n-CLUSTER TILTING SUBCATEGORIES FOR RADICAL SQUARE ZERO ALGEBRAS

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Abstract. We give a characterization of radical square zero bound quiver algebras $\mathbf{k}Q/\mathcal{J}^2$ that admit n-cluster tilting subcategories and $n\mathbb{Z}$-cluster tilting subcategories in terms of Q. We also show that if Q is not of cyclically oriented extended Dynkin type $\tilde{A}$, then the poset of n-cluster tilting subcategories of $\mathbf{k}Q/\mathcal{J}^2$ with relation given by inclusion forms a lattice isomorphic to the opposite of the lattice of divisors of an integer which depends on Q.

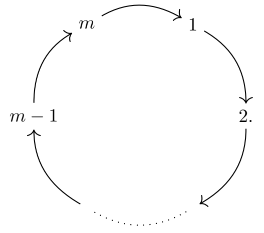
Key words and phrases. Auslander–Reiten theory, n-cluster tilting subcategory, radical square zero algebra, string algebra, representation-finite algebra

Introduction

Representation theory of algebras can be described as the study of the category $\operatorname{mod}\Lambda$ of finite-dimensional (right) modules over an algebra Λ. One of the most helpful tools in that study has been Auslander–Reiten theory. In recent years a higher-dimensional analogue of Auslander–Reiten theory has been introduced by Iyama [Iya07b, Iya07a]; see also [Iya08]. In this theory, instead of focusing on $\operatorname{mod}\Lambda$, one restricts to a suitable subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ called an *n-cluster tilting subcategory* for some positive n. If $\mathcal{C}$ has an additive generator M, then M is called an *n-cluster tilting module*. In this setting one may describe $\mathcal{C}$ using an n-dimensional version of Auslander–Reiten theory.

Every algebra Λ admits a unique 1-cluster tilting subcategory, namely $\operatorname{mod}\Lambda$ itself. On the other hand, if $n \geq 2$, then an n-cluster tilting subcategory may not exist. Generally, it is not easy to find algebras which admit n-cluster tilting subcategories. Recently there has been a lot of research in trying to find or construct n-cluster tilting subcategories, see for example [IO11, HI11, IO13, CIM19, JKPK19, CDIM20].

For simplicity, we assume that all quivers in this article are connected; the results of this paper can be straightforwardly generalised for quivers which are not connected. For an integer $m \in \mathbb{Z}_{\geq 1}$ we denote by A_m the quiver $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow m$ and by $\tilde{A}_m$ the quiver



For a quiver Q we denote by $\mathcal{J}$ the ideal of the path algebra $\mathbf{k}Q$ generated by the arrows of Q. One may then ask for which integers $l \geq 2$ does the bound quiver algebra $\Lambda = \mathbf{k}Q/\mathcal{J}^l$ admit an n-cluster tilting subcategory. Several results are known in that direction. The case $Q = A_m$ has been studied in

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[Vas19], while the case $Q = \tilde{A}_m$ has been studied in [DI20]. The case where n is the global dimension of Λ was studied in [ST21]. In this article we consider the case where $l = 2$. As a first result we have the following theorem.

Theorem A (Proposition 2.9 and Theorem 2.10). *Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and $n \geq 2$. If Λ admits an n-cluster tilting subcategory, then Λ is a representation-finite string algebra. Moreover, if X is an indecomposable Λ-module and X is not simple, then X is projective or injective.*

Theorem A shows that a radical square zero bound quiver algebra Λ which admits an n-cluster tilting subcategory is well-understood from the point of view of representation theory. In particular, since Λ is representation-finite, every n-cluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is of the form $\mathcal{C} = \operatorname{add}(M)$ for an n-cluster tilting module $M \in \operatorname{mod}\Lambda$. We then give the following characterization, which is the main result of this paper.

Theorem B (Theorem 4.1). *Let $\Lambda = \mathbf{k}Q/J^2$ and $n \geq 2$. Then Λ admits an n-cluster tilting subcategory $\mathcal{C}$ if and only if Q is an n-admissible quiver. If moreover $Q \neq \tilde{A}_m$, then $\mathcal{C}$ is unique and $\mathcal{C} = \operatorname{add}\left(\bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda)\right)$ where $\tau_n^- = \tau^-\Omega^{-(n-1)}$.*

For the definition of n-admissible quivers we refer to Definition 2.6 and Definition 3.1; we refer to Remark 4.4 for an easy way to construct n-admissible quivers. Given Theorem B, it is not hard to classify radical square zero bound quiver algebras which admit $n\mathbb{Z}$-cluster tilting subcategories in the sense of [IJ17].

Theorem C (Theorem 4.7). *Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and $n \geq 2$. Then Λ admits an $n\mathbb{Z}$-cluster tilting subcategory if and only if $Q = A_m$ and $n \mid (m-1)$ or $Q = \tilde{A}_m$ and $n \mid m$.*

Finally, we show that if $Q \neq \tilde{A}_m$, then the set of n-cluster tilting subcategories of $\mathbf{k}Q/\mathcal{J}^2$ forms a lattice isomorphic to the lattice of divisors of a certain integer which depends only on Q.

Theorem D (Theorem 4.12). *Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$. Assume that $Q \neq \tilde{A}_m$ and that Q has admissible degree N. Set*

$$\mathbf{CT}(\Lambda) := \{\mathcal{C} \subseteq \operatorname{mod}\Lambda \mid \text{there exists } n \in \mathbb{Z}_{\geq 1} \text{ such that } \mathcal{C} \text{ is } n\text{-cluster tilting}\}.$$

Then $(\mathbf{CT}(\Lambda), \subseteq)$ is a complete lattice isomorphic to the opposite of the lattice of divisors of N.

For the definition of the admissible degree of a quiver we refer to Definition 4.11.

This paper is organized as follows. In Section 1 we establish notation and include some general results about n-cluster tilting subcategories and radical square zero algebras. In Section 2 we find some necessary conditions for a radical square zero bound quiver algebra to admit an n-cluster tilting subcategories. In Section 3 we show that these necessary conditions are also sufficient. In Section 4 we state our main result and a few applications.

1. Preliminaries and notation

Let $\mathbf{k}$ be a field. By an algebra we mean a finite-dimensional associative $\mathbf{k}$-algebra with a unit and by a module we mean a finite-dimensional right module.

Let Λ be an algebra. The *(Jacobson) radical* $\operatorname{rad}(\Lambda)$ of Λ is the intersection of all the maximal right ideals of Λ. The algebra Λ is a *radical square zero* algebra if $\operatorname{rad}^2(\Lambda) = 0$. We denote by $\operatorname{mod}\Lambda$ the category of Λ-modules. A Λ-module $M \in \operatorname{mod}\Lambda$ is called *basic* if all indecomposable direct summands of M are pairwise non-isomorphic. For $M \in \operatorname{mod}\Lambda$ we denote by $\Omega(M)$ the *syzygy* of M, that is the kernel of $P(M) \twoheadrightarrow M$, where $P(M)$ is the *(minimal) projective cover* of M and by $\Omega^-(M)$ the *cosyzygy* of M, that is the cokernel of $M \hookrightarrow I(M)$ where $I(M)$ is the *injective hull* of M. Note that $\Omega(M)$ and $\Omega^-(M)$ are unique up to isomorphism.

We denote by D the duality $\operatorname{Hom}(-, \mathbf{k})$ between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$. We denote by τ and τ^- the *Auslander–Reiten translations* and we recall the *Auslander–Reiten duality*

$$\operatorname{Ext}^1_\Lambda(M, N) \cong D\underline{\operatorname{Hom}}_\Lambda(\tau^-(N), M),$$

for all $M, N \in \operatorname{mod}\Lambda$, where $\underline{\operatorname{Hom}}_\Lambda(-,-)$ denotes morphisms in the *projectively stable category* $\underline{\operatorname{mod}}\Lambda$. For more details about the representation theory of finite-dimensional algebras and Auslander–Reiten theory we refer to [ARS95, ASS06].

Throughout this article n denotes a positive integer. A subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is called *n-rigid* if $\operatorname{Ext}^i_\Lambda(\mathcal{C},\mathcal{C}) = 0$ for all $i \in \{1, \dots, n-1\}$. A functorially finite subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is called *n-cluster tilting* if

$$\begin{aligned}\mathcal{C} &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,\mathcal{C}) = 0 \text{ for all } 0 < i < n\}\\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C},X) = 0 \text{ for all } 0 < i < n\}.\end{aligned}$$

If moreover $\mathcal{C} = \operatorname{add}(M)$ for a module $M \in \operatorname{mod}\Lambda$, then M is called an *n-cluster tilting module*. Notice that any category of the form $\operatorname{add}(M)$ for some $M \in \operatorname{mod}\Lambda$ is functorially finite. In particular, if Λ is representation-finite, then any subcategory of $\operatorname{mod}\Lambda$ is functorially finite. Clearly if $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is n-cluster tilting, then $\Lambda, D(\Lambda) \in \mathcal{C}$; we use this fact throughout. We denote by τ_n and τ_n^- the *n-Auslander–Reiten translations* defined by $\tau_n = \tau\Omega^{n-1}$ and $\tau_n^- = \tau^-\Omega^{-(n-1)}$. For more details about higher dimensional Auslander–Reiten theory we refer to [Iya08].

Notice that there exists a unique 1-cluster tilting subcategory of $\operatorname{mod}\Lambda$, namely $\operatorname{mod}\Lambda$ itself. In the rest of this paper we assume that $n \geq 2$, unless otherwise stated. We also need the following observations.

Proposition 1.1. Let Λ be a finite-dimensional algebra and let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ be an n-cluster tilting subcategory.

(a) The functors $\tau_n : \mathcal{C}_\mathcal{P} \to \mathcal{C}_\mathcal{I}$ and $\tau_n^- : \mathcal{C}_\mathcal{I} \to \mathcal{C}_\mathcal{P}$ induce mutually inverse bijections, between the set $\mathcal{C}_\mathcal{P}$ of isomorphism classes of indecomposable nonprojective Λ-modules and the set $\mathcal{C}_\mathcal{I}$ of isomorphism classes of indecomposable noninjective Λ-modules.
(b) If $\mathcal{D} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory such that $\mathcal{D} \subseteq \mathcal{C}$, then $\mathcal{C} = \mathcal{D}$.
(c) Let $M = \bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda)$. Then $M \in \mathcal{C}$. If moreover M is an n-cluster tilting module, then $\mathcal{C} = \operatorname{add}(M)$.

Proof. (a) See [Iya08, Theorem 2.8].
(b) Follows directly from the definition of n-cluster tilting subcategories.
(c) Since $\Lambda \in \mathcal{C}$, we have that $M \in \mathcal{C}$ by (a). In particular we have $\operatorname{add}(M) \subseteq \mathcal{C}$. Hence if M is an n-cluster tilting module, then by (b) we conclude that $\mathcal{C} = \operatorname{add}(M)$. □

Lemma 1.2. Let Λ be a finite-dimensional algebra and $M, N \in \operatorname{mod}\Lambda$ with $M \neq 0$. Assume that $\tau_x^-(N) \cong M$ for some $x \geq 1$. Then $\operatorname{Ext}^x_\Lambda(M,N) \neq 0$.

Proof. We first consider the case $x = 1$. By additivity of τ^- and $\operatorname{Ext}^x_\Lambda(-,-)$ we may assume that M and N are indecomposable. Since $\tau^-(N) \cong M$ and M is nonzero, it follows that N is noninjective. Then there exists an almost split sequence $0 \longrightarrow N \longrightarrow F \longrightarrow \tau^-(N) \longrightarrow 0$ in $\operatorname{mod}\Lambda$ and the result follows. For $x \geq 2$ we have using dimension shift that

$$\operatorname{Ext}^x_\Lambda(M,N) \cong \operatorname{Ext}^1_\Lambda(\tau_x^-(N), \Omega^{-(x-1)}(N)) = \operatorname{Ext}^1_\Lambda(\tau^-(\Omega^{-(x-1)}(N)), \Omega^{-(x-1)}(N)) \neq 0,$$

where the last inequality follows from the case $x = 1$. □

Next we recall some background on bound quiver algebras. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of a set Q_0 of *vertices*, a set Q_1 of *arrows* and two maps $s, t : Q_1 \to Q_0$ called *source map* and *target map*. All quivers in this article are *finite*, that is, both Q_0 and Q_1 are finite sets. Moreover, for simplicity, we assume that all quivers in this article are *connected*, that is, the underlying unoriented graph of Q is connected. For a vertex $v \in Q_0$, the *incoming degree* of v, denoted by $\delta^-(v)$, is the number of arrows ending at v and the *outgoing degree* of v, denoted by $\delta^+(v)$, is the number of arrows starting at v. The *degree* of v is the tuple $(\delta^-(v), \delta^+(v))$. For a quiver Q and $k \geq 1$, a *path* $\mathbf{p}$ *of length* k in Q is a sequence of k consecutive arrows

$$\mathbf{p} = \; v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \longrightarrow v_k \xrightarrow{\alpha_k} v_{k+1},$$

in Q. We also assign a trivial path ϵ_v of length 0 to each vertex $v \in Q_0$.

Let Q be a quiver. We denote by $\mathbf{k}Q$ the *path algebra* of Q and we denote by $\mathcal{J} \subseteq \mathbf{k}Q$ the *arrow ideal* of Q, that is the ideal of $\mathbf{k}Q$ generated by the arrows of Q. An ideal $\mathcal{I} \subseteq \mathbf{k}Q$ is called *admissible* if there exists $k \geq 2$ such that $\mathcal{J}^k \subseteq \mathcal{I} \subseteq \mathcal{J}^2$. If $\mathcal{I}$ is an admissible ideal, then the *bound quiver algebra* $\Lambda = \mathbf{k}Q/\mathcal{I}$ is a finite-dimensional algebra. Throughout we identify Λ-modules and representations of Q bound by $\mathcal{I}$. For a vertex $v \in Q_0$, we denote by $P(v)$, $I(v)$ and $S(v)$ the indecomposable projective, injective and simple Λ-modules corresponding to v. When clear from context, we use composition series to denote Λ-modules. For more details on bound quiver algebras and their representation theory we refer to [ARS95, ASS06].

For radical square zero algebras we have the following easy observations.

Lemma 1.3. Let Λ be a radical square zero algebra and let M be a nonprojective Λ-module. Then $\Omega(M)$ is semisimple.

Proof. Since M is nonprojective, it follows that $\Omega(M) \neq 0$. Let $P(M)$ be the projective cover of M. Then $\mathrm{rad}^2(P(M)) = P(M)\,\mathrm{rad}^2(\Lambda) = 0$ and so $\mathrm{rad}(P(M))$ is semisimple. Since $\Omega(M)$ is a submodule of $\mathrm{rad}(P(M))$ and $\Omega(M) \neq 0$, we conclude that $\Omega(M)$ is semisimple. □

Lemma 1.4. Let Λ be a radical square zero algebra and assume that $\mathcal{C} \subseteq \mathrm{mod}\,\Lambda$ is an n-cluster tilting subcategory. Let I be an indecomposable injective Λ-module. Then $\dim(\Omega(I)) \leq 1$.

Proof. If I is projective, then $\dim(\Omega(I)) = 0$. Otherwise, assume that I is nonprojective. By Lemma 1.3 we have that $\Omega(I)$ is semisimple. By [Vas19, Corollary 3.3] and since $I \in \mathcal{C}_{\mathcal{P}}$, we have that $\Omega(I)$ is indecomposable. Since $\Omega(I)$ is semisimple and indecomposable, it follows that $\dim(\Omega(I)) \leq 1$. □

In this paper we study radical square zero bound quiver algebras. These can be easily described as in the following lemma.

Lemma 1.5. A bound quiver algebra $\mathbf{k}Q/\mathcal{I}$ is a radical square zero algebra if and only if $\mathcal{I} = \mathcal{J}^2$.

Proof. Since $\mathcal{I}$ is admissible, we have that $\mathcal{I} = \mathcal{J}^2$ if and only if $\mathcal{J}^2 \subseteq \mathcal{I}$, which is equivalent to the ideal $(\mathcal{J}/\mathcal{I})^2$ being equal to the zero ideal. Since $(\mathcal{J}/\mathcal{I})^2 = \mathrm{rad}^2(\mathbf{k}Q/\mathcal{I})$, the result follows. □

As a corollary, any radical square zero algebra over an algebraically closed field is Morita equivalent to a bound quiver algebra of the form $\mathbf{k}Q/\mathcal{J}^2$.

