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Preface

This thesis is being submitted as a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics at the Norwegian University of Science and Technology.

Firstly, I would like to express my gratitude to my supervisor Mats Ehrnström, and co-supervisor Didier Pilod and Kristoffer Varholm for their sincere help and kind guidance in my PhD study. I am also very grateful to the chances that they suggest me to join many mathematical seminars, conferences, etc. This keeps me to know the knowledge frontier and to meet with mathematicians in different areas.

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I enjoy the great working environment at the Department of Mathematical Sciences with different social activities out of the office. Technical and administrative support also enables me to perform at my best in my work.

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Jun Xue
Trondheim, August 2023

INTRODUCTION

This thesis contains three papers, where the first two of them are mainly about the theory on global weak solutions and regularity on highest cusped waves for nonlinear dispersive equations, and the third one is the investigation on existence of periodic travelling waves with point vortices for Euler equations.

Nonlinear dispersive equations play a crucial role in understanding the behavior of water waves and provide a perfect framework for studying the dynamics and evolution of water waves in various scenarios. Here dispersive (or dispersion) means that waves with different wavelengths propagate at different speeds. A couple of hydrodynamical phenomenon exists in the combination of nonlinear effects and these properties, such as wave breaking, extreme waves, peaked or cusped crests which we may discuss in Paper II for example, and periodic movement. Some models particularly attract a lot of interest. In [32], specific water-wave phenomenon like tsunamis or tidal waves, coastal oceanography are generally and extensively studied. Stokes waves are also a famous example where the regularity of highest position was investigated in [2].

In particular, Whitham equations which are nonlocal provide a mathematical framework for studying the behavior of water surfaces, where this equation has been widely studied by researchers for many years. It captures various key characteristics of shallow water waves, including solitary [13] and travelling [15] waves, and wave breaking [24] (bounded solutions with unbounded derivatives). Based on this perfect example of Whitham equation, one may choose to direct their attention towards studying a broad class of equations

$$\partial_t u + \partial_x L u + \partial_x (n(u)) = 0, \tag{1}$$

where u is on time and real-valued space variable with one dimension, the operator L describes the dispersive effect, and $n(u)$ represents the nonlinear term (usually consider as u^2). In general, the operator L is often expressed as a Fourier multiplier with a real and even symbol m, that is,

$$\widehat{(Lu)}(\xi) = m(\xi)\hat{u}(\xi). \tag{2}$$

Here $\xi \in \mathbb{R}$, and the symbol $\hat{\cdot}$ denotes the common Fourier transform for any Schwartz function. The symbol m, known as the dispersion relation, plays a crucial role in the research area of dispersion equations.

Two types of the expression $m(\xi)$ attract the particular interest, which contains homogeneous symbol

$$m(\xi) = |\xi|^s \tag{3}$$

with $Lu = |D|^s u$, and inhomogeneous symbol

$$m(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \tag{4}$$

with $Lu = (1 + |D|^2)^{\frac{s}{2}} u$, where $s \in \mathbb{R}$ and $D := -i\partial_x$. The above two kinds of symbol are usually known as fractional Korteweg–de Vries (fKdV) equation with homogeneous and inhomogeneous symbol. Below is a table to give some examples on the choices of $m(\xi)$ with different index s:

TABLE 1. Different index of $m(\xi)$ in dispersion equation (1)

Index	$m(\xi)$	Equation
-2	$(1 + \|\xi\|^2)^{-1}$	Burgers–Poisson
-2	$\|\xi\|^{-2}$	Ostrovsky–Hunter
-1	$\|\xi\|^{-1}$	Burgers–Hilbert
$-\frac{1}{2}$	$\sqrt{\frac{\tanh \xi}{\xi}}$	Whitham
$\frac{1}{2}$	$\sqrt{\frac{(1+\alpha\|\xi\|^2)\tanh \xi}{\xi}}$	Capillary–Whitham
1	$\|\xi\|$	Benjamin–Ono
2	$1 - \frac{1}{6}\|\xi\|^2$	Korteweg–de Vries
$s > 0$	$(\frac{\tanh \xi}{\xi})^s$	Generalized Whitham

Here $\alpha > 0$ is the surface tension for capillary Whitham equation. When s takes smaller or negative value, the dispersion becomes weaker, which corresponds to the suggestion of fKdV equations to measure the scale to investigate the balance of nonlinear and dispersion effects.

Let us review the development on fKdV equations briefly. Firstly, fKdV is globally well-posed for $s = 2$ [27, 28] and $s = 1$ [26, 37]. When $1 < s < 2$, we have global well-posedness in the corresponding Sobolev spaces (see [21, 23]). When $\frac{6}{7} < s < 1$, global well-posedness was recently established in [38]. But numerical result [29] suggested that we have global well-posedness for $s > \frac{1}{2}$. Since $\frac{1}{2}$ is scaling critical, this is a solid evidence to think $s = \frac{1}{2}$ is also critical for global well-posedness, which means there is no global well-posedness for $s \leq \frac{1}{2}$. Thus, one may expect the phenomenon of blowup happened for $s \leq \frac{1}{2}$. Smooth solutions will blow up for $-1 \leq s < 0$ [5] and wave breaking phenomenon happens for $-1 < s < -\frac{1}{3}$ [25], $s = -1$ [44] (Burgers–Hilbert), and $s = -2$ [35] (Ostrovsky–Hunter). It is also believed that other blowup phenomenon occur for $0 < s \leq \frac{1}{2}$ (see discussion in [29, 34, 36]). Owing to the above, two ways will be considered due to this effect, that is, the conception of weak solutions and the stronger restriction on the function spaces of initial data. The weak solution that we considered in Paper I is motivated by such consideration. One may note that the range $-1 < s < 1$ attracts many interests due to the fact that it contains Capillary–Whitham equation ($s = \frac{1}{2}$) and Whitham equation ($s = -\frac{1}{2}$). An enhanced existence time of solutions was considered in [19, 20] where the authors considered small initial data in a restricted Sobolev spaces for which blowup regime is excluded. In Paper I, we aim to show the global well-posedness of entropy solutions for fKdV equations with inhomogeneous symbol (4) with index $s < -1$. Moreover, these weak solutions are found to satisfy one-sided Hölder conditions whose coefficients will decay in time.

On the side of highest waves for fKdV equations, Whitham conjectured that the Whitham equation ($s = -\frac{1}{2}$) admits a highest, *cusped*, travelling-wave solution with $C^{\frac{1}{2}}$-regularity (Hölder regularity) at each crest. This conjecture was proved by Ehrnström and Wahlén [18] in 2019. Here $C^{\frac{1}{2}}$-regularity is given by an equivalence

$$\mu - \phi \sim |x|^{\frac{1}{2}},$$

which holds uniformly around the highest point 0 in one period $[-\pi, \pi]$. The parameter μ is the wave speed. Inspired by the work and conclusion in [18], Paper II shows the existence of highest, cusped, periodic travelling-wave solutions with exact and optimal β-Hölder regularity for fKdV equations with homogeneous symbol (3) for $0 < \beta < 1$, where

$\beta = -s$. We also give an effective way to deal with the general nonlinearity $n(u)$ for fKdV equations.

Here we introduce the travelling wave solutions to fKdV equations. We now naturally take

$$u(t,x) = \phi(x - ct)$$

for some velocity $c \in \mathbb{R}$ into (1), then equation (1) is transformed to the steady form

$$-c\phi + L\phi + n(\phi) = C,$$

where C is a constant. KdV equation is an important representative on local differential operator for L since the Fourier transform has a corresponding relation with derivative and frequency, where it performs the symbol $1 - \frac{1}{6}\xi^2$ as an approximation with nonlinearity u^2. On the nonlocal sense, the qualitative properties of water waves which does not exist in the local case can be obtained. Note that Fourier multiplier can be expressed as a convolution operator $Lu = K * u$ which leads that the nonlocal sense of water wave models presents a notable challenge, as it requires knowledge of the global wave property to compute the dispersive term locally. In Table 1, we have listed many negative indexes. These fKdV equations have a great research value to be investigated since the nonlocal property makes the pointwise estimates to become harder, and moreover, they show the different regularised properties. Paper I and Paper II just aim to investigate the related properties which benefit from the various direction in this research area.

Turning the aim to the third paper in this thesis, we consider the water wave problem on finite depth modelled by the free boundary problem for the incompressible Euler equation,

$$\begin{cases} \vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} + \nabla p + g\vec{e_2} = \vec{0}, \\ \nabla \cdot \vec{u} = 0, \end{cases} \tag{5}$$

where $\vec{u} = (u, v)$ is the velocity of the fluid, p is the pressure distribution and $-g\vec{e_2} = (0, -g)$ is the constant gravitational acceleration. Applying the transformation

$$\vec{u} = \vec{u}(x - ct, y), \quad \eta = \eta(x - ct), \quad p = p(x - ct, y),$$

the steady version of (5) is given by

$$\begin{cases} (\vec{u} \cdot \nabla)\vec{u} - c\vec{u}_x + \nabla p + g\vec{e_2} = \vec{0}, \\ \qquad\qquad\qquad\qquad \nabla \cdot \vec{u} = 0. \end{cases} \tag{6}$$

A vorticity distribution is usually a function $\gamma\colon \mathbb{R} \to \mathbb{R}$, such that

$$\Delta\psi = \gamma(\psi),$$

where ψ is the related stream function.

The research on steady water wave problem (6) mainly figures out the construction of water waves but with different types of vorticity. This may contain irrotational setting ($\gamma = 0$) where Stokes waves [41] are significant representative, and rotational setting with non-localized or localized vorticity. Rotational waves can possibly contain internal stagnation points and critical layers which are regions with closed streamlines, known as cat's eye vortices (see [11]), and this is also the main aim that many previous works tried to deal with the non-localized vorticity. Breakthrough on non-localized vorticity occurred in [7] in 2004, where a comprehensive theory of existence for two-dimensional, periodic, finite-depth, large-amplitude travelling gravity waves were constructed, but there are no interior stagnation points and critical layers. Then the progress was pushed forward to the constant vorticity with exactly one critical layer [45], affine vorticity with critical layers [12, 17, 30], or analytic vorticity with critical layers [43].

Rotational waves with localized vorticity attracted a lot of interest recently, where the pioneered work [40] investigated the existence of two-dimensional, travelling, capillary-gravity water waves with two types of compactly supported vorticity, that is, point vortex and vortex patch. Paper III mainly showed the existence of two-dimensional small amplitude periodic travelling gravity-capillary water waves on finite depth, with an arbitrary number of point vortices along one vertical line in each period. Specifically, the vorticity is concentrated to finitely many δ-distributions within a period of the fluid domain. Paper III extended the work of [40] from infinite depth to finite depth.

Next, we give a short description on each paper in this thesis below.

Paper I: One-sided Hölder regularity of global weak solutions of negative order dispersive equations.

With: Ola I.H. Mæhlen.

Published in Journal of Differential Equations, 364 (2023), pp. 412–455.

In Paper I, the focus is on analyzing global weak solutions for dispersive equations (1) of negative order with inhomogeneous symbol (4) with a u^2 nonlinearity and the range of index $s < -1$, where we represent the dispersive operator $L := G*$, and G is the convolution kernel.

When the index s becomes negative, we can find some related results by the help of the conception of entropy solutions, where it is considered in the absence of classical global solutions. Entropy solutions are weak solutions that satisfy the entropy inequalities which is also automatically satisfied by classical solutions. This solution concept allows for continuation past wave breaking and a global well-posedness theory may then be achieved. Existence and uniqueness of global entropy solutions for the Ostrovsky–Hunter equation ($s = -2$) is established in [6] with initial data in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and for the Burgers–Hilbert equation ($s = -1$) in [22] with initial data in $L^1(\mathbb{R})$. There is a difference between our paper with [22], where they consider initial data in $L^1(\mathbb{R})$ which is in the view of conservation law but we consider in $L^2(\mathbb{R})$ which is more natural and takes various analysis in details.

By the above motivation, the present paper mainly contains two parts. In the first part, we investigate the existence, uniqueness, and L^2-stability of entropy solutions for equation (1) with inhomogeneous symbol (4) for order $s < -1$. The key idea is to note that equation (1) is a combination of the two equations:

(Hyperbolic conservation law)	(Convolution evolution)
$\boxed{u_t + (u^2)_x = 0}$	$\boxed{u_t = G' * u}$

For these two (simpler) equations, we focus on analysing the properties of their corresponding two solution maps. The result of uniqueness and stability is proved through a variation of Kružkov's doubling of variables technique [31], while existence follows from an operator splitting argument. The whole analysis in the first part is firstly implemented for initial data in the setting $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then extending to $L^2(\mathbb{R})$ by a continuity argument. The benefit that we use operator splitting method is to allow for a straightforward analysis of the regularizing effect for (1) which will make perfect preparation for the part of one-sided Hölder regularity for entropy solutions of (1), and the entropy inequality below is widely used to obtain the desired conclusion of global well-posedness

$$\int_0^\infty \int_{\mathbb{R}} \eta(u)\varphi_t + q(u)\varphi_x + \eta'(u)(G' * u)\varphi \,\mathrm{d}x\mathrm{d}t \geq 0,$$

where (η, q) is entropy pair and φ is a test function.

In the second part of this paper, the operator splitting technique that we analysed in the first part is employed to establish that the obtained weak solutions satisfy explicit one-sided Hölder condition, that is,

$$u(t,x) - u(t,y) \le a(t)(x-y)^{\frac{1+b}{2}},$$

where $0 \le b \le 1$ and $a(t)$ is time decreasing. This outcome can be seen as an extension of the classical Oleĭnik estimate [10] for Burgers' equation which is given by

$$u(t,x+h) - u(t,x) \le \frac{h}{t}.$$

This one-sided Lipschitz condition resticts how fast u can grow, but not how fast it can decrease, which means jump discontinuities may happen. We mainly study the modulus of growth of ω,

$$\omega(t,h) := \sup_x \big(u(t,x+h) - u(t,x)\big)$$

for $t, h > 0$ in this part, and the difficult point is that the dispersive term in (1) has no clear effect on ω compared with the smoothing effect of nonlinearity on ω, where we need to treat it as a source term with the limitation of decreasing L^2 norm of the solution. Finally, one-sided Hölder estimate is successfully established.

Paper II: Periodic Hölder waves in a class of negative-order dispersive equations.

With: Fredrik Hildrum.

Published in Journal of Differential Equations, 343 (2023), pp. 752–789.

The second paper investigates the regularity of large amplitude travelling waves for (homogeneous) fKdV equations with generalized nonlinearity

$$n(u) = |u|^p \quad \text{or} \quad u|u|^{p-1},$$

where $p > 1$ is real. In particular, we consider the homogeneous symbol in (3) with $0 < \beta < 1$, where we set $-\beta = s$ (this is convenient to express our regularity result).

With different positive range of β, this research area was fully investigated recently. Table 2 below gives a summarization on the most related works in this area.

Before we discuss the main steps of this paper, the intuition to obtain the desired C^β-regularity is inspired from the work of [18]. Note that

TABLE 2. Works for different regularities of highest waves with homogeneous/inhomogeneous fKdV equations

Cases of fKdV \ Range of β	$0 < \beta < 1$	$\beta = 1$	$\beta > 1$
homogeneous	C^β This paper	log-Lipschitz [16]	Lipschitz [3]
inhomogeneous	C^β [1, 16, 18, 39]	log-Lipschitz [14, 16]	Lipschitz [33]

in [18], the Fourier symbol of Whitham equation corresponds to the index of $\frac{1}{2}$, then the authors obtained $C^{\frac{1}{2}}$-regularity for the periodic highest wave which led us a strong intuition to also expect C^β-regularity for the index of $\beta \in (0, 1)$.

The whole paper contains two parts. The first part is to deal with the regularity analysis on the highest wave by the help of the structure of [18]. The regularity is initially proved with global C^γ-regularity for all $\gamma < \beta$, then the exact β-Hölder estimate at 0, and finally global C^β-regularity using an interpolation argument. The difficult part is that our kernel only has algebraic but not exponential decay (unlike the kernels in [1, 18, 39]), which means an extra care must be applied to the finite-difference estimates for $|D|^{-\beta}$ when the index $\beta \to 1$. To analyse the convolution kernel K properly where $Lu = K * u$, we split the kernel into two parts, that is, $K \sim |\cdot|^{\beta-1} +$ regular part. This method is an effective way to rewrite the kernel which is already known in $\mathbb{R}$ to the period $\mathbb{T}$. This will make our analysis easier to see the structure of the kernel clearly. It is worth noting that we calculate an explicit integration representation for the singular kernel which also shows a straightforward way to derive the exact singularity.

While in the second part, which is also the most challenging part, we aim to consider the global bifurcation theory but with extending from the classical nonlinear term u^2 to a more generalized $n(u)$. To see how the solutions evolve, the local bifurcation analysis is firstly implemented with new bifurcation formulas with the new nonlinearity $n(u)$. Then it reaches to the global behavior of the solutions, where we apply an analytic theory in [4] to extend from local bifurcation. Note here the theory in [4] can not be worked with the nonsmooth $n(u)$ which makes the problem to be much harder. Thus, we employ an analytically regularisation method to regularise the nonlinearity, and show the existence of highest wave after the process of regularisation and approximation. Our method has broad applicability to many equations in Table 1 and provides an effective approach to deal with general nonlinear terms.

We finally give a direct visualization on our resulting waves with two different types of nonlinearities. When the nonlinearity performs the type $n(u) = u|u|^{p-1}$, we can see that the doubly-cusped periodic solutions were presented.

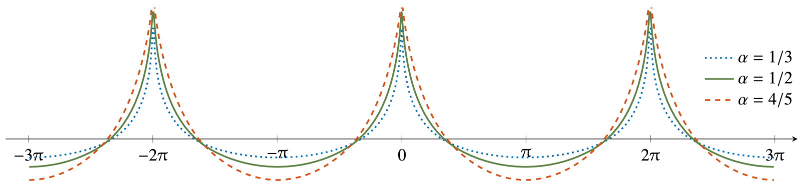

FIGURE 1. Waves with nonlinearity $|u|^p$ with $p = \sqrt{\pi}$.

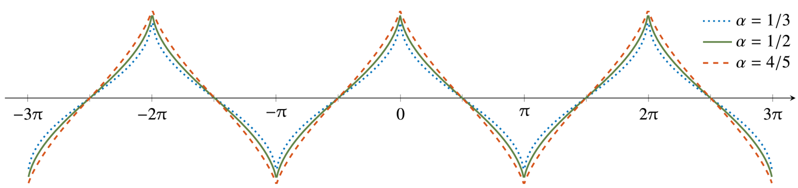

FIGURE 2. Waves with nonlinearity $u|u|^{p-1}$ with $p = e$. The antisymmetric behavior happens between the cusps.

Paper III: Periodic Travelling water waves with point vortices.

In Preparation.

The idea of this paper is inspired by [40,42]. Shatah, Walsh, and Zeng [40] constructed capillary-gravity waves of small amplitude with *a point* *vortex* for the *infinite* depth for both *solitary* and *periodic* case. Later, Varholm [42] considered the case of *finite* depth with *solitary* waves. This paper resolved the only left case, that is, *finite* depth with *periodic* case. The table below will illustrate the related works clearly and effectively.

TABLE 3. Existence of waves for different cases with finite/infinite depth for a point vortex

Cases \ Depth	Infinite Depth	Finite Depth
Solitary Case	[40]	[42]
Periodic Case	[40]	This Paper

The paper contains two parts. The procedure is mainly from [42]. In the first part, We aim to derive three preparation equations (*Bernoulli equation, Kinematic equation and point vortex equation*) so that we can prove the existence of two-dimensional small-amplitude periodic travelling gravity-capillary water waves on finite depth, with *a point vortex* along one vertical line in one period. The main method of this part is called Zakharov–Craig–Sulem formulation from [8, 9, 46]. To obtain the first *Bernoulli equation* which is also one of the difficult point in this paper, we split the velocity $\vec{u}$ as a sum of two parts,

$$\vec{u} = \vec{u}_{\mathrm{ir}} + \vec{U},$$

where $\vec{u}_{\mathrm{ir}}$ is irrotational, and both $\vec{u}_{\mathrm{ir}}$ and $\vec{U}$ are divergence free. Such decomposition is useful to our analysis, where on one hand, the rotational part $\vec{U}$ will be exactly known, and on the other hand, the Poincaré lemma can be implemented to obtain the desired representation of stream function for irrotational and rotational part, that is,

$$\vec{u}_{\mathrm{ir}} = \nabla^{\perp}\psi = (-\psi_y, \psi_x), \quad \vec{U} = \nabla^{\perp}\Psi = (-\Psi_y, \Psi_x),$$

where ψ and Ψ are the corresponding stream functions. Then from the first equation in the steady system (6), our *Bernoulli equation* is obtained by the help of *Harmonic extension operator* and *Dirichlet-to-Neumann operator*. *Kinematic boundary condition* is actually obtained from the kinematic boundary condition at surface,

$$u\eta_x + \eta_t = v,$$

where we then substitute

$$\vec{u} = \nabla^{\perp}(\psi + \Psi)$$

into the above equation to get the result. Similarly, *point vortex equation* is also derived by the above expression of $\vec{u}$ by a proper calculation.

With the aid of a stream function Φ particularly for periodic version and three corresponding equations that we have found in the first part, the implicit function theorem is applied based on an analysis of invertible operators of partial derivatives to obtain the existence result. Moreover, the leading order of surface profile function is detailedly investigated. Finally, we also deal with several point vortices along the vertical lines in every period in the last part of this paper by a generalization from one point vortex.

It is also worth noting that interestingly numerical evidence shows that the concavity and convexity of the leading order surface profile function will change at the junction between two periods depending on the choices of every period L and parameter θ, which brings a significant difference with the solitary case in [42]. We only give the proof of partial conclusions in this paper and it leaves an interesting problem to pursue later.

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Paper I

One-sided Hölder regularity of global weak solutions of negative order dispersive equations

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ONE-SIDED HÖLDER REGULARITY OF GLOBAL WEAK SOLUTIONS OF NEGATIVE ORDER DISPERSIVE EQUATIONS

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ABSTRACT. We prove global existence, uniqueness and stability of entropy solutions with L^2 initial data for a general family of negative order dispersive equations. These weak solutions are found to satisfy one-sided Hölder conditions whose coefficients decay in time. The latter result controls the height of solutions and further provides a way to bound the maximal lifespan of classical solutions from their initial data.

1. INTRODUCTION

We consider the initial value problem

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = (G * u)_x, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

for initial data $u_0 \in L^2(\mathbb{R})$ and an even convolution kernel $G \in L^1(\mathbb{R})$ admitting an integrable weak derivative $G' =: K \in L^1(\mathbb{R})$. Included in this family of equations is the Burgers–Poisson equation

$$u_t + \tfrac{1}{2}(u^2)_x = \left(\int_{-\infty}^{\infty} -\tfrac{1}{2} e^{-|x-y|} u(t,y) \mathrm{d}y \right)_x \tag{1.2}$$

which in [22] is derived as a model for shallow water waves.

1.1. **Outline of main results.** The paper can be divided in two parts: Section 3 establishes the well-posedness of entropy solutions of (1.1), while Section 4 demonstrates the one-sided Hölder regularity that the solutions enjoy. To the best of our knowledge, these results are new. It was shown in [9] that the Burgers–Poisson equation (1.2) admits unique entropy solutions with L^1 data that satisfy one sided Lipschitz conditions. Still, the results here add new insight also for Burgers–Poisson: The L^2 setting is more natural (albeit harder) to work in due to the dispersive right-hand side. For the L^2 norm of a solution is guaranteed to be non-increasing in time, which can be used to deduce a one-sided *smoothing* effect of (1.1). In particular, our Corollary 2.4 shows that the one-sided Lipschitz coefficients of solutions of Burgers–Poisson can at worst behave like 'const$+1/t$' whereas the corresponding expression in [9] takes the form $O(te^t + 1/t)$. An interesting consequence of having an explicit smoothing effect of (1.1), as given by Theorem 2.3, is that it provides a necessary condition on terminal data when seeking to solve the backward problem; this is exploited in the proof of Corollary 2.6 which bounds the lifespan of classical solutions of 1.1.

We give a brief discussion of our results which are presented in Section 2. The first main result, Theorem 2.1, provides existence, uniqueness and L^2 stability for entropy solutions of (1.1) – as defined by Def. 1.1 – for initial data in $L^2 \cap L^\infty(\mathbb{R})$. Corollary 2.2 then extends this result in a unique and continuous manner to pure L^2 data. The results are proved in Section 3. There, uniqueness and stability is proved through a variation of Kružkov's doubling of variables technique [16], while existence follows from an operator splitting argument. While there are less laborious approaches for proving existence (fixed point methods, vanishing viscosity), operator splitting has the advantage of allowing for a straight forward analysis of the regularizing effect of (1.1) which constitutes the second part of our results.

The second main result, Theorem 2.3, guarantees one-sided Hölder regularity for entropy solutions of (1.1), and it is proved in Section 4. Like the classical Oleĭnik estimate (4.1) for Burgers' equation, this one-sided regularity *improves* over time. The proof is based on an operator splitting approach, used to study the evolution of the quantity $\omega(t,h) \coloneqq \sup_x(u(t,x+h)-u(t,x))$, for $t,h>0$ and a solution u. As seen by Lemma 4.3, the nonlinearity in (1.1) has a smoothing effect on ω. The dispersive term on the other hand has no clear convenient effect on ω, and it is instead treated as a source term that we limit using the non-increasing L^2 norm of u (as done when combining Lemma 4.4 and 4.2).

The result has two interesting consequences. First, Corollary 2.5 provides an explicit height bound for a solution u in terms of $\|K\|_{L^1(\mathbb{R})}$, $\|u_0\|_{L^2(\mathbb{R})}$ and the time t. This bound decays initially like $1/t^{\frac{1}{3}}$ and converges to a positive constant for large times. Generally, the height of a solution will not tend to zero due to the existence of solitary waves [7] for several instances of (1.1). Second, Corollary 2.6 bounds the lifespan of classical solutions of (1.1) provided the initial data satisfies a skewness condition (2.6). One may wonder how these classical solutions break down, and wave breaking is the natural candidate. A proof of this is beyond the scope of this paper, but not hard to obtain; demonstrating that (1.1) is classically well posed for times $t \lesssim -1/\inf_x u_0'(x)$ (the hyperbolic lifespan) would leave wave breaking as the only type of blow up (as we already have height bounds). We point out that our skewness condition (2.6) differs from that of both [9] and [5]; neither imply the other.

1.2. **Other dispersive equations.** Central questions in the study of water wave model equations include well-posedness, persistence and non-persistence of solutions, the latter two exemplified by solitary and breaking waves. The answers depend intricately on the type of nonlinearity and dispersive term featured in the equation. In the case of a quadratic nonlinearity, the fractional Korteweg–de Vries equation (fKdV)

$$u_t + \tfrac{1}{2}(u^2)_x = (|D|^\beta u)_x \tag{1.3}$$

where $\mathcal{F}(|D|^\beta u) = |\xi|^\beta \hat{u}$ and $\beta \in \mathbb{R}$, has been suggested [17] as a scale for studying how the strength of the dispersion affects the questions of well-posedness and water-wave features. To connect (1.1) to the fKdV setting, observe that our assumption on G implies that $\widehat{G}(\xi) = o(|\xi|^{-1})$ as $|\xi| \to \infty$ and so one may place (1.1) in the region $\beta < -1$ for fKdV. However, $\widehat{G}$ will in our case be bounded, whereas $|\xi|^\beta$ blows up at zero, and thus (1.1) can not match the low-frequency effect of negative order fKdV which assigns (very) high velocities to (very) low frequencies. This qualitative difference disappears in a periodic setting; the dispersion of fKdV on the torus is for $\beta < -1$ precisely of the form assumed in (1.1). We point out that the methods in this paper can be carried out on the torus; our results can thus be extended to periodic solutions of fKdV for $\beta < -1$. With the relation between (1.1) and (1.3) accounted for, we now summarize a few results for the latter to sketch what one may expect of well-posedness and water-wave features in our case.

The fractional KdV equation of order $\beta \in (\frac{6}{7}, 2]$ is globally well-posed in appropriate function spaces. The regions $\beta \in (\frac{6}{7}, 1)$ and $\beta \in (1,2)$ are treated in [21] and [10] respectively, and there are numerous works on the well posedness for $\beta = 1$ (Benjamin–Ono equation) and $\beta = 2$ (KdV equation); see for example [13] and [14] and the references therein. For values $\beta \le \frac{6}{7}$ only local well-posedness results have been established [8, 21]. Still, numerical investigation [15] suggests that fKdV is globally well-posed for dispersion as weak as $\beta > \frac{1}{2}$, but not for $\beta \leq \frac{1}{2}$; this is also conjectured in [17]. One might expect the culprit of this loss of global well-posedness for weak dispersion, to be the appearance of breaking waves (shock formation), i.e. bounded solutions that develop infinite slope in finite time. In the negative order regime $\beta < 0$ this might be true: the occurrence of breaking waves have been proved for the case $\beta = -2$ (Ostrovsky–Hunter equation) by [18], for the case $\beta = -1$ (Burgers–Hilbert) by [23] and for the region $\beta \in (-1, -\frac{1}{3})$ by [12]. However, no such results exist in the positive order regime $\beta > 0$, and it is believed that instead other blowup phenomena occur in the range $\beta \in (0, \frac{1}{2}]$ inhibiting global well-posedness; see the discussion in [15, 17] or [20] where an example of L^∞ blowup in finite time is constructed for the modified Benjamin–Ono equation. In the absence of classical global solutions, several authors have for the $\beta < 0$ regime turned to the concept of entropy solutions. Adapted from the study of hyperbolic conservation laws, entropy solutions are weak solutions that satisfy extra conditions – the entropy inequalities – automatically satisfied by classical solutions (whenever they exist). This solution concept allows for continuation past wave breaking and a global well-posedness theory may then be achieved. In [4] existence and uniqueness of global entropy solutions for the Ostrovsky–Hunter equation ($\beta = -2$) is established for appropriate initial data. Similarly, [3] provides global entropy solutions for the Burgers–Hilbert equation ($\beta = -1$) and a partial uniqueness result. And as mentioned above, the Burgers–Poisson equation (1.2) is in [9] shown to admit unique global entropy solutions for L^1 initial data. There, the authors also provide sufficient conditions on the initial data leading to wave breaking. This equation is not an isolated instance of (1.1) featuring wave breaking; [5] shows that the phenomenon is present whenever $G \in C \cap L^1(\mathbb{R})$ is symmetric and monotone on $\mathbb{R}^+$. More generally, our Corollary 2.6 hints that every instance of (1.1) features wave breaking as explained above.

1.3. **The entropy formulation.** We define the concept of entropy solutions on the function class $L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$, which here denotes the subspace of $L^\infty_{\mathrm{loc}}([0,\infty) \times \mathbb{R})$ of functions $u(t,x)$ that are essentially bounded on $[0,T] \times \mathbb{R}$ for each $T > 0$. We will in Section 2 be more liberal in our definition of entropy solutions (allowing then for L^2 initial data) as explained after Corollary 2.2.

Necessary is the notion of an entropy pair (η, q) for (1.1), which is to say that

$$\eta \colon \mathbb{R} \to \mathbb{R} \text{ is smooth and convex, while } q'(u) = \eta'(u)u.$$

Definition 1.1. For bounded initial data $u_0 \in L^\infty(\mathbb{R})$, we say that a function $u \in L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$ is an entropy solution of (1.1) if:

(1) it satisfies for all non-negative $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ and all entropy pairs (η, q) of (1.1) the entropy inequality

$$\int_0^\infty \int_{\mathbb{R}} \eta(u)\varphi_t + q(u)\varphi_x + \eta'(u)(K * u)\varphi \,\mathrm{d}x\mathrm{d}t \geq 0, \tag{1.4}$$

(2) it assumes the initial data in L^1_{loc} sense, that is

$$\operatorname*{ess\,lim}_{t \searrow 0} \int_{-r}^{r} |u(t,x) - u_0(x)| \mathrm{d}x = 0,$$

for all $r > 0$.

The concept of entropy solutions lies between that of strong and weak solutions. If $u \in L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R})) \cap C^1(\mathbb{R}^+ \times \mathbb{R})$ is a classical solution of (1.1) then it is necessarily an entropy solution as multiplying (1.1) with $\eta'(u)\varphi$ and integrating by parts yields (1.4) as an equality. And if u is an entropy solution of (1.1) then it is necessarily a weak solution as follows from considering the two entropy pairs $(\eta(u), q(u)) = (u, \frac{1}{2}u^2)$ and $(\eta(u), q(u)) = (-u, -\frac{1}{2}u^2)$ respectively.

1.4. **A fractional variation.** The exponents of the one-sided Hölder conditions provided by Theorem 2.3 depend on the regularity of $K = G'$; the smoother K is, the higher the exponent. More precisely, we attain the Hölder exponent $\frac{1+s}{2}$ if $|K|_{TV^s} < \infty$ where the latter seminorm is for $s \in [0,1]$ defined by

$$|K|_{TV^s} = \sup_{h>0} \frac{\|K(\cdot + h) - K\|_{L^1(\mathbb{R})}}{h^s}. \tag{1.5}$$

When $s = 1$ this seminorm coincides with the classical total variation of K, while $s = 0$ gives twice the L^1 norm of K, and thus we necessarily have $|K|_{TV^0} < \infty$ as we assume $K \in L^1(\mathbb{R})$. For $s \in (0,1)$ the seminorm is a measure of intermediate regularity between $L^1(\mathbb{R})$ and $BV(\mathbb{R})$. This seminorm does not coincide with the scaling invariant fractional variation from [19] used in [2] to attain maximal smoothing effects for one-dimensional scalar conservation laws.

2. Main results

We here present the two main results, Theorem 2.1 and Theorem 2.3 and corresponding corollaries. For a general discussion of the content given here, see the end of the above introduction. We start with Theorem 2.1, which provides a global well-posedness theory for entropy solutions of (1.1) with initial data in $L^2 \cap L^\infty(\mathbb{R})$. The theorem is established in Section 3.

Theorem 2.1. *For every initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ there exists a unique entropy solution u of (1.1). The mapping $t \mapsto u(t)$ is continuous from $[0,\infty)$ to $L^2(\mathbb{R})$ and $u(t)$ satisfies for all $t \geq 0$ the bounds*

$$\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \qquad \|u(t)\|_{L^\infty(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^\infty(\mathbb{R})}, \tag{2.1}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$. Moreover, we have the following stability result: if two sequences $(t_k)_{k\in\mathbb{N}} \subset [0,\infty)$ and $(u_{0,k})_{k\in\mathbb{N}} \subset L^2 \cap L^\infty(\mathbb{R})$ admit the limits

$$\lim_{k\to\infty} t_k = t, \qquad \text{and} \qquad \lim_{k\to\infty} u_{0,k} = u_0 \quad \text{in } L^2(\mathbb{R}),$$

where $u_0 \in L^2 \cap L^\infty(\mathbb{R})$, then the corresponding entropy solutions satisfy

$$\lim_{k\to\infty} u_k(t_k) = u(t) \quad \text{in } L^2(\mathbb{R}).$$

The following corollary is an extension of the result to a pure L^2 setting which is proved at the end of Subsection 3.3.

Corollary 2.2 (Global L^2 well-posedness)**.** *Equation (1.1) is globally well-posed for $L^2(\mathbb{R})$ initial data in the following sense: The solution map $S\colon (t, u_0) \mapsto u(t)$ mapping $L^2 \cap L^\infty(\mathbb{R})$ initial data to the corresponding entropy solution evaluated at time $t \geq 0$, extends uniquely to a jointly continuous map $S\colon [0,\infty) \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$. In particular, the L^2-bound, -continuity and -stability of Theorem 2.1 carries over to all weak solutions provided by S. Moreover, for any $u_0 \in L^2(\mathbb{R})$, the corresponding weak solution $u(t,x) := S(t,u_0)(x)$ is locally bounded in $(0,\infty) \times \mathbb{R}$ and satisfies the entropy inequalities (1.4).*

For the remainder of the section, we broaden the definition of an entropy solution: for $u_0 \in L^2(\mathbb{R})$ we say that u is the corresponding entropy solution of (1.1) if, and only if, $u(t) = S(t,u_0)$, where S is as in the previous corollary.

The second theorem infers one-sided Hölder regularity for the entropy solutions. The Hölder exponent depends on the regularity of $K = G'$, here measured using the fractional variation $|K|_{TV^s}$ defined in (1.5). The theorem is proved in Section 4.

Theorem 2.3 (One-sided Hölder regularity)**.** *For initial data $u_0 \in L^2(\mathbb{R})$, let u be the corresponding entropy solution of* (1.1)*, and let $s \in [0,1]$ be such that $|K|_{TV^s} < \infty$. Then u satisfies the one-sided Hölder condition*

$$u(t,x) - u(t,y) \leq a(t)(x-y)^{\frac{1+s}{2}}, \tag{2.2}$$

for all $x \geq y$ and $t > 0$, where the Hölder coefficient $a(t)$ is given by

$$a(t) = C_1(s)|K|_{TV^s}^{\frac{2+s}{3+2s}} \|u_0\|_{L^2(\mathbb{R})}^{\frac{1+s}{3+2s}} + C_2(s)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{3}}}{t^{\frac{2+s}{3}}}, \tag{2.3}$$

for two constants $C_1(s)$ and $C_2(s)$ written out in (A.1)*.*

Since $u(t)$ is in $L^2(\mathbb{R})$ it is not necessarily true that $x \mapsto u(t,x)$ is well defined pointwise; in the previous theorem we have identified $u(t)$ with its left-continuous representation which exists due to Lemma A.1.

