ORIGINAL ARTICLE

MATHEMATISCHE
NACHRICHTEN

Existence of uncertainty minimizers for the continuous wavelet transform

Simon Halvdansson[1] | **Jan-Fredrik Olsen[2]** | **Nir Sochen[3]** | **Ron Levie[4]**

[1]Department of Mathematics, NTNU
Norwegian University of Science and
Technology, Trondheim, Norway

[2]Centre for Mathematical Sciences, Lund
University, Lund, Sweden

[3]School of Mathematical Sciences, Tel
Aviv University, Tel Aviv, Israel

[4]Faculty of Mathematics, Technion -
Israel Institute of Technology, Haifa,
Israel

Correspondence
Simon Halvdansson, Department of
Mathematics, NTNU Norwegian
University of Science and Technology,
Trondheim, Norway.
Email: simon.halvdansson@ntnu.no

Funding information
Deutsche Forschungsgemeinschaft,
Grant/Award Number: 1798; Israeli
Ministry of Agriculture's Kandel Program,
Grant/Award Number: 20-12-0030;
Ministry of Health, State of Israel,
Grant/Award Number: 3-13898

Abstract
Continuous wavelet design is the endeavor to construct mother wavelets with
desirable properties for the continuous wavelet transform (CWT). One class
of methods for choosing a mother wavelet involves minimizing a functional,
called the wavelet uncertainty functional. Recently, two new wavelet uncertainty
functionals were derived from theoretical foundations. In both approaches, the
uncertainty of a mother wavelet describes its concentration, or accuracy, as
a time-scale probe. While an uncertainty minimizing mother wavelet can be
proven to have desirable localization properties, the existence of such a mini-
mizer was never studied. In this paper, we prove the existence of minimizers for
the two uncertainty functionals.

KEYWORDS
continuous wavelet, uncertainty minimizer, uncertainty principle, wavelet design

MSC (2020)
42C40, 65T60

1 | INTRODUCTION

For a *signal s* and a *mother wavelet f* in the Hardy space

$$H^2(\mathbb{R}) = \left\{ q \in L^2(\mathbb{R}) : \text{supp}(\hat{q}) \subset \mathbb{R}^+ \right\},$$

where $\mathbb{R}^+$ denotes the positive real numbers, we consider the continuous wavelet transform (CWT) at $(\alpha, \beta) \in \mathbb{R}^2$ as

$$W_f[s](\alpha, \beta) = \left\langle s, \pi(\alpha, \beta) f \right\rangle_{L^2(\mathbb{R})}. \tag{1.1}$$

Here, π is the wavelet representation

$$\pi(\alpha, \beta) f(t) = e^{-\alpha/2} f\left(\frac{t - \beta}{e^{\alpha}}\right). \tag{1.2}$$

That is, the signal s is analyzed by taking the inner product with dilations and translations of the mother wavelet. We can consider these operations separately, by writing $\pi(\alpha, \beta) = \pi_2(\beta)\pi_1(\alpha)$, where $\pi_1(\alpha) f(t) = e^{-\alpha/2} f\left(\frac{t}{e^{\alpha}}\right)$ represents the dilations of f and $\pi_2(\beta) f(t) = f(t - \beta)$ translations. Defined in this way, the continuous wavelet transform is a bounded operator of the form

$$W_f : H^2(\mathbb{R}) \to L^2(\mathbb{R}^2; d\mu(\alpha, \beta)),$$

where $d\mu(\alpha, \beta) = e^{-\alpha} \, d\alpha d\beta$ is the Haar measure of the affine group [3, 7].

The wavelet transform of functions in $L^2(\mathbb{R})$ can be treated by analyzing the positive and negative frequency supports separately. The mother wavelet f is required to be *admissible*, namely, it must satisfy

$$\int_0^{\infty} \frac{|\hat{f}(\omega)|^2}{\omega} d\omega < \infty. \tag{1.3}$$

This guarantees that the wavelet transform is invertible [13].

In (1.1), the mother wavelet f is a free parameter, and the endeavor to construct mother wavelets with desirable properties is called *wavelet design*. In this paper, we focus on wavelet design methods based on minimizing *uncertainty functionals*. The uncertainty of a mother wavelet is typically interpreted as its *sharpness* as a time-scale probe. In the short-time Fourier transform (STFT), the sharpness of a window function is defined as its Heisenberg uncertainty, and hence, optimal windows are Gaussian functions [6]. One classical attempt to generalize this approach to wavelet analysis is to generalize the Heisenberg uncertainty principle by taking infinitesimal group generators of π as localization operators [2]. While the group generator approach for defining localization is fruitful for the STFT, this is not the case for other transforms, like the CWT, as is explained in [11, 12].

An alternative approach for defining a wavelet uncertainty, based on the concept of *observables*, was proposed and investigated in [8, 10, 11]. Observables are localization operators that enable us to define uncertainty functionals that measure the localization of mother wavelets f in time and scale. The approach was shown to be meaningful in the sense that the uncertainty of a mother wavelet is directly linked to the sparsity, or sharpness, in the corresponding coefficient space.

Two observable-based uncertainty functionals were proposed in [10] and [8]. However, the existence of minimizers of these uncertainty functionals was not proved. In this paper, we prove the existence of a wavelet uncertainty minimizer in both cases.

We note that alternative approaches to wavelet design include the method introduced by Daubechies to construct compactly supported wavelets with vanishing moments [3], and adaptive methods which aim to maximize the correlation between the mother wavelet and the signal [14].

2 | WAVELET UNCERTAINTY FUNCTIONALS

In this section, we recall the observables approach to wavelet uncertainty functionals, and the two wavelet uncertainties introduced in [8, 11].

2.1 | The wavelet transform in the frequency domain

Wavelet uncertainty functionals are represented more naturally in the frequency domain than in the time domain. Hence, in this paper, the default space in which we work is the frequency domain. Accordingly, we denote mother wavelets and signals in the signal space $L^2(\mathbb{R}^+) = \{f \in L^2(\mathbb{R}) \mid \text{supp}(f) \subset \mathbb{R}^+\}$ by f, without a hat notation. We denote signals in the time domain by $\check{f}$. The wavelet representation (1.2) is now given by

$$\pi(\alpha, \beta) f(\omega) = e^{-2\pi i \omega \beta} e^{\alpha/2} f(e^{\alpha} \omega).$$

2.2 | Wavelet localization operators

The approach for defining uncertainty functionals is based on taking *observables* as localization operators. Inspired by quantum mechanics, an observable is a symmetric operator [1, 4]. In the signal processing context, observable are interpreted as entities that measure some underlying physical quantities of signals [10]. For example, the multiplication operator

$$\check{T}_x \check{f}(t) = t \check{f}(t) \tag{2.1}$$

measures localization in *time* of signals $\check{f} \in H^2(\mathbb{R})$. That is, when treating $|\check{f}(t)|^2$ as the density of the signal $\check{f}$ at time t, the mean time of $\check{f}$ is defined as the center of mass

$$\langle \check{T}_x \check{f}, \check{f} \rangle = \int_{\mathbb{R}} t |\check{f}(t)|^2 \, dt.$$

The following definition extends the above discussion for general observables, and explains what is meant by the spread of a signal about an observed quantity.

Definition 2.1. Let T be a symmetric operator on a Hilbert space $\mathcal{H}$ and f a normalized vector in the domain $\mathrm{Dom}(T)$ of T. The *expected value* and *variance* of f with respect to T are defined, respectively, as

$$e_f(T) = \langle Tf, f \rangle,$$

$$v_f(T) = \left\| (T - e_f(T))f \right\|^2.$$

When we want to emphasize the space in which the inner product and norm are defined, we add a superscript to e and v, for example, $e_f^{\mathcal{H}}(T)$.