Proposition 1.6. Let Λ be a basic and connected finite-dimensional $\mathbf{k}$-algebra and assume that $\mathbf{k}$ is algebraically closed. Then Λ is a radical square zero algebra if and only if $\Lambda \cong \mathbf{k}Q/\mathcal{J}^2$ for some quiver Q.

Proof. Since Λ is basic and $\mathbf{k}$ is algebraically closed, there exists a quiver Q and an admissible ideal $\mathcal{I} \subseteq \mathbf{k}Q$ such that $\Lambda \cong \mathbf{k}Q/\mathcal{I}$. The result follows from Lemma 1.5. □

We also need to recall the following notion.

Definition 1.7. A bound quiver algebra $\mathbf{k}Q/\mathcal{I}$ is a *string algebra* if the following conditions hold:

(S1) For every vertex $v \in Q_0$ we have that $\delta^-(v) \leq 2$ and $\delta^+(v) \leq 2$.
(S2) For every arrow $\alpha \in Q_1$ there exists at most one arrow $\beta \in Q_1$ such that $\beta\alpha \notin \mathcal{I}$ and at most one arrow $\gamma \in Q_1$ such that $\alpha\gamma \notin \mathcal{I}$.
(S3) The ideal $\mathcal{I}$ can be generated by paths.

Indecomposable modules over string algebras are classified in [BR87] using the combinatorics of strings and bands. We briefly recall these combinatorics.

Let $\mathbf{k}Q/\mathcal{I}$ be a string algebra. For every arrow $\alpha \in Q_1$ we define a formal inverse α^- such that $s(\alpha^-) = t(\alpha)$ and $t(\alpha^-) = s(\alpha)$. We define $Q_1^- = \{\alpha^- \mid \alpha \in Q_1\}$ and we set $(\alpha^-)^- = \alpha$. We call elements of Q_1 *direct arrows* and elements of Q_1^- *inverse arrows*. A *formal path of length* $k \geq 1$ is a sequence $\boldsymbol{\ell} = \ell_k \dots \ell_1$ such that $\ell_i \in Q_1 \cup Q_1^-$ and such that for all $i \in \{1, \dots, k-1\}$ we have $t(\ell_i) = s(\ell_{i+1})$ and $\ell_i \neq \ell_{i+1}^-$. We also set $\boldsymbol{\ell}^- = \ell_1^- \dots \ell_k^-$. We say that $\boldsymbol{\ell}$ is a *string of length* k if no formal path of the form $\ell_{i+r}\dots\ell_i$ or $\ell_i^-\dots\ell_{i+r}^-$ is in $\mathcal{I}$ for $1\leq i\leq i+r\leq k$. To each vertex $v\in Q_0$ we also associate a string $\boldsymbol{e_v}$ of length 0. We say that a string $\boldsymbol{\ell}$ is a *band* if $s(\ell_1)=t(\ell_k)$ and $\boldsymbol{\ell}^q$ is a string for every $q\geq 1$, and moreover there is no string $\boldsymbol{\ell'}\neq\boldsymbol{\ell}$ such that $\boldsymbol{\ell'}\dots\boldsymbol{\ell'}=\boldsymbol{\ell}$. For each string or band $\boldsymbol{\ell}$ we can define a corresponding string or band module $M(\boldsymbol{\ell})$ which is indecomposable. Furthermore, every indecomposable $\mathbf{k}Q/\mathcal{I}$-module is isomorphic to a string or band module. For more details on the definition of $M(\boldsymbol{\ell})$ and on other facts about the representation theory of string algebras we refer to [BR87, Section 3].

2. Necessary conditions

In this section we investigate the existence of an n-cluster tilting subcategory $\mathcal{C}\subseteq \operatorname{mod}\Lambda$ where $\Lambda=\mathbf{k}Q/\mathcal{J}^2$ and $n\geq 2$. Our aim is to show that these assumptions impose some important restrictions on Q and Λ.

2.1. **n-pre-admissible quivers.** Recall that if $\mathcal{C}\subseteq\operatorname{mod}\Lambda$ is an n-cluster tilting subcategory, then $\Lambda, D(\Lambda)\in\mathcal{C}$. We start with showing that the degree of a vertex in Q is bounded.

Lemma 2.1. Let $\Lambda=\mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C}\subseteq\operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let $v\in Q_0$ be a vertex. Then $\delta^-(v)\leq 2$ and $\delta^+(v)\leq 2$.

Proof. We only show that $\delta^-(v)\leq 2$; the inequality $\delta^+(v)\leq 2$ follows dually. Consider the short exact sequence $0\longrightarrow\Omega(I(v))\longrightarrow P(I(v))\longrightarrow I(v)\longrightarrow 0$. Then

$$\dim(I(v))=\dim(P(I(v)))-\dim(\Omega(I(v)))\geq\dim(P(I(v)))-1, \tag{2.1}$$

where the last inequality follows from Lemma 1.4. By definition, the indecomposable injective module $I(v)$ has a $\mathbf{k}$-vector space basis given by the set $\{w+\mathcal{J}^2\mid w \text{ a path in } Q \text{ ending at } v\}$, see for example [ASS06, Lemma III.2.6(b)]. Therefore, a $\mathbf{k}$-vector space basis of $I(v)$ is given by all paths in Λ which end at v and have length at most 1. Since there is a unique path of length 0 ending at v (the trivial path at v) and $\delta^-(v)$ arrows ending at v, it follows that $\dim(I(v))=\delta^-(v)+1$.

Next, since $\operatorname{rad}^2(I(v))=I(v)\operatorname{rad}^2(\Lambda)=0$, it follows that $\operatorname{rad}(I(v))$ is semisimple. But since $I(v)$ is the indecomposable injective module corresponding to the vertex v, it follows that $S(v)$ is the unique simple submodule of $I(v)$ and so $\operatorname{rad}(I(v))=S(v)$. Hence $\operatorname{top}(I(v))=I(v)/S(v)=\bigoplus_{\alpha:u\to v}S(u)$. It follows that $P(I(v))=\bigoplus_{\alpha:u\to v}P(u)$. But for each arrow $\alpha:u\to v$ we have that $\dim(P(u))\geq 2$ since u is not a sink. Hence $\dim(P(I(v)))=\sum_{\alpha:u\to v}\dim(P(u))\geq\sum_{\alpha:u\to v}2=2\delta^-(v)$.

By replacing $\dim(I(v))=\delta^-(v)+1$ and $\dim(P(I(v)))\geq 2\delta^-(v)$ in (2.1) we obtain

$$\delta^-(v)+1\geq 2\delta^-(v)-1,$$

or equivalently $2\geq\delta^-(v)$. □

We continue with showing that there are no multiple arrows between two vertices of Q.

Lemma 2.2. Let $\Lambda=\mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C}\subseteq\operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let $v,u\in Q_0$ be vertices. Then $|\{\alpha\in Q_1\mid s(\alpha)=v \text{ and } t(\alpha)=u\}|\leq 1$.

Proof. By Lemma 2.1 we have that $|\{\alpha\in Q_1\mid s(\alpha)=v, t(\alpha)=u\}|\leq 2$. Assume towards a contradiction that there exist two arrows $\alpha_1:v\longrightarrow u$ and $\alpha_2:v\longrightarrow u$. Then, by Lemma 2.1 we have that the composition series of $I(u)$ is ${}^{v}{}_{u}{}^{v}$ while the composition series of $P(v)$ is ${}_{u}{}^{v}{}_{u}$. Hence the projective cover of $I(u)$ is $P(I(u))\cong P(v)\oplus P(v)$ and

$$\dim(\Omega(I(u)))=\dim(P(I(u)))-\dim(I(u))=2\dim(P(v))-\dim(I(u))=2\cdot 3-3=3,$$

which contradicts Lemma 1.4. □

Next we show that no vertex can have degree $(0,2)$ or $(2,0)$ and, moreover, that if $n>2$, then no vertex can have degree $(2,2)$ either.

Lemma 2.3. Let $\Lambda=\mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C}\subseteq\operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let $v\in Q_0$ be a vertex.

(a) If $\delta^+(v) = 2$, then $\delta^-(v) \geq 1$.
(b) If $\delta^-(v) = 2$, then $\delta^+(v) \geq 1$.
(c) If $\delta^-(v) = \delta^+(v) = 2$, then $n = 2$.

Proof. (a) Since $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and $\delta^+(v) = 2$, it follows that $\dim(P(v)) = 3$. Assume towards a contradiction that $\delta^-(v) = 0$. Then $I(v) = S(v)$ and $P(I(v)) = P(v)$. Hence

$$\dim(\Omega(I(v))) = \dim(P(I(v))) - \dim(I(v)) = 3 - 1 = 2,$$

which contradicts Lemma 1.4.

(b) Dual to (a).

(c) Let $\alpha_1 : v \longrightarrow u_1$, $\alpha_2 : v \longrightarrow u_2$, $\beta_1 : w_1 \longrightarrow v$ and $\beta_2 : w_2 \longrightarrow v$ be the arrows starting and ending at v. Then the composition series of $P(v)$ is $\begin{smallmatrix} & v & \\ u_1 & & u_2 \end{smallmatrix}$ while the composition series of $I(v)$ is $\begin{smallmatrix} w_1 & & w_2 \\ & v & \end{smallmatrix}$. For $i = 1, 2$, let $\pi_i : P(v) \longrightarrow \begin{smallmatrix} v \\ u_i \end{smallmatrix}$ be the projective cover of $\begin{smallmatrix} v \\ u_i \end{smallmatrix}$ and $\iota_i : \begin{smallmatrix} w_i \\ v \end{smallmatrix} \longrightarrow I(v)$ be the injective envelope of $\begin{smallmatrix} w_i \\ v \end{smallmatrix}$. Then it follows that

$$\operatorname{coker}\left(\begin{bmatrix}\pi_1 \\ \pi_2\end{bmatrix} : P(v) \longrightarrow \begin{smallmatrix} v \\ u_1 \end{smallmatrix} \oplus \begin{smallmatrix} v \\ u_2 \end{smallmatrix}\right) \cong S(v) \cong \ker\left(\begin{bmatrix}\iota_1 & \iota_2\end{bmatrix} : \begin{smallmatrix} w_1 \\ v \end{smallmatrix} \oplus \begin{smallmatrix} w_2 \\ v \end{smallmatrix} \longrightarrow I(v)\right).$$

Hence the sequence

$$\begin{array}{c} 0 \longrightarrow P(v) \xrightarrow{\begin{bmatrix}\pi_1 \\ \pi_2\end{bmatrix}} \begin{smallmatrix} v \\ u_1 \end{smallmatrix} \oplus \begin{smallmatrix} v \\ u_2 \end{smallmatrix} \longrightarrow \begin{smallmatrix} w_1 \\ v \end{smallmatrix} \oplus \begin{smallmatrix} w_2 \\ v \end{smallmatrix} \xrightarrow{\begin{bmatrix}\iota_1 & \iota_2\end{bmatrix}} I(v) \longrightarrow 0 \\ \searrow\!\!\!\!\searrow \quad \nearrow \\ S(v) \end{array}$$

gives a nonzero element of $\operatorname{Ext}^2_\Lambda(I(v), P(v))$. Since $I(v), P(v) \in \mathcal{C}$ and $\operatorname{Ext}^2_\Lambda(I(v), P(v)) \neq 0$, it follows that $n \leq 2$. Since by assumption we have that $n \geq 2$, we conclude that $n = 2$. □

Finally we examine how an arrow between two vertices affects the degree of the two vertices.

Lemma 2.4. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory.

(a) Let $w_1 \longrightarrow v \longleftarrow w_2$ be a subquiver of Q. Then $\delta^+(w_1) = \delta^+(w_2) = 1$.
(b) Let $u_1 \longleftarrow v \longrightarrow u_2$ be a subquiver of Q. Then $\delta^-(u_1) = \delta^-(u_2) = 1$.

Proof. We only prove (a); (b) follows dually. By symmetry it is enough to show that $\delta^+(w_1) = 1$. Since by Lemma 2.1 we have that $\delta^+(w_1) \leq 2$ and since by assumption we have that $\delta^+(w_1) \geq 1$, it is enough to show that $\delta^+(w_1) \neq 2$. Assume towards a contradiction that $\delta^+(w_1) = 2$. Since $\Lambda = \mathbf{k}Q/\mathcal{J}^2$, it follows from Lemma 2.1 that $\dim(I(v)) = 3$, $\dim(P(w_1)) = 3$, $\dim(P(w_2)) \geq 2$ and $P(I(v)) \cong P(w_1) \oplus P(w_2)$. Then

$$\dim(\Omega(I(v))) = \dim(P(I(v))) - \dim(I(v)) = \dim(P(w_1)) + \dim(P(w_2)) - \dim(I(v)) \geq 3 + 2 - 3 = 2,$$

which contradicts Lemma 1.4. □

Corollary 2.5. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let $v \longrightarrow u$ be an arrow in Q. Then $\delta^+(v) + \delta^-(u) \leq 3$.

Proof. Follows immediately by Lemma 2.1 and Lemma 2.4. □

The results of this section motivate the following definition.

Definition 2.6. A quiver Q is called *n-pre-admissible* if the following conditions are satisfied.

(i) For all vertices $v \in Q_0$ we have $\delta(v) \in \{(0,0), (0,1), (1,0), (1,1), (1,2), (2,1)\} \cup E$, where
$$E = \begin{cases} \{(2,2)\}, & \text{if } n = 2, \\ \varnothing, & \text{otherwise.} \end{cases}$$
(ii) There exist no multiple arrows between two vertices.
(iii) For all arrows $v \longrightarrow u$ in Q we have $\delta^+(v) + \delta^-(u) \leq 3$.

Remark 2.7. It follows immediately by the definition of n-pre-admissible quivers that an n-pre-admissible quiver which has no vertex of degree $(2,2)$ is n-pre-admissible for any $n \geq 2$.

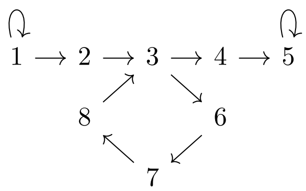
Example 2.8. (a) The quivers A_m and $\tilde{A}_m$ are n-pre-admissible for any $n \geq 2$.

(b) The quiver

$$1 \longrightarrow 2 \circlearrowright \longrightarrow 3$$

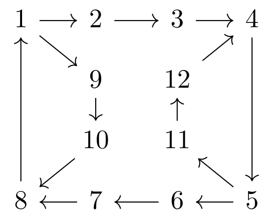
is not n-pre-admissible for any $n \geq 2$ since there exists an arrow $2 \longrightarrow 2$, but $\delta^+(2)+\delta^-(2) = 4$.

(c) The quiver



is 2-pre-admissible but not n-pre-admissible for $n \geq 3$ since $\delta(3) = (2,2)$.

(d) The quiver



is n-pre-admissible for any $n \geq 2$.

We have the following immediate result.

Proposition 2.9. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ is an n-cluster tilting subcategory. Then Q is n-pre-admissible.

Proof. Follows immediately by Lemma 2.1, Lemma 2.3, Lemma 2.2 and Corollary 2.5. □

Radical square zero bound quiver algebras with n-pre-admissible quivers are especially easy to study from the point of view of representation theory. Indeed, we have the following result.

Theorem 2.10. *Let Q be an n-pre-admissible quiver and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$.*

(a) Λ is a string algebra.
(b) If $\ell_k \dots \ell_1$ is a string in Λ, then $k \leq 2$. In particular, there are no bands in Λ.
(c) Λ is representation-finite.
(d) If M is an indecomposable Λ-module and M is not simple, then M is projective or injective.

Proof. (a) Since Q is n-pre-admissible, we have that $\delta^+(v) \leq 2$ and $\delta^-(v) \leq 2$ for every vertex $v \in Q_0$. Since $\mathcal{J}^2$ is generated by all paths of length 2, it immediately follows that Λ is a string algebra.

