi \in \mathrm{Mult}(B^N_\omega) : \varphi^n = u\psi \text{ for some } u \in \mathrm{Mult}(B^N_\omega),\ n \in \mathbb{N}\}.$$

Corollary 3.4. *If $\varphi, \psi \in \mathrm{Mult}(B^N_\omega)$, then $\varphi \in \mathrm{Rad}(\psi)$ if and only if there is $n \in \mathbb{N}$ such that $\varphi^n/\psi \in H^\infty$.*

In the course of the proof of Theorem 3.2 we will also establish some uniform norm bounds that are useful for the proof of our next Theorem. Let $\mathbb{C}_{\mathrm{stable}}[z]$ denote the stable polynomials, that is, the polynomials with no zeros in $\mathbb{B}_d$.

Theorem 3.5. *We have that $\mathbb{C}_{\mathrm{stable}}[z] \subseteq \mathcal{C}_N(B^N_\omega)$. Furthermore, if $N > 0$ and ω is of the form $d\omega(z) = u(r)2rdrd\sigma(w)$ for some $u \in L^\infty([0,1])$, then $\mathbb{C}_{\mathrm{stable}}[z] \subseteq \mathcal{C}_{N-1}(B^N_\omega)$.*

One may wonder what the smallest k is such that $\mathbb{C}_{\mathrm{stable}}[z] \subseteq \mathcal{C}_k(B^N_\omega)$. Example 2.1 shows that one cannot do any better than Theorem 3.5 for the space $D_4(\mathbb{D})$. On the other hand, for each ω there is another admissible radial measure ω' such that $B^N_\omega = B^{N+1}_{\omega'}$ (with equivalence of norms), see [4, Theorem 2.4]. Theorem 3.5 is then of course not sharp for $B^{N+1}_{\omega'}$.

For the Drury-Arveson space, represented as a Besov space via (2.4), Theorem 3.5 yields that

$$\text{if } d \text{ is odd, then } \mathbb{C}_{\text{stable}}[z] \subseteq \mathcal{C}_{\frac{d-1}{2}}(H^2_d), \tag{3.1}$$
$$\text{if } d \text{ is even, then } \mathbb{C}_{\text{stable}}[z] \subseteq \mathcal{C}_{\frac{d}{2}-1}(H^2_d).$$

In particular, for $d = 2$ every stable polynomial is cyclic. In Section 4 we will show that if $n \leq \frac{d}{4} - 1$, then $\mathbb{C}_{\text{stable}}[z] \nsubseteq \mathcal{C}_n(H^2_d)$, see Proposition 4.4 (b). Thus these inclusions are best possible for $d = 2$ and $d = 4$, but for other values of d there is potentially a gap, see Question 7.1.

If $f \in \text{Hol}(\mathbb{B}_d)$ extends to be continuous on $\overline{\mathbb{B}_d}$, then we write $Z(f) = \{z \in \overline{\mathbb{B}_d} : f(z) = 0\}$. We say that $f \in \text{Hol}(\mathbb{B}_d)$ satisfies a Lipschitz condition of order $\alpha > 0$, if there is $C > 0$ such that $|f(z) - f(w)| \leq C|z - w|^\alpha$ for all $z, w \in \mathbb{B}_d$. Note that functions that satisfy a Lipschitz condition can be extended to be continuous on $\overline{\mathbb{B}_d}$.

Theorem 3.6. *Let $f, g \in \text{Mult}(B^N_\omega)$ be such that*

(i) *$f(z) \neq 0$ and $g(z) \neq 0$ for all $z \in \mathbb{B}_d$,*
(ii) *f extends to be analytic in a neighborhood of $\overline{\mathbb{B}_d}$,*
(iii) *g satisfies a Lipschitz condition of order $\alpha > 0$.*

Assume that $Z(f) \cap \partial\mathbb{B}_d \subseteq Z(g) \cap \partial\mathbb{B}_d$. Then there is an $n \in \mathbb{N}$ such that $g^n \in [f]$. Furthermore, if we additionally assume that g is a polynomial, then $g^N \in [f^N] \subseteq [f]$ for every $N \geq 1$.

In particular, if g is cyclic, then f is cyclic. Thus, for polynomials $p \in \mathbb{C}_{\text{stable}}[z]$, the geometry of $Z(p) \cap \partial\mathbb{B}_d$ determines whether or not p is cyclic. If $w \in \partial\mathbb{B}_d$, then $p(z) = 1 - \langle z, w\rangle$ is a polynomial such that $Z(p) \cap \partial\mathbb{B}_d = \{w\}$ and it is easily seen that p is cyclic in H^2_d. This implies that if f extends to be analytic in a neighborhood of $\overline{\mathbb{B}_d}$, has no zeros in $\mathbb{B}_d$, and only finitely many zeros in $\partial\mathbb{B}_d$, then f is cyclic in H^2_d. In Examples 4.3 and 4.6 we will give examples of cyclic polynomials in H^2_d such that $Z(p) \cap \partial\mathbb{B}_d$ has 1 or 2 real dimensions. However, in Theorem 4.8 we will show that if $Z(p) \cap \partial\mathbb{B}_d$ embeds a cube of real dimension ≥ 3, then p is not cyclic in H^2_d.

If the radially weighted Besov space is also a complete Pick space, then the results of Theorem 3.1 can be partially extended to apply to arbitrary functions in B^N_ω. Recall that in these cases, by Theorem 2.6, every $f \in B^N_\omega$ has the form $f = u/v$ with $u, v \in Mult(B^N_\omega)$ with v cyclic.

Theorem 3.7. *Let $N \in \mathbb{N}$, and let B^N_ω be a radially weighted Besov space that is also a complete Pick space. Let $f, g \in B^N_\omega$ be such that $f/g \in H^\infty(\mathbb{B}_d)$.*

If $N = 1$ or if $f = u/v$ for v cyclic and $u \in \mathcal{C}_1(B^N_\omega)$, then $f \in [g]$.

It will follow that if $|f(z)| \leq |g(z)|$, and if f is cyclic, then g is cyclic. Since the constant 1 is cyclic in B^N_ω the Theorem implies that any $g \in \mathcal{H}$ that is bounded below, must be cyclic. Thus, Theorem 3.7 improves Theorem 1.5 of [41], where the Theorem was proved only for H^2_d under the additional assumptions that $f = 1$ and g be in the Bloch space.

For the Dirichlet space D of the unit disc Theorem 3.7 was known, see Corollary 5.5 of [39]. The proof here is considerably less technical than the one in [39]. Theorem 3.7 will follow from Theorem 6.4, which contains a slightly more general result.

4. Some Examples

In order to illustrate our theorems we start with some examples for H^2_d. We present two ways to embed $D_{\frac{k-1}{2}}(\mathbb{D})$ in H^2_d, where $1 \leq k \leq d$. The first one of these is well-known, and has been used before to construct functions with interesting properties in the Drury-Arveson space, see e.g. [8, Theorem 3.3], [27, Lemma 2.1], [6, Lemma 9.1], or see [10, Example 2] for a bidisc version of such an embedding.

Note that if $z = (z_1, \dots, z_d) \in \mathbb{B}_d$, then the geometric-arithmetic mean inequality implies that for each integer k with $1 \leq k \leq d$ we have

$$\left(\prod_{j=1}^{k} |z_j|^2 \right)^{1/k} \leq \frac{\sum_{j=1}^{k} |z_j|^2}{k} \leq \frac{1}{k}.$$

Hence $\tau_k(z) = k^{k/2} \prod_{j=1}^{k} z_j$ maps $\mathbb{B}_d$ into $\mathbb{D}$.

Lemma 4.1. *Let $1 \leq k \leq d$. The operator*

$$T_{k,d} : D_{\frac{k-1}{2}}(\mathbb{D}) \to H^2_d, \quad T_{k,d} f = f \circ \tau_k$$

is bounded and bounded below. Furthermore, if $\varphi \in \mathrm{Mult}(D_{\frac{k-1}{2}}(\mathbb{D}))$, then $T_{k,d}\varphi \in \mathrm{Mult}(H^2_d)$.

If $d = k$, then this is the special case of $s = 1$ of [6], Lemma 9.1 and Proposition 9.4. If $d > k$, then we combine this with use of the isometric embedding of H^2_k in H^2_d given by $f \to f \circ P$, where P is the projection from $\mathbb{C}^d \to \mathbb{C}^k, (z_1, \dots, z_d) \to (z_1, \dots, z_k)$. This embedding is also isometric as a map between multiplier algebras $\mathrm{Mult}(H^2_k) \to \mathrm{Mult}(H^2_d)$, see Lemma 6.2 of [6].

Lemma 4.2. *Let $1 \leq k \leq d$.*

(a) If $f \in D_{\frac{k-1}{2}}(\mathbb{D})$, then f is cyclic in $D_{\frac{k-1}{2}}(\mathbb{D})$, if and only if $T_{k,d} f$ is cyclic in H^2_d.