Since the wavelet transform represents signals in the time-scale space $L^2(\mathbb{R}^2; d\mu(\alpha, \beta))$, the wavelet observables are the *time* and the *scale* observables. The time observable (2.1) multiplies by the variable of the time space $H^2(\mathbb{R})$, in which $\pi_2(\beta)$ is represented as translation. Similarly, there is a *scale space* where $\pi_1(\alpha)$ is represented as a translation, and in which the scale observable is defined as a multiplication operator. The mapping between the time space and the scale space is the *scale transform U*, defined as

$$U : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}), \quad U\{f\}(\sigma) = e^{-\sigma/2} f(e^{-\sigma}) =: \tilde{f}(\sigma)$$

The motivation behind the above construction is that, in order to measure the quantity which is transformed by π_j, $j = 1, 2$, we first represent π_j as a translation operator, and then we treat the translated variable as the physical quantity corresponding to π_j.

Given an observable T, we denote its form in the time and scale spaces by $\check{T} = \mathcal{F}T\mathcal{F}^*$ and $\tilde{T} = U^*TU$, where $\mathcal{F}$ is the Fourier transform. To formally define the time and scale observables we denote by

$$Y : f(y) \mapsto y f(y) \tag{2.2}$$

the general multiplication operator in $L^2(\mathbb{R})$ with the domain $\mathrm{Dom}(Y) = \{f \in L^2(\mathbb{R}) : y \mapsto y f(y) \in L^2(\mathbb{R})\}$.

Definition 2.2. The *time* and *scale observables* T_x, T_σ are defined in the signal space $L^2(\mathbb{R}^+)$ as

$$T_x = \mathcal{F}Y\mathcal{F}^*,$$

$$T_\sigma = U^*YU.$$

Proposition 2.3 [11]. *The signal space $L^2(\mathbb{R}^+)$ is invariant under the observables T_x and T_σ. Moreover,*

$$T_x f(\omega) = i\frac{\partial}{\partial\omega}f(\omega),$$

$$T_\sigma f(\omega) = -\ln(\omega)f(\omega).$$

The domains of T_x and T_σ are the sets of $f \in L^2(\mathbb{R}^+)$ such that f is absolutely continuous in $[a, b]$ for every $-\infty < a < b < \infty$, and $\omega \mapsto -\ln(\omega)f(\omega) \in L^2(\mathbb{R}^+)$ respectively.

Note that every f in the domain of T_x must satisfy $f(0) = 0$. Indeed, f is continuous and $f(\omega) = 0$ for every $\omega < 0$. Moreover, $L^2(\mathbb{R}^+)$ is invariant under T_x, so $H^2(\mathbb{R})$ is invariant under Y.

The scale transform is canonical in the sense that it transforms dilations π_1 to translations. It can therefore be verified that the following *canonical commutation relations* [10] hold

$$\pi_1(\alpha)^* T_\sigma \pi_1(\alpha) = T_\sigma + \alpha,$$

$$\pi_2(\beta)^* T_x \pi_2(\beta) = T_x + \beta.$$

Moreover, we have

$$e_{\pi(\alpha,0)f}(T_\sigma) = e_f(T_\sigma) + \alpha,$$

$$e_{\pi(0,\beta)f}(T_x) = e_f(T_x) + \beta,$$

$$v_{\pi(\alpha,0)f}(T_\sigma) = v_f(T_\sigma),$$

$$v_{\pi(0,\beta)f}(T_x) = v_f(T_x).$$

Since admissible wavelets form a subset of $L^2(\mathbb{R}^+)$, we make the distinction between the *signal space $L^2(\mathbb{R}^+)$*, which we now denote by $\mathcal{S}$, and the *window space $\mathcal{W}$* [9]. The inner product in the window space is defined according to the admissibility condition (1.3) as

$$\left\langle f_1, f_2 \right\rangle_{\mathcal{W}} = \int_0^\infty f_1(\omega)\overline{f_2(\omega)}\frac{1}{\omega}d\omega.$$

The window space $\mathcal{W}$ is defined to be the completion to a Hilbert space of $L^2(\mathbb{R}^+)$ with the inner product $\left\langle f_1, f_2 \right\rangle_{\mathcal{W}}$. Concretely, $\mathcal{W}$ is the weighted Lebesgue space $L^2\left(\mathbb{R}^+, \frac{1}{\omega}d\omega\right)$. We call $e_f^{\mathcal{S}}(T_x)$ the *signal-expected time*, and $e_f^{\mathcal{W}}(T_x)$ the *window-expected time*, and similarly use the terms signal-expected scale, window-time variance, etc.

2.3 | Signal space uncertainty

The signal space uncertainty of $f \in L^2(\mathbb{R}^+)$, introduced in [11], is based on a combination of the time and scale variances of f.

Definition 2.4. The *signal space uncertainty $\mathcal{L}_S$* is defined as

$$\mathcal{L}_S(f) = e^{\frac{-2e_{\frac{f}{\|f\|_S}}^S(T_\sigma)}{}} v_{\frac{f}{\|f\|_S}}^S(T_x) + v_{\frac{f}{\|f\|_S}}^S(T_\sigma)$$

on the domain $\text{Dom}(\mathcal{L}_S) = \text{Dom}(T_x) \cap \text{Dom}(T_\sigma)$.

The exponential term in the definition guarantees that $\mathcal{L}_S$ is invariant under $\pi(\alpha, \beta)$. This is a consequence of the following lemma.

Lemma 2.5 [10]. *The signal space uncertainty $\mathcal{L}_S$ is invariant under $\pi(\alpha, \beta)$ and linear scalings in the sense that*

$$\mathcal{L}_S(f) = \mathcal{L}_S(c\pi(\alpha, \beta)f) \qquad \text{for all } c \in \mathbb{C} \text{ and } (\alpha, \beta) \in \mathbb{R}^2.$$

In particular, for any $f \in L^2(\mathbb{R}^+)$, the normalized signal $f_N = \frac{1}{\|f\|_S} \pi\left(e^S_{\frac{f}{\|f\|_S}}(T_x), e^S_{\frac{f}{\|f\|_S}}(T_\sigma)\right)^{-1} f$ satisfies $\|f_N\|_S = 1$, $e^S_{f_N}(T_x) = e^S_{f_N}(T_\sigma) = 0$ and

$$\mathcal{L}_S(f) = \mathcal{L}_S(f_N) = \|T_x f_N\|_S^2 + \|T_\sigma f_N\|_S^2. \tag{2.3}$$

The following proposition shows that functions in the domain of $\mathcal{L}_S$ are admissible.

Proposition 2.6. *Any element of $\mathrm{Dom}(\mathcal{L}_S)$ is admissible, that is, $\mathrm{Dom}(\mathcal{L}_S) \subset \mathcal{W} \cap S$.*

Proof. Let $f \in \mathrm{Dom}(\mathcal{L}_S)$. Then f is continuous and zero for negative ω, so $f(0) = 0$. Moreover, since f is in the domain of T_x, it holds that $f' \in L^2(\mathbb{R}^+)$. It then follows that

$$|f(\omega)| \leq \int_0^\omega |f'(\xi)|\, d\xi \leq \sqrt{\|f'\|_{L^2}} \sqrt{\omega}$$

by the Cauchy–Schwarz inequality. Hence,

$$\int_0^\infty \frac{|f(\omega)|^2}{\omega} d\omega \leq \int_0^1 \|f'\|_{L^2}\, d\omega + \|f\|_{L^2} < \infty. \qquad \square$$

2.4 | Phase space uncertainty

The *phase space uncertainty*, introduced in [8], is a way to model the spread of the 2D *ambiguity function*

$$K_f(\alpha, \beta) = W_f[f](\alpha, \beta) = \langle f, \pi(\alpha, \beta)f \rangle_S.$$

The ambiguity function determines the amount of 'blurriness' of the output of the wavelet transform in the coefficient space $L^2(\mathbb{R}^2; d\mu(\alpha, \beta))$. Indeed, K_f is the reproducing kernel of $W_f[S]$ [5, 9], meaning that

$$Q \in W_f[S] \Rightarrow Q = Q * K_f.$$

Hence, the spread of K_f poses an upper bound on the resolution of the wavelet coefficient space.