(b) Let $\ell_k \dots \ell_1$ be a string in Λ and assume towards a contradiction that $k \geq 3$. Consider the string $\ell_3\ell_2\ell_1$. Since every path of length two is in $\mathcal{J}^2$, it follows that ℓ_1 and ℓ_2 cannot be both direct or both inverse letters. Similarly ℓ_3 and ℓ_2 cannot be both direct or both inverse letters. Hence $\ell_3\ell_2\ell_1$ is either of the form $\alpha\beta^-\gamma$ or of the form $\alpha^-\beta\gamma^-$ for some arrows $\alpha, \beta, \gamma \in Q_1$ with $\alpha \neq \beta$ and $\gamma \neq \beta$. If $\alpha = \gamma$, then we readily get that $s(\alpha) = s(\beta)$ and $t(\alpha) = t(\beta)$, which contradicts Definition 2.6(ii). Otherwise, if $\alpha \neq \gamma$, then we readily get that

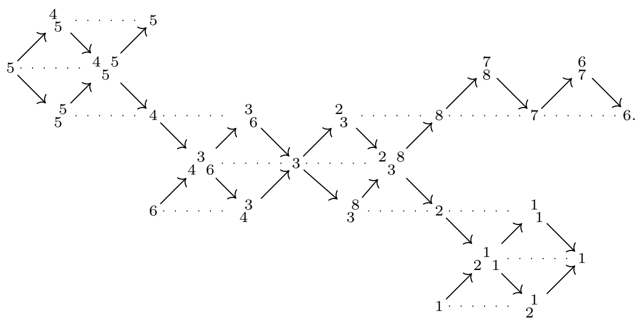
$$\delta^+(s(\beta)) + \delta^-(t(\beta)) \geq 4,$$

which contradicts Definition 2.6(iii). Hence $k \leq 2$. Since the length of a string is bounded by 2, it follows that there are no bands in Λ.

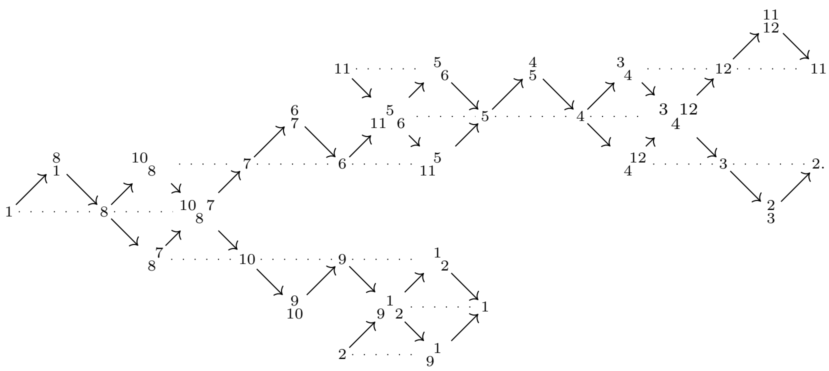
(c) Follows immediately from (b) since indecomposable Λ-modules are classified by string and band modules, see [BR87, Section 3].

(d) Let M be an indecomposable Λ-module and assume that M is not simple. From (b) it follows that M is isomorphic to a string module $M(\boldsymbol{\ell})$ where $\boldsymbol{\ell}$ has length at most 2 (for the definition of $M(\boldsymbol{\ell})$ we refer to [BR87, Section 3]). Since M is not simple, it follows that $\boldsymbol{\ell}$ has length different than 0 and so $\boldsymbol{\ell}$ has length 1 or 2. If the length of $\boldsymbol{\ell}$ is 1, then $\boldsymbol{\ell}=\alpha$ for some arrow $\alpha\in Q_1$ (the modules $M(\alpha)$ and $M(\alpha^-)$ are isomorphic). Let $\alpha: v\longrightarrow u$. Then $\delta^+(v)\in\{1,2\}$ and using Definition 2.6(iii) and the fact that Λ is a radical square zero algebra it is easy to see that
$$M(\alpha)\cong\begin{cases}P(v), & \text{if } \delta^+(v)=1,\\ I(u), & \text{if } \delta^+(v)=2.\end{cases}$$
If the length of $\boldsymbol{\ell}$ is 2, and since Λ is a radical square zero algebra, then $\boldsymbol{\ell}=\alpha\beta^-$ or $\boldsymbol{\ell}=\alpha^-\beta$ for some arrows $\alpha,\beta\in Q_1$. Let $\alpha: v\longrightarrow u$. Similarly to the case of length 1, it is easy to see that
$$M(\boldsymbol{\ell})\cong\begin{cases}P(v), & \text{if } \boldsymbol{\ell}=\alpha\beta^-,\\ I(u), & \text{if } \boldsymbol{\ell}=\alpha^-\beta,\end{cases}$$
and so in both cases M is projective or injective. □

Example 2.11. (a) Let Q be as in Example 2.8(c). The Auslander–Reiten quiver of $\mathbf{k}Q/\mathcal{J}^2$ is



(b) Let Q be as in Example 2.8(d). The Auslander–Reiten quiver of $\mathbf{k}Q/\mathcal{J}^2$ is



2.2. **Flow paths.** We have seen that if $\Lambda=\mathbf{k}Q/\mathcal{J}^2$ and $\mathcal{C}\subseteq\operatorname{mod}\Lambda$ is an n-cluster tilting subcategory, then Q is n-pre-admissible. The opposite is not true in general. It turns out that there are additional properties that Q must satisfy. To describe these properties we need to consider certain paths in Q.

Definition 2.12. Let Q be an n-pre-admissible quiver and let $k\geq 2$. A *$(k$-$)$flow path* $\mathbf{v}$ *in* Q is a path
$$\mathbf{v}=\ v_1\longrightarrow v_2\longrightarrow\cdots\longrightarrow v_{k-1}\longrightarrow v_k, \tag{2.2}$$
such that $\delta(v_s)=(1,1)$ if and only if $1<s<k$.

Notice that since Q is n-pre-admissible, there are no multiple arrows between two vertices. Hence a flow path is defined uniquely by its vertices and we do not need to label arrows in a flow path. For a flow path $\mathbf{v}$ in Q we use v_i to denote its vertices as in (2.2). Moreover, in what follows we write "k-flow path" when the length k of the flow path is important and "flow path" otherwise. Many of the results presented in this section have a dual version which although we usually omit for brevity, we sometimes use. We first study the case where there are no flow paths in Q.

Lemma 2.13. Let Q be an n-pre-admissible quiver. Then there exists a flow path in Q if and only if $Q \neq A_1$ and $Q \neq \tilde{A}_m$ for some $m \geq 1$.

Proof. It is clear by the definition of a flow path that if there exists a flow path in Q, then $Q \neq A_1$ and $Q \neq \tilde{A}_m$. For the other direction, assume that $Q \neq A_1$ and $Q \neq \tilde{A}_m$ and we show that there exists a flow path in Q. Since Q is connected and $Q \neq \tilde{A}_m$, there exists a vertex v_1 in Q with degree $\delta(v_1) \neq (1,1)$. Since $Q \neq A_1$, we have $\delta(v_1) \neq (0,0)$. Since Q is finite, any path starting or ending at v_1 eventually passes through a vertex v_k with $\delta(v_k) \neq (1,1)$; let $\mathbf{v}$ be a minimal such path. Then $\mathbf{v}$ is a flow path by definition. □

Proposition 2.14. Let Q be an n-pre-admissible quiver and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$.

(a) If $Q = A_1$, then $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ if and only if $\mathcal{C} = \operatorname{mod}\Lambda = \operatorname{add}(\Lambda)$.

(b) If $Q = \tilde{A}_m$ for some $m \geq 1$, then $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ if and only if $n \mid m$ and $\mathcal{C} = \operatorname{add}\left(\Lambda \oplus \left(\bigoplus_{j=0}^{\frac{m}{n}-1} \tau_n^{-j}(S)\right)\right)$ for some simple module $S \in \operatorname{mod}\Lambda$.

Proof. (a) In this case $\Lambda = \mathbf{k}$ and the result is clear.

(b) Follows from [DI20, Theorem 5.1]. □

By Lemma 2.13 we have that the only n-pre-admissible quivers that do not have flow paths are the quivers A_1 and $\tilde{A}_m$ for $m \geq 1$. Proposition 2.14 classifies radical square zero bound quiver algebras with such quivers that admit n-cluster tilting subcategories. Hence it remains to study n-pre-admissible quivers that have flow paths. For the rest of this section we fix an n-pre-admissible quiver Q such that $Q \neq A_1$ and $Q \neq \tilde{A}_m$ for any $m \geq 1$. It then follows that there exists a flow path in Q. We further set $\Lambda := \mathbf{k}Q/\mathcal{J}^2$. We start with some simple but important observations about flow paths.

Lemma 2.15. Let $\mathbf{v}$ be a k-flow path in Q. Let $1 \leq s \leq t \leq k$.

(a) If $1 < s$ and $t < k$, then $v_s = v_t$ if and only if $s = t$.

(b) If $s < t$ and $v_s = v_t$, then $s = 1$ and $t = k$. In particular, in this case $v_1 = v_k$.

Proof. (a) If $s = t$, then clearly $v_s = v_t$. Assume towards a contradiction that $v_s = v_t$ but $s < t$. Without loss of generality, we may assume that $s < t$ are minimal among $\{2, \dots, k-1\}$ with these properties. By the definition of a k-flow path and since $\delta(v_s) = \delta(v_t) = (1,1)$, it follows that $v_{s-1} = v_{t-1}$. By minimality of s and t we conclude that $s - 1 = 1$. Moreover, we have $1 < s \leq t-1 < t < k$ and so $\delta(v_{t-1}) = (1,1)$. Then

$$(1,1) \neq \delta(v_1) = \delta(v_{s-1}) = \delta(v_{t-1}) = (1,1),$$

which is a contradiction.

(b) Since $s < t$ and $v_s = v_t$, it follows from (a) that $s = 1$ or $t = k$. In both cases we get that $\delta(v_s) = \delta(v_t) \neq (1,1)$. It follows from the definition of a k-flow path that $s = 1$ and $t = k$. □

Lemma 2.16. Let $\mathbf{v}$ be a k-flow path in Q and let $\mathbf{u}$ be a k'-flow path in Q.

(a) Let v_s be a vertex in $\mathbf{v}$ with $\delta(v_s) = (1,1)$ and assume that $v_s = u_t$ for some vertex u_t in $\mathbf{u}$. Then $\mathbf{v} = \mathbf{u}$.

(b) Assume that $v_k = u_{k'}$ and that $v_{k-1} = u_{k'-1}$. Then $\mathbf{v} = \mathbf{u}$.

(c) Assume that $v_k = u_{k'}$ and that $\delta^-(v_k) = 1$. Then $\mathbf{v} = \mathbf{u}$.

Proof. (a) Since $\delta(v_s) = (1,1)$, it follows from the definition of a flow path that $1 < s < k$. Since $v_s = u_t$ it follows that $\delta(u_t) = (1,1)$. Without loss of generality we may assume that $s \leq t$. By the definition of a k-flow path it follows that $v_{s-1} = u_{t-1}$. Continuing inductively we see that $v_2 = u_{t-(s-2)}$ and this vertex has degree $(1,1)$. Hence the only arrow ending at $v_2 = u_{t-(s-2)}$ is the arrow coming from v_1. Since $\delta(v_1) \neq (1,1)$ and $\delta(u_{t-(s-2)}) = (1,1)$, and since there exists an arrow $v_1 \longrightarrow u_{t-(s-2)}$, it follows that $u_1 = v_1$ and $u_2 = u_{t-(s-2)}$. By Lemma 2.15(a) it follows that $2 = t - (s-2)$ and so $s = t$. A dual argument shows that $v_k = u_{k'}$. Since $v_1 = u_1$, $v_k = u_k$ and $v_s = u_s$ for some s with $1 < s < k$, it readily follows that $\mathbf{v} = \mathbf{u}$.

(b) Since $\mathbf{v}$ and $\mathbf{u}$ are flow paths, there exist arrows $\alpha : v_{k-1} \longrightarrow v_k$ and $\beta : u_{k'-1} \longrightarrow u_{k'}$. Since $v_{k-1} = u_{k'-1}$ and $v_k = u_{k'}$, and since there exist no multiple arrows between two vertices of Q, it follows that $\alpha = \beta$. If $\delta(v_{k-1}) \neq (1,1)$, then $\mathbf{v} = v_{k-1} \longrightarrow v_k = u_{k'-1} \longrightarrow u_{k'} = \mathbf{u}$, as required. Otherwise, if $\delta(v_{k-1}) = (1,1)$, then the result follows from (a).

(c) Since $\delta^-(v_k) = \delta^-(u_{k'}) = 1$, there exists a unique arrow ending at $v_k = u_{k'}$. Since there exist arrows $v_{k-1} \longrightarrow v_k$ and $u_{k'-1} \longrightarrow u_{k'}$, we conclude that $v_{k-1} = u_{k'-1}$. The result follows from (b). □

Corollary 2.17. Let $\alpha : w \longrightarrow v$ be an arrow in Q.

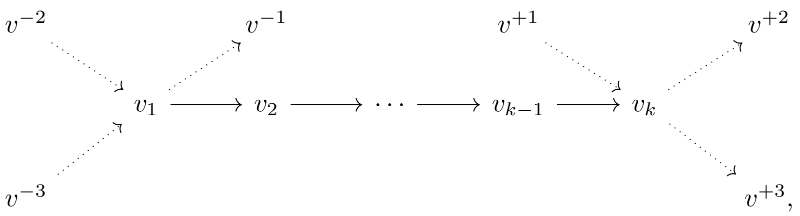
(a) If $\delta(v) = (1,1)$, then there exists a unique flow path $\mathbf{v}$ in Q through v.

(b) If $\delta(v) \neq (1,1)$, then there exists a unique flow path $\mathbf{v}$ in Q ending at v such that α is the last arrow of $\mathbf{v}$.

In both cases we have that $v = v_j$ for some $j > 1$.

Proof. The existence of the flow path is clear since $Q \neq A_1$ and $Q \neq \tilde{A}_m$. The uniqueness follows from Lemma 2.16. □

If $\mathbf{v}$ is a k-flow path in Q, then the arrows ending and starting at the vertices v_1 and v_k play an important role in our investigation. Hence we also label the following vertices



(2.3)

where a dotted arrow means that such an arrow may or may not exist. When $\delta^-(v_1) = 1$, we assume that the arrow $v^{-2} \longrightarrow v_1$ is the one that exists and when $\delta^+(v_k) = 1$ we assume that the arrow $v_k \longrightarrow v^{+2}$ is the one that exists. Notice that by Definition 2.6(ii) we have that $v^{-1} \neq v_2$ and $v^{+1} \neq v_{k-1}$, if the vertices v^{-1} and v^{+1} exist.

We also set

$$I(\mathbf{v}) := \begin{cases} I(v_1), & \text{if } \delta^+(v_1) = 1, \\ I(v^{-1}), & \text{if } \delta^+(v_1) = 2, \end{cases} \quad \text{and} \quad P(\mathbf{v}) := \begin{cases} P(v_k), & \text{if } \delta^-(v_k) = 1, \\ P(v^{+1}), & \text{if } \delta^-(v_k) = 2. \end{cases}$$

With this notation, we have the following technical results.

Lemma 2.18. Let $\mathbf{v}$ be a k-flow path in Q and let $\mathbf{u}$ be a k'-flow path in Q. Then $P(\mathbf{v})$ is not injective and $P(\mathbf{v}) \cong P(\mathbf{u})$ if and only if $\mathbf{v} = \mathbf{u}$.

Proof. That $P(\mathbf{v})$ is not injective follows immediately by the definition of $P(\mathbf{v})$ and since $\delta(v_k) \in \{(1,0),(1,2),(2,1),(2,2)\}$. That $\mathbf{v} = \mathbf{u}$ implies $P(\mathbf{v}) \cong P(\mathbf{u})$ is clear. Now assume that $P(\mathbf{v}) \cong P(\mathbf{u})$ and we show that $\mathbf{v} = \mathbf{u}$. We first claim that $\delta^-(v_k) = \delta^-(u_{k'})$. Indeed, assume towards a contradiction that $\delta^-(v_k) \neq \delta^-(u_{k'})$. Without loss of generality we may assume that $\delta^-(v_k) = 1$ and $\delta^-(u_{k'}) = 2$. Then $P(\mathbf{v}) = P(v_k)$ and $P(\mathbf{u}) = P(u^{+1})$. Hence $v_k = u^{+1}$. By Definition 2.6(iii), it follows that $\delta^+(u^{+1}) = 1$. Therefore $\delta(v_k) = (\delta^-(v_k), \delta^+(v_k)) = (1, \delta^+(u^{+1})) = (1,1)$, which contradicts the definition of a k-flow path.