Since $K \in L^1(\mathbb{R})$ and $|K|_{TV^0} = 2\|K\|_{L^1(\mathbb{R})}$, the $s = 0$ case of Theorem 2.3 is valid for any instance of (1.1). In particular, entropy solutions of (1.1) are guaranteed to admit one-sided Hölder regularity of order $\frac{1}{2}$. In the case of the Burgers–Poisson equation, where $K = \frac{1}{2}\mathrm{sgn}(x)e^{-|x|}$ we find $|K|_{TV^1} = |K|_{TV} = 2$ and so by the $s = 1$ case of Theorem 2.3 we get the following corollary.

Corollary 2.4 (One-sided Lipschitz smoothing of Burgers–Poisson)**.** *For initial data $u_0 \in L^2(\mathbb{R})$, let u be the corresponding entropy solution of the Burgers–Poisson equation* (1.2)*. Then u satisfies the one-sided Lipschitz condition*

$$u(t,x) - u(t,y) \leq \left[12^{\frac{1}{5}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{2}{5}} + \frac{1}{t}\right](x-y),$$

for all $x \geq y$ and $t > 0$.

While it was already established in [9] that entropy solutions of the Burgers–Poisson equation are one-sided Lipschitz continuous, our result has the advantage of a Lipschitz coefficient that decreases with time.

We conclude this section with two less obvious corollaries of Theorem 2.3: a decaying height bound for entropy solutions of (1.1) and a maximal lifespan estimate for classical solutions.

Corollary 2.5 (Height bound)**.** *For initial data $u_0 \in L^2(\mathbb{R})$, let u be the corresponding entropy solution of* (1.1)*. Then for all $t > 0$ we have the height bound*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \left[2^{\frac{11}{12}}3^{\frac{1}{3}}\|K\|_{L^1(\mathbb{R})}^{\frac{1}{3}} + \frac{2^{\frac{5}{4}}}{t^{\frac{1}{3}}}\right]\|u_0\|_{L^2(\mathbb{R})}^{\frac{2}{3}}. \tag{2.4}$$

Proof. See Appendix B.

Observe that together, the two height bounds (2.1) and (2.4) imply that when $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ the corresponding entropy solution of (1.1) is globally bounded.

For the final result of the section, we need to introduce the following seminorm

$$[u_0]_s \coloneqq \operatorname*{ess\,sup}_{\substack{x\in\mathbb{R}\\ h>0}} \left[\frac{u_0(x-h) - u_0(x)}{h^{\frac{1+s}{2}}}\right], \tag{2.5}$$

which is a (left) one-sided Hölder seminorm of exponent $\frac{1+s}{2}$.

Corollary 2.6 (Maximal lifespan). *There are universal constants $C, c > 0$ such that: if initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ satisfies the skewness condition*

$$[u_0]_s^{3+2s} > c|K|_{TV^s}^{2+s} \|u_0\|_{L^2(\mathbb{R})}^{1+s}, \tag{2.6}$$

for some $s \in [0,1]$ such that $|K|_{TV^s} < \infty$, then the lifespan T of a classical solution $u \in L^\infty \cap C^1((0,T) \times \mathbb{R})$ of (1.1) admitting u_0 as initial data must satisfy

$$T < C\left[\frac{\|u_0\|_{L^2(\mathbb{R})}^{1-s}}{[u_0]_s^3}\right]^{\frac{1}{2+s}}. \tag{2.7}$$

Proof. See Appendix B.

3. Well posedness of entropy solutions

In this section, we provide for (1.1) a global well-posedness theory of entropy solutions as defined by Def. 1.1. In particular, the content of Theorem 2.1 follows from Proposition 3.1, Corollary 3.6 and Proposition 3.9; see the summary at the beginning of Subsection 3.3. Corollary 2.2 is also proved here at the end of Subsection 3.3. For entropy solutions of (1.1), the proofs of existence and uniqueness is the same for $L^2 \cap L^\infty$ data as for L^∞ data; only the L^1 setting allows for 'shortcuts'. Thus for generality, many results in the two coming subsections will be presented for initial data $u_0 \in L^\infty(\mathbb{R})$. We also note that in these two subsections only Lemma 3.3 exploits the dispersive nature of (1.1), that is, that $K = G'$ is odd.

3.1. Uniqueness of entropy solutions.

It is natural to start with the proof of uniqueness, as this equips us with a weighted L^1-contraction that can further be used in the existence proof. The involved weight $w_M^r(t,x)$ can be interpreted as a bound on the propagation of information for solutions of (1.1). Its technical role in the coming proof is to serve as a subsolution of a dual equation, namely the one obtained from setting the square bracket in (3.17) to zero. A similar method can be found in [1] where nonlocal conservation laws are treated.

The weight is constructed as follows. Writing $|K|$ to denote the function $x \mapsto |K(x)|$, we introduce for a parameter $t \geq 0$ the operator $e^{t|K|*}$ mapping $L^p(\mathbb{R})$ to itself for any $p \in [1,\infty]$, defined by

$$\left(e^{t|K|*}f\right)(x) = f(x) + \sum_{n=1}^{\infty}\left((|K|*)^n f\right)(x)\frac{t^n}{n!}, \tag{3.1}$$

where $(|K|*)^n$ represents the operation of convolving with $|K|$ repeatedly n times. Observe that by repeated use of Young's convolution inequality we have for any $p \in [1,\infty]$ and $f \in L^p(\mathbb{R})$

$$\|e^{t|K|*}f\|_{L^p(\mathbb{R})} \leq e^{t\kappa}\|f\|_{L^p(\mathbb{R})}, \tag{3.2}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$. For parameters $r, M \geq 0$, we further introduce

$$\chi_M^r(t,x) = \begin{cases} 1, & |x| < r + Mt, \\ 0, & \text{else}, \end{cases} \tag{3.3}$$

and set

$$w_M^r(t,x) = \left(e^{t|K|*}\chi_M^r(t,\cdot)\right)(x). \tag{3.4}$$

By (3.2), this weight satisfies for $p \in [1,\infty]$ the bound

$$\|w_M^r(t,\cdot)\|_{L^p(\mathbb{R})} \leq e^{t\kappa}(2r + 2Mt)^{\frac{1}{p}}, \tag{3.5}$$

where the case $p = \infty$ is evaluated in a limit sense. Thus, $w_M^r(t,\cdot) \in L^1 \cap L^\infty(\mathbb{R})$ for all $t, r, M \geq 0$. With w_M^r defined, we are ready to state Proposition 3.1 establishing the uniqueness of entropy solutions. Although the following result is stated to hold for a.e. $t \geq 0$, it can be extended to all $t \geq 0$, as we later prove that entropy solutions of (1.1) are continuous when viewed as $L^1_{\mathrm{loc}}(\mathbb{R})$-valued time-dependent functions.

Proposition 3.1. *Let $u, v \in L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$ be entropy solutions of (1.1) with $u_0, v_0 \in L^\infty(\mathbb{R})$ as initial data. Then, for any $r > 0$ and a.e. $t \geq 0$ we have the weighted L^1-contraction*

$$\int_{-r}^{r} |u(t,x) - v(t,x)| dx \leq \int_{-\infty}^{\infty} |u_0(x) - v_0(x)| w_M^r(t,x) dx, \tag{3.6}$$

where w_M^r is given by (3.4), and M is any parameter satisfying

$$M \geq \frac{\|u\|_{L^\infty([0,t]\times\mathbb{R})} + \|v\|_{L^\infty([0,t]\times\mathbb{R})}}{2}. \tag{3.7}$$

Thus, there is at most one entropy solution of (1.1) for each $u_0 \in L^\infty(\mathbb{R})$.

Proof. We begin by reformulating (1.4) in terms of the Kružkov entropies; parameterized over $k \in \mathbb{R}$, they are given by $(\eta_k(u), q_k(u)) = (|u-k|, F(u,k))$ where

$$F(u,k) \coloneqq \tfrac{1}{2}\operatorname{sgn}(u-k)(u^2 - k^2).$$

These entropy pairs lack the required smoothness, but are still applicable in (1.4) as they can be smoothly approximated. Indeed, consider for $\delta > 0$ and $k \in \mathbb{R}$ the entropy pairs $\eta_k^\delta(u) = \sqrt{(u-k)^2 + \delta^2}$ and $q_k^\delta(u) = \int_k^u (\eta_k^\delta)'(y) y \mathrm{d}y$. As we have the pointwise limits

$$\lim_{\delta\to 0} \eta_k^\delta(u) = |u-k|, \qquad \lim_{\delta\to 0} q_k^\delta(u) = F(u,k), \qquad \lim_{\delta\to 0} (\eta_k^\delta)'(u) = \operatorname{sgn}(u-k),$$

we can substitute $(\eta, q) \mapsto (\eta_k^\delta, q_k^\delta)$ in (1.4) and let $\delta \to 0$ to conclude through dominated convergence that u satisfies

$$0 \leq \int_0^\infty \int_{\mathbb{R}} |u-k|\varphi_t + F(u,k)\varphi_x + \operatorname{sgn}(u-k)(K * u)\varphi \mathrm{d}x\mathrm{d}t, \tag{3.8}$$

for all $k \in \mathbb{R}$ and all non-negative $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$. For brevity, we set $U = \mathbb{R}^+ \times \mathbb{R}$ for use throughout the proof. Let $\psi \in C_c^\infty(U \times U)$ be non-negative, and consider u and v as functions in (t,x) and (s,y) respectively. For fixed $(s,y) \in U$, we can in (3.8) insert the test-function $\varphi\colon (t,x) \mapsto \psi(t,x,s,y)$ and the constant $k = v(s,y)$ so to obtain

$$0 \leq \int_U |u-v|\psi_t + F(u,v)\psi_x + \operatorname{sgn}(u-v)(K *_x u)\psi \mathrm{d}x\mathrm{d}t, \tag{3.9}$$

where we write $K *_x u$ to stress that the operator $K*$ is applied with respect to the x-variable. As (3.9) holds for all $(s,y) \in U$ we can integrate (3.9) over $(s,y) \in U$ giving

$$0 \leq \int_U \int_U |u-v|\psi_t + F(u,v)\psi_x + \operatorname{sgn}(u-v)(K *_x u)\psi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \tag{3.10}$$

Swapping the roles of $u(t,x)$ and $v(s,y)$ we similarly find

$$0 \leq \int_U \int_U |u-v|\psi_s + F(v,u)\psi_y + \operatorname{sgn}(v-u)(K *_y v)\psi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \tag{3.11}$$

As $F(u,v) = F(v,u)$ and $\mathrm{sgn}(v-u) = -\mathrm{sgn}(u-v)$ we can add (3.10) to (3.11) so to further obtain

$$\begin{aligned} 0 \leq & \int_U \int_U |u-v|(\psi_t + \psi_s) + F(u,v)(\psi_x + \psi_y) \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ & + \int_U \int_U \mathrm{sgn}(u-v)(K *_x u - K *_y v)\psi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.12}$$

Let $\rho \in C_c^\infty(\mathbb{R}^2)$ be non-negative and satisfy $\|\rho\|_{L^1(\mathbb{R}^2)} = 1$, and let ρ_ε denote the expression

$$\rho_\varepsilon = \rho_\varepsilon(t-s, x-y) = \frac{1}{\varepsilon^2}\rho\left(\frac{t-s}{\varepsilon}, \frac{x-y}{\varepsilon}\right),$$

for $\varepsilon > 0$. For a fixed $T \in (0,\infty)$, we further let φ denote a non-negative element of $C_c^\infty((0,T)\times\mathbb{R})$ and set

$$\psi(t,x,s,y) = \varphi(t,x)\rho_\varepsilon(t-s,x-y),$$

or simply $\psi = \varphi\rho_\varepsilon$ for short. Note that, for $\varepsilon > 0$ sufficiently small, this ψ is non-negative, smooth and of compact support in $U \times U$; in particular, it satisfies the prior assumptions posed on it. Using that $(\partial_t + \partial_s)\rho_\varepsilon = 0 = (\partial_x + \partial_y)\rho_\varepsilon$, we can conclude

$$(\psi_t + \psi_s) = \varphi_t \rho_\varepsilon, \qquad\qquad (\psi_x + \psi_y) = \varphi_x \rho_\varepsilon,$$

and so inserting for ψ in (3.12) we get

$$\begin{aligned} 0 \leq & \int_U \int_U \Big[|u-v|\varphi_t + F(u,v)\varphi_x\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ & + \int_U \int_U \mathrm{sgn}(u-v)(K *_x u - K *_y v)\varphi\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.13}$$

We now wish to 'go to the diagonal' by taking $\limsup_{\varepsilon\to 0}$ of (3.13); for simplicity we study each line separately. For the first one we pick $M \in (0,\infty)$ satisfying the inequality (3.7) with T replacing t, and use $(u^2 - v^2) = (u+v)(u-v)$ to compute

$$\begin{aligned} & \int_U \int_U \Big[|u-v|\varphi_t + F(u,v)\varphi_x\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ \leq & \int_U \int_U |u-v|\Big[\varphi_t + M|\varphi_x|\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ \leq & \int_U |u(t,x) - v(t,x)|\Big[\varphi_t + M|\varphi_x|\Big]\mathrm{d}x\mathrm{d}t \\ & + \int_U \int_U |v(t,x) - v(s,y)|\Big[\varphi_t + M|\varphi_x|\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.14}$$

As $\rho_\varepsilon(t-s,x-y)$ is supported in the region $|(t-s,x-y)| \leq \varepsilon$ and satisfies $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^2)} = 1$, the very last integral in (3.14) is bounded by

$$\sup_{|(\epsilon,\delta)|\leq\varepsilon} \int_U |v(t,x) - v(t+\epsilon, x+\delta)|\Big[\varphi_t + M|\varphi_x|\Big]\mathrm{d}x\mathrm{d}t \to 0, \quad \varepsilon \to 0,$$

where the limit holds as translation is a continuous operation on $L^1_{\mathrm{loc}}(\mathbb{R})$ and $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$. Thus we have established

$$\begin{aligned} & \limsup_{\varepsilon\to 0} \int_U \int_U \Big[|u-v|\varphi_t + F(u,v)\varphi_x\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ \leq & \int_U |u(t,x) - v(t,x)|\Big[\varphi_t + M|\varphi_x|\Big]\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.15}$$

Turning our attention to the second line of (3.13), we start by observing

$$\begin{aligned}
&\int_U\int_U \operatorname{sgn}(u-v)(K *_x u - K *_y v)\varphi\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\
\le &\int_U\int_U\int_{\mathbb{R}} |K(z)||u(t,x-z)-v(s,y-z)|\varphi(t,x)\rho_\varepsilon(t-s,x-y)\mathrm{d}z\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\
= &\int_U\int_U\int_{\mathbb{R}} |K(z)||u(t,x)-v(s,y)|\varphi(t,x+z)\rho_\varepsilon(t-s,x-y)\mathrm{d}z\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\
= &\int_U\int_U |u-v|\Big[|K| *_x \varphi\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

where the third line holds by the substitution $(x,y)\mapsto(x+z,y+z)$ and the last by the symmetry of $z\mapsto|K(z)|$. By similar reasoning used to attain (3.14), we conclude

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_U\int_U \operatorname{sgn}(u-v)(K *_x u - K *_y v)\rho_\varepsilon\varphi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\
&\le \int_U |u(t,x)-v(t,x)|(|K|*\varphi)\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{3.16}$$

Combining (3.13) with (3.15) and (3.16), yields the inequality

$$0 \le \int_U |u-v|\Big[\varphi_t + M|\varphi_x| + |K|*\varphi\Big]\mathrm{d}x\mathrm{d}t, \tag{3.17}$$

where both u and v are now functions in (t,x). By density, we may extend (3.17) to hold for all non-negative $\varphi\in W_0^{1,1}((0,T)\times\mathbb{R})$. Thus, we can set $\varphi(t,x)=\theta(t)\phi(t,x)$ for two non-negative functions $\theta\in W_0^{1,1}((0,T))$ and $\phi\in W^{1,1}((0,T)\times\mathbb{R})$ where we note that ϕ need not vanish at $t=0$ and $t=T$. In doing so, (3.17) yields

$$0 \le \int_U |u-v|\theta'\phi\mathrm{d}x\mathrm{d}t + \int_U |u-v|\theta\Big[\phi_t + M|\phi_x| + |K|*\phi\Big]\mathrm{d}x\mathrm{d}t, \tag{3.18}$$

To rid ourselves of the second integral, we now construct a particular ϕ such that the square bracket in (3.18) is non-positive in $(0,T)\times\mathbb{R}$. Let $f\colon\mathbb{R}\to[0,1]$ be smooth, non-increasing and satisfy $f(x)=1$ for $x\le 0$ and $f(x)=0$ for sufficiently large x, and define

$$g(t,x) = f(|x| + M(t-T)). \tag{3.19}$$

It is readily checked that $g\in C_c^\infty([0,T]\times\mathbb{R})$. We now define the function ϕ to be

$$\phi(t,x) = \Big(e^{(T-t)|K|*}g(t,\cdot)\Big)(x), \tag{3.20}$$

where we used the operator defined in (3.1). Observe that ϕ is non-negative and smooth on $[0,T]\times\mathbb{R}$ with integrable derivatives; this last part follows when using (3.2). That the square bracket in (3.18) is non-positive, can be seen as follows: note first from (3.19) that

$$\begin{aligned}
g_t(t,x) &= Mf'(|x|+M(t-T)), \\
g_x(t,x) &= \operatorname{sgn}(x)f'(|x|+M(t-T)).
\end{aligned}$$

As f' is non-positive, we find $g_t=-M|g_x|$. Thus, using (3.20) we calculate for $t\in(0,T)$

$$\begin{aligned}
\phi_t + |K|*\phi &= e^{(T-t)|K|*}g_t, \\
&= -M\Big(e^{(T-t)|K|*}|g_x|\Big), \\
&\le -M\Big|e^{(T-t)|K|*}g_x\Big| \\
&= -M|\phi_x|,
\end{aligned}$$

where the last equality holds as differentiation commutes with convolution. In conclusion, the second integral in (3.18) is non-positive. Next, for a small parameter $\epsilon > 0$ we set $\theta = \theta_\epsilon$ where θ_ϵ is given by

$$\theta_\epsilon(t) = \begin{cases} t/\epsilon, & t \in (0,\epsilon), \\ 1, & t \in (\epsilon, T-\epsilon), \\ (T-t)/\epsilon, & t \in (T-\epsilon, T). \end{cases} \tag{3.21}$$

Inserting this in (3.18), removing the non-positive integral and letting $\epsilon \to 0$, we conclude

$$\begin{aligned} &\liminf_{\epsilon\to 0} \int_{T-\epsilon}^{T} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|\phi(t,x)\mathrm{d}x \right) \frac{\mathrm{d}t}{\epsilon} \\ \leq\ &\limsup_{\epsilon\to 0} \int_{0}^{\epsilon} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|\phi(t,x)\mathrm{d}x \right) \frac{\mathrm{d}t}{\epsilon} \end{aligned} \tag{3.22}$$

where we moved the negative term over to the left-hand side. As u and v are bounded on $(0,T)\times\mathbb{R}$ and continuous at $t=0$ in L^1_{loc} sense, it is easy to see that $|u(t,\cdot)-v(t,\cdot)|\phi(t,\cdot) \to |u_0(\cdot) - v_0(\cdot)|\phi(0,\cdot)$ in $L^1(\mathbb{R})$ when $t \to 0$ since the same is true for $\phi(t,x)$ and $\phi(0,x)$. Thus the right-hand side of (3.22) is given by

$$\limsup_{\epsilon\to 0} \int_{0}^{\epsilon} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|\phi(t,x)\mathrm{d}x \right) \frac{\mathrm{d}t}{\epsilon} = \int_{\mathbb{R}} |u_0 - v_0|\phi(0,x)\mathrm{d}x.$$

As for the left-hand side, we wish to apply the Lebesgue differentiation theorem so to get convergence for a.e. $T > 0$, but this can not be directly done due to the implicit T-dependence of ϕ. Instead, we observe from (3.19) and (3.20) that $\phi(T,x) = g(T,x) = f(|x|)$ where the latter function is independent of T. Since $\varphi(t,\cdot) \to f(|\cdot|)$ in $L^1(\mathbb{R})$ as $t \to T$, the boundness of u and v means that $|u(t,\cdot) - v(t,\cdot)|(\varphi(t,\cdot) - f(|\cdot|)) \to 0$ in $L^1(\mathbb{R})$ as $t \to T$ and so we may estimate

$$\begin{aligned} &\limsup_{\epsilon\to 0} \int_{T-\epsilon}^{T} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|\phi(t,x)\mathrm{d}x \right) \frac{\mathrm{d}t}{\epsilon} \\ =\ &\limsup_{\epsilon\to 0} \int_{T-\epsilon}^{T} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|f(|x|)\mathrm{d}x \right) \frac{\mathrm{d}t}{\epsilon} \\ =\ &\int_{\mathbb{R}} |u(T,x) - v(T,x)|f(|x|)\mathrm{d}x, \quad \text{a.e. } T \geq 0, \end{aligned}$$

where the last equality used the Lebesgue differentiation theorem. Thus we conclude from (3.22) that we for a.e. $T \geq 0$ have

$$\begin{aligned} &\int_{\mathbb{R}} |u(T,x) - v(T,x)|f(|x|)\mathrm{d}x \\ \leq\ &\int_{\mathbb{R}} |u_0(x) - v_0(x)| \Big(e^{T|K|*} f(|\cdot| - MT) \Big)(x)\mathrm{d}x, \end{aligned} \tag{3.23}$$

where we inserted for $\phi(0,x)$ using (3.19) and (3.20). As f was any smooth, non-negative, non-increasing function satisfying $f(x) = 1$ for $x \leq 0$ and $f(x) = 0$ for sufficiently large x, we may in (3.23) set $f = \mathbb{1}_{(-\infty,r)}$ through a standard approximation argument. Doing this, we observe that $f(|x| - MT) = \chi^r_M(T,x)$ where the latter is defined in (3.3), and so we obtain from (3.23) exactly (3.6), with T substituting for t. This concludes the proof. $\square$

While we in this paper are concerned with global entropy solutions, one may wish to study entropy solutions on a time-bounded domain $(0,T)\times\mathbb{R}$. Such solutions would be defined as in Def. 1.1, but with the test-functions in (1.4) restricted to $C_c^\infty((0,T)\times\mathbb{R})$.

Still, no new solutions are attained this way: the uniqueness of entropy solutions on a time-bounded domain follows from the same argument as above, and thus an entropy solution on $(0,T)\times\mathbb{R}$ is the restriction of a global one which the following section establishes the existence of.

3.2. **Existence of entropy solutions.** In this subsection, we prove the existence of an entropy solution of (1.1) for arbitrary initial data $u_0 \in L^\infty(\mathbb{R})$. The strategy goes as follows: we first introduce for a parameter $\varepsilon > 0$ an approximate solution map $S_{\varepsilon,t}\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ whose key properties are collected in Proposition 3.2. Next, we show in Lemma 3.4 that when $S_{\varepsilon,t}$ is applied to sufficiently regular initial data u_0, we attain approximate entropy solutions. Further, in Proposition 3.5 we establish the convergence (as $\varepsilon \to 0$) of these approximations to an entropy solution, and the result is extended to general L^∞ data in Corollary 3.6.

By an operator splitting argument, we aim to build entropy solutions of (1.1) from those of Burgers' equation, $u_t + \frac{1}{2}(u^2)_x = 0$, and the linear convolution equation, $u_t = K * u$. On that note, we introduce two families of operators $(S_t^B)_{t\geq 0}$ and $(S_t^K)_{t\geq 0}$ parameterized over $t \geq 0$. The operator $S_t^B\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is the solution map for Burgers' equation restricted to L^∞ data at time t; that is,

$$S_t^B\colon f \mapsto u^f(t,\cdot), \tag{3.24}$$

where $(t,x) \mapsto u^f(t,x)$ is the unique bounded entropy solution for the problem

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R}. \end{cases}$$

As demonstrated in [6], this solution lies in $C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$, the space of functions $u \in L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$ such that $t \mapsto u(t,\cdot)$ is continuous from $[0,\infty)$ to $L^1_{\mathrm{loc}}(\mathbb{R})$. Note that S_t^B is a flow map in the sense that $S_{t_1}^B \circ S_{t_2}^B = S_{t_1+t_2}^B$ for all $t_1, t_2 \geq 0$. The second map $S_t^K\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is for $t \geq 0$ defined by

$$S_t^K\colon f \mapsto f + tK * f. \tag{3.25}$$

The actual solution map for the equation $u_t = K * u$ is the operator e^{tK*} defined similarly to (3.1); the reason we have instead chosen S_t^K as (3.25) (which can be seen as a first order approximation of e^{tK*}) is for our calculations to be slightly tidier. Note however, S_t^K is not a flow mapping. With these two families of operators, we build a third family of operators $S_{\varepsilon,t}$: for fixed parameters $\varepsilon > 0$ and $t \geq 0$, pick the unique pair $n \in \mathbb{N}_0$ and $s \in [0,\varepsilon)$ such that $t = s + n\varepsilon$, and define

$$S_{\varepsilon,t} = S_s^B \circ \left[S_\varepsilon^K \circ S_\varepsilon^B\right]^{\circ n}, \tag{3.26}$$

where the notation $\circ n$ implies that the square bracket is composed with itself $(n-1)$ times; if $n = 0$, then the square bracket should be replaced by the identity. We shall demonstrate that as $\varepsilon \to 0$ the map $S_{\varepsilon,t}$ converges in an appropriate sense to the solution map for entropy solutions of (1.1). We begin by collecting a few properties of $S_{\varepsilon,t}$ when applied to the space $BV(\mathbb{R})$; this subspace of $L^1(\mathbb{R})$ is equipped with the norm $\|\cdot\|_{BV(\mathbb{R})} = \|\cdot\|_{L^1(\mathbb{R})} + |\cdot|_{TV}$, where the total variation seminorm $|\cdot|_{TV}$ coincides with $|\cdot|_{TV^1}$ as defined in (1.5). A short and effective discussion of $BV(\mathbb{R})$ can be found in either [6] or [11]; we note that functions in $BV(\mathbb{R})$ have essential right and left limits at each point, and their height is bounded by their total variation, thus $BV(\mathbb{R}) \hookrightarrow L^1 \cap L^\infty(\mathbb{R})$.

Proposition 3.2. *With $S_{\varepsilon,t}$ as defined in (3.26), we have for all $\varepsilon > 0$, $t \geq \tilde{t} \geq 0$, $f \in BV(\mathbb{R})$ and $p \in [1,\infty]$*

$$\begin{aligned} \|S_{\varepsilon,t}(f)\|_{L^p(\mathbb{R})} &\leq e^{t\kappa}\|f\|_{L^p(\mathbb{R})}, && (L^p \text{ bound}),\\ \|S_{\varepsilon,t}(f)\|_{TV} &\leq e^{t\kappa}\|f\|_{TV}, && (TV \text{ bound}),\\ \|S_{\varepsilon,t}(f) - S_{\varepsilon,\tilde{t}}(f)\|_{L^1(\mathbb{R})} &\leq (t-\tilde{t}+\varepsilon)C_f(t), && (\text{Approximate time continuity}), \end{aligned}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$ and where the factor $C_f(t)$ only depends on f and t.

Proof. Consider $\varepsilon > 0$ fixed. We will be using the following properties of the mappings S_t^B and S_t^K

$$\|S_t^B(f)\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}, \qquad \|S_t^K(f)\|_{L^p(\mathbb{R})} \leq e^{t\kappa}\|f\|_{L^p(\mathbb{R})}, \tag{3.27}$$

$$|S_t^B(f)|_{TV} \leq |f|_{TV}, \qquad |S_t^K(f)|_{TV} \leq e^{t\kappa}|f|_{TV}, \tag{3.28}$$

$$\|S_t^B(f) - f\|_{L^1(\mathbb{R})} \leq t|f|_{TV}^2, \qquad \|S_t^K(f) - f\|_{L^1(\mathbb{R})} \leq t\kappa\|f\|_{L^1(\mathbb{R})}, \tag{3.29}$$

valid for all $t \geq 0$, $p \in [1,\infty]$ and $f \in BV(\mathbb{R})$. The inequalities involving S_t^B are well known and can be found for example in [11]. As for the inequalities involving S_t^K, these estimates follow directly from the definition of S_t^K (3.25) together with Young's convolution inequality and $1 + t\kappa \leq e^{t\kappa}$. We start by proving the L^p and TV bound of the proposition. For this we fix $t \geq 0$ and pick $n \in \mathbb{N}_0$ and $s \in [0,\varepsilon)$ such that $t = s + n\varepsilon$, and pick an arbitrary $f \in BV(\mathbb{R})$. By iteration of the two inequalities in (3.27) we attain

$$\|S_{\varepsilon,t}(f)\|_{L^p(\mathbb{R})} = \|S_s^B \circ [S_\varepsilon^K \circ S_\varepsilon^B]^{\circ n}(f)\|_{L^p(\mathbb{R})} \leq e^{n\varepsilon\kappa}\|f\|_{L^p(\mathbb{R})}, \tag{3.30}$$

for all $p \in [1,\infty]$, and by iteration of the inequalities in (3.28) we similarly get

$$|S_{\varepsilon,t}(f)|_{TV} = |S_s^B \circ [S_\varepsilon^K \circ S_\varepsilon^B]^{\circ n}(f)|_{TV} \leq e^{n\varepsilon\kappa}|f|_{TV}. \tag{3.31}$$

This gives the first two bounds of the proposition. For the time continuity, we pick $\tilde{t} \in [0,t]$ and $\tilde{n} \in \mathbb{N}$ and $\tilde{s} \in [0,\varepsilon)$ such that $\tilde{t} = \tilde{s} + \tilde{n}\varepsilon$. Suppose first that $t - \tilde{t} \leq \varepsilon$, and set $\tilde{f} = S_{\varepsilon,\tilde{n}\varepsilon}(f)$. Then either $S_{\varepsilon,t}(f) = S_{s-\tilde{s}}^B(\tilde{f})$ or $S_{\varepsilon,t}(f) = S_s^B \circ S_\varepsilon^K \circ S_{\varepsilon-\tilde{s}}^B(\tilde{f})$ corresponding to the two situations $n = \tilde{n}$ and $n = \tilde{n}+1$; we will only deal with the latter as the other case is dealt with similarly. By the triangle inequality we then have

$$\begin{aligned} \|S_{\varepsilon,t}(f) - S_{\varepsilon,\tilde{t}}(f)\|_{L^1(\mathbb{R})} \leq{}& \|S_s^B \circ S_\varepsilon^K \circ S_{\varepsilon-\tilde{s}}^B(\tilde{f}) - S_\varepsilon^K \circ S_{\varepsilon-\tilde{s}}^B(\tilde{f})\|_{L^1(\mathbb{R})}\\ &+ \|S_\varepsilon^K \circ S_{\varepsilon-\tilde{s}}^B(\tilde{f}) - S_{\varepsilon-\tilde{s}}^B(\tilde{f})\|_{L^1(\mathbb{R})} + \|S_{\varepsilon-\tilde{s}}^B(\tilde{f}) - \tilde{f}\|_{L^1(\mathbb{R})}. \end{aligned}$$

The three terms on the right-hand side can be directly dealt with using the two inequalities (3.29) followed by the estimates (3.30) and (3.31). Doing so in a straight forward manner results in the bound

$$se^{2n\varepsilon\kappa}|f|_{TV}^2 + \varepsilon\kappa e^{\tilde{n}\varepsilon\kappa}\|f\|_{L^1(\mathbb{R})} + (\varepsilon - \tilde{s})e^{2\tilde{n}\varepsilon\kappa}|f|_{TV}^2 \leq \varepsilon e^{2t\kappa}(2|f|_{TV}^2 + \kappa\|f\|_{L^1(\mathbb{R})}).$$

Thus, setting for example $C_f(t) = e^{2t\kappa}(2|f|_{TV}^2 + \kappa\|f\|_{L^1(\mathbb{R})})$ the time continuity estimate holds whenever $t - \tilde{t} \leq \varepsilon$. By breaking any large time step into steps of size no larger than ε, the general case follows by the triangle inequality. $\square$

The L^p bound provided by the previous proposition was attained by applying Young's convolution inequality on the operator $K*$; in doing so, we miss possible cancellations that might take place as K, after all, is an odd function. While efficient L^p bounds might not be feasible for general $p \geq 1$, these cancellations are easily exploited for the L^2 norm as seen from the following lemma. This L^2 control is crucial for the analysis of Section 4.

Lemma 3.3. *With $S_{\varepsilon,t}$ as defined in (3.26), we have for all $\varepsilon > 0$, $t \geq 0$ and $f \in L^2 \cap L^\infty(\mathbb{R})$*

$$\|S_{\varepsilon,t}(f)\|_{L^2(\mathbb{R})} \leq e^{\frac{1}{2}\varepsilon t\kappa^2}\|f\|_{L^2(\mathbb{R})},$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$.

Proof. Consider $\varepsilon > 0$ and $t \geq 0$ fixed. As K is odd, real valued and in $L^1(\mathbb{R})$, it is readily checked that $K*$ is a skew-symmetric operator on $L^2(\mathbb{R})$, and consequently $\langle f, K * f\rangle = 0$ for all $f \in L^2(\mathbb{R})$. In particular,

$$\|S_\varepsilon^K(f)\|^2_{L^2(\mathbb{R})} = \langle f, f\rangle + \varepsilon^2\langle K * f, K * f\rangle \leq (1 + \varepsilon^2\kappa^2)\|f\|^2_{L^2(\mathbb{R})}.$$

Combined with $1 + \varepsilon^2\kappa^2 \leq e^{\varepsilon^2\kappa^2}$ and the fact that $\|S_\varepsilon^B(f)\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}$ (left-most inequality in (3.27)), the result follows by iteration. □

When $u_0 \in BV(\mathbb{R})$, we can use $S_{\varepsilon,t}$ to construct a family of approximate entropy solutions of (1.1) as follows. For an arbitrary, but fixed, $u_0 \in BV(\mathbb{R})$, let the family $(u^\varepsilon)_{\varepsilon>0} \subset L^\infty_{\text{loc}}([0,\infty), L^\infty(\mathbb{R}))$ be defined by

$$u^\varepsilon(t) = S_{\varepsilon,t}(u_0), \tag{3.32}$$

where $u^\varepsilon(t)$ is compact notation for $x \mapsto u^\varepsilon(t,x)$. Although $(u^\varepsilon)_{\varepsilon>0}$ is considered a family in $L^\infty_{\text{loc}}([0,\infty), L^\infty(\mathbb{R}))$, we stress that each member is for all $t \geq 0$ well defined in $L^\infty(\mathbb{R})$. For small $\varepsilon > 0$ these functions are not far off from satisfying the entropy inequality (1.4), as we now show.