The phase space uncertainty is based on the variance of the *phase space scale* and *phase space time observables*. These are defined, respectively, for $F : \mathbb{R}^2 \to \mathbb{C}$, by

$$AF(\alpha, \beta) = \alpha F(\alpha, \beta),$$

$$BF(\alpha, \beta) = \beta F(\alpha, \beta).$$

These operators are self-adjoint on their domains

$$\mathrm{Dom}(A) = \left\{ F \in L^2(\mathbb{R}^2, d\mu(\alpha, \beta)) \ : \ (\alpha, \beta) \mapsto \alpha F(\alpha, \beta) \text{ is in } L^2(\mathbb{R}^2, d\mu(\alpha, \beta)) \right\},$$

$$\mathrm{Dom}(B) = \left\{ F \in L^2(\mathbb{R}^2, d\mu(\alpha, \beta)) \ : \ (\alpha, \beta) \mapsto \beta F(\alpha, \beta) \text{ is in } L^2(\mathbb{R}^2, d\mu(\alpha, \beta)) \right\}.$$

Definition 2.7. The *phase space uncertainty* associated to the window f is defined to be

$$\mathcal{L}_P(f) = v_{\frac{K_f}{\|K_f\|_2}}(A) + v_{\frac{K_f}{\|K_f\|_2}}(B),$$

where $\frac{K_f}{\|K_f\|_2} = W_{\frac{f}{\|f\|_{\mathcal{W}}}}\left[\frac{f}{\|f\|_S}\right]$ is the normalized ambiguity function, and the variance is taken in the space $L^2(\mathbb{R}^2, d\mu(\alpha, \beta))$. The domain $\mathrm{Dom}(\mathcal{L}_P)$ is the set of all $f \in L^2(\mathbb{R}^+)$ such that $K_f \in \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$.

A main result in [8] is a pull-back of the calculation of the phase space uncertainty to the window function, based on the wavelet-Plancherel theory [9], which makes it considerably easier to work with.

Proposition 2.8 [8]. *Let $\mathcal{D}_P$ be the set of $f \in L^2(\mathbb{R}^+)$ such that f is absolutely continuous in every compact interval, $\|f\|_S = 1$, $e_f^S(T_x) = e_f^S(T_\sigma) = 0$, and the functions*

$$\omega \mapsto f'(\omega), \qquad \omega \mapsto \sqrt{\omega}f'(\omega), \qquad \omega \mapsto \frac{f(\omega)}{\omega}, \qquad \omega \mapsto \ln(\omega)f(\omega)$$

are square integrable. Then $\mathcal{D}_P \subset \mathrm{Dom}(\mathcal{L}_P)$ and for all $f \in \mathcal{D}_P$,

$$\mathcal{L}_P(f) = \left\|T_x f\right\|_S^2 + \left\|T_\sigma f\right\|_S^2 + v_{\frac{\mathcal{W}}{\|f\|_{\mathcal{W}}}}\left(i\omega\frac{\partial}{\partial\omega}\right)\left\|\frac{f}{\omega}\right\|_S^2 + v_{\frac{\mathcal{W}}{\|f\|_{\mathcal{W}}}}(-\ln(\omega)). \tag{2.4}$$

Formula (2.4) is similar to the signal space uncertainty (2.3), with two added terms. The constraint $e_f^S(T_x) = e_f^S(T_\sigma) = 0$ in $\mathcal{D}_P$ is taken for its signal processing utility. It assures that f is centered at time and scale 0, so that $W_f[s](\alpha, \beta)$ can be interpreted as the content of s at the time-scale (α, β). The following proposition is analogous to Proposition 2.6.

Proposition 2.9. *Any element of $\mathcal{D}_P$ is admissible, that is, $\mathcal{D}_P \subset \mathcal{W} \cap S$.*

Proof. This follows by the fact that $\mathcal{D}_P \subset \mathrm{Dom}(\mathcal{L}_S)$. $\qquad\qquad\square$

3 | EXISTENCE OF SIGNAL SPACE UNCERTAINTY MINIMIZERS

In this section, we prove our main result on the existence of minimizers of the signal space uncertainty (Definition 2.4).

Theorem 3.1. *There exists a minimizer of $\mathcal{L}_S$ in $\mathrm{Dom}(\mathcal{L}_S)$.*

We first note that by Lemma 2.5, we can restrict our attention to the set

$$\mathcal{D}_S = \left\{ f \in \mathrm{Dom}(\mathcal{L}_S) : \|f\|_S = 1, e_f(T_x) = 0, e_f(T_\sigma) = 0 \right\},$$

where the uncertainty simplifies to (2.3), that is,

$$\mathcal{L}_S(f) = \left\|T_x f\right\|_S^2 + \left\|T_\sigma f\right\|_S^2.$$

The following proposition is the key to proving existence.

Proposition 3.2. *Let $(f_n)_n \subset \mathcal{D}_S$ be a minimizing sequence of $\mathcal{L}_S(f)$ in the sense that*

$$\lim_{n\to\infty} \mathcal{L}_S(f_n) = \inf_{y\in\mathcal{D}_S} \mathcal{L}_S(y).$$

Then, there exist $N \in \mathbb{N}$ and a compact subset $\mathcal{K}_S \subset \mathcal{D}_S$ such that $f_n \in \mathcal{K}_S$ for $n > N$.

In the above proposition, note that a minimizing sequence exists since $\mathcal{D}_S$ is non-empty and $\mathcal{L}_S(\mathcal{D}_S)$ consists of non-negative real numbers. In the following analysis, we fix a value $K > 0$ such that

$$K/2 > \inf_{y \in \mathcal{D}_S} \mathcal{L}_S(y). \tag{3.1}$$

Definition 3.3. We define the subset $\mathcal{K}_S \subset \text{Dom}(\mathcal{L}_S)$ to be

$$\mathcal{K}_S = \left\{ f \in \text{Dom}(\mathcal{L}_S) : \|f\|_S = 1, \, e_f(T_x) = 0, \, e_f(T_\sigma) = 0, \right.$$

$$\left. \|T_x f\|_S^2 \le K, \, \|T_\sigma f\|_S^2 \le K \right\}. \tag{3.2}$$

The following lemma is now easy to verify.

Lemma 3.4. *Let $(f_n)_n \subset \mathcal{D}_S$ be a minimizing sequence of $\mathcal{L}_S$. Then there exists an $N \in \mathbb{N}$ such that $f_n \in \mathcal{K}_S$ for $n \ge N$.*

Proof. By (3.1), there exists an $N \in \mathbb{N}$ such that $f_n < K$ for $n \ge N$. In particular, this means that $f_n \in \mathcal{K}_S$ for large enough n since both terms of $\mathcal{L}_S$ are non-negative. $\qquad\square$

We prove that $\mathcal{K}_S$ is compact by showing that it is both closed and pre-compact. For the closedness, we begin by stating two auxiliary lemmas, the proofs of which we leave to the reader.

Lemma 3.5. *For any $M > 0$, the set*

$$\left\{ q \in L^2(\mathbb{R}) : \int_{\mathbb{R}} y^2 |q(y)|^2 \, dy \le M \right\}$$

is closed in $L^2(\mathbb{R})$.

