

AG
T

*Algebraic & Geometric
Topology*

Volume 23 (2023)

The $\mathrm{Sp}_{k,n}$ -local stable homotopy category

DREW HEARD



The $\mathrm{Sp}_{k,n}$ -local stable homotopy category

DREW HEARD

We study the category of $(K(k) \vee K(k+1) \vee \cdots \vee K(n))$ -local spectra, following a suggestion of Hovey and Strickland. When $k = 0$, this is equivalent to the category of $E(n)$ -local spectra, while for $k = n$, this is the category of $K(n)$ -local spectra, both of which have been studied in detail by Hovey and Strickland. Based on their ideas, we classify the localizing and colocalizing subcategories, and give characterizations of compact and dualizable objects. We construct an Adams-type spectral sequence and show that when $p \gg n$ it collapses with a horizontal vanishing line above filtration degree $n^2 + n - k$ at the E_2 -page for the sphere spectrum. We then study the Picard group of $(K(k) \vee K(k+1) \vee \cdots \vee K(n))$ -local spectra, showing that this group is algebraic, in a suitable sense, when $p \gg n$. We also consider a version of Gross–Hopkins duality in this category. A key concept throughout is the use of descent.

55P42, 55P60; 55T15

1. Introduction	3656
2. The category of $\mathrm{Sp}_{k,n}$ -local spectra	3660
3. Thick subcategories and (co)localizing subcategories	3667
4. Descent theory and the $E(n, J_k)$ -local Adams spectral sequence	3679
5. Dualizable objects in $\mathrm{Sp}_{k,n}$	3687
6. The Picard group of the $\mathrm{Sp}_{k,n}$ -local category	3693
7. $E(n, J_k)$ -local Brown–Comenetz duality	3699
References	3702

1 Introduction

In their memoir [32] Hovey and Strickland studied the categories of $K(n)$ –local and E_n –local spectra in great detail. Here $K(n)$ is the n^{th} Morava K –theory; the spectrum whose homotopy groups are the graded field

$$K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}], \quad |v_n| = 2(p^n - 1)$$

and E_n is the n^{th} Lubin–Tate spectrum, or Morava E –theory, with

$$(E_n)_* \cong W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]], \quad |u_i| = 0, |u| = 2.$$

As explained in the introduction of [32], the Morava K –theories are the prime field objects in the stable homotopy category — for a way to make that precise, see Hopkins and Smith [27], or more specifically, Balmer [4, Corollary 9.5] — and are one of the fundamental objects in the chromatic approach to stable homotopy theory.

A deep result of Hopkins and Ravenel [48] is that Bousfield localization with respect to E_n is smashing, which simplifies the study of the category of E_n –local spectra considerably. On the other hand, localization with respect to $K(n)$ is not smashing [32, Lemma 8.1], and the monoidal unit $L_{K(n)}S^0 \in \text{Sp}_{K(n)}$ is dualizable, but not compact. In the language of tensor-triangulated geometry, $\text{Sp}_{K(n)}$ is a nonrigidly compactly generated category. Because of this, much of the work in [32] is therefore dedicated to understanding the more complicated category of $K(n)$ –local spectra.

By a Bousfield class argument, the category of E_n –local spectra is equivalent to the category of $(K(0) \vee \dots \vee K(n))$ –local spectra. In this paper we study the categories of $(K(k) \vee \dots \vee K(n))$ –local spectra for $0 \leq k \leq n$, which were suggested as “interesting to investigate” by Hovey and Strickland; see the remark after Corollary B.9 in [32]. We write $L_{k,n}$ for the associated Bousfield localization functor. As we shall see, when $k \neq 0$, the category $\text{Sp}_{k,n}$ of $(K(k) \vee \dots \vee K(n))$ –local spectra behaves much like the category $\text{Sp}_{K(n)} = \text{Sp}_{n,n}$ of $K(n)$ –local spectra. For example, it is an example of a nonrigidly compactly generated category; as soon as $k \neq 0$, the monoidal unit $L_{k,n}S^0 \in \text{Sp}_{k,n}$ is dualizable, but not compact. However, the categories $\text{Sp}_{k,n}$ for $k \neq n$ are in some sense more complicated than the case $k = n$; for example, $\text{Sp}_{K(n)}$ has no nontrivial (co)localizing subcategories, while this is not true for $\text{Sp}_{k,n}$ as long as $k \neq n$.

1A Contents of the paper

We now describe the contents of the paper in more detail. We begin with a study of Bousfield classes, constructing some other spectra which are Bousfield equivalent to $K(k) \vee \cdots \vee K(n)$. In particular, we show that there is a Bousfield equivalence between a localized quotient of BP , denoted by $E(n, J_k)$, and $K(k) \vee \cdots \vee K(n)$. For this reason, as well as brevity, we often say that X is $E(n, J_k)$ -local, instead of $(K(k) \vee \cdots \vee K(n))$ -local.

As was already noted by Hovey and Strickland [32, Corollary B.9], $\mathrm{Sp}_{k,n}$ is an algebraic stable homotopy theory in the sense of Hovey, Palmieri and Strickland [30] with compact generator $L_{k,n}F(k)$, the localization of a finite spectrum of type k . We investigate some consequences of this; for example, analogous to Hovey and Strickland's formulas for $L_{K(n)}X$, in Proposition 2.24, we prove some formulas for $L_{k,n}X$ in terms of towers of finite type k Moore spectra. Some of these results had previously been obtained by the author and Barthel and Valenzuela [11].

In Section 3 we investigate the tensor-triangulated geometry of $\mathrm{Sp}_{k,n}$. We begin by characterizing the compact objects in $\mathrm{Sp}_{k,n}$, culminating in Theorem 3.8 which is a natural extension of Hovey and Strickland's results in the cases $k = 0, n$. A classification of the thick ideals of $\mathrm{Sp}_{k,n}^\omega$ is an almost immediate consequence of this classification; see Theorem 3.16 for the precise result. Of course, here we rely on the deep thick subcategory theorem in stable homotopy [27] and its consequences. Finally, we classify the localizing and colocalizing subcategories of $\mathrm{Sp}_{k,n}$ in Theorem 3.33. We obtain the following.

Theorem 1.1 *There is an order-preserving bijection between (co)localizing subcategories of $\mathrm{Sp}_{k,n}$ and subsets of $\{k, \dots, n\}$. Moreover, the map that sends a localizing subcategory \mathcal{C} of $\mathrm{Sp}_{k,n}$ to its left orthogonal \mathcal{C}^\perp induces a bijection between the set of localizing and colocalizing subcategories of $\mathrm{Sp}_{k,n}$. The inverse map sends a colocalizing subcategory \mathcal{U} to its right orthogonal ${}^\perp\mathcal{U}$.*

We also compute the Bousfield lattice of $\mathrm{Sp}_{k,n}$ (Proposition 3.39) and show that a form of the telescope conjecture holds (Theorem 3.46).

In Section 4 we show that, as a consequence of the Hopkins–Ravenel smash product theorem, the commutative algebra object $E_n \in \mathrm{Sp}_{k,n}$ is descendable, in the sense of Mathew [41]. This has a number of immediate consequences. For example, it

implies the existence of a strongly convergent Adams-type spectral sequence, which we call the $E(n, J_k)$ -local E_n -Adams spectral sequence, computing $\pi_*(L_{k,n}X)$ for any spectrum X . Moreover, descendability implies this collapses with a horizontal vanishing line at a finite stage (independent of X). In the case of $X = S^0$ it is known that, in the cases $k = 0$ and $k = n$, this vanishing line already occurs on the E_2 -page so long as $p \gg n$. In order to generalize this result, we first show that when $X = S^0$, the E_2 -term of the $E(n, J_k)$ -local E_n -Adams spectral sequence can be given as the inverse limit of certain Ext groups computed in the category of $(E_n)_*E_n$ -comodules; see [Proposition 4.16](#) for the precise result. We are then able to utilize a chromatic spectral sequence and Morava's change of rings theorem to show the following ([Theorem 4.24](#)):

Theorem 1.2 *Suppose $p - 1$ does not divide $k + s$ for $0 \leq s \leq n - k$ (for example, if $p > n + 1$), then in the E_2 -term of the $E(n, J_k)$ -local E_n -Adams spectral sequence converging to $L_{k,n}S^0$, we have $E_2^{s,t} = 0$ for $s > n^2 + n - k$.*

In the case $k = 0$, this recovers a result of Hovey and Sadofsky [[31](#), Theorem 5.1].

As noted previously, so long as $k \neq 0$, the categories of dualizable and compact spectra do not coincide in $\mathrm{Sp}_{k,n}$; every compact spectrum is dualizable, but the converse does not hold, with the unit $L_{k,n}S^0$ being an example. In [Section 5](#) we study the category of dualizable objects in $\mathrm{Sp}_{k,n}$. As a consequence of descendability, we show that $X \in \mathrm{Sp}_{k,n}$ is dualizable if and only if $L_{k,n}(E_n \wedge X)$ is dualizable in the category of $E(n, J_k)$ -local E_n -modules. In turn, we show that this holds if and only if $L_{k,n}(E_n \wedge X)$ is dualizable (equivalently, compact) in the category of E_n -modules. We deduce that $X \in \mathrm{Sp}_{k,n}$ is dualizable if and only if its Morava module $(E_{k,n})_*^\vee(X) := \pi_*L_{k,n}(E_n \wedge X)$ is finitely generated as an $(E_n)_*$ -module; see [Theorem 5.11](#). This generalizes a result of Hovey and Strickland, but even in this case our proof differs from theirs.

It is an observation of Hopkins that the Picard group of invertible $K(n)$ -local spectra is an interesting object to study; see Hopkins, Mahowald and Sadofsky [[26](#)]. Likewise, Hovey and Sadofsky [[31](#)] have studied the Picard group of $E(n)$ -local spectra. In [Section 6](#) we study the Picard group $\mathrm{Pic}_{k,n}$ of $E(n, J_k)$ -local spectra. Our first result, which is a consequence of descent, is that $X \in \mathrm{Sp}_{k,n}$ is invertible if and only if its Morava module $(E_{k,n})_*^\vee(X)$ is free of rank 1. We then study the Picard spectrum—see Mathew and Stojanoska [[44](#)]—of the category $\mathrm{Sp}_{k,n}$. Using descent again, we construct a spectral sequence whose abutment for π_0 is exactly $\mathrm{Pic}_{k,n}$. The existence of this spectral sequence in the case $k = n$ is folklore. We say that this spectral sequence

is algebraic if the only nonzero terms in the spectral sequence occur in filtration degree 0 and 1. Using [Theorem 1.2](#) we deduce the following result ([Theorem 6.8](#)). In the case $k = n$, this is a theorem of Pstrągowski [\[45\]](#).

Theorem 1.3 *If $2p - 2 \geq n^2 + n - k$ and $p - 1$ does not divide $k + s$ for $0 \leq s \leq n - k$, then $\mathrm{Pic}_{k,n}$ is algebraic. For example, this holds if $2p - 2 > n^2 + n$.*

There is an interesting element in the $K(n)$ -local Picard group, namely the Brown–Comenetz dual of the monochromatic sphere [\[32, Theorem 10.2\]](#). In [Section 7](#) we extend Brown–Comenetz duality to the $E(n, J_k)$ -local category. We do not know when the Brown–Comenetz dual of the monochromatic sphere defines an element of $\mathrm{Pic}_{k,n}$; this is not true when $k = 0$, and we provide a series of equivalent conditions for the general case in [Proposition 7.10](#).

The case $n = 2$ and $k = 1$

The first example that has essentially not been studied in the literature is when $n = 2$ and $k = 1$, ie the category of $(K(1) \vee K(2))$ -local spectra. In [Section 5B](#) we give a computation of the Balmer spectrum of $(K(1) \vee K(2))$ -locally dualizable spectra. For this, we recall that Hovey and Strickland have conjectured a description of the Balmer spectrum $\mathrm{Spc}(\mathrm{Sp}_{K(n)}^{\mathrm{dual}})$ of dualizable objects in $K(n)$ -local spectra [\[32, page 61\]](#). This was investigated by the author, along with Barthel and Naumann, in [\[10\]](#). This admits a natural generalization to $\mathrm{Sp}_{k,n}^{\mathrm{dual}}$. For $i \leq n$, let \mathcal{D}_i denote the category of $X \in \mathrm{Sp}_{k,n}^{\mathrm{dual}}$ such that X is a retract of $Y \wedge Z$ for some $Y \in \mathrm{Sp}_{k,n}^{\mathrm{dual}}$ and some finite spectrum Z of type at least i . We also set $\mathcal{D}_{n+1} = (0)$. The conjecture is that these exhaust all the thick tensor-ideals of $\mathrm{Sp}_{k,n}^{\mathrm{dual}}$. We show in [Theorem 5.21](#) that if this holds $K(n)$ -locally (ie in $\mathrm{Sp}_{n,n}^{\mathrm{dual}}$), then it holds for all $\mathrm{Sp}_{k,n}^{\mathrm{dual}}$. In particular, since it is known to hold $K(2)$ -locally by [\[10, Theorem 4.15\]](#), we obtain the following; see [Corollary 5.22](#).

Theorem 1.4 *The Balmer spectrum of $K(1) \vee K(2)$ -locally dualizable spectra*

$$\mathrm{Spc}(\mathrm{Sp}_{1,2}^{\mathrm{dual}}) = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$$

with topology determined by $\overline{\{\mathcal{D}_j\}} = \{\mathcal{D}_i \mid i \geq j\}$. In particular, if \mathcal{C} is a thick tensor-ideal of $\mathrm{Sp}_{1,2}^{\mathrm{dual}}$, then $\mathcal{C} = \mathcal{D}_k$ for $0 \leq k \leq 3$.

Conventions and notation

We let $\langle X \rangle$ denote the Bousfield class of a spectrum X . The smallest thick tensor-ideal containing an object A will be denoted by $\mathrm{thick}_{\otimes} \langle A \rangle$ (it will always be clear

in which category this thick subcategory should be taken in). Likewise, the smallest thick (resp. localizing) subcategory containing an object A will be written as $\text{Thick}(A)$ (resp. $\text{Loc}(A)$).

Acknowledgements

It goes without saying that this paper owes a tremendous intellectual debt to Hovey and Strickland, in particular for the wonderful manuscript [32]. We also thank Neil Strickland for a helpful conversation, as well as his comments on a draft version of this document.

2 The category of $\text{Sp}_{k,n}$ -local spectra

2A Chromatic spectra

We begin by introducing some of the main spectra that we will be interested in.

Definition 2.1 Let BP denote the Brown–Peterson homotopy ring spectrum with coefficient ring

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

with $|v_i| = 2(p^i - 1)$.

Remark 2.2 The classes v_i are not intrinsically defined, and so the definition of BP depends on a choice of sequence of generators; for example, they could be the Hazewinkel generators or the Araki generators. However, the ideals $I_n = (p, v_1, \dots, v_{n-1})$ for $0 \leq n \leq \infty$ do not depend on this choice.

By taking quotients and localizations of BP —for example, using the theory of structured ring spectra [19, Chapter V]—we can form new homotopy ring spectra. In particular, let J_k denote a fixed invariant regular sequence $p^{i_0}, v_1^{i_1}, \dots, v_{k-1}^{i_{k-1}}$ of length k . Then we can form the homotopy associative ring spectrum BPJ_k with

$$(BPJ_k)_* \cong BP_*/J_k.$$

These were first studied by Johnson and Yosimura [35]. A detailed study on the product structure one obtains via this method can be found in [51].

Definition 2.3 We let $E(n, J_k)$ for $n \geq k$ denote the Landweber exact spectrum with

$$E(n, J_k)_* \cong v_n^{-1}((BPJ_k)_*/(v_{n+1}, v_{n+2}, \dots)).$$

Here Landweber exact means over BPJ_k (as studied by Yosimura [56]), that is, there is an isomorphism

$$E(n, J_k)_*(X) \cong (BPJ_k)_*(X) \otimes_{BP_*/J_k} E(n, J_k)_*.$$

Example 2.4 If $k = 0$ (so that J_k is the trivial sequence), then $E(n, J_0) \simeq E(n)$, Johnson–Wilson theory. For the other extreme, if $J_n = p, v_1, \dots, v_{n-1}$, then BPJ_n is the spectrum known as $P(n)$, and $E(n, J_n) \simeq K(n)$ is Morava K -theory [32].

Definition 2.5 For $k \leq n < \infty$, we let $\mathrm{Sp}_{k,n} \subseteq \mathrm{Sp}$ denote the full subcategory of $(K(k) \vee K(k+1) \vee \dots \vee K(n))$ -local spectra.

Lemma 2.6 The inclusion $\mathrm{Sp}_{k,n} \hookrightarrow \mathrm{Sp}$ has a left adjoint $L_{k,n}$, and $\mathrm{Sp}_{k,n}$ is a pre-*sentable*, stable ∞ -category.

Proof This is a consequence of [39, Proposition 5.5.4.15]. □

Remark 2.7 The category $\mathrm{Sp}_{k,n}$ and localization functor $L_{k,n}$ only depend on the Bousfield class $\langle K(k) \vee \dots \vee K(n) \rangle$.

Notation 2.8 We will follow standard conventions and write $\mathrm{Sp}_{0,n}$ as Sp_n and $\mathrm{Sp}_{n,n}$ as $\mathrm{Sp}_{K(n)}$. Similarly, the corresponding Bousfield localization functors will be denoted by $L_{0,n} = L_n$ and $L_{n,n} = L_{K(n)}$, respectively.

Remark 2.9 By [46, Theorem 2.1] we have $\langle E(n) \rangle = \langle K(0) \vee \dots \vee K(n) \rangle$. In fact, let E be a BP -module spectrum that is Landweber exact over BP , and is v_n -periodic, in the sense that $v_n \in BP_*$ maps to a unit in $E_*/(p, v_1, \dots, v_{n-1})$. Then Hovey has shown that $\langle E \rangle = \langle K(0) \vee \dots \vee K(n) \rangle$ [28, Corollary 1.12]. In particular, this applies to the Lubin–Tate E -theory spectrum E_n — see [49] — with

$$(E_n)_* \cong W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$$

or the completed version of E -theory used in [32] with

$$E_* \cong (E(n)_*)_{I_n}^\wedge \cong \mathbb{Z}_p[v_1, \dots, v_{n-1}, v_n^{\pm 1}]_{I_n}^\wedge.$$

2B Bousfield decomposition

In the previous section we introduced the spectra $E(n, J_k)$ for $n \geq k$ and an invariant regular sequence $p^{i_0}, \dots, v_{k-1}^{i_{k-1}}$ of length k . We now give Bousfield decompositions for $E(n, J_k)$ -local spectra.

Proposition 2.10 *There are equivalences of Bousfield classes:*

- (1) (Johnson–Yosimura) $\langle v_n^{-1}BPJ_k \rangle = \langle E(n, J_k) \rangle$.
- (2) (Yosimura) $\langle E(n, J_k) \rangle = \bigvee_{i=k}^n \langle K(i) \rangle$.
- (3) $\langle E_n/I_k \rangle = \bigvee_{i=k}^n \langle K(i) \rangle$.

Proof Part (1) is [35, Corollary 4.11]. Part (2) can be deduced from [57] as we now explain. First, by [57, Corollary 1.3 and Proposition 1.4] along with (1),

$$\langle E(n, J_k) \rangle = \langle v_n^{-1}BPJ_k \rangle = \langle L_nBPJ_k \rangle = \bigvee_{i=k}^n \langle v_i^{-1}P(i) \rangle.$$

By [57, Corollary 1.8] we have $\langle v_i^{-1}P(i) \rangle = \langle K(i) \rangle$, and hence (2) follows. For (3), we first note that by the thick subcategory theorem $\langle E_n/I_k \rangle = \langle E_n \wedge F(k) \rangle$ for some finite type k spectrum. Since $E_n = \bigvee_{i=0}^n K(i)$, (3) then follows from the definition of a type k spectrum. □

Remark 2.11 In other words, the category of $E(n, J_k)$ –local spectra is equivalent to the category of $K(k) \vee \dots \vee K(n)$ –local spectra. Note that this implies this category only depends on the length of the sequence, and not the integers i_0, \dots, i_{n-1} . We will therefore sometimes say that a spectrum X is $E(n, J_k)$ –local if $X \in \text{Sp}_{k,n}$.