Hence we have shown that $\delta^-(v_k) = \delta^-(u_{k'})$. Next we consider the cases $\delta^-(v_k) = 1$ and $\delta^-(v_k) = 2$ separately.

Case $\delta^-(v_k) = 1$. In this case $\delta^-(u_{k'}) = 1$ and so $P(v_k) \cong P(u_{k'})$. It follows that $v_k = u_{k'}$. Therefore we have that $\mathbf{v} = \mathbf{u}$ by Lemma 2.16(c).

Case $\delta^-(v_k) = 2$. In this case $\delta^-(u_{k'}) = 2$ and so $P(v^{+1}) \cong P(u^{+1})$. It follows that $v^{+1} = u^{+1}$. Since $\delta^+(v^{+1}) = 1$ and there exist an arrow $v^{+1} \longrightarrow v_k$ and an arrow $v^{+1} = u^{+1} \longrightarrow u_{k'}$, it follows that $v_k = u_{k'}$. Since $v^{+1} = u^{+1} \neq u_{k'-1}$, it follows that $u_{k'-1} = v_{k-1}$. Therefore we have that $\mathbf{v} = \mathbf{u}$ by Lemma 2.16(b). □

Lemma 2.19. Let $v \in Q_0$ be a vertex. Then exactly one of the following three conditions hold:

(i) $P(v)$ is injective.
(ii) $\delta(v) = (2, 2)$.
(iii) $P(v) = P(\mathbf{v})$ for some flow path $\mathbf{v}$.

Proof. Notice that conditions (i) and (iii) cannot hold simultaneously since by Lemma 2.18 we have that $P(\mathbf{v})$ is not injective. Moreover, by the definition of $P(\mathbf{v})$, conditions (ii) and (iii) also cannot hold simultaneously. It is also clear that conditions (i) and (ii) cannot hold simultaneously, since if $\delta(v) = (2, 2)$, then $P(v)$ does not have simple socle. Hence it is enough to show that one of the conditions (i),(ii) or (iii) holds. We consider the cases $\delta^+(v) = 0$, $\delta^+(v) = 1$ and $\delta^+(v) = 2$ separately.

Case $\delta^+(v) = 0$. In this case $\delta(v) = (1, 0)$ and by Corollary 2.17(b) there exists a unique flow path $\mathbf{v}$ ending at v. It follows from the definition of $P(\mathbf{v})$ that $P(\mathbf{v}) = P(v)$ and so condition (iii) holds.

Case $\delta^+(v) = 1$. Let $\alpha : v \longrightarrow u$ be the unique arrow starting at v. We consider the subcases $\delta^-(u) = 1$ and $\delta^-(u) = 2$ separately.

- Subcase $\delta^-(u) = 1$. In this case $P(v) = I(u)$ is injective and so condition (i) holds.
- Subcase $\delta^-(u) = 2$. Let $\beta : w \longrightarrow u$ be the other arrow ending at u. By Corollary 2.17(b) and since $\delta(u) \neq (1, 1)$, it follows that there exists a unique flow path $\mathbf{v}$ such that the last arrow of $\mathbf{v}$ is β. It follows from the definition of $P(\mathbf{v})$ that $P(\mathbf{v}) = P(v)$ and so condition (iii) holds.

Case $\delta^+(v) = 2$. We consider the subcases $\delta(v) = (1, 2)$ and $\delta(v) = (2, 2)$ separately.

- Subcase $\delta(v) = (1, 2)$. By Corollary 2.17(b) there exists a unique flow path $\mathbf{v}$ ending at v. It follows from the definition of $P(\mathbf{v})$ that $P(\mathbf{v}) = P(v)$ and so condition (iii) holds.
- Subcase $\delta(v) = (2, 2)$. In this case condition (ii) holds. □

Let $v \in Q_0$ be a vertex. If there exists an n-cluster titling subcategory $\mathcal{C} \subseteq \operatorname{mod} \Lambda$, then we have that $P(v) \in \mathcal{C}$. By Proposition 1.1(a) we then have that $\tau_n^{-j}(P(v)) \in \mathcal{C}$ for all $j \geq 0$. By Lemma 2.19 there are three different cases for $P(v)$. If $P(v)$ belongs to the first case, that is if $P(v)$ is injective, then $\tau_n^{-j}(P(v)) = 0$ for $j \geq 1$. Our aim now is to compute $\tau_n^{-j}(P(v))$ for the two remaining cases. To this end we need the following lemma.

Lemma 2.20. Let $v \in Q_0$ be a vertex.

(a) If $\delta^-(v) = 0$, then $\Omega^-(S(v)) = \tau^-(S(v)) = 0$.
(b) If $\delta^-(v) = 1$, let $w \longrightarrow v$ be the unique arrow ending at v. Then $\Omega^-(S(v)) \cong S(w)$ and $\tau^-(S(v)) \cong \operatorname{coker}(S(v) \hookrightarrow P(w))$.
(c) If $\delta^-(v) = 2$, let $w_1 \longrightarrow v$ and $w_2 \longrightarrow v$ be the two arrows ending at v. Then $\Omega^-(S(v)) \cong S(w_1) \oplus S(w_2)$ and $\tau^-(S(v)) \cong I(v)$.

Proof. By Theorem 2.10 the algebra Λ is a string algebra. The Auslander–Reiten translations for modules over string algebras are computed in [BR87]. We include here a simple proof in this special case.

(a) If $\delta^-(v) = 0$, then $S(v) = I(v)$ is injective and so $\Omega^-(S(v)) = \tau^-(S(v)) = 0$.

(b) Since Λ is a radical square zero algebra and $\delta^-(v) = 1$, we have that $I(v) = \begin{smallmatrix} w \\ v \end{smallmatrix}$. Hence there exists a minimal injective presentation of $S(v)$ of the form

$$\begin{array}{ccccccc} 0 \longrightarrow S(v) & \xrightarrow{i_0} & I(v) & & \xrightarrow{\quad i_1 \quad} & & I(w), \\ & & & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\hookrightarrow & \\ & & & & S(w) & & \end{array}$$

from which it follows that $\Omega^-(S(v)) \cong S(w)$. Furthermore, by applying the inverse Nakayama functor ν^- to the above presentation we obtain an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \nu^-(S(v)) & \xrightarrow{\nu^-(i_0)} & P(v) & & \xrightarrow{\quad \nu^-(i_1) \quad} & & P(w) \longrightarrow \tau^-(S(v)), \\ & & & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\hookrightarrow & \\ & & & & S(v) & & \end{array}$$

from which it follows that $\tau^-(S(v)) \cong \operatorname{coker}(S(v) \hookrightarrow P(w))$.

(c) Since Λ is a radical square zero algebra and $\delta^-(v) = 2$, we have that $I(v) = {}^{w_1}{}_{v}{}^{w_2}$. Hence there exists a minimal injective presentation of $S(v)$ of the form

$$\begin{array}{ccccccc} 0 \longrightarrow S(v) & \xrightarrow{i_0} & I(v) & & \xrightarrow{\qquad i_1 \qquad} & & I(w_1) \oplus I(w_2), \\ & & & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\hookrightarrow & \\ & & & & S(w_1) \oplus S(w_2) & & \end{array}$$

from which it follows that $\Omega^-(S(v)) \cong S(w_1) \oplus S(w_2)$. By applying the inverse Nakayama functor ν^- to the above presentation we obtain an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \nu^-(S(v)) & \xrightarrow{\nu^-(i_0)} & P(v) & & \xrightarrow{\quad \nu^-(i_1) \quad} & & P(w_1) \oplus P(w_2) \longrightarrow \tau^-(S(v)). \\ & & & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\hookrightarrow & \\ & & & & S(v) & & \end{array}$$

By Definition 2.6(iii) we have that $P(w_1) = \begin{smallmatrix} w_1 \\ v \end{smallmatrix}$ and $P(w_2) = \begin{smallmatrix} w_2 \\ v \end{smallmatrix}$. Then $\operatorname{coker}(S(v) \hookrightarrow P(w_1) \oplus P(w_2)) \cong I(v)$ and the result follows. $\square$

We also include the following dual version of Lemma 2.20 since it is often used in the sequel.

Lemma 2.21. Let $v \in Q_0$ be a vertex.

(a) If $\delta^+(v) = 0$, then $\Omega(S(v)) = \tau(S(v)) = 0$.

(b) If $\delta^+(v) = 1$, let $v \longrightarrow w$ be the unique arrow starting at v. Then $\Omega(S(v)) \cong S(w)$ and $\tau(S(v)) \cong \ker(I(w) \twoheadrightarrow S(v))$.

(c) If $\delta^+(v) = 2$, let $v \longrightarrow w_1$ and $v \longrightarrow w_2$ be the two arrows starting at v. Then $\Omega(S(v)) \cong S(w_1) \oplus S(w_2)$ and $\tau(S(v)) \cong P(v)$.

We can now compute $\tau_n^{-j}(P(v))$ in the second case of Lemma 2.19, that is when $\delta(v) = (2,2)$. Notice that in this case we have $n = 2$ by Definition 2.6(i).

Corollary 2.22. Let $v \in Q_0$ be a vertex with $\delta(v) = (2,2)$. Then $\tau_2^-(P(v)) \cong I(v)$.

Proof. Let $v \longrightarrow u_1$ and $v \longrightarrow u_2$ be the arrows starting at v. By Definition 2.6(iii) we have that $\delta^-(u_1) = \delta^-(u_2) = 1$. It follows that $\Omega^-(P(v)) \cong S(v)$. By Lemma 2.20(c) we have that $\tau^-(S(v)) \cong I(v)$. Hence

$$\tau_2^-(P(v)) = \tau^-\Omega^-(P(v)) \cong \tau^-(S(v)) \cong I(v),$$

as required. $\square$

Before continuing with the computation of $\tau_n^{-j}(P(v))$ in the last case, that is when $P(v) = P(\mathbf{v})$ for a flow path $\mathbf{v}$ in Q, let us introduce one more piece of notation.

Definition 2.23. Let $\mathbf{v} = v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_k$ be a k-flow path. We define

$$q_1 = q_1(\mathbf{v}) := \begin{cases} 1, & \text{if } \delta(v_1) = (2,1), \\ 0, & \text{if } \delta(v_1) \neq (2,1), \end{cases} \text{ and } q_k = q_k(\mathbf{v}) := \begin{cases} 1, & \text{if } \delta(v_k) = (1,2), \\ 0, & \text{if } \delta(v_k) \neq (1,2). \end{cases}$$

We also define

$$q(\mathbf{v}) := -1 + q_1 + q_k = \begin{cases} 1, & \text{if } \delta(v_1) = (2,1) \text{ and } \delta(v_k) = (1,2), \\ 0, & \text{if either } \delta(v_1) = (2,1) \text{ or } \delta(v_k) = (1,2), \\ -1, & \text{if } \delta(v_1) \neq (2,1) \text{ and } \delta(v_k) \neq (1,2). \end{cases}$$

With this definition we can write some of the next results in a more compact way. First we have the following statement.

Lemma 2.24. Let $\mathbf{v}$ be a k-flow path in Q. Let $s \in \mathbb{Z}$ and assume that $2 \leq s \leq k-1+q_k$. Then $\delta^-(v_s) = 1$.

Proof. We have $s \leq k-1+q_k \leq k$. We consider the cases $s \leq k-1$ and $s = k$ separately. If $s \leq k-1$, then $\delta(v_s) = (1,1)$ by the definition of flow paths and so the result holds. If $s = k$, then $k-1+q_k = k$ and so $q_k = 1$. Then by the definition of q_k we have $\delta(v_s) = \delta(v_k) = (1,2)$, and so the result holds again. □

With this we are ready to make the following computations.

Lemma 2.25. Let $\mathbf{v}$ be a k-flow path in Q. Let $s, x \in \mathbb{Z}_{\geq 0}$ and assume that $1 \leq s \leq k-1+q_k$.

(a) If $s - x \geq 1$, then $\Omega^{-x}(S(v_s)) \cong S(v_{s-x})$.

(b) If $1 \leq x \leq s-1+q_1$, then $\tau_x^-(S(v_s)) \cong \begin{cases} S(v_{s-x}), & \text{if } 1 \leq x < s-1+q_1, \\ I(\mathbf{v}), & \text{if } x = s-1+q_1. \end{cases}$

Proof. (a) We use induction on x. If $x = 0$, then the result holds trivially. Assume now that the result holds for $x-1 \geq 0$ and we show that it holds for x. Since $s - x \geq 1$, we have that $s-(x-1) \geq 1$. Hence by induction hypothesis we have that $\Omega^{-(x-1)}(S(v_s)) \cong S(v_{s-(x-1)})$. Then

$$2 = 1+1 \leq (s-x)+1 = s-(x-1) \leq s \leq k-1+q_k,$$

and so $\delta^-(v_{s-(x-1)}) = 1$ by Lemma 2.24. Then by the definition of flow paths and Lemma 2.20(b) applied on $v_{s-(x-1)}$ it follows that

$$\Omega^{-x}(S(v_s)) \cong \Omega^-(S(v_{s-(x-1)})) \cong S(v_{s-x}),$$

as required.

(b) Since $x \leq s-1+q_1$, we have that $s-(x-1) \geq 2-q_1 \geq 1$. Therefore, by (a) we have that

$$\tau_x^-(S(v_s)) = \tau^-\Omega^{-(x-1)}(S(v_s)) \cong \tau^-(S(v_{s-(x-1)})).$$

Hence it is enough to show that

$$\tau^-(S(v_{s-(x-1)})) \cong \begin{cases} S(v_{s-x}), & \text{if } 1 \leq x < s-1+q_1, \\ I(\mathbf{v}), & \text{if } x = s-1+q_1. \end{cases}$$

We consider the cases $1 \leq x < s-1+q_1$ and $x = s-1+q_1$ separately.

Case $1 \leq x < s-1+q_1$. In this case we want to show that $\tau^-(S(v_{s-(x-1)})) \cong S(v_{s-x})$. We have

$$2 - q_1 < s-(x-1) \leq s \leq k-1+q_k \leq k. \tag{2.4}$$

Hence $2 \leq s-(x-1) \leq k-1+q_k$ and so by Lemma 2.24 we have that $\delta^-(v_{s-(x-1)}) = 1$. It follows from Lemma 2.20(b) that it is enough to show that $\delta^+(v_{s-x}) = 1$. We consider the subcases $q_1 = 0$ and $q_1 = 1$ separately.

- Subcase $q_1 = 0$. Then by (2.4) we conclude that $2 \leq s-x \leq k-1$ and so $\delta^+(v_{s-x}) = 1$.
- Subcase $q_1 = 1$. Then by (2.4) we conclude that $1 \leq s-x \leq k-1$. Since in this case we have $\delta(v_1) = (2,1)$, it follows that $\delta^+(v_{s-x}) = 1$.

Case $x = s - 1 + q_1$. In this case we have $s - (x - 1) = 2 - q_1$ and we want to show that $\tau^-(S(v_{2-q_1})) \cong I(\mathbf{v})$. We consider the cases $q_1 = 0$ and $q_1 = 1$ separately.

- Subcase $q_1 = 0$. Then the result follows immediately by Lemma 2.20(b) and by considering the possibilities $\delta(v_1) = (0,1)$, $\delta(v_1) = (1,2)$ and $\delta(v_1) = (2,2)$ separately.
- Subcase $q_1 = 1$. Then $\delta(v_1) = (2,1)$ and by Lemma 2.20(c) we have $\tau^-(S(v_1)) \cong I(v_1) = I(\mathbf{v})$. □

Lemma 2.26. Let $\mathbf{v}$ be a k-flow path in Q. Let $x \in \mathbb{Z}_{\geq 1}$.