Proof. Let $(q_n)_n$ be a sequence converging to q in $L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} y^2 |q_n(y)|^2 \, dy \le M$ for all n and write $q = q_n + \delta_n$. We show that $\int_{-N}^{N} y^2 |q(y)|^2 \, dy \le M$ for all $N \in \mathbb{N}$ by noting that for all n,

$$\int_{-N}^{N} y^2 |q(y)|^2 \, dy = \int_{-N}^{N} y^2 |q_n(y)|^2 \, dy + 2 \operatorname{Re} \int_{-N}^{N} y^2 q_n(y) \overline{\delta_n(y)} \, dy + \int_{-N}^{N} y^2 |\delta_n(y)|^2 \, dy$$

$$\le M + 2 \underbrace{\sqrt{\int_{-N}^{N} y^2 |q_n(y)|^2 \, dy}}_{< \sqrt{M}} \underbrace{\sqrt{N^2 \int_{-N}^{N} |\delta_n(y)|^2 \, dy}}_{\xrightarrow{n \to \infty} 0} + N^2 \underbrace{\int_{-N}^{N} |\delta_n(y)|^2 \, dy}_{\xrightarrow{n \to \infty} 0} \xrightarrow{n \to \infty} M.$$

Letting $N \to \infty$, we obtain the desired inequality. $\qquad\square$

Lemma 3.6. *For any $M > 0$, the set*

$$\left\{ q \in L^2(\mathbb{R}) : \int_{\mathbb{R}} y^2 |q(y)|^2 \, dy \le M, \, \int_{\mathbb{R}} y |q(y)|^2 \, dy = 0 \right\} \tag{3.3}$$

is closed in $L^2(\mathbb{R})$.

Proof. We remark that since $q \in L^2(\mathbb{R})$ and $\int_{\mathbb{R}} y^2 |q(y)|^2 \, dy < \infty$, it follows that $y|q(y)|^2$ is integrable. Now, let $(q_n)_n$ be a sequence in the set (3.3) which converges to q in $L^2(\mathbb{R})$ and write $q = q_n + \delta_n$. By Lemma 3.5, $\int_{\mathbb{R}} y^2 |q(y)|^2 \, dy \le M$. For

the second condition in the set (3.3), note first that by the inequality $|a - b|^2 \leq 2|a|^2 + 2|b|^2$, and by Lemma 3.5,

$$\int_{\mathbb{R}} y^2 |\delta_n(y)|^2 \, dy \leq 2 \int_{\mathbb{R}} y^2 |q(y)|^2 \, dy + 2 \int_{\mathbb{R}} y^2 |q_n(y)|^2 \, dy \leq 4M.$$

Also, for all $a > 0$,

$$\int_{\mathbb{R}} |y| |\delta_n(y)|^2 dy = \int_{|y| \leq a} |x| |\delta_n(y)|^2 dy + \int_{|y| > a} |y| |\delta_n(y)|^2 dy$$

$$\leq a \underbrace{\int_{|y| \leq a} |\delta_n(y)|^2 \, dy}_{\to 0} + \frac{1}{a} \underbrace{\int_{|y| > a} |y|^2 |\delta_n(y)|^2 \, dy}_{\leq 4M} \xrightarrow{n \to \infty} L \leq \frac{4M}{a}$$

and since this holds for arbitrarily large a, we have that

$$\int_{\mathbb{R}} |y| |\delta_n(y)|^2 \, dy \xrightarrow{n \to \infty} 0.$$

Next, note that

$$\int_{\mathbb{R}} y |q(y)|^2 \, dy = \int_{\mathbb{R}} y |q_n(y) + \delta_n(y)|^2 \, dx$$

$$= \underbrace{\int_{\mathbb{R}} y |q_n(y)|^2 \, dy}_{=0} + 2 \operatorname{Re} \int_{\mathbb{R}} y q_n(y) \overline{\delta_n(y)} \, dy + \int_{\mathbb{R}} y |\delta_n(y)|^2 \, dy.$$

Therefore, using the Cauchy–Schwarz inequality, we can estimate

$$\left| \int_{\mathbb{R}} y |q(y)|^2 \, dy \right| \leq 2 \sqrt{\int_{\mathbb{R}} |y| |q_n(y)|^2 \, dy} \underbrace{\sqrt{\int_{\mathbb{R}} |y| |\delta_n(y)|^2 \, dy}}_{\to 0} + \underbrace{\int_{\mathbb{R}} |y| |\delta_n(y)|^2 \, dy}_{\to 0}$$

which yields the desired conclusion once we show that $\int_{\mathbb{R}} |y| |q_n(y)|^2 \, dy$ is bounded in n. Indeed,

$$\int_{\mathbb{R}} |y| |q_n(y)|^2 \, dy = \int_{|y| \leq 1} |y| |q_n(y)|^2 \, dy + \int_{|y| > 1} |y| |q_n(y)|^2 \, dy$$

$$\leq \int_{|y| \leq 1} |q_n(y)|^2 \, dy + \int_{|y| > 1} |y|^2 |q_n(y)|^2 \, dy \leq 1 + M.$$ □

We are now ready to prove that each of the conditions in (3.2) defining $\mathcal{K}_S$ corresponds to a closed subset.

Lemma 3.7. *For any $M \geq 0$, the following subsets are closed in $L^2(\mathbb{R}^+)$:*

$$A = \left\{ f \in \operatorname{Dom}(T_x) \, : \, \|T_x f\|_S^2 \leq M, \, e_f(T_x) = 0 \right\},$$

$$B = \left\{ f \in \operatorname{Dom}(T_\sigma) \, : \, \|T_\sigma f\|_S^2 \leq M, \, e_f(T_\sigma) = 0 \right\}.$$

Proof. By Proposition 2.3, $H^2(\mathbb{R})$ is invariant under multiplication by t for $\check{f}$ in the domain of $\mathcal{L}_S$. Hence, we can express the restrictions in the time and scale spaces by

$$A = \left\{ f \in \mathrm{Dom}(T_x) : \int_{\mathbb{R}} t^2 |\check{f}(t)|^2 \, dt \leq M, \int_{\mathbb{R}} t |\check{f}(t)|^2 \, dt = 0 \right\},$$

$$B = \left\{ f \in \mathrm{Dom}(T_\sigma) : \int_{\mathbb{R}} \sigma^2 |\tilde{f}(\sigma)|^2 \, d\sigma \leq M, \int_{\mathbb{R}} \sigma |\tilde{f}(\sigma)|^2 \, d\sigma = 0 \right\}.$$

By Lemma 3.6, we obtain that both A and B are closed. $\qquad\square$

Proposition 3.8. *The set $\mathcal{K}_S$ is closed in $L^2(\mathbb{R}^+)$.*

Proof. By writing $\mathcal{K}_S$ as an intersection of sets corresponding to the conditions in (3.2), and noting that these sets are closed by Lemma 3.7, it follows that $\mathcal{K}_S$ is closed too. $\qquad\square$

To establish that $\mathcal{K}_S$ is pre-compact, we show that this set can be approximated by compact sets with arbitrary small error.

Lemma 3.9. *For any $\varepsilon > 0$, there exists a compact subset $C_{a,b}$ of $L^2(\mathbb{R}^+)$ such that for any $f \in \mathcal{K}_S$, there is a $y \in C_{a,b}$ such that*

$$\|f - y\|_S < \varepsilon.$$

Proof. For $f \in \mathcal{K}_S$, we consider $|\tilde{f}(\sigma)|^2$ as a probability distribution with mean value 0 and variance $\|T_\sigma f\|_S^2 \leq K$. Applying Chebyshev's inequality to the associated random variable, we have that for any $\alpha > 0$,

$$\int_{[e^{-\alpha}, e^{\alpha}]^c} |f(\omega)|^2 \, d\omega = \int_{[-\alpha, \alpha]^c} |\tilde{f}(\sigma)|^2 \, d\sigma \leq \frac{K}{\alpha^2}.$$

Now, fix $\varepsilon > 0$ and choose α so large that $\frac{K}{\alpha^2} < \varepsilon$. Then by the above inequality with $a = e^{-\alpha}$ and $b = e^{\alpha}$, it holds that $\|f - f|_{[a,b]}\| < \varepsilon$.