2C Algebraic stable homotopy categories

We now begin by recalling the basics on algebraic stable homotopy theories; see [30] in the triangulated setting.

Definition 2.12 A stable homotopy theory is a presentable, symmetric monoidal stable ∞ –category $(\mathcal{C}, \otimes, \mathbf{1})$ where the tensor product commutes with all colimits. It is algebraic if there is a set \mathcal{G} of compact objects such that the smallest localizing subcategory of \mathcal{C} containing all $G \in \mathcal{G}$ is \mathcal{C} itself.

Remark 2.13 The assumptions on \mathcal{C} imply that it the functor $-\wedge Y$ has a right adjoint $F(Y, -)$, ie the symmetric monoidal structure on \mathcal{C} is closed.

Remark 2.14 The associated homotopy category $\text{Ho}(\mathcal{C})$ is then an algebraic stable homotopy theory in the sense of [30]. We note that compactness can be checked at the level of the homotopy category; see [40, Remark 1.4.4.3].

Applying [32, Corollary B.9; 30, Theorem 3.5.1], we have the following.

Proposition 2.15 (Hovey, Palmieri and Strickland) $\mathrm{Sp}_{k,n}$ is an algebraic stable homotopy category with compact generator $L_{k,n}F(k)$. The symmetric monoidal structure in $\mathrm{Sp}_{k,n}$ is given by

$$X \bar{\wedge} Y := L_{k,n}(X \wedge Y).$$

Colimits are computed by taking the colimit in spectra and then applying $L_{k,n}$, while function objects and limits are computed in the category of spectra.

Remark 2.16 The most difficult part of the above proposition is that $L_{k,n}F(k)$ is a compact generator of $\mathrm{Sp}_{k,n}$. Indeed, one must show that the conditions of [32, Proposition B.7] are satisfied and to do this, one at some point needs to invoke the thick subcategory theorem [27], or one its consequences (such as the Hopkins–Ravenel smash product theorem [48]).

Remark 2.17 The localization $L_n = L_{0,n}$ is smashing (that is $L_n X \simeq L_n S^0 \wedge X$) by the Hopkins–Ravenel smash product theorem [48] and in this case $X \bar{\wedge} Y \simeq X \wedge Y$. However, if $k \neq 0$, then localization $L_{k,n}$ is not smashing as the following lemma shows, and so $X \bar{\wedge} Y \not\simeq X \wedge Y$ in general.

Lemma 2.18 If $k \neq 0$, then $L_{k,n}$ is not smashing, and $L_{k,n}S^0$ is not compact in $\mathrm{Sp}_{k,n}$.

Proof We first claim that $\langle L_{k,n}S^0 \rangle = \langle E(n) \rangle$. To see this, note that we have ring maps $L_n S^0 \rightarrow L_{k,n}S^0 \rightarrow L_{K(n)}S^0$, so $\langle L_{K(n)}S^0 \rangle \leq \langle L_{k,n}S^0 \rangle \leq \langle L_n S^0 \rangle$. However, $\langle L_{K(n)}S^0 \rangle = \langle L_n S^0 \rangle = \langle E(n) \rangle$ [32, Corollary 5.3], so these inequalities are actually equalities, and all three are Bousfield equivalent to $E(n)$.

Suppose now that $L_{k,n}$ were smashing, so that $\langle L_{k,n}S^0 \rangle = \bigvee_{i=k}^n \langle K(i) \rangle$; see [46, Proposition 1.27]. Then, since $\langle E(n) \rangle \not\supseteq \bigvee_{i=k}^n \langle K(i) \rangle$ as soon as $k \neq 0$, we have obtained a contradiction.

The second part is then a consequence of [30, Theorem 3.5.2]. □

Remark 2.19 Using the periodicity theorem of Hopkins and Smith [27], Hovey and Strickland [32, Section 4] constructed a sequence of ideals $\{I_j\}_j \subseteq \mathfrak{m} \subseteq E_0$ and type k spectra $\{M_k(j)\}_j$ with the following properties (see also [7, Remark 2.1]):

- (1) $I_{j+1} \subseteq I_j$ and $\bigcap_j I_j = 0$;
- (2) E_0/I_j is finite; and

- (3) $E_*(M_k(j)) \cong E_*/I_j$ and there are spectrum maps $q: M_k(j + 1) \rightarrow M_k(j)$ realizing the quotient $E_*/M_k(j + 1) \rightarrow E_*/M_k(j)$.

We call such a tower $\{M_k(j)\}_j$ a tower of generalized Moore spectrum of type k .

Remark 2.20 The tower as above is constructed in the homotopy category of spectra. However, as explained in [27, page 9, equation (15)], such sequential diagrams can always be lifted to a sequence of cofibrations between cofibrant objects, and in particular to a diagram in the ∞ -category of spectra (the point is that such diagrams have no nontrivial homotopy coherence data). Then, the (co)limit in the ∞ -categorical sense, agrees with the homotopy (co)limit used in [30, Definitions 2.2.3 and 2.2.10].

Notation 2.21 We write $M_{k,n}X$ for the fiber of the localization map $L_n X \rightarrow L_{k-1} X$. By definition, we set $M_{0,n} = L_n$.

Lemma 2.22 We have an equality of Bousfield classes

$$\langle M_{k,n}S^0 \rangle = \bigvee_{i=k}^n \langle K(i) \rangle.$$

Proof Recall that, by definition, there is a cofiber sequence

$$C_{k-1}S^0 \rightarrow S^0 \rightarrow L_{k-1}S^0.$$

Applying L_n to this and using $L_i L_j \simeq L_{\min(i,j)}$ we see that

$$M_{k,n}S^0 \simeq L_n C_{k-1}S^0 \simeq C_{k-1}L_n S^0,$$

where the last equivalence follows as both functors are smashing. It follows from [32, Proposition 5.3] that $\langle M_{k,n}S^0 \rangle = \bigvee_{i=k}^n \langle K(i) \rangle$ as claimed. □

Remark 2.23 In [32, Proposition 7.10(e)] Hovey and Strickland give a formula for $L_{K(n)}X$ in terms of towers of generalized Moore spectra. We show now that their proof extends to $L_{k,n}X$.

Proposition 2.24 There are equivalences

$$L_{k,n}X \simeq L_{F(k)}L_n X \simeq \varprojlim_j (L_n X \wedge M_k(j)) \simeq F(M_{k,n}S^0, L_n X),$$

where the limit is taken over a tower $\{M_k(j)\}$ of generalized Moore spectra of type k .

Proof We first note that $L_{k-1}X \simeq L_n L_{k-1}X \simeq L_n L_{k-1}^f X$, the latter by Corollary 6.10 of [32]. It follows that $M_{k,n}X \simeq L_n C_{k-1}^f X$, where C_{k-1}^f is the acyclization

functor associated to L_{k-1}^f . By [32, Proposition 7.10(a)] (and Remark 2.20) $C_{k-1}^f X \simeq \varinjlim_j D(M_k(j)) \wedge X$, so $M_{k,n}X \simeq \varinjlim_j D(M_k(j)) \wedge L_n X$. It follows that

$$\varprojlim_j (L_n X \wedge M_k(j)) \simeq F(M_{k,n}S^0, L_n X).$$

Moreover, by [32, Proposition 7.10(a)] this is equivalent to $L_{F(k)}L_n X$.

To finish the proof, we will show that $L_{k,n}X \simeq L_{F(k)}L_n X$. First, note that $X \rightarrow L_n X$ is an $L_n S^0$ -equivalence, and $L_n X \rightarrow L_{F(k)}L_n X$ is an $F(k)$ -equivalence, so $X \rightarrow L_{F(k)}L_n X$ is an $L_n S^0 \wedge F(k)$ -equivalence. But $L_n S^0 \wedge F(k) \simeq L_n F(k)$ and $\langle L_n F(k) \rangle = \bigvee_{i=k}^n \langle K(i) \rangle$ [32, Proposition 5.3]. Therefore $X \rightarrow L_{F(k)}L_n X$ is a $(K(k) \vee \dots \vee K(n))$ -equivalence, and we only need show that $L_{F(k)}L_n X$ is $(K(k) \vee \dots \vee K(n))$ -local. But $L_{F(k)}L_n X \simeq F(M_{k,n}S^0, L_n X)$ and so it follows from Lemma 2.22 that $L_{F(k)}L_n X$ is $(K(k) \vee \dots \vee K(n))$ -local. We conclude that $L_{k,n}X \simeq L_{F(k)}L_n X$, as required. \square

Remark 2.25 The equivalence

$$L_{k,n}X \simeq \varprojlim_j (L_n X \wedge M_k(j))$$

has also been obtained in [11, Proposition 6.21] using the theory and complete and torsion objects in a stable ∞ -category. The next result is also contained in [11, Corollary 6.17].

Proposition 2.26 For any spectrum X there is a pullback square

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{k,n}X \\ \downarrow & & \downarrow \\ L_{k-1}X & \longrightarrow & L_{k-1}L_{k,n}X \end{array}$$

Proof This is a standard consequence of the Bousfield decomposition

$$\langle E(n) \rangle = \langle E(k-1) \rangle \vee \langle E(n, J_k) \rangle$$

using for example [13, Proposition 2.2] (or, in the ∞ -categorical setting, see [1]). \square

Remark 2.27 Use [13, Proposition 2.2] one can deduce various other chromatic fracture squares. For example, we have a pullback square

$$\begin{array}{ccc} L_{k,n}X & \longrightarrow & L_{k,h}X \\ \downarrow & & \downarrow \\ L_{h+1,n}X & \longrightarrow & L_{k,h}L_{h+1,n}X \end{array}$$

for $k \leq h \leq n-1$.

Remark 2.28 These types of iterated chromatic localizations have been investigated by Bellumat and Strickland [53]. Results such as the chromatic fracture square can be recovered from their work; however we do not investigate this in detail.

Corollary 2.29 Suppose M_j is a generalized Moore spectrum of type at least k . Then $L_n M_j \simeq L_{k,n} M_j$.

Proof By definition, $L_{k-1} M_j \simeq *$, and so by the pullback square of Proposition 2.26 we must show that $L_{k-1} L_{k,n} M_j$ is contractible. Because M_j is a finite complex, this is equivalent to $L_{k-1}((L_{k,n} S^0) \wedge M_j) \simeq *$, and the result follows. \square

Definition 2.30 Let $\mathcal{M}_{k,n}$ denote the essential image of the functor $M_{k,n}: \text{Sp} \rightarrow \text{Sp}$.

Theorem 2.31 For any spectrum X , we have natural equivalences

$$M_{k,n} L_{k,n} X \simeq M_{k,n} X, \quad L_{k,n} X \simeq L_{k,n} M_{k,n} X.$$

It follows that there is an equivalence of categories $\mathcal{M}_{k,n} \simeq \text{Sp}_{k,n}$ given by $L_{k,n}$, with inverse given by $M_{k,n}$.

Proof The proof of Hovey and Strickland in the case $k = n$ generalizes essentially without change.

By definition, $M_{k,n} X$ fits into a cofiber sequence

$$M_{k,n} X \rightarrow L_n X \rightarrow L_{k-1} X,$$

so applying $L_{k,n}$ gives a cofiber sequence

$$L_{k,n} M_{k,n} X \rightarrow L_{k,n} L_n X \rightarrow L_{k,n} L_{k-1} X.$$

But $\langle E(k-1) \rangle = \bigvee_{i=0}^{k-1} \langle K(k) \rangle$, so $L_{k,n} L_{k-1} X \simeq 0$, while clearly $L_{k,n} L_n X \simeq L_{k,n} X$. It follows that $L_{k,n} M_{k,n} X \simeq L_{k,n} X$.

Using Proposition 2.24, $L_{k,n} X \simeq F(M_{k,n} S^0, L_n X)$, and so applying $F(-, L_n X)$ to the defining cofiber sequence for $M_{k,n} S^0$ we obtain a cofiber sequence

$$F(L_{k-1} S^0, L_n X) \rightarrow F(L_n S^0, L_n X) \rightarrow F(M_{k,n} S^0, L_n X),$$

or, equivalently,

$$F(L_{k-1} S^0, L_n X) \rightarrow L_n X \rightarrow L_{k,n} X.$$

It is easy to check that $F(L_{k-1} S^0, L_n X)$ is $E(k-1)$ -local, and so by Lemma 2.22 we have $M_{k,n} F(L_{k-1} S^0, L_n X) \simeq *$. It follows that $M_{k,n} L_n X \simeq M_{k,n} X \simeq M_{k,n} L_{k,n} X$ as claimed. \square

Remark 2.32 Once again, this result was obtained (by different methods) in [11, Proposition 6.21].

3 Thick subcategories and (co)localizing subcategories

In this section we compute the thick subcategories of compact objects in $\mathrm{Sp}_{k,n}$ and (co)localizing subcategories of $\mathrm{Sp}_{k,n}$. When $k = 0$ or $k = n$ both results have been obtained by Hovey and Strickland. Along the way we give a classification of the compact objects in $\mathrm{Sp}_{k,n}$.

3A Compact objects in $\mathrm{Sp}_{k,n}$

In this section we characterize the compact objects in $\mathrm{Sp}_{k,n}$. We will use this in the next section to compute the thick subcategories of $\mathrm{Sp}_{k,n}^\omega$.

We begin by recalling the notions of thick and (co)localizing subcategories.

Definition 3.1 Let $(\mathcal{C}, \wedge, \mathbf{1})$ be an algebraic stable homotopy category, and let \mathcal{D} be a full, stable subcategory.

- (1) \mathcal{D} is called thick if it is closed under extensions and retracts.
- (2) \mathcal{D} is called localizing if it is thick and closed under arbitrary colimits.
- (3) \mathcal{D} is called colocalizing if it is thick and closed under arbitrary limits.
- (4) \mathcal{D} is a tensor-ideal if $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ implies $X \wedge Y \in \mathcal{D}$.
- (5) \mathcal{D} is a coideal if $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ implies $F(X, Y) \in \mathcal{D}$.

We will also speak of localizing (or thick) tensor-ideals and colocalizing coideals.

Remark 3.2 In Sp_n the dualizable and compact objects coincide, and are precisely those that lie in the thick subcategory generated by the tensor unit $L_n S^0$. In categories whose tensor unit is not compact, such as $\mathrm{Sp}_{k,n}$ for $k \neq 0$, the dualizable and compact objects do not coincide—for example, the tensor unit is always dualizable, but is not compact (Lemma 2.18). In [32] Hovey and Strickland gave numerous characterizations of compact objects in $\mathrm{Sp}_{K(n)}$. In this section we extend some of these characterizations to $\mathrm{Sp}_{k,n}$. We first recall the concept of a nilpotent object in a symmetric monoidal category.

Definition 3.3 We say that X is R -nilpotent if X lies in the thick \otimes -ideal generated by R , ie $X \in \mathrm{Thick}_\otimes \langle R \rangle$.

Lemma 3.4 *The category of E_n/I_k -nilpotent spectra is the same in $\mathrm{Sp}_{k,n}$, Sp_n and Sp .*

Proof Using that $\langle E_n/I_k \rangle = \langle E(n, J_k) \rangle$ (Proposition 2.10) we see that $E_n/I_k \wedge X$ is always $E(n, J_k)$ -local, from which the result easily follows. \square

Remark 3.5 In other words, we can talk unambiguously about the category of E_n/I_k -nilpotent spectra.

We will also need the following generalization of [32, Lemma 6.15].

Lemma 3.6 *If X is a finite spectrum of type at least k , then $L_n X \simeq L_{k,n} X$ is E_n/I_k -nilpotent.*

Proof The argument is only a slight adaptation of that given by Hovey and Strickland. By a thick subcategory argument, we can assume that $X = M_k$ is a generalized Moore spectrum of type k . By [48], $L_n S^0 \in \mathrm{thick}_{\otimes} \langle E_n \rangle$, and it follows that $L_n S^0 \wedge M_k \simeq L_n M_k \in \mathrm{thick}_{\otimes} \langle E_n \wedge M_k \rangle$. But it is easy to see that $\mathrm{thick}_{\otimes} \langle E_n \wedge M_k \rangle \simeq \mathrm{thick}_{\otimes} \langle E_n/I_k \rangle$ and we are done. \square

Remark 3.7 The fact that $L_n S^0 \in \mathrm{thick}_{\otimes} \langle E_n \rangle$ is equivalent to the claim that $E_n \in \mathrm{Sp}_n$ is *descendable*, a condition we investigate further in Section 4A.

The compact objects in $\mathrm{Sp}_{k,n}$ can be characterized in the following ways, partially generalizing [32, Theorem 8.5]. We note that every compact object in $\mathrm{Sp}_{k,n}$ is automatically dualizable by [30, Theorem 2.1.3]; we investigate the dualizable objects in $\mathrm{Sp}_{k,n}$ in more detail in Section 5.

Theorem 3.8 *The following are equivalent for $X \in \mathrm{Sp}_{k,n}$:*

- (1) X is compact.
- (2) $X \in \mathrm{thick} \langle L_n F(k) \rangle$.
- (3) X is a retract of $L_n X' \simeq L_{k,n} X'$ for a finite spectrum X' of type at least k .
- (4) X is a retract of $Y \wedge X'$, where Y is dualizable and X' is a finite spectrum of type at least k .
- (5) X is dualizable and E_n/I_k -nilpotent.

The category $\mathrm{Sp}_{k,n}^{\omega} \subseteq \mathrm{Sp}_{k,n}^{\mathrm{dual}}$ is thick. Moreover, if $X \in \mathrm{Sp}_{k,n}^{\omega}$ and $Y \in \mathrm{Sp}_{k,n}^{\mathrm{dual}}$, then $X \wedge Y$, $F(X, Y)$ and $F(Y, X)$ lie in $\mathrm{Sp}_{k,n}^{\omega}$. In particular, $F(X, L_{k,n} S^0) \in \mathrm{Sp}_n^{\omega}$.

Proof The equivalence of (1) and (2) is [30, Theorem 2.1.3] along with Proposition 2.15. Item (3) implies (2) because every finite spectrum of type at least k lies in the thick subcategory generated by $F(k)$. That (1) implies (3) is the same as given by Hovey and Strickland [32, Theorem 8.5]. Namely, suppose that $X \in \mathrm{Sp}_{k,n}$ is compact. By Proposition 2.24, $X \simeq \varinjlim_j (X \wedge DM_k(j))$, so $[X, X] \simeq \varinjlim_j [X, X \wedge DM_k(j)]$. In particular, X is a retract of $Y := X \wedge DM_k(j)$. We claim that such a Y is compact in Sp_n . Indeed, let $\{Z_i\}$ be a filtered diagram of E_n -local spectra. Then we have equivalences

$$\begin{aligned} [Y, \varinjlim_i Z_i]_* &\simeq [X, L_n M_k(j) \wedge \varinjlim_i Z_i]_* \\ &\simeq [X, L_{k,n} \varinjlim_i (M_k(j) \wedge Z_i)]_* \\ &\simeq \varprojlim_i [X, M_k(j) \wedge Z_i]_* \\ &\simeq \varprojlim_i [Y, Z_i]_* . \end{aligned}$$

The first and last equivalence follow by adjunction, the second because $\langle L_n M_k(j) \rangle = \bigvee_{i=k}^n \langle K(i) \rangle$ [32, Proposition 5.3], so

$$L_n M_k(j) \wedge \varinjlim_i Z_i \simeq L_n M_k(j) \wedge L_{k,n} \varinjlim_i Z_i \simeq L_{k,n} \varinjlim_i (M_k(j) \wedge Z_i),$$

while the third equivalence follows because $X \in \mathrm{Sp}_{k,n}^\omega$ by assumption and because $L_{k,n} \varinjlim_i$ is the colimit in $\mathrm{Sp}_{k,n}$. We have $K(i)_* Y = 0$ for $i < k$ and so Corollary 6.11 of [32] implies that Y , and hence X , is a retract of $L_n Z \simeq L_{k,n} Z$ for a finite spectrum Z of type at least k . This shows that (1), (2) and (3) are equivalent.