(b) Let $n \in \mathbb{N}$ *and* $f \in \mathrm{Mult}(D_{\frac{k-1}{2}}(\mathbb{D}))$*, then* $f \in \mathcal{C}_n(D_{\frac{k-1}{2}}(\mathbb{D}))$*,if and only if* $T_{k,d}f \in \mathcal{C}_n(H^2_d)$*.*

(c) If $f \in D_{\frac{k-1}{2}}(\mathbb{D})$ *is an outer function, then for each* $z \in \partial\mathbb{B}_d$ *the slice function* $(T_{k,d}f)_z$ *is outer. Here* $(T_{k,d}f)_z(\lambda) = T_{k,d}f(\lambda z)$*,* $\lambda \in \mathbb{D}$*.*

Proof. We prove (b) and (c). The proof of (a) is similar to (b).

(b) Fix $n \in \mathbb{N}$ and $f \in \mathrm{Mult}(D_{\frac{k-1}{2}}(\mathbb{D}))$. Then by Lemma 4.1 $T_{k,d}f \in \mathrm{Mult}(H^2_d)$. If $f \in \mathcal{C}_n(D_{\frac{k-1}{2}}(\mathbb{D}))$, then there is a sequence of polynomials $\{p_j\}$ such that $p_j f^{n+1} \to f^n$ in $D_{\frac{k-1}{2}}(\mathbb{D})$. Then for each j we have $q_j = T_{k,d}p_j$ is a polynomial and by Lemma 4.1

$$(T_{k,d}p_j)(T_{k,d}f)^{n+1} = T_{k,d}(p_j f^{n+1}) \to T_{k,d}(f^n) = (T_{k,d}f)^n$$

in H^2_d. Hence $T_{k,d}f \in \mathcal{C}_n(H^2_d)$.

Conversely, if $T_{k,d}f \in \mathcal{C}_n(H^2_d)$, then there are polynomials $q_j \in \mathrm{Mult}(H^2_d)$ such that $q_j(T_{k,d}f)^{n+1} \to (T_{k,d}f)^n$ in H^2_d.

For $n \in \mathbb{N}_0$ define $\alpha_n = (n, \dots, n, 0, \dots, 0) \in \mathbb{N}_0^d$, where the first k components of α_n equal n and the remaining components are 0. If $q_j(z) = \sum_\alpha \hat{q}_j(\alpha) z^\alpha$, then let $P_k q_j(z) = \sum_{n\geq 0} \hat{q}_j(\alpha_n) \prod_{k=1}^d z_k^n$. Note that $P_k q_j = T_{k,d}p_j$ for some polynomial p_j and that

$$(q_j - P_k q_j)T_{k,d}(f^{n+1}) \perp (P_k q_j)T_{k,d}(f^{n+1}) - T_{k,d}(f^n)$$

by the orthogonality of the monomials in H^2_d. Hence

$$\begin{aligned}
&\|p_j f^{n+1} - f^n\|^2_{D_{(k-1)/2}} \approx \|T_{k,d}(p_j f^{n+1}) - T_{k,d}(f^n)\|^2_{H^2_d}\\
&= \|(P_k q_j)T_{k,d}(f^{n+1}) - T_{k,d}(f^n)\|^2_{H^2_d}\\
&\leq \|(q_j - P_k q_j)T_{k,d}(f^{n+1})\|^2 + \|(P_k q_j)T_{k,d}(f^{n+1}) - T_{k,d}(f^n)\|^2_{H^2_d}\\
&= \|q_j(T_{k,d}f)^{n+1} - (T_{k,d}f)^n\|^2_{H^2_d} \to 0
\end{aligned}$$

as $j \to \infty$. Thus, $f \in \mathcal{C}_n(D_{(k-1)/2})$.

(c) If $f \in D_{\frac{k-1}{2}}(\mathbb{D})$ is an outer function, then

$$\log|f(0)| = \int_0^{2\pi} \log|f(e^{it})|\frac{dt}{2\pi}.$$

Let $z \in \partial\mathbb{B}_d$, then $(T_{k,d}f)_z(e^{it}) = f(e^{ikt}\tau_k(z))$ and hence

$$\begin{aligned}
\int_0^{2\pi} \log|(T_{k,d}f)_z(e^{it})|\frac{dt}{2\pi} &= \int_0^{2\pi} \log|f(e^{ikt}\tau_k(z))|\frac{dt}{2\pi}\\
&= \log|f(0)| = \log|(T_{k,d}f)_z(0)|.
\end{aligned}$$

Hence $(T_{k,d}f)_z$ is outer. ■

Example 4.3. If $1 \leq k \leq 3$, $f(z) = 1 - z$, then $T_{k,d}f$ is cyclic in H_d^2. Furthermore, the set $Z(T_{k,d}f) \cap \partial\mathbb{B}_d$ equals an embedded $k-1$ dimensional cube.

It is well-known that f is cyclic in $D_\alpha(\mathbb{D})$ for $\alpha \leq 1$, see [14]. It also follows from the second part of Theorem 3.5, which will be proved later. Hence the cyclicity of $T_{k,d}f$ follows from Lemma 4.2 (a). The statement about the zero set is also easily seen. For example, if $k = 3$, then

$$Z(T_{k,d}f) \cap \partial\mathbb{B}_d = \{3^{-1/2}(e^{it}, e^{is}, e^{-i(t+s)}, 0, \dots, 0) : t, s \in [0, 2\pi]\}.$$

Similarly one sees that for $k \geq 4$ the set $Z(T_{k,d}f) \cap \partial\mathbb{B}_d$ embeds a cube of dimension $k-1 \geq 3$, hence it will follow from Theorem 4.8 that $T_{k,d}f$ is not cyclic for any $k \geq 4$. Alternatively, that will also follow from part (b) of the following proposition.

Proposition 4.4. *(a) If $d \geq 2$, then there is non-cyclic $f \in H_d^2$ such that every slice function f_z is outer in $H^2(\mathbb{D})$.*

(b) If $d \geq k \geq 4n > 0$ and $p(z) = 1 - z$, then the polynomial $T_{k,d}p \notin \mathcal{C}_{n-1}(H_d^2)$. Hence $\mathbb{C}_{\text{stable}}[z] \nsubseteq \mathcal{C}_{n-1}(H_d^2)$.

Proof. (a) By Lemma 4.2 it will be enough to show that there is a noncyclic outer function $f \in D_{1/2}(\mathbb{D})$, because then $T_{2,d}f$ will be the required example. The existence of the required function is known; let us briefly describe the ideas that go into the construction. The proof uses Carleson sets and α-capacity for $\alpha = 1/2$. Indeed, using Theorem 3 of Section IV of [16] one constructs a "generalized Cantor" set E with positive 1/2-capacity (also see [22], Section 4). Generalized Cantor sets are Carleson sets. Thus, by results of Carleson for any $n \in \mathbb{N}$ there is an outer function $f \in C^n(\overline{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$ such that $f = 0$ on E ([15]). Then f is not cyclic in $D_{1/2}$, see e.g. [22] Theorem 1.1.

(b) If $d \geq k \geq 4n > 0$, then $\frac{k-1}{2} > 2n - 1$. Then, as noted in the Introduction, it follows that $f \to f^{(j)}(z)$ defines a bounded linear functional on $D_{(k-1)/2}(\mathbb{D})$ for all $z \in \partial\mathbb{D}$ and all $j = 0, 1, \dots, n-1$. But then the functional of evaluation of the $n-1$-derivative at 1 annihilates every function in $[(1-z)^n]$, but it does not annihilate $(1-z)^{n-1}$. Hence $(1-z)^{n-1} \notin [(1-z)^n]$. This implies that $1 - z \notin \mathcal{C}_{n-1}(D_{(k-1)/2}(\mathbb{D}))$. Then part (b) of the proposition follows from Lemma 4.2. ∎

The second way to embed $D_{(k-1)/2}$ in H_d^2 is given by the following lemma.