We now show that $f|_{[a,b]}$ is contained in a compact subset for each $f \in \mathcal{K}_S$. First, we note that for every $f \in \mathcal{K}_S$, we have

$$\int_a^b |f'(\omega)|^2 \, d\omega \leq \|f'\|_S^2 = \|T_x f\|_S^2 \leq K.$$

Therefore, if we mirror $f|_{[a,b]}$ around $\omega = b$ to $[a, 2b - a]$ and let f_e denote the absolutely continuous periodic extension of the resulting function, it will hold that

$$\|f_e'\|^2 \leq 2K \Rightarrow \|f_e'\|^2 = \sum_n \frac{\pi}{(b-a)^2} n^2 |c_n|^2 \leq 2K$$

$$\Rightarrow |c_n| \leq \frac{\sqrt{2K}(b-a)}{\pi} \frac{1}{|n|},$$

where $(c_n)_n$ are the Fourier coefficients of f_e. Next, we define

$$H_{a,b} = \left\{ \text{Periodic functions with period } 2(b-a) \text{ such that } |c_n| \leq \frac{\sqrt{2K}(b-a)}{\pi} \frac{1}{|n|} \right\}$$

and note that by Parseval's formula and Tychonoff's theorem, this set, known as the Hilbert cube, is compact. Moreover, we obtain that

$$C_{a,b} = \left\{ q \in L^2(a,b) : \exists\, y \in H_{a,b} : q = y\big|_{[a,b]} \right\}$$

is compact. Indeed, let $(q_n)_n$ be a sequence in $C_{a,b}$. Then each q_n can be mirrored and extended to yield a sequence in $(q_{e_n})_n \subset H_{a,b}$ which has a convergent subsequence. Restricting back to (a,b), we obtain a convergent subsequence of $(q_n)_n$.

Finally, since $f\big|_{[a,b]} \in C_{a,b}$, we obtain the desired conclusion with $y = f\big|_{[a,b]}$. □

Proposition 3.10. *The set $\mathcal{K}_S$ is pre-compact.*

Proof. Fix a sequence $(f_n)_n \subset \mathcal{K}_S$. We prove that $(f_n)_n$ has a convergent subsequence by constructing a Cauchy subsequence. By Lemma 3.9, we can choose a_m, b_m for each m such that any function in $\mathcal{K}_S$ can be approximated by a function in C_{a_m,b_m} with error less than $\frac{1}{m}$. For each f_n, we let f_n^m denote these approximations. That is,

$$f_n^m \in C_{a_m,b_m}, \qquad \|f_n - f_n^m\|_S < \frac{1}{m}. \tag{3.4}$$

For fixed m, the sequence $(f_n^m)_n$ is in the compact set C_{a_m,b_m}. There is, thus, a convergent subsequence

$$(f_{n_j^m}^m)_j \subset C_{a_m,b_m}, \qquad f_{n_j^m}^m \xrightarrow[j\to\infty]{} f^m \text{ in } C_{a_m,b_m}.$$

We choose subsequences so that $(n_j^m)_j$ is a subsequence of $(n_j^{m'})_j$ for every $m > m'$.

Keeping m fixed, we have, by (3.4), that

$$\|f_{n_j^m} - f_{n_j^m}^m\|_S < \frac{1}{m}.$$

Now, since $f_{n_j^m}^m \xrightarrow[j\to\infty]{} f^m$ in C_{a_m,b_m}, we can choose j_m so large that

$$\left\|f_{n_j^m}^m - f^m\right\|_S < \frac{1}{m} \Rightarrow \left\|f_{n_j^m} - f^m\right\|_S < \frac{2}{m}$$

for all $j \geq j_m$. This implies that $(f_{n_{j_m}^m})_m$ is a Cauchy sequence. Indeed, for every $m > m'$ it holds that

$$\left\|f_{n_{j_m}^m} - f^{m'}\right\|_S < \frac{2}{m'} \Rightarrow \left\|f_{n_{j_{m_1}}^{m_1}} - f_{n_{j_{m_2}}^{m_2}}\right\|_S < \frac{4}{m'}$$

for every $m_1, m_2 > m'$. By the completeness of $L^2(\mathbb{R}^+)$, the proof is complete. □

We can now prove Proposition 3.2 and Theorem 3.1.

Proof of Proposition 3.2. The tail of any minimizing sequence is in $\mathcal{K}_S$ by Lemma 3.4. The compactness of $\mathcal{K}_S$ follows from Proposition 3.8 ($\mathcal{K}_S$ closed) and Proposition 3.10 ($\mathcal{K}_S$ pre-compact). □

Proof of Theorem 3.1. By Proposition 3.2, there is a minimizing sequence in the compact subset $\mathcal{K}_S \subset \mathcal{D}_S$. This sequence, therefore, converges to some point $f_0 \in \mathcal{K}_S$. Moreover, by the compactness of $[0, K]$, we can pass to a subsequence such that

$$\|T_x f_n\|_S^2 \xrightarrow[n\to\infty]{} \sigma_1, \qquad \|T_\sigma f_n\|_S^2 \xrightarrow[n\to\infty]{} \sigma_2,$$

for some $\sigma_1, \sigma_2 \leq K$ such that $\sigma_1 + \sigma_2 = \inf_{y \in \mathcal{D}_S} \mathcal{L}_S(y)$.

Next, we show that f_0 is a minimizer of $\mathcal{L}_S$. Define for every $\alpha, \beta > 0$

$$\mathcal{K}_{\alpha,\beta} = \left\{ q \in \mathcal{K}_S \,:\, \|T_x q\|_S^2 \leq \alpha, \, \|T_\sigma q\|_S^2 \leq \beta \right\}$$

and note that this set is closed by Lemma 3.7. Since $\mathcal{K}_{\alpha,\beta} \subset \mathcal{K}_S$, this implies that $\mathcal{K}_{\alpha,\beta}$ is compact. Now, for each $\varepsilon > 0$, it holds that the tail of the minimizing sequence $(f_n)_n$ is contained in $\mathcal{K}_{\sigma_1+\varepsilon, \sigma_2+\varepsilon}$. Thus, for any $\varepsilon > 0$, we have $\mathcal{L}_S(f_0) \leq \sigma_1 + \sigma_2 + 2\varepsilon$, which implies that $\mathcal{L}_S(f_0) = \sigma_1 + \sigma_2$. Thus, f_0 is a minimizer. $\qquad\square$

4 | EXISTENCE OF PHASE SPACE UNCERTAINTY MINIMIZERS

In this section, we prove our main result on the existence of minimizers of the phase space uncertainty (Definition 2.7). For the convenience of the reader, we recall that from Proposition 2.8, for $f \in \mathcal{D}_P$, the phase space uncertainty (2.4) can be expressed as

$$\mathcal{L}_P(f) = \|T_x f\|_S^2 + \|T_\sigma f\|_S^2 + v_{\frac{\mathcal{W}_f}{\|f\|_\mathcal{W}}}\left(i\omega \frac{\partial}{\partial \omega} \right) \left\| \frac{f}{\omega} \right\|_S^2 + v_{\frac{\mathcal{W}_f}{\|f\|_\mathcal{W}}}(-\ln(\omega)), \tag{4.1}$$

where $\mathcal{D}_P$ is the set of $f \in L^2(\mathbb{R}^+)$ such that f is absolutely continuous in every compact interval, $\|f\|_S = 1$, $e_f^S(T_x) = e_f^S(T_\sigma) = 0$, and the functions

$$\omega \mapsto f'(\omega), \qquad \omega \mapsto \sqrt{\omega} f'(\omega), \qquad \omega \mapsto \frac{f(\omega)}{\omega}, \qquad \omega \mapsto \ln(\omega) f(\omega)$$

are square integrable.