Lemma 3.4. *With $(u^\varepsilon)_{\varepsilon>0}$ as defined in (3.32) for some $u_0 \in BV(\mathbb{R})$, we have for every entropy pair (η, q) of (1.1) and non-negative $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ the approximate entropy inequality*

$$\int_0^\infty \int_{\mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x + \eta'(u^\varepsilon)(K * u^\varepsilon)\varphi \mathrm{d}x\mathrm{d}t \geq O(\varepsilon).$$

Proof. Fixing $\varepsilon > 0$, we observe from the definition of $S_{\varepsilon,t}$ (3.26) that u^ε is an entropy solution of Burgers' equation on the open sets $(t^\varepsilon_{n-1}, t^\varepsilon_n) \times \mathbb{R}$ for $n \in \mathbb{N}$, where $t^\varepsilon_n = n\varepsilon$; thus

$$\int_{t^\varepsilon_{n-1}}^{t^\varepsilon_n} \int_{\mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x \mathrm{d}x\mathrm{d}t \geq 0, \tag{3.33}$$

for every non-negative $\varphi \in C_c^\infty((t^\varepsilon_{n-1}, t^\varepsilon_n) \times \mathbb{R})$ and every entropy pair (η, q) of Burgers' equation, which coincides with the entropy pairs of (1.1) as the convection term of the two equations agree. Moreover, by the time continuity of S_t^B (3.28) and the TV bound from Proposition 3.2, we see that $u^\varepsilon \in C([t^\varepsilon_{n-1}, t^\varepsilon_n), L^1_{\text{loc}}(\mathbb{R}))$; at $t = t^\varepsilon_n$ it is discontinuous from the left, as the left limit is given by $u^\varepsilon(t^\varepsilon_n-) = S_\varepsilon^B(u^\varepsilon(t^\varepsilon_{n-1}))$, while we have defined

$$u^\varepsilon(t^\varepsilon_n) = u^\varepsilon(t^\varepsilon_n-) + \varepsilon K * u^\varepsilon(t^\varepsilon_n-). \tag{3.34}$$

The continuity in time allows us, by a similar trick used to attain (3.22), to extend (3.33) to

$$\begin{aligned}\int_{t^\varepsilon_{n-1}}^{t^\varepsilon_n} \int_{\mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x \mathrm{d}x\mathrm{d}t \geq{}& \int_{\mathbb{R}} \eta(u^\varepsilon(t^\varepsilon_n-))\varphi(t^\varepsilon_n, x)\mathrm{d}x \\ &- \int_{\mathbb{R}} \eta(u^\varepsilon(t^\varepsilon_{n-1}))\varphi(t^\varepsilon_{n-1}, x)\mathrm{d}x,\end{aligned} \tag{3.35}$$

for all non-negative $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$. For the remainder of the proof, consider the entropy pair (η, q) and $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ fixed. Summing (3.35) over $n \in \mathbb{N}$ and using $\varphi(0,x) = 0$, we get

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}} &\eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x \mathrm{d}x\mathrm{d}t \\ &\geq \sum_{n=1}^{\infty} \int_{\mathbb{R}} \Big[\eta(u^\varepsilon(t_n^\varepsilon -)) - \eta(u^\varepsilon(t_n^\varepsilon))\Big]\varphi(t_n^\varepsilon, x)\mathrm{d}x.\end{aligned} \tag{3.36}$$

By Proposition 3.2, the family $(u^\varepsilon)_{\varepsilon>0}$ is uniformly bounded on the support of φ, and so we can assume without loss of generality that $|\eta'|, |\eta''| < C_1$ for some large C_1. Using the relation (3.34), the square bracket from (3.36) can thus be estimated

$$\begin{aligned}&\eta(u^\varepsilon(t_n^\varepsilon -)) - \eta(u^\varepsilon(t_n^\varepsilon)) \\ &\geq -\varepsilon\eta'(u^\varepsilon(t_n^\varepsilon -))\Big[K * u^\varepsilon(t_n^\varepsilon -)\Big] - \frac{C_1\varepsilon^2}{2}|K * u^\varepsilon(t_n^\varepsilon -)|^2,\end{aligned}$$

which, again by the uniform bound of u^ε on the compact support of φ, further implies

$$\begin{aligned}\int_{\mathbb{R}} &\Big[\eta(u^\varepsilon(t_n^\varepsilon -)) - \eta(u^\varepsilon(t_n^\varepsilon))\Big]\varphi(t_n^\varepsilon, x)\mathrm{d}x \\ &\geq -\varepsilon\int_{\mathbb{R}} \eta'(u^\varepsilon(t_n^\varepsilon -))\Big[K * u^\varepsilon(t_n^\varepsilon -)\Big]\varphi(t_n, x)\mathrm{d}x - C_2\varepsilon^2,\end{aligned} \tag{3.37}$$

for some $C_2 > 0$ independent of n and ε. Combining the uniform time regularity of Proposition 3.2 and the compact support of φ, we see that the function

$$g_\varepsilon(t) := \int_{\mathbb{R}} \eta'(u^\varepsilon(t))\Big[K * u^\varepsilon(t)\Big]\varphi(t,x)\mathrm{d}x, \tag{3.38}$$

satisfies for all $t \geq \tilde{t} \geq 0$ an inequality $|g_\varepsilon(t) - g_\varepsilon(\tilde{t})| \leq C_3(t - \tilde{t} + \varepsilon)$ for some sufficiently large C_3 independent of ε. Thus, the integral on the right-hand side of (3.37) can be bounded from below as such

$$\begin{aligned}&-\varepsilon\int_{\mathbb{R}} \eta'(u^\varepsilon(t_n^\varepsilon -))\Big[K * u^\varepsilon(t_n^\varepsilon -)\Big]\varphi(t_n, x)\mathrm{d}x \\ &= -\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon}\int_{\mathbb{R}} \eta'(u^\varepsilon(t_n^\varepsilon -))\Big[K * u^\varepsilon(t_n^\varepsilon -)\Big]\varphi(t_n, x)\mathrm{d}x\mathrm{d}t \\ &\geq -\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon}\int_{\mathbb{R}} \eta'(u^\varepsilon(t))\Big[K * u^\varepsilon(t)\Big]\varphi(t, x)\mathrm{d}x\mathrm{d}t - 2C_3\varepsilon^2.\end{aligned} \tag{3.39}$$

Picking the smallest $N(\varepsilon) \in \mathbb{N}$ such that $\operatorname{supp}\varphi \cap (\varepsilon N(\varepsilon), \infty) \times \mathbb{R} = \emptyset$, we combine (3.36), (3.37) and (3.39) to deduce

$$\int_{\mathbb{R}^+\times\mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x + \eta'(u^\varepsilon)(K * u^\varepsilon)\varphi \mathrm{d}x\mathrm{d}t \geq CN(\varepsilon)\varepsilon^2,$$

for some sufficiently large $C > 0$. And as $N(\varepsilon)\varepsilon^2 \sim \varepsilon$ the proof is complete. □

With the previous result at hand, it is natural to look for a limit function of $(u^\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$; this would be a suitable candidate for an entropy solution of (1.1) with initial data $u_0 \in BV(\mathbb{R})$. In the next proposition, we do exactly this and collect a few properties about the resulting solution.

Proposition 3.5. *For any initial data $u_0 \in BV(\mathbb{R})$, let $(u^\varepsilon)_{\varepsilon>0}$ be as defined in (3.32). Then, for all $t \geq 0$ the following limit holds in $L^1_{\mathrm{loc}}(\mathbb{R})$*

$$u^\varepsilon(t) \to u(t), \quad \varepsilon \to 0, \tag{3.40}$$

where u is an entropy solution of (1.1) with initial data u_0. Moreover, u is an element of $C([0,\infty), L^1(\mathbb{R})) \cap L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$ and satisfies for all $t \geq 0$

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^\infty(\mathbb{R})}, \tag{3.41}$$

$$\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \tag{3.42}$$

where $\kappa \coloneqq \|K\|_{L^1(\mathbb{R})}$.

Proof. We first prove the limit (3.40) for a special subsequence of $(u^\varepsilon)_{\varepsilon>0}$ and then generalize afterwards. Fixing $t \geq 0$, we see from Proposition 3.2 that the functions $(u^\varepsilon(t))_{\varepsilon>0}$ satisfy for any $p \in [1,\infty]$

$$\|u^\varepsilon(t)\|_{L^p(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^p(\mathbb{R})}, \tag{3.43}$$

and in particular, they are uniformly bounded in $L^1(\mathbb{R})$. Moreover, they are equicontinuous with respect to translation

$$\|u^\varepsilon(t, \cdot + h) - u^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leq h e^{t\kappa}|u_0|_{TV},$$

for all $h > 0$, and so by the Kolmogorov–Riesz compactness Theorem, any infinite subset of $(u^\varepsilon(t))_{\varepsilon>0}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R})$; as we have skipped developing a tightness estimate for $(u^\varepsilon(t))_{\varepsilon>0}$, we can not claim the family to be relatively compact in $L^1(\mathbb{R})$. The family $(u^\varepsilon)_{\varepsilon>0}$ is not equicontinuous in time and so we can not directly apply the Arzelà-Ascoli theorem, however, the family is for small ε arbitrary close to be equicontinuous and so the proof of the theorem is still applicable; for clarity we perform the steps. By a standard diagonalization argument, we can select a sub-sequence $(u^{\varepsilon_j})_{j\in\mathbb{N}} \subset (u^\varepsilon)_{\varepsilon>0}$ such that $\lim_{j\to\infty}\varepsilon_j = 0$ and $u^{\varepsilon_j}(t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $t \in E$ with E being a countable dense subset of $\mathbb{R}^+$. Next, we claim that $u^{\varepsilon_j}(t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $t \geq 0$. Indeed, fix $r > 0$ for locality and pick $s \in E$ such that $|s-t| < \epsilon$ for some arbitrary $\epsilon > 0$. By the time regularity estimate of Proposition 3.2, we have

$$\begin{aligned}
&\limsup_{j,i\to\infty}\int_{-r}^{r}|u^{\varepsilon_j}(t) - u^{\varepsilon_i}(t)|\mathrm{d}x\\
&\leq \limsup_{j,i\to\infty}\int_{-r}^{r}|u^{\varepsilon_j}(t) - u^{\varepsilon_j}(s)| + |u^{\varepsilon_j}(s) - u^{\varepsilon_i}(s)| + |u^{\varepsilon_i}(s) - u^{\varepsilon_i}(t)|\mathrm{d}x\\
&\leq \limsup_{j,i\to\infty}(2\epsilon + \varepsilon_j + \varepsilon_i)C_{u_0}(t+\epsilon) + \limsup_{j,i\to\infty}\int_{-r}^{r}|u^{\varepsilon_j}(s) - u^{\varepsilon_i}(s)|\mathrm{d}x\\
&= 2\epsilon C_{u_0}(t+\epsilon),
\end{aligned}$$

and since r and ϵ were arbitrary, we conclude that $u^{\varepsilon_j}(t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ to some $u(t)$. Moreover, as $u^{\varepsilon_j}(t)$ converges locally to $u(t)$, the bound (3.43) necessarily carries over to $u(t)$, and so in particular

$$\|u(t)\|_{L^p(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^p(\mathbb{R})},$$

and further by Fatou's lemma we infer for all $t \geq \tilde{t} \geq 0$

$$\begin{aligned}
\|u(t) - u(\tilde{t})\|_{L^1(\mathbb{R})} &\leq \liminf_{j\to\infty}\|u^{\varepsilon_j}(t) - u^{\varepsilon_j}(\tilde{t})\|_{L^1(\mathbb{R})}\\
&\leq \liminf_{j\to\infty}(t - \tilde{t} + \varepsilon_j)C_{u_0}(t)\\
&= (t-\tilde{t})C_{u_0}(t).
\end{aligned} \tag{3.44}$$

Thus $u \in C([0,\infty), L^1(\mathbb{R})) \cap L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$. Next, we prove that u is, in accordance with Def. 1.1, an entropy solution of (1.1) with initial data u_0; the latter part follows from $u(0) = u_0$ and (3.44). To see that u satisfies the entropy inequalities (1.4), we pick an arbitrary entropy pair (η, q) of (1.1) and a non-negative $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ and recall Lemma 3.4 to calculate

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \eta(u)\varphi_t + q(u)\varphi_x + \eta'(u)(K * u)\varphi \mathrm{d}x\mathrm{d}t \\
&= \lim_{j\to 0} \int_0^\infty \int_{\mathbb{R}} \eta(u^{\varepsilon_j})\varphi_t + q(u^{\varepsilon_j})\varphi_x + \eta'(u^{\varepsilon_j})(K * u^{\varepsilon_j})\varphi \mathrm{d}x\mathrm{d}t \\
&\geq \lim_{j\to 0} O(\varepsilon_j) = 0,
\end{aligned} \tag{3.45}$$

where the second line holds as the integrand converges in $L^1(\mathbb{R})$; after all, $(u^{\varepsilon_j})_{j\in\mathbb{N}}$ is uniformly bounded on the compact support of φ. By Proposition 3.1 we conclude that u is the unique entropy solution of (1.1) with u_0 as initial data. What remains to show, is the general limit (3.40) and the L^2 bound of u (3.42); the latter follow by Lemma 3.3 and Fatou's lemma. We prove (3.40) by contradiction; if this limit does not exist, then there is a subsequence $(u^{\varepsilon_j})_{j\in\mathbb{N}} \subset (u^\varepsilon)_{\varepsilon>0}$, a $t > 0$ and an $r > 0$ such that

$$\liminf_{j\to\infty} \int_{-r}^{r} |u(t) - u^{\varepsilon_j}(t)|\mathrm{d}x > 0.$$

But as argued above, the infinite set $(u^{\varepsilon_j})_{j\in\mathbb{N}}$ must be precompact in $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $t \geq 0$, and thus we can pick a subsequence converging for every $t \geq 0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ to the unique (Proposition 3.1) entropy solution u which contradicts the above limit inferior. $\square$

The existence of entropy solutions for general L^∞ data now follows from the previous proposition together with the weighted L^1-contraction provided by Proposition 3.1. As entropy solutions with BV data are L^1-continuous in time, said contraction extends to all $t \geq 0$.

Corollary 3.6. *For any initial data $u_0 \in L^\infty(\mathbb{R})$, there exists a corresponding entropy solution $u \in C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$ of (1.1) satisfying for all $t \geq 0$*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^\infty(\mathbb{R})}, \tag{3.46}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$. If $u_0 \in L^2 \cap L^\infty(\mathbb{R})$, it also satisfies for all $t \geq 0$

$$\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}. \tag{3.47}$$

Proof. For $u_0 \in L^\infty(\mathbb{R})$, let $(u^j)_{j\in\mathbb{N}}$ be a sequence of entropy solutions of (1.1) whose corresponding initial data $(u_0^j)_{j\in\mathbb{N}} \subset BV(\mathbb{R})$ satisfies $\sup_j \|u_0^j\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$ and $u_0^j \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $j \to \infty$. For a fixed $T > 0$, set

$$M = e^{T\kappa}\|u_0\|_{L^\infty(\mathbb{R})},$$

and observe from (3.41) that $\sup_j \|u^j(t)\|_{L^\infty(\mathbb{R})} \leq M$ for all $t \in [0,T]$. Using (3.6), we find for any $r > 0$

$$\begin{aligned}
&\limsup_{j,i\to\infty} \sup_{0\leq t\leq T} \int_{-r}^{r} |u^j(t,x) - u^i(t,x)|\mathrm{d}x \\
&\leq \limsup_{j,i\to\infty} \int_{\mathbb{R}} |u_0^j(x) - u_0^i(x)| w_M^r(T,x)\mathrm{d}x = 0,
\end{aligned}$$

where we used that w_M^r is increasing in t. This shows that $(u^j)_{j\in\mathbb{N}}$ is Cauchy in the Fréchet space $C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$ and so the sequence converges to some $u \in C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$. Moreover,

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \liminf_{j\to\infty} \|u^j(t)\|_{L^\infty(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^\infty(\mathbb{R})},$$

by (3.41), and so $u \in L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$ too. That u takes u_0 as initial data in L^1_{loc}-sense follows from the time-continuity of u and $u(0) = \lim_{j\to\infty} u_0^j = u_0$ where the limit is taken in $L^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, as each member $(u^j)_{j\in\mathbb{N}}$ satisfies the entropy inequalities (1.4), the same can be said for u by a similar calculation as (3.45). Thus the corollary is proved, save for the L^2 estimate; this is attained through Fatou's lemma and (3.42) as we may assume $\sup_j \|u_0^j\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}$. □

3.3. L^2 **continuity and stability of entropy solutions.** For clarity, we summarize what of Theorem 2.1 has been proved so far and what remains to be proved. Combining Proposition 3.1 and Corollary 3.6, we conclude that there exists a unique entropy solution of (1.1) in accordance with Def. 1.1 for every initial data $u_0 \in L^\infty(\mathbb{R})$ and thus also for $u_0 \in L^2 \cap L^\infty(\mathbb{R})$. Furthermore, Corollary 3.6 guarantees that these solutions are continuous from $[0,\infty)$ to $L^1_{\mathrm{loc}}(\mathbb{R})$ so that the restriction $u(t) := u(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R})$ makes sense for all $t \ge 0$. The same corollary also provides the bounds (2.1) of Theorem 2.1.

It remains to prove that entropy solutions with $L^2 \cap L^\infty$ data are continuous from $[0,\infty)$ to $L^2(\mathbb{R})$ and that they satisfy the stability result of Theorem 2.1. To do so, we shall exploit the height bound of Corollary 2.5. As explained at the beginning of Section 4, Corollary 2.5 can be proved for the case $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ independently of this subsection; thus we may here use the height bound (2.4) for entropy solutions of (1.1) without risking a circular argument. From here til the end of the section, we take the above properties of entropy solutions for granted. We begin with a variant of Proposition 3.1 which makes use of the discussed height bound.

Lemma 3.7. *There is a function $\Psi\colon [0,\infty)^3 \to [0,\infty)$, increasing in all arguments, such that for any pair of entropy solutions u, v of (1.1) with respective initial data $u_0, v_0 \in L^2 \cap L^\infty(\mathbb{R})$ one has for any $t, r \ge 0$ and $N \ge \max\{\|u_0\|_{L^2(\mathbb{R})}, \|v_0\|_{L^2(\mathbb{R})}\}$ the inequality*

$$\|u(t) - v(t)\|_{L^1([-r,r])} \le \Psi(t,N,r)\|u_0 - v_0\|_{L^2(\mathbb{R})}. \tag{3.48}$$

Proof. Let u, v, u_0, v_0 and N be as described in the lemma. By (2.4) from Corollary 2.5, and the property of N, we have for all $t > 0$

$$\frac{\|u(t)\|_{L^\infty(\mathbb{R})} + \|v(t)\|_{L^\infty(\mathbb{R})}}{2} \le CN^{\frac{2}{3}}\Big(1 + \frac{1}{t^{\frac{1}{3}}}\Big) =: m(t), \tag{3.49}$$

where $C := \max\{2^{\frac{11}{12}} 3^{\frac{1}{3}} \|K\|_{L^1(\mathbb{R})}^{\frac{1}{3}}, 2^{\frac{5}{4}}\}$. With $F(u,v) := \frac{1}{2}\mathrm{sgn}(u-v)(u^2 - v^2)$, we have for any non-negative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ the inequality

$$0 \le \int_0^\infty \int_{\mathbb{R}} |u-v|\varphi_t + F(u,v)\varphi_x + |u-v|(|K| * \varphi)\mathrm{d}x\mathrm{d}t. \tag{3.50}$$

This is attained by following the first half of the proof of Proposition 3.1 without using the bound $|F(u,v)| \le M|u-v|$ as done in the first inequality of (3.14); one may instead, when 'going to the diagonal', subtract $F(u(t,x), v(t,x))$ from $F(u(t,x), v(s,y))$ and use

$$|F(u(t,x), v(s,y)) - F(u(t,x), v(x,y))| \lesssim |v(s,y) - v(t,x)|,$$

which follows from local Lipschitz continuity of F and the fact that u and v are globally bounded (as pointed out after Corollary 2.5). With (3.50) established, we may *now* filter out $(u+v)/2$ from F using the more precise bound (3.49), that is

$$|F(u(t,x), v(t,x))| \le m(t)|u(t,x) - v(t,x)|.$$

Doing so, and additionally setting $\varphi(t,x) = \theta(t)\phi(t,x)$ for two arbitrary non-negative functions $\theta \in C_c^\infty((0,T))$ and $\phi \in C_c^\infty((0,T)\times\mathbb{R})$, with $T > 0$ also arbitrary, we conclude from (3.50) that

$$0 \leq \int_0^T \int_{\mathbb{R}} |u-v|\theta' \phi \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}} |u-v|\theta \Big[\phi_t + m(t)|\phi_x| + |K| * \phi\Big] \mathrm{d}x\mathrm{d}t. \tag{3.51}$$

Observe that (3.51) resembles (3.18); for brevity, we skip minor details in the following steps due to their similarity of those following (3.18). Let $f\colon \mathbb{R} \to [0,1]$ be a smooth and non-increasing function satisfying $f(x) = 1$ for $x \leq 0$ and $f(x) = 0$ for sufficiently large x, and set

$$g(t,x) \coloneqq f(|x| + M(t) - M(T)),$$

where we have here defined $M(t)$ by

$$M(t) \coloneqq \int_0^t m(s)\mathrm{d}s = CN^{\frac{2}{3}}\Big(t + \tfrac{3}{2}t^{\frac{2}{3}}\Big).$$

Analogous to (3.20), we then set

$$\phi(t,x) = \Big(e^{(T-t)|K|*} g(t,\cdot)\Big)(x), \tag{3.52}$$

and while this ϕ is not of compact support, both it, and its derivatives, are integrable on $(0,T)\times\mathbb{R}$ and so by an approximation argument it can be used in (3.51). By similar arguments as those following (3.20) we find also here that the second integral in (3.51) is non-positive, and so we may remove it. Letting then θ approximate $\mathbb{1}_{(0,T)}$ in a similar (smooth) manner as done by the sequence (3.21), we may from (3.51) conclude

$$\int_{\mathbb{R}} |u(T,x) - v(T,x)|\phi(T,x)\mathrm{d}x \leq \int_{\mathbb{R}} |u_0(x) - v_0(x)|\phi(0,x)\mathrm{d}x, \tag{3.53}$$

where we used that $t \mapsto |u(t,\cdot) - v(t,\cdot)|\phi(t,\cdot)$ is L^1-continuous which can be seen by a triangle inequality argument. Note that $\phi(0,x) = f(|x|)$, and so letting $f \to \mathbb{1}_{(-\infty,r)}$ in L^1 sense, the left-hand side of (3.53) becomes the left-hand side of (3.48). When $f \to \mathbb{1}_{(-\infty,r)}$ we also get from (3.52) that

$$\phi(0,x) \to \Big(e^{T|K|*}\mathbb{1}_{(-\infty,r)}(|\cdot| - M(T))\Big)(x), \tag{3.54}$$

in L^1 sense. Denoting the right-hand side of (3.54) also by $\phi(0,x)$, it follows by Young's convolution inequality that

$$\|\phi(0,x)\|_{L^2(\mathbb{R})} \leq e^{T\kappa}[2r + 2M(T)]^{\frac{1}{2}} = e^{T\kappa}\Big[2r + 2CN^{\frac{2}{3}}\Big(T + \tfrac{3}{2}T^{\frac{2}{3}}\Big)\Big]^{\frac{1}{2}}, \tag{3.55}$$

where $\kappa \coloneqq \|K\|_{L^1(\mathbb{R})}$. Applying then the Cauchy–Schwarz inequality to the right-hand side of (3.53), and using the above L^2 bound for $\phi(0,x)$, we attain (3.48) (with T substituting for t) for $\Psi(T,N,r)$ given by the right-hand side of (3.55). □

We follow up with a tightness bound for entropy solutions with $L^2 \cap L^\infty$ data.

Lemma 3.8. *There is a function $\Phi\colon [0,\infty)^2 \times \mathbb{R} \to [0,\infty)$, increasing in all arguments, such that if u is an entropy solution of (1.1) with initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$, then for any $t, r \geq 0$ and $N \geq \|u_0\|_{L^2(\mathbb{R})}$*

$$\int_{|x|>r} u^2(t,x)\mathrm{d}x \leq \int_{\mathbb{R}} u_0^2(x)\Phi(t,N,|x|-r)\mathrm{d}x. \tag{3.56}$$

Moreover,

$$\lim_{\xi\to-\infty} \Phi(t,N,\xi) = 0, \qquad \Phi(t,N,\xi) = e^{2t\kappa}, \quad \xi > 0,$$

where $\kappa \coloneqq \|K\|_{L^1(\mathbb{R})}$, and in particular, $\xi \mapsto \Phi(t,M,\xi)$ is a bounded function.

Proof. Pick arbitrary initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ and let u denote the corresponding entropy solution of (1.1). Writing out the entropy inequality (1.4) for u using the entropy pair $(\eta(u), q(u)) = (u^2, \frac{2}{3}u^3)$ and a non-negative test function $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$, with $T \in (0,\infty)$ fixed, we get

$$0 \le \int_0^T \int_{\mathbb{R}} u^2\varphi_t + \tfrac{2}{3}u^3\varphi_x + 2u(K*u)\varphi \,\mathrm{d}x\mathrm{d}t. \tag{3.57}$$

By the height bound (2.4) of Corollary 2.5, we have $\|u(t)\|_{L^\infty(\mathbb{R})} \le m(t)$ where $m(t)$ is as defined in (3.49), and so the second term of the above integrand satisfies

$$\tfrac{2}{3}u^3\varphi_x \le u^2\Big[\tfrac{2}{3}m(t)|\varphi_x|\Big].$$

Additionally, the third part of the integrand satisfies

$$\begin{aligned}\int_{\mathbb{R}} 2u(K*u)\varphi \,\mathrm{d}x &= \int_{\mathbb{R}}\int_{\mathbb{R}} 2u(t,x)u(t,y)K(x-y)\varphi(t,x)\,\mathrm{d}y\mathrm{d}x \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \Big[|u(t,x)|^2 + |u(t,y)|^2\Big]|K(x-y)|\varphi(t,x)\,\mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbb{R}} u^2\Big[\kappa\varphi + |K|*\varphi\Big]\mathrm{d}x.\end{aligned}$$

Inserting these two bounds in (3.57) we get for any non-negative $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$

$$0 \le \int_0^T \int_{\mathbb{R}} u^2\Big[\varphi_t + \tfrac{2}{3}m(t)|\varphi_x| + \mathcal{K}*\varphi\Big]\,\mathrm{d}x\mathrm{d}t, \tag{3.58}$$

where we introduced the measure $\mathcal{K} := \kappa\delta + |K|$, where δ is the Dirac measure. Like in the previous proof, we proceed in a manner similar to the second half of the proof of Proposition 3.1, though some necessary changes are made. We set $\varphi(t,x) = \theta(t)\rho(x)\phi(t,x)$ for three smooth non-negative functions on $[0,T]\times\mathbb{R}$ with θ and ρ having compact support in $(0,T)$ and $\mathbb{R}$ respectively. Additionally, while ϕ need not be compactly supported, we require ϕ and its derivatives to be bounded. Inserting this in (3.58) we get

$$0 \le \int_0^T \int_{\mathbb{R}} u^2\theta'\rho\phi \mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}} u^2\theta\Big[\rho\phi_t + \tfrac{2}{3}m(t)|(\rho\phi)_x| + \mathcal{K}*(\rho\phi)\Big]\,\mathrm{d}x\mathrm{d}t. \tag{3.59}$$

Letting θ approximate $\mathbb{1}_{(0,T)}$ in a similar (smooth) manner as done by the sequence (3.21), we may from (3.59) conclude that

$$\begin{aligned}\int_{\mathbb{R}} u^2(T,x)\rho(x)\phi(T,x)\mathrm{d}x \le& \int_{\mathbb{R}} u_0^2(x)\rho(x)\phi(0,x)\mathrm{d}x \\ &+ \int_0^\infty \int_{\mathbb{R}} u^2\Big[\rho\phi_t + \tfrac{2}{3}m(t)|(\rho\phi)_x| + \mathcal{K}*(\rho\phi)\Big]\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.60}$$

where we used that $t \mapsto u^2(t,\cdot)\rho(\cdot)\phi(t,\cdot)$ is L^1-continuous which can be seen by a triangle inequality argument. Next, we set $\rho(x) = \tilde{\rho}(x/N)$ where $\tilde{\rho} \in C_c^\infty(\mathbb{R})$ is non-negative and satisfies $\tilde{\rho}(0) = 1$. Letting $N \to \infty$, (3.60) yields by the dominated convergence theorem

$$\begin{aligned}\int_{\mathbb{R}} u^2(T,x)\phi(T,x)\mathrm{d}x \le& \int_{\mathbb{R}} u_0^2(x)\phi(0,x)\mathrm{d}x \\ &+ \int_0^\infty \int_{\mathbb{R}} u^2\Big[\phi_t + \tfrac{2}{3}m(t)|\phi_x| + \mathcal{K}*\phi\Big]\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.61}$$

where the convergence of the integrals follows from the boundness of ϕ (and its derivatives) combined with $\|u(t)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}$ for all $t \in [0,T]$. To rid ourselves of the last integral in (3.61), we perform a similar trick as done for (3.18) and (3.51), but with a different f;

we here let $f\colon \mathbb{R}\to[0,1]$ be a *non-decreasing* function with bounded derivatives. Define further g by

$$g(t,x) := f(|x| + M(T) - M(t)),$$

where $M(t)$ denotes

$$M(t) := \int_0^t \tfrac{2}{3}m(s)\mathrm{d}s = CN^{\frac{2}{3}}\Big(\tfrac{2}{3}t + t^{\frac{2}{3}}\Big), \tag{3.62}$$

and analogues to (3.20), we set ϕ to be

$$\phi(t,x) = \Big(e^{(T-t)\mathcal{K}*}g(t,\cdot)\Big)(x).$$

We conclude by similar arguments as those following (3.20) that the square bracket in (3.61) is non-positive. Thus, removing the non-positive integral in (3.61) we get

$$\int_{\mathbb{R}} u^2(T,x)f(|x|)\mathrm{d}x \le \int_{\mathbb{R}} u_0^2(x)\Big(e^{T\mathcal{K}*}f(|\cdot| + M(T))\Big)(x)\mathrm{d}x, \tag{3.63}$$

where we used the explicit expressions for $\phi(T,x)$ and $\phi(0,x)$. Letting $f\to\mathbb{1}_{(r,\infty)}$ pointwise a.e. it is clear that the left-hand side of (3.63) converges to $\int_{|x|>r} u^2(T)\mathrm{d}x$. As for the right-hand side, we get the cumbersome term $e^{T\mathcal{K}*}\mathbb{1}_{(r,\infty)}(|\cdot| + MT)$ which we now simplify. Let the Borel measure ν_T be defined by the relation $\nu_T* = e^{T\mathcal{K}*}$ and observe that we for $x\in\mathbb{R}$ have

$$\Big(\nu_T * \mathbb{1}_{(r,\infty)}(|\cdot| + M(T))\Big)(x) = \int_{|x-y|+M(T)>r} \mathrm{d}\nu_T(y) \le \int_{|x|-r+M(T)>-|y|} \mathrm{d}\nu_T(y). \tag{3.64}$$

Thus, we define $\Phi(T,N,|x|-r)$ to be the latter expression after substituting for $M(T)$ using (3.62). Inserting this in (3.63) we get exactly (3.56) with T substituting for t. The properties of Φ stated in the lemma can be read directly from (3.64) when setting $\xi = |x|-r$ together with the fact that $T\mapsto\nu_T$ is increasing (in the canonical sense) and $\int_{\mathbb{R}} d\nu_T = e^{T\mathcal{K}*}1 = e^{2T\kappa}$. □

We may now prove the remaining part of Theorem 2.1.

Proposition 3.9. *Let two sequences $(t_k)_{k\in\mathbb{N}}\subset[0,\infty)$ and $(u_{0,k})_{k\in\mathbb{N}}\subset L^2\cap L^\infty(\mathbb{R})$ admit limits*

$$\lim_{k\to\infty}|t_k - t| = 0, \qquad \lim_{k\to\infty}\|u_{0,k} - u_0\|_{L^2(\mathbb{R})} = 0,$$

with $t\in[0,\infty)$ and $u_0\in L^2\cap L^\infty(\mathbb{R})$. Letting $(u_k)_{k\in\mathbb{N}}$ and u denote the entropy solutions of (1.1) corresponding to the initial data $(u_{0,k})_{k\in\mathbb{N}}$ and u_0 respectively, we have

$$\lim_{k\to\infty}\|u_k(t_k) - u(t)\|_{L^2(\mathbb{R})} = 0.$$

In particular, entropy solutions of (1.1) with $L^2\cap L^\infty$ data are continuous from $[0,\infty)$ to $L^2(\mathbb{R})$.

Proof. Suppose first that $t>0$. As $t_k\to t$ there is a $T\in(0,\infty)$ such that $(t_k)_{k\in\mathbb{N}}\subset[0,T]$. Similarly, there is an N such that $N\ge\|v_0\|_{L^2(\mathbb{R})}$ for every $v_0\in\{u_{0,1},u_{0,2},\dots,u_0\}$; observe that such an N satisfies $N\ge\|v(t)\|_{L^2}$ for all $t\in[0,T]$ and v ranging over the corresponding entropy solutions. As the function Φ from Lemma 3.8 was increasing in its arguments, we infer for all $k\in\mathbb{N}$ and $r>0$ that

$$\int_{|x|>r} u_k^2(t_k,x)\mathrm{d}x \le \int_{\mathbb{R}} u_{0,k}^2(x)\Phi(T,N,|x|-r).$$

Furthermore, as $\xi \mapsto \Phi(T, M, \xi)$ is bounded while $u_{0,k}^2 \to u_0^2$ in $L^1(\mathbb{R})$ as $k \to \infty$, it follows that

$$\limsup_{k\to\infty} \int_{|x|>r} u_k^2(t_k, x)\mathrm{d}x \le \int_{\mathbb{R}} u_0^2(x)\Phi(T, M, |x| - r), \tag{3.65}$$

for any $r > 0$. Since u_0^2 is integrable and $\lim_{\xi\to-\infty} \Phi(T, M, \xi) = 0$, we may for any $\varepsilon > 0$ pick a sufficiently large $r > 0$ such that the right-hand side of (3.65) is smaller than ε^2. For such a couple of constants $\varepsilon, r > 0$ we find

$$\limsup_{k\to\infty} \|u_k(t_k) - u(t)\|_{L^2(\mathbb{R})} \le 2\varepsilon + \limsup_{k\to\infty} \|u_k(t_k) - u(t)\|_{L^2([-r,r])}. \tag{3.66}$$

To deal with the rightmost term in (3.66), we yet again let m be the function defined in (3.49) using the above N. As $t > 0$, there are only a finite number of elements in $(t_k)_{k\in\mathbb{N}}$ smaller than $t/2$; without loss of generality, we shall assume there are none. By the height bound (2.4) from Corollary 2.5 and m being decreasing in t, it then follows that $\|v\|_{L^\infty(\mathbb{R})} \le m(t/2)$ for every $v \in \{u_1(t_1), u_2(t_2), \dots, u(t)\}$. Thus,

$$\|u_k(t_k) - u(t)\|_{L^2([-r,r])}^2 \le 2m(t/2)\|u_k(t_k) - u(t)\|_{L^1([-r,r])},$$

and by the triangle inequality, we further have

$$\|u_k(t_k) - u(t)\|_{L^1([-r,r])} \le \|u_k(t_k) - u(t_k)\|_{L^1([-r,r])} + \|u(t_k) - u(t)\|_{L^1([-r,r])} \to 0,$$

as $k \to 0$. Here we used the L^1_{loc}-continuity of $y \mapsto u(t)$ and Lemma 3.7. Thus, the last term of (3.66) is zero, and as $\varepsilon > 0$ was arbitrary, we conclude $\limsup_{k\to\infty} \|u_k(t_k) - u(t)\|_{L^2(\mathbb{R})} = 0$.

Suppose next $t = 0$. As above, $u_k(t_k)$ converges to $u(0) = u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$, and in particular, the convergence holds in the sense of distributions. Moreover, we have norm convergence as

$$\limsup_{k\to\infty} \|u_k(t_k)\|_{L^2(\mathbb{R})} \le \limsup_{k\to\infty} \|u_{0,k}\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})},$$

while $\|u_0\|_{L^2(\mathbb{R})} \le \liminf_{k\to\infty} \|u_k(t_k)\|_{L^2(\mathbb{R})}$ follows from Fatou's lemma. Thus, we conclude $\|u_k(t_k) - u_0\|_{L^2(\mathbb{R})} \to 0$ as $k \to \infty$. With the stability result proved, the L^2-continuity of $t \mapsto u(t)$ follows by setting $u_{0,k} = u_0$ for all $k \in \mathbb{N}$. □

We end the section by proving Corollary 2.2.

Proof of Corollary 2.2. The solution mapping S is by Proposition 3.9 jointly continuous from $[0, \infty) \times (L^2 \cap L^\infty(\mathbb{R}))^*$ to $L^2(\mathbb{R})$, where $(L^2 \cap L^\infty(\mathbb{R}))^*$ denotes the set $L^2 \cap L^\infty(\mathbb{R})$ equipped with its L^2 subspace-topology. Seeking to extend S to all of $[0, \infty) \times L^2(\mathbb{R})$ in a continuous manner, we note that we have only one choice: whenever a sequence $(u_{0,k})_{k\in\mathbb{N}} \in L^2 \cap L^\infty(\mathbb{R})$ converges in $L^2(\mathbb{R})$, it follows from Lemma 3.7 that the corresponding entropy solutions $(u_k)_{k\in\mathbb{N}}$ form a Cauchy sequence in the Fréchet space $C([0, \infty), L^1_{\mathrm{loc}}(\mathbb{R}))$, and thus they converge to a unique element $u \in C([0, \infty), L^1_{\mathrm{loc}}(\mathbb{R}))$ in the appropriate topology. We now argue that u inherits all the nice properties of entropy solutions of (1.1) established so far, apart from being bounded at $t = 0$. Denoting $u_0 \in L^2(\mathbb{R})$ for the L^2 limit of $(u_{0,k})_{k\in\mathbb{N}}$, we have by Fatou's lemma

$$\|u(t)\|_{L^2(\mathbb{R})} \le \liminf_{k\to\infty} \|u_k(t)\|_{L^2(\mathbb{R})} \le \liminf_{k\to\infty} \|u_{0,k}\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}.$$

Moreover, as each u_k satisfy the height bound (2.4) this bound also carries over to u, and thus u is locally bounded in $(0, \infty)\times\mathbb{R}$. Similarly, as each u_k satisfy the entropy inequalities (1.4), the same is true for u by a limit argument exploiting the uniform bound of $(u_k)_{k\in\mathbb{N}}$ on the support of φ and the fact that η and q are smooth; in particular, u is a weak solution of (1.1). Even Lemma 3.7 and Lemma 3.8 carries over to u by approximation.

In conclusion, u – and all other weak solutions obtained this way – satisfy every property used for entropy solutions in the proof of Proposition 3.9, and so the proposition extends to these weak solutions. Consequently, S is continuous on the larger set $[0,\infty)\times L^2(\mathbb{R})$, and the proof is complete. □

4. One-sided Hölder regularity for entropy solutions

In this section we show that entropy solutions of (1.1) with $L^2\cap L^\infty$ data satisfy one-sided Hölder conditions with time-decreasing coefficients. As Subsection 3.3 exploits Corollary 2.5, which is proved using the results established here, we stress that the coming analysis will only depend on the results of Subsection 3.1 and 3.2, thus avoiding a circular argument. In Subsection 4.1 we introduce the necessary building blocks for Subsection 4.2 where the Hölder conditions are constructed; Theorem 2.3 is proved at the very end of this section. Central in this section is the following object, which in classical terms can be described as a modulus of right upper semi-continuity.