Lemma 4.5. *Let $1 \leq k \leq d$. Then the operator $S_k : D_{\frac{k-1}{2}}(\mathbb{D}) \to H_d^2, S_k f(z) = f(\sum_{j=1}^k z_j^2)$ is bounded and bounded below.*

Proof. As above, using the embeeding $H^2_k \subseteq H^2_d$ it suffices to prove the case $k = d$. If $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$, then

$$S_d f(z) = \sum_{n=0}^{\infty} a_n (\sum_{j=1}^{d} z_j^2)^n = \sum_{n=0}^{\infty} a_n \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!} z^{(2\alpha)}.$$

Then

$$\|S_d f\|^2_{H^2_d} = \sum_{n=0}^{\infty} |a_n|^2 \frac{(n!)^2}{(2n)!} \sum_{|\alpha|=n} \frac{(2\alpha)!}{(\alpha!)^2}.$$

Thus we have to prove that

$$(n+1)^{(d-1)/2} \approx \frac{(n!)^2}{(2n)!} \sum_{|\alpha|=n} \frac{(2\alpha)!}{(\alpha!)^2},$$

where the implied constants may depend on d, but not on n. For $n = 0$ we have equality, so it will be enough to consider $n \geq 1$. We will prove the statement by induction on d. For $d = 1$ the statement holds with equality, and we will also explicitly verify the case $d = 2$. Note by Stirling's formula we have that $(n!)^2/(2n)! \approx \sqrt{n}/2^{2n}$ for $n \geq 1$. Then

$$\begin{aligned}
\frac{(n!)^2}{(2n)!} \sum_{|\alpha|=n} \frac{(2\alpha)!}{(\alpha!)^2} &= \frac{(n!)^2}{(2n)!} \sum_{k=0}^{n} \frac{(2k)!}{(k!)^2} \frac{(2(n-k))!}{((n-k)!)^2} \\
&\approx 2 + \frac{\sqrt{n}}{2^{2n}} \sum_{k=1}^{n-1} \frac{(2k)!}{(k!)^2} \frac{(2(n-k))!}{((n-k)!)^2} \\
&\approx 2 + \sqrt{n} \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}\sqrt{n-k}} \\
&\approx \sqrt{n} \sum_{1 \leq k \leq n/2} \frac{1}{\sqrt{k}\sqrt{n-k}} \\
&\approx \sum_{1 \leq k \leq n/2} \frac{1}{\sqrt{k}} \approx \sqrt{n}.
\end{aligned}$$

Now assume that $d \geq 2$ and the statement holds for d. We will show that it also holds for $d + 1$. Note that for each $n \geq 0$ we have

$$\{\beta \in \mathbb{N}_0^{d+1} : |\beta| = n\} = \{(\alpha, n - |\alpha|) : \alpha \in \mathbb{N}_0^d, 0 \leq |\alpha| \leq n\}$$

and hence by the induction hypothesis

$$\begin{aligned}\frac{(n!)^2}{(2n)!}\sum_{\beta\in N_0^{d+1},|\beta|=n}\frac{(2\beta)!}{(\beta!)^2} &= \frac{(n!)^2}{(2n)!}\sum_{k=0}^{n}\sum_{\alpha\in\mathbb{N}_0^d,|\alpha|=k}\frac{(2\alpha)!(2(n-k))!}{(\alpha!)^2((n-k)!)^2}\\ &\approx \frac{(n!)^2}{(2n)!}\sum_{k=0}^{n}\frac{(2k)!}{(k!)^2}\frac{(2(n-k))!}{((n-k)!)^2}(k+1)^{(d-1)/2}\\ &\le (n+1)^{(d-1)/2}\frac{(n!)^2}{(2n)!}\sum_{k=0}^{n}\frac{(2k)!}{(k!)^2}\frac{(2(n-k))!}{((n-k)!)^2}\\ &\approx (n+1)^{(d-1)/2}\sqrt{n}\le (n+1)^{d/2},\end{aligned}$$

where we applied the case when $d=2$ in the last step. Thus, we have the required upper bound. For the lower bound note that by symmetry

$$\begin{aligned}&2\sum_{k=0}^{n}\frac{(2k)!}{(k!)^2}\frac{(2(n-k))!}{((n-k)!)^2}(k+1)^{(d-1)/2}\\ &\quad=\sum_{k=0}^{n}\frac{(2k)!}{(k!)^2}\frac{(2(n-k))!}{((n-k)!)^2}((k+1)^{(d-1)/2}+(n-k+1)^{(d-1)/2})\\ &\quad\ge\sum_{k=0}^{n}\frac{(2k)!}{(k!)^2}\frac{(2(n-k))!}{((n-k)!)^2}(\frac{n}{2}+1)^{(d-1)/2},\end{aligned}$$

and now we can substitute this into the previous formula and obtain the lower bound with a similar calculation as before. ■

Example 4.6. If $d\ge 3$, then $p(z)=1-(z_1^2+z_2^2+z_3^2)$ is cyclic in H_d^2 and $Z(p)\cap\partial\mathbb{B}_d$ is a 2-dimensional cube.

Again we note that for $k=3$ we have $D_{(k-1)/2}(\mathbb{D})=D$ is the classical Dirichlet space and the polynomial $1-z$ is cyclic in D. Hence the statement follows from Lemma 4.5 with arguments that are analogous to the proof of Lemma 4.2 (a) by use of Lemma 4.1. In this case

$$Z(p)\cap\partial\mathbb{B}_d=\{(\cos t\cos s,\cos t\sin s,\sin t,0,\dots,0): t,s\in[0,2\pi]\}.$$

To complete this section, we will follow the ideas of Brown and Shields [14] (see also [44] for a $\mathbb{B}_2$-version) to obtain a necessary condition for cyclicity in H_d^2, proving that the zero set of a cyclic function $f\in H_d^2\cap C(\overline{\mathbb{B}_d})$ cannot embed a 3-dimensional cube. To prove this we want to construct bounded linear functionals of the form

$$f\to\int_{\partial\mathbb{B}_d} f\,d\mu,\quad f\in H_d^2\cap C(\overline{\mathbb{B}_d}),$$

for appropriate finite Borel measures μ on $\partial\mathbb{B}_d$, such that these functionals annihilate nontrivial multiplier-invariant subspaces. The argument applies directly to Hilbert function spaces whose kernel has the form $(1-\langle z,w\rangle)^{-\alpha}$, but here we shall focus on H^2_d.

Lemma 4.7. *Let μ be a finite Borel measure on $\partial\mathbb{B}_d$ and*

$$f_\mu(z) = \int \frac{1}{1-\langle z,w\rangle} d\mu(w), \quad z \in \mathbb{B}_d.$$

(i) If

$$E(\mu) = \int\int \frac{1}{|1-\langle z,w\rangle|} d\mu(w)d\mu(z) < \infty, \tag{4.1}$$

then $f_\mu \in H^2_d$ and $\|f_\mu\|^2_{H^2_d} \leq E(\mu)$.
(ii) If $E(\mu) < \infty$ and $f \in H^2_d \cap C(\overline{\mathbb{B}_d})$, then

$$\langle f, f_\mu\rangle_{H^2_d} = \int f d\mu.$$

Proof. (i) For fixed $r \in (0,1)$, the H^2_d- valued function $u_r(z) = k_{rz}$ is continuous on the closed unit ball $\overline{\mathbb{B}_d}$. For a measure μ as in the statement, consider the Bochner integral $\int u_r d\mu$. Evaluating at $z \in \mathbb{B}_d$ and using elementary properties of Bochner integrals yields

$$\left(\int u_r d\mu\right)(z) = \int \langle u_r, k_z\rangle_{H^2_d} d\mu = f_\mu(rz) = (f_\mu)_r(z).$$

But then the same properties of the Bochner integral yield for $f \in H^2_d$

$$\langle f, (f_\mu)_r\rangle_{H^2_d} = \int f_r d\mu. \tag{4.2}$$

In particular, for $f = (f_\mu)_r$ we obtain

$$\|(f_\mu)_r\|^2_{H^2_d} = \int\int \frac{1}{1-r^2\langle z,w\rangle} d\mu(w)d\mu(z)$$
$$\leq \int\int \frac{1}{|1-r^2\langle z,w\rangle|} d\mu(w)d\mu(z).$$

Now let $r \to 1$, use the inequality

$$|1-r^2\zeta| \geq \frac{1}{2}|1-\zeta|,$$

together with the dominated convergence theorem to conclude that

$$\limsup_{r\to 1} \|(f_\mu)_r\|^2_{H^2_d} \leq E(\mu).$$

Since $(f_\mu)_r(z) \to f_\mu(z)$, $z \in \mathbb{B}_d$, it follows that $f_\mu \in H^2_d$ and $\|f_\mu\|^2_{H^2_d} \leq E(\mu)$, which proves (i). Part (ii) follows from (i) and (4.2), since

$$\int f_r d\mu = \langle f, (f_\mu)_r \rangle = \langle f_r, f_\mu \rangle.$$

If $f \in H^2_d \cap C(\overline{\mathbb{B}_d})$ we obtain the result letting $r \to 1$. ∎

Given a set $S \subseteq \mathbb{C}^d$ and an integer $m \geq 0$, we say that S contains an embedded cube of dimension m if there exists a diffeomorphism ϕ from $(-1,1)^m$ into S.