Theorem 4.1. *There exists a minimizer of $\mathcal{L}_P$ in $\mathcal{D}_P$.*

The proof follows a similar path to that of Theorem 3.1. In particular, the following proposition, which is analogous to Proposition 3.2, is a key step.

Proposition 4.2. *Let $(f_n)_n \subset \mathcal{D}_P$ be a minimizing sequence of $\mathcal{L}_P(f)$ in the sense that*

$$\lim_{n \to \infty} \mathcal{L}_P(f_n) = \inf_{y \in \mathcal{D}_P} \mathcal{L}_P(y).$$

Then there exist $N \in \mathbb{N}$ and a compact subset $\mathcal{K}_P \subset \mathcal{D}_P$ such that $f_n \in \mathcal{K}_P$ for $n > N$.

Note that when the signal-expected time $e_f^S(T_x)$ is zero, so is $e_f^\mathcal{W}\left(i\omega \frac{\partial}{\partial \omega}\right)$ since

$$e_{\frac{\mathcal{W}_f}{\|f\|_\mathcal{W}}}\left(i\omega \frac{\partial}{\partial \omega} \right) = \frac{1}{\|f\|_\mathcal{W}^2} e_f\left(i\frac{\partial}{\partial \omega} \right).$$

We can, therefore, further simplify the uncertainty (4.1) for $f \in \mathcal{D}_P$ to

$$\mathcal{L}_P(f) = \|T_x f\|_S^2 + \|T_\sigma f\|_S^2 + \frac{1}{\|f\|_\mathcal{W}^2} \|i\omega f'\|_\mathcal{W}^2 \left\| \frac{f}{\omega} \right\|_S^2 + \frac{1}{\|f\|_\mathcal{W}^2} v_f^\mathcal{W}(-\ln(\omega)).$$

From here, we proceed as in Section 3. First, we define $\mathcal{K}_P$ analogously to Definition 3.3. We fix a value $K > 0$ such that

$$K/2 > \inf_{y \in \mathcal{D}_S} \mathcal{L}_P(y). \tag{4.2}$$

Definition 4.3. The domain $\mathcal{K}_\mathrm{P} \subset \mathcal{D}_\mathrm{P}$ is defined to be

$$
\mathcal{K}_\mathrm{P} = \left\{ f \in \mathrm{Dom}(\mathcal{L}_\mathrm{P}) : \|f\|_S = 1, \, e_f(T_x) = 0, \, e_f(T_\sigma) = 0, \right.
$$

$$
\|T_x f\|_S^2 \leq K, \, \frac{1}{\|f\|_{\mathcal{W}}^2} \|i\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|_S^2 \leq K, \tag{4.3}
$$

$$
\left. \|T_\sigma f\|_S^2 \leq K, \, \frac{1}{\|f\|_{\mathcal{W}}^2} v_f^{\mathcal{W}}(-\ln(\omega)) \leq K \right\}.
$$

The following lemma follows by the same argument as for Lemma 3.4.

Lemma 4.4. *Let $(f_n)_n \subset \mathcal{D}_\mathrm{P}$ be a minimizing sequence of $\mathcal{L}_\mathrm{P}$. Then there exists an $N \in \mathbb{N}$ such that $f_n \in \mathcal{K}_\mathrm{P}$ for $n \geq N$.*

To prove Proposition 4.2, we show that $\mathcal{K}_\mathrm{P}$ is compact. Since $\mathcal{K}_\mathrm{P} \subset \mathcal{K}_S$ and $\mathcal{K}_S$ is compact, it only remains to show that $\mathcal{K}_\mathrm{P}$ is closed. Lemma 3.7 already shows that two of the conditions in (4.3) correspond to closed sets. For the remaining two conditions, we need the following auxiliary lemmas.

Lemma 4.5. *Let $M > 0$ and $f \in \mathrm{Dom}(T_x)$ be such that $\|T_x f\|_S^2 \leq M$ and $f(0) = 0$. Then*

$$
|f(\omega)| \leq \sqrt{M}\sqrt{\omega}.
$$

Proof. This follows from an application of the Cauchy–Schwarz inequality. $\square$

Lemma 4.6. *Let $M > 0$ and $f \in L^2(\mathbb{R}^+)$ be such that $\|T_x f\|_S^2 \leq M$ and $\frac{1}{\|f\|_{\mathcal{W}}^2} \|\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|_S^2 \leq M$. Then*

$$
\|\omega f'\|_{\mathcal{W}}^2 \leq 2e^2 M(M+1).
$$

Proof. Note first that $\|\omega f'\|_{\mathcal{W}}^2$ is the second factor of $\frac{1}{\|f\|_{\mathcal{W}}^2} \|\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|^2 \leq M$. Therefore, by bounding $\frac{1}{\|f\|_{\mathcal{W}}^2}$ and $\left\| \frac{f}{\omega} \right\|^2$ from below, we obtain a bound of $\|\omega f'\|_{\mathcal{W}}^2$ from above. For $\frac{1}{\|f\|_{\mathcal{W}}^2}$, we bound

$$
\|f\|_{\mathcal{W}}^2 = \int_0^\infty \frac{|f(\omega)|^2}{\omega} d\omega \leq \int_0^1 \frac{|f(\omega)|^2}{\omega} d\omega + \underbrace{\|f\|^2}_{=1} \leq \int_0^1 \frac{M\omega}{\omega} d\omega + 1 = M + 1,
$$

where we made use of Lemma 4.5. Hence,

$$
\frac{1}{\|f\|_{\mathcal{W}}^2} \|\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|_S^2 \leq M \Rightarrow \|\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|_S^2 \leq \frac{M}{\frac{1}{\|f\|_{\mathcal{W}}^2}} \leq M(M+1).
$$

To bound $\left\| \frac{f}{\omega} \right\|_S^2$ from below, we consider the following two cases separately.
Case 1: $\int_e^\infty |f(\omega)|^2 d\omega \geq 1/2$.
By the fact that $1/\omega^2 > |\ln(\omega)|$ in (0,1) and by the fact that

$$
e_f^S(T_\sigma) = 0 \Rightarrow \int_0^1 |\ln(\omega)||f(\omega)|^2 d\omega = \int_1^\infty |\ln(\omega)||f(\omega)|^2 d\omega,
$$

we have

$$\left\|\frac{f}{\omega}\right\|_S^2 = \int_0^\infty \frac{|f(\omega)|^2}{\omega^2}d\omega \geq \int_0^1 \frac{|f(\omega)|^2}{\omega^2}d\omega \geq \int_0^1 |\ln(\omega)||f(\omega)|^2 d\omega$$

$$= \int_1^\infty |\ln(\omega)||f(\omega)|^2 d\omega \geq \int_e^\infty \underbrace{|\ln(\omega)|}_{\geq 1}|f(\omega)|^2 d\omega \geq \int_e^\infty |f(\omega)|^2 d\omega \geq 1/2.$$

Case 2: $\int_0^e |f(\omega)|^2 d\omega \geq 1/2$.