Assume now that (4) holds. Note that $Y \wedge X'$ is $E(n, J_k)$ -local, and moreover $Y \wedge X' \simeq Y \wedge L_{k,n} X'$, where $L_{k,n} X' \in \mathrm{Sp}_{k,n}^\omega$. By [30, Theorem 2.1.3] the smash product of a dualizable and compact object is compact, and so X is a retract of a compact $E(n, J_k)$ -local spectrum, and thus is also compact; ie (1) holds.

To see that (3) implies (5), we use a thick subcategory argument to reduce to the case that $X = L_n M_k \simeq L_{k,n} M_k$ is a localized generalized Moore spectrum of type k . Such an X is clearly dualizable and is additionally E_n/I_k -nilpotent by Lemma 3.6.

Now suppose that X satisfies (5). Following Hovey and Strickland [32, Proof of Corollary 12.16] let \mathcal{J} be the collection of spectra $Z \in \mathrm{Sp}_{k,n}$ such that Z is a module over a generalized Moore spectrum of type i (for a fixed i with $k \leq i \leq n$). By [32, Proposition 4.17], \mathcal{J} forms an ideal. Because $K(i) \wedge Z$ is nonzero and a wedge of suspensions of $K(i)$, \mathcal{J} contains the ideal of $K(i)$ -nilpotent spectra. Moreover, it follows

from the Bousfield decomposition $\langle E_n/I_k \rangle = \bigvee_{i=k}^n \langle K(i) \rangle$ that $K(i) \wedge E_n/I_k \neq 0$, and so $\text{thick}_{\otimes} \langle K(i) \rangle \subseteq \text{thick}_{\otimes} \langle E_n/I_k \rangle$; ie every $K(i)$ -nilpotent spectrum (for $k \leq i \leq n$) is also E_n/I_k -nilpotent. In particular, $X \in \mathcal{J}$, so X is retract of a spectrum of the form $Y \wedge X$ where Y is a generalized Moore spectrum of type i , and thus (4) holds.

Finally, we prove the subsidiary claims. It is immediate from (2) that $\text{Sp}_{k,n}^{\omega} \subseteq \text{Sp}_{k,n}^{\text{dual}}$ is thick, and it is an ideal by [30, Theorem 2.1.3(a)]. Because generalized Moore spectra are self-dual — see [32, Proposition 4.18] — (c) implies that $\text{Sp}_{k,n}^{\omega}$ is closed under Spanier–Whitehead duality. Therefore, $F(X, Y) \simeq F(X, L_{k,n}S^0) \bar{\wedge} Y$ and $F(Y, X) \simeq X \bar{\wedge} F(Y, L_{k,n}S^0)$ lie in $\text{Sp}_{k,n}^{\omega}$. □

Remark 3.9 When $k = 0$, then X is compact if and only if X is dualizable [32, Theorem 6.2]. To reconcile this with (5) of the previous theorem, we note that every spectrum $X \in \text{Sp}_n$ is $E_n/I_0 \simeq E_n$ -nilpotent [48, Theorem 5.3].

3B The thick subcategory theorem

We now give a thick subcategory theorem for $\text{Sp}_{k,n}^{\omega}$. As we shall see, given Theorem 3.8 this is an immediate consequence of the classification of thick subcategories of Sp_n^{ω} , which ultimately relies on the Devinatz–Hopkins–Smith nilpotence theorem.

Definition 3.10 For $0 \leq j \leq n + 1$ let \mathcal{C}_j denote the thick subcategory of Sp_n consisting of all compact spectra X such that $K(i)_*X = 0$ for all $i < j$; ie

$$\mathcal{C}_j = \{X \in \text{Sp}_n^{\omega} \mid K(i)_*X = 0 \text{ for all } i < j\}.$$

Remark 3.11 By [32, Proposition 6.8], we equivalently have

$$\mathcal{C}_j = \{X \in \text{Sp}_n^{\omega} \mid K(j - 1)_*X = 0\}.$$

Remark 3.12 We have

$$\mathcal{C}_0 \supsetneq \mathcal{C}_1 \supsetneq \cdots \supsetneq \mathcal{C}_{n+1} = (0),$$

and moreover $L_n F(j)$ is in \mathcal{C}_j , but not \mathcal{C}_{j+1} .

We now present the result of Hovey and Strickland [32, Theorem 6.9].

Theorem 3.13 (Hovey–Strickland) *If \mathcal{C} is a thick subcategory of Sp_n^{ω} , then $\mathcal{C} = \mathcal{C}_j$ for some j such that $0 \leq j \leq n + 1$.*

Remark 3.14 This result can be restated in terms of the Balmer spectrum of Sp_n^{ω} [3]. In particular,

$$\text{Spc}(\text{Sp}_n^{\omega}) \cong \{\mathcal{C}_1, \dots, \mathcal{C}_{n+1}\}$$

with topology determined by the closure operator $\overline{\{C_j\}} = \{C_i \mid i \geq j\}$. This is in fact equivalent to [Theorem 3.13](#), essentially by the same argument as in [\[10, Proposition 3.5\]](#).

The Balmer support of $X \in \mathrm{Sp}_n^\omega$, defined in [\[3, Definition 2.1\]](#), is given by

$$\mathrm{supp}(X) = \{\mathcal{P} \in \mathrm{Spc}(\mathrm{Sp}_n^\omega) \mid X \notin \mathcal{P}\}.$$

Note that $X \notin C_j$ if and only if $K(j-1)_*X \neq 0$. Therefore, by [Theorem 3.13](#),

$$\mathrm{supp}(X) = \{i \in \{0, \dots, n\} \mid K(i)_*X \neq 0\} = \{i \in \{0, \dots, n\} \mid K(i) \wedge X \neq 0\}.$$

For a thick subcategory \mathcal{J} , we define $\mathrm{supp}(\mathcal{J}) = \bigcup_{X \in \mathcal{J}} \mathrm{supp}(X)$. Then, Balmer’s classification result [\[3, Theorem 4.10\]](#) shows that there is a bijection

$$\{\text{thick subcategories of } \mathrm{Sp}_n^\omega\} \xrightarrow[\mathrm{supp}]{} \{\text{specialization closed subsets of } \{0, \dots, n\}\}.$$

with the topology on $\{0, \dots, n\}$ determined by $\overline{\{k\}} = \{k, k+1, \dots, n\}$, with inverse given by sending a specialization closed subset Y to $\{X \in \mathrm{Sp}_n^\omega \mid \mathrm{supp}(X) \subseteq Y\}$. Note that there are exactly $n+2$ such specialization closed subsets, namely \emptyset and the subsets $\{k, \dots, n\}$ for $k = 0, \dots, n$. The thick subcategory \mathcal{C}_{n+1} corresponds to \emptyset under this bijection, while \mathcal{C}_k corresponds to $\{k, \dots, n\}$ for $0 \leq k \leq n$.

Given the classification of compact $E(n, J_k)$ -local spectra in [Theorem 3.8](#), we deduce the following.

Lemma 3.15 *The category of compact $E(n, J_k)$ -local spectra, $\mathrm{Sp}_{k,n}^\omega$, is equivalent to the thick subcategory $\mathcal{C}_k \subseteq \mathrm{Sp}_n^\omega$.*

Proof By [\[32, Corollary 6.11\]](#) if $X \in \mathcal{C}_k$, then X is a retract of $L_n Y \simeq L_{k,n} Y$ for some finite spectrum Y of type of least k . Then X is a compact $E(n, J_k)$ -local spectrum by [Theorem 3.8](#). Conversely, if X is a compact $E(n, J_k)$ -local spectrum, then X is a retract of $L_n Y \simeq L_{k,n} Y$ for Y a finite spectrum Y of type of least k , again by [Theorem 3.8](#). Therefore $K(i)_*X = 0$ for $i < k$ and $X \in \mathcal{C}_k$. □

Theorem 3.16 (thick subcategory theorem) *There is a bijection*

$$\{\text{thick subcategories of } \mathrm{Sp}_{k,n}^\omega\} \xrightarrow[\mathrm{supp}]{} \{\text{specialization closed subsets of } \{k, \dots, n\}\},$$

with inverse given by sending a specialization closed subset Y to

$$\{X \in \mathrm{Sp}_{k,n}^\omega \mid \mathrm{supp}(X) \subseteq Y\}.$$

In particular, if \mathcal{C} is a thick subcategory of $\mathrm{Sp}_{k,n}^\omega$, then $\mathcal{C} = \mathcal{C}_j$ for some j such that $k \leq j \leq n+1$.

Proof This follows by combining [Theorem 3.13](#) and [Lemma 3.15](#). □

Remark 3.17 Note that $\mathrm{Sp}_{k,n}^\omega$ is not a tensor-triangulated category when $k \neq 0$, as it does not have a tensor unit. Therefore, we cannot speak of the Balmer spectrum of $\mathrm{Sp}_{k,n}^\omega$.

We also have a nilpotence theorem.

Proposition 3.18 *Let $X \in \mathrm{Sp}_{k,n}^\omega$, and $u: \Sigma^d X \rightarrow X$ a self-map such that $K(i)_*u$ is nilpotent for $k \leq i \leq n$. Then u is nilpotent; ie the j -fold composite*

$$u \circ \dots \circ u: \Sigma^{jd} X \rightarrow X$$

is trivial for large enough j .

Proof In light of [Lemma 3.15](#), this follows from [[32](#), Corollary 6.6]. □

3C Localizing and colocalizing subcategories

In this section we calculate the (co)localizing (co)ideals of $\mathrm{Sp}_{k,n}$. We first observe that every (co)localizing subcategory is automatically a (co)ideal, so it suffices in fact to concentrate on (co)localizing subcategories.

Lemma 3.19 *Every (co)localizing subcategory of $\mathrm{Sp}_{k,n}$ is a (co)ideal.*

Proof We prove the case of localizing subcategories — the case of colocalizing subcategories is similar.¹ To that end, let $\mathcal{C} \subseteq \mathrm{Sp}_{k,n}$ be a localizing subcategory, and consider the collection $\mathcal{D} = \{X \in \mathrm{Sp} \mid X \bar{\wedge} \mathcal{C} \subseteq \mathcal{C}\}$. This is a localizing subcategory of Sp containing \mathbb{S} , and hence $\mathcal{D} = \mathrm{Sp}$ itself. It follows that \mathcal{C} is a localizing ideal. □

Remark 3.20 We remind the reader that $\mathbf{1}$ is not compact in $\mathrm{Sp}_{k,n}$ unless $k = 0$ (see [Lemma 2.18](#)). Therefore, in all other cases, $\mathbf{1}$ is a noncompact generator of $\mathrm{Sp}_{k,n}$.

Notation 3.21 Throughout this section we let $\mathcal{Q} = \{k, \dots, n\}$.

We begin by defining a notion of support and cosupport in $\mathrm{Sp}_{k,n}$, extending the notion of support defined previously for Sp_n^ω .

Definition 3.22 For a spectrum $X \in \mathrm{Sp}_{k,n}$, we define the support and cosupport of X by

$$\mathrm{supp}(X) = \{i \in \mathcal{Q} \mid K(i) \wedge X \neq 0\}, \quad \mathrm{cosupp}(X) = \{i \in \mathcal{Q} \mid F(K(i), X) \neq 0\}.$$

¹We thank Neil Strickland for providing this argument, which simplifies a previous argument.

Example 3.23 Because $K(i) \wedge K(j) = 0$ if $i \neq j$, and $K(i) \wedge K(i) \neq 0$ [46, Theorem 2.1], we have

$$\mathrm{supp}(K(i)) = i$$

for $i \in \mathcal{Q}$. On the other hand, $K(i)^* K(j) = \mathrm{Hom}_{K(i)_*}(K(i)_* K(j), K(i)_*)$, and so

$$\mathrm{cosupp}(K(i)) = i$$

as well.

Remark 3.24 The notion of support is slightly ambiguous, as objects can live in multiple categories. For example $L_{K(n)} S^0 \in \mathrm{Sp}_{i,n}$ for all $0 \leq i \leq n$, and in fact has different support in each category. However, it should also be clear in which category we are considering the support.

Remark 3.25 Because $K(i) \wedge X$ is always $K(i)$ -local, we equivalently have

$$\mathrm{supp}(X) = \{i \in \mathcal{Q} \mid K(i) \bar{\wedge} X \neq 0\}.$$

Remark 3.26 In [32, Definition 6.7] Hovey and Strickland define the support of an object by

$$\mathrm{supp}_{\mathrm{HS}}(X) = \{i \mid K(i) \wedge X \neq 0\}.$$

By definition then, $\mathrm{supp}(X) = \mathrm{supp}_{\mathrm{HS}}(X) \cap \mathcal{Q}$.

Support and cosupport are well behaved with respect to products and function objects in $\mathrm{Sp}_{k,n}$.

Lemma 3.27 For any $X, Y \in \mathrm{Sp}_{k,n}$ there are equalities

$$\mathrm{supp}(X \bar{\wedge} Y) = \mathrm{supp}(X) \cap \mathrm{supp}(Y), \quad \mathrm{cosupp}(F(X, Y)) = \mathrm{supp}(X) \cap \mathrm{cosupp}(Y).$$

Proof Because $K(i) \wedge X$ is always $K(i)$ -local, $K(i) \wedge (X \bar{\wedge} Y) \simeq K(i) \wedge X \wedge Y$, and it is clear that $\mathrm{supp}(X \bar{\wedge} Y) \subseteq \mathrm{supp}(X) \cap \mathrm{supp}(Y)$. The converse follows because $K(i)_*$ is a graded field; if $K(i) \wedge X \wedge Y \simeq *$ then either $K(i) \wedge X \simeq *$ or $K(i) \wedge Y \simeq *$.

For the cosupport, suppose $i \in \mathrm{cosupp}(F(X, Y))$, ie $F(K(i), F(X, Y)) \neq 0$. By adjunction we must have $F(K(i) \wedge X, Y) \neq 0$ as well as $F(X, F(K(i), Y)) \neq 0$, so that $K(i) \wedge X \neq 0$ and $F(K(i), Y) \neq 0$. This shows that $\mathrm{cosupp}(F(X, Y)) \subseteq \mathrm{supp}(X) \cap \mathrm{cosupp}(Y)$. For the converse, let $i \in \mathrm{supp}(X) \cap \mathrm{cosupp}(Y)$, and consider $F(K(i), F(X, Y)) \simeq F(K(i) \wedge X, Y)$. Because $i \in \mathrm{supp}(X)$, and $K(i)_*$ is a graded

field, $K(i) \wedge X$ is a wedge of suspensions of $K(i)$, and it suffices to show that $F(K(i), Y) \neq 0$, which is precisely the statement that $i \in \text{cosupp}(Y)$. Therefore, $i \in \text{cosupp}(F(X, Y))$, as required. \square

Notation 3.28 For an arbitrary collection \mathcal{C} of objects we set

$$\text{supp}(\mathcal{C}) = \bigcup_{X \in \mathcal{C}} \text{supp}(X), \quad \text{cosupp}(\mathcal{C}) = \bigcup_{X \in \mathcal{C}} \text{cosupp}(X).$$

For a subset $\mathcal{T} \subseteq \mathcal{Q}$ we also define

$$\begin{aligned} \text{supp}^{-1}(\mathcal{T}) &= \{X \in \text{Sp}_{k,n} \mid \text{supp}(X) \subseteq \mathcal{T}\}, \\ \text{cosupp}^{-1}(\mathcal{T}) &= \{X \in \text{Sp}_{k,n} \mid \text{cosupp}(X) \subseteq \mathcal{T}\}. \end{aligned}$$

Lemma 3.29 For a subset $\mathcal{T} \subseteq \mathcal{Q}$, $\text{supp}^{-1}(\mathcal{T})$ and $\text{cosupp}^{-1}(\mathcal{T})$ are localizing and colocalizing subcategories of $\text{Sp}_{k,n}$, respectively.

Proof We simply note that

$$\begin{aligned} \text{supp}^{-1}(\mathcal{T}) &= \{X \in \text{Sp}_{k,n} \mid K(i) \wedge X = 0 \text{ for all } i \in \mathcal{Q} \setminus \mathcal{T}\}, \\ \text{cosupp}^{-1}(\mathcal{T}) &= \{X \in \text{Sp}_{k,n} \mid F(K(i), X) \simeq 0 \text{ for all } i \in \mathcal{Q} \setminus \mathcal{T}\}, \end{aligned}$$

which are clearly (co)localizing subcategories of $\text{Sp}_{k,n}$. \square

We thus obtain maps

$$(3-1) \quad \{\text{localizing subcategories of } \text{Sp}_{k,n}\} \xrightleftharpoons[\text{supp}^{-1}]{\text{supp}} \{\text{subsets of } \mathcal{Q}\},$$

$$(3-2) \quad \{\text{colocalizing subcategories of } \text{Sp}_{k,n}\} \xrightleftharpoons[\text{cosupp}^{-1}]{\text{cosupp}} \{\text{subsets of } \mathcal{Q}\}.$$

We will see that these are bijections. We need the following local–global principle, which is a slight variant of that given by Hovey and Strickland [32, Proposition 6.18].

Proposition 3.30 (local–global principle) For any $X \in \text{Sp}_{k,n}$,

$$\begin{aligned} X \in \text{Loc}_{\text{Sp}_{k,n}}(X) &= \text{Loc}_{\text{Sp}_{k,n}}(K(i) \mid i \in \text{supp}(X)), \\ X \in \text{Coloc}_{\text{Sp}_{k,n}}(X) &= \text{Coloc}_{\text{Sp}_{k,n}}(K(i) \mid i \in \text{cosupp}(X)). \end{aligned}$$

Proof Because $X \in \text{Sp}_{k,n} \subseteq \text{Sp}_n$, applying [32, Proposition 6.18] we have

$$(3-3) \quad \begin{aligned} X \in \text{Loc}_{\text{Sp}}(X) &= \text{Loc}_{\text{Sp}}(K(i) \mid i \in \text{supp}_{\text{HS}}(X)), \\ X \in \text{Coloc}_{\text{Sp}}(X) &= \text{Coloc}_{\text{Sp}}(K(i) \mid i \in \text{cosupp}(X)). \end{aligned}$$

The result for colocalizing subcategories is then clear, as we get the same result taking the colocalizing subcategories in $\mathrm{Sp}_{k,n}$. For localizing subcategories we apply [8, Lemma 2.5] to (3-3) with the colimit-preserving functor $F = L_{k,n} : \mathrm{Sp} \rightarrow \mathrm{Sp}_{k,n}$ to see that

$$L_{k,n}X \simeq X \in \mathrm{Loc}_{\mathrm{Sp}_{k,n}}(X) = \mathrm{Loc}_{\mathrm{Sp}_{k,n}}(K(i) \mid i \in \mathrm{supp}_{\mathrm{HS}}(X) \cap \mathcal{Q}),$$

where we have used that

$$L_{k,n}K(i) \simeq \begin{cases} K(i) & \text{if } i \in \mathcal{Q}, \\ 0 & \text{if } i \notin \mathcal{Q}. \end{cases}$$

As noted in Remark 3.26, $\mathrm{supp}_{\mathrm{HS}}(X) \cap \mathcal{Q} = \mathrm{supp}(X)$, and the result follows. \square

Remark 3.31 It follows from the local–global principle that both support and cosupport detect trivial objects:

$$\mathrm{supp}(X) = \emptyset \iff X \simeq 0 \iff \mathrm{cosupp}(X) = \emptyset.$$

Corollary 3.32 We have

$$\mathrm{supp}^{-1}(\mathcal{T}) = \mathrm{Loc}_{\mathrm{Sp}_{k,n}}(K(i) \mid i \in \mathcal{T}), \quad \mathrm{cosupp}^{-1}(\mathcal{T}) = \mathrm{Coloc}_{\mathrm{Sp}_{k,n}}(K(i) \mid i \in \mathcal{T}).$$

Proof Let $\mathcal{A} = \mathrm{Loc}_{\mathrm{Sp}_{k,n}}(K(i) \mid i \in \mathcal{T})$. Because $\mathrm{supp}(K(i)) = i$ (Example 3.23), it is clear that $\mathcal{A} \subseteq \mathrm{supp}^{-1}(\mathcal{T})$. Conversely, if $X \in \mathrm{supp}^{-1}(\mathcal{T})$, then Proposition 3.30 shows that

$$X \in \mathrm{Loc}_{\mathrm{Sp}_{k,n}}(K(i) \mid i \in \mathcal{T}) = \mathcal{A},$$

so $\mathrm{supp}^{-1}(\mathcal{T}) = \mathcal{A}$, as claimed. The argument for colocalizing categories is similar. \square

We now give the promised classification of localizing and colocalizing subcategories.