Theorem 4.8. *Let $f \in H^2_d \cap C(\overline{\mathbb{B}_d})$ and assume that $Z(f) \cap \partial\mathbb{B}_d$ contains an embedded cube of dimension $m \geq 3$. Then f is not cyclic in H^2_d.*

Proof. Let $\phi : (-1,1)^m \to U \subseteq Z(f) \cap \partial\mathbb{B}_d$ be a diffeomorphism. Then ϕ satisfies for some $c > 0$ that

$$|\phi(t) - \phi(s)| \geq c|t - s|, \quad t, s \in (-1,1)^m. \tag{4.3}$$

Consider the pushforward measure

$$\mu(E) = \lambda_m(\phi^{-1}(E \cap U))$$

on $\partial\mathbb{B}_d$, where λ_m denotes the $m-$dimensional Lebesgue measure on $(-1,1)^m$. According to Lemma 4.7 it will be sufficient to show that (4.2) holds. Indeed, in this case part (ii) of the lemma gives that $f_\mu \in [f]^\perp$, and clearly $f_\mu \neq 0$. To demonstrate this, we observe that

$$|1 - \langle z, w \rangle| \geq \text{Re}\,(1 - \langle z, w \rangle) = \frac{|z - w|^2}{2}, \qquad z, w \in \partial\mathbb{B}_d,$$

to obtain that

$$E(\mu) \leq \frac{2}{c} \int_{(-1,1)^m} \int_{(-1,1)^m} \frac{1}{|t - s|^2} d\lambda_m(s) d\lambda_m(t) < \infty,$$

since $m \geq 3$. ∎

5. Radially weighted Besov spaces

Throughout this section, ω will denote an admissible radial measure, see Section 2.

5.1. **Lemmas about ratios of multipliers.** We start with a lemma, which is basically from [4], and which says that all radially weighted Besov spaces satisfy the "multiplier inclusion condition" with constant 1.

Lemma 5.1. *For each* $k \in \mathbb{N}$ *the space* $\mathrm{Mult}(B^k_\omega)$ *is contractively contained in* $\mathrm{Mult}(B^{k-1}_\omega)$*, that is,*

$$\|\varphi\|_{\mathrm{Mult}(B^{k-1}_\omega)} \leq \|\varphi\|_{\mathrm{Mult}(B^k_\omega)} \quad \textit{for all } \varphi \in \mathrm{Mult}(B^k_\omega).$$

Proof. If the measure $d\omega(z) = d\mu(r)d\sigma(w)$ is such that μ is absolutely continuous, then this follows directly from Theorem 1.2 or Corollary 3.4 of [4]. But the proof given in [4] actually applies to the more general situation considered here, where μ is not assumed to be absolutely continuous. Indeed, it follows from (2.3) that the reproducing kernel K^k for B^k_ω is given by

$$K^k_w(z) = \frac{1}{\omega(\mathbb{B}_d)} + \sum_{n=1}^{\infty} \frac{1}{n^{2k}\omega_n} \langle z, w\rangle^n.$$

Thus, we can use Proposition 3.3 of [4] with the sequences $\{a_n\}$ and $\{b_n\}$ where $a_0 = b_0 = \frac{1}{\omega(\mathbb{B}_d)}$, $a_n = \frac{1}{n^{2k}\omega_n}$ and $b_n = \frac{1}{n^{2(k-1)}\omega_n}$ for $n \geq 1$. ∎

For us, the multiplier inclusion condition is important because the following lemma is now elementary.

Lemma 5.2. *We have that* $\varphi \in \mathrm{Mult}(B^k_\omega)$ *if and only if* $\varphi \in \mathrm{Mult}(B^{k-1}_\omega)$ *and* $R\varphi \in \mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)$*. Furthermore,*

$$\|\varphi\|_{\mathrm{Mult}(B^k_\omega)} \leq 2(\|\varphi\|_{\mathrm{Mult}(B^{k-1}_\omega)} + \|R\varphi\|_{\mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)}) \textit{ and} \tag{5.1}$$

$$\|R\varphi\|_{\mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)} \leq 2\|\varphi\|_{\mathrm{Mult}(B^k_\omega)} \tag{5.2}$$

Proof. Since $f \in B^k_\omega$ if and only if $Rf \in B^{k-1}_\omega$, we see that $\varphi \in \mathrm{Mult}(B^k_\omega)$, if and only if $(R\varphi)f + \varphi Rf \in B^{k-1}_\omega$ for each $f \in B^k_\omega$. Thus, if $\varphi \in \mathrm{Mult}(B^{k-1}_\omega)$ and $R\varphi \in \mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)$, then $\varphi \in \mathrm{Mult}(B^k_\omega)$. Conversely, if $\varphi \in \mathrm{Mult}(B^k_\omega)$, then by the multiplier inclusion condition $\varphi \in \mathrm{Mult}(B^{k-1}_\omega)$, and hence the identity $(R\varphi)f = R(\varphi f) - \varphi Rf$ implies that $R\varphi \in \mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)$. That argument can be used to get the estimates. Let $f \in B^k_\omega$. Then by equation (2.2) we have

$$\begin{aligned}
\|\varphi f\|^2_{B^k_\omega} &= \omega(\mathbb{B}_d)|(\varphi f)(0)|^2 + \|(R\varphi)f + \varphi Rf\|^2_{B^{k-1}_\omega} \\
&\leq \|\varphi\|^2_{\mathrm{Mult}(B^{k-1}_\omega)}\omega(\mathbb{B}_d)|f(0)|^2 \\
&\quad + 2(\|R\varphi\|^2_{\mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)}\|f\|^2_{B^k_\omega} + \|\varphi\|^2_{\mathrm{Mult}(B^{k-1}_\omega)}\|Rf\|^2_{B^{k-1}_\omega}) \\
&\leq 2(\|R\varphi\|^2_{\mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)} + \|\varphi\|^2_{\mathrm{Mult}(B^{k-1}_\omega)})\|f\|^2_{B^k_\omega}
\end{aligned}$$

This proves (5.1). Furthermore, since all functions in $(R\varphi)f = R(\varphi f) - \varphi Rf$ are 0 at the origin we have

$$\begin{aligned}
\|(R\varphi)f\|_{B^{k-1}_\omega} &\le \|R(\varphi f)\|_{B^{k-1}_\omega} + \|\varphi Rf\|_{B^{k-1}_\omega}\\
&\le \|\varphi f\|_{B^k_\omega} + \|\varphi\|_{\mathrm{Mult}(B^{k-1}_\omega)}\|Rf\|_{B^{k-1}_\omega}\\
&\le (\|\varphi\|_{\mathrm{Mult}(B^k_\omega)} + \|\varphi\|_{\mathrm{Mult}(B^{k-1}_\omega)})\|f\|_{B^k_\omega}\\
&\le 2\|\varphi\|_{\mathrm{Mult}(B^k_\omega)}\|f\|_{B^k_\omega},
\end{aligned}$$

where the last inequality followed from Lemma 5.1. Hence (5.2) holds. ■

The following lemma is key to proving Theorem 3.1. Note that it immediately implies Theorem 3.2.

Lemma 5.3. *If $M > 0$ and $\varphi, \psi \in \mathrm{Mult}(B^N_\omega)$ with*

(i) $\|\varphi\|_{\mathrm{Mult}(B^N_\omega)}, \|\psi\|_{\mathrm{Mult}(B^N_\omega)} \le M$ *and*
(ii) $\frac{\varphi}{\psi} \in H^\infty(\mathbb{B}_d)$ *with* $\|\frac{\varphi}{\psi}\|_\infty \le 1$,

then for all $s \in \mathbb{N}$ and integers k with $0 \le k \le N$ we have $\frac{\varphi^{s+k}}{\psi^s} \in \mathrm{Mult}(B^k_\omega)$ with

$$\left\|\frac{\varphi^{s+k}}{\psi^s}\right\|_{\mathrm{Mult}(B^k_\omega)} \le 8^k(s+k)^k M^k \tag{5.3}$$

Furthermore, if the functions φ, ψ are nonzero in $\mathbb{B}_d$, then the conclusion and inequality (5.3) hold for all real $s > 0$.

Proof. Note that if s is a positive integer or if the functions φ, ψ are nonzero in $\mathbb{B}_d$ and $s \in (0, \infty)$, then $\varphi^s/\psi^s \in H^\infty(\mathbb{B}_d)$. This is the only place that the different hypotheses on s and φ, ψ are used, and in the following we will treat these cases simultaneously.

We start by noting that Lemma 5.1 implies that $\|\varphi\|_{\mathrm{Mult}(B^k_\omega)} \le M$ and $\|\psi\|_{\mathrm{Mult}(B^k_\omega)} \le M$ for all k with $0 \le k \le N$ and hence by Lemma 5.2 we have $\|R\varphi\|_{\mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)} \le 2M$ and $\|R\psi\|_{\mathrm{Mult}(B^k_\omega, B^{k-1}_\omega)} \le 2M$, if $1 \le k \le N$.