(a) If $k - x + q_k \geq 1$, then $\Omega^{-x}(P(\mathbf{v})) \cong S(v_{k-x+q_k})$.

(b) If $1 \leq x \leq k + q(\mathbf{v})$, then $\tau_x^-(P(\mathbf{v})) \cong \begin{cases} S(v_{k-x+q_k}), & \text{if } 1 \leq x < k + q(\mathbf{v}), \\ I(\mathbf{v}), & \text{if } x = k + q(\mathbf{v}). \end{cases}$

Proof. (a) If $x = 1$, then the result follows immediately by considering the cases $\delta(v_k) = (1,0)$, $\delta(v_k) = (1,2)$, $\delta(v_k) = (2,1)$ and $\delta(v_k) = (2,2)$ separately (recall that if $\delta(v_k) = (1,2)$, then $\delta^-(v^{+2}) = \delta^-(v^{+3}) = 1$ by Definition 2.6(iii)). For $x \geq 2$ notice that $1 \leq k - x + q_k$ implies that $k - 1 + q_k - (x-1) \geq 1$. Hence we can apply Lemma 2.25(a) to obtain

$$\Omega^{-x}(P(\mathbf{v})) = \Omega^{-(x-1)}\Omega^-(P(\mathbf{v})) \cong \Omega^{-(x-1)}(S(v_{k-1+q_k})) \cong S(v_{k-x+q_k}),$$

as required.

(b) We first show the result for $x = 1$. We consider the cases $1 = x < k + q(\mathbf{v})$ and $1 = x = k + q(\mathbf{v})$ separately.

Case $1 = x < k + q(\mathbf{v})$. In this case we want to show that $\tau^-(P(\mathbf{v})) \cong S(v_{k-1+q_k})$. We consider the subcases $\delta(v_k) = (1,0)$, $\delta(v_k) = (1,2)$ and $\delta(v_k) \in \{(2,1),(2,2)\}$ separately.

- Subcase $\delta(v_k) = (1,0)$. In this case $q_k = 0$ and $P(\mathbf{v}) = S(v_k)$ and so we want to show that $\tau^-(S(v_k)) \cong S(v_{k-1})$. We claim that $\delta^+(v_{k-1}) = 1$. Indeed, assume towards a contradiction that $\delta^+(v_{k-1}) = 2$. Then $v_{k-1} = v_1$ and so $k = 2$ and $q_1 = 0$. Hence $1 = x < 2 + q(\mathbf{v}) = 2 - 1 = 1$, which is a contradiction. We conclude that we are in the situation

$$v_x = v_1 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow v_k$$

with $\delta(v_k) = (1,0)$ and $\delta^+(v_{k-1}) = 1$. By Lemma 2.20(b), it follows that $\tau^-(S(v_k)) \cong S(v_{k-1})$.

- Subcase $\delta(v_k) = (1,2)$. In this case we are in the situation

$$\begin{array}{ccccccc} & & & & & \nearrow & v^{+2} \\ v_x = v_1 & \longrightarrow & \cdots & \longrightarrow & v_k & & \\ & & & & & \searrow & v^{+3}. \end{array}$$

Since $\delta(v_k) = (1,2)$, we have $q_k = 1$ and $P(\mathbf{v}) = P(v_k)$ and so we want to show that $\tau^-(P(v_k)) \cong S(v_k)$. By Lemma 2.21(c) we have that $\tau(S(v_k)) \cong P(v_k)$. By applying τ^- we obtain $\tau^-(P(v_k)) \cong \tau^-\tau(S(v_k)) \cong S(v_k)$.

- Subcase $\delta(v_k) \in \{(2,1),(2,2)\}$. In this case we are in the situation

$$\begin{array}{ccccccc} & & & & v^{+1} & \searrow & \\ v_x = v_1 & \longrightarrow & \cdots & \longrightarrow & v_{k-1} & \longrightarrow & v_k. \end{array}$$

Since $\delta(v_k) \in \{P(2,1),(2,2)\}$, we have $q_k = 0$ and $P(\mathbf{v}) = P(v^{+1})$ and so we want to show that $\tau^-(P(v^{+1})) \cong S(v_{k-1})$. By Definition 2.6(iii) we have that $\delta^+(v_{k-1}) = \delta^+(v^{+1}) = 1$. By Lemma 2.21(b) we then have that $\tau(S(v_{k-1})) \cong \ker(I(v_k) \twoheadrightarrow S(v_{k-1})) \cong P(v^{+1})$. By applying τ^- we obtain $\tau^-(P(v^{+1})) \cong \tau^-\tau(S(v_{k-1})) \cong S(v_{k-1})$.

Case $1 = x = k + q(\mathbf{v})$. In this case we have that $k = 2$ and $q(\mathbf{v}) = -1$ and we want to show that $\tau^-(P(\mathbf{v})) \cong I(\mathbf{v})$. Since $q(\mathbf{v}) = -1$, we have $\delta(v_1) \neq (2,1)$ and $\delta(v_2) \neq (1,2)$. We consider the subcases $\delta(v_2) = (1,0)$ and $\delta(v_2) \in \{(2,1),(2,2)\}$ separately.

- Subcase $\delta(v_2) = (1,0)$. In this case we have $P(\mathbf{v}) = S(v_2)$ and so we want to show that $\tau^-(S(v_2)) \cong I(\mathbf{v})$.
  If $\delta^+(v_1) = 1$, since $\delta(v_1) \neq (2,1)$ and since by the definition of a flow path we have $\delta(v_1) \neq (1,1)$, we conclude that $\delta(v_1) = (0,1)$. In this case we are in the situation
  $$v_x = v_1 \longrightarrow v_2 = v_k,$$
  where $\delta(v_1) = (0,1)$ and $\delta(v_k) = (1,0)$ (in other words, this is simply an A_2 situation). Hence by Lemma 2.20(b) we have $\tau^-(S(v_2)) \cong S(v_1) = I(\mathbf{v})$, where the last equality follows from the definition of $I(\mathbf{v})$.
  If $\delta^+(v_1) = 2$, then we are in the situation
  $$\begin{array}{ccc} & \nearrow & v^{-1} \\ v_x = v_1 & \longrightarrow & v_2 = v_k. \end{array}$$
  Then by Lemma 2.20(b) and Definition 2.6(iii) we have $\tau^-(S(v_2)) \cong I(v^{-1}) = I(\mathbf{v})$, where the last equality again follows from the definition of $I(\mathbf{v})$.
- Subcase $\delta(v_2) \in \{(2,1),(2,2)\}$. In this case we have $P(\mathbf{v}) = P(v^{+1})$ and so we want to show that $\tau^-(P(v^{+1})) \cong I(\mathbf{v})$. Since $k = 2$ and $\delta^-(v_2) = 2$, by Definition 2.6(iii) we have that $\delta^+(v_1) = 1$. Since $\delta(v_1) \neq (2,1)$ and since by the definition of a flow path we have $\delta(v_1) \neq (1,1)$, we conclude that $\delta(v_1) = (0,1)$. Hence we are in the situation
  $$\begin{array}{ccc} v^{+1} & \searrow & \\ v_x = v_1 & \longrightarrow & v_2 = v_k, \end{array}$$
  with $\delta(v_1) = (0,1)$. It follows that $I(\mathbf{v}) = S(v_1)$. By Lemma 2.21(b) we then have that $\tau(S(v_1)) \cong \ker(I(v_2) \twoheadrightarrow S(v_1)) \cong P(v^{+1})$. By applying τ^- we obtain $\tau^-(P(v^{+1})) \cong \tau^-\tau(S(v_1)) \cong S(v_1) = I(\mathbf{v})$.

Now let $x \geq 2$. Then $2 \leq k + q(\mathbf{v})$ gives $k - 1 + q_k \geq 1$. Hence by (a) we have that
$$\tau_x^-(P(\mathbf{v})) = \tau_{x-1}^-\Omega^-(P(\mathbf{v})) \cong \tau_{x-1}^-(S(v_{k-1+q_k})).$$
Moreover, since $1 \leq k - 1 + q_k$ and
$$(k - 1 + q_k) - 1 + q_1 = k + q(\mathbf{v}) - 1 \geq x - 1 \geq 1,$$
we can apply Lemma 2.25(b) to obtain
$$\tau_{x-1}^-(S(v_{k-1+q_k})) \cong \begin{cases} S(v_{k-1+q_k-(x-1)}), & \text{if } 1 \leq x - 1 < (k-1+q_k) - 1 + q_1, \\ I(\mathbf{v}), & \text{if } x - 1 = (k-1+q_k) - 1 + q_1. \end{cases}$$
After simplifying the above expression, we get
$$\tau_x^-(P(\mathbf{v})) \cong \tau_{x-1}^-(S(v_{k-1+q_k})) \cong \begin{cases} S(v_{k-x+q_k}), & \text{if } 2 \leq x < k + q(\mathbf{v}), \\ I(\mathbf{v}), & \text{if } x = k + q(\mathbf{v}), \end{cases}$$
which proves the case $x \geq 2$. □

With the above computation we can show the following important results about flow paths in Q.

Proposition 2.27. Let $\mathbf{v}$ be a k-flow path in Q and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Then $n \mid (k + q(\mathbf{v}))$.

Proof. We write $k + q(\mathbf{v}) = pn + r$ where $p \in \mathbb{Z}_{\geq 0}$ and $0 \leq r \leq n - 1$. We first claim that $p \geq 1$. Indeed, assume towards a contradiction that $p = 0$. Then $1 \leq k + q(\mathbf{v}) = r$. Hence by Lemma 2.26(b) we have that $\tau_r^-(P(\mathbf{v})) \cong I(\mathbf{v})$. By Lemma 1.2 we obtain $\operatorname{Ext}_\Lambda^r(I(\mathbf{v}), P(\mathbf{v})) \neq 0$. But this contradicts the fact that $\mathcal{C}$ is an n-cluster tilting subcategory, since $I(\mathbf{v}), P(\mathbf{v}) \in \mathcal{C}$ and $1 \leq r \leq n - 1$.

Hence $p \geq 1$ and it remains to show that $r = 0$. Assume towards a contradiction that $r \geq 1$. Then
$$1 \leq n \leq pn = k + q(\mathbf{v}) - r < k + q(\mathbf{v}).$$

Hence we can apply Lemma 2.26(b) to obtain that $\tau_n^-(P(\mathbf{v})) \cong S(v_{k-n+q_k})$. Then we can apply Lemma 2.25(b) repeatedly $p-1$ more times to obtain

$$\tau_n^{-p}(P(\mathbf{v})) \cong \tau_n^{-(p-1)}(S(v_{k-n+q_k})) \cong \tau_n^{-(p-2)}(S(v_{k-2n+q_k})) \cong \cdots \cong S(v_{k-pn+q_k}) = S(v_{r+1-q_1}).$$

By Proposition 1.1(a) and since $P(\mathbf{v}) \in \mathcal{C}$, it follows that $S(v_{r+1-q_1}) \in \mathcal{C}$. By Lemma 2.25(b) we have $\tau_r^-(S(v_{r+1-q_1})) \cong I(\mathbf{v})$. By Lemma 1.2 we obtain $\operatorname{Ext}^r_\Lambda(I(\mathbf{v}), S(v_{r+1-q_1})) \neq 0$. But this contradicts the fact that $\mathcal{C}$ is an n-cluster tilting subcategory, since $I(\mathbf{v}), S(v_{r+1-q_1}) \in \mathcal{C}$ and $1 \leq r \leq n-1$. ☐

Corollary 2.28. Let $\mathbf{v}$ be a k-flow path in Q. Assume that $k + q(\mathbf{v}) = pn$ for some $p \geq 1$ and let $j \in \mathbb{Z}$ with $0 \leq j \leq p$. Then

$$\tau_n^{-j}(P(\mathbf{v})) \cong \begin{cases} P(\mathbf{v}), & \text{if } j = 0, \\ S(v_{k-jn+q_k}), & \text{if } 1 \leq j \leq p-1, \\ I(\mathbf{v}), & \text{if } j = p. \end{cases} \tag{2.5}$$

Moreover, if $1 \leq j \leq p-1$, then $\delta(v_{k-jn+q_k}) = (1,1)$. In particular, the module $\tau_n^{-j}(P(\mathbf{v}))$ is indecomposable and not projective-injective.

Proof. We first prove (2.5). For $j = 0$ the result is clear. For $1 \leq j \leq p-1$ we use induction on j, where the base case $j = 1$ follows from Lemma 2.26(b), while the induction step follows from Lemma 2.25(b).

Next, if $1 \leq j \leq p-1$, then

$$2 = 2 + 1 - 1 \leq n + 1 - q_1 = k - (p-1)n + q_k \leq k - jn + q_k \leq k - n + q_k \leq k - 2 + 1 = k - 1,$$

from which it follows that $\delta(v_{k-jn+q_k}) = (1,1)$ and so $S(v_{k-jn+q_k})$ is neither projective nor injective.

Finally, if $j = 0$, then $\tau_n^{-j}(P(\mathbf{v})) \cong P(\mathbf{v})$ is not injective by Lemma 2.18, while if $j = p$, then $\tau_n^{-j}(P(\mathbf{v})) \cong I(\mathbf{v})$ is not projective by the dual of Lemma 2.18. ☐

3. Sufficient conditions

Motivated by Proposition 2.14 and Proposition 2.27 we give the following definition.

Definition 3.1. Let Q be an n-pre-admissible quiver. We say that Q is *n-admissible* if one of the following conditions hold:

(a) $Q = \tilde{A}_m$ and $n \mid m$, or
(b) $Q \neq \tilde{A}_m$ and for every k-flow path $\mathbf{v}$ in Q we have that $n \mid (k + q(\mathbf{v}))$.

Example 3.2. (a) The quiver A_m is n-admissible if and only if $n \mid (m-1)$. In particular, the quiver A_1 is n-admissible for all $n \geq 2$.
(b) The quiver of Example 2.8(c) is 2-admissible.
(c) The quiver of Example 2.8(d) is 3-admissible but not n-admissible for any $n \neq 3$.

Remark 3.3. (a) When studying n-admissible quivers, the cases $Q = A_1$ and $Q = \tilde{A}_m$ for $m \geq 1$ usually behave differently from the rest of the cases; the reason for this is that the quivers A_1 and $\tilde{A}_m$ are the only n-pre-admissible quivers that do not have flow paths as Lemma 2.13 shows. Hence many times in the rest of this paper we will exclude one or both of the cases $Q = A_1$ and $Q = \tilde{A}_m$ from our statements. We remind the reader that this does not present a problem in our aim of classification of n-cluster tilting subcategories for radical square zero bound quiver algebras since such a classification in these exceptional cases is given in Proposition 2.14.
(b) If Q is an n-admissible quiver and n' is an integer such that $n' \geq 2$ and $n' \mid n$, then it follows directly from Remark 2.7 and Definition 3.1 that Q is also an n'-admissible quiver.

By Proposition 2.14 and Proposition 2.27 it follows that if Q is a quiver and there exists an n-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}(\mathbf{k}Q/\mathcal{J}^2)$, then Q is n-admissible. The aim of this section is to show that the opposite is also true. We also want to show that if $Q \neq \tilde{A}_m$, then $\mathcal{C}$ is unique and give a description of $\mathcal{C}$.

For the rest of this section we fix an n-admissible quiver Q with $Q \neq A_1$ and $Q \neq \tilde{A}_m$ and we set $\Lambda := \mathbf{k}Q/\mathcal{J}^2$. We denote by $\mathbf{V}$ the set of all flow paths in Q. Note that by Lemma 2.13 we have that $\mathbf{V} \neq \varnothing$. For a k-flow path $\mathbf{v} \in \mathbf{V}$ we set $p(\mathbf{v}) = \frac{k+q(\mathbf{v})}{n}$; since Q is n-admissible, it follows that $p(\mathbf{v})$ is an integer. We define

$$M(\mathbf{v}) := \bigoplus_{j=0}^{p(\mathbf{v})} \tau_n^{-j}(P(\mathbf{v})) \cong P(\mathbf{v}) \oplus \left(\bigoplus_{j=1}^{p(\mathbf{v})-1} S(v_{k-jn+q_k}) \right) \oplus I(\mathbf{v}),$$

where the last isomorphism follows from Corollary 2.28. We also set $M(\mathbf{V}) := \bigoplus_{\mathbf{v}\in\mathbf{V}} M(\mathbf{v})$. With this notation we have the following lemmas.