Theorem 3.33 (1) The maps (3-1) give an order-preserving bijection between localizing subcategories of $\mathrm{Sp}_{k,n}$ and subsets of $\mathcal{Q} = \{k, \dots, n\}$.

(2) The maps (3-2) give an order-preserving bijection between colocalizing subcategories of $\mathrm{Sp}_{k,n}$ and subsets of $\mathcal{Q} = \{k, \dots, n\}$.

Proof Let $\mathcal{C} \subseteq \mathrm{Sp}_{k,n}$ be a localizing subcategory and $\mathcal{T} \subseteq \{k, \dots, n\}$ a subset. Then via Corollary 3.32 and basic properties of support,

$$\mathrm{supp}(\mathrm{supp}^{-1}(\mathcal{T})) = \bigcup_{i \in \mathcal{T}} \mathrm{supp}(K(i)) = \mathcal{T}.$$

Now suppose that $X \in \mathcal{C}$, so that $\mathrm{supp}(X) \subseteq \mathrm{supp}(\mathcal{C})$. It follows from the definitions that $X \in \mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{C}))$, and so $\mathcal{C} \subseteq \mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{C}))$. We are therefore reduced

to showing that $\text{supp}^{-1}(\text{supp}(\mathcal{C})) \subseteq \mathcal{C}$. To that end, let $Y \in \text{supp}^{-1}(\text{supp}(\mathcal{C}))$, so that $\text{supp}(Y) \subseteq \text{supp}(\mathcal{C})$. Using the local–global principle, [Proposition 3.30](#), we then have

$$Y \in \text{Loc}_{\text{Sp}_{k,n}}(K(i) \mid i \in \text{supp}(Y)) \subseteq \text{Loc}_{\text{Sp}_{k,n}}(K(i) \mid i \in \text{supp}(\mathcal{C})) = \mathcal{C},$$

where the last equality follows from [Proposition 3.30](#) again. The proof for colocalizing subcategories is analogous. □

Notation 3.34 For the following, we recall that for $\mathcal{C} \subseteq \text{Sp}_{k,n}$ the right orthogonal \mathcal{C}^\perp is defined as

$$\mathcal{C}^\perp = \{Y \in \text{Sp}_{k,n} \mid F(X, Y) = 0 \text{ for all } X \in \mathcal{C}\}.$$

Similarly, the left orthogonal ${}^\perp\mathcal{C}$ is

$${}^\perp\mathcal{C} = \{Y \in \text{Sp}_{k,n} \mid F(Y, X) = 0 \text{ for all } X \in \mathcal{C}\}.$$

Moreover, the right orthogonal is a colocalizing subcategory, and the left orthogonal is a localizing subcategory.

Corollary 3.35 *The map that sends a localizing subcategory \mathcal{C} of $\text{Sp}_{k,n}$ to \mathcal{C}^\perp induces a bijection*

$$(3-4) \quad \{\text{Localizing subcategories of } \text{Sp}_{k,n}\} \xrightarrow{\sim} \{\text{Colocalizing subcategories of } \text{Sp}_{k,n}\}.$$

The inverse map sends a colocalizing subcategory \mathcal{U} to ${}^\perp\mathcal{U}$.

Proof We follow [\[14, Corollary 9.9\]](#). Let \mathcal{C} be a localizing subcategory; then, using [Remark 3.31](#) and [Lemma 3.27](#),

$$\begin{aligned} \mathcal{C}^\perp &= \{Y \in \text{Sp}_{k,n} \mid F(X, Y) = 0 \text{ for all } X \in \mathcal{C}\} \\ &= \{Y \in \text{Sp}_{k,n} \mid \text{cosupp}(Y) \cap \text{supp}(\mathcal{C}) = \emptyset\} \\ &= \{Y \in \text{Sp}_{k,n} \mid \text{cosupp}(Y) \subseteq \mathcal{Q} \setminus \text{supp}(\mathcal{C})\} \\ &= \text{cosupp}^{-1}(\mathcal{Q} \setminus \text{supp}(\mathcal{C})). \end{aligned}$$

Similarly, if \mathcal{U} is a colocalizing subcategory, then

$$\begin{aligned} {}^\perp\mathcal{U} &= \{X \in \text{Sp}_{k,n} \mid F(X, Y) = 0 \text{ for all } Y \in \mathcal{U}\} \\ &= \{X \in \text{Sp}_{k,n} \mid \text{cosupp}(\mathcal{U}) \cap \text{supp}(X) = \emptyset\} \\ &= \{X \in \text{Sp}_{k,n} \mid \text{supp}(X) \subseteq \mathcal{Q} \setminus \text{cosupp}(\mathcal{U})\} \\ &= \text{supp}^{-1}(\mathcal{Q} \setminus \text{cosupp}(\mathcal{U})). \end{aligned}$$

It follows that under the equivalences of [Theorem 3.33](#), the maps $\mathcal{C} \mapsto \mathcal{C}^\perp$ and $\mathcal{U} \mapsto {}^\perp\mathcal{U}$ correspond to the map $\mathcal{Q} \rightarrow \mathcal{Q}$ sending a subset to its complement, and are thus mutually inverse bijections. □

3D The Bousfield lattice

We recall the basics on the Bousfield lattice of an algebraic stable homotopy theory. In order to avoid confusion with the (localized) categories of spectra considered previously we let $(\mathcal{C}, \wedge, \mathbf{1})$ denote a tensor triangulated category.

Definition 3.36 The Bousfield class of an object $X \in \mathcal{C}$ is the full subcategory of objects

$$\langle X \rangle = \{W \in \mathcal{C} \mid X \wedge W = 0\}.$$

Remark 3.37 We always assume that our categories are compactly generated and hence there is a set of Bousfield classes [34, Theorem 3.1].

Remark 3.38 We let $\mathrm{BL}(\mathcal{C})$ denote the set of Bousfield classes of \mathcal{C} . As is known, this has a lattice structure, which we now describe. We say that $\langle X \rangle \leq \langle Y \rangle$ if $Y \wedge W = 0$ implies $X \wedge W = 0$. Hence, $\langle 0 \rangle$ is the minimum Bousfield class, and $\langle \mathbf{1} \rangle$ is the maximum. The join is defined by $\bigvee_{i \in I} \langle X_i \rangle = \langle \bigsqcup_{i \in I} X_i \rangle$, and the meet is the join of all lower bounds.

Proposition 3.39 The Bousfield lattice $\mathrm{BL}(\mathrm{Sp}_{k,n})$ is isomorphic to the lattice of subsets of \mathcal{Q} via the map sending $\langle X \rangle$ to $\mathrm{supp}(X)$.

Proof Define a map that sends $\mathcal{T} \subseteq \mathcal{Q}$ to $\langle \bigvee_{i \in \mathcal{T}} K(i) \rangle$ in $\mathrm{BL}(\mathrm{Sp}_{k,n})$. We claim that this gives the necessary inverse map. By the local–global principle (Proposition 3.30),

$$\mathrm{Loc}_{\mathrm{Sp}_{k,n}}(X) = \mathrm{Loc}_{\mathrm{Sp}_{k,n}}(K(i) \mid i \in \mathrm{supp}(X)).$$

In particular, $X \bar{\wedge} W \simeq 0$ if and only if $K(i) \bar{\wedge} W \simeq 0$ for all $i \in \mathrm{supp}(X)$, so

$$(3-5) \quad \langle X \rangle = \left\langle \bigvee_{i \in \mathrm{supp}(X)} K(i) \right\rangle.$$

The result then follows by direct computation. □

3E The telescope conjecture and variants

We begin by considering variants of the telescope conjecture in the localized categories $\mathrm{Sp}_{k,n}$ using work of Wolcott [55].

Definition 3.40 For $i \in \mathcal{Q}$, let $l_i^f : \mathrm{Sp}_{k,n} \rightarrow \mathrm{Sp}_{k,n}$ denote finite localization away from $L_{k,n}F(i+1)$.

Remark 3.41 Because $L_{k,n}F(i + 1)$ is in $\text{Sp}_{k,n}^\omega$ by [Theorem 3.8](#), this is a smashing localization.

Remark 3.42 By [[55](#), Proposition 3.8] we have an equivalence of endofunctors of $\text{Sp}_{k,n}$ (recall that $\langle \text{Tel}(n) \rangle$ is the Bousfield class of a telescope of a finite type n spectrum),

$$l_i^f \simeq L_{L_{k,n} \text{Tel}(0) \vee L_{k,n} \text{Tel}(1) \vee \dots \vee L_{k,n} \text{Tel}(i)}.$$

We note that $L_{k,n} \text{Tel}(j)$ is trivial when $j \notin \mathcal{Q}$ by [[48](#), Proposition A.2.13]. In particular,

$$l_i^f \simeq L_{L_{k,n} \text{Tel}(k) \vee \dots \vee L_{k,n} \text{Tel}(i)}.$$

We also consider the following Bousfield localization on $\text{Sp}_{k,n}$.

Definition 3.43 For $i \in \mathcal{Q}$, let $l_i: \text{Sp}_{k,n} \rightarrow \text{Sp}_{k,n}$ denote Bousfield localization at $K(k) \vee K(k + 1) \vee \dots \vee K(i)$.

Remark 3.44 Following Wolcott [[55](#)], we consider the following variants of the telescope conjecture on $\text{Sp}_{k,n}$ for $i \in \mathcal{Q}$:

LTC1_i $\langle L_{k,n} \text{Tel}(i) \rangle = \langle K(i) \rangle$ in $\text{BL}(\text{Sp}_{k,n})$.

LTC2_i $l_i^f X \xrightarrow{\sim} l_i X$ for all X , or equivalently,

$$\left\langle \bigvee_{j=k}^i L_{k,n} \text{Tel}(j) \right\rangle = \left\langle \bigvee_{j=k}^i K(j) \right\rangle$$

in $\text{BL}(\text{Sp}_{k,n})$.

LTC3_i If X is a type i spectrum and f is a v_i self-map, $l_i(L_{k,n}X) \cong L_{k,n}(f^{-1}X)$.

GSC Every smashing localization is generated by a set of compact objects.

SDGSC Every smashing localization is generated by a set of dualizable objects.

Here LTC stands for the localized telescope conjecture, GSC is the generalized smashing conjecture, and SDGSC is the strongly dualizable generalized smashing conjecture. We emphasize the difference here because compact and dualizable objects do not coincide in $\text{Sp}_{k,n}$ when $k \neq 0$.

Proposition 3.45 On $\text{Sp}_{k,n}$, we have that LTC1_i, LTC2_i and LTC3_i hold for all $i \in \mathcal{Q}$.

Proof By [[55](#), Theorem 3.12] it suffices to prove that LTC1_i holds. By [Proposition 3.39](#) this will follow if we show that $L_{k,n} \text{Tel}(i)$ and $K(i)$ have the same support in $\text{Sp}_{k,n}$. To see this, we have $\text{supp}(K(i)) = \{i\}$ by [Example 3.23](#), while $\text{supp}(\text{Tel}(i)) = \{i\}$ by [[55](#), Lemmas 2.10 and 3.7]. □

We now classify all smashing localizations on $\mathrm{Sp}_{k,n}$ and show that all variants of the telescope conjecture hold.

Theorem 3.46 *Let L be a nontrivial smashing localization functor on $\mathrm{Sp}_{k,n}$. Then $L \simeq l_j^f \simeq l_j$ for some $j \in \mathcal{Q}$. In particular, the GSC and SDGSC both hold in $\mathrm{Sp}_{k,n}$.*

Proof We closely follow [55, Theorem 4.4]. Throughout the proof we let $\mathbf{1}$ denote $L_{k,n}S^0$, the monoidal unit in $\mathrm{Sp}_{k,n}$, so that $\langle L \rangle = \langle L\mathbf{1} \rangle$. By (3-5),

$$\langle L\mathbf{1} \rangle = \left\langle \bigvee_{i \in \mathrm{supp}(L\mathbf{1})} K(i) \right\rangle.$$

Note that $\mathrm{supp}(L\mathbf{1})$ is nonempty because we assume $L \neq 0$. Hence, we can fix $j \in \mathrm{supp}(L\mathbf{1})$ such that $\langle K(j) \rangle \leq \langle L\mathbf{1} \rangle$ in $\mathrm{BL}(\mathrm{Sp}_{k,n})$. It follows that $L_{K(j)}L \simeq LL_{K(j)} \simeq L_{K(j)}$, and $\langle L_{K(j)}\mathbf{1} \rangle = \langle L_{K(j)}\mathbf{1} \bar{\wedge} L\mathbf{1} \rangle \leq \langle L\mathbf{1} \rangle$ in $\mathrm{BL}(\mathrm{Sp}_{k,n})$. We also note that $L_{K(j)}\mathbf{1} = L_{K(j)}L_{k,n}S^0 \simeq L_{K(j)}S^0$.

By [32, Proposition 5.3], $\langle L_{K(j)}S^0 \rangle = \bigvee_{i=0}^j \langle K(i) \rangle$ in $\mathrm{BL}(\mathrm{Sp})$, and it follows easily that $\langle L_{K(j)}S^0 \rangle = \bigvee_{i=k}^j \langle K(i) \rangle$ in $\mathrm{BL}(\mathrm{Sp}_{k,n})$. It follows that $\langle L\mathbf{1} \rangle \geq \bigvee_{i=k}^j \langle K(i) \rangle$ in $\mathrm{BL}(\mathrm{Sp}_{k,n})$. We deduce that $\langle L\mathbf{1} \rangle = \bigvee_{i=k}^j \langle K(i) \rangle$, where $j = \max\{\mathrm{supp}(L\mathbf{1})\}$, and hence by Proposition 3.45 that $L \simeq l_j^f \simeq l_j$. Finally, because $L_n F(j+1)$ is compact and therefore also dualizable in $\mathrm{Sp}_{k,n}$, both the GSC and SDGSC hold in $\mathrm{Sp}_{k,n}$. \square

Remark 3.47 Using [30, Proposition 3.8.3] and Theorems 3.33 and 3.46 one can prove the thick subcategory theorem Theorem 3.16.

4 Descent theory and the $E(n, J_k)$ -local Adams spectral sequence

In this section we use descent theory to construct an Adams-type spectral sequence in the $E(n, J_k)$ -local category. Using descent, we shall see that this has a vanishing line at some finite stage. Moreover, for $p \gg n$, we show that the $E(n, J_k)$ -local Adams spectral sequence computing $\pi_* L_{k,n}S^0$ has a horizontal vanishing line on the E_2 -page, and there are no nontrivial differentials.

4A Descendability

We begin with the notion of a descendable object in an algebraic stable homotopy category.

Remark 4.1 We recall that in \mathcal{C} there is an ∞ -category $\text{CAlg}(\mathcal{C})$ of commutative algebra objects; see [40, Chapter 3]. Moreover, given $A \in \text{CAlg}(\mathcal{C})$ we can define a stable, presentable, symmetric monoidal ∞ -category $\text{Mod}_A(\mathcal{C})$ of A -modules internal to \mathcal{C} , with the relative A -linear tensor product [40, Section 4.5]. We will mainly focus on the case $A = E_n$ and $\mathcal{C} = \text{Sp}_{k,n}$, so that $\text{Mod}_{E_n}(\text{Sp}_{k,n})$ denotes the ∞ -category of $E(n, J_k)$ -local E_n -modules, that is E_n -modules whose underlying spectrum is $E(n, J_k)$ -local, with monoidal structure $A \wedge B = L_{k,n}(A \wedge_E B)$.

Remark 4.2 Note that $E_n \in \text{CAlg}(\text{Sp})$ by the Goerss–Hopkins–Miller [21] theorem, and so $E_n \in \text{CAlg}(\text{Sp}_{k,n})$ as well. On the other hand, $E(n, J_k)$ will not, in general, be a commutative ring spectrum (for example, $K(n)$ is never a commutative ring spectrum).

Definition 4.3 [41, Definition 3.18] A commutative algebra object $A \in \text{CAlg}(\mathcal{C})$ is said to be descendable if $\mathbf{1} \in \mathcal{C}$ is A -nilpotent (Definition 3.3), or equivalently $\mathcal{C} = \text{thick}_{\otimes} \langle A \rangle$.

One reason to be interested in descendable objects is the following [41, Proposition 3.22].

Proposition 4.4 (Mathew) *Let $A \in \text{CAlg}(\mathcal{C})$ be descendable. Then the adjunction $\mathcal{C} \rightleftarrows \text{Mod}_{\mathcal{C}}(A)$ given by tensoring with A and forgetting is comonadic. In particular, the natural functor from \mathcal{C} to the totalization*

$$\mathcal{C} \rightarrow \text{Tot} \left(\text{Mod}_A(\mathcal{C}) \rightrightarrows \text{Mod}_{A \wedge A}(\mathcal{C}) \rightrightarrows \dots \right)$$

is an equivalence.

We also note the following [41, Proposition 3.19].

Proposition 4.5 (Mathew) *If $A \in \text{CAlg}(\mathcal{C})$ is descendable, then the functor*

$$\mathcal{C} \rightarrow \text{Mod}_A(\mathcal{C}), \quad M \mapsto M \wedge A,$$

is conservative.

4B Morava modules and L -complete comodules

The following theorem, essentially due to Hopkins–Ravenel [48], shows that the results of the previous section can be applied in $\text{Sp}_{k,n}$. We note that $E_n \in \text{Sp}$ is a commutative algebra object; this is the Goerss–Hopkins–Miller theorem [21]. It follows that $E_n \in \text{CAlg}(\text{Sp}_{k,n})$.

Theorem 4.6 $E_n \in \mathrm{CAlg}(\mathrm{Sp}_{k,n})$ is descendable, and there is an equivalence of symmetric-monoidal stable ∞ -categories

$$(4-1) \quad \mathrm{Sp}_{k,n} \simeq \mathrm{Tot}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n}) \rightrightarrows \mathrm{Mod}_{E_n \bar{\wedge} E_n}(\mathrm{Sp}_{k,n}) \rightrightarrows)$$

Proof It is consequence of the Hopkins–Ravenel smash product theorem that $E_n \in \mathrm{CAlg}(\mathrm{Sp}_n)$ is descendable; see [41, Theorem 4.18]. It follows from [41, Corollary 3.21] that $L_{k,n}E_n \simeq E_n$ is descendable in $\mathrm{Sp}_{k,n}$. The equivalence then follows from Proposition 4.4. \square

By Proposition 4.5 we deduce the following.

Corollary 4.7 The functor $E_n \bar{\wedge} (-) : \mathrm{Sp}_{k,n} \rightarrow \mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})$ is conservative.

We therefore define the following.

Definition 4.8 For $X \in \mathrm{Sp}_{k,n}$ the Morava module of X is $(E_{k,n})^\vee_* X := \pi_*(E_n \bar{\wedge} X)$.

We recall that $L_{k,n}X \simeq \varprojlim_j (L_n X \wedge M_k(j))$ (Proposition 2.24). The Milnor sequence then gives the following.