We will now establish the lemma by induction on k. Since $\mathrm{Mult}(B^0_\omega) = H^\infty(\mathbb{B}_d)$ it is clear that the case $k = 0$ holds. Now suppose that $1 \le k \le N$ and the inequality (5.3) holds for $k-1$ for all $s > 0$ and for some φ, ψ that satisfy (i) and (ii). Then in particular for a fixed s it also holds for $s+1$, that is,

$$\left\|\frac{\varphi^{s+k-1}}{\psi^s}\right\|_{\mathrm{Mult}(B_\omega^{k-1})} \leq 8^{k-1}(s+k-1)^{k-1}M^{k-1} \leq 8^{k-1}(s+k)^{k-1}M^{k-1}$$

$$\left\|\frac{\varphi^{s+k}}{\psi^{s+1}}\right\|_{\mathrm{Mult}(B_\omega^{k-1})} \leq 8^{k-1}(s+k)^{k-1}M^{k-1}.$$

We compute

$$R\frac{\varphi^{s+k}}{\psi^s} = (s+k)\frac{\varphi^{s+k-1}}{\psi^s}R\varphi - s\frac{\varphi^{s+k}}{\psi^{s+1}}R\psi,$$

and hence

$$\begin{aligned}\left\|R\frac{\varphi^{s+k}}{\psi^s}\right\|_{\mathrm{Mult}(B_\omega^k,B_\omega^{k-1})} &\leq (s+k)\left\|\frac{\varphi^{s+k-1}}{\psi^s}\right\|_{\mathrm{Mult}(B_\omega^{k-1})}\|R\varphi\|_{\mathrm{Mult}(B_\omega^k,B_\omega^{k-1})}\\ &\quad + s\left\|\frac{\varphi^{s+k}}{\psi^{s+1}}\right\|_{\mathrm{Mult}(B_\omega^{k-1})}\|R\psi\|_{\mathrm{Mult}(B_\omega^k,B_\omega^{k-1})}\\ &\leq (2s+k)8^{k-1}(s+k)^{k-1}M^{k-1}(2M)\\ &= 2(2s+k)8^{k-1}(s+k)^{k-1}M^k\end{aligned}$$

by the induction hypothesis as stated above. Hence by inequality (5.1) and the induction hypothesis we obtain

$$\begin{aligned}\left\|\frac{\varphi^{s+k}}{\psi^s}\right\|_{B_\omega^k} &\leq 2\left(\left\|\varphi\frac{\varphi^{s+k-1}}{\psi^s}\right\|_{B_\omega^{k-1}} + 2(2s+k)8^{k-1}(s+k)^{k-1}M^k\right)\\ &\leq 2(M8^{k-1}(s+k)^{k-1}M^{k-1} + 2(2s+k)8^{k-1}(s+k)^{k-1}M^k)\\ &\leq 2(1+2(2s+k))8^{k-1}(s+k)^{k-1}M^k\\ &\leq 8^k(s+k)^kM^k.\end{aligned}$$

∎

5.2. **The proof of Theorem 3.5.** If $f \in \mathrm{Hol}(\mathbb{B}_d)$ and $0 < r < 1$, we write $f_r(z) = f(rz)$.

Theorem 5.4. *Let $N \in \mathbb{N}_0$. If $M > 0$ and $\varphi \in \mathrm{Mult}(B_\omega^N)$ such that $\varphi(z) \neq 0$ for each $z \in \mathbb{B}_d$ and $\|\frac{\varphi}{\varphi_r}\|_\infty \leq M$ for all $0 < r < 1$, then $[\varphi^N] = [\varphi^{N+1}]$, that is, $\varphi \in \mathcal{C}_N(B_\omega^N)$.*

Proof. By hypothesis $\frac{1}{\varphi_r} \in \mathrm{Mult}(B_\omega^N)$, and thus $\frac{\varphi^{N+1}}{\varphi_r} \in [\varphi^{N+1}]$ for each r. By Lemma 5.3 with $\psi = C\varphi_r$, $k = N$, and $s = 1$ there is a constant $K > 0$ such that $\|\frac{\varphi^{N+1}}{\varphi_r}\|_{\mathrm{Mult}(B_\omega^N)} \leq K$ for all $0 < r < 1$. This implies that $\frac{\varphi^{N+1}}{\varphi_r} \to \varphi^N$ weakly in B_ω^N, and thus that $\varphi^N \in [\varphi^{N+1}]$. ∎

Theorem 5.4 applies to polynomials in $\mathbb{C}_{\text{stable}}[z]$, and thus implies the first part of Theorem 3.5. In order to prove the remaining part of Theorem 3.5 we need estimates for p^n/p_r and its derivatives for $n \geq 1$. We will prove the needed results by looking at slice functions. Thus, we start with a single variable lemma (see Lemma 1 of [31] for the case $n = 1$).

Theorem 5.5. *Let $n, m \in \mathbb{N}$. There is a constant $c = c(n,m)$ such that whenever p is a polynomial of degree $\leq m$ that has no zeros in $\mathbb{D}$, then for all $0 \leq r < 1$ we have*

$$\left|\frac{d^k}{dz^k}\frac{p(z)^n}{p(rz)}\right| \leq c\ |p(0)|^{n-1} \quad \textit{for all } 0 \leq k < n, z \in \mathbb{D}, \quad \textit{and}$$
$$\int_{|z|<1}\left|\frac{d^n}{dz^n}\frac{p(z)^n}{p(rz)}\right|^2 \frac{dA(z)}{\pi} \leq c\ |p(0)|^{2n-2}.$$

Proof. The statement is obviously true for constant polynomials, thus we may assume that the degree of p equals $m \geq 1$. By dividing through by $p(0)$ we may also assume that $p(0) = 1$. Then there are $A_1, \dots A_m \in \overline{\mathbb{D}}$ such that

$$\frac{p(z)^n}{p(rz)} = \prod_{j=1}^{m} \frac{(1 - A_j z)^n}{(1 - A_j rz)}, \quad z \in \mathbb{D},\ 0 \leq r \leq 1.$$

Write $g_{A,r}(z) = \frac{(1-Az)^n}{(1-Arz)}$, then by the multi-product Leibniz formula we have for each $0 \leq k \leq n$

$$\frac{d^k}{dz^k}\frac{p(z)^n}{p(rz)} = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} \prod_{j=1}^{m} g_{A_j,r}^{(\alpha_j)}(z).$$

The Theorem will follow, if we show that there is $c > 0$ that is independent of $A \in \overline{\mathbb{D}}$ and $0 \leq r \leq 1$ such that $|g_{A,r}^{(j)}(z)| \leq c$, whenever $0 \leq j < n$ and $z \in \mathbb{D}$, and $\int_{|z|<1} |g_{A,r}^{(n)}(z)|^2 \frac{dA(z)}{\pi} \leq c$.

Note that $|g_{A,r}^{(j)}(z)| = |g_{1,r}^{(j)}(Az)A^j| \leq |g_{1,r}^{(j)}(Az)|$. Hence it suffices to prove the statement for $A = 1$. If $0 \leq r \leq 1/2$, then the function $\frac{(1-z)^n}{1-rz}$ and all of its derivatives are rational functions with poles in $\{|z| \geq 2\}$ that are continuous as functions of the parameter r, hence by compactness there is $c > 0$ such that $|g_{1,r}^{(j)}(z)| \leq c$ for all $0 \leq j \leq n$, $|z| \leq 1$ and all $0 \leq r \leq 1/2$.

Now let $1/2 \leq r < 1$. For $z \in \mathbb{D}$ set $w = 1 - rz$. Then

$$\begin{aligned} g_{1,r}(z) &= \frac{((r-1)+w)^n}{r^n w} \\ &= \frac{1}{r^n} \sum_{k=0}^{n} \binom{N}{\alpha} (r-1)^{n-k} w^{k-1} \\ &= \frac{1}{r^n} \left(\frac{(r-1)^n}{1-rz} + q(r,z) \right) \end{aligned}$$

where $q(r,z)$ is a polynomial expression in the variables r and z.

Let $0 \leq j \leq n$ and take the j-th derivative

$$\begin{aligned} |g_{1,r}^{(j)}(z)| &= r^{-n} \left| \frac{j! r^j (r-1)^n}{(1-rz)^{j+1}}) + \frac{\partial^j}{\partial z^j} q(r,z) \right| \\ &\leq 2^{-n} \left(\frac{j!(1-r)^n}{|1-rz|^{j+1}} + |\frac{\partial^j}{\partial z^j} q(r,z)| \right) \end{aligned}$$

By compactness the expressions $|\frac{\partial^j}{\partial z^j} q(r,z)| \leq C$, where C is independent of $0 \leq j \leq n$, $|z| \leq 1$ and $1/2 \leq r \leq 1$. Furthermore, if $j < n$, then $\frac{j!(1-r)^n}{|1-rz|^{j+1}} \leq (n-1)!$, hence the boundedness statement follows in those cases. Finally, we have that

$$\int_{\mathbb{D}} |g_{1,r}^{(n)}(z)|^2 \frac{dA(z)}{\pi} \leq 2 \cdot 2^{-2n} \left(\int_{\mathbb{D}} \frac{(n!)^2 (1-r)^{2n}}{|1-rz|^{2n+2}} \frac{dA(z)}{\pi} + C^2 \right) \leq C(n)$$

see for example [28, Theorem 1.7]. ■

We can now prove the second part of Theorem 3.5. For convenience we restate it here.