Lemma 3.4. (a) The module $M(\mathbf{v})$ is basic and has no projective-injective direct summand.
(b) The module $M(\mathbf{V})$ is basic and has no projective-injective direct summand.

Proof. (a) Follows immediately by Corollary 2.28 and Lemma 2.15(a).
(b) By (a) we have that $M(\mathbf{V})$ has no projective-injective direct summand. It remains to show that $M(\mathbf{V})$ is basic. Since the module $M(\mathbf{v})$ for $\mathbf{v} \in \mathbf{V}$ is basic by (a), it is enough to show that if $\mathbf{v}$ and $\mathbf{u}$ are two flow paths in Q with $\mathbf{v} \neq \mathbf{u}$, then $M(\mathbf{v})$ and $M(\mathbf{u})$ have no isomorphic direct summands. Assume towards a contradiction that there exist indecomposable direct summands V of $M(\mathbf{v})$ and U of $M(\mathbf{u})$ such that $V \cong U$ but $\mathbf{v} \neq \mathbf{u}$. Then $V \cong \tau_n^{-j_v}(P(\mathbf{v}))$ and $U \cong \tau_n^{-j_u}(P(\mathbf{u}))$ for some $j_v, j_u \in \mathbb{Z}_{\geq 0}$ with $j_v \leq p(\mathbf{v})$ and $j_u \leq p(\mathbf{u})$. Without loss of generality we assume that $j_u \geq j_v$. It follows that

$$\tau_n^{-(p(\mathbf{v})-j_v+j_u)}(P(\mathbf{u})) = \tau_n^{-(p(\mathbf{v})-j_v)}\tau_n^{-j_u}(P(\mathbf{u})) \cong \tau_n^{-(p(\mathbf{v})-j_v)}\tau_n^{-j_v}(P(\mathbf{v})) = \tau_n^{-p(\mathbf{v})}(P(\mathbf{v})) \cong I(\mathbf{v}),$$

where the last isomorphism follows from Corollary 2.28. In particular, we have that the module $\tau_n^{-(p(\mathbf{v})-j_v+j_u)}(P(\mathbf{u}))$ is injective and nonzero. By Corollary 2.28 we have that $\tau_n^{-j'}(P(\mathbf{u})) = 0$ for $j' > p(\mathbf{u})$ and $\tau_n^{-j'}(P(\mathbf{u}))$ is not injective for $j' < p(\mathbf{u})$. We conclude that $p(\mathbf{v}) - j_v + j_u = p(\mathbf{u})$ and so $I(\mathbf{u}) \cong \tau_n^{-p(\mathbf{u})}(P(\mathbf{u})) = \tau_n^{-(p(\mathbf{v})-j_v+j_u)}(P(\mathbf{u})) \cong I(\mathbf{v})$. Then by the dual of Lemma 2.18 it follows that $\mathbf{v} = \mathbf{u}$, which contradicts our assumption $\mathbf{v} \neq \mathbf{u}$. □

Lemma 3.5. Let $i \in \{1, \dots, n-1\}$. Then $\operatorname{Ext}^i_\Lambda(M(\mathbf{V}), M(\mathbf{V})) = 0$.

Proof. Let $\mathbf{v}$ be a k-flow path in Q and let $\mathbf{u}$ be a k'-flow path in Q. By the definition of $M(\mathbf{V})$ and additivity of $\operatorname{Ext}^i_\Lambda(-,-)$ it is enough to show that $\operatorname{Ext}^i_\Lambda(M(\mathbf{u}), M(\mathbf{v})) = 0$. By the definition of $M(\mathbf{u})$ and $M(\mathbf{v})$ and additivity of $\operatorname{Ext}^i_\Lambda(-,-)$ it is enough to show that

$$\operatorname{Ext}^i_\Lambda(\tau_n^{-x}(P(\mathbf{u})), \tau_n^{-y}(P(\mathbf{v}))) = 0 \tag{3.1}$$

for any $x \in \{0, 1, \dots, p(\mathbf{u})\}$ and $y \in \{0, 1, \dots, p(\mathbf{v})\}$. If $x = 0$, then $\tau_n^{-x}(P(\mathbf{u})) = P(\mathbf{u})$ is projective and so (3.1) holds. If $y = p(\mathbf{v})$, then by Corollary 2.28 we have that $\tau_n^{-p(\mathbf{v})}(P(\mathbf{v})) \cong I(\mathbf{v})$ is injective and so (3.1) holds again. Hence we may assume that $x > 0$ and $y < p(\mathbf{v})$.

Using dimension shift and the Auslander–Reiten duality we compute

$$\begin{aligned}\operatorname{Ext}^i_\Lambda(\tau_n^{-x}(P(\mathbf{u})), \tau_n^{-y}(P(\mathbf{v}))) &\cong \operatorname{Ext}^1_\Lambda(\tau_n^{-x}(P(\mathbf{u})), \Omega^{-(i-1)}\tau_n^{-y}(P(\mathbf{v}))) \\ &\cong D\underline{\operatorname{Hom}}_\Lambda(\tau^{-}\Omega^{-(i-1)}\tau_n^{-y}(P(\mathbf{v})), \tau_n^{-x}(P(\mathbf{u}))) \\ &\cong D\underline{\operatorname{Hom}}_\Lambda(\tau_i^{-}\tau_n^{-y}(P(\mathbf{v})), \tau_n^{-x}(P(\mathbf{u}))) \\ &\cong D\underline{\operatorname{Hom}}_\Lambda(S(v_{k-yn-i+q_k(\mathbf{v})}), \tau_n^{-x}(P(\mathbf{u}))),\end{aligned}$$

where the last isomorphism follows from Lemma 2.26(b) if $y = 0$ and by Corollary 2.28 and Lemma 2.25(b) if $y > 0$. Hence it is enough to show that

$$D\underline{\operatorname{Hom}}_\Lambda(S(v_{k-yn-i+q_k(\mathbf{v})}), \tau_n^{-x}(P(\mathbf{u}))) = 0. \tag{3.2}$$

Assume towards a contradiction that (3.2) does not hold. We consider the cases $0 < x < p(\mathbf{u})$ and $x = p(\mathbf{u})$ separately and reach a contradiction in each case.

Case $0 < x < p(\mathbf{u})$. In this case by Corollary 2.28 we have that $\tau_n^{-x}(P(\mathbf{u})) \cong S(u_{k'-xn+q_{k'}(\mathbf{u})})$. Then it follows that $\mathrm{Hom}_\Lambda(S(v_{k-yn-i+q_k(\mathbf{v})}), S(u_{k'-xn+q_{k'}(\mathbf{u})})) \neq 0$. Since both modules are simple, we conclude that $v_{k-yn-i+q_k(\mathbf{v})} = u_{k'-xn+q_{k'}(\mathbf{u})}$. By Corollary 2.28 and since $0 < x < p(\mathbf{u})$, it follows that $\delta(u_{k'-xn+q_{k'}(\mathbf{u})}) = (1,1)$. Thus by Lemma 2.16(a) we obtain $\mathbf{v} = \mathbf{u}$. In particular, we have that $k = k'$ and $q_k(\mathbf{v}) = q_{k'}(\mathbf{u})$ and so $v_{k-yn-i+q_k(\mathbf{v})} = v_{k-xn+q_k(\mathbf{v})}$. Hence by Lemma 2.15(a) it follows that $k - yn - i + q_k(\mathbf{v}) = k - xn + q_k(\mathbf{v})$. Equivalently we get $(x-y)n = i$, which contradicts $1 \leq i \leq n-1$.

Case $x = p(\mathbf{u})$. In this case by Corollary 2.28 we have that $\tau_n^{-x}(P(\mathbf{u})) \cong I(\mathbf{u})$. Since we assume that (3.2) does not hold, and since $I(\mathbf{u})$ is indecomposable and injective, it follows that $S(v_{k-yn-i+q_k(\mathbf{v})}) \cong \mathrm{soc}(I(\mathbf{u}))$. We consider the subcases $\delta^+(u_1) = 1$ and $\delta^+(u_1) = 2$ separately.

- Subcase $\delta^+(u_1) = 1$. In this case we have $I(\mathbf{u}) = I(u_1)$ by definition. Hence $v_{k-yn-i+q_k(\mathbf{v})} = u_1$ and so $\delta(v_{k-yn-i+q_k(\mathbf{v})}) \neq (1,1)$. By the definition of a k-flow path we obtain that $k - yn - i + q_k(\mathbf{v}) \in \{1, k\}$. We claim that $k - yn - i + q_k(\mathbf{v}) = 1$. Indeed, assume towards a contradiction that $k - yn - i + q_k(\mathbf{v}) = k$. Since $0 \leq y \leq p(\mathbf{v}) - 1$, $1 \leq i \leq n-1$ and $0 \leq q_k(\mathbf{v}) \leq 1$, it follows that $y = 0$, $i = 1$ and $q_k(\mathbf{v}) = 1$. But then $(1,2) = \delta(v_k) = \delta(v_{k-yn-i+q_k(\mathbf{v})}) = \delta(u_1)$ contradicts the fact that $\delta^+(u_1) = 1$.

  Hence we have $k - yn - i + q_k(\mathbf{v}) = 1$. Using this equality together with $k + q(\mathbf{v}) = p(\mathbf{v})n$, we obtain that $(p(\mathbf{v}) - y)n = i + q_1(\mathbf{v})$. Since $y < p(\mathbf{v})$ and $1 \leq i \leq n-1$, it follows that $q_1(\mathbf{v}) = 1$. Hence we have $v_1 = v_{k-yn-i+q_k(\mathbf{v})} = u_1$ and $\delta(v_1) = (2,1)$. Then any morphism from $S(v_{k-yn-i+q_k(\mathbf{v})}) = S(v_1) = {\scriptstyle v_1}$ to $\tau_n^{-x}(P(\mathbf{u})) \cong I(u_1) = I(v_1) = \begin{smallmatrix} v^{-2} & & v^{-3} \\ & v_1 & \end{smallmatrix}$ clearly factors through $P(v^{-2}) = \begin{smallmatrix} v^{-2} \\ v_1 \end{smallmatrix}$. But this shows that (3.2) holds, which is a contradiction.
- Subcase $\delta^+(u_1) = 2$. In this case we have $I(\mathbf{u}) = I(u^{-1})$ by definition. Hence $v_{k-yn-i+q_k(\mathbf{v})} = u^{-1}$. Then any morphism from $S(v_{k-yn-i+q_k(\mathbf{v})}) = S(u^{-1}) = {\scriptstyle u^{-1}}$ to $\tau_n^{-x}(P(\mathbf{u})) \cong I(u^{-1}) = \begin{smallmatrix} u_1 \\ u^{-1} \end{smallmatrix}$ clearly factors through $P(u_1) = \begin{smallmatrix} & u_1 & \\ u^{-1} & & u_2 \end{smallmatrix}$. But this shows that (3.2) holds, which is a contradiction. □

Lemma 3.6. Let $v, u \in Q_0$ be such that $\delta(v) = \delta(u) = (2,2)$.

(a) We have $\mathrm{Ext}^1_\Lambda(M(\mathbf{V}), P(v)) = 0$ and $\mathrm{Ext}^1_\Lambda(I(v), M(\mathbf{V})) = 0$.

(b) We have $\mathrm{Ext}^1_\Lambda(I(u), P(v)) = 0$.

Proof. (a) We only show that $\mathrm{Ext}^1_\Lambda(M(\mathbf{V}), P(v)) = 0$; the other equality follows dually. Let $\mathbf{w}$ be a k-flow path in Q. By additivity of $\mathrm{Ext}^1_\Lambda(-,-)$ it is enough to show that

$$\mathrm{Ext}^1_\Lambda(\tau_n^{-x}(P(\mathbf{w})), P(v)) = 0$$

for any $x \in \{0, 1, \ldots, p(\mathbf{w})\}$. If $x = 0$, then $\tau_n^{-x}(P(\mathbf{w})) = P(\mathbf{w})$ is projective and so the result follows. Otherwise, assume that $1 \leq x \leq p(\mathbf{w})$. By Lemma 2.21(c) we have that $\tau^-(P(v)) \cong S(v)$. Then by the Auslander–Reiten duality, it is enough to show that

$$D\underline{\mathrm{Hom}}_\Lambda(S(v), \tau_n^{-x}(P(\mathbf{w}))) = 0. \tag{3.3}$$

We consider the cases $1 \leq x \leq p(\mathbf{w}) - 1$ and $x = p(\mathbf{w})$ separately.

Case $1 \leq x \leq p(\mathbf{w}) - 1$. In this case by Corollary 2.28 we have that $\tau_n^{-x}(P(\mathbf{w})) \cong S(w_{k-xn+q_k})$. Assume towards a contradiction that (3.3) does not hold. Then $S(v) \cong S(w_{k-xn+q_k})$ from which it follows that $v = w_{k-xn+q_k}$. By Corollary 2.28 we have that $\delta(w_{k-xn+q_k}) = (1,1)$, which contradicts $\delta(v) = (2,2)$.

Case $x = p(\mathbf{w})$. In this case by Corollary 2.28 we have that $\tau_n^{-x}(P(\mathbf{w})) \cong I(\mathbf{w})$. Assume towards a contradiction that (3.3) does not hold. Then $S(v) \cong \mathrm{soc}(I(\mathbf{w}))$ from which it follows that $I(v) \cong I(\mathbf{w})$. But this contradicts the dual of Lemma 2.19 since $\delta(v) = (2,2)$.

(b) By Lemma 2.21(c) we have that $\tau^-(P(v)) \cong S(v)$. Then by the Auslander–Reiten duality it is enough to show that

$$D\underline{\mathrm{Hom}}_\Lambda(S(v), I(u)) = 0.$$

If $v \neq u$, then $\mathrm{Hom}_\Lambda(S(v), I(u)) = 0$ and the result follows. Otherwise, assume that $v = u$. Since $\delta^-(v) = 2$, we have arrows $v^{-2} \longrightarrow v$ and $v^{-3} \longrightarrow v$ ending at v. Then any morphism from $S(v) = {\scriptstyle v}$ to $I(u) = I(v) = \begin{smallmatrix} v^{-2} & & v^{-3} \\ & v & \end{smallmatrix}$ clearly factors through $P(v^{-2}) = \begin{smallmatrix} v^{-2} \\ v \end{smallmatrix}$, which shows that $D\underline{\mathrm{Hom}}_\Lambda(S(v), I(u)) = 0$. □

Next, let $\{R_t\}_{t=1}^{f}$ be a complete collection of representatives of pairwise non-isomorphic projective-injective Λ-modules. Set

$$M \coloneqq M(\mathbf{V}) \oplus \left(\bigoplus_{t=1}^{f} R_t \right) \oplus \left(\bigoplus_{\substack{v \in Q_0 \\ \delta(v)=(2,2)}} (P(v) \oplus I(v)) \right). \tag{3.4}$$

The main aim of this section is to show that M is the unique n-cluster tilting module of Λ. We start by giving an alternate description of M.

Corollary 3.7. The module M is basic and $M \cong \bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda)$. In particular, we have that $D(\Lambda) \in M$.

Proof. We set

$$R \coloneqq \bigoplus_{t=1}^{f} R_t, \text{ and } M_{(2,2)} \coloneqq \bigoplus_{\substack{v \in Q_0 \\ \delta(v)=(2,2)}} (P(v) \oplus I(v)).$$

By Lemma 2.19 we have that

$$\Lambda \cong \bigoplus_{v\in Q_0} P(v) \cong \left(\bigoplus_{\mathbf{v}\in\mathbf{V}} P(\mathbf{v}) \right) \oplus R \oplus \left(\bigoplus_{\substack{v \in Q_0 \\ \delta(v)=(2,2)}} P(v) \right).$$

Then by (3.4) and Corollary 2.22 it follows that $M \cong \bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda)$.