Lemma 4.9 There is a short exact sequence

$$0 \rightarrow \varprojlim_j^1 (E_n)_{*+1}(X \wedge M_j(k)) \rightarrow (E_{k,n})^\vee_* X \rightarrow \varprojlim_j E_*(X \wedge M_j(k)) \rightarrow 0.$$

Example 4.10 If $(E_n)_* X$ is a free $(E_n)_*$ -module, then the \varprojlim^1 term vanishes and it follows that $(E_{k,n})^\vee_* X = (E_* X)_{I_k}^\wedge$.

Remark 4.11 As the short exact sequence shows, $(E_{k,n})^\vee_* X$ is not always complete with respect to the I_k -adic topology. However, it is always $L_0^{I_k}$ -complete in the sense of [32, Appendix A]—this the same argument as given in [32, Proposition 8.4(a)].

4C The $E(n, J_k)$ -local E_n -Adams spectral sequence

In this section we construct an Adams-type spectral sequence in the $E(n, J_k)$ -local category. When $k = 0$, this is the E_n -Adams spectral sequence, while when $k = n$ this is the $K(n)$ -local E_n -Adams spectral sequence considered in [16, Appendix A].

To begin, we recall that the cobar (or Amitsur) complex for E_n in $\mathrm{Sp}_{k,n}$ is

$$CB^\bullet(E_n): E_n \rightrightarrows E_n \bar{\wedge} E_n \rightrightarrows \cdots .$$

Definition 4.12 Let $\widehat{\mathrm{Ext}}_{(E_{k,n})_*^\vee(E_n)}^{s,*}((E_n)_*, (E_n)_*) := H^s(\pi_*(CB^\bullet(E_n)))$, ie it is the cohomology of the complex

$$(E_n)_* \rightrightarrows (E_{k,n})_*^\vee(E_n) \rightrightarrows \cdots .$$

More generally, we let

$$\widehat{\mathrm{Ext}}_{(E_{k,n})_*^\vee(E_n)}^{s,*}((E_n)_*, (E_{k,n})_*^\vee(X)) := H^s(\pi_*(X \bar{\wedge} CB^\bullet(E_n))).$$

Proposition 4.13 For any spectrum X there is a strongly convergent spectral sequence

$$E_2^{s,t} \cong \widehat{\mathrm{Ext}}_{(E_{k,n})_*^\vee(E_n)}^{s,*}((E_n)_*, (E_{k,n})_*^\vee(X)) \Rightarrow \pi_*(L_{k,n}X)$$

which has a horizontal vanishing line at a finite stage (independent of X).

Proof This is the Bousfield–Kan spectral sequence associated to the tower

$$X \bar{\wedge} CB^\bullet(E_n).$$

The claimed results are a consequence of descendability (Theorem 4.6); see [41, Corollary 4.4; 42, Example 2.11, Propositions 2.12 and 2.14]. □

Remark 4.14 For the $\mathrm{Sp}_{K(n)}$ –local homotopy category, this completed Ext can be interpreted as an Ext group in the category of $L_0^{I_n}$ –complete comodules [9]. In the case of $X = S^0$, Morava’s change of rings theorem, in the form [9, Theorem 4.3], shows that

$$E_2^{s,t} \cong H_c^s(\mathbb{G}_n, (E_n)_t),$$

the continuous cohomology of the Morava stabilizer group \mathbb{G}_n , and this spectral sequence is isomorphic to that considered by Devinatz and Hopkins in [16, Appendix A]. The key point is the computation that

$$(E_n)_*^\vee(E_n) \cong \mathrm{Hom}^c(\mathbb{G}_n, (E_n)_*),$$

for which see [29]. We remark that we do not know what $(E_{k,n})_*^\vee(E_n)$ is for $k \neq n$. However, the same arguments as in [9] go through; the pair $((E_n)_*, (E_{k,n})_*^\vee(E_n))$ is an $L_0^{I_k}$ –complete comodule, and if $(E_{k,n})_*^\vee(X)$ is either a finitely generated $(E_n)_*$ –module, is the I_k –adic completion of a free-module, or has bounded I_k –torsion, then $(E_{k,n})_*^\vee(X)$ is a comodule over this Hopf algebroid — see [9, Lemma 1.17 and

Proposition 1.22]. The relative homological algebra studied in [9, Section 2] also goes through to see that $\widehat{\mathrm{Ext}}^{s,*}$ as used above is a relative Ext group in the category of $L_0^{I^k}$ -complete comodules. We will not use this in what follows, so we leave the details to the interested reader.

Remark 4.15 In [26, Section 7] the authors construct the $K(n)$ -local Adams spectral sequence for dualizable $K(n)$ -local X as the inverse limit of the E_n -Adams spectral sequences for $X \wedge M_n(j)$. The following result recovers the identification of the E_2 -term in the case $k = n$.

Proposition 4.16 *Let $M_k(j)$ be a tower of generalized Moore spectra of height k . Then there is an isomorphism*

$$\widehat{\mathrm{Ext}}_{(E_{k,n})_*^\vee(E_n)}^{s,t}((E_n)_*, (E_n)_*) \cong \varprojlim_j \mathrm{Ext}_{(E_n)_*E_n}^{s,t}((E_n)_*, (E_n)_*(M_k(j))).$$

Proof By definition, $\widehat{\mathrm{Ext}}_{(E_{k,n})_*^\vee(E_n)}^{s,t}((E_n)_*, (E_n)_*)$ is the cohomology of the complex

$$(E_n)_* \rightrightarrows (E_{k,n})_*^\vee(E_n) \rightrightarrows \cdots$$

The t^{th} term of this complex is the homotopy of $L_{k,n}(E_n^{\wedge t}) \simeq \varprojlim_j (E_n^{\wedge t} \wedge M_k(j))$ by Proposition 2.24, and there is a corresponding Milnor exact sequence of the form

$$0 \rightarrow \varprojlim_j^1 \pi_{q+1}(E_n^{\wedge t} \wedge M_k(j)) \rightarrow \pi_q(L_{k,n}(E_n^{\wedge t})) \rightarrow \varprojlim_j \pi_q(E_n^{\wedge t} \wedge M_k(j)) \rightarrow 0.$$

We note that $E_n^{\wedge t}$ is Landweber exact, as the smash product of Landweber exact spectra; see [12, Lemma 4.3]. It follows that $\pi_*(E_n^{\wedge t} \wedge M_k(j)) \cong \pi_*(E_n^{\wedge t})/(p^{i_0}, \dots, u_{k-1}^{i_{k-1}})$ for suitable integers i_0, \dots, i_{k-1} . In particular, the maps in the tower are surjections by the construction of the tower $\{M_k(j)\}$ (see Remark 2.19), and so the \varprojlim_j^1 -term vanishes, and

$$(E_{k,n})_*^\vee(E_n^{t-1}) \cong \pi_*(L_{k,n}(E_n^{\wedge t})) \cong \varprojlim_j \pi_q(E_n^{\wedge t} \wedge M_k(j)).$$

Note that the cohomology of the complex $\{\pi_q(E_n^t \wedge M_k(j))\}_t$ is

$$\mathrm{Ext}_{(E_n)_*E_n}^{*,*}((E_n)_*, (E_n)_*(M_k(j))).$$

Therefore, there is an exact sequence

$$\begin{aligned} 0 \rightarrow \varprojlim_j^1 \mathrm{Ext}_{(E_n)_*E_n}^{q-1,*}((E_n)_*, (E_n)_*(M_k(j))) &\rightarrow \widehat{\mathrm{Ext}}_{(E_{k,n})_*^\vee(E_n)}^{q,*}((E_n)_*, (E_n)_*) \\ &\rightarrow \varprojlim_j \mathrm{Ext}_{(E_n)_*E_n}^{q,*}((E_n)_*, (E_n)_*(M_k(j))) \rightarrow 0. \end{aligned}$$

We will see below in [Corollary 4.22](#) that $\text{Ext}_{(E_n)_*E_n}^{q,*}((E_n)_*, (E_n)_*(M_k(j)))$ is finite, and so the \varprojlim_j^1 -term vanishes in the exact sequence, and the result follows. \square

Remark 4.17 It follows that when $k \neq 0$, the groups $\widehat{\text{Ext}}_{(E_{k,n})^*(E_n)}^{s,t}((E_n)_*, (E_n)_*)$ are profinite, ie either finite or uncountable. Contrast the case $k = 0$, where

$$\text{Ext}_{(E_n)_*E_n}^{s,t}((E_n)_*, (E_n)_*)$$

is countable [[28](#), Proof of Lemma 5.4].

4D Vanishing lines in the $E(n, J_k)$ -local E_n -Adams spectral sequence

In [Proposition 4.13](#) we constructed a spectral sequence

$$\widehat{\text{Ext}}_{(E_{k,n})^*(E_n)}^{s,t}((E_n)_*, (E_n)_*) \Rightarrow \pi_{t-s}(L_{k,n}S^0),$$

and showed that, as a consequence of descendability, this has a horizontal vanishing line at some finite stage. In the extreme cases of $k = 0$ and $k = n$ it is known that when $p \gg n$ and $X = S^0$, this vanishing line occurs on the E_2 -page, and occurs at $s = n^2 + n$ and $s = n^2$, respectively; see [[31](#), Theorem 5.1] and [[47](#), Theorem 6.2.10]. In this section, we show ([Theorem 4.24](#)) that the analogous result occurs in general; for $p \gg n$ there is a vanishing line on the E_2 -page of the spectral sequence of [Proposition 4.13](#) above, and $s = n^2 + n - k$ in the case $X = S^0$. The proof relies on a variant of the chromatic spectral sequence [[47](#), Chapter 5], which we now construct. Along the way we prove [Corollary 4.22](#), which also completes the proof of [Proposition 4.16](#).

Remark 4.18 (the chromatic spectral sequence) Fix $k \leq n$, and for $0 \leq s \leq n - k$ let M^s denote the $(E_n)_*(E_n)$ -comodule

$$u_{k+s}^{-1}(E_n)_*/(p, u_1, \dots, u_{k-1}, u_k^\infty, \dots, u_{k+s-1}^\infty).$$

Arguing as in [[47](#), Lemma 5.1.6], there is an exact sequence of $(E_n)_*(E_n)$ -comodules

$$(E_n)_*/I_k \rightarrow M^0 \rightarrow M^1 \rightarrow \dots \rightarrow M^{n-k} \rightarrow 0.$$

Applying [[47](#), Theorem A.1.3.2], there is then a chromatic spectral sequence of the form

$$(4-2) \quad E_1^{s,r,*} \cong \text{Ext}_{(E_n)_*(E_n)}^{r,*}((E_n)_*, M^s) \Rightarrow \text{Ext}_{(E_n)_*(E_n)}^{r+s,*}((E_n)_*, (E_n)_*/I_k).$$

Proposition 4.19 In the chromatic spectral sequence (4-2),

$$E_1^{s,r,*} \cong \begin{cases} \text{Ext}_{(E_{k+s})^*(E_{k+s})}^{r,*}((E_{k+s})_*, (E_{k+s})_*/(p, \dots, u_{k-1}, u_k^\infty, \dots, u_{k+s-1}^\infty)) & \text{if } s \leq n-k, \\ 0 & \text{if } s > n-k. \end{cases}$$

If particular, if $p - 1$ does not divide $k + s$, we have $E_1^{s,r,*} = 0$ for $r > (s + k)^2$. Thus, if $p - 1$ does not divide $k + s$ for all $0 \leq s \leq n - k$,² then

$$\mathrm{Ext}_{(E_n)_*(E_n)}^{s,*}((E_n)_*, (E_n)_*/I_k) = 0$$

for $s > n^2 + n - k$.

Proof This is similar to the proof by Hovey and Sadofsky [31, Theorem 5.1], which is the case where $k = 0$. We first recall the change of rings theorem of Hovey and Sadofsky [31, Theorem 3.1]; if M is a BP_*BP -comodule, on which v_j acts isomorphically, and $n \geq j$, then there is an isomorphism³

$$\mathrm{Ext}_{BP_*BP}^{*,*}(BP_*, M) \cong \mathrm{Ext}_{(E_n)_*(E_n)}^{*,*}((E_n)_*, (E_n)_* \otimes_{BP_*} M).$$

Applying this change of rings theorem twice to the BP_*BP -comodule

$$u_{k+s}^{-1}BP_*/(p, \dots, u_{k-1}, u_k^\infty, \dots, u_{k+s-1}^\infty)$$

with $j = k + s$ and $j = n$ shows that the E_1 -term has the claimed form.

For brevity, let us denote $I = (p, \dots, u_{k-1}, u_k^\infty, \dots, u_{k+s-1}^\infty)$. By Morava's change of rings theorem,

$$\mathrm{Ext}_{(E_{k+s})_*(E_{k+s})}^{r,*}((E_{k+s})_*, (E_{k+s})_*/I) \cong H^r(\mathbb{G}_{k+s}, (E_{k+s})_*/I).$$

Morava's vanishing theorem [47, Theorem 6.2.10] shows that if $p - 1$ does not divide $k + s$, then

$$H^r(\mathbb{G}_{k+s}, (E_{k+s})_*) = 0$$

for $r > (k + s)^2$. Along with an argument similar to that given by Hovey and Sadofsky's, using standard exact sequences and taking direct limits we find that

$$\mathrm{Ext}_{(E_{k+s})_*(E_{k+s})}^{r,*}((E_{k+s})_*, (E_{k+s})_*/I) = 0$$

for $r > (k + s)^2$ as well. □

Remark 4.20 Let M_k denote a generalized Moore spectrum of type k . Then there is an obvious analog of this spectral sequence, whose abutment is

$$\mathrm{Ext}_{(E_n)_*(E_n)}^{r+s,*}((E_n)_*, (E_n)_*(M_k)) \cong \mathrm{Ext}_{(E_n)_*(E_n)}^{r+s,*}((E_n)_*, (E_n)_*/(p^{i_0}, \dots, u_{k-1}^{i_{k-1}}))$$

²Taking $p > n + 1$ suffices, but may not be optimal.

³Hovey and Sadofsky work with $E(n)$ instead of E_n , but this does not change anything in light of [33, Theorem C].

with E_1 -term of the form

$$E_1^{s,r,*} \cong \begin{cases} \text{Ext}_{(E_{k+s})^*(E_{k+s})}^{r,*}((E_{k+s})^*, (E_{k+s})^*/(p^{i_0}, \dots, u_{k-1}^{i_{k-1}}, u_k^\infty, \dots, u_{k+s-1}^\infty)) & \text{if } s \leq n-k, \\ 0 & \text{if } s > n-k. \end{cases}$$

Remark 4.21 The following completes the proof of [Proposition 4.16](#).

Corollary 4.22 *Let M_k denote a generalized Moore spectrum of type k . Then the group $\text{Ext}_{(E_n)^*(E_n)}^{r,*}((E_n)^*, (E_n)^*(M_k))$ is finite.*

Proof By taking appropriate exact sequences it suffices to show this for $(E_n)^*/I_k$ (alternatively, one can argue directly using the spectral sequence of [Remark 4.20](#)). Given the chromatic spectral sequence, we can reduce to showing that $H^r(\mathbb{G}_{k+s}, (E_{k+s})^*/I)$ is finite, with I as in the proof of the previous proposition. For this, see [Proposition 4.2.2](#) of [\[54\]](#). □

Corollary 4.23 *Let M_k denote a generalized Moore spectrum of type k . Then if $p - 1$ does not divide $k + s$ for $0 \leq s \leq n - k$,*

$$\text{Ext}_{(E_n)^*(E_n)}^{s,*}((E_n)^*, (E_n)^*(M_k)) = 0$$

for $s > n^2 + n - k$.

Proof Recall that $(E_n)^*(M_k) \cong (E_n)^*/(p^{i_0}, \dots, u_{k-1}^{i_{k-1}})$ for a suitable sequence of integers (i_0, \dots, i_{k-1}) . The result for the sequence $(1, \dots, 1)$ holds by [Proposition 4.19](#), and therefore in general by taking appropriate exact sequences. □

Theorem 4.24 *Suppose $p - 1$ does not divide $k + s$ for $0 \leq s \leq n - k$. Then*

$$\widehat{\text{Ext}}_{(E_{k,n})^*(E_n)}^{s,t}((E_n)^*, (E_n)^*) = 0$$

for $s > n^2 + n - k$.

Proof Combine [Proposition 4.16](#) and [Corollary 4.23](#). □

Remark 4.25 The condition on the prime is always satisfied if p is large enough compared to n (in fact $p > n + 1$ suffices). This suggests the following, which we do not attempt to make precise: for large enough primes, the cohomological dimension of $(E_n)^*$ in a suitable category of (completed) $(E_{k,n})^*(E_n)$ -comodules is finite, and equal to $n^2 + n - k$.

We also have the following expected sparseness result.

Proposition 4.26 *Let $q = 2(p - 1)$. Then*

$$\widehat{\mathrm{Ext}}_{(E_n)_* \vee (E_n)_*}^{s,t}((E_n)_*, (E_n)_*) = 0$$

for all s and t unless $t \equiv 0 \pmod q$. Consequently, in the spectral sequence of [Proposition 4.13](#), d_r is nontrivial only if $r \equiv 1 \pmod q$ and $E_{mq+2}^{*,*} = E_{mq+q+1}^{*,*}$ for all $m \geq 0$.

Proof Using [Proposition 4.16](#) it suffices to show the first statement for the E_1 -term of the chromatic spectral sequence of [Proposition 4.19](#). Again using the Hovey–Sadofsky change of rings theorem, this E_1 -term is isomorphic to

$$\mathrm{Ext}_{BP_*BP}^{s,*,*}(BP_*, u_{k+s}^{-1}BP_*/(p, \dots, u_{k-1}, u_k^\infty, \dots, u_{k+s-1}^\infty)).$$

Now apply [\[47, Proposition 4.4.2\]](#). □

5 Dualizable objects in $\mathrm{Sp}_{k,n}$

In this section we use descendability to characterize the dualizable objects in $\mathrm{Sp}_{k,n}$. As noted previously, as long as $k \neq 0$, these differ from the compact objects studied in [Section 3A](#).

Definition 5.1 Let $(\mathcal{C}, \wedge, \mathbf{1})$ be a symmetric-monoidal ∞ -category. Then $X \in \mathcal{C}$ is dualizable if there exists an object $D_{\mathcal{C}}X$ and a pair of morphisms

$$e: D_{\mathcal{C}}X \wedge X \rightarrow \mathbf{1}, \quad c: \mathbf{1} \rightarrow X \wedge D_{\mathcal{C}}X$$

such that the composites

$$X \xrightarrow{c \wedge \mathrm{id}} X \wedge D_{\mathcal{C}}X \wedge X \xrightarrow{\mathrm{id} \wedge e} X, \quad D_{\mathcal{C}}X \xrightarrow{\mathrm{id} \wedge c} D_{\mathcal{C}}X \wedge X \wedge D_{\mathcal{C}}X \xrightarrow{e \wedge \mathrm{id}} D_{\mathcal{C}}X$$

are the identity on X and $D_{\mathcal{C}}X$, respectively.

Remark 5.2 The definition makes it clear that $X \in \mathcal{C}$ is dualizable if and only if it is dualizable in the homotopy category of \mathcal{C} . Moreover, a formal argument shows that, if it exists, we must have $D_{\mathcal{C}}X \simeq F(X, \mathbf{1})$. Finally, for the equivalence with other definitions of dualizability the reader may have seen, see [\[17, Theorem 1.3\]](#).

Definition 5.3 We let $\mathcal{C}^{\mathrm{dual}} \subseteq \mathcal{C}$ denote the full subcategory consisting of the dualizable objects of \mathcal{C} .

Remark 5.4 The full subcategory $\mathcal{C}^{\mathrm{dual}}$ is a thick tensor ideal [\[30, Theorem A.2.5\]](#).

We have the following relationship between descent theory and dualizability.