Theorem 5.6. *Let $N \in \mathbb{N}$ and let ω be an admissible radial measure of the type $d\omega(w) = u(r) 2r dr d\sigma(z)$, $u \in L^\infty(0,1)$. Then every polynomial p without zeros in $\mathbb{B}_d$ satisfies $[p^{N-1}] = [p^N]$ in B^N_ω, that is, $\mathbb{C}_{\text{stable}}[z] \subseteq \mathcal{C}_{N-1}(B^N_\omega)$.*

Proof. Let $f \in \text{Hol}(\mathbb{B}_d)$, and let $f = \sum_{n=0}^\infty f_n$ be the representation as sum of homogeneous polynomials of degree n. For $\lambda \in \mathbb{C}$ let $f_z(\lambda) = f(\lambda z)$ be a slice function at $z \in \partial \mathbb{B}_d$. Write $D_\lambda = \lambda \frac{\partial}{\partial \lambda}$, then for $f = \sum_{n=0}^\infty f_n$ we have $Rf(\lambda z) = \sum_{n=1}^\infty n f_n(z) \lambda^n = D_\lambda f_z(\lambda)$. Then

$$R^N f(\lambda z) = \sum_{k=1}^{N} a_k \lambda^k \frac{\partial^k}{\partial \lambda^k} f_z(\lambda)$$

for some coefficients $a_1, \dots, a_N$, and

$$\begin{aligned}\int_{\mathbb{B}_d} |R^N f|^2 d\omega &= \int_{\partial\mathbb{B}_d} \int_0^1 \int_0^{2\pi} |R^N f(re^{it}z)|^2 \frac{dt}{2\pi} w(r) 2r dr dt d\sigma(z) \\ &\leq \|w\|_\infty \int_{\partial\mathbb{B}_d} \int_{\mathbb{D}} \left| \sum_{k=1}^N a_k \lambda^k \frac{\partial^k}{\partial\lambda^k} f_z(\lambda) \right|^2 \frac{dA(\lambda)}{\pi} d\sigma(z) \\ &\leq N\|w\|_\infty \sum_{k=1}^N |a_k| \int_{\partial\mathbb{B}_d} \int_{\mathbb{D}} \left| \frac{\partial^k}{\partial\lambda^k} f_z(\lambda) \right|^2 \frac{dA(\lambda)}{\pi} d\sigma(z).\end{aligned}$$

Hence, by the lemma,

$$\int_{\mathbb{B}_d} \left| R^N \frac{p^N}{p_r} \right|^2 d\omega \leq N\|w\|_\infty \sum_{k=1}^N |a_k| \; c|p(0)|^{2N-2}$$

This implies that $\frac{p^N}{p_r} \to p^{N-1}$ weakly in B^N_ω as $r \to 1$, and therefore that $p^{N-1} \in [p^N]$. ∎

5.3. **The proof of Theorem 3.1.** In Lemma 5.3 we proved that for $\varphi, \psi \in \mathrm{Mult}(B^N_\omega)$ with $\varphi/\psi \in H^\infty$ we have $\varphi^{N+k}/\psi^k \in \mathrm{Mult}(B^N_\omega)$. That immediately implies that $\varphi^{N+k} \in [\psi^k]$. We will now take advantage of the fact that ω is a radial measure to show that actually $\varphi^{N+k-1} \in [\psi^k]$. That is, we will prove Theorem 3.1. We start with some preliminaries.

Lemma 5.7. *Let $N \geq 1$.For any $f \in B^N_\omega$ we have*

$$\|f - f_r\|_{B^{N-1}_\omega} \leq (1-r)\|f\|_{B^N_\omega}, 0 < r < 1.$$

Proof. Let $f = \sum_{n=0}^\infty f_n$ be the expansion of f into a sum of homogeneous polynomials of degree n. Then for $0 < r < 1$ we have

$$\begin{aligned}\|f - f_r\|^2_{B^{N-1}_\omega} &= \sum_{n=0}^\infty \|(1-r^n) f_n\|^2_{B^{N-1}_\omega} \\ &\leq (1-r)^2 \sum_{n=0}^\infty n^2 \|f_n\|^2_{B^{N-1}_\omega} \\ &\leq (1-r)^2 \|f\|^2_{B^N_\omega}.\end{aligned}$$

∎

Lemma 5.8. *Let $k \geq 0$ and $\varphi \in \mathrm{Mult}(B^k_\omega)$. Then*

$$\|R\varphi_r\|_{\mathrm{Mult}(B^k_\omega)} \leq \frac{\|\varphi\|_{\mathrm{Mult}(B^k_\omega)}}{1-r^2}.$$

Proof. For $\lambda \in \mathbb{D}, z \in \mathbb{B}_d$ define $F(\lambda)(z) = \varphi(\lambda z)$. Then $F : \mathbb{D} \to \mathrm{Mult}(B^k_\omega)$ is an analytic function with $\|F(\lambda)\|_{\mathrm{Mult}(B^k_\omega)} \leq \|\varphi\|_{\mathrm{Mult}(B^k_\omega)}$ for all $\lambda \in \mathbb{D}$. Then by the Cauchy formula we obtain $\|F'(\lambda)\|_{\mathrm{Mult}(B^k_\omega)} \leq \frac{\|\varphi\|}{1-|\lambda|^2}$. The Lemma follows, because $(R\varphi_r)(z) = rF'(r)(z)$. ∎

Lemma 5.9. *If $N \geq 1$ and if $f, g \in B^N_\omega$ such that $\varphi = f/g \in \mathrm{Mult}(B^{N-1}_\omega)$, then for all $0 \leq r < 1$ we have*

$$\|\varphi_r g\|_{B^N_\omega} \leq 3\|\varphi\|_{\mathrm{Mult}(B^{N-1}_\omega)}\|g\|_{B^N_\omega} + \|f\|_{B^N_\omega}$$

and hence $f \in [g]$, since $\varphi_r g \to f$ weakly in B^N_ω.

Proof. We have

$$\|\varphi_r g\|_{B^N_\omega} \leq \|\varphi_r(g - g_r)\|_{B^N_\omega} + \|\varphi_r g_r\|_{B^N_\omega} \leq \|\varphi_r(g - g_r)\|_{B^N_\omega} + \|f\|_{B^N_\omega}.$$

Since $g - g_r$ vanishes at the origin we have $\|\varphi_r(g-g_r)\|_{B^N_\omega} = \|R(\varphi_r(g-g_r))\|_{B^{N-1}_\omega}$. Using the previous two lemmas, we have

$$\begin{aligned}\|R(\varphi_r(g - g_r))\|_{B^{N-1}_\omega} &\leq \|(g - g_r)R\varphi_r\|_{B^{N-1}_\omega} + \|\varphi_r R(g - g_r)\|_{B^{N-1}_\omega}\\ &\leq \frac{\|\varphi\|_{\mathrm{Mult}(B^{N-1}_\omega)}}{1 - r^2}(1 - r)\|g\|_{B^N_\omega} + \|\varphi\|_{\mathrm{Mult}(B^{N-1}_\omega)}\|R(g - g_r)\|_{B^{N-1}_\omega}\\ &\leq 3\|\varphi\|_{\mathrm{Mult}(B^{N-1}_\omega)}\|g\|_{B^N_\omega},\end{aligned}$$

concluding the proof. ∎

Proof of Theorem 3.1. Let $N \in \mathbb{N}$ and $\varphi, \psi \in \mathrm{Mult}(B^N_\omega)$ with $\varphi/\psi \in H^\infty$. Let $k \in \mathbb{N}$, and set $f = \varphi^{N+k-1}$ and $g = \psi^k$. Then by Lemma 5.3 we have $\frac{f}{g} = \frac{\varphi^{N+k-1}}{\psi^k} \in \mathrm{Mult}(B^{N-1}_\omega)$. Hence Lemma 5.9 implies $\varphi^{N+k-1} \in [\psi^k]$. ∎

5.4. **Cyclicity and zero sets.** If $f : U \to \mathbb{R}$ is a function, then let $Z(f) = \{x \in U : f(x) = 0\}$ be the zero locus of f. As in [31] we will use the Łojasiewicz inequality from real algebraic geometry, (see [36], Chapter IV.7).

Lemma 5.10. *Let $U \subseteq \mathbb{R}^{2d}$ be open, and let $f : U \to \mathbb{R}$ be a real analytic function such that $Z(f) \neq \emptyset$. Then for every compact set $K \subseteq U$ there are positive constants p and C such that*

$$\mathrm{dist}(x, Z(f))^p \leq C|f(x)| \quad \text{for all } x \in K.$$

We obtain the following.