Definition 4.1. We say that a smooth and strictly increasing function $\omega\colon(0,\infty)\to(0,\infty)$ is a *modulus of growth* for $v\colon\mathbb{R}\to\mathbb{R}$ if for all $h>0$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}}\Big[v(x+h)-v(x)\Big]\le\omega(h).$$

The requirement that ω be smooth and strictly increasing is for technical convenience. Note also that we did not require $\omega(0+)=0$; this is to include the expression (4.10) when $s=0$.

4.1. Preliminary results.

The classical Oleĭnik estimate [6] for entropy solutions of Burgers' equation is for $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ and $h\ge 0$ given by

$$u(t,x+h)-u(t,x)\le\frac{h}{t}. \tag{4.1}$$

For a fixed $t>0$, this one-sided Lipschitz condition (or modulus of growth) restricts how fast $x\mapsto u(t,x)$ can grow, but not how fast it can decrease, thus allowing for jump discontinuities (shocks) whose left limit is above the right. Interestingly, when the initial data of Burgers' equation satisfies $u_0\in L^p(\mathbb{R})$ for some $p\in[1,\infty)$, one can for the corresponding entropy solution u use (4.1) to attain

$$\|u(t)\|^{p+1}_{L^\infty(\mathbb{R})}\le\tfrac{p+1}{t}\|u(t)\|^p_{L^p(\mathbb{R})}\le\tfrac{p+1}{t}\|u_0\|^p_{L^p(\mathbb{R})}, \tag{4.2}$$

where the rightmost inequality is just the classical L^p bound for Burgers' equation, and thus, the height of $u(t)=u(t,\cdot)$ tends to zero as $t\to\infty$. We omit the proof of (4.2), which is similar to that of the next lemma where we provide a general method for bounding the height of a function $u\in L^2(\mathbb{R})$ admitting a modulus of growth ω. We focus on $L^2(\mathbb{R})$ because other L^p norms might fail to be non-increasing for entropy solutions of (1.1); the generalization of (4.1) will require a generalization of (4.2), so $p=2$ is the natural choice as $\|u(t)\|_{L^2(\mathbb{R})}\le\|u_0\|_{L^2(\mathbb{R})}$ for entropy solutions of (1.1). In the coming lemma we also provide for later convenience a bound on the following seminorm defined for $v\in L^\infty(\mathbb{R})$ by

$$|v|_\infty:=\operatorname*{ess\,sup}_{x,y\in\mathbb{R}}\frac{v(x)-v(y)}{2}. \tag{4.3}$$

As $|v|_\infty\le\|v\|_{L^\infty(\mathbb{R})}$, any bound on $\|v\|_{L^\infty(\mathbb{R})}$ obviously carries over to $|v|_\infty$. Note however, that the next lemma bounds $|v|_\infty$ sharper than it does $\|v\|_{L^\infty(\mathbb{R})}$. Finally, we mention that the extra assumptions posed on ω in the lemma are only for technical simplicity, as the lemma holds more generally.

Lemma 4.2. *Let $v \in L^2(\mathbb{R})$ admit a modulus of growth ω that satisfies $\omega(0+) = 0$ and $\omega(\infty) = \infty$. Then $v \in L^2 \cap L^\infty(\mathbb{R})$ and moreover*

$$\|v\|^2_{L^2(\mathbb{R})} \geq F\Big(\|v\|_{L^\infty(\mathbb{R})}\Big), \tag{4.4}$$

$$\tfrac{1}{2}\|v\|^2_{L^2(\mathbb{R})} \geq F\Big(|v|_\infty\Big), \tag{4.5}$$

where F is the strictly increasing and convex function

$$F(y) := 2\int_0^y \int_0^{y_1} \omega^{-1}(y_2)\mathrm{d}y_2\mathrm{d}y_1. \tag{4.6}$$

Proof. By Lemma A.1 from the appendix we may assume v to be left-continuous, and in particular, well defined at every point. Then, for all $x \in \mathbb{R}$ such that $v(x) \geq 0$ we have for $h \in (0, \omega^{-1}(v(x))]$

$$v(x-h) \geq v(x) - \omega(h) \geq 0,$$

and similarly, for all $x \in \mathbb{R}$ such that $v(x) < 0$ we have for $h \in (0, \omega^{-1}(-v(x))]$

$$v(x+h) \leq v(x) + \omega(h) \leq 0.$$

Squaring each of these inequalities (the bottom one would flip direction) and integrating over $h \in (0, \omega^{-1}(|v(x)|)]$, yields in both cases

$$\|v\|^2_{L^2(\mathbb{R})} \geq \int_0^{\omega^{-1}(|v(x)|)} (|v(x)| - \omega(h))^2 \mathrm{d}h, \tag{4.7}$$

where the left-hand side has been replaced by the upper bound $\|v\|^2_{L^2(\mathbb{R})}$. Performing the change of variables $h = \omega^{-1}(y)$ the right-hand side of (4.7) can further be written

$$\begin{aligned}\int_0^{|v(x)|} (|v(x)| - y)^2 \mathrm{d}\omega^{-1}(y) &= 2\int_0^{|v(x)|} (|v(x)| - y)\omega^{-1}(y)\mathrm{d}y \\ &= 2\int_0^{|v(x)|}\int_0^y \omega^{-1}(z)\mathrm{d}z\mathrm{d}y,\end{aligned}$$

where we integrated by parts twice. This last expression is exactly $F(|v(x)|)$, and so letting this replace the right-hand side of (4.7) followed by taking the supremum with respect to $x \in \mathbb{R}$ yields (4.4). For (4.5), we write v_+ and v_- for the positive and negative part of v respectively, and observe that $v \in L^2 \cap L^\infty(\mathbb{R})$ implies $|v|_\infty = \frac{1}{2}(\|v_+\|_{L^\infty(\mathbb{R})} + \|v_-\|_{L^\infty(\mathbb{R})})$ and $\|v\|^2_{L^2(\mathbb{R})} = \|v_+\|^2_{L^2(\mathbb{R})} + \|v_-\|^2_{L^2(\mathbb{R})}$. Furthermore, as both v_+ and $-v_-$ admit ω as a modulus of growth, we can use (4.4) followed by Jensen's inequality to calculate

$$\begin{aligned}\tfrac{1}{2}\|v\|^2_{L^2(\mathbb{R})} &= \tfrac{1}{2}\Big[\|v_+\|^2_{L^2(\mathbb{R})} + \|v_-\|^2_{L^2(\mathbb{R})}\Big] \\ &\geq \tfrac{1}{2}\Big[F\Big(\|v_+\|_{L^\infty(\mathbb{R})}\Big) + F\Big(\|v_-\|_{L^\infty(\mathbb{R})}\Big)\Big] \\ &\geq F\Big(\tfrac{1}{2}\Big[\|v_+\|_{L^\infty(\mathbb{R})} + \|v_-\|_{L^\infty(\mathbb{R})}\Big]\Big) \\ &= F\Big(|v|_\infty\Big).\end{aligned}$$

□

The calculations of the next subsection, where Theorem 2.3 is proved, can be boiled down to the three lemmas of this subsection (Lemma 4.2 being the first). The remaining Lemma 4.3 and Lemma 4.4, induce a natural evolution of a modulus of growth from the mappings S_t^B and S_t^K, introduced in (3.24) and (3.25). The relevance of these results should come as no surprise; the previous section showed that entropy solutions could be approximated by repeated compositions of said mappings.

Lemma 4.3. *Suppose $v \in BV(\mathbb{R})$ admits a concave modulus of growth ω. Then for any $\varepsilon > 0$, the function $w = S^B_\varepsilon(v)$, admits the modulus of growth*

$$h \mapsto \frac{\omega(h)}{1 + \varepsilon\omega'(h)}. \tag{4.8}$$

Proof. As S^B_t maps BV to itself, both v and w admits essential left and right limits at point. Thus, we assume without loss of generality that they are left continuous. For $x \in \mathbb{R}$, $h > 0$ and $t \in [0, \varepsilon]$, introduce the two (minimal) backward characteristics of $S^B_t(v)$ emanating from (ε, x) and $(\varepsilon, x + h)$ respectively

$$\begin{aligned} \xi_1(t) &= x + (t - \varepsilon)w(x), \\ \xi_2(t) &= x + h + (t - \varepsilon)w(x + h). \end{aligned}$$

As v and w are left continuous, it follows from Theorem 11.1.3. in [6] that

$$v(\xi_1(0)) \le w(x), \qquad\qquad w(x + h) \le v(\xi_2(0)+).$$

Moreover, by the Oleĭnik estimate of w (4.1), we find

$$\xi_2(0) - \xi_1(0) = h - \varepsilon[w(x + h) - w(x)] \ge 0,$$

and so exploiting ω we can calculate

$$\begin{aligned} w(x + h) - w(x) &\le v(\xi_2(0)+) - v(\xi_1(0)) \\ &\le \omega(h - \varepsilon[w(x + h) - w(x)]) \\ &\le \omega(h) - \varepsilon\omega'(h)(w(x + h) - w(x)), \end{aligned} \tag{4.9}$$

where the last inequality holds as ω is concave. We conclude that

$$w(x + h) - w(x) \le \frac{\omega(h)}{1 + \varepsilon\omega'(h)},$$

for all $x \in \mathbb{R}$ and $h > 0$. That (4.8) is positive, smooth and strictly increasing follows from ω being positive, smooth, strictly increasing and concave. □

We follow immediately with a similar result for the operator S^K_t, which will depend on the fractional variation $|K|_{TV^s}$ as defined in (1.5) and the seminorm $|\cdot|_\infty$ defined in (4.3).

Lemma 4.4. *Let $s \in [0, 1]$ and assume $|K|_{TV^s} < \infty$. Suppose $v \in L^\infty(\mathbb{R})$ admits a modulus of growth ω. Then for any $\varepsilon > 0$, the function $w = S^K_\varepsilon(v)$ admits the modulus of growth*

$$h \mapsto \omega(h) + \varepsilon|K|_{TV^s}|v|_\infty h^s. \tag{4.10}$$

Proof. For simple notation, we introduce the shift operator $T_h \colon f \mapsto f(\cdot + h)$. As shifts commute with convolution, and since $\int_{\mathbb{R}} T_h K - K \mathrm{d}x = 0$, we start by noting that for any $k \in \mathbb{R}$

$$(T_h - 1)(K * v) = [(T_h - 1)K] * (v - k).$$

Next, we introduce $\overline{v} = \operatorname{ess\,sup}_x v(x)$ and $\underline{v} = \operatorname{ess\,inf}_x v(x)$, and we observe that

$$\|v - k\|_{L^\infty(\mathbb{R})} = \max\{\overline{v} - k, k - \underline{v}\}.$$

Thus, we minimize by setting $k = \frac{1}{2}(\overline{v} + \underline{v})$ and get $\|v - k\|_{L^\infty(\mathbb{R})} = \frac{1}{2}(\overline{v} - \underline{v}) = |v|_\infty$. By Young's convolution inequality and the above calculations we infer

$$\|(T_h - 1)(K * v)\|_{L^\infty(\mathbb{R})} \le \|K(\cdot + h) - K\|_{L^1(\mathbb{R})}\|v - k\|_{L^\infty(\mathbb{R})}$$

$$\leq |K|_{TV^s}|v|_\infty h^s.$$

For any $h > 0$ we then conclude

$$(T_h - 1)w = (T_h - 1)v + \varepsilon(T_h - 1)(K * v) \leq \omega(h) + \varepsilon|K|_{TV^s}|v|_\infty h^s,$$

where the last inequality holds pointwise a.e. in $\mathbb{R}$. □

4.2. **Deriving a modulus of growth for entropy solutions.** Throughout this subsection we consider $s \in [0, 1]$ fixed and assume that $|K|_{TV^s}$ is finite. Further, we let $\mu, \kappa_s \in (0,\infty)$ denote arbitrary fixed values, though we impose the requirement $\kappa_s \geq |K|_{TV^s}$. The role of μ and κ_s will essentially be that of placeholders for the L^2 norm of the initial data and of $|K|_{TV^s}$ respectively, but note that μ and κ_s are strictly positive (even if the quantities they represent might be zero). This positivity is for technical convenience as some of the coming expressions would otherwise need a limit sense interpretation.

We shall for an arbitrary entropy solution u of (1.1) with $L^2 \cap L^\infty$ data, seek an expression $a(t)$ such that $h \mapsto a(t)h^{\frac{1+s}{2}}$ serves as a modulus of growth (Def. 4.1) for $x \mapsto u(t, x)$. We begin with an important result, which among other things rephrases Lemma 4.2 for the more explicit case $\omega(h) = ah^{\frac{1+s}{2}}$. For this purpose, we introduce the constant

$$c_s = \left[\frac{(2+s)(3+s)}{2(1+s)^2}\right]^{\frac{1+s}{4+2s}}, \tag{4.11}$$

and the function

$$H(a) = \frac{(2c_s)^{\frac{2}{1+s}}\mu^{\frac{2}{2+s}}}{a^{\frac{2}{2+s}}}, \tag{4.12}$$

defined for all $a > 0$. We also recall definition (4.3) of the seminorm $|\cdot|_\infty$. The essential part of the next lemma is in allowing us to extend the domain for which a homogeneous modulus of growth is valid. This will be vital when proving the following proposition.

Lemma 4.5. *With fixed $a > 0$, define $\omega(h) = ah^{\frac{1+s}{2}}$. Suppose $v \in L^2(\mathbb{R})$ satisfies $\|v\|_{L^2(\mathbb{R})} \leq \mu$ and admits ω as a modulus of growth for $h \in (0, H(a))$. Then v admits ω as a modulus of growth for all $h \in (0,\infty)$ and moreover*

$$\|v\|_{L^\infty(\mathbb{R})} \leq 2^{\frac{1+s}{4+2s}} c_s \mu^{\frac{1+s}{2+s}} a^{\frac{1}{2+s}}, \tag{4.13}$$

$$|v|_\infty \leq c_s \mu^{\frac{1+s}{2+s}} a^{\frac{1}{2+s}}. \tag{4.14}$$

Proof. We begin by proving the two inequalities, so let us assume for now that v admits ω as a modulus of growth for all $h \in (0,\infty)$. Since $\omega^{-1}(y) = a^{-\frac{2}{1+s}} y^{\frac{2}{1+s}}$ the function F from (4.6) can here be written

$$F(y) = \left[\frac{2(1+s)^2}{(3+s)(4+2s)}\right] \frac{y^{\frac{4+2s}{1+s}}}{a^{\frac{2}{1+s}}} = \frac{1}{2}\left(\frac{y}{c_s a^{\frac{1}{2+s}}}\right)^{\frac{4+2s}{1+s}},$$

with inverse

$$F^{-1}(y) = 2^{\frac{1+s}{4+2s}} c_s a^{\frac{1}{2+s}} y^{\frac{1+s}{4+2s}}.$$

Combined with $\|v\|_{L^2(\mathbb{R})} \leq \mu$, (4.4) and (4.5) give $\|v\|_{L^\infty(\mathbb{R})} \leq F^{-1}(\mu^2)$ and $|v|_\infty \leq F^{-1}(\frac{1}{2}\mu^2)$, which coincides with (4.13) and (4.14) respectively. Next, assume we only know that v admits ω as a modulus of growth for $h \in (0, H(a))$. The steps in the proof of Lemma 4.2 can still be carried out if one lets the role of $\omega^{-1}(y) = a^{-\frac{2}{1+s}}y^{\frac{2}{1+s}}$ be taken by the truncated version

$$y \mapsto \min\left\{a^{-\frac{2}{1+s}}y^{\frac{2}{1+s}}, H(a)\right\},$$

to yield the inequalities $\|v\|_{L^\infty(\mathbb{R})} \leq \tilde{F}^{-1}(\mu^2)$ and $|v|_\infty \leq \tilde{F}^{-1}(\frac{1}{2}\mu^2)$ with

$$\tilde{F}(y) := 2\int_0^y \int_0^{y_1} \min\left\{a^{-\frac{2}{1+s}}y_2^{\frac{2}{1+s}}, H(a)\right\}\mathrm{d}y_2\mathrm{d}y_1.$$

As $\tilde{F}$ is strictly increasing and agrees with F on $(0, aH(a)^{\frac{1+s}{2}})$, we necessarily have both $\tilde{F}^{-1}(\mu^2) = F^{-1}(\mu^2)$ and $\tilde{F}^{-1}(\frac{1}{2}\mu^2) = F^{-1}(\frac{1}{2}\mu^2)$ provided $F^{-1}(\mu^2) < aH(a)^{\frac{1+s}{2}}$. As $F^{-1}(\mu^2)$ is exactly the right-hand side of (4.13), we see that the latter inequality holds since

$$F^{-1}(\mu^2) = 2^{\frac{1+s}{4+2s}}c_s\mu^{\frac{1+s}{2+s}}a^{\frac{1}{2+s}} < 2c_s\mu^{\frac{1+s}{2+s}}a^{\frac{1}{2+s}} = aH(a)^{\frac{1+s}{2}}.$$

Thus, the bounds for $\|v\|_{L^\infty(\mathbb{R})}$ and $|v|_\infty$ attained now coincides again with (4.13) and (4.14). It then follows that v admits ω as a modulus of growth for all $h \in (0,\infty)$; indeed, for any $h \in [H(a),\infty)$ we have the two trivial inequalities

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}}\Big[v(x+h) - v(x)\Big] \leq 2|v|_\infty, \qquad\qquad aH(a)^{\frac{1+s}{2}} \leq ah^{\frac{1+s}{2}},$$

and so we would be done if $2|v|_\infty \leq aH(a)^{\frac{1+s}{2}}$, which is precisely the already established inequality (4.14) multiplied by two. $\square$

The next proposition combines Lemma 4.3 and 4.4 to attain a corresponding result for the operator $S_\varepsilon^B \circ S_\varepsilon^K$. While it in Section 3 was natural to work with iterations of $S_\varepsilon^K \circ S_\varepsilon^B$, it will here be easier to work with its counterpart $S_\varepsilon^B \circ S_\varepsilon^K$. We now introduce the useful limit value $\underline{a}$ defined by

$$\underline{a} = \left(\frac{2c_s\kappa_s}{1+s}\right)^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}}. \tag{4.15}$$

This quantity will naturally occur in our calculations to come; it relates to the sought coefficient $a(t)$ through the relation $\lim_{t\to\infty} a(t) = \underline{a}$.

Proposition 4.6. *For every $A > \underline{a}$, there are constants $C_A, \varepsilon_A > 0$ such that: if $v \in BV(\mathbb{R})$ satisfies $\|v\|_{L^2(\mathbb{R})} \leq \mu$ and admits the modulus of growth $h \mapsto ah^{\frac{1+s}{2}}$ for some $a \in [\underline{a}, A]$, then for every $\varepsilon \in (0,\varepsilon_A]$ the function $w = S_\varepsilon^B \circ S_\varepsilon^K(v)$ admits the modulus of growth*

$$h \mapsto \Big(a - \varepsilon f(a) + \varepsilon^2 C_A\Big)h^{\frac{1+s}{2}}, \tag{4.16}$$

where $f(a) \geq 0$ is given by

$$f(a) = \left[\frac{(1+s)a^{\frac{2-s}{2+s}}}{2^{\frac{2}{1+s}}c_s^{\frac{1-s}{1+s}}\mu^{\frac{1-s}{2+s}}}\right]\left[a^{\frac{3+2s}{2+s}} - \underline{a}^{\frac{3+2s}{2+s}}\right]. \tag{4.17}$$

Proof. For fixed $A > \underline{a}$, let $v \in BV(\mathbb{R})$ and $a \in [\underline{a}, A]$ be as described in the lemma. We fix the pair v and a for convenience, but it should be clear from the proof that the construction of C_A and ε_A do not in fact depend on said pair. Introduce for $\varepsilon > 0$ the auxiliary function $\tilde{v} = S_\varepsilon^K(v)$. Combining Lemma 4.4 and (4.14), $\tilde{v}$ admits the concave modulus of growth

$$\tilde{\omega}(h) = ah^{\frac{1+s}{2}} + \varepsilon c_s\kappa_s a^{\frac{1}{2+s}}\mu^{\frac{1+s}{2+s}}h^s,$$

where $|K|_{TV^s}$ was replaced by the larger κ_s introduced at the beginning of this subsection. And since $\tilde{v} \in BV(\mathbb{R})$, as follows from (3.27) and (3.28), we can further apply Lemma 4.3 to $w = S_\varepsilon^B(\tilde{v})$, which combined with $\tilde{\omega}'(h) > (\frac{1+s}{2})ah^{\frac{s-1}{2}}$, allows us to conclude that w admits the modulus of growth

$$\begin{aligned}\omega(h) &= \frac{ah^{\frac{1+s}{2}} + \varepsilon c_s\kappa_s a^{\frac{1}{2+s}}\mu^{\frac{1+s}{2+s}}h^s}{1+\varepsilon(\frac{1+s}{2})ah^{\frac{s-1}{2}}} \\ &= ah^{\frac{1+s}{2}} + \frac{-\varepsilon(\frac{1+s}{2})a^2h^s + \varepsilon c_s\kappa_s a^{\frac{1}{2+s}}\mu^{\frac{1+s}{2+s}}h^s}{1+\varepsilon(\frac{1+s}{2})ah^{\frac{s-1}{2}}} \\ &= ah^{\frac{1+s}{2}} - \varepsilon \underbrace{\left[\frac{(1+s)a^2 - 2c_s\kappa_s a^{\frac{1}{2+s}}\mu^{\frac{1+s}{2+s}}}{2h^{\frac{1-s}{2}} + \varepsilon(1+s)a}\right]}_{B(a,h,\varepsilon)} h^{\frac{1+s}{2}},\end{aligned} \tag{4.18}$$

where $B(a,h,\varepsilon)$ denotes the square bracket. With $\underline{a}$ as given by (4.15), this square bracket can further be factored

$$B(a,h,\varepsilon) = \left[\frac{(1+s)a^{\frac{1}{2+s}}}{2h^{\frac{1-s}{2}} + \varepsilon(1+s)a}\right]\left[a^{\frac{3+2s}{2+s}} - \underline{a}^{\frac{3+2s}{2+s}}\right]. \tag{4.19}$$

Since $a \geq \underline{a}$ it follows that $B(a,h,\varepsilon)$ is non-negative and thus non-increasing in $h > 0$. Consequently, we read from (4.18) the inequality

$$\omega(h) \leq \Big(a - \varepsilon B(a,\overline{h},\varepsilon)\Big)h^{\frac{1+s}{2}}, \qquad 0 < h < \overline{h}. \tag{4.20}$$

Since (4.20) can be viewed as implying that w admits a homogeneous modulus of growth on bounded intervals, we would like to make use of Lemma 4.5; however, we do not necessarily have $\|w\|_{L^2(\mathbb{R})} \leq \mu$ (as is assumed by said lemma). We deal with this small inconvenience as follows: define $\tilde{w}$ by

$$\tilde{w} := \rho^{-1}w, \qquad \rho := \max\Big\{1, \mu^{-1}\|w\|_{L^2(\mathbb{R})}\Big\}, \tag{4.21}$$

that is, $\tilde{w}$ is the renormalized version of w if the L^2 norm of w exceeds μ. We proceed by proving the proposition for $\tilde{w}$ and then extend the result to w. Observe that ω must serve as a modulus of growth also for $\tilde{w}$ since $\rho \geq 1$, and consequently by (4.20), $\tilde{w}$ further admits for any fixed $\overline{h} > 0$ the modulus of growth

$$h \mapsto \Big(a - \varepsilon B(a,\overline{h},\varepsilon)\Big)h^{\frac{1+s}{2}}, \tag{4.22}$$

for the restricted values $h \in (0,\overline{h})$. Lemma 4.5 then tells us that $\tilde{w}$ must additionally admit (4.22) as a modulus of growth for all $h > 0$ provided

$$H\Big(a - \varepsilon B(a,\overline{h},\varepsilon)\Big) \leq \overline{h}, \tag{4.23}$$

where the function H is as defined by (4.12). We now show that there is an appropriate constant D_A so that $\overline{h} = H(a) + \varepsilon D_A$ satisfies (4.23). To do so, we start by introducing the closed set of points (a,h,ε) defined by

$$S_A = [\underline{a}, A] \times [H(A),\infty) \times [0,\infty),$$

where we abuse notation slightly by reusing a as a dummy variable for referring to elements in $[\underline{a}, A]$ (although the original $a \in [\underline{a}, A]$ is fixed). From (4.19) we see that both $(a,h,\varepsilon) \mapsto$

$B(a,h,\varepsilon)$ and its partial derivatives are bounded on the set S_A. We exploit the additional smoothness of B later; for now we need only $\|B\|_{L^\infty(S_A)} < \infty$. Pick $\varepsilon_A > 0$ such that

$$\varepsilon_A \|B\|_{L^\infty(S_A)} \le \tfrac{1}{2}\underline{a},$$

and observe that the argument of H in (4.23) must then lie in $[\frac{1}{2}\underline{a}, A]$ for all $(a,\overline{h},\varepsilon) \in [\underline{a}, A] \times [H(a),\infty) \times [0,\varepsilon_A] \subset S_A$. Moreover, as H is smooth on $[\frac{1}{2}\underline{a}, A]$ we conclude for any such triplet $(a,\overline{h},\varepsilon)$ that

$$H\Big(a - \varepsilon B(a,\overline{h},\varepsilon)\Big) \le H(a) + \varepsilon \|H'\|_{L^\infty([\frac{1}{2}\underline{a},A])}\|B\|_{L^\infty(S_A)} =: H(a) + \varepsilon D_A.$$

Thus, the choice $\overline{h} := H(a) + \varepsilon D_A$ satisfies (4.23) for every $a \in [\underline{a}, A]$ and $\varepsilon \in (0,\varepsilon_A]$, and so substituting for $\overline{h}$ in (4.22), we conclude that $\tilde{w}$ admits the modulus of growth

$$h \mapsto \Big(a - \varepsilon B(a, H(a) + \varepsilon D_A, \varepsilon)\Big) h^{\frac{1+s}{2}}, \tag{4.24}$$

for all $h > 0$, provided $\varepsilon \in (0,\varepsilon_A]$ and $a \in [\underline{a}, A]$ (the latter already assumed). Recalling that the partial derivatives of B are bounded on S_A, we can write

$$B(a, H(a) + \varepsilon D_A, \varepsilon) \ge B(a, H(a), 0) - \varepsilon\Big[D_A \|\tfrac{\partial B}{\partial h}\|_{L^\infty(S_A)} + \|\tfrac{\partial B}{\partial \varepsilon}\|_{L^\infty(S_A)}\Big], \tag{4.25}$$

and so letting C_A denote a constant no smaller than the square bracket in (4.25), we combine this inequality with (4.24) to further conclude that

$$h \mapsto \Big(a - \varepsilon B(a, H(a), 0) + \varepsilon^2 C_A\Big) h^{\frac{1+s}{2}}, \tag{4.26}$$

also serves as a modulus of growth for $\tilde{w}$, again with $\varepsilon \in (0,\varepsilon_A]$ and $a \in [\underline{a}, A]$. Using the explicit expressions (4.19) and (4.12) one attains the identity $B(a, H(a), 0) = f(a)$, where f is defined in (4.17), and so the proposition has been proved for the renormalized function $\tilde{w}$. It remains to extend the result to w; assume from here on out that $\varepsilon \in (0,\varepsilon_A]$. Introducing $\tilde{a} = (a - \varepsilon f(a) + \varepsilon^2 C_A)$ for brevity, it is clear from the relation $w = \rho\tilde{w}$, where ρ is as defined in (4.21), that w admits $h \mapsto \rho\tilde{a}h^{\frac{1+s}{2}}$ as a modulus of growth, as the same can be said for $\tilde{w}$ and $h \mapsto \tilde{a}h^{\frac{1+s}{2}}$. Moreover, by a similar and coarser calculation as in the proof of Lemma 3.3, we have $\|w\|_{L^2(\mathbb{R})} \le (1+\varepsilon^2\kappa^2)\|u\|_{L^2(\mathbb{R})}$ where $\kappa = \|K\|_{L^1(\mathbb{R})}$, and so $\rho \le 1 + \varepsilon^2\kappa^2$. Thus

$$\rho\tilde{a} \le (1+\varepsilon^2\kappa^2)\tilde{a} = a - \varepsilon f(a) + \varepsilon^2[C_A + \kappa^2\tilde{a}] \le a - \varepsilon f(a) + \varepsilon^2\tilde{C}_A,$$

where $\tilde{C}_A := [C_A + \kappa^2(A + \varepsilon_A^2 C_A)]$, and so this calculation shows that the proposition also holds for w after choosing a larger constant C_A. □

Together with a few results from Section 3, the previous proposition equips us with all we need to construct moduli of growth for entropy solutions of (1.1). Roughly speaking, we can for small $\varepsilon > 0$ iterate Proposition 4.6 repeatedly to construct a modulus of growth for an approximate entropy solution (3.32), and further letting $\varepsilon \to 0$ this construction carries over to the entropy solution itself. To formalize, we shall introduce some notation and assume from here on that a pair of constants ε_A, C_A, as described by Proposition 4.6, has been chosen for each $A > \underline{a}$. Define the function

$$g_A^\varepsilon(a) := a - \varepsilon f(a) + \varepsilon^2 C_A, \tag{4.27}$$

which is parameterized over $A > \underline{a}$ and $\varepsilon \in (0,\varepsilon_A]$ and where

$$f(a) = \gamma a^{\frac{2-s}{2+s}}\Big(a^{\frac{3+2s}{2+s}} - \underline{a}^{\frac{3+2s}{2+s}}\Big), \qquad\qquad \gamma = \frac{1+s}{2^{\frac{2}{1+s}} c_s^{\frac{1-s}{1+s}} \mu^{\frac{1-s}{2+s}}}. \tag{4.28}$$

The function f in (4.28) is indeed the same as in (4.17), and so $g_A^\varepsilon(a)$ is the 'new Hölder coefficient' that Proposition 4.6 provides. In the coming proposition, we carry out the above sketched argument consisting in part of repeated iterations of Proposition 4.6, and consequently, we will encounter repeated compositions of g_A^ε. We point out two relevant facts about g_A^ε. First off, for any $A > \underline{a}$ and sufficiently small $\varepsilon > 0$, the function g_A^ε maps $[\underline{a}, A]$ to itself. To see this, note from (4.27) that $(g_A^\varepsilon)'$ is strictly positive on $[\underline{a}, A]$ for small $\varepsilon > 0$. Moreover, we have

$$g_A^\varepsilon(\underline{a}) = \underline{a}, \qquad\qquad g_A^\varepsilon(A) = A - \varepsilon f(A) + \varepsilon^2 C_A,$$

and since $f(A) > 0$, it is clear that $\varepsilon > 0$ can be made sufficiently small such that

$$\underline{a} = g_A^\varepsilon(\underline{a}) \le g_A^\varepsilon(a) \le g_A^\varepsilon(A) \le A, \tag{4.29}$$

for all $a \in [\underline{a}, A]$. Our second fact, rigorously justified in the coming proposition, is that repeated compositions of g_A^ε applied to the starting value $a = A$ will, as $\varepsilon \to 0$, result in a smooth function $a_A \colon [0, \infty) \to (\underline{a}, A]$, implicitly defined by

$$t = \int_{a_A(t)}^{A} \frac{\mathrm{d}a}{f(a)}. \tag{4.30}$$

That (4.30) yields a unique value $a_A(t) \in (\underline{a}, A]$ for each $t \in [0, \infty)$ follows as the positive integrand has a non-integrable singularity at $a = \underline{a}$. Alternatively, the function a_A can be viewed as the solution of the differential equation

$$\begin{cases} a'(t) = -f(a(t)), & t > 0, \\ a(0) = A. \end{cases} \tag{4.31}$$

For the next proposition, we shall exploit the two constants

$$M_A = \max_{a \in [\underline{a}, A]} |f'(a)|, \qquad\qquad \tilde{M}_A = \max_{a \in [\underline{a}, A]} |f(a) f'(a)|, \tag{4.32}$$

both well defined as f is smooth on $\mathbb{R}^+$. Note that the latter serves as a bound on $(a_A)'' = f(a_A) f'(a_A)$, and so by Taylor expansion, we infer

$$|a_A(t + \varepsilon) - a_A(t) + \varepsilon f(a_A(t))| \le \frac{\varepsilon^2}{2} \tilde{M}_A, \tag{4.33}$$

for all $t \ge 0$ and $\varepsilon \ge 0$.

Proposition 4.7. *Let u be an entropy solution of (1.1), whose initial data $u_0 \in BV(\mathbb{R})$ satisfies $\|u_0\|_{L^2(\mathbb{R})} \le \mu$ and admits a modulus of growth $h \mapsto A h^{\frac{1+s}{2}}$ for some $A > \underline{a}$. Then for all $t > 0$, the function $x \mapsto u(t, x)$ admits the modulus of growth*

$$h \mapsto a_A(t) h^{\frac{1+s}{2}},$$

with a_A given by (4.30).

Proof. Consider $t > 0$ fixed, and assume without loss of generality that $\|u_0\|_{L^2(\mathbb{R})} < \mu$; if the proposition holds in this case, it necessarily also holds in the case $\|u_0\|_{L^2(\mathbb{R})} \le \mu$ as the implicit μ-dependence of $a_A(t)$ is a continuous one. Pick a large $n \in \mathbb{N}$, set $\varepsilon = \frac{t}{n}$ and consider the family of functions $u_n^k \in BV(\mathbb{R})$ defined inductively by

$$\begin{cases} u_n^0 = S_\varepsilon^B(u_0), \\ u_n^k = S_\varepsilon^B \circ S_\varepsilon^K(u_n^{k-1}), \quad & k = 1, 2, \ldots, n, \end{cases}$$

As u_0 admits $h \mapsto A h^{\frac{1+s}{2}}$ as a modulus of growth, so does u_n^0 by Lemma 4.3. Observe also that each $u_n^k \in BV(\mathbb{R})$ as follows by induction and the properties of S_ε^B and S_ε^K listed at the very beginning in the proof of Proposition 3.2. Moreover, by similar reasoning as in the proof of Lemma 3.3, we have

$$\|u_n^k\|_{L^2(\mathbb{R})} \le e^{\frac{k}{2}\varepsilon^2\kappa^2}\|u_0\|_{L^2(\mathbb{R})} \le e^{\frac{t}{2n}\kappa^2}\|u_0\|_{L^2(\mathbb{R})}, \quad k = 0, 1, \dots, n,$$

where $\kappa = \|K\|_{L^1(\mathbb{R})}$. Since we have a strict inequality $\|u_0\|_{L^2(\mathbb{R})} < \mu$, we can assume n large enough such that $\|u_n^k\|_{L^2(\mathbb{R})} \le \mu$ for every k. We define further the coefficients a_n^k inductively by

$$\begin{cases} a_n^0 = A, \\ a_n^k = g_A^\varepsilon(a_n^{k-1}), \quad k = 1, 2, \dots, n, \end{cases}$$

where g_A^ε is given by (4.27). We assume n large enough such that $\varepsilon = \frac{t}{n}$ is both less than $\varepsilon_A > 0$ and small enough such that g_A^ε maps $[\underline{a}, A]$ to itself (see the discussion leading up to (4.29)). In particular, each a_n^k is in $[\underline{a}, A]$. We may now apply Proposition 4.6 inductively to each pair (u_n^k, a_n^k), starting with (u_n^0, a_n^0). As u_n^0 admits $h \mapsto a_n^0 h^{\frac{1+s}{2}}$ as a modulus of growth, Proposition 4.6 infers the same relationship for the pair (u_n^1, a_n^1), and by repeating the argument, the same can be said for all pairs (u_n^k, a_n^k). Most importantly, u_n^n admits $h \mapsto a_n^n h^{\frac{1+s}{2}}$ as a modulus of growth. The proposition will now follow if we can, as $n \to \infty$, establish the limits

$$a_n^n \to a_A(t), \tag{4.34}$$
$$u_n^n \to u(t), \tag{4.35}$$

where $u(t) = u(t, \cdot)$ and the latter limit is taken in $L^1_{\text{loc}}(\mathbb{R})$. Indeed, in this scenario we can let φ denote any non-negative smooth function of compact support that satisfies $\int_{\mathbb{R}} \varphi \mathrm{d}x = 1$ so to calculate for $h > 0$

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in R} \Big[u(t, x+h) - u(t, x)\Big] &= \sup_{\varphi} \langle u(t, \cdot + h) - u(t, \cdot), \varphi\rangle \\ &= \sup_{\varphi} \lim_{n\to\infty} \langle u_n^n(\cdot + h) - u_n^n, \varphi\rangle \\ &\le \sup_{\varphi} \lim_{n\to\infty} a_n^n h^{\frac{1+s}{2}} \\ &= a_A(t) h^{\frac{1+s}{2}}. \end{aligned} \tag{4.36}$$

We first prove (4.34). Using the explicit form (4.27) of g_A^ε with $\varepsilon = \frac{t}{n}$, the constants (4.32) and the inequality (4.33) we can calculate for $k \ge 1$,

$$\begin{aligned} &\Big|a_n^k - a_A\Big(\tfrac{kt}{n}\Big)\Big| \\ &= \Big|g_A^\varepsilon\Big(a_n^{k-1}\Big) - a_A\Big(\tfrac{(k-1)t}{n} + \tfrac{t}{n}\Big)\Big| \\ &\le \Big|a_n^{k-1} - a_A\Big(\tfrac{(k-1)t}{n}\Big)\Big| + (\tfrac{t}{n})\Big|f\Big(a_n^{k-1}\Big) - f\Big(a_A\Big(\tfrac{(k-1)t}{n}\Big)\Big)\Big| + (\tfrac{t}{n})^2\Big(C_A + \tfrac{1}{2}\tilde{M}_A\Big) \\ &\le \Big[1 + (\tfrac{t}{n})M_A\Big]\Big|a_n^{k-1} - a_A\Big(\tfrac{(k-1)t}{n}\Big)\Big| + (\tfrac{t}{n})^2 D_A, \end{aligned} \tag{4.37}$$

with $D_A \coloneqq C_A + \frac{1}{2}\tilde{M}_A$. By repeated use of (4.37), and the fact that $a_n^0 = a_A(0) = A$, we conclude

$$|a_n^n - a_A(t)| \le (\tfrac{t}{n})^2 D_A \sum_{k=0}^{n-1} \Big[1 + (\tfrac{t}{n})M_A\Big]^k \le \tfrac{1}{n}\Big[t^2 D_A e^{tM_A}\Big],$$

and thus (4.34) is established. To prove (4.35), we recall definition (3.26) of the approximate solution map $S_{\varepsilon,t}$ and observe the relation

$$u_n^n = S_\varepsilon^B \circ S_{\varepsilon,t}(u_0) =: S_\varepsilon^B(u^\varepsilon(t)), \tag{4.38}$$

where the definition of u^ε coincides with (3.32), although we now work with a particular u_0 and $\varepsilon = \frac{t}{n}$. As Proposition 3.5 ensures that $\lim_{\varepsilon\to 0} u^\varepsilon(t) = u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$, the same limit then carries over to u_n^n (as $n\to\infty$) by (4.38) and the time continuity of the map S_ε^B (3.29) together with the TV bound of u^ε provided by Proposition 3.2. With the two limits (4.34) and (4.35) established, the proof is complete. $\square$

We may now prove Theorem 2.3.