Proposition 5.5 *Let $A \in \text{CAlg}(\mathcal{C})$ be descendable. Then the adjunction $\mathcal{C} \rightleftarrows \text{Mod}_A(\mathcal{C})$ gives rise to an equivalence of symmetric monoidal ∞ -categories*

$$\mathcal{C}^{\text{dual}} \rightarrow \text{Tot} \left(\text{Mod}_A(\mathcal{C})^{\text{dual}} \rightrightarrows \text{Mod}_{A \wedge A}(\mathcal{C})^{\text{dual}} \rightrightarrows \right).$$

In particular, $M \in \mathcal{C}$ is dualizable if and only if $M \wedge A \in \text{Mod}_A(\mathcal{C})$ is dualizable.

Proof The first claim follows from Proposition 4.4 because passing to dualizable objects commutes with limits of ∞ -categories [40, Proposition 4.6.1.11]. The second is then an easy consequence, using that all the maps in the totalization are symmetric monoidal. □

5A Dualizable objects in the $E(n, J_k)$ -local category

Using Theorem 4.6 and Proposition 5.5 we deduce the following.

Proposition 5.6 *The adjunction $\text{Sp}_{k,n} \rightleftarrows \text{Mod}_{E_n}(\text{Sp}_{k,n})$ gives rise to an equivalence of symmetric monoidal ∞ -categories*

$$\text{Sp}_{k,n}^{\text{dual}} \rightarrow \text{Tot} \left(\text{Mod}_{E_n}(\text{Sp}_{k,n})^{\text{dual}} \rightrightarrows \text{Mod}_{E_n \bar{\wedge} E_n}(\text{Sp}_{k,n})^{\text{dual}} \rightrightarrows \right).$$

In particular, $X \in \text{Sp}_{k,n}$ is dualizable if and only if $E_n \bar{\wedge} X \in \text{Mod}_{E_n}(\text{Sp}_{k,n})$ is dualizable.

This proposition suggests we begin by studying dualizable objects in the category $\text{Mod}_{E_n}(\text{Sp}_{k,n})$. Fortunately, these have a nice characterization. We begin with the following.

Lemma 5.7 *If X is dualizable in $\text{Mod}_{E_n}(\text{Sp}_{k,n})$ then the spectrum underlying X is $K(n)$ -local.*

Proof We first note that for any $M \in \text{Mod}_{E_n}$ (in particular, for $M = X$), the Bousfield localization $L_{K(n)}M$ is the spectrum underlying $L_{E_n \wedge X}^{E_n}$, where the latter denotes the Bousfield localization with respect to $E_n \wedge X$ internal to the category of E_n -modules. In particular, the localization map $M \rightarrow L_{K(n)}M$ is a map in Mod_{E_n} ; see [19, Chapter VIII], particularly [19, Proposition VIII.1.8]. It follows that $K(n)$ -localization defines a localization

$$L_{K(n)} : \text{Mod}_{E_n}(\text{Sp}_{k,n}) \rightarrow \text{Mod}_{E_n}(\text{Sp}_{K(n)}).$$

Because $X \in \mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})^{\mathrm{dual}}$, using [30, Lemma 3.3.1], we see that there are equivalences

$$L_{K(n)}X \simeq L_{k,n}((L_{K(n)}E_n) \wedge_{E_n} X) \simeq L_{k,n}(E_n \wedge_{E_n} X) \simeq L_{k,n}X \simeq X. \quad \square$$

Remark 5.8 For the following, we let $K_n \cong E_n/I_n$. This is a 2-periodic form of Morava K -theory; indeed,

$$(K_n)_*X \cong (K_n)_* \otimes_{(K(n)_*} K(n)_*X,$$

and so $\langle K(n) \rangle = \langle K_n \rangle$. We use this only because K_n is naturally an E_n -module.

Proposition 5.9 For $X \in \mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})$ the following are equivalent:

- (1) X is dualizable in $\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})$.
- (2) X is compact (equivalently, dualizable) in $\mathrm{Mod}_{E_n}(\mathrm{Sp})$.
- (3) The spectrum underlying X is $K(n)$ -local and the homotopy groups $\pi_*(K_n \wedge_{E_n} X)$ are finite.

Proof We first show that (2) implies (1). The compact objects in $\mathrm{Mod}_{E_n}(\mathrm{Sp})$ are precisely those in the thick subcategory generated by E_n ; see, for example, [40, Proposition 7.2.4.2]. Since $E_n \in \mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})^{\mathrm{dual}}$, and the collection of dualizable objects is thick, the implication (2) implies (1) follows.

Conversely, assume that (1) holds. As above, we have a symmetric-monoidal localization

$$L_{K(n)}: \mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n}) \rightarrow \mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)}),$$

which preserves dualizable objects (as any symmetric-monoidal functor does). Using Lemma 5.7 it follows that $L_{K(n)}X \simeq X$ is dualizable in $\mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)})$, which implies by [41, Proposition 10.11] that X is compact in $\mathrm{Mod}_{E_n}(\mathrm{Sp})$.

Finally, the equivalence of (2) and (3) is well known; see for example Proposition 2.9.4 of [25].⁴ □

Remark 5.10 Suppose $X \in \mathrm{Sp}_{k,n}^{\mathrm{dual}}$, so that $L_{k,n}(E_n \wedge X) \in \mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})^{\mathrm{dual}}$. The previous proposition then implies that

$$L_{k,n}(E_n \wedge X) \simeq L_{K(n)}L_{k,n}(E_n \wedge X) \simeq L_{K(n)}(E_n \wedge X).$$

In other words, for dualizable X , there is an isomorphism $(E_{k,n})_*^\vee(X) \cong (E_{n,n})_*^\vee(X)$.

⁴Lurie has confirmed via private communication that the cited proposition [25, Proposition 2.9.4] should additionally have the assumption X is $K(n)$ -local in condition (3).

We now give our characterization of dualizable spectra in $\text{Sp}_{k,n}$ — see [32, Theorem 8.6] for the case $k = n$. We note that even in this case our proof, which uses descendability, differs from that of Hovey and Strickland.

Theorem 5.11 *The following are equivalent for $X \in \text{Sp}_{k,n}$:*

- (1) X is dualizable.
- (2) X is F -small, ie for any collection of objects $\{Z_i\}$, the natural map

$$L_{k,n}\left(\bigvee_i F(X, Z_i)\right) \rightarrow F\left(X, L_{k,n}\left(\bigvee_i Z_i\right)\right)$$

is an equivalence.

- (3) $E_n \bar{\wedge} X \in \text{Mod}_{E_n}(\text{Sp}_{k,n})$ is dualizable.
- (4) $E_n \bar{\wedge} X \in \text{Mod}_{E_n}(\text{Sp})$ is dualizable (equivalently, compact).
- (5) $E_n \bar{\wedge} X$ is $K(n)$ -local and $(K_n)_* X$ is finite.
- (6) $(E_{k,n})_*^\vee(X)$ is a finitely generated E_* -module.

Proof The equivalences between the first five items come from [30, Theorem 2.1.3(c)] ((1) \iff (2)), Proposition 5.5 ((1) \iff (3)) and Proposition 5.9 ((3) \iff (4) \iff (5)). We note that if M is an E_n -module, then M is compact if and only if $\pi_* M$ is finitely generated over $(E_n)_*$; see [22, Lemma 10.2(i)]. Applying this with $M = E_n \bar{\wedge} X$ gives the equivalence between (4) and (6). □

Finally, we show that there is only a set of isomorphism classes of dualizable objects.

Lemma 5.12 *There are at most 2^{\aleph_0} isomorphism classes of objects in $\text{Sp}_{k,n}^{\text{dual}}$.*

Proof This is the same as the argument given in [32, Proposition 12.17]. Namely, there are only countably many finite spectra X' of type at least k , and for each one $[L_n X', L_n X']$ is finite, so $L_n X'$ has only finitely many retracts. By Theorem 3.8 it follows that there is a countable set of isomorphism classes of objects in $\text{Sp}_{k,n}^\omega$. If U and V are finite, then $[U, V]$ is finite, and so there are at most $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ different towers of spectra in $\text{Sp}_{k,n}^\omega$. For $X \in \text{Sp}_{k,n}^{\text{dual}}$, write $X \simeq \varprojlim_j X \wedge M_k(j)$, as in Proposition 2.24. Because X is dualizable and $M_k(j)$ is compact, $X \wedge M_k(j)$ is compact [30, Theorem 2.1.3(a)]. Therefore, X is the inverse limit of a tower of spectra in $\text{Sp}_{k,n}^\omega$, and hence there are at most 2^{\aleph_0} isomorphism classes of objects in $\text{Sp}_{k,n}^{\text{dual}}$. □

5B The spectrum of dualizable objects

In [Theorem 3.16](#) we computed the thick subcategories (equivalently, thick tensor-ideals) of compact objects in $\mathrm{Sp}_{k,n}$. One could also ask for a classification of the thick tensor-ideals of dualizable objects in $\mathrm{Sp}_{k,n}$, or equivalently a computation of the Balmer spectrum $\mathrm{Spc}(\mathrm{Sp}_{k,n}^{\mathrm{dual}})$ (which is well defined by [Lemma 5.12](#)). Based on a conjecture of Hovey and Strickland, the author, along with Barthel and Naumann, investigated $\mathrm{Spc}(\mathrm{Sp}_{K(n)}^{\mathrm{dual}})$ in detail in [\[10\]](#), showing that the Hovey–Strickland conjecture holds when $n = 1$ and 2 , and that in general it is implied by a hope of Chai in arithmetic geometry. In this section, we make some general comments regarding $\mathrm{Spc}(\mathrm{Sp}_{k,n}^{\mathrm{dual}})$.

Remark 5.13 The following full subcategories were considered in the case $k = n$ by Hovey and Strickland [\[32, Definition 12.14\]](#).

Definition 5.14 For $i \leq n$, let \mathcal{D}_i denote the category of spectra $X \in \mathrm{Sp}_{k,n}^{\mathrm{dual}}$ such that X is a retract of $Y \wedge Z$ for some $Y \in \mathrm{Sp}_{k,n}^{\mathrm{dual}}$ and some finite spectrum Z of type at least i . It is also useful to set $\mathcal{D}_{n+1} = (0)$.

Remark 5.15 We note that $\mathcal{D}_k \simeq \mathrm{Sp}_{k,n}^{\omega}$; this is a consequence of the characterization of compact objects given in [Theorem 3.8](#), and that $\mathcal{D}_0 = \mathrm{Sp}_{k,n}^{\mathrm{dual}}$.

The following is [\[32, Proposition 4.17\]](#).

Lemma 5.16 X is in \mathcal{D}_k if and only if X is a module over a generalized Moore spectrum of type k . Moreover, $\mathcal{D}_k \subseteq \mathrm{Sp}_{k,n}^{\mathrm{dual}}$ is a thick tensor ideal.

Hovey and Strickland conjecture that in the case $k = n$ these exhaust the thick-tensor ideals of $\mathrm{Sp}_{K(n)}^{\mathrm{dual}}$. This has been investigated in detail in [\[10\]](#). We conjecture this holds more generally in $\mathrm{Sp}_{k,n}$.

Conjecture 5.17 If \mathcal{C} is a thick tensor-ideal of $\mathrm{Sp}_{k,n}^{\mathrm{dual}}$, then $\mathcal{C} = \mathcal{D}_i$ for some $0 \leq i \leq n + 1$. Equivalently,

$$\mathrm{Spc}(\mathrm{Sp}_{k,n}^{\mathrm{dual}}) = \{\mathcal{D}_1, \dots, \mathcal{D}_{n+1}\}$$

with topology determined by the closure operator $\overline{\{\mathcal{D}_i\}} = \{\mathcal{D}_j \mid j \geq i\}$.

In this section we show that if [Conjecture 5.17](#) holds $K(n)$ -locally, ie for $\mathrm{Sp}_{n,n}^{\mathrm{dual}}$, then it holds for all $\mathrm{Sp}_{k,n}^{\mathrm{dual}}$. We first recall the following definition.

Definition 5.18 Suppose $F: \mathcal{K} \rightarrow \mathcal{L}$ is an exact tensor triangulated functor between tensor-triangulated categories. We say that F detects tensor-nilpotence of morphisms if every morphism $f: X \rightarrow Y$ in \mathcal{K} such that $F(f) = 0$ satisfies $f^{\otimes m} = 0$ for some $m \geq 1$.

We will use the following.

Proposition 5.19 Suppose $A \in \text{CAlg}(\mathcal{C})$ is descendable. Then extension of scalars $\mathcal{C} \rightarrow \text{Mod}_A(\mathcal{C})$ detects tensor-nilpotence of morphisms.

Proof Let I denote the fiber of $\mathbf{1} \xrightarrow{\eta} A$, and let $\xi: I \rightarrow \mathbf{1}$ denote the induced map. By [43, Proposition 4.7] if A is descendable, then there exists $m \geq 1$ such that $I^{\otimes m} \rightarrow \mathbf{1}$ is null-homotopic, ie ξ is tensor-nilpotent. We can now argue as in (ii) implies (iii) of [5]: suppose we are given $f: X \rightarrow Y$, a morphism in \mathcal{C} , with $A \otimes f: A \otimes X \rightarrow A \otimes Y$ null-homotopic. Now consider the diagram of fiber sequences:

$$\begin{array}{ccccc}
 I \otimes X & \xrightarrow{\xi \otimes \text{id}_X} & X & \xrightarrow{\eta \otimes \text{id}_X} & A \otimes X \\
 \text{id}_I \otimes f \downarrow & & \downarrow f & & \downarrow \text{id}_A \otimes f \\
 I \otimes Y & \xrightarrow{\xi \otimes \text{id}_Y} & Y & \xrightarrow{\eta \otimes \text{id}_Y} & A \otimes Y
 \end{array}$$

We see that $(\eta \otimes \text{id}_Y) f$ is null-homotopic, so f factors through $\xi \otimes \text{id}_Y$, which is tensor-nilpotent. □

The following is our key observation.

Proposition 5.20 If $i > k$, then the map induced by localization

$$\text{Spc}(\text{Sp}_{i,n}^{\text{dual}}) \rightarrow \text{Spc}(\text{Sp}_{k,n}^{\text{dual}})$$

is surjective.

Proof By [6, Theorem 1.3] it suffices to show that the functor $L_{i,n}: \text{Sp}_{k,n}^{\text{dual}} \rightarrow \text{Sp}_{i,n}^{\text{dual}}$ detects tensor-nilpotence of morphisms. To that end, let $f: X \rightarrow Y$ be a morphism in $\text{Sp}_{k,n}^{\text{dual}}$ with $L_{i,n}(f) = 0$, so that we must show $f^{\wedge m} = 0$ for some $m \geq 1$. Because $E_n \in \text{Sp}_{k,n}$ is descendable, Proposition 5.19 shows that

$$L_{k,n}(E_n \wedge -): \text{Sp}_{k,n} \rightarrow \text{Mod}_{E_n}(\text{Sp}_{k,n})$$

detects tensor-nilpotence of morphisms, and hence so does its restriction to dualizable objects; ie $L_{k,n}(E_n \wedge f) = 0$ implies $f^{\wedge m} = 0$ for some $m \geq 1$. In other words,

it suffices to show that $L_{k,n}(E_n \wedge f)$ is trivial. By Lemma 5.7 however, this is a morphism in $\mathrm{Mod}_{E_n}(\mathrm{Sp}_{n,n})$. In particular,

$$L_{k,n}(E_n \wedge f) \simeq L_{i,n}(E_n \wedge f) \simeq L_{i,n}(E_n \wedge L_{i,n}(f)) = 0$$

because $L_{i,n}(f) = 0$ by assumption. □

Theorem 5.21 Suppose Conjecture 5.17 holds for $\mathrm{Sp}_{n,n}^{\mathrm{dual}}$. Then it holds for all $\mathrm{Sp}_{k,n}^{\mathrm{dual}}$.

Proof By [10, Proposition 3.5], Conjecture 5.17 holds for $\mathrm{Sp}_{n,n}^{\mathrm{dual}}$ if and only if $L_{n,n}: \mathrm{Sp}_{0,n}^{\mathrm{dual}} \rightarrow \mathrm{Sp}_{n,n}^{\mathrm{dual}}$ induces a homeomorphism on Balmer spectra. In other words, the composite, induced by the localization maps,

$$\mathrm{Spc}(\mathrm{Sp}_{n,n}^{\mathrm{dual}}) \rightarrow \mathrm{Spc}(\mathrm{Sp}_{n-1,n}^{\mathrm{dual}}) \rightarrow \cdots \rightarrow \mathrm{Spc}(\mathrm{Sp}_{0,n}^{\mathrm{dual}})$$

is a homeomorphism. It follows that $\mathrm{Spc}(L_{n-1,n}): \mathrm{Spc}(\mathrm{Sp}_{n,n}^{\mathrm{dual}}) \rightarrow \mathrm{Spc}(\mathrm{Sp}_{n-1,n}^{\mathrm{dual}})$ is an injection, and hence a bijection by Proposition 5.20. Using that $\mathrm{Spc}(L_{n-1,n})$ is continuous and the topologies on each space, we see that it is fact a homeomorphism. It follows that $\mathrm{Spc}(\mathrm{Sp}_{n-1,n}^{\mathrm{dual}}) \rightarrow \mathrm{Spc}(\mathrm{Sp}_{0,n}^{\mathrm{dual}})$ is a homeomorphism, and we can now repeat the argument. □

By [10, Theorem 4.15], Conjecture 5.17 holds for $\mathrm{Sp}_{2,2}^{\mathrm{dual}}$. Along with Theorem 5.21 we deduce the following.

Corollary 5.22 The Balmer spectrum $\mathrm{Spc}(\mathrm{Sp}_{1,2}^{\mathrm{dual}}) = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ with

$$(0) = \mathcal{D}_3 \subsetneq \mathcal{D}_2 \subsetneq \mathcal{D}_1 = \mathrm{Sp}_{1,2}^{\omega}.$$

In particular, if \mathcal{C} is a thick tensor-ideal of $\mathrm{Sp}_{1,2}^{\mathrm{dual}}$, then $\mathcal{C} = \mathcal{D}_k$ for $0 \leq k \leq 3$.

6 The Picard group of the $\mathrm{Sp}_{k,n}$ -local category

In this section we study invertible objects in the $\mathrm{Sp}_{k,n}$ -local category. We show that invertibility of an object can be detected by its Morava module. We construct a spectral sequence computing the homotopy groups of the Picard spectrum of $\mathrm{Sp}_{k,n}$ and use this to show that if p is large compared to n , then the Picard group of $\mathrm{Sp}_{k,n}$ is entirely algebraic, in a sense we make precise.

6A Invertible objects and Picard spectra

We recall that if \mathcal{C} is a symmetric monoidal category, we denote by $\mathrm{Pic}(\mathcal{C})$ the group of isomorphism classes of invertible objects; a priori this could be a proper class, but if

\mathcal{C} is a presentable stable ∞ -category (which it will always be in our cases), then it is actually a set [44, Remark 2.1.4].

The following standard lemma will be useful for us. Here we write $D_{\mathcal{C}}(X)$ for the dual of an object in a category \mathcal{C} , ie $D_{\mathcal{C}}(X) = F(X, \mathbf{1})$. Note that an invertible object is always dualizable [30, Proposition A.2.8].