To see that M is basic, we have that $M(\mathbf{V})$ is basic by Lemma 3.4(b), that R is basic by definition and that $M_{(2,2)}$ is basic since $P(v)$ is never injective if $\delta^+(v) = 2$. By Corollary 2.28 and by Lemma 2.19 and its dual and by comparing direct summands of $M(\mathbf{V})$, R and $M_{(2,2)}$, it easily follows that M is basic.

Finally, we show that $D(\Lambda) \in \mathrm{add}(M)$. It is enough to show that for every vertex $v \in Q_0$, the indecomposable injective Λ-module $I(v)$ corresponding to the vertex $v \in Q_0$ belongs to $\mathrm{add}(M)$. If $\delta(v) = (2,2)$ or $I(v)$ is projective, then clearly $I(v) \in \mathrm{add}(M)$ by the definition of M. Otherwise, by the dual of Lemma 2.19 it follows that $I(v) \cong I(\mathbf{v})$ for some flow path $\mathbf{v}$ in Q. Then by Corollary 2.28 and Proposition 1.1(a) we have

$$I(v) \cong \tau_n^{-p(\mathbf{v})}(P(\mathbf{v})) \in \mathrm{add}(M),$$

as required. □

Next we want to show that M is n-rigid.

Proposition 3.8. Let $i \in \{1, \dots, n-1\}$. Then $\mathrm{Ext}^i_\Lambda(M, M) = 0$.

Proof. By Lemma 3.5 and since R_t is projective-injective for every $t \in \{1, \dots, f\}$, it follows that the module $M(\mathbf{V}) \oplus \left(\bigoplus_{t=1}^{f} R_t \right)$ is n-rigid. Hence if there exists no vertex $v \in Q_0$ with degree $\delta(v) = (2,2)$, the result follows immediately, while if there exists a vertex $v \in Q_0$ with degree $\delta(v) = (2,2)$, the result follows from Lemma 3.6. □

We are now ready to show that M is n-cluster tilting.

Proposition 3.9. The module M is an n-cluster tilting Λ-module and any basic n-cluster tilting Λ-module is isomorphic to M.

Proof. To show that M is an n-cluster tilting module we need to show that

$$\begin{aligned}\operatorname{add}(M) &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M,X) = 0 \text{ for all } 0 < i < n\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,M) = 0 \text{ for all } 0 < i < n\}.\end{aligned}$$

We only show the first equality; the other follows dually. Since by Proposition 3.8 the module M is n-rigid, the inclusion

$$\operatorname{add}(M) \subseteq \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M,X) = 0 \text{ for all } 0 < i < n\}$$

holds. It remains to show the opposite inclusion, that is that if $\operatorname{Ext}^i_\Lambda(M,X) = 0$ for all $0 < i < n$, then $X \in \operatorname{add}(M)$. We show the contrapositive statement that if $X \notin \operatorname{add}(M)$, then $\operatorname{Ext}^i_\Lambda(M,X) \neq 0$ for some $i \in \{1,\dots,n-1\}$. By additivity of $\operatorname{Ext}^i_\Lambda(-,-)$ we may assume that X is indecomposable. Since by Corollary 3.7 we have that $\Lambda \in \operatorname{add}(M)$ and $D(\Lambda) \in \operatorname{add}(M)$, it follows that X is neither projective nor injective. Since X is neither projective nor injective, it follows from Theorem 2.10(d) that X is simple. Then $X \cong S(v)$ for some vertex $v \in Q_0$. Clearly $\delta(v) \neq (0,1)$ and $\delta(v) \neq (1,0)$ because in the first case we have that $S(v)$ is injective while in the second case we have that $S(v)$ is projective. We consider the cases $\delta^-(v) = 2$ and $\delta^-(v) = 1$ separately.

Case $\delta^-(v) = 2$. In this case we have by Lemma 2.20(c) that $\tau^-(S(v)) \cong I(v)$. By Lemma 1.2 it follows that $\operatorname{Ext}^1_\Lambda(I(v),S(v)) \neq 0$. Since by Corollary 3.7 we have that $I(v) \in \operatorname{add}(M)$, we conclude that $\operatorname{Ext}^1_\Lambda(M,S(v)) \neq 0$, as required.

Case $\delta^-(v) = 1$. In this case we have that $\delta(v) = (1,1)$ or $\delta(v) = (1,2)$. By Corollary 2.17 there exists a unique k-flow path $\mathbf{v}$ in Q such that $v = v_j$ for some $j > 1$. Notice that $j < k + q_k$ also holds. We first claim that n does not divide $k - j + q_k$.

To show this, assume towards a contradiction that $k - j + q_k = mn$ for some $m \in \mathbb{Z}_{\geq 0}$. Then $j = k - mn + q_k$. Since we have $1 < j < k + q_k$, we obtain

$$1 < k - mn + q_k < k + q_k.$$

Using $k + q(\mathbf{v}) = p(\mathbf{v})n$ and $q(\mathbf{v}) = -1 + q_1 + q_k$, we obtain that

$$0 < m < p(\mathbf{v}) - \frac{q_1}{n},$$

which implies $0 < m < p(\mathbf{v})$. But then by Corollary 2.28 we have

$$X \cong S(v) = S(v_j) = S(v_{k-mn+q_k}) \cong \tau_n^{-m}(P(\mathbf{v})) \in \operatorname{add}(M),$$

which contradicts $X \notin \operatorname{add}(M)$.

Hence n does not divide $k - j + q_k$. Let m be the unique integer such that $m < \frac{k-j+q_k}{n} < m+1$. Using $1 < j < k + q_k$, we obtain that $0 \leq m \leq p(\mathbf{v}) - 1$. Then by Lemma 2.26, Corollary 2.28 and Lemma 2.25 it follows that

$$\Omega^{-(k-mn+q_k-j)}\tau_n^{-m}(P(\mathbf{v})) \cong S(v_j). \tag{3.5}$$

Set $i := (m+1)n - k - q_k + j$. Since $m < \frac{k-j+q_k}{n} < m+1$, we obtain that $0 < i < n$. Then, using (3.5), we compute

$$\begin{aligned}\tau_i^-(S(v_j)) &\cong \tau_i^-\Omega^{-(k-mn+q_k-j)}\tau_n^{-m}(P(\mathbf{v})) \\ &= \tau^-\Omega^{-(i-1+k-mn+q_k-j)}\tau_n^{-m}(P(\mathbf{v})) \\ &= \tau^-\Omega^{-((m+1)n-k-q_k+j-1+k-mn+q_k-j)}\tau_n^{-m}(P(\mathbf{v})) \\ &= \tau^-\Omega^{-(n-1)}\tau_n^{-m}(P(\mathbf{v})) \\ &= \tau_n^{-(m+1)}(P(\mathbf{v})).\end{aligned}$$

By Corollary 2.28 and since $0 \leq m \leq p(\mathbf{v}) - 1$, it follows that $\tau_n^{-(m+1)}(P(\mathbf{v})) \neq 0$. Then by Lemma 1.2 we have that $\operatorname{Ext}^i_\Lambda(\tau_n^{-(m+1)}(P(\mathbf{v})),S(v_j)) \neq 0$, which shows that $\operatorname{Ext}^i_\Lambda(M,S(v)) \neq 0$ since $\tau_n^{-(m+1)}(P(\mathbf{v})) \in \operatorname{add}(M)$.

Finally, the fact that M is the unique basic n-cluster tilting module up to isomorphism follows from Proposition 1.1(c). □

4. Main result and applications

We are now ready to state our main result.

Theorem 4.1. *Let Q be a quiver, let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and let $n \in \mathbb{Z}_{\geq 2}$. Then the algebra Λ admits an n-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ if and only if Q is an n-admissible quiver. If moreover $Q \neq \tilde{A}_m$ for any $m \geq 1$, then $\mathcal{C}$ is unique and $\mathcal{C} = \operatorname{add}\left(\bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda)\right)$.*

Proof. The statement that if Λ admits an n-cluster tilting subcategory, then Q is an n-admissible quiver follows from Proposition 2.9, Proposition 2.14 and Proposition 2.27. The statement that if Q is an n-admissible quiver, then Λ admits an n-cluster tilting subcategory follows from Proposition 2.14 and Proposition 3.9. The description of $\mathcal{C}$ in the case $Q \neq \tilde{A}_m$ follows from Proposition 3.9. □

Remark 4.2. In Theorem 4.1 we classify n-cluster tilting subcategories for bound quiver algebras of the form $\mathbf{k}Q/\mathcal{J}^2$ when $n \geq 2$. We also find that all of them are of the form $\operatorname{add}(M)$ for an n-cluster tilting module M. If $n = 1$, then the algebra $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ admits a unique 1-cluster tilting subcategory, namely the whole module category $\operatorname{mod}\Lambda$. Moreover, the module category $\operatorname{mod}\Lambda$ is of the form $\operatorname{add}(M)$ if and only if Λ is a representation-finite algebra. A result of Gabriel [Gab72] classifies representation-finite algebras with radical square zero in terms of their *separated quiver*; see also [ARS95, Section X.2].

Using Theorem 4.1 we can construct many examples of algebras that admit n-cluster tilting modules and have many interesting properties. As an example, Erdmann and Holm ask in paragraph 5.5 of [EH08] whether there can exist n-cluster tilting modules for non-selfinjective algebras which admit modules of complexity bigger than 1. A positive answer to this question is given in [MV22] using radical square zero bound quiver algebras.

Example 4.3. (a) Let $Q = A_m$, let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and let $n \geq 2$ be such that $n \mid (m-1)$. Then

$$\Lambda \oplus \left(\bigoplus_{j=1}^{\frac{m-1}{n}} S(m - jn) \right)$$

is the unique basic n-cluster tilting Λ-module.

(b) Let Q be as in Example 2.8(c) and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$. Then the module

$$\begin{aligned} M &= \Lambda \oplus \tau_2^{-}(\Lambda) \oplus \tau_2^{-2}(\Lambda) \\ &\cong \Lambda \oplus \left(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 7 \oplus \begin{smallmatrix}2&&8\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}4&&5\\&5&\end{smallmatrix} \oplus \begin{smallmatrix}3\\6\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\end{smallmatrix} \right) \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \end{aligned}$$

is the unique basic 2-cluster tilting Λ-module.

(c) Let Q be as in Example 2.8(d) and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$. Then the module

$$\begin{aligned} M &= \Lambda \oplus \tau_3^{-}(\Lambda) \\ &\cong \Lambda \oplus \left(\begin{smallmatrix}10&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&12\\&4&\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}&5\\11&\end{smallmatrix} \oplus \begin{smallmatrix}&1\\9&\end{smallmatrix} \right) \end{aligned}$$

is the unique basic 3-cluster tilting Λ-module.

In the rest of this section we further investigate some properties of radical square zero bound quiver algebras which admit n-cluster tilting subcategories. We start with describing a method to construct n-admissible quivers.

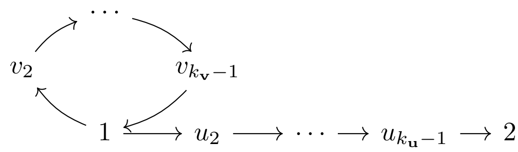
Remark 4.4. Starting from any n-pre-admissible quiver Q, it is not difficult to construct an n-admissible quiver by adjusting the lengths of flow paths in Q appropriately. For example, if Q is the quiver

$$\overset{\circlearrowright}{1} \longrightarrow 2,$$

then Q is n-pre-admissible for any $n \geq 2$ and there are two flow paths in Q, namely

$$\mathbf{v} : 1 \longrightarrow 1,$$
$$\mathbf{u} : 1 \longrightarrow 2.$$

In particular, we have $q(\mathbf{v}) = 0$ and $q(\mathbf{u}) = -1$. Now let us fix an $n \geq 2$ and construct an n-admissible quiver. We pick $k_\mathbf{v}, k_\mathbf{u} \geq 2$ such that $n \mid k_\mathbf{v}$ and $n \mid (k_\mathbf{u} - 1)$. Then the quiver



is n-admissible.

4.1. **$n\mathbb{Z}$-cluster tilting subcategories.** We recall the definition of $n\mathbb{Z}$-cluster tilting subcategories from [IJ17].

Definition 4.5. [IJ17, Definition-Proposition 2.15] Let Λ be an algebra and let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ be an n-cluster tilting subcategory. We say that $\mathcal{C}$ is an *$n\mathbb{Z}$-cluster tilting subcategory* if one of the two equivalent conditions

(a) $\Omega^n(\mathcal{C}) \subseteq \mathcal{C}$, and
(b) $\Omega^{-n}(\mathcal{C}) \subseteq \mathcal{C}$

holds.

In this subsection we classify radical square zero bound quiver algebras which admit $n\mathbb{Z}$-cluster tilting subcategories. We start with the following proposition.

Proposition 4.6. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that there exists an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$. Let $v \in Q_0$ be a vertex of Q. Then $\delta(v) \in \{(0,0),(0,1),(1,0),(1,1)\}$.

Proof. Since $\mathcal{C}$ is an $n\mathbb{Z}$-cluster tilting subcategory, it follows that Q is an n-admissible quiver. Hence $\delta^+(v) \leq 2$ and $\delta^-(v) \leq 2$ and it is enough to show that $\delta^+(v) \neq 2$ and $\delta^-(v) \neq 2$. We show that $\delta^+(v) \neq 2$; the fact that $\delta^-(v) \neq 2$ follows dually.

Assume towards a contradiction that $\delta^+(v) = 2$ and let $v \longrightarrow u_1$ and $v \longrightarrow u_2$ be the two arrows starting at v. Then $P(v) \in \mathcal{C}$ and $P(v)$ is not injective. It follows from Proposition 1.1(a) that $P(v) \cong \tau_n(X)$ for some nonprojective indecomposable module $X \in \mathcal{C}$. In particular, the module $\Omega^{n-1}(X)$ is nonprojective and so we have

$$\Omega^{n-1}(X) \cong \tau^-\tau\Omega^{n-1}(X) = \tau^-\tau_n(X) \cong \tau^-(P(v)) \cong S(v),$$

where the last isomorphism follows from Lemma 2.21(c). Since by Lemma 2.21(c) we have $\Omega(S(v)) \cong S(u_1) \oplus S(u_2)$, we obtain that

$$\Omega^n(X) = \Omega\Omega^{n-1}(X) \cong \Omega(S(v)) \cong S(u_1) \oplus S(u_2).$$

Since $\mathcal{C}$ is an $n\mathbb{Z}$-cluster tilting subcategory, it follows that $S(u_1) \oplus S(u_2) \in \mathcal{C}$. But then a direct computation shows that $\Omega(I(u_2)) \cong S(u_1)$, from which we conclude that $\operatorname{Ext}^1_\Lambda(I(u_2), S(u_1) \oplus S(u_2)) \neq 0$. This contradicts the fact that $\mathcal{C}$ is an n-cluster tilting subcategory since $I(u_2), S(u_1) \oplus S(u_2) \in \mathcal{C}$. $\square$

We can now give the classification of $n\mathbb{Z}$-cluster tilting subcategories for radical square zero bound quiver algebras.

Theorem 4.7. *Let Q be a quiver, let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and let $n \in \mathbb{Z}_{\geq 2}$. Then the algebra Λ admits an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ if and only if $Q = A_m$ and $n \mid (m-1)$ or $Q = \tilde{A}_m$ and $n \mid m$.*

Proof. If $Q = A_m$ and $n \mid (m-1)$ or $Q = \tilde{A}_m$ and $n \mid m$, then Λ admits an n-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ by Theorem 4.1. Moreover, in this case, it is easy to see that $\tau(M) \cong \Omega(M)$ for any $M \in \operatorname{mod}\Lambda$ and hence $\mathcal{C}$ is also an $n\mathbb{Z}$-cluster tilting subcategory by Proposition 1.1(a).

For the other direction, assume that Λ admits an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C}$. Then by Proposition 4.6 we have that if $v \in Q_0$, then $\delta(v) \in \{(0,0),(1,0),(0,1),(1,1)\}$. Since Q is connected, we conclude that there exists some $m \in \mathbb{Z}_{\geq 1}$ such that $Q = A_m$ or $Q = \tilde{A}_m$. Since $\mathcal{C}$ is n-cluster tilting, it follows that Q is n-admissible from Theorem 4.1. Hence we conclude that if $Q = A_m$, then $n \mid (m-1)$, while if $Q = \tilde{A}_m$, then $n \mid m$, as required. □

In particular we see that the only radical square zero bound quiver algebras which admit $n\mathbb{Z}$-cluster tilting subcategories are Nakayama algebras.