Lemma 5.11. *Suppose that $f, g \in \mathrm{Hol}(\mathbb{B}_d)$ such that*

(i) *$f(z) \neq 0, g(z) \neq 0$ for all $z \in \mathbb{B}_d$,*
(ii) *f extends to be analytic in a neighborhood of $\overline{\mathbb{B}_d}$,*
(iii) *g satisfies a Lipschitz condition of order $\alpha > 0$.*

Let $Z(g) \subseteq \partial\mathbb{B}_d$ denote the zero set of the Lipschitz extension of g.

If $Z(f) \cap \partial\mathbb{B}_d \subseteq Z(g) \cap \partial\mathbb{B}_d$, then there is a constant $C > 0$ and an integer $j > 0$ such that

$$|g(z)|^j \leq C|f(z)|$$

for all $z \in \mathbb{B}_d$.

Proof. If $Z(f) \cap \partial\mathbb{B}_d = \emptyset$, then $|f|$ is bounded below on $\overline{\mathbb{B}_d}$ and the conclusion of the lemma follows with $j = 1$. Thus, we assume $Z(f) \cap \partial\mathbb{B}_d \neq \emptyset$. Since f has no zeroes in $\mathbb{B}_d$ we have

$$1 - \sum_{i=1}^{d} |z_i|^2 \leq 2\,\mathrm{dist}(z, Z(f) \cap \partial\mathbb{B}_d) \text{ for all } z \in \mathbb{B}_d.$$

Then by the Łojasiewicz inequality applied with $K = \overline{\mathbb{B}_d}$ there is an even integer n and a $C_1 > 0$ such that

$$\left(1 - \sum_{i=1}^{d} |z_i|^2\right)^n \leq 2^n\,\mathrm{dist}(z, Z(f))^n \leq C_1|f(z)|^2, \quad z \in \mathbb{B}_d.$$

Next we apply the Łojasiewicz inequality to the function

$$r(z) = |f(z)|^2 + \left(1 - \sum_{i=1}^{d} |z_i|^2\right)^n,$$

and we find that there is an integer m and $C_2 > 0$ such that for all $z \in \mathbb{B}_d$

$$\mathrm{dist}(z, Z(f) \cap \partial\mathbb{B}_d)^m \leq C_2 r(z) \leq C_2(1 + C_1)|f(z)|^2.$$

On the other hand, by the Lipschitz property of g, we have that there is $C_3 > 0$ such that

$$|g(z)| \leq C_3\,\mathrm{dist}(z, Z(g))^\alpha \leq C_3\,\mathrm{dist}(z, Z(f) \cap \partial\mathbb{B}_d)^\alpha \text{ for all } z \in \mathbb{B}_d.$$

This proves the lemma with $j \geq \frac{m}{2\alpha}$. ■

Proof of Theorem 3.6. Let $f, g \in \mathrm{Mult}(B^N_\omega)$ as in the hypothesis of the theorem. Then by Lemma 5.11 there is $j \in \mathbb{N}$ such that $\frac{g^j}{f} \in H^\infty$. Then Theorem 3.1 implies that for each $k \in \mathbb{N}$ we have $g^{j(N+k-1)} \in [f^k] \subseteq [f]$. This proves the first part of the theorem with $n = jN$. If g is a polynomial, then $[g^N] = [g^m]$ for all $m \geq N$. Hence taking $k = N$ we obtain $g^N \in [g^N] = [g^{j(2N-1)}] \subseteq [f^N]$. ■

6. Complete Pick spaces

6.1. **Cyclic subspaces.** The aim of this subsection is to show that for radial Besov spaces which are also complete Pick spaces there is variant of Theorem 3.1 which refers to functions in $\mathcal{H}$ rather than multipliers. To this end, we will use frequently a direct application of Theorem 2.6 which asserts that each complete Pick space $\mathcal{H}$ is contained in the corresponding Pick-Smirnov class

$$N^+(\mathcal{H}) = \{\varphi/\psi : \varphi, \psi \in \mathrm{Mult}(\mathcal{H}), \psi \text{ cyclic}\},$$

see Lemma 6.1 below.

We begin by listing three observations regarding subspaces of the form $[f]$.

Lemma 6.1. *Let $\mathcal{H}$ be a separable Hilbert function space on X. If $f = \varphi/(1-\psi) \in \mathcal{H}$, where $\varphi, \psi \in \mathrm{Mult}(\mathcal{H})$ and $\psi \neq 1$, $\|\psi\|_{\mathrm{Mult}(\mathcal{H})} \leq 1$, then $1-\psi$ is cyclic in $\mathcal{H}$ and $[f] = [\varphi]$.*

Proof. This is a is a straightforward combination of Lemma 2.3 of [2] and Lemma 3.6 (a) of [5]. ■

Lemma 6.2. *Let $\mathcal{H}$ be a separable Hilbert function space on X. If $f = \frac{u}{v} = \frac{u_1}{v_1} \in N^+(\mathcal{H})$, where $u, v, u_1, v_1 \in \mathrm{Mult}(\mathcal{H})$, v, v_1 cyclic, then $[u^n] = [u_1^n]$, for all $n \in \mathbb{N}$.*

Proof. We have $u^n v_1^n = v^n u_1^n$, hence $u^n v_1^n \in [v^n u_1^n] \subseteq [u_1^n]$. Since v_1 is cyclic, so is v_1^n, and since u^n is a multiplier, it easily follows that $u^n \in [u_1^n]$. By symmetry $u_1^n \in [u^n]$, hence $[u^n] = [u_1^n]$. ■

Lemma 6.3. *Let $\mathcal{H}$ be a complete Pick space. If $f = \frac{u}{v} \in \mathcal{H}$, where $u, v \in \mathrm{Mult}(\mathcal{H})$ and v is cyclic, then $[u] = [f]$.*

Proof. Let $f = \frac{u}{v}$. By Theorem 2.6 there are $\varphi, \psi \in \mathrm{Mult}(\mathcal{H})$ such that $\|\psi\|_{\mathrm{Mult}(\mathcal{H})} \leq 1$, $\psi \neq 1$, and $f = \frac{\varphi}{1-\psi}$. By Lemma 6.1 we have $[f] = [\varphi]$, hence we have to show $[u] = [\varphi]$ and that follows from Lemma 6.2. ■

With these lemmas in hand we can turn to the main result of this subsection, which contains Theorem 3.7 as a special case.

Theorem 6.4. *Let $N \in \mathbb{N}$, and let B^N_ω be a radially weighted Besov space that is also a complete Pick space. Let $f, g \in B^N_\omega$ be such that $f/g \in H^\infty$.*

If $f = \frac{u}{v}$, where $u, v \in \mathrm{Mult}(B^N_\omega)$, v is cyclic, then $u^{N-1} f \in [g]$. If $u \in \mathcal{C}_n(B^N_\omega)$ for some $1 \leq n \leq N$, then $u^{n-1} f \in [g]$.

Proof. Write $f = \frac{u}{v}$, $g = \frac{a}{b}$ with $a, b, u, v \in \mathrm{Mult}(B^N_\omega)$, v and b cyclic. By Lemma 6.3 we have that $[f] = [u]$ and $[g] = [a]$. Theorem 3.1 with

$k = 1$ gives us that $b^N u^N \in [va] = [a]$, and thus that $u^N \in [a] = [g]$. Now Lemma 6.3 implies $u^{N-1} f = \frac{u^N}{v} \in [u^N] \subseteq [g]$.

If $u \in \mathcal{C}_n(B^N_\omega)$, then $u^n \in [u^{n+k}]$ for all nonnegative integers k, and hence $u^n \in [u^N] \subseteq [g]$. As above this and Lemma 6.3 implies that $u^{n-1} f \in [g]$. ■

6.2. **Inner factors of slices.** As usual, for a function $u : \mathbb{B}_d \to \mathbb{C}$ and $z \in \partial\mathbb{B}_d$ the corresponding slice function $u_z : \mathbb{D} \to \mathbb{C}$ is given by

$$u_z(\lambda) = u(\lambda z), \quad \lambda \in \mathbb{D}.$$

The following simple observation follows directly from Lemma 2.5.

Lemma 6.5. *Let $N \in \mathbb{N}$, and let B^N_ω be a radially weighted Besov space that is also a complete Pick space. Then there is $c > 0$ such that whenever $f \in B^N_\omega$, then every slice $f_z \in H^2(\mathbb{D})$ with*

$$\|f_z\|_{H^2(\mathbb{D})} \leq c\|f\|_{B^N_\omega}, \quad z \in \partial\mathbb{B}_d.$$

Proof. Let $f \in B^N_\omega$, $z \in \partial\mathbb{B}_d$. By Lemma 2.5 we have for all $\lambda \in \mathbb{D}$

$$|f_z(\lambda)|^2 \leq 2\mathrm{Re}\langle f, k_{\lambda w} f\rangle - \|f\|^2,$$

for a suitable equivalent norm on B^N_ω and the induced scalar product. The right-hand side is harmonic in λ and its value at 0 equals $\|f\|^2$. Thus $|f_z|^2$ has a harmonic majorant in the unit disc, that is, $f_z \in H^2(\mathbb{D})$ with $\|f_z\|^2_{H^2(\mathbb{D})} \leq \|f\|^2 \leq c\|f\|^2_{B^N_\omega}$, where the constant c appears, because of the equivalence of norms. ■

An application of the lemma yields the following result.