Lemma 6.1 *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a symmetric-monoidal conservative functor between stable ∞ -categories. Then $X \in \mathcal{C}$ is invertible if and only if $F(X) \in \mathcal{D}$ is invertible.*

Proof We first note that X is invertible if and only if the natural morphism

$$X \otimes_{\mathcal{C}} D_{\mathcal{C}}(X) \rightarrow \mathbf{1}_{\mathcal{C}}$$

is an equivalence; see [30, Proposition A.2.8]. Because F is assumed to be symmetric monoidal and conservative, this is an equivalence if and only if it is so after applying F ; ie if and only if $F(X) \otimes_{\mathcal{D}} F(D_{\mathcal{C}}(X)) \rightarrow \mathbf{1}_{\mathcal{D}}$ is an equivalence. But $F(D_{\mathcal{C}}(X)) \simeq D_{\mathcal{D}}(F(X))$, as symmetric-monoidal functors preserve dualizable objects, and the result follows. □

Remark 6.2 To our symmetric monoidal category \mathcal{C} we can instead associate the Picard spectrum $\text{pic}(\mathcal{C})$ [44, Definition 2.2.1]; this is a connective spectrum with the property that

$$\pi_i(\text{pic}(\mathcal{C})) = \begin{cases} \text{Pic}(\mathcal{C}) & \text{if } i = 0, \\ (\pi_0(\text{End}_{\mathcal{C}}(\mathbf{1}))^\times & \text{if } i = 1, \\ \pi_{i-1}(\text{End}_{\mathcal{C}}(\mathbf{1})) & \text{if } i > 1. \end{cases}$$

The key advantage of using the Picard spectrum is that, as a functor from the ∞ -category of symmetric monoidal ∞ -categories to the ∞ -category of connective spectra, pic commutes with limits [44, Proposition 2.2.3].

Example 6.3 Let \mathcal{C} be a category and $A \in \text{CAlg}(\mathcal{C})$. Then the Picard spectrum of the category $\text{Mod}_A(\mathcal{C})$ of A -modules internal to \mathcal{C} satisfies

$$\pi_i(\text{pic}(\text{Mod}_A(\mathcal{C}))) = \begin{cases} \text{Pic}(\text{Mod}_A(\mathcal{C})) & \text{if } i = 0, \\ (\pi_0(\text{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, A)))^\times & \text{if } i = 1, \\ \pi_{i-1}(\text{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, A)) & \text{if } i > 1. \end{cases}$$

This follows because A is the tensor unit in $\text{Mod}_A(\mathcal{C})$. Indeed, writing $F : \mathcal{C} \rightarrow \text{Mod}_{\mathcal{C}}(A)$ for the extension of scalars functor (so that $A \simeq F(\mathbf{1}_{\mathcal{C}})$), we have

$$\text{End}_{\text{Mod}_A(\mathcal{C})}(A) = \text{Hom}_{\text{Mod}_A(\mathcal{C})}(F(\mathbf{1}_{\mathcal{C}}), A) \simeq \text{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, A)$$

by adjunction.

6B Invertible objects in the $\mathrm{Sp}_{k,n}$ -local category

Our main group of interest is the Picard group of $K(k) \vee \cdots \vee K(n)$ -local spectra.

Definition 6.4 Let $\mathrm{Pic}_{k,n} = \mathrm{Pic}(\mathrm{Sp}_{k,n})$, the group of invertible $K(k) \vee \cdots \vee K(n)$ -local spectra.

Remark 6.5 By [36, Lemma 2.2] the localization functors induce natural morphisms $\mathrm{Pic}_{0,n} \rightarrow \mathrm{Pic}_{1,n} \rightarrow \cdots \rightarrow \mathrm{Pic}_{n,n}$.

Remark 6.6 The morphism $X \mapsto E_n \bar{\wedge} X$ induces a functor

$$\mathrm{Pic}_{k,n} \rightarrow \mathrm{Pic}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})).$$

We can fully understand the latter Picard group.

Lemma 6.7 For all $0 \leq k \leq n$,

$$\mathrm{Pic}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})) \cong \mathrm{Pic}(\mathrm{Mod}_{E_n}) \cong \mathrm{Pic}(E_{n*}) \cong \mathbb{Z}/2.$$

Proof We always have $\mathrm{Pic}(\mathrm{Mod}_{E_n}) \subseteq \mathrm{Pic}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n}))$ because any invertible E_n -module is compact, and hence $E(n, J_k)$ -local. The other inclusion follows if any $M \in \mathrm{Pic}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n}))$ is compact as an E_n -module. Such an M is automatically dualizable in $\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})$, and hence compact in Mod_{E_n} by Proposition 5.9. This gives the first of the above isomorphisms, and the others hold by work of Baker and Richter [2]. □

We now give criteria for when $X \in \mathrm{Pic}_{k,n}$ is invertible. This (partially) extends work of Hopkins, Mahowald and Sadofsky [26], who considered the case $k = n$.

Theorem 6.8 Let $X \in \mathrm{Sp}_{k,n}$. The following are equivalent:

- (1) $X \in \mathrm{Pic}_{k,n}$.
- (2) $E_n \bar{\wedge} X \in \mathrm{Pic}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n}))$.
- (3) $E_n \bar{\wedge} X \in \mathrm{Pic}(\mathrm{Mod}_{E_n})$.
- (4) $(E_{k,n})_*^\vee X \cong (E_n)_*$, up to suspension.

Proof The equivalence of (1) and (2) follows from Corollary 4.7 and Lemma 6.1, while the equivalence of (2) and (3) follows from Lemma 6.7, which also shows that

(3) implies (4). Finally, to see that (4) implies (3), we note that if M is any E_n -module whose homotopy groups are free of rank one over $(E_n)_*$, then M is equivalent to a suspension of E_n ; for the elementary proof, see [23, Proposition 2.2]. Thus, (4) implies that $E_n \bar{\wedge} X \simeq E_n$, up to suspension, and hence (3) holds. \square

Remark 6.9 When $n = 1$, there are two possibilities, the $K(1)$ and $E(1)$ -local Picard groups, both of which are known:

$$\text{Pic}_{0,1} = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } p = 2, \\ \mathbb{Z} & \text{if } p > 2, \end{cases} \quad \text{Pic}_{1,1} = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \text{if } p = 2, \\ \mathbb{Z}_p \oplus \mathbb{Z}/(p-1) \oplus \mathbb{Z}/2 & \text{if } p > 2. \end{cases}$$

These are due to Hovey and Sadofsky [31] and Hopkins, Mahowald and Sadofsky [26], respectively.

When $n = 2$, we have three possibilities, the $K(2)$, $K(1) \vee K(2)$, and $E(2)$ -local Picard groups. The first and last are known for $p > 2$:

$$\text{Pic}_{0,2} = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 & \text{if } p = 3, \\ \mathbb{Z} & \text{if } p > 3, \end{cases}$$

$$\text{Pic}_{2,2} = \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 & \text{if } p = 3, \\ \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}/(2(p^2 - 1)) & \text{if } p > 3. \end{cases}$$

These are due to a combination of authors: Hovey and Sadofsky [31], Lader [38], Goerss, Henn, Mahowald and Rezk [20], Karamanov [37], and Hopkins (unpublished).

This leaves the remaining case of $\text{Pic}_{1,1}$. We note the following.

Proposition 6.10 *If $p \geq 3$, then the morphism $\text{Pic}_{0,2} \rightarrow \text{Pic}_{1,1}$ of Remark 6.5 is an injection.*

Proof The morphism in question factors through the morphism $\text{Pic}_{0,2} \rightarrow \text{Pic}_{2,2}$ and so it suffices to show that this is an injection. When $p > 3$ this is clear, and so we focus on the case $p = 3$. In this case, the calculations of Goerss, Henn, Mahowald and Rezk [20] show that this map is an injection. \square

Remark 6.11 As noted in the proof, the interesting case in the above proposition is the case $p = 3$. In fact, for all n and $p \gg n$ we have that $\text{Pic}_{0,n} \rightarrow \text{Pic}_{i,n}$ is an injection for $i \geq 0$. However, here $\text{Pic}_{0,n} \cong \mathbb{Z}$ (by [31]), so this is not particularly helpful.

6C Descent and Picard groups

In Remark 6.2 we recalled that we can associate a connective Picard spectrum $\text{pic}(\mathcal{C})$ to a symmetric-monoidal ∞ -category \mathcal{C} . Using descent for the $E(n, J_k)$ -local category,

we now construct a spectral sequence whose π_0 computes $\mathrm{Pic}_{k,n}$. We need to introduce another algebraic gadget to describe the spectral sequence.

Definition 6.12 We let $\mathrm{Pic}_{k,n}^{\mathrm{alg}}$ denote the first cohomology of the complex

$$(E_n)_0^\times \rightrightarrows ((E_{k,n})_0^\vee(E_n))^\times \rightrightarrows \cdots$$

induced by taking the units in degree 0 in the cobar complex.

Theorem 6.13 *There exists a spectral sequence with*

$$E_2^{s,t} \cong \begin{cases} \mathbb{Z}/2 & \text{if } s = t = 0, \\ \mathrm{Pic}_{k,n}^{\mathrm{alg}} & \text{if } s = t = 1, \\ \widehat{\mathrm{Ext}}_{(E_{k,n})^\vee(E_n)}^{s,t-1}((E_n)_*, (E_n)_*) & t \geq 2, \end{cases}$$

which converges for $t-s \geq 0$ to $\pi_{t-s}\mathrm{pic}(\mathrm{Sp}_{k,n})$. In particular, when $t = s$, this computes $\mathrm{Pic}_{k,n}$. The differentials in the spectral sequence run $d_r : E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$.

Proof Because pic commutes with limits (Remark 6.2), Theorem 4.6 implies that

$$(6-1) \quad \mathrm{pic}(\mathrm{Sp}_{k,n}) \simeq \tau_{\geq 0} \mathrm{Tot} \left(\mathrm{pic}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})) \rightrightarrows \mathrm{pic}(\mathrm{Mod}_{E_n \bar{\wedge} E_n}(\mathrm{Sp}_{k,n})) \rightrightarrows \right).$$

We have (compare Example 6.3)

$$(6-2) \quad \pi_t(\mathrm{pic}(\mathrm{Mod}_{E_n \bar{\wedge} i}(\mathrm{Sp}_{k,n}))) \cong \begin{cases} \mathrm{Pic}(\mathrm{Mod}_{E_n \bar{\wedge} i}) & \text{if } t = 0, \\ \pi_0(E_n \bar{\wedge} i)^\times & \text{if } t = 1, \\ \pi_{t-1}(E_n \bar{\wedge} i) & \text{if } t \geq 2. \end{cases}$$

The Bousfield–Kan spectral sequence associated to (6-1) has the form

$$E_2^{s,t} \cong H^s \left(\pi_t \mathrm{pic}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})) \rightrightarrows \pi_t \mathrm{pic}(\mathrm{Mod}_{E_n \bar{\wedge} E_n}(\mathrm{Sp}_{k,n})) \rightrightarrows \right).$$

By (6-2) when $t \geq 2$, the spectral sequence is just a shift of the $E(n, J_k)$ -local Adams spectral for $X = S^0$ sequence considered in Proposition 4.13.

When $t = 0$ and $i = 0$, we have $\mathrm{Pic}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{k,n})) \cong \mathbb{Z}/2$ by Lemma 6.7. We do not know the higher terms, but this does not matter as only the $\mathbb{Z}/2$ is relevant for the $s = t = 0$ part of the spectral sequence.

Finally, we consider the $t = 1$ part of the spectral sequence. Again using (6-2),

$$E_2^{s,1} \cong H^s \left((E_n)_0^\times \rightrightarrows ((E_{k,n})_0^\vee(E_n))^\times \rightrightarrows \cdots \right).$$

By definition, when $s = 1$ this is $\mathrm{Pic}_{k,n}^{\mathrm{alg}}$. □

Remark 6.14 The proof shows that when $t = 1$, we can compute $E_2^{s,1}$ as the s^{th} cohomology of the complex in [Definition 6.12](#). However, unless $k = 0$ or n we do not have a convenient description of this group (for the case $k = n$, see [Example 6.18](#) below).

Definition 6.15 We will say that $\text{Pic}_{k,n}$ is algebraic if the only contributions come from the $s = 0$ and $s = 1$ lines of the spectral sequence.

Remark 6.16 The $E_2^{0,0}$ term of the spectral sequence always survives the spectral sequence, as it is the Picard group of E_n -modules. It is however possible that there is a nontrivial differential in the $E_r^{1,1}$ spot.

Theorem 6.17 Suppose that $2p - 2 \geq n^2 + n - k$ and $p - 1$ does not divide $k + s$ for $0 \leq s \leq n - k$. Then $\text{Pic}_{k,n}$ is algebraic. For example, this holds if $2p - 2 > n^2 + n$.

Proof For all primes p and $t \geq 2$ we have that $E_2^{s,t} = 0$ unless $t - 1 \equiv 0 \pmod{2(p - 1)}$ by [Proposition 4.26](#). In particular, for $s > 2$, $E_2^{s,s} = 0$ unless $s \equiv 1 \pmod{2p - 2}$, and the lowest possible nonalgebraic term is in filtration degree $2p - 1$.

By [Theorem 4.24](#) and the assumption that $p - 1$ does not divide $k + s$ for $0 \leq s \leq n - k$ we have that $E_2^{s,s} = 0$ for $s > n^2 + n - k$. Therefore, if additionally $2p - 2 \geq n^2 + n - k$, there can be no nonalgebraic contributions to the spectral sequence.

Finally, if $2p - 2 > n^2 + n$, then $p > n + 1$, and in particular $p - 1$ does not divide $k + s$ for $0 \leq s \leq n - k$. □

Example 6.18 Let us spell out the details in the case $k = n$. We first claim that the spectral sequence of [Theorem 6.13](#) takes the form

$$E_2^{s,t} \cong \begin{cases} \mathbb{Z}/2 & \text{if } s = t = 0, \\ H_c^s(\mathbb{G}_n, (E_n)_0^\times) & \text{if } t = 1, \\ H_c^s(\mathbb{G}_n, \pi_{t-1} E_n) & \text{if } t \geq 2, \end{cases}$$

and converges for $t - s \geq 0$ to $\pi_{t-s} \text{pic}(\text{Sp}_{K(n)})$.

The identification is much as in [Remark 4.14](#). For the $t = 1$ term, we note that $\pi_0(E_n^{\bar{i}})^\times \cong \text{Hom}^c(\mathbb{G}_n^{\times(i-1)}, (E_n)_0^\times) \cong \text{Hom}^c(\mathbb{G}_n^{\times(i-1)}, (E_n)_0^\times)$.

The existence of such a spectral sequence is folklore; see [[18](#), [Remark 6.10](#)] or [[45](#), [Remark 2.6](#)]. In fact, the latter also proves [Theorem 6.17](#) in the case $k = n$.

7 $E(n, J_k)$ -local Brown–Comenetz duality

We recall the classical definition of Brown–Comenetz duality. The group \mathbb{Q}/\mathbb{Z} is an injective abelian group, and so the functor

$$X \mapsto \mathrm{Hom}(\pi_0 X, \mathbb{Q}/\mathbb{Z})$$

defines a cohomology theory on spectra represented by a spectrum $I_{\mathbb{Q}/\mathbb{Z}}$; this is the Brown–Comenetz dual of the sphere. The Brown–Comenetz dual of a spectrum X is then defined as $I_{\mathbb{Q}/\mathbb{Z}}X := F(X, I_{\mathbb{Q}/\mathbb{Z}})$, and satisfies

$$[Y, I_{\mathbb{Q}/\mathbb{Z}}X]_0 \cong \mathrm{Hom}(\pi_0(X \wedge Y), \mathbb{Q}/\mathbb{Z}).$$

It is an insight of Hopkins [24] that there is a good notion of Brown–Comenetz duality (also known as Gross–Hopkins duality) internal to the $K(n)$ -local category, given by defining $I_n X = F(M_n X, I_{\mathbb{Q}/\mathbb{Z}})$ for a $K(n)$ -local spectrum X . For details on this, see [52]. As we will see, this definition can also naturally be made in the $E(n, J_k)$ -local category. We begin with the following generalization of a result of Stojanoska [50, Proposition 2.2]. We recall that, by definition, $M_{0,n} = L_n$. In this case, the following lemma just says that $F(L_n X, Y)$ is already L_n -local.

Proposition 7.1 *For any X and Y , the natural map $F(L_n X, Y) \rightarrow F(M_{k,n} X, Y)$ is an $E(n, J_k)$ -localization.*

Proof We can repeat Stojanoska’s argument. First, we show that $F(M_{k,n} X, Y)$ is $E(n, J_k)$ -local. Indeed, let Z be $E(n, J_k)$ -acyclic. Then we must show that

$$F(Z, F(M_{k,n} X, Y)) \simeq F(Z \wedge M_{k,n} X, Y) \simeq F(M_{k,n} Z \wedge X, Y)$$

is contractible. Here we have used that $M_{k,n}$ is smashing. But $M_{k,n} Z \simeq M_{k,n} L_{k,n} Z$ by Theorem 2.31 and this is contractible because Z is $E(n, J_k)$ -acyclic.

We now show that the fiber $F(L_{k-1} X, Y)$ is $E(n, J_k)$ -acyclic. By Proposition 2.15 it suffices by a localizing subcategory argument to show this after smashing with a generalized Moore spectrum $M(k)$ of type k . Then (up to suspension),

$$\begin{aligned} F(L_{k-1} X, Y) \wedge M(k) &\simeq F(L_{k-1} X, Y) \wedge DM(k) \\ &\simeq F(M(k), F(L_{k-1} X, Y)) \\ &\simeq F((L_{k-1} M(k)) \wedge X, Y) \simeq *. \end{aligned}$$

□

Definition 7.2 The $E(n, J_k)$ -local Brown–Comenetz dual of X is

$$I_{k,n}X = I_{\mathbb{Q}/\mathbb{Z}}(M_{k,n}X) = F(M_{k,n}X, I_{\mathbb{Q}/\mathbb{Z}}).$$

We let $I_{k,n}$ denote the $E(n, J_k)$ -local Brown–Comenetz dual of $L_{k,n}S^0$.

Remark 7.3 It does not matter if we ask that X be $E(n, J_k)$ -local in the previous definition, as $I_{k,n}X$ only depends on the $E(n, J_k)$ -localization of X . Indeed, we have equivalences

$$I_{k,n}X = F(M_{k,n}X, I_{\mathbb{Q}/\mathbb{Z}}) \simeq F(M_{k,n}L_{k,n}X, I_{\mathbb{Q}/\mathbb{Z}}) = I_{k,n}(L_{k,n}X).$$

In particular, $I_{k,n} = I_{k,n}(L_{k,n}S^0) \simeq I_{k,n}S^0$.

From the definition of $I_{\mathbb{Q}/\mathbb{Z}}$, we deduce the following.

Lemma 7.4 *There is a natural isomorphism*

$$[Y, I_{k,n}X]_0 \cong \text{Hom}(\pi_0(M_{k,n}(X) \wedge Y), \mathbb{Q}/\mathbb{Z}).$$

As a consequence of [Proposition 7.1](#) we deduce the following.

Lemma 7.5 $I_{k,n}X$ is always $E(n, J_k)$ -local. In fact, $I_{k,n}X \simeq L_{k,n}I_{\mathbb{Q}/\mathbb{Z}}(L_nX)$ and moreover, $I_{k,n}X \cong L_{k,n}I_{j,n}X$ for any $j \leq k$.

It follows that we have natural maps given by localization,

$$I_{0,n} \rightarrow I_{1,n} \rightarrow \cdots \rightarrow I_{n,n}.$$

Example 7.6 Let $n = 1$ and $p > 2$. Then $I_{0,1} \simeq L_1(S_p^2)$, the localization of the p -completion of S^2 . On the other hand, when $p = 2$ we have $I_{0,1} \simeq \Sigma^2 L_1(DQ_2^\wedge)$ where DQ is the dual question mark complex [[15](#), Remark 1.5]. Similarly, $I_{1,1} \simeq L_{K(1)}S^2$ if $p > 2$, while $I_{1,1} \simeq \Sigma^2 L_{K(1)}DQ$.

Example 7.7 We always have $I_{k,n}(K_n) \simeq K_n$. Indeed, first note that $L_{k,n}K_n \simeq K_n$, so [Lemma 7.5](#) allows us to reduce the case where $k = 0$ (although the proof is no more difficult in the other cases — just note that $M_{k,n}K_n \simeq K_n$). Using the fact that

$$[Y, K_n]_* \cong \text{Hom}_{(K_n)_*}((K_n)_*X, (K_n)_*),$$

we argue as in [[32](#), Theorem 10.2(a)] to see that

$$[Y, K_n]_0 \cong \text{Hom}((K_n)_0X, \mathbb{F}_p) \cong \text{Hom}((K_n)_0X, \mathbb{Q}/\mathbb{Z}) \cong [Y, I_{0,n}(K_n)]_0.$$

This implies that $I_{0,n}(K_n) \simeq K_n$, as claimed.