Proof of Theorem 2.3. We prove the theorem first for $u_0 \in C_c^\infty(\mathbb{R})$, and without loss of generality we assume $u_0 \neq 0$. As the positive constant μ (introduced at the beginning of the subsection) was arbitrary, we may assume $\mu = \|u_0\|_{L^2(\mathbb{R})}$. As $u_0 \in C_c^\infty(\mathbb{R})$, we infer from Proposition 4.7 the existence of a sufficiently large A such that u, the entropy solution of (1.1) corresponding to u_0, admits $h \mapsto a_A(t)h^{\frac{1+s}{2}}$ as a modulus of growth for all $t > 0$.

Observe that we have the following elementary inequality if $a - \underline{a} \geq 0$

$$\left(a-\underline{a}\right)^{\frac{5+s}{2+s}} = \left(a-\underline{a}\right)^{\frac{2-s}{2+s}}\left(a-\underline{a}\right)^{\frac{3+2s}{2+s}} \leq a^{\frac{2-s}{2+s}}\left(a^{\frac{3+2s}{2+s}} - \underline{a}^{\frac{3+2s}{2+s}}\right),$$

where we for the second factor used that $x \mapsto x^p$ is super-additive when $x \geq 0$ and $p \geq 1$ (giving the desired conclusion for $x = a - \underline{a}$ and $p = \frac{3+2s}{2+s}$). Using this in (4.30) gives

$$t \leq \int_{a_A(t)}^{A} \frac{\mathrm{d}a}{\gamma\left(a-\underline{a}\right)^{\frac{5+s}{2+s}}} = \frac{2+s}{3\gamma\left(a_A(t)-\underline{a}\right)^{\frac{3}{2+s}}} - \frac{2+s}{3\gamma\left(A-\underline{a}\right)^{\frac{3}{2+s}}}. \tag{4.39}$$

Removing the negative term on the right hand side and then rewriting (4.39), further gives

$$a_A(t) \leq \underline{a} + \left(\frac{2+s}{3\gamma}\right)^{\frac{2+s}{3}} \frac{1}{t^{\frac{2+s}{3}}} =: a(t).$$

In particular, u must also admit $h \mapsto a(t)h^{\frac{1+s}{2}}$ as a modulus of growth. By Lemma A.2, we see that $a(t)$ may equivalently be written

$$a(t) = C_1(s)\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}} + C_2(s)\frac{\mu^{\frac{1-s}{3}}}{t^{\frac{2+s}{3}}}, \tag{4.40}$$

where $C_1(s)$ and $C_2(s)$ are given by (A.1). This expression is exactly (2.3) save for the fact that we have required κ_s (introduced at the beginning of the subsection) to be greater or equal to $|K|_{TV^s}$ *and* positive. Thus, we may not directly set $\kappa_s = |K|_{TV^s}$ if $K = 0$. However, by $a(t)$'s continuous dependence on κ_s, it is clear that no problem may occur. Thus, Theorem 2.3 follows for C_c^∞-initial data.

Next, consider $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ and let u denote the corresponding entropy solution of (1.1). Pick a sequence of entropy solutions $(u_k)_{k\in\mathbb{N}}$ whose initial data $(u_{0,k})_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ satisfies

$$\|u_{0,k}\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \qquad \|u_{0,k}\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})},$$

and yields the limit $\lim_{k\to\infty} u_{0,k} = u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. By Proposition 3.1, we then also get $\lim_{k\to\infty} u_k(t) = u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Now Theorem 2.3 carries over to u by a calculation similar to (4.36), and so the theorem has been proved for $L^2 \cap L^\infty$-initial data.

Finally, that this result can be extended to all weak solutions provided by Corollary 2.2 follows by a density argument as above (using the continuity of the solution map S of Corollary 2.2 instead of the weighted L^1-contraction of Proposition 3.1). ☐

Appendix A. Auxiliary results

In the coming lemma we work with the concept of a modulus of growth as defined by Def. 4.1.

Lemma A.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ admit a modulus of growth ω that satisfies $\omega(0+) = 0$. Then f admits essential left and right limits at each point $x \in \mathbb{R}$. In particular, there are functions f^- and f^+, respectively left- and right-continuous, that coincides a.e. with f.*

Proof. For any $x \in \mathbb{R}$ the existence of an essential left limit $f(x-)$ of f at x, follows from the calculation

$$\begin{aligned}
&\operatorname*{ess\,lim\,sup}_{\substack{y<0\\ y\to 0}} f(x+y) - \operatorname*{ess\,lim\,inf}_{\substack{y<0\\ y\to 0}} f(x+y)\\
&= \operatorname*{ess\,lim\,sup}_{\substack{y_2<y_1<0\\ y_2,y_1\to 0}} \Big[f(x+y_1) - f(x+y_2)\Big]\\
&\leq \limsup_{\substack{y_2<y_1<0\\ y_2,y_1\to 0}} \omega(y_1-y_2) = 0.
\end{aligned}$$

By the Lebesgue differentiation theorem, the function $f^-(x) := f(x-)$ can only differ from f on a null set, and moreover, must be left continuous as the above calculation could be repeated for f^- with essential limits replaced by limits. A similar argument yields the existence of an essential right limit $f(x+)$ of f at each $x \in \mathbb{R}$ and further that $f^+(x) := f(x+)$ is a right-continuous function agreeing a.e. with f. ☐

The next lemma deals with quantities appearing throughout the paper and the relations between them. For convenience, we here list the definition of each relevant quantity; some of them given for the first time. The quantities c_s and γ were in (4.11) and (4.28) defined to be

$$c_s = \left[\frac{(2+s)(3+s)}{2(1+s)^2}\right]^{\frac{1+s}{2(2+s)}}, \qquad \gamma = \frac{1+s}{2^{\frac{2}{1+s}} c_s^{\frac{1-s}{1+s}} \mu^{\frac{1-s}{2+s}}}.$$

We also introduce the expressions $C_1(s)$ and $C_2(s)$ by

$$C_1(s) := \frac{2^{\frac{3+s}{6+4s}}[(2+s)(3+s)]^{\frac{1+s}{6+4s}}}{1+s}, \qquad C_2(s) := \frac{2^{\frac{4+2s}{3+3s}}(2+s)^{\frac{5+s}{6}}(3+s)^{\frac{1-s}{6}}}{2^{\frac{1-s}{6}}3^{\frac{2+s}{3}}(1+s)}. \tag{A.1}$$

In the coming lemma, we will also see the quantities μ and κ_s; these are simply placeholders for the expressions $\|u_0\|_{L^2(\mathbb{R})}$ and $|K|_{TV^s}$ respectively and will not affect the algebra in any non-trivial way.

Lemma A.2. *With $c_s, \gamma, C_1(s), C_2(s), \mu$ and κ_s as they appear above, we have the relations*

$$\underline{a} := \left(\frac{2c_s\kappa_s}{1+s}\right)^{\frac{2+s}{3+2s}} \mu^{\frac{1+s}{3+2s}} = C_1(s)\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}}, \tag{A.2}$$

$$\left(\frac{2+s}{3\gamma}\right)^{\frac{2+s}{3}} = C_2(s)\mu^{\frac{1-s}{3}}. \tag{A.3}$$

Proof. We start with (A.2): inserting for c_s on the left-hand side of (A.2) we get

$$\left(\frac{2}{1+s}\right)^{\frac{2+s}{3+2s}}\left(\frac{(2+s)(3+s)}{2(1+s)^2}\right)^{\frac{1+s}{2(3+2s)}}\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}}$$
$$=\underbrace{\left[\frac{2^{\frac{3+s}{6+4s}}[(2+s)(3+s)]^{\frac{1+s}{6+4s}}}{1+s}\right]}_{C_1(s)}\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}},$$

and so (A.2) is established. Second, we prove (A.3): if we on the left-hand side of (A.3) insert for γ we get

$$\left(\frac{(2+s)^{\frac{2+s}{3}}2^{\frac{2(2+s)}{3(1+s)}}}{3^{\frac{2+s}{3}}(1+s)^{\frac{2+s}{3}}}\right)c_s^{\frac{(1-s)(2+s)}{3(1+s)}}\mu^{\frac{1-s}{3}}.$$

Further inserting for c_s we obtain

$$\left(\frac{(2+s)^{\frac{2+s}{3}}2^{\frac{2(2+s)}{3(1+s)}}}{3^{\frac{2+s}{3}}(1+s)^{\frac{2+s}{3}}}\right)\left(\frac{(2+s)(3+s)}{2(1+s)^2}\right)^{\frac{1-s}{6}}\mu^{\frac{1-s}{3}}$$
$$=\underbrace{\left[\frac{2^{\frac{4+2s}{3+3s}}(2+s)^{\frac{5+s}{6}}(3+s)^{\frac{1-s}{6}}}{2^{\frac{1-s}{6}}3^{\frac{2+s}{3}}(1+s)}\right]}_{C_2(s)}\mu^{\frac{1-s}{3}},$$

and so (A.3) is established. □

Appendix B. Proof of Corollary 2.5 and Corollary 2.6

We prove Corollary 2.5 which provides a decaying L^∞ bound for entropy solutions of (1.1).

Proof of Corollary 2.5. By the $s=0$ case of Theorem 2.3 we know that $u(t)$ admits the modulus of growth (Def. 4.1) $h\mapsto a(t)h^{\frac{1}{2}}$, where $a(t)$ is given by

$$a(t)=2^{\frac{4}{3}}3^{\frac{1}{6}}\|K\|_{L^1(\mathbb{R})}^{\frac{2}{3}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{1}{3}}+\frac{4\|u_0\|_{L^2(\mathbb{R})}^{\frac{1}{3}}}{3^{\frac{1}{2}}t^{\frac{2}{3}}}.$$

This expression is precisely what is provided by (2.3) when using $C_1(0)=2^{\frac{2}{3}}3^{\frac{1}{6}}$, $C_2(0)=4/3^{\frac{1}{2}}$ and $|K|_{TV^0}=2\|K\|_{L^1}$. Setting $\mu=\|u_0\|_{L^2(\mathbb{R})}$ in Lemma 4.5 and using $\|u(t)\|_{L^2(\mathbb{R})}\le\|u_0\|_{L^2(\mathbb{R})}$ we infer from said lemma – more specifically (4.13) – that

$$\|u(t)\|_{L^\infty(\mathbb{R})}\le 2^{\frac{1}{4}}3^{\frac{1}{4}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{1}{2}}a(t)^{\frac{1}{2}},$$

for all $t>0$, where we used that $c_s=3^{\frac{1}{4}}$ when $s=0$. Using the sub-additivity of $y\mapsto|y|^{\frac{1}{2}}$ we infer that

$$a(t)^{\frac{1}{2}}\le 2^{\frac{2}{3}}3^{\frac{1}{12}}\|K\|_{L^1(\mathbb{R})}^{\frac{1}{3}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{1}{6}}+\frac{2\|u_0\|_{L^2(\mathbb{R})}^{\frac{1}{6}}}{3^{\frac{1}{4}}t^{\frac{1}{3}}},$$

and so inserting this in the above inequality we get

$$\|u(t)\|_{L^\infty(\mathbb{R})}\le 2^{\frac{11}{12}}3^{\frac{1}{3}}\|K\|_{L^1(\mathbb{R})}^{\frac{1}{3}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{2}{3}}+\frac{2^{\frac{5}{4}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{2}{3}}}{t^{\frac{1}{3}}},$$

for all $t>0$. □

Next, we prove Corollary 2.6 which established a maximal lifespan for classical solutions of (1.1) with $L^2 \cap L^\infty$ data.

Proof of Corollary 2.6. Consider $s \in [0,1]$ fixed for now, and assume $|K|_{TV^s} < \infty$. As (bounded) classical solutions are entropy solutions, we may associate $u \in L^\infty \cap C^1((0,T)\times\mathbb{R})$ with the global entropy solution admitting u_0 as initial data, provided by Theorem 2.1; the discussion following the proof of Proposition 3.1 justifies this viewpoint. Referring to this solution also as u, we have by (2.1) that $x \mapsto u(T,x)$ is a well defined element of $L^2 \cap L^\infty(\mathbb{R})$ approximated in L^2 sense by $u(t)$ as $t \nearrow T$. Setting $v(t,x) \coloneqq u(T-t,-x)$, we see through pointwise evaluation that v also is a classical solution of (1.1) (and thus an entropy solution) on $(0,T)\times\mathbb{R}$ with initial data $v_0(x) \coloneqq u(T,-x)$. From (2.1) we then infer $\|v_0\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$ since

$$\|v_0\|_{L^2(\mathbb{R})} = \|u(T)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})} = \|v(T)\|_{L^2(\mathbb{R})} \le \|v_0\|_{L^2(\mathbb{R})}.$$

Using the identity $u_0(x) = v(T,-x)$ for a.e. $x \in \mathbb{R}$ and applying Theorem 2.3 to v we further find for all $h > 0$ and a.e. $x \in \mathbb{R}$ that

$$u_0(x-h) - u_0(x) = v(T,-x+h) - v(T,-x) \le a(T)h^{\frac{1+s}{2}}, \tag{B.1}$$

where $a(T)$ is given by

$$a(T) = C_1(s)|K|_{TV^s}^{\frac{2+s}{3+2s}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{1+s}{3+2s}} + C_2(s)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{3}}}{T^{\frac{2+s}{3}}} =: \underline{a} + \frac{q}{T^{\frac{2+s}{3}}}, \tag{B.2}$$

and where we have substituted $\|u_0\|_{L^2(\mathbb{R})}$ for $\|v_0\|_{L^2(\mathbb{R})}$ as the two quantities agree. Dividing each side of (B.1) by $h^{\frac{1+s}{2}}$ and taking the essential supremum with respect to $x \in \mathbb{R}$ we get

$$[u_0]_s \coloneqq \operatorname*{ess\,sup}_{\substack{x\in\mathbb{R}\\ h>0}}\left[\frac{u_0(x-h)-u_0(x)}{h^{\frac{1+s}{2}}}\right] \le \underline{a} + \frac{q}{T^{\frac{2+s}{3}}}, \tag{B.3}$$

and if $[u_0]_s > \underline{a}$ then (B.3) can be rewritten as

$$T \le \left[\frac{q}{[u_0]_s - \underline{a}}\right]^{\frac{3}{2+s}} = \left(\frac{C_2(s)}{1-\frac{\underline{a}}{[u_0]_s}}\right)^{\frac{3}{2+s}}\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{2+s}}}{[u_0]_s^{\frac{3}{2+s}}} =: F\Big(\tfrac{\underline{a}}{[u_0]_s}\Big)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{2+s}}}{[u_0]_s^{\frac{3}{2+s}}}, \tag{B.4}$$

where the first equality replaced q by its explicit expression as given by (B.2). We now show that this gives for any $\rho \in (0,1)$ the following implication

$$[u_0]_s^{3+2s} > \left(\frac{C_1(s)}{\rho}\right)^{3+2s}|K|_{TV^s}^{2+s}\|u_0\|_{L^2(\mathbb{R})}^{1+s}, \quad\implies\quad T \le F(\rho)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{2+s}}}{[u_0]_s^{\frac{3}{2+s}}}. \tag{B.5}$$

Indeed, using the explicit expression (B.2) for $\underline{a}$ we see that the left-hand side of (B.5) is equivalent to $[u_0]_s > \underline{a}/\rho$ which, as $\rho \in (0,1)$, implies that $[u_0]_s > \underline{a}$ and so (B.4) holds. By observing that $\rho \mapsto F(\rho)$ is strictly increasing on $(0,1)$ and that $\rho > \underline{a}/[u_0]_s$ we see that the right-hand side of (B.5) then follows from (B.4). With (B.5) established, the corollary follows: for any $\rho \in (0,1)$ we get such universal constants c and C by setting

$$c = \sup_{s\in[0,1]}\left(\frac{C_1(s)}{\rho}\right)^{3+2s}, \qquad C = \sup_{s\in[0,1]} F(\rho) = \sup_{s\in[0,1]}\left(\frac{C_2(s)}{1-\rho}\right)^{\frac{3}{2+s}}. \tag{B.6}$$

The free parameter ρ allows us to shrink one of the two constants at the cost of enlarging the other; in particular, c is at its smallest for $\rho \to 1$ while C is at its smallest for $\rho \to 0$. $\square$

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Paper II

Periodic Hölder waves in a class of negative-order dispersive equations

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PERIODIC HÖLDER WAVES IN A CLASS OF NEGATIVE-ORDER DISPERSIVE EQUATIONS

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Abstract. We prove the existence of highest, cusped, periodic travelling-wave solutions with exact and optimal α-Hölder continuity in a class of fractional negative-order dispersive equations of the form

$$u_t + (|\mathrm{D}|^{-\alpha} u + n(u))_x = 0$$

for every $\alpha \in (0,1)$ with homogeneous Fourier multiplier $|\mathrm{D}|^{-\alpha}$. We tackle nonlinearities $n(u)$ of the type $|u|^p$ or $u|u|^{p-1}$ for all real $p > 1$, and show that when n is odd, the waves also feature antisymmetry and thus contain inverted cusps. Tools involve detailed pointwise estimates in tandem with analytic global bifurcation, where we resolve the issue with nonsmooth n by means of regularisation. We believe that both the construction of highest antisymmetric waves and the regularisation of nonsmooth terms to an analytic bifurcation setting are new in this context, with direct applicability also to generalised versions of the Whitham, the Burgers–Poisson, the Burgers–Hilbert, the Degasperis–Procesi, the reduced Ostrovsky, and the bidirectional Whitham equations.

1. Introduction

1.1. **Main result.** In this paper, we shall be concerned with singular periodic travelling-wave solutions to a class of nonlinear and dispersive evolution equations of the form

$$u_t + (|\mathrm{D}|^{-\alpha} u + n(u))_x = 0. \tag{1}$$

This family may be viewed as a kind of generalised fractional Korteweg–de Vries (KdV) equations of negative-order, where we refer to [3] for a classical description of nonlocal variants of the KdV equation in the mathematical modelling of long-wave phenomena. The dispersive properties occur in the homogeneous negative-order (spatial) Fourier multiplier $|\mathrm{D}|^{-\alpha}$ for $\alpha \in (0,1)$ defined by

$$\mathscr{F}(|\mathrm{D}|^{-\alpha} u)(\xi) := |\xi|^{-\alpha}\, \widehat{u}(\xi),$$

with $\mathrm{D} := -\mathrm{i}\partial_x$, whereas the nonlinear effects originate from either of the generally nonsmooth nonlinearities

$$n(x) := \left\{ \begin{array}{l} |x|^p \text{ or} \\ x|x|^{p-1} \end{array} \right\} \qquad \text{with } p > 1 \text{ real.} \qquad \begin{array}{r} (2_{\mathrm{abs}}) \\ (2_{\mathrm{sgn}}) \end{array}$$

Our main contributions are to

i) prove the existence of highest, exactly α-Hölder continuous periodic steady solutions of the negative-order dispersive family (1) for all $\alpha \in (0,1)$ on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, and
ii) initiate a study of nonsmooth nonlinearities and antisymmetric features in the large-amplitude theory for negative-order dispersive evolution equations.

Precisely, we obtain the following result, with corresponding numerical illustrations in Fig. 1.

Theorem 1 (Existence)**.** *Let $\alpha \in (0,1)$ and $p > 1$ be real. Then there exists a nontrivial periodic travelling-wave solution φ of* (1) *with positive speed $c < \frac{p}{p-1}\|\mathscr{F}^{-1}(|\cdot|^{-\alpha})\|_{\mathrm{L}^1(\mathbb{T})}$. The solution is even (about $2\pi\mathbb{Z}$), has zero mean, and satisfies*

$$\max \varphi = \varphi(0) = \mu \qquad \textit{and} \qquad \varphi \in \mathrm{C}^\alpha(\mathbb{T}),$$

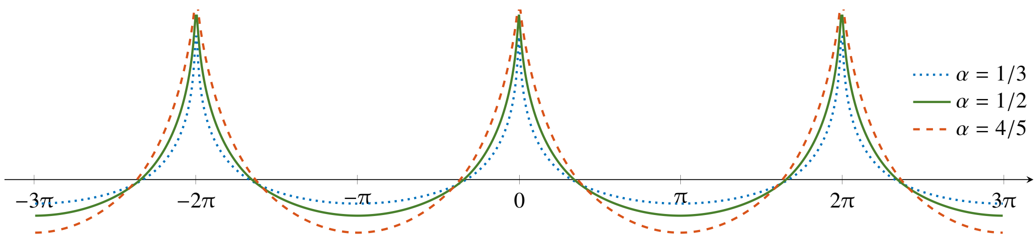
where $\mu := (c/p)^{1/(p-1)}$. It is also smooth (except possibly at the point where it vanishes) and strictly increasing on $(-\pi, 0)$ and exactly α-Hölder continuous at $x \in 2\pi\mathbb{Z}$, that is,

$$\mu - \varphi(x) \eqsim |x - 2\pi\ell|^\alpha$$

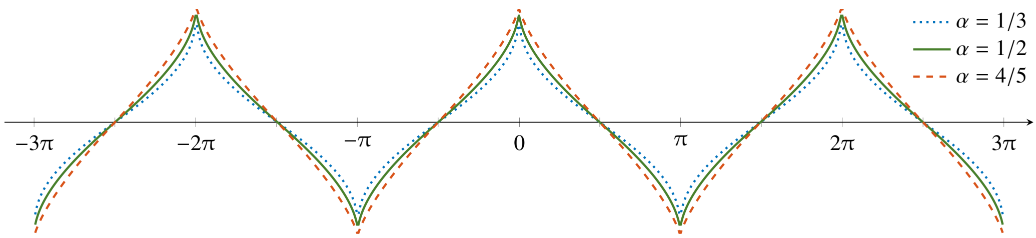
uniformly around $2\pi\ell$ for $\ell \in \mathbb{Z}$.

One has that φ is smooth around $-\pi$ in case (2_{abs})*, while φ is antisymmetric about $-\frac{\pi}{2}$ in case* (2_{sgn}) *and therefore also exactly α-Hölder continuous at $\pi\mathbb{Z}$ with $\min \varphi = \varphi(-\pi) = -\mu$.*

Remark 2. *$A \eqsim B$ is short for $A \lesssim B \lesssim A$, where $A \lesssim B$ symbolises that $A \leqslant \lambda B$ for a constant $\lambda > 0$. We say that $A(x) \lesssim B(x)$ (etc.) holds* uniformly *over a region if λ does not depend on x there.*



(a) Waves for the nonlinearity (2_{abs}) with $p = \sqrt{\pi} \approx 1.77$.



(b) Waves for the nonlinearity (2_{sgn}) with $p = \mathrm{e} \approx 2.71$.

Figure 1. Numerical approximations of large-amplitude, C^α-regular periodic waves for various α's and nonlinearities (2) in Theorem 1 using the SpecTraVVave software [20] for bifurcation in nonlinear dispersive evolution equations.

1.2. **Background.** Full-dispersion nonlinear evolution equations such as (1) have seen a keen interest in the recent years as nonlocal improvements of classical local equations. In particular, surface-wave models in shallow water of this class with various dispersive operators approximate the full water-wave equations [9, 10] and capture singular features not found in their local counterparts. Equation (1) with $n(u) = u^2$ may be seen as a dispersive perturbation of Burgers' equation and [18] outlines how this case for $\alpha = \frac{1}{2}$ is perhaps the simplest model incorporating the linear behaviour and characteristic nonlinearity of the water-waves problem. Both [18] $\left(\text{for } \alpha = \frac{1}{2}\right)$ and [7, Theorem 3.1] (for $\alpha \in (0,1)$) prove that solutions of (1) blow up in finite time for certain initial data. The resulting singularities occur in at least two ways: wave breaking [19], in which the spatial derivative of a bounded solution of (1) blows up, or sharp crests in travelling-wave solutions – reminiscent of the highest Stokes' wave [31] – which is the subject of this paper. We refer also to [25, 22, 8, 23, 29] for other results concerning singularities, well-posedness, persistence, and existence time of solutions.

Classical Fourier analysis shows that $|\mathrm{D}|^{-\alpha}$ constitutes a singular convolution operator on $\mathbb{R}$ with kernel $|\cdot|^{\alpha-1}$ and describes an eigenfunction of $\mathscr{F}$ when $\alpha = \frac{1}{2}$. Related to this case is the recent work by Ehrnström & Wahlén [15] on the Whitham equation [33], being a shallow-water model of type (1) with inhomogeneous dispersion $\sqrt{\tanh(\mathrm{D})/\mathrm{D}}$ and $n(u) = u^2$. Its corresponding symbol behaves as that of the KdV equation for small frequencies and decays like $|\xi|^{-\frac{1}{2}}$ as $|\xi| \to \infty$, for which the kernel may be written as $|x|^{-\frac{1}{2}}$ plus a regular term. The existence of a highest, cusped steady solution whose behaviour at the crest is like $1 - |x|^{\frac{1}{2}}$ modulo constants was conjectured by Whitham [34, p. 479], and the authors of [15] found this exactly $\mathrm{C}^{\frac{1}{2}}$ wave on $\mathbb{T}$ based on properties of the kernel, precise regularity estimates, and global bifurcation theory, building on preliminary analysis from [12, 13]. New work [32] also proves the existence of an extreme $\mathrm{C}^{\frac{1}{2}}$ solitary-wave solution by means of nonlocal center-manifold theory for the global bifurcation.

Table 1. Exact and global regularity of extreme periodic waves in negative-order equations with inhomogeneous or homogeneous dispersion. This paper also treats nonsmooth nonlinearities (2) for any real order $p > 1$, with applicability to the other works (that considered smooth $n(u) = u^2$).

DISPERSIVE OPERATOR	NEGATIVE ORDER AND REGULARITY		
	$\alpha \in (0,1)$ α-Hölder	$\alpha = 1$ log-Lipschitz	$\alpha > 1$ Lipschitz
Inhomogeneous: $(\tanh(\mathrm{D})/\mathrm{D})^{\alpha}$ or $(1 + \lvert\mathrm{D}\rvert^2)^{-\frac{\alpha}{2}}$	[15, 14, 1, 27]	[11, 14]	[24]
Homogeneous: $\lvert\mathrm{D}\rvert^{-\alpha}$	This paper	[14]	[5]

Inspired by the results for the Whitham equation, there has been a series of papers concerned with exact and global regularity of extreme periodic waves in similar negative-order equations with prototypical inhomogeneous or homogeneous dispersion. As shown in Table 1, one obtains C^{α} waves for $\alpha \in (0,1)$, noting that [15] consider $(\tanh(\mathrm{D})/\mathrm{D})^{\alpha}$ for $\alpha = \frac{1}{2}$ and [1] the same dispersion for $\alpha \in (0,1)$, and that [27] studies $(1 + |\mathrm{D}|^2)^{-\alpha/2}$ for $\alpha \in (0,1)$. When $\alpha = 1$, the extreme waves turn out to be log-Lipschitz [11, 14] in the sense that the behaviour at the crests is like $1 - |x \log|x||$. In [14], they even provide exact asymptotics by new techniques, with applicability also to a subregime of $\alpha \in (0,1)$ including the Whitham equation. We note that global existence of weak solutions are guaranteed by [4] in the homogeneous case of $\alpha = 1$ known as the Burgers–Hilbert equation. Finally, the waves are all Lipschitz [5, 24] when $\alpha > 1$, and one naturally conjectures that the Lipschitz angles vanish as $\alpha \searrow 1$. See further [16] for uniqueness and instability of the highest wave when $\alpha = 2$, corresponding to the reduced Ostrovsky equation, and also [2] for the existence of extreme Lipschitz waves for Degasperis–Procesi equation.

1.3. **Contributions.** As promised and illustrated in Table 1, we complete the regularity picture for extreme periodic waves in the given negative-order class of dispersive equations. The regularity analysis emerges from the overall structure of [15] with the following key differences:

i) Whereas [15, 5, 1] obtain monotonicity properties of the kernels based on a general characterisation of completely monotone functions or sequences, we establish monotonicity of the singular kernel
$$K_\alpha(x) \coloneqq \mathscr{F}^{-1}(|\cdot|^{-\alpha})(x)$$
on $\mathbb{T}$ by computing an explicit integral representation valid for all $\alpha > 0$, and use the Poisson summation formula to derive its precise singular behaviour ($\eqsim |x|^{\alpha-1}$ as $|x| \to 0$) from the situation on $\mathbb{R}$.

ii) Since K_α has only algebraic but not exponential decay (unlike the kernels in [15, 1, 27]), extra care must be applied to the finite-difference estimates for $|\mathrm{D}|^{-\alpha}u$ when α is arbitrarily close to 1. In fact, we must exercise order-optimal estimates in order for the integrals to converge.

iii) We treat a class of nonlinearities, including sign-dependent ones (2_{sgn}), in the regularity estimates, which amongst others requires the use of suitable properties of composition operators on Hölder spaces.

The study of nonsmooth nonlinearities – with both slow ($p \gtrapprox 1$) and arbitrary (polynomially) fast growth in (2) – also poses new challenges since analytic bifurcation theory cannot be applied directly. We resolve this issue by analytically regularising the nonlinearities and proving that important features related to wave regularity and speed hold uniformly as the regularisation vanishes. The approach is strikingly simple (see (35)).

In the special case of smooth $n(x) = x^p$ with $2 \leqslant p \in \mathbb{N}$, we also compute in Theorem 20 local bifurcation formulas for all p, which may be of independent interest. We are also able to deduce the overall structure of the bifurcation formulas along the entire local bifurcation curve when p is odd. As for the case of general sign-dependent nonlinearities (2_{sgn}), we establish that the highest waves exhibit antisymmetry and thus also contain an inverted cusp at the troughs, as illustrated in Fig. 1. This construction appears to be completely new in the context of large-amplitude singular waves and sheds light on underlying symmetry principles.

With appropriate modifications, these results are also transferable to other nonlocal dispersive equations. In particular, one may obtain such “doubly-cusped” periodic solutions (with zero mean) in the full scale of equations in Table 1 with generalised nonlinearities of type (2). Specifically, consider the evolution equation
$$u_t + (\mathcal{L}_\alpha u + n(u))_x = 0, \tag{3}$$
where $\mathcal{L}_\alpha$ is any of the dispersive operators
$$(\tanh(\mathrm{D})/\mathrm{D})^\alpha, \qquad \left(1 + |\mathrm{D}|^2\right)^{-\frac{\alpha}{2}} \qquad \text{or} \qquad |\mathrm{D}|^{-\alpha}$$
for $\alpha \in (0, \infty)$ and n is as in (2). By readily adapting the regularity estimates in [15, 11, 24, 5, 1, 27, 14] with the estimates for general nonlinearities considered here and applying the

regularisation procedure in the global bifurcation analysis, we can also deduce the following analogous result of Theorem 1. Here $C \coloneqq 1$ for the inhomogeneous operators (the value at the origin for their symbol) and $C \coloneqq \|\mathscr{F}^{-1}(|\cdot|^{-\alpha})\|_{\mathrm{L}^1(\mathbb{T})}$ in the homogeneous case.

Theorem 3. *For all $\alpha \in (0,\infty)$ and $p > 1$, the dispersive equation (3) admits a nontrivial periodic travelling-wave solution φ with speed $c < \frac{p}{p-1}C$, with zero mean in the homogeneous case and in case (2_{sgn}). The solution is even (about $x \in 2\pi\mathbb{Z}$) and strictly increasing on $(-\pi, 0)$, and satisfies*

$$\max \varphi = \varphi(0) = \mu \qquad \textit{and} \qquad \varphi \in \begin{cases} \mathrm{C}^\alpha(\mathbb{T}) & \textit{if } \alpha \in (0,1); \\ \text{log-Lipschitz}(\mathbb{T}) & \textit{if } \alpha = 1; \\ \text{Lipschitz}(\mathbb{T}) & \textit{if } \alpha > 1, \end{cases}$$

where again $\mu = (c/p)^{1/(p-1)}$. Moreover, the estimate

$$\mu - \varphi(x) \eqsim \begin{cases} |x|^\alpha & \textit{if } \alpha \in (0,1); \\ |x \log|x|| & \textit{if } \alpha = 1; \\ |x| & \textit{if } \alpha > 1, \end{cases}$$

holds uniformly around the extreme point $x = 0$.

One has that φ is smooth around $-\pi$ in case (2_{abs}), while φ is antisymmetric about $-\frac{\pi}{2}$ in case (2_{sgn}), thereby featuring inverted cusps/peakons at $\pi\mathbb{Z}$.

A similar statement may be formed for extreme Lipschitz waves in a generalised version of the Degasperis–Procesi equation with the nonlinearities (2) by combining the analysis here with that of [2, 27].

1.4. **Outline of the analysis.** For homogeneous dispersion one has a choice as to what class of functions $|\mathrm{D}|^{-\alpha}$ should act upon in the interpretation of (1). We restrict our attention to functions with zero mean ($\int_{\mathbb{T}} u(\cdot, x)\,\mathrm{d}x = 0$), but note that other alternatives such as equivalence classes of functions that differ by a constant are possible; see for instance [26] on homogeneous Sobolev-type spaces.

We set up (1) in steady variables $u(t,x) \coloneqq \varphi(x - ct)$ with wave speed $c > 0$, so that, after integration, (1) takes the form

$$|\mathrm{D}|^{-\alpha}\varphi = N(\varphi; c) + ⨍_{\mathbb{T}} n(\varphi), \tag{4}$$

where we have introduced $N(\varphi; c) \coloneqq c\varphi - n(\varphi)$, and the mean $⨍_{\mathbb{T}} n(\varphi) \coloneqq \frac{1}{2\pi}\int_{\mathbb{T}} n(\varphi(y))\,\mathrm{d}y$ of $n(\varphi)$ is the constant of integration. One may observe that

$$N'(\varphi) > 0 \quad \Leftrightarrow \quad n'(\varphi) < c \quad \Leftrightarrow \quad \begin{cases} \varphi \ < \mu \text{ in case } (2_{\mathrm{abs}}); \\ |\varphi| < \mu \text{ in case } (2_{\mathrm{sgn}}), \end{cases} \tag{5}$$

in which the value

$$\mu \coloneqq (c/p)^{1/(p-1)}, \tag{6}$$

being the first positive critical point for $N(\varphi)$, turns out to be the maximum of the highest wave. As the regularity analysis will reveal, the quadratic nature near $\varphi = \mu$, where N'' is strictly negative, causes in partnership with $|\mathrm{D}|^{-\alpha}$ the singular behaviour of φ at the crest (and through, in case (2_{sgn})).

With regards to the precise C^α regularity estimates, we consider as in [15] fine details of local regularity and first- and second-order differences of both u, $|\mathrm{D}|^{-\alpha}u$, and $n(u)$ in connection with the Hölder seminorm. We first establish global C^β regularity for all $\beta < \alpha$, then the exact α-Hölder estimate at 0, and finally global C^α regularity with help of an interpolation argument. A key property in this setting is that $|\mathrm{D}|^{-\alpha}$ is α-smoothing on the scale of Hölder–Zygmund spaces, and that if $|\mathrm{D}|^{-\alpha}u$ is (2α)-Hölder continuous at a point, then u is α-Hölder continuous at that point for $\alpha \in \left(0, \frac{1}{2}\right]$. However, when $\alpha > \frac{1}{2}$ (remember that [15] corresponds to $\alpha = \frac{1}{2}$), $|\mathrm{D}|^{-\alpha}u$ passes index 1 on the Hölder–Zygmund scale, and we must partially work with derivatives as in [11].