In this case, we use the estimate

$$\left\|\frac{f}{\omega}\right\|_S^2 = \int_0^\infty \frac{|f(\omega)|^2}{\omega^2}d\omega \geq \int_0^e \frac{|f(\omega)|^2}{\omega^2}d\omega \geq \int_0^e \frac{|f(\omega)|^2}{e^2}d\omega \geq \frac{1}{2e^2}.$$

Hence, $\left\|\frac{f}{\omega}\right\|_S^2 \geq \frac{1}{2e^2}$ uniformly. As a result,

$$\|\omega f'\|_{\mathcal{W}}^2 \leq \frac{M(M+1)}{\left\|\frac{f}{\omega}\right\|_S^2} \leq 2e^2 M(M+1),$$

which completes the proof. □

Lemma 4.7. *Let $M > 0$ and $(f_n)_n$ be a sequence in $L^2(\mathbb{R}^+)$ with $\|T_x f_n\|_S^2 \leq M$ for each n such that $\|f_n - f\|_S \to 0$ for some $f \in L^2(\mathbb{R}^+)$. Then $\|f_n - f\|_{\mathcal{W}} \to 0$.*

Proof. Fix $\varepsilon > 0$ and choose n so large that $\|f_n - f\|_S^2 < \varepsilon^2/M$. Then

$$\|f_n - f\|_{\mathcal{W}}^2 = \int_0^{\varepsilon/M} \frac{|f_n(\omega) - f(\omega)|^2}{\omega}d\omega + \int_{\varepsilon/M}^\infty \frac{|f_n(\omega) - f(\omega)|^2}{\omega}d\omega$$

$$< 4M\frac{\varepsilon}{M} + \frac{M}{\varepsilon}\|f_n - f\|_S^2 < 5\varepsilon,$$

where we used Lemma 4.5 for the first term. □

We are now ready to prove two lemmas corresponding to the remaining two conditions of (4.3).

Lemma 4.8. *For any $M > 0$, the set*

$$\left\{ f \in L^2(\mathbb{R}^+) : \frac{1}{\|f\|_{\mathcal{W}}^2}\|i\omega f'\|_{\mathcal{W}}^2 \left\|\frac{f}{\omega}\right\|_S^2 \leq M, \|T_x f\|_S^2 \leq M \right\}$$

is closed in $L^2(\mathbb{R}^+)$.

Proof. By writing

$$A(f) = \|i\omega f'\|_{\mathcal{W}}^2, \qquad B(f) = \left\|\frac{f}{\omega}\right\|_S^2,$$

we have that

$$\frac{1}{\|f\|_{\mathcal{W}}^2}\|i\omega f'\|_{\mathcal{W}}^2 \left\|\frac{f}{\omega}\right\|_S^2 \leq M \iff A(f)B(f) \leq M\|f\|_{\mathcal{W}}^2.$$

Using the change of variables $\omega = e^s$, we obtain

$$A(f) = \int_0^\infty \omega |f'(\omega)|^2 d\omega = \int_{-\infty}^\infty e^s |f'(e^s)|^2 e^s ds = \int_{\mathbb{R}} |e^s f'(e^s)|^2 ds = \int_{\mathbb{R}} |f^{\flat'}(s)|^2 ds$$

where $f^\flat(s) = f(e^s)$. It can be verified that $\|f^\flat\|_S = \|f\|_{\mathcal{W}}$, which allows us to apply the Fourier transform to $f^\flat$ to get

$$A(f) = 4\pi^2 \int_{\mathbb{R}} t^2 |\widehat{f^\flat}(\xi)|^2 d\xi.$$

Next, let (f_n) be a sequence such that $\|f_n - f\|_S \to 0$ for some $f \in L^2(\mathbb{R}^+)$ and write $\delta_n = f - f_n$. Then it holds that

$$\int_{\mathbb{R}} |\widehat{\delta_n^\flat}(t)|^2 dt = \int_{\mathbb{R}} |\delta_n^\flat(s)|^2 ds = \int_{\mathbb{R}} |\delta_n(e^s)|^2 ds$$

$$= \int_0^\infty |\delta_n(\omega)|^2 \frac{1}{\omega} d\omega = \|f_n - f\|_{\mathcal{W}}^2 \xrightarrow[n \to \infty]{} 0,$$

where we used the Plancherel theorem for the first step, and Lemma 4.7 for the last step.

We now estimate

$$\left(4\pi^2 \int_{-N}^N \xi^2 |\widehat{f^\flat}(\xi)|^2 d\xi \right) \left(\int_{1/N}^N \frac{1}{\omega^2} |f(\omega)|^2 d\omega \right)$$

$$\leq \left(4\pi^2 \underbrace{\int_{-N}^N \xi^2 |\widehat{f_n^\flat}(\xi)|^2 d\omega}_{\leq A(f_n)} + 2 \cdot 4\pi^2 \sqrt{\underbrace{\int_{-N}^N \xi^2 |\widehat{f_n^\flat}(\xi)|^2 d\xi}_{\leq A(f_n) \leq M(M+1)2e^2}} \sqrt{N^2 \int_{-N}^N |\widehat{\delta_n^\flat}(\xi)|^2 d\xi} \right.$$

$$+ 4\pi^2 N^2 \int_{-N}^N |\widehat{\delta_n^\flat}(\xi)|^2 d\xi \bigg) \left(\underbrace{\int_{1/N}^N \frac{1}{\omega^2} |f_n(\omega)|^2 d\omega}_{\leq B(f_n)} \right.$$

$$+ 2 \sqrt{N^2 \underbrace{\int_{1/N}^N |f_n(\omega)|^2 d\omega}_{\leq \|f_n\|^2 \leq 1}} \sqrt{N^2 \int_{1/N}^N |\delta_n(\omega)|^2 d\omega} + N^2 \int_{1/N}^N |\delta_n(\omega)|^2 d\omega \bigg)$$

$$= A(f_n)B(f_n) + o_n(1) \leq M\|f_n\|_{\mathcal{W}}^2 + o_n(1)$$

where we in the last step used the fact that $\|\delta_n\|_S \to 0$ and $\|\widehat{\delta_n^\flat}\|_S \to 0$ as $n \to \infty$.

Finally, since $\|f_n\|_{\mathcal{W}} \to \|f\|_{\mathcal{W}}$, by Lemma 4.7 and by letting $n \to \infty$, we conclude, for all N, that

$$\left(\int_{-N}^N t^2 |\widehat{f^\flat}(\xi)|^2 d\xi \right) \left(\int_{1/N}^N \frac{1}{\omega^2} |f(\omega)|^2 d\omega \right) \leq M\|f\|_{\mathcal{W}}^2. \tag{4.4}$$

The left-hand side of (4.4) converges to $A(f)B(f)$ as $N \to \infty$, which yields

$$A(f)B(f) = \|i\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|_S^2 \leq M\|f\|_{\mathcal{W}}^2.$$

This concludes the proof. □

Lemma 4.9. *For any $M > 0$, the set*

$$\left\{ f \in L^2(\mathbb{R}^+) : \frac{1}{\|f\|_{\mathcal{W}}^2} v_f^{\mathcal{W}}(-\ln(\omega)) \leq M, \ \|T_x f\|_S^2 \leq M \right\}$$

is closed in $L^2(\mathbb{R}^+)$.