Proposition 6.6. *Let $N \in \mathbb{N}$, and let B^N_ω be a radially weighted Besov space that is also a complete Pick space. If $\varphi \in \mathcal{C}_n(B^N_\omega)$ for some $n \geq 0$, then every slice φ_z, $z \in \partial\mathbb{B}_d$, is an outer function in H^∞.*

Proof. By assumption we have that there is a sequence (u_k) in $\mathrm{Mult}(B^N_\omega)$ such that $(u_k\varphi^{n+1})$ converges to φ^n in B^N_ω. Then Lemma 2.5 implies that for every $z \in \partial\mathbb{B}_d$, $(u_k)_z\varphi_z^{n+1} \to \varphi_z^n$ in $H^2(\mathbb{D})$. Since $(u_k)_z \in H^\infty$, this shows that $\varphi_z \in C_n(H^2(\mathbb{D}))$, i.e. it is outer. ■

It is natural to ask whether this result continues to hold under the weaker assumption that the multiplier φ belongs to $\mathcal{C}_\infty(B^N_\omega)$? Of course the answer is affirmative in the one variable case. However, it turns out that this is no longer the case when $d > 1$. Our counterexample is based on the following result about the structure of the Drury-Arveson space H^2_d. It is a precise version of the decomposition used in the remarks after Theorem 4.3. of [24]. For $n \geq 0$ let $\mathcal{K}_n$ be the space of analytic functions on the unit disc with reproducing kernel $k_\lambda(z) = (1-\overline{\lambda}z)^{-n-1}$.

Lemma 6.7. *Let $d \in \mathbb{N}$, $d > 1$. For $\alpha \in \mathbb{N}_0^{d-1}$, $z = (z_1, \ldots, z_{d-1}) \in \mathbb{C}^{d-1}$, let $e_\alpha(z_1, \ldots, z_{d-1}) = \sqrt{\frac{\alpha!}{|\alpha|!}} z^\alpha$. Then the map from $\mathcal{K}_{|\alpha|}$ to H^2_d,*

$$f \mapsto g, \quad g(z) = e_\alpha(z_1, \ldots, z_{d-1}) f(z_d),$$

is an isometry, and

$$H^2_d = \bigoplus_{\alpha \in \mathbb{N}_0^{d-1}} e_\alpha \mathcal{K}_{|\alpha|}.$$

Proof. A direct computation reveals that for all $k \in \mathbb{N}$, we have

$$\|w^k\|_{\mathcal{K}_{|\alpha|}} = \|e_\alpha(z_1, \ldots, z_{d-1}) z_d^k\|_{H^2_d}$$

and since normalized monomials form an orthonormal basis in $\mathcal{K}_{|\alpha|}$, the first assertion follows. It is also clear that the subspaces $e_\alpha \mathcal{K}_{|\alpha|} \subseteq H^2_d$ are pairwise orthogonal and since their sum contains all monomials it must equal the whole space H^2_d. ■

It is important to note that $\mathcal{K}_0 = H^2$, while for $|\alpha| > 0$ $\mathcal{K}_{|\alpha|}$ is a weighted Bergman space with $\mathcal{K}_1 = L^2_a$, the unweighted Bergman space on the unit disc. In particular, if $u \in H^\infty$ then the function

$$v(z_1, \ldots, z_d) = u(z_d), \quad (z_1, \ldots, z_d) \in \mathbb{B}_d,$$

is a multiplier of H^2_d with

$$\|v\|_{\mathrm{Mult}(H^2_d)} = \|u\|_\infty.$$

Finally, for the result below we shall use the well known fact that there exist singular inner functions $\theta \in H^\infty$ such that θ is cyclic in each of the spaces $\mathcal{K}_{|\alpha|}$, $|\alpha| > 0$ (see [28], Ch. 7, theorems 7.3 and 7.12).

Proposition 6.8. *Let $\theta \in H^\infty$ be singular inner such that θ is cyclic in each of the spaces $\mathcal{K}_{|\alpha|}$ for $|\alpha| > 0$, and for $d > 1$ set*

$$\varphi(z_1, \ldots, z_d) = \theta(z_d), \quad (z_1, \ldots, z_d) \in \mathbb{B}_d,$$

Then

$$[\varphi^n] = \varphi^n e_{(0,\ldots,0)} \mathcal{K}_0 \oplus (H^2_d \ominus e_{(0,\ldots,0)} \mathcal{K}_0). \tag{6.1}$$

In particular, $\varphi \in \mathcal{C}_\infty(H^2_d)$ but for $z = (0, \ldots, 0, 1) \in \partial\mathbb{B}_d$ we have that $\varphi_z(\lambda) = \theta(\lambda)$ is an inner function.

Proof. The statement is self-explanatory, since $\varphi^n e_\alpha \mathcal{K}_{|\alpha|}$ is contained and dense in $e_\alpha \mathcal{K}_{|\alpha|}$ when $|\alpha| > 0$, which immediately leads to (6.1). Clearly,

$$\bigcap_{n \geq 1} [\varphi^n] = H^2_d \ominus e_{(0,\ldots,0)} \mathcal{K}_0,$$

i.e. $\varphi \in \mathcal{C}_\infty(H^2_d)$. ■

7. Further questions

We start with the obvious question that we have left open if $d = 3$ or $d \geq 5$.

Question 7.1. *If $d \in \mathbb{N}$, then what is the smallest n such that $\mathbb{C}_{\text{stable}}[z] \subseteq \mathcal{C}_n(H^2_d)$?*

We think of functions in the classes $\mathcal{C}_n(\mathcal{H})$ as H^2_d-analogues of functions without inner factors. With this in mind we formulate a weakened form of the Brown-Shields conjecture for H^2_d, see [14].

Question 7.2. *Given $d \in \mathbb{N}$, is there is $N \in \mathbb{N}$ such that whenever $f \in \mathcal{C}_n(H^2_d)$ for some n, then $f \in \mathcal{C}_N(H^2_d)$?*

There are natural related questions.

Question 7.3. *If $f \in \operatorname{Mult}(H^2_d)$ such that every slice f_z is outer, then is $f \in \mathcal{C}_N(H^2_d)$ for some N?*

We mentioned in the introduction that the analogous questions for the Dirichlet space D have a positive answer. We finish by providing some further evidence that this might extend to H^2_d. Let $A^\infty(\mathbb{D}) = \{f \in C^\infty(\overline{\mathbb{D}}) : f|\mathbb{D} \text{ analytic}\}$. Then $A^\infty(\mathbb{D}) \subseteq \operatorname{Mult}(D_\alpha)$ for all $\alpha \in \mathbb{R}$.

Proposition 7.4. *If $f \in A^\infty(\mathbb{D})$ is outer and if $1 \leq k \leq d$, then $T_{k,d} f \in \mathcal{C}_n(H^2_d)$ for every $n \geq \frac{k-1}{4}$.*

Proof. Let $1 \leq k \leq d$, $n \geq \frac{k-1}{4}$, and let $f \in A^\infty(\mathbb{D})$ be outer. By Lemma 4.2 it suffices to show that $f \in \mathcal{C}_n(D_{(k-1)/2}(\mathbb{D}))$. The choice of n implies that $D_{2n}(\mathbb{D}) \subseteq D_{(k-1)/2}$ with $\|g\|_{D_{(k-1)/2}} \leq \|g\|_{D_{2n}(\mathbb{D})}$ for all $g \in D_{2n}(\mathbb{D})$. Hence it will be enough to show that $f \in \mathcal{C}_n(D_{2n}(\mathbb{D}))$. For the spaces $D_{2n}(\mathbb{D})$ Korenblum determined the invariant subspaces, [32]. Indeed, each invariant subspace that contains an outer function is of the form

$$I(E_0, E_1, \dots, E_{n-1}) = \{f \in D_{2n}(\mathbb{D}) : f^{(j)}(z) = 0 \text{ for } z \in E_j, j \leq n-1\},$$

where $\partial\mathbb{D} \supseteq E_0 \supseteq E_1 \supseteq \dots \supseteq E_{n-1}$ are compact sets such that E_0 is a Carleson set and $E_0 \setminus E_{n-1}$ is discrete. If $E = \{z \in \partial\mathbb{D} : f(z) = 0\}$, then it follows that $[f^n] = [f^{n+1}] = I(E, \dots, E)$. Hence $f \in \mathcal{C}_n(D_{2n}(\mathbb{D}))$. ∎

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