When it comes to the bifurcation analysis, we first establish small-amplitude waves by the local Crandall–Rabinowitz bifurcation theorem [6, Theorems 8.3.1 and 8.4.1]. It is interesting to note that the regularisation of n lightens the computation of the local bifurcation formulas; case (2_{abs}) acts essentially as u^2 and case (2_{sgn}) behaves like u^3. As for the construction of the highest waves, we make use of the analytic global bifurcation theory of Buffoni and Toland [6]. One obtains, after ruling out certain possibilities, a global, locally analytic curve $s \mapsto \varphi^\epsilon(s)$ of smooth sinusoidal waves, along which $\max_{x\in\mathbb{T}} \varphi^\epsilon(s)(x)$ approaches a particular value μ^ϵ depending on the wave speed and the regularisation parameter ϵ (see (6) and (36)). Although we are not able to establish unconditional antisymmetry in case (2_{sgn}), we enforce this property along the global branch by working in a subspace. Coupled with the a priori regularity estimates for solutions touching μ^ϵ from below, it is then possible to extract a subsequence of $(\varphi^\epsilon(s))_s$ converging to a solution φ^ϵ with both $\max\varphi^\epsilon = \mu^\epsilon$ and the exact, global α-Hölder continuity. Finally, we show that φ^ϵ converges to a solution of (1) with the same properties as $\epsilon \searrow 0$.

The outline of the paper is as follows. In Section 2 we focus on properties and representations of $|\mathrm{D}|^{-\alpha}$ and K_α on $\mathbb{T}$ together with the relevant function spaces. In Section 3 we study a priori properties of solutions—especially, what concerns the α-Hölder continuity when $\max\varphi = \mu$, which is the most technical part. Finally, in Section 4 we first consider the local bifurcation analysis, and then study what happens at the end of the global bifurcation curve, supported by the theory in Section 3.

2. Properties of $|\mathrm{D}|^{-\alpha}$ and functional-analytic setting

2.1. Representations of the kernel.

On the real line it is well known that the inverse Fourier transform of the symbol $|\cdot|^{-\alpha}$ for $\alpha \in (0,1)$ equals

$$\mathscr{F}^{-1}(|\cdot|^{-\alpha})(x) = \gamma_\alpha |x|^{\alpha-1} \tag{7}$$

in the sense of (tempered) distributions, with $\gamma_\alpha^{-1} := 2\Gamma(\alpha)\sin\left(\frac{\pi}{2}(1-\alpha)\right)$ using the normalisation

$$(\mathscr{F}\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x)\, \mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x,$$

so that $\mathscr{F}^{-1}(\varphi)(x) = \frac{1}{2\pi}\mathscr{F}(\varphi)(-x)$. Here Γ is the gamma function, and we observe that $\gamma_\alpha \searrow 0$ as $\alpha \searrow 0$ and $\gamma_\alpha \nearrow \infty$ as $\alpha \nearrow 1$. We are interested in the action of $|\mathrm{D}|^{-\alpha}$ in the periodic setting, and by the convolution theorem this amounts to understanding the periodic kernel

$$K_\alpha(x) := \frac{1}{2\pi}\sum_{k\neq 0} |k|^{-\alpha}\mathrm{e}^{\mathrm{i}kx}$$

for which

$$\widehat{K_\alpha} = |\cdot|^{-\alpha} \quad \text{on } \mathbb{Z}\setminus\{0\} \qquad \text{and} \qquad |\mathrm{D}|^{-\alpha}\varphi = K_\alpha * \varphi \quad \text{on } \mathbb{T}.$$

Here φ has zero mean so that $\widehat{\varphi}(0) = 0$, and the normalisation is again $\widehat{\varphi}(k) = \int_{\mathbb{T}} \varphi(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$. A naïve application of the Poisson summation formula yields that

$$\tfrac{1}{2\pi}\sum_{k\in\mathbb{Z}} |k|^{-\alpha}\mathrm{e}^{\mathrm{i}kx} \text{ ``$=$'' } \sum_{k\in\mathbb{Z}} \gamma_\alpha |x+2\pi k|^{\alpha-1},$$

which, although it is nonsense due to divergence on both sides, nevertheless suggests that the kernel on $\mathbb{T}$ mimics the singularity of the kernel on $\mathbb{R}$. In fact, [30, Theorem 2.17] establishes the following result with help of a cut-off argument in the Poisson summation formula. We include a proof of this formula for two reasons. First, it is essential in our work. Second, the technique may be useful to define the analogues on $\mathbb{T}$ of other known kernels on $\mathbb{R}$. The latter can be used to study other nonlocal weakly dispersive equations.

Proposition 4 ([30, Theorem 2.17])**.** *The periodic convolution kernel may be written as*

$$K_\alpha = \gamma_\alpha |\cdot|^{\alpha-1} + K_{\alpha,\mathrm{reg}}$$

on $(-\pi,\pi)$*, where* $K_{\alpha,\mathrm{reg}}$ *is an even, smooth function. In particular,* $K_\alpha \in \mathrm{L}^1(\mathbb{T})$*.*

Proof. Let ϱ be an even, smooth cut-off function that vanishes in a neighbourhood of $\xi = 0$ and equals 1 for $|\xi| \geqslant 1$, and define $F(\xi) := \varrho(\xi)|\xi|^{-\alpha}$ for $\xi \in \mathbb{R}$. Then F is the Fourier transform of an integrable function of the form

$$f(x) := \gamma_\alpha |x|^{\alpha-1} + f_{\mathrm{reg}}(x),$$

where $f_{\mathrm{reg}} \in \mathrm{C}^\infty(\mathbb{R})$ and $|f(x)| = \mathcal{O}(|x|^{-m})$ as $|x| \to \infty$ for all $m \geqslant 1$. Indeed, writing

$$F = |\cdot|^{-\alpha} + (\varrho - 1)|\cdot|^{-\alpha},$$

it follows directly from (7) that $f = \mathscr{F}^{-1}F$ has the given form, where we remember that the inverse Fourier transform of an integrable function of bounded support (in this case $(\varrho-1)|\cdot|^{-\alpha}$) is smooth. Since $\mathscr{F}(x^m f(x)) \sim F^{(m)}$ is integrable for all $m \geqslant 1$, it must be the case that $x^m f(x)$ is bounded. In particular, $f \in \mathrm{L}^1(\mathbb{R})$, and the Poisson summation formula then gives

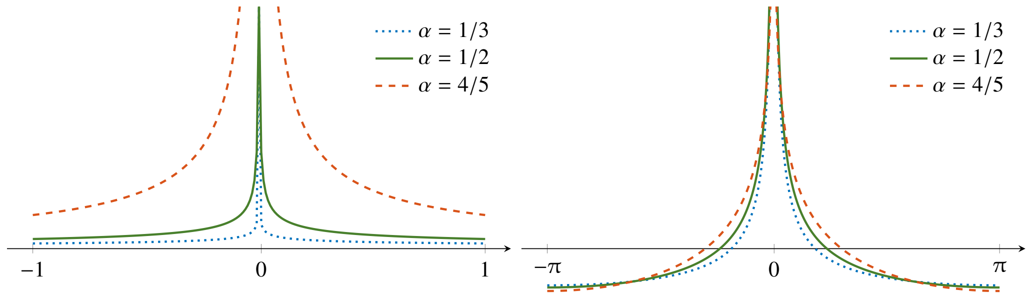
$$\sum_{k\in\mathbb{Z}} f(x-2\pi k) = \tfrac{1}{2\pi}\sum_{k\in\mathbb{Z}} F(k)\,\mathrm{e}^{\mathrm{i}kx} = \tfrac{1}{2\pi}\sum_{k\neq 0} |k|^{-\alpha}\mathrm{e}^{\mathrm{i}kx} = K_\alpha(x).$$

Since

$$\sum_{k\in\mathbb{Z}} f(x-2\pi k) = f(x) + \sum_{k\neq 0} f(x-2\pi k),$$

this proves the result with $K_{\alpha,\mathrm{reg}} := f_{\mathrm{reg}} + \sum_{k\neq 0} f(\cdot - 2\pi k)$. ◻

In Fig. 2 we display the integral kernels on both $\mathbb{R}$ and $\mathbb{T}$ for various values of α. Whereas the kernels on $\mathbb{R}$ are all nonnegative, those on $\mathbb{T}$ become negative away from the positive singularity at 0, because K_α has zero mean. In both cases the profiles are monotone on either side of the singularity; this is obvious on $\mathbb{R}$, and on $\mathbb{T}$ we deduce this by means of the following integral representation of K_α, which is valid for all $\alpha \in (0,\infty)$. Although the formula is known [28, Section 5.4.3], we include a slick computation of it using the gamma distribution. We remark that we shall only need the monotonicity of K_α in our work, and for that property one may alternatively use the theory of completely monotone sequences and the discrete analogue of Bernstein's theorem; see [5, Theorem 3.6 b)].



(a) Kernels on $\mathbb{R}$.

(b) Kernels on $\mathbb{T}$.

Figure 2. Illustrating the differences between the singular kernels on $\mathbb{R}$ (cut at $x = \pm 1$) and $\mathbb{T}$ for various α. (The vertical axes also have different scaling.) We compute the kernels on $\mathbb{T}$ by numerically integrating the formula in Theorem 5.

Proposition 5. *For all $\alpha \in (0, \infty)$ the periodic kernel has the integral representation*

$$K_\alpha(x) = \frac{1}{\pi\,\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}(\mathrm{e}^t \cos x - 1)}{1 - 2\mathrm{e}^t \cos x + \mathrm{e}^{2t}}\,\mathrm{d}t$$

for $x \notin 2\pi\mathbb{Z}$. In particular, K_α is strictly increasing on $(-\pi, 0)$.

Proof. By recognising $k^{-\alpha}$ in the definition of the gamma distribution with shape α and rate k, whose probability density function is $t \mapsto k^\alpha t^{\alpha-1}\mathrm{e}^{-kt}/\Gamma(\alpha)$ on $(0, \infty)$, we find that

$$\begin{aligned}
\pi\,\Gamma(\alpha)\,K_\alpha(x) &= \Gamma(\alpha) \sum_{k=1}^\infty k^{-\alpha} \cos(kx) \\
&= \sum_{k=1}^\infty \int_0^\infty t^{\alpha-1}\mathrm{e}^{-kt} \cos(kx)\,\mathrm{d}t \\
&= \int_0^\infty t^{\alpha-1}\,\mathrm{Re}\left(\sum_{k=1}^\infty \left(\mathrm{e}^{-t}\mathrm{e}^{\mathrm{i}x}\right)^k\right)\mathrm{d}t \\
&= \int_0^\infty t^{\alpha-1}\,\mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}x}}{\mathrm{e}^t - \mathrm{e}^{\mathrm{i}x}}\right)\mathrm{d}t = \int_0^\infty \frac{t^{\alpha-1}(\mathrm{e}^t \cos x - 1)}{1 - 2\mathrm{e}^t \cos x + \mathrm{e}^{2t}}\,\mathrm{d}t
\end{aligned}$$

using the dominated convergence theorem and a trick with geometric series. Leibniz's integral rule next yields that

$$K_\alpha'(x) = -\frac{\sin x}{\pi\,\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}\mathrm{e}^t(\mathrm{e}^{2t} - 1)}{(1 - 2\mathrm{e}^t \cos x + \mathrm{e}^{2t})^2}\,\mathrm{d}t,$$

which shows that K_α is strictly increasing on $(-\pi, 0)$. ∎

Remark 6. *For $\alpha = 1$ in Theorem 5 one finds the explicit form*

$$\begin{aligned} K_1(x) &= \tfrac{1}{\pi}\Big[\tfrac{1}{2}\log\big(1 - 2\mathrm{e}^t\cos x + \mathrm{e}^{2t}\big) - t\Big]_{t=0}^{\infty} \\ &= -\tfrac{1}{2\pi}\log\big(2(1-\cos x)\big) \\ &= -\tfrac{1}{\pi}\log|x| + \mathcal{O}(x^2) \end{aligned}$$

as $x \to 0$. Such logarithmic singularities occur for all the kernels in Table 1 when $\alpha = 1$; see for instance [11, Lemma 2.3 (iii)] *for the kernel in the bidirectional Whitham equation with inhomogeneous dispersion* $\tanh(\mathrm{D})/\mathrm{D}$.

2.2. **Action of $|\mathrm{D}|^{-\alpha}$ on Hölder–Zygmund spaces.** As regards the functional-analytic framework, it is desirable to work with spaces which both capture the precise regularity of cusps and interact well with Fourier multipliers. These turn out to be the so-called Hölder–Zygmund spaces, which we explain next.

Let $\mathrm{C}^m(\mathbb{T})$, for $m = 0, 1, \ldots$, denote the space of m times continuously differentiable functions φ on $\mathbb{T}$ with norm

$$\|\varphi\|_{\mathrm{C}^m(\mathbb{T})} \coloneqq \|\varphi\|_\infty + \|\varphi^{(m)}\|_\infty,$$

where $\|\cdot\|_\infty$ is the supremum norm on $\mathbb{T}$. Furthermore, define $\mathrm{C}^{m,\beta}(\mathbb{T})$ with $\beta \in (0,1]$ to be the class of Hölder spaces consisting of all $\varphi \in \mathrm{C}^m(\mathbb{T})$ for which $\varphi^{(m)}$ is β-Hölder continuous with norm

$$\|\varphi\|_{\mathrm{C}^{m,\beta}(\mathbb{T})} \coloneqq \|\varphi\|_{\mathrm{C}^m(\mathbb{T})} + |\varphi^{(m)}|_{\mathrm{C}^\beta(\mathbb{T})},$$

where

$$|\psi|_{\mathrm{C}^\beta(\mathbb{T})} \coloneqq \sup_{x\neq y} \frac{|\psi(x)-\psi(y)|}{|x-y|^\beta}$$

is a seminorm. We write $\mathrm{C}(\mathbb{T}) \coloneqq \mathrm{C}^0(\mathbb{T})$ and $\mathrm{C}^\beta(\mathbb{T}) \coloneqq \mathrm{C}^{0,\beta}(\mathbb{T})$ for simplicity, and note that $\mathrm{C}^{m,\beta}(\mathbb{T})$ is compactly embedded in $\mathrm{C}^{m,\widetilde{\beta}}(\mathbb{T})$ when $\beta > \widetilde{\beta}$. Moreover, the Fourier series of $\varphi \in \mathrm{C}^\beta(\mathbb{T})$ for $\beta > \frac{1}{2}$ converges both uniformly to φ and absolutely.

While the standard Hölder norms provide an accurate description of the modulus of continuity of a function (and its derivatives), an alternative, frequency-based characterisation by means of the Littlewood–Paley decomposition is more suitable for Fourier multipliers. To this end, let $\sum_{j=0}^\infty \varrho_j(\xi) = 1$ be a partition of unity of smooth functions ϱ_j on $\mathbb{R}$ supported on $2^j \leqslant |\xi| \leqslant 2^{j+1}$ for $j \geqslant 1$ and on $|\xi| \leqslant 2$ for $j = 0$. We then define the Hölder–Zygmund space $\mathrm{C}^s_*(\mathbb{T})$ for $s \in [0,\infty)$ to consist of those functions φ for which

$$\|\varphi\|_{\mathrm{C}^s_*(\mathbb{T})} \coloneqq \sup_{j\geqslant 0} 2^{js}\|\varrho_j(\mathrm{D})\varphi\|_\infty$$

is finite, where $\varrho_j(\mathrm{D})$ is the Fourier multiplier with symbol ϱ_j, that is,

$$\varrho_j(\mathrm{D})\varphi(x) \coloneqq \sum_{k\in\mathbb{Z}} \varrho_j(k)\,\widehat{\varphi}(k)\,\mathrm{e}^{\mathrm{i}kx}.$$

One has that

$$\mathrm{C}^s_*(\mathbb{T}) = \mathrm{C}^{\lfloor s\rfloor, s-\lfloor s\rfloor}(\mathbb{T}) \quad \text{for } s \neq 1, 2, \ldots,$$

in the sense of equivalent norms, whereas there are strict inclusions

$$\mathrm{C}^s(\mathbb{T}) \subsetneq \mathrm{C}^{s-1,1}(\mathbb{T}) \subsetneq \mathrm{C}^s_*(\mathbb{T}) \quad \text{for } s = 1, 2, \ldots$$

Since $|\mathrm{D}|^{-\alpha}$ is homogeneous, we restrict from now on to the corresponding subspaces $\mathring{\mathrm{C}}^m(\mathbb{T})$, $\mathring{\mathrm{C}}^{m,\beta}(\mathbb{T})$, and $\mathring{\mathrm{C}}^s_*(\mathbb{T})$ of functions with zero mean in the above spaces, with identical norms. Note that the seminorm $|\cdot|_{\mathring{\mathrm{C}}^\beta(\mathbb{T})}$ is now a norm equivalent to $\|\cdot\|_{\mathring{\mathrm{C}}^{0,\beta}(\mathbb{T})}$ by the zero-mean restriction and compactness of $\mathbb{T}$. We observe in this setting that

$$|\mathrm{D}|^{-\alpha}\colon \mathring{\mathrm{C}}^s_*(\mathbb{T}) \to \mathring{\mathrm{C}}^{s+\alpha}_*(\mathbb{T})$$

is α-smoothing for all $s \in [0,\infty)$ (and therefore also on the Hölder-space scale for $s \neq 1, 2, \ldots$), because in terms of Fourier multipliers we have, with $j \geqslant 1$, that

$$\mathscr{F}\left(\varrho_j(\mathrm{D})|\mathrm{D}|^{-\alpha}\varphi\right) = \varrho_j|\cdot|^{-\alpha}\widehat{\varphi} \sim 2^{-j\alpha}\varrho_j\widehat{\varphi} = 2^{-j\alpha}\widehat{\varrho_j(\mathrm{D})\varphi}.$$

Finally, the subscript "even" attached to any space on $\mathbb{T}$ means the subspace of symmetric functions about 0 (mod 2π).

Lemma 7. *The smoothing property $|\mathrm{D}|^{-\alpha}\colon \mathring{\mathrm{C}}^s_*(\mathbb{T}) \to \mathring{\mathrm{C}}^{s+\alpha}_*(\mathbb{T})$ extends to a local version. More precisely, if $\varphi \in \mathring{\mathrm{C}}(\mathbb{T})$ lies in $\mathrm{C}^s_{*,\mathrm{loc}}(U)$ for an open subset $U \subset \mathbb{T}$, in the sense that $\rho\varphi \in \mathrm{C}^s_*(\mathbb{T})$ for any compactly supported $\rho \in \mathrm{C}^\infty_\mathrm{c}(U)$, then we still have $|\mathrm{D}|^{-\alpha}\varphi \in \mathrm{C}^{s+\alpha}_{*,\mathrm{loc}}(U)$.*

Proof. To see this, let $\rho \in \mathrm{C}^\infty_\mathrm{c}(U)$ and let $\eta \in \mathrm{C}^\infty_\mathrm{c}(U)$ satisfy $\eta = 1$ in a neighbourhood $V \Subset U$ of $\operatorname{supp}\rho$. Then

$$\rho|\mathrm{D}|^{-\alpha}\varphi = \rho|\mathrm{D}|^{-\alpha}(\eta\varphi) + \rho|\mathrm{D}|^{-\alpha}\big((1-\eta)\varphi\big),$$

and the first term on the right-hand side is globally $\mathrm{C}^{s+\alpha}_*$ regular. Moreover, since the integrand in

$$\rho(x)|\mathrm{D}|^{-\alpha}\big((1-\eta)\varphi\big)(x) = \int_{-\pi}^{\pi} K_\alpha(x-y)\rho(x)(1-\eta(y))\varphi(y)\,\mathrm{d}y$$

vanishes for y near x and K_α is smooth away from 0, it follows that $\rho|\mathrm{D}|^{-\alpha}\big((1-\eta)\varphi\big)$ is smooth. Hence, $|\mathrm{D}|^{-\alpha}\varphi \in \mathrm{C}^{s+\alpha}_{*,\mathrm{loc}}(U)$, as claimed. □

Finally, we include a monotonicity property of $|\mathrm{D}|^{-\alpha}$ which will be useful in establishing a priori nodal properties of highest waves in Section 3.

Proposition 8 ([15, Lemma 3.6])**.** *$|\mathrm{D}|^{-\alpha}$ is a parity-preserving operator, and $|\mathrm{D}|^{-\alpha}f > 0$ on $(-\pi, 0)$ for odd $f \in \mathring{\mathrm{C}}(\mathbb{T})$ satisfying $f \geqslant 0$ on $(-\pi,0)$ with $f(y_0) > 0$ for some $y_0 \in (-\pi, 0)$.*

Proof. Since K_α is even, one immediately obtains that $|\mathrm{D}|^{-\alpha}$ is parity-preserving from

$$|\mathrm{D}|^{-\alpha}f(x) \pm |\mathrm{D}|^{-\alpha}f(-x) = \int_\mathbb{T} K_\alpha(x-y)\left(f(y) \pm f(-y)\right)\mathrm{d}y.$$

Next consider odd $f \in \mathring{\mathrm{C}}(\mathbb{T})$ satisfying $f(y) \geqslant 0$ on $(-\pi, 0)$ with $f(y_0) > 0$ for some $y_0 \in (-\pi, 0)$. Then

$$|\mathrm{D}|^{-\alpha}f(x) = \int_{-\pi}^{\pi} K_\alpha(x-y)f(y)\,\mathrm{d}y = \int_{-\pi}^{0}\left(K_\alpha(x-y) - K_\alpha(x+y)\right)f(y)\,\mathrm{d}y$$

for $x \in (-\pi, 0)$. When y also lies in $(-\pi, 0)$, it follows that $-2\pi < x+y < x-y < \pi$ and

$$\operatorname{dist}(x-y, 0) < \min\big\{\operatorname{dist}(x+y,0), \operatorname{dist}(x+y,-2\pi)\big\}.$$

The latter inequality is a consequence of $|x-y| < |x+y|$ for $x, y < 0$ and $|x-y| < x+y+2\pi$ for $x, y > -\pi$. Since K_α is strictly decreasing as a function of the distance from the origin to $\pm\pi$ by Theorem 5 and is even and periodic, we therefore obtain that

$$K_\alpha(x-y) > K_\alpha(x+y)$$

for every $y \in (-\pi, 0) \setminus \{x\}$. In particular, $|\mathrm{D}|^{-\alpha} f(x) > 0$ for all $x \in (-\pi, 0)$ as f is strictly positive in an interval around y_0 by continuity. □

3. A priori properties of travelling-wave solutions

In this section we establish many a priori bounds and regularity properties of continuous solutions of (4). Most importantly, we prove exact, global α-Hölder regularity in Theorem 16 for solutions touching the highest point μ (see (6)) from below at $x = 0$. This is obtained with help of a nodal pattern of solutions in Theorem 12. We remind the reader of (5): $n'(\varphi) < c$ corresponds to solutions which stay away from $(\pm)\mu$, and $n'(\varphi) \leqslant c$ includes the possibility of also touching $(\pm)\mu$.

Our first result is a uniform upper bound on both the size of solutions and the wave speed that we will use in Section 4 in compactness arguments.

Lemma 9. *For all solutions $\varphi \in \mathring{\mathrm{C}}(\mathbb{T})$ of (4) satisfying $n'(\varphi) \leqslant c$ one has the uniform estimate*

$$\|\varphi\|_\infty \lesssim (1+c)^{1/(p-1)}.$$

Moreover, if $c \geqslant \frac{p}{p-1}\|K_\alpha\|_{\mathrm{L}^1(\mathbb{T})}$, then there are no nontrivial such solutions.

Proof. In case (2_{sgn}) for $n'(\varphi) \leqslant c$, the bound in L^∞ is immediate since $\mu \sim c^{1/(p-1)}$. As regards (2_{abs}), we need to control the minimum of a nontrivial φ. Let $x_{\min}$ and $x_{\max}$ be points where φ attains its global minimum $\varphi_{\min}$ and maximum $\varphi_{\max}$, respectively, where we note that $\varphi_{\max} > 0 > \varphi_{\min}$ as φ has zero mean. From (4) we then find that

$$\begin{aligned} c\left(\varphi_{\max} - \varphi_{\min}\right) - \left(n(\varphi_{\max}) - n(\varphi_{\min})\right) &= |\mathrm{D}|^{-\alpha}\varphi(x_{\max}) - |\mathrm{D}|^{-\alpha}\varphi(x_{\min}) \\ &\leqslant \|K_\alpha\|_{\mathrm{L}^1(\mathbb{T})}\left(\varphi_{\max} - \varphi_{\min}\right), \end{aligned} \tag{8}$$

which leads to

$$\begin{aligned} n(\varphi_{\min}) &\leqslant n(\varphi_{\max}) + (\|K_\alpha\|_{\mathrm{L}^1(\mathbb{T})} - c)(\varphi_{\max} + |\varphi_{\min}|) \\ &\lesssim \max\left\{n(\varphi_{\max}), (1+c)(\varphi_{\max} + |\varphi_{\min}|)\right\}. \end{aligned}$$

In the first situation, $n(\varphi_{\min}) \lesssim n(\varphi_{\max})$, so that $|\varphi_{\min}| \lesssim \varphi_{\max} \leqslant \mu$. In the second situation, it suffices to investigate the case $\varphi_{\max} < |\varphi_{\min}|$ (otherwise we freely get $|\varphi_{\min}| \leqslant \varphi_{\max} \leqslant \mu$). Then

$$|\varphi_{\min}|^p = n(\varphi_{\min}) \lesssim (1+c)|\varphi_{\min}|,$$

implying that $|\varphi_{\min}| \lesssim (1+c)^{1/(p-1)}$.

For the last part, we use in case (2_{abs}) that for $a > 0 > b$ one has

$$a^p - |b|^p = a^{p-1}(a-b) - |b|(a^{p-1} + |b|^{p-1}) < a^{p-1}(a-b).$$

Applied to $a = \varphi_{\max}$ and $b = \varphi_{\min}$, we reorder (8) and find that

$$c - \|K_\alpha\|_{\mathrm{L}^1(\mathbb{T})} \leqslant \frac{n(\varphi_{\max}) - n(\varphi_{\min})}{\varphi_{\max} - \varphi_{\min}} < \varphi_{\max}^{p-1} \leqslant \mu^{p-1} = \frac{c}{p},$$

which gives $c < \frac{p}{p-1}\|K_\alpha\|_{\mathrm{L}^1(\mathbb{T})}$. One may similarly treat case (2_{sgn}). $\square$

Next, we want to establish regularity for solutions which stay away from $(\pm)\mu$. To this end, we need the inverse function theorem for the Hölder scale. We could not find a proof in the literature and therefore provide a short argument.

Proposition 10 (Inverse function theorem for $\mathrm{C}^{m,\beta}$). *Let $m \geqslant 1$ be an integer and $\beta \in (0,1]$, and assume that a function f is $\mathrm{C}^{m,\beta}$ regular and strictly monotone on a compact interval $I \subset \mathbb{R}$. Then f^{-1} is $\mathrm{C}^{m,\beta}$ regular on $f(I)$.*

Proof. We only establish β-Hölder regularity of g', where $g \coloneqq f^{-1}$; the rest follows by the standard inverse function theorem and similar estimates for the higher-order derivatives. To this end, let $x, y \in f(I)$ be different and observe that

$$\begin{aligned}\frac{|g'(x)-g'(y)|}{|x-y|^\beta} &= \frac{\left|\dfrac{1}{f'(g(x))} - \dfrac{1}{f'(g(y))}\right|}{|x-y|^\beta} \\ &= \frac{1}{|f'(g(x))\,f'(g(y))|}\cdot\frac{|f'(g(x))-f'(g(y))|}{|g(x)-g(y)|^\beta}\cdot\left|\frac{g(x)-g(y)}{x-y}\right|^\beta \\ &= |g'(x)\,g'(y)|\cdot\frac{|f'(g(x))-f'(g(y))|}{|g(x)-g(y)|^\beta}\cdot|g'(z)|^\beta\end{aligned}$$

for some z between x and y by the mean value theorem. Hence, $|g'|_{\mathrm{C}^\beta} \leqslant \|g'\|_\infty^{2+\beta}|f'|_{\mathrm{C}^\beta} < \infty$, where we note that g' is bounded on the compact set $f(I)$ due to its continuity by the standard inverse function theorem. $\square$

Lemma 11. *Let $\varphi \in \mathring{\mathrm{C}}(\mathbb{T})$ be a solution of (4). Then*

i) φ is smooth on any open set where $n'(\varphi) < c$ and that does not contain the boundary $\partial(\varphi^{-1}(0))$ of the set $\varphi^{-1}(0)$; and
ii) φ has at least the same regularity in the Hölder scale around $\partial(\varphi^{-1}(0))$ as the nonlinearity n around 0.

In particular, if n is smooth, then so is φ on any open set where $n'(\varphi) < c$.

Proof. Note first by translation invariance ($|\mathrm{D}|^{-\alpha}$ is a convolution operator) that if φ is a solution of (4), then so is $\varphi(\cdot + h)$ for any $h \in \mathbb{R}$. Accordingly, it suffices to consider open sets $U \subseteq (-\pi, \pi)$ where $n'(\varphi) < c$, so that (5) holds uniformly on every compact interval $I \subset U$. By the inverse function theorem (Theorem 10), it follows that N^{-1} exists on $\varphi(I)$ and has the same regularity as n in the Hölder scale. As such, we may then invert (4) to get the pointwise relation

$$\varphi(x) = G(\varphi, c)(x) \coloneqq N^{-1}\big(|\mathrm{D}|^{-\alpha}\varphi(x) - ⨍_{\mathbb{T}} n(\varphi)\big) \tag{9}$$

for $x \in I$, where G is a nonlinear composition operator, depending implicitly on c via N^{-1}.

It is clear from the (higher-order) chain rule that an operator $f \mapsto F \circ f$ maps the space $\mathrm{C}^m(I)$ into itself provided that $F \in \mathrm{C}^m_{\mathrm{loc}}(\mathbb{R})$. More generally, the same remains true for $\mathrm{C}^{m,\beta}(I)$ for all $m \in \mathbb{N}_0$ and $\beta \in (0,1]$ if $F \in \mathrm{C}^{m,\beta}_{\mathrm{loc}}(\mathbb{R}) \cap \mathrm{C}^{0,1}_{\mathrm{loc}}(\mathbb{R})$ by [17, Theorems 2.1, 4.1 and 5.1], and the composition operator is also bounded (maps bounded sets to bounded sets). Therefore, since $\varphi \in \mathring{\mathrm{C}}(\mathbb{T}) \hookrightarrow \mathrm{C}^0_{*,\mathrm{loc}}(U)$ and $|\mathrm{D}|^{-\alpha}$ is locally C^α_* smoothing by Theorem 7, it follows by bootstrapping of $G(\cdot, c)$ in (9) that φ has at least the same $\mathrm{C}^{m,\beta}$ Hölder regularity as N^{-1} (that is, as n) on I. In particular, φ has the given regularity around $\partial(\varphi^{-1}(0))$ by applying this result to a covering $\{I_j\}_j$ of compact intervals I_j such that $(-\pi, \pi) \supset \bigcup_j I_j \supset \partial(\varphi^{-1}(0))$, which proves property ii.

Similarly, when U does not contain $\partial(\varphi^{-1}(0))$, we know that N^{-1} is smooth on $\varphi(I) \not\ni 0$, and so bootstrapping (9) yields that φ is smooth on I. As $I \subset U$ was arbitrary, this establishes property i. □

We continue by proving a nodal pattern for solutions which stay away from $\pm\mu$. The result will be crucial in establishing that the global bifurcation branch of solutions in Section 4 is not periodic.

Theorem 12 (Nodal pattern)**.** *Let $\varphi \in \mathring{\mathrm{C}}_{\mathrm{even}}(\mathbb{T})$ be a nontrivial solution of (4) that is increasing on $(-\pi, 0)$. If $\varphi \in \mathring{\mathrm{C}}^1_{\mathrm{even}}(\mathbb{T})$, then*

$$\varphi' > 0 \quad \text{and} \quad n'(\varphi) < c \quad \text{on} \quad (-\pi, 0),$$

and φ has the regularity specified in Theorem 11. Moreover, if φ is also C^2 regular around 0 and $-\pi$ (in the sense of $\mathbb{T}$), then $n'(\varphi) < c$ everywhere,

$$\varphi''(0) < 0, \quad \text{and} \quad \varphi''(-\pi) > 0.$$

Conversely, if $n'(\varphi) \leqslant c$ everywhere, then φ features the regularity in Theorem 11, with

$$\varphi' > 0 \quad \text{on} \quad (-\pi, 0).$$

Remark 13. *We write "in-/decreasing" instead of "nonde-/increasing", so that constant functions are both increasing and decreasing, and add the prefix "strictly" for the nontrivial cases.*

Proof. In the first case, φ' is odd and satisfies $\varphi' \geqslant 0$ on $(-\pi, 0)$ with $\varphi'(y_0) > 0$ for some $y_0 \in (-\pi, 0)$, as φ is nonconstant and increasing, so that $|\mathrm{D}|^{-\alpha}\varphi' > 0$ on $(-\pi, 0)$ by Theorem 8. We then differentiate in (4) to find that

$$N'(\varphi)\varphi' = |\mathrm{D}|^{-\alpha}\varphi' > 0 \quad \text{on} \quad (-\pi, 0),$$

which implies that both $N'(\varphi) = c - n'(\varphi)$ and φ' are strictly positive on that interval.

If φ is also C^2 around 0, we differentiate (4) twice and use that $\varphi'(0) = 0$ to obtain

$$\begin{aligned} N'(\varphi(0))\, \varphi''(0) &= (|\mathrm{D}|^{-\alpha}\varphi')'(0) \\ &= \lim_{r \searrow 0} \frac{\mathrm{d}}{\mathrm{d}x} \left(\int\limits_{|y|<r} K_\alpha(y)\varphi'(x-y)\,\mathrm{d}y + \int\limits_{\pi \geqslant |y| \geqslant r} K_\alpha(y)\varphi'(x-y)\,\mathrm{d}y \right)\Bigg|_{x=0}, \end{aligned}$$

where we have also isolated the singularity of K_α and interchanged limits (which is legitimate since $(|\mathrm{D}|^{-\alpha}\varphi')'$ is continuous around 0). Leibniz's integral rule now gives

$$\frac{\mathrm{d}}{\mathrm{d}x} \int\limits_{|y|<r} K_\alpha(y)\varphi'(x-y)\,\mathrm{d}y \Bigg|_{x=0} = \int\limits_{|y|<r} K_\alpha(y)\varphi''(y)\,\mathrm{d}y,$$

and the latter integral vanishes as $r \searrow 0$ because K_α is integrable and φ'' is continuous around 0. By Leibniz's rule once more, we also find that

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_{\pi\geqslant|y|\geqslant r} K_\alpha(y)\varphi'(x-y)\,\mathrm{d}y\bigg|_{x=0} &= -\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_{\pi\geqslant|x-y|\geqslant r} K_\alpha(x-y)\varphi'(y)\,\mathrm{d}y\bigg|_{x=0}\\ &= K_\alpha(r)\varphi'(r) - K_\alpha(\pi)\underbrace{\varphi'(\pi)}_{=0} - \int_r^\pi K_\alpha'(y)\varphi'(y)\,\mathrm{d}y,\end{aligned}$$

where we have utilised that K_α is even and φ' is odd. Observe that $K_\alpha(r) \eqsim |r|^{\alpha-1}$ and $\varphi'(r) = \mathcal{O}(r)$ (because φ'' is continuous around 0) as $r \searrow 0$, which means that $K_\alpha(r)\varphi'(r)$ vanishes in the limit. Since K_α' and φ' are strictly negative on $(0,\pi)$ by Theorem 5 and the assumption, respectively, we further infer that $-\int_r^\pi K_\alpha'(y)\varphi'(y)\,\mathrm{d}y$ is both negative and strictly decreasing as $r \searrow 0$. As such, we obtain

$$\underbrace{(c - n'(\varphi(0)))}_{=N'(\varphi(0))}\varphi''(0) = (|\mathrm{D}|^{-\alpha}\varphi')'(0) = -2\lim_{r\searrow 0}\int_r^\pi K_\alpha'(y)\varphi'(y)\,\mathrm{d}y < 0.$$

Since $n'(\varphi) < c$ on $(-\pi,\pi)\setminus\{0\}$ and $n'(\varphi)$ is continuous, it follows that $n'(\varphi(0)) < c$ also, and consequently $\varphi''(0) < 0$. By similar calculcations one finds that $n'(\varphi(-\pi)) < c$ (for free in case (2_{abs})) and $\varphi''(-\pi) > 0$.