Proof. We begin by writing the variance as

$$v_f^{\mathcal{W}}(-\ln(\omega)) = e_f^{\mathcal{W}}(\ln(\omega)^2) - e_f^{\mathcal{W}}(-\ln(\omega))^2.$$

As a result, $\frac{1}{\|f\|_{\mathcal{W}}^2} v_f^{\mathcal{W}}(-\ln(\omega)) \leq M$ can be written as

$$\int_0^\infty \frac{\ln(\omega)^2}{\omega} |f(\omega)|^2 \, d\omega \leq M\|f\|_{\mathcal{W}}^2 + \left(\int_0^\infty \frac{-\ln(\omega)}{\omega} |f(\omega)|^2 \, d\omega \right)^2 \tag{4.5}$$

$$= M\|f\|_{\mathcal{W}}^2 + e_f^{\mathcal{W}}(-\ln(\omega))^2.$$

To show that the inequality (4.5) corresponds to a closed subset, we let $(f_n)_n$ converge to f in $L^2(\mathbb{R}^+)$, such that each f_n satisfies (4.5), and show that f also satisfies (4.5). We first show that

$$e_{f_n}^{\mathcal{W}}(-\ln(\omega)) \to e_f^{\mathcal{W}}(-\ln(\omega)) \tag{4.6}$$

as $n \to \infty$. To see this, let $\varepsilon > 0$ be so small that $\varepsilon/M < 1/2$, and choose n so large that $\left| \|f_n\|_S^2 - \|f\|_S^2 \right| < \varepsilon^2$. Then

$$\left| e_{f_n}^{\mathcal{W}}(-\ln(\omega)) - e_f^{\mathcal{W}}(-\ln(\omega)) \right| = \left| \int_0^\infty \frac{-\ln(\omega)}{\omega} \left(|f_n(\omega)|^2 - |f(\omega)|^2 \right) d\omega \right|$$

$$\leq \int_0^{\varepsilon/M} |\ln(\omega)| 2M \, d\omega + \int_{\varepsilon/M}^\infty \underbrace{\frac{|\ln(\omega)|}{\omega}}_{\leq \frac{\ln(\varepsilon/M)}{\varepsilon/M}} \left| |f_n(\omega)|^2 - |f(\omega)|^2 \right| d\omega$$

$$\leq \frac{\varepsilon}{M} (\ln(\varepsilon/M) - 1) 2M + M \frac{\ln(\varepsilon/M)}{\varepsilon} \left| \|f_n\|_S^2 - \|f\|_S^2 \right|$$

$$\leq \varepsilon (3 \ln(\varepsilon/M) - 2).$$

Next, with $\delta_n = f - f_n$, we have that for each N,

$$\int_{1/N}^N \frac{\ln(\omega)^2}{\omega} |f(\omega)|^2 \, d\omega = \int_{1/N}^N \frac{\ln(\omega)^2}{\omega} |f_n(\omega)|^2 \, d\omega + 2\,\mathrm{Re} \int_{1/N}^N \frac{\ln(\omega)^2}{\omega} f_n(\omega) \overline{\delta_n(\omega)} \, d\omega + \int_{1/N}^N \frac{\ln(\omega)^2}{\omega} |\delta_n(\omega)|^2 \, d\omega$$

$$\leq M\|f_n\|_{\mathcal{W}}^2 + e_{f_n}^{\mathcal{W}}(-\ln(\omega))^2 + 2\sqrt{\frac{\ln(1/N)^2}{N} \int_{1/N}^N |f_n(\omega)|^2 \, d\omega} \sqrt{\frac{\ln(1/N)^2}{N} \int_{1/N}^N |\delta_n(\omega)|^2 \, d\omega}$$

$$+ \frac{\ln(1/N)^2}{N} \int_{1/N}^N |\delta_n(\omega)|^2 \, d\omega.$$

By Lemma 4.7 and (4.6), the first two terms of the upper bound, above, converge to $M\|f\|_{\mathcal{W}}^2 + e_f^{\mathcal{W}}(-\ln(\omega))^2$ as $n \to \infty$, while the remaining terms all vanish as $n \to \infty$ since $\|\delta_n\|_S \to 0$. We, therefore, have that

$$\int_0^\infty \frac{\ln(\omega)^2}{\omega}|f(\omega)|^2\,d\omega = \lim_{N \to \infty} \int_{1/N}^N \frac{\ln(\omega)^2}{\omega}|f(\omega)|^2\,d\omega \le M\|f\|_{\mathcal{W}}^2 + e_f^{\mathcal{W}}(-\ln(\omega))^2,$$

which concludes the proof. □

Proposition 4.10. *The set $\mathcal{K}_P$ is closed in $L^2(\mathbb{R}^+)$.*

Proof. This follows directly from Lemmas 3.7, 4.8, and 4.9 by taking the intersections of the sets appearing in these lemmas. □

Proof of Proposition 4.2. By Lemma 4.4, the tail of any minimizing sequence is in $\mathcal{K}_P$. Moreover, $\mathcal{K}_P$ is compact since, by Lemma 4.10, it is a closed subset of $\mathcal{K}_S$. □

  With Proposition 4.2 established, the proof of Theorem 4.1 follows the same procedure used to prove Theorem 3.1. The only modifications are adjustments for Equation (4.1) defining $\mathcal{L}_P(f)$ consisting of four terms and replacing $\mathcal{D}_S$ and $\mathcal{K}_S$ by $\mathcal{D}_P$ and $\mathcal{K}_P$.

5 | CONCLUSION

We have proven non-constructively the existence of minimizers for the signal space uncertainty and phase space uncertainty functionals $\mathcal{L}_S$ and $\mathcal{L}_P$. In [8], a numerical gradient descent scheme was presented, which estimates a minimizer of the uncertainty functional $\mathcal{L}_P$. However, no approximation results were proven for the numerical scheme. In a future work, we prove that discrete numerical estimates of wavelet uncertainty minimizers indeed approximate true minimizers in $L^2(\mathbb{R}^+)$, for both $\mathcal{L}_S$ and $\mathcal{L}_P$.

ORCID
Simon Halvdansson ⓘ https://orcid.org/0000-0002-4589-3960

REFERENCES
[1] P. Busch, P. Lahti, J. Pellonpaa, and K. Ylinen, *Quantum measurement*, Theoretical and Mathematical Physics, Springer, Cham, Switzerland, 2016.
[2] S. Dahlke and P. Maass, *The affine uncertainty principle in one and two dimensions*, Comput. Math. Appl. **30** (1995), no. 3, 293–305.
[3] I. Daubechies, *Ten lectures on wavelets*, Society for Industrial and Applied Mathematics, Philadelphia, 1992.
[4] G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, New Jersey, 1989.
[5] H. Führ, *Abstract harmonic analysis of continuous wavelet transforms*, Lecture Notes in Mathematics, vol. 1863, Springer, Berlin, 2005, pp. 1–201.
[6] K. Gröchenig, *Foundations of time-frequency analysis*, Birkhäuser Boston, 2001.
[7] A. Grossmann, J. Morlet, and T. Paul, *Transforms associated to square integrable group representations, I, General results*, J. Math. Phys. **26** (1985), no. 10, 2473–2479.
[8] R. Levie, E. K. Avraham, and N. Sochen, *Wavelet design with optimally localized ambiguity function: a variational approach*, preprint 2021, arXiv:2104.01654.
[9] R. Levie and N. Sochen, *A wavelet plancherel theory with application to sparse continuous wavelet transform*, Numer. Funct. Anal. Optim. **43** (2017), 1303.
[10] R. Levie and N. Sochen, *Uncertainty principles and optimally sparse wavelet transforms*, Appl. Comput. Harmon. Anal. **48** (2020), no. 3, 811–867.

[11] R. Levie, H. Stark, F. Lieb, and N. Sochen, *Adjoint translation, adjoint observable and uncertainty principles*, Adv. Comput. Math. **40** (2013), no. 3, 609–627.

[12] P. Maass, C. Sagiv, N. Sochen, and H. Stark, *Do uncertainty minimizers attain minimal uncertainty?*, J. Fourier Anal. Appl. **16** (2010), no. 3, 448–469.

[13] S. Mallat, *A wavelet tour of signal processing*, 3rd ed., Elsevier, Amsterdam, 2009.

[14] R. Rao and J. Chapa, *Algorithms for designing wavelets to match a specified signal*, IEEE Trans. Signal Process. **48** (2000), no. 12, 3395–3406.

How to cite this article: S. Halvdansson, J.-F. Olsen, N. Sochen, and R. Levie, *Existence of uncertainty minimizers for the continuous wavelet transform*, Math. Nachr. 296 (2023), 1156–1172. https://doi.org/10.1002/mana.202100466