4.2. **A lattice of n-cluster tilting subcategories.** Before giving our next result, let us recall the following classical definition.

Definition 4.8. A *poset* is a partially ordered set. A *lattice* is a partially ordered set in which every two elements have a *meet*, that is a greatest lower bound and a *join*, that is a least upper bound. A *complete lattice* is a lattice in which any subset has a greatest lower bound and a least upper bound.

Example 4.9. Let N be a positive integer. Then the set $D(N) = \{x \in \mathbb{Z} \mid x \geq 1 \text{ and } x \mid N\}$ forms a complete lattice called the *lattice of divisors of* N under the relation $x \leq y$ if $x \mid y$. If $x, y \in D(N)$, then their meet corresponds to their greatest common divisor $\gcd(x,y)$ and their join corresponds to their least common multiple $\operatorname{lcm}(x,y)$.

For the rest of this article, we drop our assumption that we consider n-cluster tilting subcategories for $n \geq 2$ and we assume that $n \geq 1$ instead. Let $Q \neq \tilde{A}_m$ be a quiver and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$. Our aim is to show that the collection of n-cluster tilting subcategories (for varying n) of $\operatorname{mod}\Lambda$ forms a lattice with respect to inclusion of subcategories. We start with the following result.

Proposition 4.10. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ be a radical zero bound quiver algebra and assume that $Q \neq A_1$ and $Q \neq \tilde{A}_m$ for any $m \geq 1$. Let $\mathcal{C}_n \subseteq \operatorname{mod}\Lambda$ be an n-cluster tilting subcategory and $\mathcal{C}_{n'} \subseteq \operatorname{mod}\Lambda$ be an n'-cluster tilting subcategory. Then $\mathcal{C}_n \subseteq \mathcal{C}_{n'}$ if and only if $n' \mid n$.

Proof. If $n' = 1$, then the result is clear since $\mathcal{C}_{n'} = \mathcal{C}_1 = \operatorname{mod}\Lambda$. If $n = 1$, then the result is also clear since $\operatorname{mod}\Lambda$ is an n-cluster tilting subcategory if and only if $n = 1$ (since $Q \neq A_1$).

Hence we may assume that $n > 1$ and $n' > 1$. Then Q is n-admissible and n'-admissible by Theorem 4.1. Moreover, we have that $\mathcal{C}_n = \operatorname{add}(M_n)$ and $\mathcal{C}_{n'} = \operatorname{add}(M_{n'})$ where

$$M_n = \bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda) \text{ and } M_{n'} = \bigoplus_{j\geq 0} \tau_{n'}^{-j}(\Lambda).$$

Assume first that $n' \mid n$. Then $n = hn'$ for some $h \geq 1$. Let $X \in \mathcal{C}_n$ and we show that $X \in \mathcal{C}_{n'}$. Since $\mathcal{C}_n$ and $\mathcal{C}_{n'}$ are closed under direct sums and summands, we may assume that X is indecomposable. If X is projective or injective, then $X \in \mathcal{C}_{n'}$ since $\mathcal{C}_{n'}$ is an n'-cluster tilting subcategory. Otherwise we have by (3.4) and Corollary 3.7 that $X \cong \tau_n^{-j}(P(\mathbf{v}))$ for some k-flow path $\mathbf{v}$ in Q and some $j \geq 1$. Since Q is n-admissible and n'-admissible, we have that $k + q(\mathbf{v}) = pn$ and $k + q(\mathbf{v}) = p'n'$ for some $p, p' \in \mathbb{Z}_{\geq 1}$. In particular, we have $p = \frac{p'n'}{n}$. Moreover, by Corollary 2.28 and since X is not injective, we have that $1 \leq j \leq p-1$. Hence we obtain

$$1 \leq jh \leq (p-1)h = ph - h = \frac{p'n'}{n}\frac{n}{n'} - h = p' - h \leq p' - 1,$$

and so $1 \leq jh \leq p' - 1$. Hence by Corollary 2.28 we have

$$X \cong \tau_n^{-j}(P(\mathbf{v})) \cong S(v_{k-jn+q_k}) = S(v_{k-jhn'+q_k}) \cong \tau_{n'}^{-jh}(P(\mathbf{v})) \in \operatorname{add}(M_{n'}) = \mathcal{C}_{n'},$$

as required.

Assume now that $\mathcal{C}_n \subseteq \mathcal{C}_{n'}$. Then by Lemma 2.13 there exists a k-flow path $\mathbf{v}$ in Q. Since Q is n-admissible and n'-admissible, we have that $k + q(\mathbf{v}) = pn$ and $k + q(\mathbf{v}) = p'n'$ for some $p, p' \in \mathbb{Z}_{\geq 1}$. If $p = 1$, then $n = p'n'$ and $n' \mid n$ as required. Otherwise, assume that $p > 1$. Then by Corollary 2.28 and Proposition 1.1(a) we have that

$$\tau_n^{-}(P(\mathbf{v})) \cong S(v_{k-n+q_k}) \in \mathcal{C}_n.$$

Since by assumption we have $\mathcal{C}_n \subseteq \mathcal{C}_{n'}$, we conclude that $S(v_{k-n+q_k}) \in \mathcal{C}_{n'}$. Write $n = hn' + r$ with $h \in \mathbb{Z}_{\geq 0}$ and $0 \leq r \leq n' - 1$. We first claim that $1 \leq h \leq p' - 1$.

First assume towards a contradiction that $h = 0$. Then $n = r$ and

$$k - n + q_k = (p-1)n + 1 - q_1 \geq (2-1) \cdot 1 + 1 - 1 = 1.$$

Hence by Lemma 2.26(a) we have that $\Omega^{-n}(P(\mathbf{v})) \cong S(v_{k-n+q_k})$. Since $0 < n = r < n'$, we have $0 < n' - n < n'$. Hence by Proposition 1.1(a) we obtain

$$\tau_{n'}^{-}(P(\mathbf{v})) = \tau_{n'-n}^{-}\Omega^{-n}(P(\mathbf{v})) \cong \tau_{n'-n}^{-}(S(v_{k-n+q_k})) \in \mathcal{C}_{n'}.$$

It follows from Lemma 1.2 that $\operatorname{Ext}_{\Lambda}^{n'-n}(\tau_{n'}^{-}(P(\mathbf{v})), S(v_{k-n+q_k})) \neq 0$. This contradicts the fact that $\mathcal{C}_{n'}$ is n'-cluster tilting since $\tau_{n'}^{-}(P(\mathbf{v})), S(v_{k-n+q_k}) \in \mathcal{C}_{n'}$.

Next assume towards a contradiction that $h \geq p'$. Then we have

$$n < pn = k + q(\mathbf{v}) = p'n' \leq hn' \leq hn' + r = n,$$

which is a contradiction.

We conclude that $n = hn' + r$ with $1 \leq h \leq p' - 1$ and we now claim that $r = 0$. Assume towards a contradiction that $r > 0$. By Corollary 2.28 and Proposition 1.1(a) we have that

$$\tau_{n'}^{-h}(P(\mathbf{v})) \cong S(v_{k-hn'+q_k}) \in \mathcal{C}_{n'}.$$

Then $1 \leq k - hn' + q_k \leq k - 1 + q_k$ and $(k - hn' + q_k) - r = k - (hn' + r) + q_k = k - n + q_k \geq 1$, and so by Lemma 2.25(a) we have that

$$\Omega^{-r}(S(v_{k-hn'+q_k})) \cong S(v_{k-hn'-r+q_k}) = S(v_{k-n+q_k}).$$

But then we have that

$$\operatorname{Ext}_{\Lambda}^{r}(S(v_{k-n+q_k}), S(v_{k-hn'+q_k})) \cong \operatorname{Ext}_{\Lambda}^{r}(\Omega^{-r}(S(v_{k-hn'+q_k})), S(v_{k-hn'+q_k})) \neq 0.$$

This contradicts the fact that $\mathcal{C}_{n'}$ is n'-cluster tilting since $S(v_{k-n+q_k}), S(v_{k-hn'+q_k}) \in \mathcal{C}_{n'}$ and $1 \leq r \leq n' - 1$. We conclude that $r = 0$ and so $n = hn'$, as required. □

We also need the following definition.

Definition 4.11. Let Q be a quiver. We define the *admissible degree* of Q to be

$$N(Q) \coloneqq \begin{cases} \max\left(\{n \in \mathbb{Z}_{\geq 2} \mid Q \text{ is } n\text{-admissible}\} \cup \{1\}\right), & \text{if } Q \neq A_1, \\ 1, & \text{if } Q = A_1. \end{cases}$$

Since Q is finite, it follows that $N(Q)$ is well-defined. We now give the main result for this section.

Theorem 4.12. *Let Q be a quiver with admissible degree $N = N(Q)$ and let $D(N) = \{n \in \mathbb{Z} \mid n \geq 1$ and $n \mid N\}$. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$.*

(a) If $Q \neq A_1$, then there exists an n-cluster tilting subcategory $\mathcal{C}_n \subseteq \operatorname{mod}\Lambda$ if and only if $n \in D(N)$.

(b) If $Q \neq \tilde{A}_m$, set

$$\mathbf{CT}(\Lambda) \coloneqq \{\mathcal{C} \subseteq \operatorname{mod}\Lambda \mid \text{there exists } n \in \mathbb{Z}_{\geq 1} \text{ such that } \mathcal{C} \text{ is } n\text{-cluster tilting}\}.$$

Then for every $n \in D(N)$ there exists a unique n-cluster tilting subcategory $\mathcal{C}_n$. Moreover, the pair $(\mathbf{CT}(\Lambda), \subseteq)$ is a poset isomorphic to the opposite of the poset of divisors of N. In particular, $(\mathbf{CT}(\Lambda), \subseteq)$ forms a complete lattice where the meet of $\mathcal{C}_n$ and $\mathcal{C}_{n'}$ is given by $\mathcal{C}_{\operatorname{lcm}(n,n')}$ and the join of $\mathcal{C}_n$ and $\mathcal{C}_{n'}$ is given by $\mathcal{C}_{\gcd(n,n')}$.

Proof. (a) For $n = 1$ the result is clear since $\operatorname{mod}\Lambda$ is a 1-cluster tilting subcategory. Assume now that $n > 1$. If $n \in D(N)$, then it follows from Remark 3.3(b) and Theorem 4.1 that Λ admits an n-cluster tilting subcategory, which proves one direction.

For the other direction assume that there exists an n-cluster tilting subcategory $\mathcal{C}_n \subseteq \operatorname{mod}\Lambda$ and we show that $n \in D(N)$. It follows from Theorem 4.1 that Q is n-admissible and so

$1 < n \leq N$. Hence by Definition 4.11 it follows that Q is N-admissible too. We consider the cases $Q = \tilde{A}_m$ and $Q \neq \tilde{A}_m$ separately.

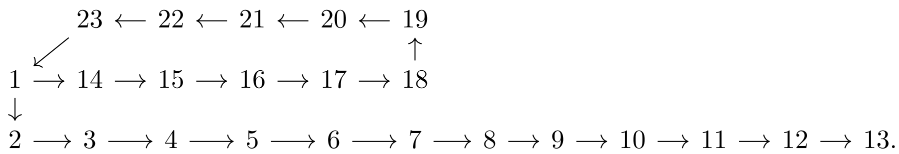
If $Q = \tilde{A}_m$ for some $m \geq 1$, then we have by Definition 3.1 that $N \mid m$ and so $N \leq m$. Moreover, the quiver $\tilde{A}_m$ is always m-admissible and so $m \leq N$. It follows that $m = N$. Since Q is also n-admissible, we have that $n \mid m = N$ and so $n \in D(N)$.

If $Q \neq \tilde{A}_m$, then there exists a flow path $\mathbf{v}$ in Q by Lemma 2.13. Moreover, for every $k_{\mathbf{v}}$-flow path $\mathbf{v}$ we have that $n \mid (k_{\mathbf{v}} + q(\mathbf{v}))$ and $N \mid (k_{\mathbf{v}} + q(\mathbf{v}))$. It follows that $\operatorname{lcm}(n, N) \mid (k_{\mathbf{v}} + q(\mathbf{v}))$ for every flow path $\mathbf{v}$ in Q. Hence Q is $\operatorname{lcm}(n, N)$-admissible and so $\operatorname{lcm}(n, N) \leq N$. We conclude that $n \mid N$ and so $n \in D(N)$.

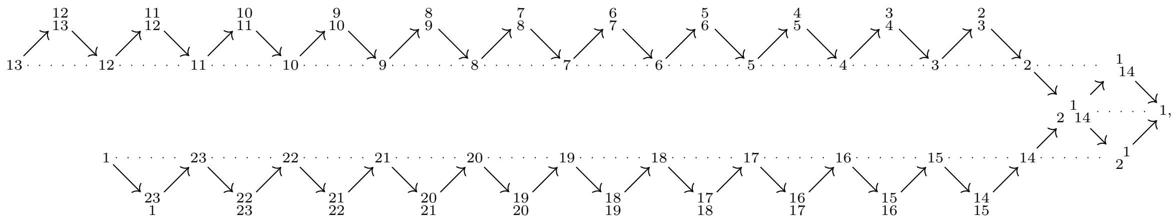
(b) If $N(Q) = 1$, then $\mathbf{CT}(\Lambda) = \{\operatorname{mod}\Lambda\}$ and the result is clear. If $N(Q) > 1$, then existence of $\mathcal{C}_n$ follows from (a) and uniqueness by Theorem 4.1. Then $(\mathbf{CT}(\Lambda), \subseteq)$ is a poset isomorphic to the opposite of the poset of divisors of N by Proposition 4.10. That $\mathbf{CT}(\Lambda)$ forms a complete lattice with the given meet and join follows from Example 4.9. □

We finish with an example which illustrates Theorem 4.12.

Example 4.13. Let Q be the quiver



Then $N(Q) = 12$. The Auslander–Reiten quiver of $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ is



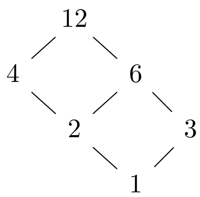
where the simple module $S(1)$ appears twice. The divisors of 12 are $D(12) = \{1, 2, 3, 4, 6, 12\}$. For $n \in D(12)$ we set $M_n = \bigoplus_{j \geq 0} \tau_n^{-j}(\Lambda)$ and $\mathcal{C}_n = \operatorname{add}(M_n)$. Then we have

$\mathcal{C}_1 = \operatorname{mod}\Lambda$, $\quad \mathcal{C}_2 = \operatorname{add}\{\Lambda, 11, 9, 7, 5, 3, \begin{smallmatrix}1\\14\end{smallmatrix}, 23, 21, 19, 17, 15, \begin{smallmatrix}1\\2\end{smallmatrix}\}$,

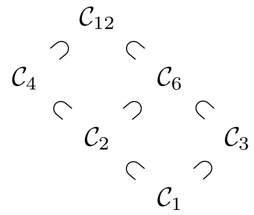
$\mathcal{C}_3 = \operatorname{add}\{\Lambda, 10, 7, 4, \begin{smallmatrix}1\\14\end{smallmatrix}, 22, 19, 16, \begin{smallmatrix}1\\2\end{smallmatrix}\}$, $\quad \mathcal{C}_4 = \operatorname{add}\{\Lambda, 9, 5, \begin{smallmatrix}1\\14\end{smallmatrix}, 21, 17, \begin{smallmatrix}1\\2\end{smallmatrix}\}$,

$\mathcal{C}_6 = \operatorname{add}\{\Lambda, 7, \begin{smallmatrix}1\\14\end{smallmatrix}, 19, \begin{smallmatrix}1\\2\end{smallmatrix}\}$, $\quad \mathcal{C}_{12} = \operatorname{add}\{\Lambda, \begin{smallmatrix}1\\14\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\}$,

and $\mathcal{C}_n$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ by Theorem 4.1. Then the lattice



of divisors of 12 corresponds to the lattice



of inclusions of n-cluster tilting subcategories of $\operatorname{mod}\Lambda$.

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