Conversely, suppose that $n'(\varphi) \leqslant c$ everywhere. If in fact $n'(\varphi) < c$ uniformly, then Theorem 11 implies that $\varphi \in \mathring{\mathrm{C}}^1(\mathbb{T})$, which leads to $\varphi' > 0$ on $(-\pi,0)$ by the first case of Theorem 12. When $n'(\varphi)$ touches c, however, we must use a different approach. Note that φ is differentiable almost everywhere on $(-\pi,0)$ by Lebesgue's theorem for increasing functions, and that we may also use central differences to compute φ'. To this end, observe that

$$|\mathrm{D}|^{-\alpha}\varphi(x+h) - |\mathrm{D}|^{-\alpha}\varphi(x-h) = \int_{-\pi}^0 \big(K_\alpha(y-x) - K_\alpha(y+x)\big)\big(\varphi(y+h) - \varphi(y-h)\big)\,\mathrm{d}y \tag{10}$$

for $x \in (-\pi,0)$ and $h \in (0,\pi)$ by periodicity and evenness of K_α and φ. The second factor in the integrand is nonnegative by assumption, whereas the first factor is strictly positive by Theorem 5. Consequently, since φ is nontrivial, $|\mathrm{D}|^{-\alpha}\varphi$ and therefore also $N(\varphi)$ are strictly increasing on $(-\pi,0)$. Then for all $-\pi < y < x < 0$ we find that

$$0 < N(\varphi(x)) - N(\varphi(y)) = (\varphi(x) - \varphi(y))\underbrace{N'(\varphi(\xi))}_{>0} \tag{11}$$

for some $\xi \in (y,x)$ by the mean value and intermediate value theorems, which yields that φ is strictly increasing on $(-\pi,0)$. Moreover, (10) and (11) together show that

$$\begin{aligned}N'(\varphi(x))\,\varphi'(x) &= \lim_{h\searrow 0}\frac{N(\varphi(x+h)) - N(\varphi(x-h))}{2h}\\ &= \lim_{h\searrow 0}\frac{|\mathrm{D}|^{-\alpha}\varphi(x+h) - |\mathrm{D}|^{-\alpha}\varphi(x-h)}{2h} \qquad (12)\\ &\geqslant \int_{-\pi}^0 \big(K_\alpha(y-x) - K_\alpha(y+x)\big)\,\varphi'(y)\,\mathrm{d}y,\end{aligned}$$

where we have applied Fatou's lemma in the last transition. Focusing on $(-\pi, 0)$, we know that both the first factor in the integrand is strictly positive, $N'(\varphi) = c - n'(\varphi) > 0$ (because φ is strictly increasing), and $\varphi' \gneq 0$. Thus $\varphi' > 0$ on $(-\pi, 0)$. $\square$

We next start to investigate what happens if solutions touch μ, and begin with a one-sided α-Hölder estimate around 0.

Lemma 14. *Let $\varphi \in \mathring{C}_{\mathrm{even}}(\mathbb{T})$ be a nontrivial solution of* (4) *that is increasing in $(-\pi, 0)$ and satisfies $n'(\varphi) \leqslant c$ everywhere. Then uniformly around 0 one has*

$$\mu - \varphi(x) \gtrsim |x|^\alpha.$$

Proof. Let $x \in (-\pi, 0)$ be close to 0 and let $\xi \in \left(x, \frac{x}{2}\right)$. Monotonicity yields that $N'(\varphi(x)) \geqslant N'(\varphi(\xi))$, and since $\varphi' > 0$ on $(-\pi, 0)$ by Theorem 12, we may compute

$$\begin{aligned} N'(\varphi(x))\, \varphi'(\xi) &\geqslant N'(\varphi(\xi))\, \varphi'(\xi) \\ &\geqslant \int_{-\pi}^{0} \big(K_\alpha(\xi - y) - K_\alpha(\xi + y)\big)\, \varphi'(y)\, \mathrm{d}y \end{aligned}$$

using Fatou's lemma as in (12). By strict positivity of the integrand and the mean value theorem with $\zeta \in (\xi, \xi - 2y)$, this may be continued as

$$\begin{aligned} N'(\varphi(x))\, \varphi'(\xi) &\geqslant \int_{x}^{\frac{x}{2}} K'_\alpha(\zeta + y)(-2y)\, \varphi'(y)\, \mathrm{d}y \\ &\geqslant |x| \min_{y \in \left[x, \frac{x}{2}\right]} K'_\alpha(\zeta + y)\, \left(\varphi\left(\tfrac{x}{2}\right) - \varphi(x)\right) \\ &\gtrsim |x|^{\alpha - 1} \left(\varphi\left(\tfrac{x}{2}\right) - \varphi(x)\right), \end{aligned}$$

where we have used that $\min K'_\alpha(\zeta + y) \geqslant K'_\alpha(2x) \eqsim |x|^{\alpha-2}$ as $x \to 0$ by Theorem 4. We then integrate over $\left(x, \frac{x}{2}\right)$ in ξ and divide by $\varphi\left(\frac{x}{2}\right) - \varphi(x)$ on both sides, which is valid since φ is strictly increasing on $(-\pi, 0)$. This gives

$$N'(\varphi(x)) \gtrsim |x|^\alpha$$

uniformly around 0. The stated bound is now a consequence of

$$N'(\varphi(x)) = c - n'(\varphi(x)) \sim \mu^{p-1} - (\varphi(x))^{p-1} \eqsim \mu - \varphi(x), \tag{13}$$

where the latter uniform equivalence around 0 follows from continuity of φ and the observation by L'Hôpital's rule that

$$\lim_{t \nearrow \mu} \frac{\mu^{p-1} - t^{p-1}}{\mu - t} = (p-1)\mu^{p-2} > 0.$$

$\square$

Lemma 15. *The wave speed c is uniformly bounded away from 0 over the class of solution pairs (φ, c) for which $\varphi \in \mathring{C}_{\mathrm{even}}(\mathbb{T})$ is nontrivial, increasing in $(-\pi, 0)$, and satisfies $n'(\varphi) \leqslant c$ everywhere, where we in case (2_{sgn}) also assume that $\varphi\left(-\frac{\pi}{2}\right) = 0$. The estimate $c - n'(\varphi(-\pi)) \gtrsim 1$ holds in case (2_{abs}), implying that φ is smooth around $-\pi$.*

Proof. Let $x \in I := \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right]$ and consider first case (2_{abs}). Monotonicity of N' and φ plus (12) show that

$$\begin{aligned} N'(\varphi(-\pi))\,\varphi'(x) &\geqslant N'(\varphi(x))\,\varphi'(x) \\ &\geqslant \int_{-\pi}^{0} \big(K_\alpha(x-y) - K_\alpha(x+y)\big)\,\varphi'(y)\,\mathrm{d}y \\ &\geqslant \int_{I} \big(K_\alpha(x-y) - K_\alpha(x+y)\big)\,\varphi'(y)\,\mathrm{d}y \\ &\geqslant M_\alpha \left(\varphi\big(-\tfrac{\pi}{4}\big) - \varphi\big(-\tfrac{3\pi}{4}\big)\right), \end{aligned}$$

where we have used that $M_\alpha := \min\big\{K_\alpha(x-y) - K_\alpha(x+y) \,:\, x, y \in I\big\} > 0$ by the extreme value theorem and the fact that K_α is even and strictly increasing on $(-\pi, 0)$ by Theorem 5. Integrating over I in x then yields

$$c - n'(\varphi(-\pi)) = N'(\varphi(-\pi)) \geqslant \tfrac{\pi}{2} M_\alpha > 0 \tag{14}$$

after cancelling $\varphi\big(-\frac{\pi}{4}\big) - \varphi\big(-\frac{3\pi}{4}\big) > 0$ on both sides. Suppose to the contrary that there exists a sequence $\{(\varphi_j, c_j)\}_j$ of such solution pairs for which $c_j \searrow 0$. Then $n'(\varphi_j(-\pi)) \leqslant c_j \searrow 0$ as well, contradicting (14). Thus $c \gtrsim 1$ uniformly and $n'(\varphi(-\pi))$ does not touch c, so φ is smooth around $-\pi$ by Theorem 11.

In case (2_{sgn}) we consider $-\frac{\pi}{2}$ instead of $-\pi$ and similarly obtain $c = N'\big(\varphi\big(-\frac{\pi}{2}\big)\big) \gtrsim 1$. □

Finally we come to the main result in this section, which concerns both the global regularity of solutions and the exact α-Hölder regularity at 0 for solutions that touch μ. This is the most technical part of the paper.

Theorem 16 (Regularity)**.** *Let $\varphi \in \mathring{\mathrm{C}}_{\mathrm{even}}(\mathbb{T})$ be a nontrivial solution of (4) that is increasing in $(-\pi, 0)$ and satisfies $n'(\varphi) \leqslant c$, with maximum $\varphi(0) = \mu$. Then*

i) φ is strictly increasing on $(-\pi, 0)$, smooth except at 0 and possibly the point $\varphi^{-1}(0)$, and features at least the same regularity in the Hölder scale around $\varphi^{-1}(0)$ as n around 0;
ii) $\varphi \in \mathring{\mathrm{C}}^{\alpha}_{\mathrm{even}}(\mathbb{T})$; and
iii) φ is exactly α-Hölder continuous at 0, that is, uniformly around 0 we have

$$\mu - \varphi(x) \eqsim |x|^\alpha.$$

Proof. Property i and the lower bound in property iii follow directly from Theorems 11, 12 and 14. As a consequence, it remains to establish global α-Hölder regularity and the upper bound in property iii. Note that Hölder regularity at a point plus smoothness everywhere except at that point does *not* in general imply global Hölder regularity—one additionally needs uniform Hölder regularity *around* the point. In particular, in order to obtain property ii, it suffices to prove C^α regularity in a small interval around 0.

To this end, we first establish C^β regularity (around 0) for all $\beta < \alpha$. Let $-r \leqslant y < x \leqslant 0$ with $0 < r \ll 1$, and observe from Taylor's theorem that

$$N(\varphi(x)) - N(\varphi(y)) = (\varphi(x) - \varphi(y))\,N'(\varphi(x)) - \tfrac{1}{2}\,(\varphi(x) - \varphi(y))^2\,N''(\varphi(\xi))$$

for some $\xi \in (y, x)$ due to the intermediate value theorem. By (13) we know that

$$N'(\varphi(x)) \eqsim \mu - \varphi(x),$$

and $-N''(\varphi(\xi)) = n''(\varphi(\xi)) \gtrsim 1$ independently of ξ by choosing r so small that $\varphi(-r) > 0$ and remembering that φ is monotone ($\varphi(\xi) \geqslant \varphi(-r)$). This gives

$$N(\varphi(x)) - N(\varphi(y)) \gtrsim (\varphi(x) - \varphi(y))\,(\mu - \varphi(x)) + (\varphi(x) - \varphi(y))^2$$

uniformly over $-r \leqslant y < x \leqslant 0$, and by evenness of φ, also uniformly over $x, y \in (-r, r)$ with $|y| > |x|$. Thus (4) implies both that

$$|\mathrm{D}|^{-\alpha}\varphi(x) - |\mathrm{D}|^{-\alpha}\varphi(y) \gtrsim (\varphi(x) - \varphi(y))\,(\mu - \varphi(x)) \tag{15}$$

and

$$|\mathrm{D}|^{-\alpha}\varphi(x) - |\mathrm{D}|^{-\alpha}\varphi(y) \gtrsim (\varphi(x) - \varphi(y))^2 \tag{16}$$

uniformly over $x, y \in (-r, r)$ with $|y| > |x|$. Since $|\mathrm{D}|^{-\alpha}$ is locally C^{α}_{*} smoothing by Theorem 7, we deduce by a bootstrapping argument in (16) that φ, being a priori only continuous around 0, is in fact C^{β} regular around 0 for all $\beta < \min\{\frac{1}{2}, \alpha\}$.

If $\alpha \in \left(\frac{1}{2}, 1\right)$, however, then $|\mathrm{D}|^{-\alpha}\varphi$ will eventually pass index 1 in the Hölder scale, and we must instead work with derivatives. Specifically, take any β sufficiently close to $\frac{1}{2}$ from the previous argument such that now $\alpha + \beta > 1$ and $|\mathrm{D}|^{-\alpha}\varphi \in \mathring{\mathrm{C}}^{\alpha+\beta}_{*}(\mathbb{T}) = \mathring{\mathrm{C}}^{1,\alpha+\beta-1}(\mathbb{T})$. Then, since $|\mathrm{D}|^{-\alpha}\varphi'(0) = 0$, we find from the mean value theorem that

$$\begin{aligned} |\mathrm{D}|^{-\alpha}\varphi(x) - |\mathrm{D}|^{-\alpha}\varphi(y) &= |x - y|\,\big||\mathrm{D}|^{-\alpha}\varphi'(\xi) - |\mathrm{D}|^{-\alpha}\varphi'(0)\big| \\ &\lesssim |x - y|\,|\xi|^{\alpha+\beta-1} \\ &< |x - y|\,|y|^{\alpha+\beta-1} \end{aligned} \tag{17}$$

for some $\xi \in (y, x)$, where $-r \leqslant y < x \leqslant 0$, as above. If $|x| < |x - y|$, then (16), (17), and the triangle inequality imply that

$$\varphi(x) - \varphi(y) \lesssim |x - y|^{\frac{\alpha+\beta}{2}}. \tag{18}$$

In particular, for $x = 0$, this gives

$$\mu - \varphi(y) \lesssim |y|^{\frac{\alpha+\beta}{2}} \tag{19}$$

for all $y \in [-r, 0]$. Otherwise, if $|x - y| \leqslant |x|$, then (15), (17), and the triangle inequality yield

$$(\varphi(x) - \varphi(y))\,(\mu - \varphi(x)) \lesssim |x - y|\,|x|^{\alpha+\beta-1}.$$

We next use that $\mu - \varphi(x) \gtrsim |x|^{\alpha}$ by Theorem 14 and get

$$\varphi(x) - \varphi(y) \lesssim \frac{|x - y|}{|x|^{1-\beta}}. \tag{20}$$

Interpolating between (19) and (20) with index $\gamma \in (0, 1)$, and using that $|y| \leqslant 2|x|$, then subsequently show that

$$\begin{aligned} \frac{\varphi(x) - \varphi(y)}{|x - y|^{\gamma}} &\leqslant \frac{(\varphi(x) - \varphi(y))^{\gamma}}{|x - y|^{\gamma}}\,(\mu - \varphi(y))^{1-\gamma} \\ &\lesssim |x|^{(\beta-1)\gamma + \frac{\alpha+\beta}{2}(1-\gamma)} \end{aligned}$$

is uniformly bounded over $-r \leqslant y < x \leqslant 0$ in the case $|x - y| \leqslant |x|$ provided the last exponent is nonnegative, that is, if

$$\gamma \leqslant \frac{\alpha + \beta}{2 + \alpha - \beta}.$$

As such, by choosing the maximal γ, we obtain the estimate

$$\varphi(x) - \varphi(y) \lesssim |x-y|^{(\alpha+\beta)/(2+\alpha-\beta)}$$

when $|x-y| \leqslant |x|$, so that, together with (18) it is true that

$$\varphi(x) - \varphi(y) \lesssim \max\big\{|x-y|^{\frac{\alpha+\beta}{2}}, |x-y|^{(\alpha+\beta)/(2+\alpha-\beta)}\big\}$$

uniformly over $-r \leqslant y < x \leqslant 0$. Since both exponents are strictly increasing in β and converge to α as $\beta \nearrow \alpha$, it follows by bootstrapping that φ is C^β regular around 0 for all $\beta < \alpha$.

We next establish the upper C^α estimate at 0 in property iii. In fact, with $u(x) \coloneqq \mu - \varphi(x)$, we shall prove that

$$u(x) \lesssim |x|^\beta \tag{21}$$

uniformly over $x \in (-r, r)$ and $\beta \in [0, \alpha)$, from which the desired estimate follows by letting $\beta \nearrow \alpha$. On this route, note from (16) that

$$\begin{aligned}
(u(x))^2 \lesssim |\mathrm{D}|^{-\alpha}\varphi(0) - |\mathrm{D}|^{-\alpha}\varphi(x) = &\int_{\mathbb{T}} \big(K_\alpha(y) - K_\alpha(x-y)\big)\,\varphi(y)\,\mathrm{d}y \\
= &\int_{\mathbb{T}} \big(K_\alpha(x-y) - K_\alpha(y)\big)\,u(y)\,\mathrm{d}y \qquad (22)\\
= \tfrac{1}{2} &\int_{\mathbb{T}} \diamondsuit_y K_\alpha(x)\,u(y)\,\mathrm{d}y,
\end{aligned}$$

where $\diamondsuit_y f(x) \coloneqq f(x+y) - 2f(y) + f(x-y)$ denotes the second-order central difference operator. Here we have utilised periodicity of K_α in the first transition between the integrals, and averaging, variable change $y \mapsto -y$, and evenness of u (from φ) in the last step. Since we have already established that $u \in \mathrm{C}^\beta(\mathbb{T})$ for all $\beta \in [0, \alpha)$, it is clear that $u|\cdot|^{-\beta}$ is bounded on $(-r, r)$. Thus (22) shows that

$$\sup_{|x|<r} \big|u(x)|x|^{-\beta}\big| \lesssim \sup_{|x|<r} |x|^{-2\beta} \int_{\mathbb{T}} \big|\diamondsuit_y K_\alpha(x)\big|\,|y|^\beta\,\mathrm{d}y$$

after cancelling $\sup_{|x|<r}\big|u(x)|x|^{-\beta}\big|$ once on each side. Now remember that $K_\alpha = \gamma_\alpha|\cdot|^{\alpha-1} + K_{\alpha,\mathrm{reg}}$ from Theorem 4. In particular, for the regular part we may Taylor expand around y to see that

$$\big|\diamondsuit_y K_{\alpha,\mathrm{reg}}(x)\big| = \mathcal{O}(x^2)$$

uniformly over $y \in \mathbb{T}$, because $K_{\alpha,\mathrm{reg}}$ is even and $K''_{\alpha,\mathrm{reg}}$ and $K'''_{\alpha,\mathrm{reg}}$ are bounded on $\mathbb{T}$. As such, using that $|y|^\beta \lesssim 1$ on $\mathbb{T}$ independently of β, we obtain that

$$|x|^{-2\beta} \int_{\mathbb{T}} \big|\diamondsuit_y K_{\alpha,\mathrm{reg}}(x)\big|\,|y|^\beta\,\mathrm{d}y = \mathcal{O}\big(|x|^{2(1-\beta)}\big) = \mathcal{O}(1)$$

since $\beta < 1$. For the singular part, one has with $y = xs$ that

$$|x|^{-2\beta} \int\limits_{|y|\leqslant\pi} \big|\diamondsuit_y|\cdot|^{\alpha-1}(x)\big|\,|y|^\beta\,\mathrm{d}y \leqslant |x|^{\alpha-\beta} \int\limits_{|s|<\infty} \big|\diamondsuit_1|\cdot|^{\alpha-1}(s)\big|\,|s|^\beta\,\mathrm{d}s \qquad \text{(note the 1 in } \diamondsuit_1\text{)}.$$

The right-hand side is $\mathcal{O}(1)$ over $x \in (-r, r)$ because of $\alpha - \beta \geqslant 0$ and the following observation: $|\cdot|^{\alpha-1}$ is locally integrable, and

$$\diamondsuit_1|\cdot|^{\alpha-1}(s) = |s|^{\alpha-1}\big[(\alpha-1)(\alpha-2)\,s^{-2} + \mathcal{O}(s^{-4})\big]$$

as $|s| \to \infty$, so that

$$\left|\diamondsuit_1| \cdot |^{\alpha-1}(s)\right| |s|^{\beta} \lesssim |s|^{\alpha+\beta-3}$$

for $|s| \gg 1$, where $\alpha + \beta - 3 < -1$ uniformly over $\beta < \alpha$ because $\alpha \in (0,1)$ is fixed, thereby guaranteeing integrability at infinity. Hence, $\sup_{|x|<r} \left|u(x)|x|^{-\beta}\right| \lesssim 1$ uniformly over $\beta < \alpha$, which is (21).

It remains to establish C^α continuity around 0. Since φ is increasing on $[-\pi, 0]$ and even, it suffices to show that

$$\sup_{\substack{x\in[-r,0);\\ h\in(0,|x|]}} \frac{\Delta_h \varphi(x)}{h^\alpha} < \infty,$$

where we have introduced the (scaled) symmetric difference $\Delta_h f(x) \coloneqq f(x+h) - f(x-h)$. To this end, we shall extract $\Delta_h \varphi(x)$ from $\Delta_h[N(\varphi)](x)$ and estimate each side of the relation

$$\Delta_h[N(\varphi)](x) = \Delta_h[|\mathrm{D}|^{-\alpha}\varphi](x), \tag{23}$$

which comes straight from (4). On this path, we let $x \in [-r, 0)$ and $h \in (0, |x|]$, and then choose $a \coloneqq \varphi(x+h)$ and $b \coloneqq \varphi(x-h)$ in the Taylor expansion

$$N(b) = N(a) + N'(a)(b-a) + \tfrac{1}{2} N''(\zeta)(b-a)^2,$$

with ζ between a and b, to see that

$$\Delta_h[N(\varphi)](x) = \left(N'(\varphi(x+h)) - \tfrac{1}{2} N''(\varphi(\xi))\, \Delta_h \varphi(x)\right) \Delta_h \varphi(x) \tag{24}$$

for some $\xi \in (x-h, x+h)$ (satisfying $\varphi(\xi) = \zeta$) by the intermediate value theorem. Here

$$-N''(\varphi(\xi)) = n''(\varphi(\xi)) \eqsim 1$$

uniformly over $x \in [-r, 0)$ and $h \in (0, |x|]$. Now note that

$$\begin{aligned} &N'(\varphi(x+h)) - p\left(\varphi(x)^{p-1} - \varphi(x+h)^{p-1}\right) \\ &\qquad\qquad \sim \mu^{p-1} - \varphi(x)^{p-1} \eqsim \mu - \varphi(x) \eqsim |x|^\alpha \end{aligned}$$

in light of (13) and the exact C^α estimate at $x = 0$. Since $\varphi(x)^{p-1} - \varphi(x+h)^{p-1}$ and $\Delta_h \varphi(x)$ both vanish as $h \searrow 0$, we see that

$$\sup_{h\in(0,|x|]} \left(N'(\varphi(x+h)) - \tfrac{1}{2} N''(\varphi(\xi))\, \Delta_h \varphi(x)\right) \gtrsim |x|^\alpha.$$

Thus (24) yields that

$$\sup_{h\in(0,|x|]} \frac{\Delta_h \varphi(x)}{h^\beta} \lesssim |x|^{-\alpha} \sup_{h\in(0,|x|]} \frac{\Delta_h[N(\varphi)](x)}{h^\beta} \tag{25}$$

for all $\beta < \alpha$, where we postpone taking the supremum over $x \in [-r, 0)$ until we have estimates for $\Delta_h[|\mathrm{D}|^{-\alpha}\varphi](x)$ in (23). With that in mind, we first consider the regular part in $|\mathrm{D}|^{-\alpha}\varphi$

and compute

$$\begin{aligned}
\Delta_h\big[K_{\alpha,\mathrm{reg}} * \varphi\big](x) = & \int_{\mathbb{T}} \Delta_h K_{\alpha,\mathrm{reg}}(x-y)\,\varphi(y)\,\mathrm{d}y \\
= & \; h \int_{\mathbb{T}} 2\,K'_{\alpha,\mathrm{reg}}(x-y)\,\varphi(y)\,\mathrm{d}y \\
= & \; h \int_{\mathbb{T}} \Delta_x K'_{\alpha,\mathrm{reg}}(y)\,\varphi(y)\,\mathrm{d}y \qquad \text{(note the } x \text{ in } \Delta_x) \\
= & \; xh \int_{\mathbb{T}} \int_{-1}^{1} K''_{\alpha,\mathrm{reg}}(y+tx)\,\mathrm{d}t\,\varphi(y)\,\mathrm{d}y
\end{aligned}$$

by the mean value theorem and repeated use of parity and periodicity of $K_{\alpha,\mathrm{reg}}$ and φ. Consequently,

$$\big|\Delta_h\big[K_{\alpha,\mathrm{reg}} * \varphi\big](x)\big| \lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}|x|h < \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}|x|^{\alpha}h^{\beta},$$

for any $\theta \in (0,1)$ because $K''_{\alpha,\mathrm{reg}}$ is bounded and $\|\varphi\|_\infty \lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}\|\varphi\|_\infty^{1-\theta} \lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}$. Hence,

$$|x|^{-\alpha} \sup_{h\in(0,|x|]} \frac{\Delta_h\big[K_{\alpha,\mathrm{reg}} * \varphi\big](x)}{h^{\beta}} \lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}. \tag{26}$$

Switching to the singular part, one finds by parity and periodicity that

$$\begin{aligned}
\Delta_h\big[|\cdot|^{\alpha-1} * \varphi\big](x) = & \int_{-\pi}^{0} \Delta_h|\cdot|^{\alpha-1}(y)\,\Delta_{|x|}\varphi(y)\,\mathrm{d}y \\
= & \; h^{\alpha} \int_{-\pi/h}^{0} \Delta_1|\cdot|^{\alpha-1}(s)\,\Delta_{|x|}\varphi(hs)\,\mathrm{d}s \qquad \text{(note the subscripts).}
\end{aligned} \tag{27}$$

Since $\varphi \in \mathring{\mathrm{C}}^{\beta}(\mathbb{T})$, we have

$$|\Delta_{|x|}\varphi(y)| \lesssim \|\varphi\|_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})} \min\{|x|^{\beta}, |y|^{\beta}\} \quad \text{for } \beta < \alpha, \tag{28}$$

and furthermore,

$$|\Delta_{|x|}\varphi(y)| \lesssim \max\{|x|^{\alpha}, |y|^{\alpha}\} \tag{29}$$

by the already established estimate $\mu - \varphi(\xi) \lesssim |\xi|^{\alpha}$ for $|\xi| \ll 1$. Interpolating between (28) and (29) with parameter

$$\theta := \frac{\alpha}{\alpha+\beta} \in \left(\tfrac{1}{2}, 1\right), \quad \text{so that} \quad \theta\beta = (1-\theta)\alpha,$$

then yields

$$\begin{aligned}
|\Delta_{|x|}\varphi(y)| &\lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})} \min\big\{|x|^{\theta\beta}, |y|^{\theta\beta}\big\} \max\big\{|x|^{(1-\theta)\alpha}, |y|^{(1-\theta)\alpha}\big\} \\
&= \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}|xy|^{\theta\beta}.
\end{aligned} \tag{30}$$

This estimate, with $y = hs$, is appropriate for small s in (27), but becomes problematic for large s when $\alpha > 2/3$ since

$$\Delta_1|\cdot|^{\alpha-1}(s)\,|s|^{\theta\beta} \eqsim |s|^{\alpha-2+\theta\beta}$$

for $|s| \gg 1$ (at scale $s \sim h^{-1}$), thus failing to be integrable in (27) as $h \searrow 0$. As a remedy, we use the estimate

$$\begin{aligned} \big|\Delta_{|x|}\varphi(hs)\big| &\leqslant \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}|x|^{\theta\beta}\big|\Delta_{|x|}\varphi(hs)\big|^{1-\theta} \\ &\lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}|x|^{\theta\beta+1-\theta}\max_{|t-hs|\leqslant|x|}|\varphi'(t)|^{1-\theta} \end{aligned} \tag{31}$$

when $s \sim h^{-1}$, where one observes that the given maximum of $\varphi'(t)$ is uniformly bounded over $h \in (0,|x|]$ and $x \in [-r,0)$ since t stays away from the singularity at 0. We then note that

$$\min\Big\{|hs|^{\theta\beta}, |x|^{1-\theta}\max_{|t-hs|\leqslant|x|}|\varphi'(t)|^{1-\theta}\Big\} \lesssim \max\big\{h^{\theta\beta}, |x|^{1-\theta}\big\} \leqslant |x|^{\min\{\theta\beta,1-\theta\}}$$

uniformly over $s \in (-\pi/h, 0)$, where we have utilised that $h \leqslant |x|$. In particular, combining (30) and (31) implies that

$$|\Delta_{|x|}\varphi(hs)| \lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}|x|^{\theta\beta}|x|^{\min\{\theta\beta,1-\theta\}}$$

uniformly over $s \in (-\pi/h, 0)$. Now (27) may be estimated as

$$\begin{aligned} \big|\Delta_h\big[|\cdot|^{\alpha-1} * \varphi\big](x)\big| &\lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}h^{\alpha}|x|^{\theta\beta+\min\{\theta\beta,1-\theta\}}\int_{-\infty}^{0}\big|\Delta_1|\cdot|^{\alpha-1}(s)\big|\,\mathrm{d}s \\ &\lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}h^{\alpha}|x|^{\theta\beta+\min\{\theta\beta,1-\theta\}}, \end{aligned}$$

where the integral converges because $\Delta_1|\cdot|^{\alpha-1}(s) \lesssim |s|^{\alpha-2}$ for $|s| \gg 1$ with $\alpha - 2 < -1$. Therefore, as $h \leqslant |x|$,

$$\begin{aligned} |x|^{-\alpha}\sup_{h\in(0,|x|]}\frac{\Delta_h\big[|\cdot|^{\alpha-1} * \varphi\big](x)}{h^{\beta}} &\lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})}|x|^{\min\{(2\theta-1)\beta,(1-\theta)(1-\beta)\}} \\ &\lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})} \end{aligned} \tag{32}$$

uniformly over $x \in [-r,0)$ and all $\beta < \alpha$ sufficiently close to α, since in that case

$$\min\big\{(2\theta-1)\beta, (1-\theta)(1-\beta)\big\} > 0 \quad \text{(uniformly)}.$$

We then put (23), (25), (26), and (32) together and find that

$$\sup_{\substack{x\in[-r,0);\\ h\in(0,|x|]}}\frac{\Delta_h\varphi(x)}{h^{\beta}} \lesssim \|\varphi\|^{\theta}_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})} \tag{33}$$

uniformly over all β sufficiently close to α. By smoothness away from 0, one has

$$\|\varphi\|_{\mathring{\mathrm{C}}^{\beta}(\mathbb{T})} \lesssim \max\left\{1, \sup_{\substack{x\in[-r,0);\\ h\in(0,|x|]}}\frac{\Delta_h\varphi(x)}{h^{\beta}}\right\}$$

for all $\beta \leqslant \alpha$, and so (33) implies that

$$\left(\sup_{\substack{x\in[-r,0);\\ h\in(0,|x|]}}\frac{\Delta_h\varphi(x)}{h^{\beta}}\right)^{1-\theta} \lesssim 1$$

uniformly over β sufficiently close to α. In particular, letting $\beta \nearrow \alpha$ (for which $\theta \searrow \frac{1}{2}$ stays away from 1), it follows that φ is indeed C^{α} continuous around 0. $\square$

In case (2_{sgn}) we could have assumed that $\varphi(-\pi) = -\mu$ instead of $\varphi(0) = \mu$ in Theorem 16 and then proved exact α-Hölder continuity at $-\pi$. We conjecture that both assumptions imply the other and more generally imply antisymmetry of waves about $-\frac{\pi}{2}$ whenever one deals with antisymmetric nonlinearities. This is also the reason why we assumed that $\varphi\big(-\frac{\pi}{2}\big) = 0$ in Theorem 15. As a remedy to the lack of proof of the general property, we shall in Section 4 instead *construct* solutions which are antisymmetric about $-\frac{\pi}{2}$.

4. Global bifurcation analysis

We first establish nontrivial small-amplitude travelling waves around the line $c \mapsto (0, c)$ of trivial solutions by means of local bifurcation theory and then extend the bifurcation curve globally using the analytic theory of Buffoni and Toland [6]. By carefully examining the structure of the global curve in connection with the a priori nodal properties in Section 3, we finally deduce the existence of a limiting sequence along the curve which converges to a highest wave satisfying Theorem 16. This establishes Theorem 1 when the nonlinearities (2) are smooth, that is, when they equal $n(x) = x^p$ for $2 \leqslant p \in \mathbb{N}$, and in Fig. 3 we provide a sketch of the analysis.

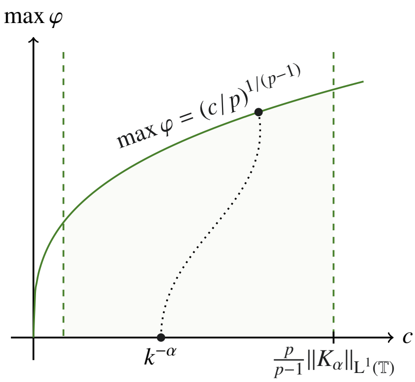


Figure 3. Illustrating the global bifurcation diagram in the smooth case $n(x) = x^p$ for $2 \leqslant p \in \mathbb{N}$ of $2\pi/k$-periodic even solutions obtained in Theorem 24 bifurcating from the trivial solution $(0, k^{-\alpha})$ and reaching a limiting extreme wave. The dashed vertical lines mark the bounds for the wave speed in Theorems 9 and 27, whereas the solid curve displays the possible maximal height from (6) for these waves (plotted for $p = 3$). Along the dotted bifurcation curve, one may extract a sequence for which possibilities i) and ii) in Theorem 24 occur simultaneously, converging to a solution of (4) with the C^α properties of Theorem 16.

In the general nonsmooth situation, however, one cannot use the analytic bifurcation theory directly. We resolve this issue by regularising n analytically around 0 (where its regularity is only of order p in the Hölder scale) and instead study global bifurcation for the regularised equation

$$0 = F^\epsilon(\varphi, c) := |\mathrm{D}|^{-\alpha}\varphi - N^\epsilon(\varphi; c) - \fint_{\mathbb{T}} n^\epsilon(\varphi) \tag{34}$$

of (4) for every $0 < \epsilon \ll 1$. This leads to solutions $(\varphi^\epsilon, c^\epsilon)$ at the end of the bifurcation curves, with the optimal α-Hölder continuity of Section 3, that will be shown to converge (up to a subsequence) to a solution of (4) with the same Hölder properties as $\epsilon \searrow 0$. Here

$N^\epsilon(\varphi; c) := c\varphi - n^\epsilon(\varphi)$ and

$$n^\epsilon(x) := \begin{cases} \left(x^2 + \epsilon^2\right)^{p/2} - \epsilon^p & \text{in case } (2_{\mathrm{abs}}); \\ x\left(\left(x^2 + \epsilon^2\right)^{(p-1)/2} - \epsilon^{p-1}\right) & \text{in case } (2_{\mathrm{sgn}}) \end{cases} \tag{35}$$

is a natural analytic regularisation with the same monotonicity properties as n and that converges uniformly to n on compact sets as $\epsilon \searrow 0$. In particular, the regularity theory in Section 3 carries over to the new setting by replacing n and N with n^ϵ and N^ϵ, noting that the extreme value corresponding to the first positive critical point for N^ϵ is a continuous function

$$\mu^\epsilon := \mu(p, c, \epsilon) \tag{36}$$

that converges to μ in (6) as $\epsilon \searrow 0$ by the implicit function theorem.

In the remainder, we focus on the analysis of the nonsmooth situation, leaving the appropriate modifications ("$\epsilon = 0$") when $n(x) = x^p$ for $2 \leqslant p \in \mathbb{N}$ to the reader, but shall nevertheless provide details for the bifurcation formulas in the smooth case as they may be of independent interest.

According to the above, we study F^ϵ from (34) as an operator $\mathcal{X}^\beta \times \mathbb{R}_+ \to \mathcal{X}^\beta$, where $\mathbb{R}_+ := [0, \infty)$ and $\mathcal{X}^\beta := \mathring{\mathrm{C}}^\beta_{\mathrm{even}}(\mathbb{T})$, noting that $N^\epsilon(\cdot, c)$ acts on $\mathcal{X}^\beta$ in light of [17, Theorem 2.1]. We also let $\beta \in \left(\max\{\alpha, \frac{1}{2}\}, 1\right)$; the choice $\beta > \frac{1}{2}$ guarantees that the Fourier series of $\varphi \in \mathcal{X}^\beta$ converges uniformly to φ, whereas the requirement $\beta \in (\alpha, 1)$ avoids the technicalities of the Hölder–Zygmund space of order 1 and assures that $\mathcal{X}^\beta$ contains the sought-after extreme wave in Theorem 16.

Observe that F^ϵ is analytic due to the regularisation and that its linearisation around the line of trivial solutions equals

$$\partial_\varphi F^\epsilon(0, c) = |\mathrm{D}|^{-\alpha} - c\,\mathrm{id}\,.$$

Hence, for $c > 0$ the kernel of $\partial_\varphi F^\epsilon(0, c)$ is trivial unless $c = k^{-\alpha}$ for some integer $k \geqslant 1$, being a simple eigenvalue of $|\mathrm{D}|^{-\alpha}$, in which case

$$\ker \partial_\varphi F^\epsilon(0, k^{-\alpha}) = \operatorname*{span}_{k \geqslant 1}\{\cos(k\cdot)\}$$

is one-dimensional. Furthermore, $|\mathrm{D}|^{-\alpha}$ is a compact operator $\mathcal{X}^\beta \to \mathcal{X}^\beta$ since it is α-smoothing and $\mathcal{X}^{\beta'}$ is compactly embedded in $\mathcal{X}^\beta$ for $\beta' > \beta$. Thus $\partial_\varphi F^\epsilon(0, c)$ is a compact perturbation of the identity and therefore constitutes a Fredholm operator of index zero. We may therefore apply the (analytic) Crandall–Rabinowitz theorem [6, Theorems 8.3.1 and 8.4.1] and obtain the following local bifurcation result.