Theorem 7.8 *Let $X \in \mathrm{Sp}_{k,n}$. Then the natural map $X \rightarrow I_{k,n}^2 X$ is an isomorphism when $\pi_*(F(k) \wedge X) \cong \pi_*(L_{k,n} F(k) \wedge X)$ is finite in each degree. In particular, this holds for $X = L_{k,n} S^0$.*

Proof Let $\kappa_X : X \rightarrow I_{k,n}^2 X$ denote the natural map. We first note that

$$I_{k,n}^2(F(k) \bar{\wedge} X) \simeq I_{k,n}^2(F(k) \wedge X) \simeq F(k) \wedge I_{k,n}^2(X),$$

because $F(k)$ is compact (and hence dualizable) in Sp . As in the proof of Theorem 10.2 in [32], this identifies $\kappa_{F(k) \wedge X} \simeq \mathrm{id}_F \wedge \kappa_X$, and so it is enough to show that κ_Y is an equivalence, where $Y = F(k) \wedge X$.

Because $F(k)$ has type k , $L_{k-1} F(k) \simeq *$, and $M_{k,n} F(k) \simeq L_n F(k)$, so

$$M_{k,n} Y = M_{k,n}(F(k) \wedge X) \simeq M_{k,n} F(k) \wedge X \simeq L_n F(k) \wedge X \simeq Y.$$

Likewise, $M_{k,n}(I_{k,n} Y) \simeq M_{k,n}(DF(k) \wedge I_{k,n} X) \simeq DF(k) \wedge I_{k,n} X \simeq I_{k,n} Y$. This implies that $\pi_* I_{k,n}^2 Y \cong \mathrm{Hom}(\mathrm{Hom}(\pi_* Y, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, which is the same as $\pi_* Y$ because $\pi_* Y$ is finite in each degree. Therefore κ_Y is an equivalence, as required. \square

Remark 7.9 The Gross–Hopkins dual $I_{n,n}$ is always an invertible $K(n)$ -local spectrum. We do not know what happens for $I_{k,n}$ in general; however we note the following result.

Proposition 7.10 *The following are equivalent:*

- (1) $I_{k,n} \in \mathrm{Sp}_{k,n}^{\mathrm{dual}}$,
- (2) $I_{k,n} \in \mathrm{Pic}_{k,n}$;
- (3) $(E_{k,n})_*^\vee(I_{k,n})$ is a finitely generated E_* -module;
- (4) $E_n \bar{\wedge} I_{k,n}$ is $K(n)$ -local.

Proof Suppose first that (1) holds. Then, $F(I_{k,n}, I_{k,n}) \simeq DI_{k,n} \bar{\wedge} I_{k,n}$, but on the other hand $F(I_{k,n}, I_{k,n}) \simeq I_{k,n}^2(L_{k,n} S^0) \simeq L_{k,n} S^0$ by Theorem 7.8. It follows that $I_{k,n} \in \mathrm{Pic}_{k,n}$, ie that (2) holds. The converse, (2) \implies (1), always holds; see for example [30, Proposition A.2.8].

The equivalence of (1) and (3) is just Theorem 5.11.

Finally to see that (1) \iff (4), we note that it suffices to show that $(K_n)_* I_{k,n}$ is finite. In fact, because $(K_n)_*$ is a graded field, it suffices to see that $(K_n)^* I_{k,n}$ is finite. For this, we compute, using [Example 7.7](#) and [Theorem 7.8](#),

$$\begin{aligned} [I_{k,n}, K_n]_* &\simeq [I_{k,n}, I_{k,n}(K_n)]_* \\ &\simeq [I_{k,n}, F(K_n, I_{k,n})]_* \\ &\simeq [K_n, F(I_{k,n}, I_{k,n})]_* \\ &\simeq [K_n, L_{k,n} S^0]_* . \end{aligned}$$

By [\[32, Lemma 10.4\]](#) if M_n denotes a generalized Moore spectrum of type n , then $[E_n, L_{K(n)} M_n]_* \simeq [E_n, L_{k,n} M_n]_*$ is finite (the last equivalence follows, for example, from the fact that $L_n M_n \simeq L_{K(n)} M_n$ for a generalized Moore spectrum of type n). As in [\[32, Corollary 10.5\]](#) it follows that $[E_n \bar{\wedge} DM_n, L_{k,n} S^0]$ is finite, and hence so is $[K_n, L_{k,n} S^0]$, as K_n lies in the thick subcategory generalized by $E_n \bar{\wedge} DM_n$ (note that DM_n is also the localization of a generalized Moore spectrum of type n ; see [\[32, Proposition 4.18\]](#)). □

Question 7.11 *For which values of k and n do the conditions of [Proposition 7.10](#) hold?*

Remark 7.12 Condition (4) clearly holds in the case $k = n$. Of course, [Proposition 7.10](#) is precisely Hovey and Strickland’s proof in this case. However, due to the p -completion, this does not hold for $n = 1$ and $k = 0$ ([Example 7.6](#)). In fact, this fails at all heights when $k = 0$, as we now explain.

Remark 7.13 Fix $n > 1$ and $k = 0$, and take $p \gg n$. Then $\text{Pic}_{0,n} \cong \mathbb{Z}$, generated by $L_n S^1$ [\[31, Theorem 5.4\]](#). Therefore, if [Proposition 7.10](#) held for $k = 0$, we must have $I_{0,n} \simeq L_n S^k$ for some $k \in \mathbb{Z}$. On the other hand, work of Hopkins and Gross [\[24\]](#), as written up by Strickland [\[52\]](#), and known results about the $K(n)$ -local Picard group [\[26, Proposition 7.5\]](#) show that $I_{n,n} \simeq \Sigma^{n^2-n} S \langle \det \rangle$, where $S \langle \det \rangle$ is the determinant sphere spectrum [\[7\]](#). It cannot then be the case that $L_{K(n)} I_{0,n} \simeq I_{n,n}$; a contradiction to [Lemma 7.5](#). We do not know what occurs in the cases $k \neq 0, n$.

References

[1] **O Antolín-Camarena, T Barthel**, *Chromatic fracture cubes*, from “Equivariant topology and derived algebra” (S Balchin, D Barnes, M Kędziorek, M Szymik, editors), London Math. Soc. Lecture Note Ser. 474, Cambridge Univ. Press (2022) 100–118 [MR](#) [Zbl](#)

- [2] **A Baker, B Richter**, *Invertible modules for commutative \mathbb{S} -algebras with residue fields*, Manuscripta Math. 118 (2005) 99–119 [MR](#) [Zbl](#)
- [3] **P Balmer**, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. 588 (2005) 149–168 [MR](#) [Zbl](#)
- [4] **P Balmer**, *Spectra, spectra, spectra: tensor triangular spectra versus Zariski spectra of endomorphism rings*, Algebr. Geom. Topol. 10 (2010) 1521–1563 [MR](#) [Zbl](#)
- [5] **P Balmer**, *Separable extensions in tensor-triangular geometry and generalized Quillen stratification*, Ann. Sci. Éc. Norm. Supér. 49 (2016) 907–925 [MR](#) [Zbl](#)
- [6] **P Balmer**, *On the surjectivity of the map of spectra associated to a tensor-triangulated functor*, Bull. Lond. Math. Soc. 50 (2018) 487–495 [MR](#) [Zbl](#)
- [7] **T Barthel, A Beaudry, P G Goerss, V Stojanoska**, *Constructing the determinant sphere using a Tate twist*, Math. Z. 301 (2022) 255–274 [MR](#) [Zbl](#)
- [8] **T Barthel, N Castellana, D Heard, G Valenzuela**, *Stratification and duality for homotopical groups*, Adv. Math. 354 (2019) art. id. 106733 [MR](#) [Zbl](#)
- [9] **T Barthel, D Heard**, *The E_2 -term of the $K(n)$ -local E_n -Adams spectral sequence*, Topology Appl. 206 (2016) 190–214 [MR](#) [Zbl](#)
- [10] **T Barthel, D Heard, N Naumann**, *On conjectures of Hovey–Strickland and Chai*, Selecta Math. 28 (2022) art. no. 56 [MR](#) [Zbl](#)
- [11] **T Barthel, D Heard, G Valenzuela**, *Local duality in algebra and topology*, Adv. Math. 335 (2018) 563–663 [MR](#) [Zbl](#)
- [12] **T Barthel, N Stapleton**, *Centralizers in good groups are good*, Algebr. Geom. Topol. 16 (2016) 1453–1472 [MR](#) [Zbl](#)
- [13] **T Bauer**, *Bousfield localization and the Hasse square*, from “Topological modular forms” (CL Douglas, J Francis, A G Henriques, MA Hill, editors), Mathematical Surveys and Monographs 201, Amer. Math. Soc., Providence, RI (2014) 79–88 [MR](#) [Zbl](#)
- [14] **D J Benson, S B Iyengar, H Krause**, *Colocalizing subcategories and cosupport*, J. Reine Angew. Math. 673 (2012) 161–207 [MR](#) [Zbl](#)
- [15] **E S Devinatz**, *K -theory and the generating hypothesis*, Amer. J. Math. 112 (1990) 787–804 [MR](#) [Zbl](#)
- [16] **E S Devinatz, M J Hopkins**, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology 43 (2004) 1–47 [MR](#) [Zbl](#)
- [17] **A Dold, D Puppe**, *Duality, trace, and transfer*, from “Proceedings of the international conference on geometric topology” (K Borsuk, editor), PWN, Warsaw (1980) 81–102 [MR](#) [Zbl](#)
- [18] **W G Dwyer, J P C Greenlees, S B Iyengar**, *Gross–Hopkins duality and the Gorenstein condition*, J. K-Theory 8 (2011) 107–133 [MR](#) [Zbl](#)

- [19] **A D Elmendorf, I Kriz, M A Mandell, J P May**, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs 47, Amer. Math. Soc., Providence, RI (1997) [MR](#) [Zbl](#)
- [20] **P Goerss, H-W Henn, M Mahowald, C Rezk**, *On Hopkins' Picard groups for the prime 3 and chromatic level 2*, *J. Topol.* 8 (2015) 267–294 [MR](#) [Zbl](#)
- [21] **P G Goerss, M J Hopkins**, *Moduli spaces of commutative ring spectra*, from “Structured ring spectra” (A Baker, B Richter, editors), London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press (2004) 151–200 [MR](#) [Zbl](#)
- [22] **J Greenlees**, *Homotopy invariant commutative algebra over fields*, from “Building bridges between algebra and topology” (D Herbera, W Pitsch, S Zarzuela, editors), Springer (2018) 103–169 [MR](#) [Zbl](#)
- [23] **D Heard, V Stojanoska**, *K–theory, reality, and duality*, *J. K–Theory* 14 (2014) 526–555 [MR](#) [Zbl](#)
- [24] **M J Hopkins, B H Gross**, *The rigid analytic period mapping, Lubin–Tate space, and stable homotopy theory*, *Bull. Amer. Math. Soc.* 30 (1994) 76–86 [MR](#) [Zbl](#)
- [25] **M Hopkins, J Lurie**, *On Brauer groups of Lubin–Tate spectra, I*, preprint (2017) Available at <https://www.math.ias.edu/~lurie/papers/Brauer.pdf>
- [26] **M J Hopkins, M Mahowald, H Sadofsky**, *Constructions of elements in Picard groups*, from “Topology and representation theory” (E M Friedlander, M E Mahowald, editors), *Contemp. Math.* 158, Amer. Math. Soc., Providence, RI (1994) 89–126 [MR](#) [Zbl](#)
- [27] **M J Hopkins, J H Smith**, *Nilpotence and stable homotopy theory, II*, *Ann. of Math.* 148 (1998) 1–49 [MR](#) [Zbl](#)
- [28] **M Hovey**, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, from “The Čech centennial” (M Cenk, H Miller, editors), *Contemp. Math.* 181, Amer. Math. Soc., Providence, RI (1995) 225–250 [MR](#) [Zbl](#)
- [29] **M Hovey**, *Operations and co-operations in Morava E–theory*, *Homology Homotopy Appl.* 6 (2004) 201–236 [MR](#) [Zbl](#)
- [30] **M Hovey, J H Palmieri, N P Strickland**, *Axiomatic stable homotopy theory*, *Mem. Amer. Math. Soc.* 610, Amer. Math. Soc., Providence, RI (1997) [MR](#) [Zbl](#)
- [31] **M Hovey, H Sadofsky**, *Invertible spectra in the E(n)–local stable homotopy category*, *J. London Math. Soc.* 60 (1999) 284–302 [MR](#) [Zbl](#)
- [32] **M Hovey, N P Strickland**, *Morava K–theories and localisation*, *Mem. Amer. Math. Soc.* 666, Amer. Math. Soc., Providence, RI (1999) [MR](#) [Zbl](#)
- [33] **M Hovey, N Strickland**, *Comodules and Landweber exact homology theories*, *Adv. Math.* 192 (2005) 427–456 [MR](#) [Zbl](#)
- [34] **S B Iyengar, H Krause**, *The Bousfield lattice of a triangulated category and stratification*, *Math. Z.* 273 (2013) 1215–1241 [MR](#) [Zbl](#)

- [35] **DC Johnson, Z Yosimura**, *Torsion in Brown–Peterson homology and Hurewicz homomorphisms*, Osaka Math. J. 17 (1980) 117–136 [MR](#) [Zbl](#)
- [36] **Y Kamiya, K Shimomura**, *Picard groups of some local categories*, Publ. Res. Inst. Math. Sci. 43 (2007) 303–314 [MR](#) [Zbl](#)
- [37] **N Karamanov**, *On Hopkins’ Picard group Pic_2 at the prime 3*, Algebr. Geom. Topol. 10 (2010) 275–292 [MR](#) [Zbl](#)
- [38] **O Lader**, *Une résolution projective pour le second groupe de Morava pour $p \geq 5$ et applications*, PhD thesis, Université de Strasbourg (2013) Available at <https://tel.archives-ouvertes.fr/tel-00875761>
- [39] **J Lurie**, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton Univ. Press (2009) [MR](#) [Zbl](#)
- [40] **J Lurie**, *Higher algebra*, book project (2017) Available at <https://url.msp.org/Lurie-HA>
- [41] **A Mathew**, *The Galois group of a stable homotopy theory*, Adv. Math. 291 (2016) 403–541 [MR](#) [Zbl](#)
- [42] **A Mathew**, *Examples of descent up to nilpotence*, from “Geometric and topological aspects of the representation theory of finite groups” (JF Carlson, SB Iyengar, J Pevtsova, editors), Springer Proc. Math. Stat. 242, Springer (2018) 269–311 [MR](#) [Zbl](#)
- [43] **A Mathew, N Naumann, J Noel**, *Nilpotence and descent in equivariant stable homotopy theory*, Adv. Math. 305 (2017) 994–1084 [MR](#) [Zbl](#)
- [44] **A Mathew, V Stojanoska**, *The Picard group of topological modular forms via descent theory*, Geom. Topol. 20 (2016) 3133–3217 [MR](#) [Zbl](#)
- [45] **P Pstragowski**, *Chromatic Picard groups at large primes*, Proc. Amer. Math. Soc. 150 (2022) 4981–4988 [MR](#) [Zbl](#)
- [46] **DC Ravenel**, *Localization with respect to certain periodic homology theories*, Amer. J. Math. 106 (1984) 351–414 [MR](#) [Zbl](#)
- [47] **DC Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics 121, Academic, Orlando, FL (1986) [MR](#) [Zbl](#)
- [48] **DC Ravenel**, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies 128, Princeton Univ. Press (1992) [MR](#) [Zbl](#)
- [49] **C Rezk**, *Notes on the Hopkins–Miller theorem*, from “Homotopy theory via algebraic geometry and group representations” (M Mahowald, S Priddy, editors), Contemp. Math. 220, Amer. Math. Soc., Providence, RI (1998) 313–366 [MR](#) [Zbl](#)
- [50] **V Stojanoska**, *Duality for topological modular forms*, Doc. Math. 17 (2012) 271–311 [MR](#) [Zbl](#)
- [51] **NP Strickland**, *Products on MU-modules*, Trans. Amer. Math. Soc. 351 (1999) 2569–2606 [MR](#) [Zbl](#)

- [52] **N P Strickland**, *Gross–Hopkins duality*, *Topology* 39 (2000) 1021–1033 [MR](#) [Zbl](#)
- [53] **N Strickland**, **N Bellumat**, *Iterated chromatic localisation*, preprint (2019) [arXiv 1907.07801](#)
- [54] **P Symonds**, **T Weigel**, *Cohomology of p -adic analytic groups*, from “New horizons in pro- p groups” (M du Sautoy, D Segal, A Shalev, editors), *Progr. Math.* 184, Birkhäuser, Boston, MA (2000) 349–410 [MR](#) [Zbl](#)
- [55] **F L Wolcott**, *Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra*, *Pacific J. Math.* 276 (2015) 483–509 [MR](#) [Zbl](#)
- [56] **Z Yosimura**, *BP operations and homological properties of BP_*BP -comodules*, *Osaka J. Math.* 20 (1983) 123–136 [MR](#) [Zbl](#)
- [57] **Z Yosimura**, *Acyclicity of BP -related homologies and cohomologies*, *Osaka J. Math.* 22 (1985) 875–893 [MR](#) [Zbl](#)

*Department of Mathematical Sciences, Norwegian University of Science and Technology
Trondheim, Norway*

drew.k.heard@ntnu.no

Received: 13 August 2021

Revised: 15 February 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Norihiko Minami	Nagoya Institute of Technology nori@nitech.ac.jp
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Thomas Nikolaus	University of Münster nikolaus@uni-muenster.de
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris 13 bobol@math.univ-paris13.fr
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jessica S Purcell	Monash University jessica.purcell@monash.edu
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
David Futer	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Vesna Stojanoska	Univ. of Illinois at Urbana-Champaign vesna@illinois.edu
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Zoltán Szabó	Princeton University szabo@math.princeton.edu
Matthew Hedden	Michigan State University mhedden@math.msu.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Nathalie Wahl	University of Copenhagen wahl@math.ku.dk
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu	Chris Wendl	Humboldt-Universität zu Berlin wendl@math.hu-berlin.de
Thomas Koberda	University of Virginia thomas.koberda@virginia.edu	Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr		


See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2023 is US \$650/year for the electronic version, and \$940/year (+ \$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2023 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23

Issue 8 (pages 3417–3908)

2023

Partial Torelli groups and homological stability	3417
ANDREW PUTMAN	
On symplectic fillings of small Seifert 3–manifolds	3497
HAKHO CHOI and JONGIL PARK	
Milnor–Witt motivic cohomology of complements of hyperplane arrangements	3531
KEYAO PENG	
Connective models for topological modular forms of level n	3553
LENNART MEIER	
Asymptotic dimension of graphs of groups and one-relator groups	3587
PANAGIOTIS TSELEKIDIS	
Pressure metrics for deformation spaces of quasifuchsian groups with parabolics	3615
HARRISON BRAY, RICHARD CANARY and LIEN-YUNG KAO	
The $\mathrm{Sp}_{k,n}$ –local stable homotopy category	3655
DREW HEARD	
Smooth one-dimensional topological field theories are vector bundles with connection	3707
DANIEL BERWICK-EVANS and DMITRI PAVLOV	
Round fold maps on 3–manifolds	3745
NAOKI KITAZAWA and OSAMU SAEKI	
The epsilon invariant at 1 of 3–braid knots	3763
PAULA TRUÖL	
Cusps and commensurability classes of hyperbolic 4–manifolds	3805
CONNOR SELL	
The group of quasi-isometries of the real line cannot act effectively on the line	3835
SHENGGUI YE and YANXIN ZHAO	
Simplicial model structures on pro-categories	3849
THOMAS BLOM and IEKE MOERDIJK	