Theorem 17 (Local bifurcation). *For all $\epsilon > 0$ and $k \geqslant 1$ there exists a local, analytic curve*

$$\mathfrak{C}^\epsilon_{\mathrm{loc},k} \colon s \mapsto (\varphi^\epsilon_k(s), c^\epsilon_k(s)) \in \mathcal{X}^\beta \times \mathbb{R}_+,$$

defined around $s = 0$, of nontrivial $2\pi/k$-periodic solutions of (34) that bifurcates from the line of trivial solutions $c \mapsto (0, c)$ at $\mathfrak{C}^\epsilon_{\mathrm{loc},k}(0) = (0, k^{-\alpha})$. In a neighbourhood of $(0, k^{-\alpha})$ these are all the nontrivial solutions of $F^\epsilon(\varphi, c) = 0$ in $\mathcal{X}^\beta \times \mathbb{R}_+$.

Since we have an analytic curve in $\mathcal{X}^\beta \times \mathbb{R}_+$, we may compute the associated asymptotic formulas for $\mathfrak{C}^\epsilon_{\mathrm{loc},k}(s)$ as $s \to 0$ by means of direct expansions in the regularised steady equation (34). Alternatively, one could use the more general theory in [21, Section I.6].

Theorem 18 (Bifurcation formulas). *$\mathfrak{C}^{\epsilon}_{\mathrm{loc},k}$ can be parametrised in such a way that $s \mapsto c^{\epsilon}_k(s)$ is even, and with this choice the bifurcation formulas are as follows as $s \to 0$:*

$$\text{In case } (2_{\mathrm{abs}}): \qquad \begin{cases} \varphi^{\epsilon}_k(s)(x) = s\cos kx + s^2\, C^{\epsilon}_{k,\mathrm{abs}} \cos 2kx + \mathcal{O}(s^3); \\ \quad c^{\epsilon}_k(s) = k^{-\alpha} + s^2\, 2C^{\epsilon}_{k,\mathrm{abs}} + \mathcal{O}(s^4); \end{cases}$$

$$\text{in case } (2_{\mathrm{sgn}}): \qquad \begin{cases} \varphi^{\epsilon}_k(s)(x) = s\cos kx + s^3\, C^{\epsilon}_{k,\mathrm{sgn}} \cos 3kx + \mathcal{O}(s^5); \\ \quad c^{\epsilon}_k(s) = k^{-\alpha} + s^2\, \tfrac{3}{4}(p-1)\epsilon^{p-3} + \mathcal{O}(s^4), \end{cases}$$

with $C^{\epsilon}_{k,\mathrm{abs}} := \dfrac{\frac{1}{4}p\epsilon^{p-2}}{k^{-\alpha} - (2k)^{-\alpha}}$ *and* $C^{\epsilon}_{k,\mathrm{sgn}} := \dfrac{\frac{1}{8}(p-1)\epsilon^{p-3}}{k^{-\alpha} - (3k)^{-\alpha}}$.

Remark 19. *It suffices to study the case $k = 1$ of 2π-periodic solutions in the bifurcation analysis, since $F^{\epsilon}(\varphi, c) = 0$ is invariant under the scaling*

$$\varphi \mapsto k^{\alpha/(p-1)}\varphi(k\cdot), \qquad c \mapsto k^{\alpha}c, \qquad \epsilon \mapsto k^{\alpha/(p-1)}\epsilon.$$

Thus we focus on $\mathfrak{C}^{\epsilon}_{\mathrm{loc}} := \mathfrak{C}^{\epsilon}_{\mathrm{loc},1}$, $\varphi^{\epsilon} := \varphi^{\epsilon}_1$, and $c^{\epsilon} := c^{\epsilon}_1$ from now on.

Proof. As in the proof of [15, Theorem 6.1], we parametrise $\mathfrak{C}^{\epsilon}_{\mathrm{loc}}$ with the requirement $[\![\varphi^{\epsilon}(s)]\!]_1 = s$, where

$$[\![\varphi]\!]_m := \tfrac{1}{\pi}\int_{\mathbb{T}} \varphi(x)\cos(mx)\,\mathrm{d}x, \qquad m = 1, 2, \dots$$

are the coefficients in the cosine expansion $\varphi = \sum_{m=1}^{\infty} [\![\varphi]\!]_m \cos(m\cdot)$. Since $(\varphi^{\epsilon}(\cdot + \pi), c^{\epsilon})$ also constitutes a solution pair whenever $(\varphi^{\epsilon}, c^{\epsilon})$ is, and

$$[\![\varphi^{\epsilon}(s)(\cdot + \pi)]\!]_1 = -[\![\varphi^{\epsilon}(s)]\!]_1 = -s = [\![\varphi^{\epsilon}(-s)]\!]_1,$$

it follows by uniqueness that $\varphi^{\epsilon}(s)(\cdot + \pi) = \varphi^{\epsilon}(-s)$ and $c^{\epsilon}(s) = c^{\epsilon}(-s)$, proving the symmetry.

Switching to the bifurcation formulas, we analytically expand $\varphi^{\epsilon}(s)$ and $c^{\epsilon}(s)$ into

$$\varphi^{\epsilon}(s) = \sum\nolimits_{\ell=1}^{\infty} \varphi_{\ell}\, s^{\ell} \qquad \text{and} \qquad c^{\epsilon}(s) = \sum\nolimits_{\ell=0}^{\infty} \varsigma_{2\ell}\, s^{2\ell} \tag{37}$$

and observe that the coefficients may be found by plugging the expansions into (34) and identifying terms of equal order in s by uniqueness. Note that the Taylor expansion

$$n^{\epsilon}(x) = \begin{cases} \frac{1}{2}p\epsilon^{p-2}x^2 + \mathcal{O}(x^4) & \text{in case } (2_{\mathrm{abs}}); \\ \frac{1}{2}(p-1)\epsilon^{p-3}x^3 + \mathcal{O}(x^5) & \text{in case } (2_{\mathrm{sgn}}) \end{cases} \tag{38}$$

holds in an ϵ-dependent interval around $x = 0$, which simplifies the analysis for all sufficiently small s. With $L := |\mathrm{D}|^{-\alpha} - \varsigma_0\,\mathrm{id}$, this gives the following in case (2_{abs}):

$$\begin{aligned} s&\colon\ L\varphi_1 = 0; \\ s^2&\colon\ L\varphi_2 = -\tfrac{1}{2}p\epsilon^{p-2}\big(\varphi_1^2 - ⨍_{\mathbb{T}} \varphi_1^2\big); \\ s^3&\colon\ L\varphi_3 = \varsigma_2\varphi_1 - p\epsilon^{p-2}\big(\varphi_1\varphi_2 - ⨍_{\mathbb{T}} \varphi_1\varphi_2\big). \end{aligned}$$

The first-order case yields that $\varphi_1 = \cos$ and $\varsigma_0 = 1$ (more generally, $\varsigma_0 = k^{-\alpha}$), whence

$$\varphi_2(x) = \frac{\frac{1}{4}p\epsilon^{p-2}}{1 - 2^{-\alpha}}\cos 2x.$$

Since $2\cos x\cos 2x = \cos x + \cos 3x$, it follows that

$$\varsigma_2 = \frac{\frac{1}{8}p^2\epsilon^{2(p-2)}}{1-2^{-\alpha}} \qquad \text{and} \qquad \varphi_3(x) = \frac{\frac{1}{8}p^2\epsilon^{2(p-2)}}{(1-2^{-\alpha})(1-3^{-\alpha})}\cos 3x.$$

As for case (2_{sgn}), we find that

$$\begin{aligned} s\colon\; & L\varphi_1 = 0,\\ s^2\colon\; & L\varphi_2 = 0,\\ \text{and } s^3\colon\; & L\varphi_3 = \varsigma_2\varphi_1 - \tfrac{1}{2}(p-1)\epsilon^{p-3}\big(\varphi_1^3 - ⨍_{\mathbb{T}}\varphi_1^3\big), \end{aligned}$$

leading to $\varphi_1 = \cos$ and $\varsigma_0 = 1$. Moreover, $\varphi_2 = 0$ by choice of parametrisation ($[\![\varphi_\ell]\!]_1 = 0$ for $\ell \geqslant 2$). Finally,

$$\varsigma_2 = \tfrac{3}{4}(p-1)\epsilon^{p-3} \qquad \text{and} \qquad \varphi_3(x) = \frac{\frac{1}{8}(p-1)\epsilon^{p-3}}{1-3^{-\alpha}}\cos 3x$$

with help of the identity $4\cos^3 x = 3\cos x + \cos 3x$. □

We also include asymptotic formulas in the smooth case $n(x) = x^p$ for any $2 \leqslant p \in \mathbb{N}$ (with no regularisation). Formulas with arbitrary order in p seem to be new, and the result adapts easily to other dispersive operators as well.

Theorem 20 (Bifurcation formulas for smooth n)**.** *Consider smooth $n(x) = x^p$ for $2 \leqslant p \in \mathbb{N}$. Then the bifurcation formulas for $\mathfrak{C}^{\epsilon=0}_{\mathrm{loc},k}$ with the parametrisation that $s \mapsto c_k^{\epsilon=0}(s)$ is even are as follows as $s \to 0$:*

$$\varphi_k^{\epsilon=0}(s)(x) = s\cos kx + s^p\Phi_k(x) + \mathcal{O}(s^{2p-1}),$$

where $\Phi_k \in \{\Phi_k^{\mathrm{even}}, \Phi_k^{\mathrm{odd}}\}$ depending on whether $p \geqslant 2$ is even or odd, with corresponding speed

$$\begin{aligned} & c_k^{\epsilon=0}(s) = k^{-\alpha} + s^{2p-2}C_k^{\mathrm{even}} + \mathcal{O}(s^{2p})\\ \textit{or}\quad & c_k^{\epsilon=0}(s) = k^{-\alpha} + s^{p-1}\,C_k^{\mathrm{odd}} + \mathcal{O}(s^{2p-2}). \end{aligned}$$

Here

$$\begin{aligned} \Phi_k^{\mathrm{even}} &:= \textstyle\sum_{j=0}^{\frac{p}{2}-1}\Phi_{k,j}, & C_k^{\mathrm{even}} &:= \tfrac{p}{2^{p-1}}\Big(C_{k,0} + \textstyle\sum_{j=1}^{\frac{p}{2}-1}\Big(\binom{p-1}{j} + \binom{p-1}{j-1}\Big)C_{k,j}\Big),\\ \Phi_k^{\mathrm{odd}} &:= \textstyle\sum_{j=0}^{\frac{p-3}{2}}\Phi_{k,j}, \quad \textit{and} & C_k^{\mathrm{odd}} &:= \tfrac{1}{2^{p-1}}\binom{p}{\frac{p-1}{2}}, \end{aligned}$$

with $C_{k,j} := \dfrac{\binom{p}{j}/2^{p-1}}{k^{-\alpha} - ((p-2j)k)^{-\alpha}}$ *and* $\Phi_{k,j}(x) := C_{k,j}\cos\big((p-2j)kx\big)$.

Proof. The cases $p = 2, 3$ are similar to those in the proof of Theorem 18, and we only examine $k = 1$ by Theorem 19. Thus let $L := |\mathrm{D}|^{-\alpha} - \varsigma_0\,\mathrm{id}$ and consider first even $p \geqslant 4$:

$$\begin{aligned} s\colon\quad & L\varphi_1 = 0; & & s^{p-1}\colon\quad L\varphi_{p-1} = \textstyle\sum_{i=1}^{\frac{p}{2}-1}\varsigma_{2i}\varphi_{p-1-2i};\\ s^2\colon\quad & L\varphi_2 = 0; & \dots & \\ s^3\colon\quad & L\varphi_3 = \varsigma_2\varphi_1; & & s^p\colon\quad L\varphi_p = \textstyle\sum_{i=1}^{\frac{p}{2}-1}\varsigma_{2i}\varphi_{p-2i} - \varphi_1^p + ⨍_{\mathbb{T}}\varphi_1^p. \end{aligned}$$

The first-order case yields that $\varphi_1 = \cos$ and $\varsigma_0 = 1$, and we successively find that

$$\varphi_2 = \cdots = \varphi_{p-1} = 0 \qquad \text{and} \qquad \varsigma_2 = \cdots = \varsigma_{p-2} = 0. \tag{39}$$

This leads to

$$\varphi_p = -L^{-1}\left(\varphi_1^p - \fint_{\mathbb{T}} \varphi_1^p\right) = \Phi_1^{\text{even}}$$

by means of the even power-reduction formula

$$\cos^p x - \fint_{\mathbb{T}} \cos^p = \frac{1}{2^{p-1}} \sum\nolimits_{j=1}^{\frac{p}{2}-1} \binom{p}{j} \cos\left((p-2j)x\right).$$

Taking the prior results into account, we next examine higher-order coefficients for even $p \geqslant 4$ with help of successive cancellations of terms that vanish:

$$\begin{aligned}
s^{p+1}\colon &\quad L\varphi_{p+1} &&= \varsigma_p \varphi_1 &&\Rightarrow &&\varphi_{p+1} &&= 0, \quad \varsigma_p = 0;\\
s^{p+2}\colon &\quad L\varphi_{p+2} &&= 0 &&\Rightarrow &&\varphi_{p+2} &&= 0;\\
&\quad\vdots && &&\vdots\\
s^{2p-3}\colon &\quad L\varphi_{2p-3} &&= \varsigma_{2p-2}\varphi_1 &&\Rightarrow &&\varphi_{2p-3} &&= 0, \quad \varsigma_{2p-2} = 0;\\
s^{2p-2}\colon &\quad L\varphi_{2p-2} &&= 0 &&\Rightarrow &&\varphi_{2p-2} &&= 0;\\
s^{2p-1}\colon &\quad L\varphi_{2p-1} &&= \varsigma_{2p-2}\varphi_1 - p\varphi_1^{p-1}\varphi_p + \fint_{\mathbb{T}}\left(p\varphi_1^{p-1}\varphi_p\right).
\end{aligned}$$

From the last equation it follows that

$$\varsigma_{2p-2} = \text{the coefficient of } (\varphi_1 =) \cos \text{ in } p\left(\varphi_1^{p-1}\varphi_p - \fint_{\mathbb{T}}\left(\varphi_1^{p-1}\varphi_p\right)\right) = C_1^{\text{even}}$$

with help of the odd power-reduction formula $\cos^q x = 2^{-q} \sum_{j=1}^{\frac{q-1}{2}} \binom{q}{j} \cos\left((q-2j)x\right)$ for $q = p-1$ and the product-to-sum identity for cosine.

Switching to odd $p \geqslant 5$, we similarly obtain that $\varphi_1 = \cos$ and $\varsigma_0 = 1$ and that (39) is true. Moreover, from

$$s^p\colon \quad L\varphi_p = \varsigma_{p-1}\varphi_1 - \varphi_1^p + \underbrace{\fint_{\mathbb{T}} \varphi_1^p}_{=0 \text{ for odd } p}$$

we finally deduce that

$$\varsigma_{p-1} = \text{the coefficient of } \cos \text{ in } \varphi_1^p = C_1^{\text{odd}}$$

and

$$\varphi_p = L^{-1}\left(\varsigma_{p-1}\varphi_1 - \varphi_1^p\right) = \Phi_1^{\text{odd}},$$

again by the odd power-reduction formula. □

For odd p we can improve upon Theorem 20 and obtain the overall structure of the bifurcation formulas near the line of trivial solutions. This shows that $\varphi^{\epsilon=0}(s)$ is antisymmetric about $-\frac{\pi}{2}$, and agrees with the general conjecture set forth in Section 3.

Proposition 21 (Local antisymmetry)**.** *Consider $n(x) = x^p$ for odd $p \geqslant 3$ and the choice of parametrisation in Theorem 20. Then the analytic structure* (37) *of the local bifurcation formulas equals*

$$\varphi^{\epsilon=0}(s) = \sum_{j=0}^{\infty} \varphi_{j(p-1)+1}\, s^{j(p-1)+1} \qquad \textit{and} \qquad c^{\epsilon=0}(s) = \sum_{j=0}^{\infty} \varsigma_{j(p-1)}\, s^{j(p-1)},$$

on $\mathfrak{C}_{\text{loc}}^{\epsilon=0}$, where all the $\varphi_{j(p-1)+1}$ functions lie in $\mathcal{W} := \operatorname*{span}_{\text{odd } k \geqslant 1}\{\cos(k\cdot)\}$. Hence $\varphi_{j(p-1)+1}$ and thus also $\varphi^{\epsilon=0}$ are antisymmetric about $-\frac{\pi}{2}$.

Remark 22. *Theorem 21 also hold in case* (2_{sgn}) *of* (35) *with* $\epsilon > 0$*, provided* s *is sufficiently small. In this case the representations become*

$$\varphi^\epsilon(s) = \textstyle\sum_{j=0}^\infty \varphi_{2j+1}\, s^{2j+1} \qquad \textit{and} \qquad c^\epsilon(s) = \textstyle\sum_{j=0}^\infty \varsigma_{2j}\, s^{2j},$$

as indicated by Theorem 18.

Proof. We use strong induction, where the base case is given by Theorem 20. Let $q := p-1$ and suppose the result is true for $\{0q, \dots, jq\}$ for some $j \geqslant 0$. Now consider case $(j+1)q$. As in the proof of Theorems 18 and 20, we insert (37) into (4), identify terms of equal order in s, and simplify by means of successive cancellations, with $L := |\mathrm{D}|^{-\alpha} - \varsigma_0\,\mathrm{id}$:

$$\begin{aligned}
s^{jq+2}\colon &\quad L\varphi_{jq+2} &&= 0 &&\Rightarrow &&\varphi_{jq+2} &&= 0;\\
s^{jq+3}\colon &\quad L\varphi_{jq+3} &&= \varsigma_{jq+2}\varphi_1 &&\Rightarrow &&\varphi_{jq+3} &&= 0, \quad \varsigma_{jq+2} = 0;\\
&\quad\vdots && && \vdots\\
s^{(j+1)q-1}\colon &\quad L\varphi_{(j+1)q-1} &&= \varsigma_{(j+1)q-2}\varphi_1 &&\Rightarrow &&\varphi_{(j+1)q-1} &&= 0, \quad \varsigma_{(j+1)q-2} = 0;\\
s^{(j+1)q}\colon &\quad L\varphi_{(j+1)q} &&= 0 &&\Rightarrow &&\varphi_{(j+1)q} &&= 0;\\
s^{(j+1)q+1}\colon &\quad L\varphi_{(j+1)q+1} &&= \textstyle\sum_{i=1}^{j+1} \varsigma_{iq}\varphi_{(j-i+1)q+1} - \varLambda,
\end{aligned}$$

where

$$\varLambda := \Bigg\{ \tbinom{p}{1}\varphi_1^q\varphi_{jq+1} + \tbinom{p}{2}\varphi_1^{q-1}\Big(\varphi_{q+1}\varphi_{(j-1)q+1} + \varphi_{2q+1}\varphi_{(j-2)q+1} + \dots + \varphi_{\lfloor\frac{j}{2}\rfloor q+1}\varphi_{\lceil\frac{j}{2}\rceil q+1}\Big)$$
$$+ \dots + \tbinom{p}{j}\varphi_1^{q-j+1}\varphi_{q+1}^{j}\Bigg\}$$

follows by expanding $(\varphi^{\epsilon=0}(s))^p$. Then we obtain

$$\varsigma_{(j+1)q} = \text{coefficient of } (\varphi_1 =) \cos \text{ in } \varLambda \qquad \text{and} \qquad \varphi_{(j+1)q+1} = L^{-1}\Big(\textstyle\sum_{i=1}^{j+1} \varsigma_{iq}\varphi_{(j-i+1)q+1} - \varLambda\Big).$$

By the induction hypothesis we know that $\varphi_{\tilde{j}q+1} \in \mathcal{W}$ for all $0 \leqslant \tilde{j} \leqslant j$. Moreover, each term in $\varLambda$ is the product of an *odd* number of (some of) the terms $\varphi_{\tilde{j}q+1}$, with repetitions allowed. This establishes the result by noting that $\mathcal{W}$ is algebraically closed under an odd number of multiplications, which can be deduced from the identity

$$4\cos u\cos v\cos w = \cos(\underbrace{u+v+w}_{=\,\text{odd}}) + \cos(\underbrace{-u+v+w}_{=\,\text{odd}}) + \cos(\underbrace{u-v+w}_{=\,\text{odd}}) + \cos(\underbrace{u+v-w}_{=\,\text{odd}})$$

whenever u, v, and w are odd. General products reduce iteratively to triple products. □

Although Theorem 21 is promising, it is not clear to us how one can prove antisymmetry everywhere along $\mathfrak{C}^\epsilon_{\mathrm{loc}}$ and its upcoming global extension. Thus we instead redefine $\mathcal{X}^\beta$ in case (2_{sgn}) as the subspace

$$\Big\{\varphi \in \mathring{\mathrm{C}}^\beta_{\mathrm{even}}(\mathbb{T}) \;:\; \varphi \text{ is antisymmetric about } -\tfrac{\pi}{2}, \text{ that is, } \varphi(\cdot + \pi) = -\varphi\Big\},$$

for which correspondingly $\ker \partial_\varphi F^\epsilon(0, k^{-\alpha}) = \mathcal{W}$ and Theorems 17 and 18 hold for odd k.

We proceed to analyse the global structure of an extension of $\mathfrak{C}^{\epsilon}_{\mathrm{loc}}$ in Theorem 17. To this end, let

$$\mathcal{S}^{\epsilon} := \big\{(\varphi, c) \in \mathcal{U}^{\epsilon} \ : \ F^{\epsilon}(\varphi, c) = 0\big\}$$

be the set of admissible solution pairs, where

$$\mathcal{U}^{\epsilon} := \big\{(\varphi, c) \in \mathcal{X}^{\beta} \times \mathbb{R}_{+} \ : \ (n^{\epsilon})'(\varphi) < c\big\}$$

is an open set whose boundary contains any solution pair of (34) with the desirable regularity features in Theorem 16. We first note the following property of $\mathcal{S}^{\epsilon}$.

Lemma 23. *Bounded, closed subsets of $\mathcal{S}^{\epsilon}$ are compact in $\mathcal{X}^{\beta} \times \mathbb{R}_{+}$.*

Proof. It follows from Theorem 11 and its proof, that the operator G in (9)—adapted with N^{ϵ} replacing the nonsmooth N—sends (φ, c) to φ on $\mathcal{S}^{\epsilon}$ and boundedly maps $\mathcal{S}^{\epsilon}$ into C^{m} for any $m \geqslant 1$. Since $\mathcal{X}^{\beta'}$ is compactly embedded in $\mathcal{X}^{\beta}$ for $\beta' > \beta$, we find that

$$G \text{ maps bounded subsets of } \mathcal{S}^{\epsilon} \text{ into relatively compact subsets of } \mathcal{X}^{\beta}.$$

In particular, if $\{(\varphi_j, c_j)\}_j$ is a sequence in a bounded subset $\mathcal{B} \subseteq \mathcal{S}^{\epsilon}$, then a subsequence of $\{\varphi_j\}_j$ converges in $\mathcal{X}^{\beta}$, which together with the Bolzano–Weierstrass theorem imply that a subsequence of $\{(\varphi_j, c_j)\}_j$ converges in the $\mathcal{X}^{\beta} \times \mathbb{R}_{+}$-topology. Thus if $\mathcal{B}$ also is closed, it is compact. □

By means of Theorem 23 and the fact that $c^{\epsilon}(s)$ is not identically constant due to Theorem 18, we may appeal to [6, Theorem 9.1.1] and obtain a global extension of $\mathfrak{C}^{\epsilon}_{\mathrm{loc}}$. Note that we do not distinguish between a curve and its image.

Theorem 24 (Global bifurcation)**.** *$\mathfrak{C}^{\epsilon}_{\mathrm{loc}}$ extends to a global continuous curve $\mathfrak{C}^{\epsilon} \colon \mathbb{R}_{+} \to \mathcal{S}^{\epsilon}$ of solution pairs $\mathfrak{C}^{\epsilon}(s) = (\varphi^{\epsilon}(s), c^{\epsilon}(s))$, and either*

i) $\lim_{s\to\infty} \|\mathfrak{C}^{\epsilon}(s)\|_{\mathcal{X}^{\beta}\times\mathbb{R}_{+}} = \infty$, *ii)* $\operatorname{dist}(\mathfrak{C}^{\epsilon}, \partial\mathcal{U}^{\epsilon}) = 0$, *or iii)* $\mathfrak{C}^{\epsilon}$ *is periodic.*

We shall prove that possibility iii) does not happen and that possibilities i) and ii) occur simultaneously, from which it will follow that one finds a highest, α-Hölder continuous wave as a limit along $\mathfrak{C}^{\epsilon}$.

In order to eliminate the possibility that $\mathfrak{C}^{\epsilon}$ is periodic, we make use of a conic refinement of the global bifurcation theorem [6, Theorem 9.1.1]. Specifically, let

$$\mathcal{K} := \big\{\varphi \in \mathcal{X}^{\beta} \ : \ \varphi \text{ is increasing on } (-\pi, 0)\big\}$$

be a closed cone in $\mathcal{X}^{\beta}$ and observe that $\mathfrak{C}^{\epsilon}(s) \in \mathcal{K} \times \mathbb{R}_{+}$ for sufficiently small s. Indeed, cosine is strictly increasing on $(-\pi, 0)$ and strict monotonicity is stable under C^1-perturbations on a compact set (here, $\mathbb{T}$). Therefore, since $\varphi^{\epsilon}(s) = s\cos + \mathcal{O}(s^2)$ from Theorems 17 and 18 is smooth on $\mathbb{T}$ by Theorem 11 (adapted to (34) with n^{ϵ}), it holds that $\varphi^{\epsilon}(s) \in \mathcal{K} \setminus \{0\}$ for small $s = o(\epsilon)$. In fact, this is true globally.

Proposition 25. *$\varphi^{\epsilon}(s) \in \mathcal{K} \setminus \{0\}$ for all $s > 0$ and $0 < \epsilon \ll 1$. In particular, $\mathfrak{C}^{\epsilon}$ never returns (for finite s) to the line of trivial solutions, thereby ruling out possibility iii) in Theorem 24.*

Proof. According to [6, Theorem 9.2.2], it suffices to show that each $(\varphi^{\epsilon}, c^{\epsilon})$ on $\mathfrak{C}^{\epsilon}$ which also belongs to $(\mathcal{K} \setminus \{0\}) \times \mathbb{R}_{+}$ lies in the interior of $(\mathcal{K} \setminus \{0\}) \times \mathbb{R}_{+}$ in the topology of $\mathcal{S}^{\epsilon}$. To this end, observe by Theorems 11 and 12 that such φ^{ϵ} with speed c^{ϵ} is smooth and satisfies $(\varphi^{\epsilon})' > 0$ on $(-\pi, 0)$, with $(\varphi^{\epsilon})''(0) < 0$ and $(\varphi^{\epsilon})''(-\pi) > 0$. Now let $(\phi, d) \in \mathcal{S}^{\epsilon}$ be another solution (not necessarily on $\mathfrak{C}^\epsilon$) lying within δ-distance to $(\varphi^\epsilon, c^\epsilon)$ in $\mathcal{X}^\beta \times \mathbb{R}_+$. Then ϕ and d are nonzero, and ϕ is also smooth (Theorem 11). Moreover, $(N^\epsilon)^{-1}$ is smooth—also as a function of the wave speed. Hence, it follows from [17, Theorems 2.2, 4.2 and 5.2] and iteration of the smoothing effect of $|\mathrm{D}|^{-\alpha}$ that G in (9) (with N^ϵ replacing N) is a continuous map $\mathcal{S}^\epsilon \to \mathcal{S}_1^\epsilon \cap \mathcal{X}^m$ for any integer $m \geqslant 1$, where $\mathcal{S}_1^\epsilon$ is the functional component of $\mathcal{S}^\epsilon$. As such,

$$\|\phi - \varphi^\epsilon\|_{\mathring{\mathrm{C}}^2(\mathbb{T})} = \|G(\phi, d) - G(\varphi^\epsilon, c^\epsilon)\|_{\mathring{\mathrm{C}}^2(\mathbb{T})} < \tau(\delta)$$

when $\|(\phi, d) - (\varphi^\epsilon, c^\epsilon)\|_{\mathcal{X}^\beta \times \mathbb{R}_+} < \delta$. Thus for sufficiently small δ, one deduces that ϕ is strictly increasing on $(-\pi, 0)$, so that $\phi \in \mathcal{K} \setminus \{0\}$. $\square$

Remark 26. *Proofs of similar results (for instance* [15, Theorem 6.7]*,* [5, Proposition 5.9]*, and* [2, Theorem 4.6]*) as Theorem 25 seem to disregard that G depends on the wave speed. But $G(\varphi, d)$ does not necessarily equal φ when $d \neq c$ and $(\varphi, c) \in \mathcal{S}^\epsilon$, and it is key to work with open δ-balls around solution* pairs *$(\varphi, c) \in \mathcal{S}^\epsilon$ and not only around solutions φ.*

Theorem 15 (adapted to (34) with n^ϵ) and Theorem 25 immediately imply the following result.

Corollary 27. *The wave speed $c^\epsilon(s)$ is uniformly bounded away from* 0 *along $\mathfrak{C}^\epsilon$ and $0 < \epsilon \ll 1$.*

In the remainder, we let $\{(\varphi_j^\epsilon, c_j^\epsilon)\}_j \coloneqq \{(\varphi^\epsilon(s_j), c^\epsilon(s_j))\}_j$ denote a generic sequence along $\mathfrak{C}^\epsilon$ with $s_j \to \infty$ as $j \to \infty$.

Proposition 28. *Any sequence $\{(\varphi_j^\epsilon, c_j^\epsilon)\}_j$ with $\{c_j^\epsilon\}_j$ bounded has a subsequence converging to a solution of* (34) *in $\mathcal{X}^0 \times \mathbb{R}_+$.*

Proof. Note from Theorem 9 (adapted to (34) with n^ϵ) that $\{\varphi_j^\epsilon\}_j$ is bounded in $\mathcal{X}^0$. Since K_α is integrable and translation in $\mathrm{L}^1(\mathbb{T})$ is uniformly continuous, it follows from

$$\left||\mathrm{D}|^{-\alpha}\varphi_j^\epsilon(x) - |\mathrm{D}|^{-\alpha}\varphi_j^\epsilon(y)\right| \leqslant \|K_\alpha(x - \cdot) - K_\alpha(y - \cdot)\|_{\mathrm{L}^1(\mathbb{T})} \sup_j \|\varphi_j^\epsilon\|_\infty$$

that $\{|\mathrm{D}|^{-\alpha}\varphi_j^\epsilon\}_j$ is (uniformly) equicontinuous on $\mathbb{T}$. Moreover,

$$\begin{aligned} |\mathrm{D}|^{-\alpha}\varphi_j^\epsilon(x) - |\mathrm{D}|^{-\alpha}\varphi_j^\epsilon(y) &= N^\epsilon(\varphi_j^\epsilon(x)) - N^\epsilon(\varphi_j^\epsilon(y)) \\ &= \big(\varphi_j^\epsilon(x) - \varphi_j^\epsilon(y)\big)(N^\epsilon)'(\varphi_j^\epsilon(\xi_j)) \end{aligned}$$

for some ξ_j between $x, y \in \mathbb{T}$, which since $(N^\epsilon)'(\varphi_j^\epsilon(\xi_j)) > 0$, implies equicontinuity of $\{\varphi_j^\epsilon\}_j$ strictly away from 0 (and $-\pi$ in case (2_{sgn})). Patched together with (16) around 0, we infer that $\{\varphi_j^\epsilon\}_j$ is equicontinuous on all of $\mathbb{T}$. Thus a subsequence converges in $\mathcal{X}^0$ by the Arzelà–Ascoli theorem. Continuity of $|\mathrm{D}|^{-\alpha}$ and n^ϵ on $\mathcal{X}^0$ together with the existence of a convergent subsequence of $\{c_j^\epsilon\}_j$ (by the Bolzano–Weierstrass theorem), then show that a subsequence of $\{(\varphi_j^\epsilon, c_j^\epsilon)\}_j$ converges to a solution of (34) in $\mathcal{X}^0 \times \mathbb{R}_+$. $\square$

Proposition 29. *Possibilities i) and ii) in Theorem 24 occur simultaneously.*

Proof. In light of Theorem 25, we know that either possibility i) or possibility ii) takes place, and that $\varphi^\epsilon(s)$ is nontrivial and increasing on $(-\pi, 0)$ for $s > 0$ by Theorem 25. If possibility i) occurs, then either $\|\varphi^\epsilon(s)\|_{\mathcal{X}^\beta} \to \infty$ or $c^\epsilon(s) \to \infty$ as $s \to \infty$. Since the wave speed cannot blow up due to Theorem 9 (adapted to (34)) and $\varphi^\epsilon(s)$ being nontrivial, it must be that $\|\varphi^\epsilon(s)\|_{\mathcal{X}^\beta}$ explodes. But then

$$c^\epsilon(s) - (n^\epsilon)'(\varphi^\epsilon(s)(x)) \xrightarrow[s \to \infty]{} 0$$

at $x = 0$ (and at $x = -\pi$ in case (2_{sgn})) by Theorem 11 adapted to (34), demonstrating that possibility ii) holds.

Conversely, suppose that possibility ii) but not possibility i) occurs. Then there exists a sequence $\{(\varphi_j^\epsilon, c_j^\epsilon)\}_j$ along $\mathfrak{C}^\epsilon$, with φ_j^ϵ increasing on $(-\pi, 0)$, satisfying $(n^\epsilon)'(\varphi_j^\epsilon) < c_j^\epsilon$ everywhere and

$$c_j^\epsilon - (n^\epsilon)'(\varphi_j^\epsilon(0)) \xrightarrow[j\to\infty]{} 0, \qquad \text{equiv.that} \qquad \mu_j^\epsilon - \varphi_j^\epsilon(0) \xrightarrow[j\to\infty]{} 0,$$

while $\{\varphi_j^\epsilon\}_j$ remains bounded in $\mathcal{X}^\beta$, where $\mu_j^\epsilon := \mu(p, c_j^\epsilon, \epsilon)$ as in (36). By compactness we may extract a convergent subsequence in $\mathcal{X}^{\beta'}$ for $\beta' \in (\alpha, \beta)$, which yields a contradiction to Theorem 14 (adapted to (34)) with respect to the one-sided α-Hölder rate at 0. Hence, possibility i) is true. □

In order to conclude the proof of Theorem 1, let $\{(\varphi_j^\epsilon, c_j^\epsilon)\}_j$ be any sequence along $\mathfrak{C}^\epsilon$ for fixed $0 < \epsilon \ll 1$. By Theorem 9 (adapted to (34)) we know that $\{c_j^\epsilon\}_j$ is bounded, and so Theorem 28 shows that $\{(\varphi_j^\epsilon, c_j^\epsilon)\}_j$ converges, up to a subsequence, to a solution $(\varphi^\epsilon, c^\epsilon) \in \mathcal{X}^0 \times \mathbb{R}_+$ of (34) with $\varphi^\epsilon \neq 0$ increasing on $(-\pi, 0)$ by Theorem 25 and $c^\epsilon \neq 0$ due to Theorem 27. It is then clear from Theorem 29 that $(n^\epsilon)'(\varphi^\epsilon(0)) = c^\epsilon$ or equivalently, that $\varphi^\epsilon(0) = \mu^\epsilon$ by (36).

Now let $\epsilon \searrow 0$. Theorems 9 and 15 (adapted to (34)) imply that $\{c^\epsilon\}_\epsilon$ converges, up to a subsequence, to some $c \neq 0$, from which we also find that $\{\varphi^\epsilon\}_\epsilon$ is bounded in $\mathcal{X}^0$. As in the proof of Theorem 28, there exists a uniformly convergent subsequence (not relabeled) with limit $\varphi \in \mathcal{X}^0$ by the Arzelà–Ascoli theorem. Since also $n^\epsilon \to n$ uniformly (locally in $\mathbb{R}$) by its construction (35), we infer that

$$n^\epsilon(\varphi^\epsilon) \to n(\varphi) \quad \text{in } \mathcal{X}^0.$$

Coupled with continuity of $|\mathrm{D}|^{-\alpha}$ on $\mathcal{X}^0$, it follows that $\{(\varphi^\epsilon, c^\epsilon)\}_\epsilon$ converges, up to a subsequence, to a solution $(\varphi, c) \in \mathcal{X}^0 \times \mathbb{R}_+$ of the original equation (4), with $n'(\varphi) \leqslant c$ and φ being increasing on $(-\pi, 0)$, and with φ also being antisymmetric about $-\frac{\pi}{2}$ in case (2_{sgn}). Observe finally that φ is nontrivial, because

$$\varphi(0) = \lim_{\epsilon\searrow 0} \varphi^\epsilon(0) = \lim_{\epsilon\searrow 0} \mu^\epsilon = \mu \neq 0,$$

where μ is as in (6). This then finishes the proof in light of Theorem 16.

5. Conclusion

In this paper, we have established the existence of large-amplitude periodic travelling-wave solutions with exact and optimal α-Hölder regularity in a class of evolution equations with negative-order homogeneous dispersion of order $-\alpha$ for all $\alpha \in (0, 1)$. Techniques include elaborate local estimates for nonlocal operators and global bifurcation analysis. A main novelty is the inclusion of generally nonsmooth, power-type nonlinearities in the considered class of equations, which we analyse using a regularisation process. We also obtain that antisymmetric nonlinearities lead to the first existence result of "doubly-cusped" extreme waves with antisymmetry.

These results open up for new investigations. One may, for instance, consider inhomogeneous nonlinearities and also study associated symmetry principles for the existence of large-amplitude waves. Another line of research may seek to establish the convexity of the highest waves and its connection to the order of the dispersive operator and the growth and regularity of the nonlinearity.

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