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ORIGINAL PAPER

A regularized system for the nonlinear variational wave equation

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Abstract

We present a new generalization of the nonlinear variational wave equation. We prove existence of local, smooth solutions for this system. As a limiting case, we recover the nonlinear variational wave equation.

Keywords Nonlinear variational wave equation · Regularization · Conservative solutions

Mathematics Subject Classification Primary 35L51; Secondary 35B35 · 35B65

Contents

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1 Introduction

The nonlinear variational wave equation (NVW) is given by

$$u_{tt} - c(u)(c(u)u_x)_x = 0, \tag{1.1}$$

where $u = u(t, x)$, $t \geq 0$ and $x \in \mathbb{R}$, with initial data

$$u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1. \tag{1.2}$$

In this paper we modify (1.1) by adding two transport equations and coupling terms. The resulting system is given by

$$u_{tt} - c(u)(c(u)u_x)_x = -\frac{c'(u)}{4}(\rho^2 + \sigma^2), \tag{1.3a}$$

$$\rho_t - (c(u)\rho)_x = 0, \tag{1.3b}$$

$$\sigma_t + (c(u)\sigma)_x = 0, \tag{1.3c}$$

with initial data

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad \rho|_{t=0} = \rho_0, \quad \text{and} \quad \sigma|_{t=0} = \sigma_0. \tag{1.4}$$

It is clear that when $\rho = \sigma = 0$ we recover (1.1). We are interested in studying conservative solutions of the initial value problem (1.3), (1.4) for initial data $u_0, u_{0,x}, u_1, \rho_0, \sigma_0 \in L^2(\mathbb{R})$.

We assume that $c \in C^2(\mathbb{R})$ and satisfies

$$\frac{1}{\kappa} \leq c(u) \leq \kappa \tag{1.5}$$

for some $\kappa \geq 1$. In addition, we assume that

$$\max_{u\in\mathbb{R}} |c'(u)| \leq k_1 \quad \text{and} \quad \max_{u\in\mathbb{R}} |c''(u)| \leq k_2 \tag{1.6}$$

for positive constants k_1 and k_2.

The NVW equation was introduced by Saxton in [16], where it is derived by applying the variational principle to the functional

$$\int_0^\infty \int_{-\infty}^\infty (u_t^2 - c^2(u)u_x^2)\, dx\, dt.$$

The equation appears in the study of liquid crystals, where it describes the director field of a nematic liquid crystal, and where the function c is given by

$$c^2(u) = \alpha \sin^2(u) + \beta \cos^2(u),$$

where α and β are positive physical constants. We refer to [13, 16] for information about liquid crystals, and the derivation of the equation.

A key property of (1.1), and hence also (1.3), is that solutions can loose regularity in finite time, even for smooth initial data, see [10]. The loss of regularity is due to the formation of singularities in the derivatives of u. A singularity means that either u_x or u_t becomes unbounded pointwise while u remains continuous. Therefore, one has to consider weak solutions of (1.1). For smooth solutions of the NVW equation, the energy

$$\frac{1}{2}\int_{\mathbb{R}}(u_t^2+c^2(u)u_x^2)\,dx$$

is independent of time. In the general case, the singularities in the derivatives are characterized by the fact that $u_x(t,\cdot)$ and $u_t(t,\cdot)$ remain in $L^2(\mathbb{R})$ after they become pointwise unbounded. Moreover, we have concentration of energy at points where the derivative blows up. Thus, it is reasonable to look for weak solutions with bounded energy. This naturally leads to the two following notions of solutions: dissipative and conservative solutions. The difference between these solutions comes from the continuation after the formation of a singularity. For dissipative solutions the energy decreases at the blow-up time, see [6, 17–19], while for conservative solutions the energy remains unchanged. In the latter case a semigroup of solutions has been constructed in [8, 12].

Uniqueness of weak solutions to the NVW equation is a delicate subject, as the characteristics in general are not unique. The uniqueness of conservative solutions has been studied in [2, 5], where uniqueness is established for the solutions constructed in [8] given that certain conditions hold, which yield unique characteristics.

A result on the regularity of conservative solutions to (1.1) is established in [3]. For initial data satisfying certain smoothness conditions, it is shown that the solution u is piecewise smooth and that the derivative u_x can become pointwise unbounded at finitely many characteristics. An asymptotic description of these solutions in a neighborhood of the singularities is shown in [7]. Moreover, in this setting a Lipschitz metric for conservative solutions has been constructed in [4].

We will use the approach from [12] for (1.1) to obtain conservative solutions of (1.3).

For the moment, we assume that u, ρ, and σ are smooth solutions of (1.3). Then the energy is given by

$$\frac{1}{2}\int_{\mathbb{R}}\Big(u_t^2+c^2(u)u_x^2+\frac{1}{2}c(u)\rho^2+\frac{1}{2}c(u)\sigma^2\Big)\,dx, \tag{1.7}$$

and is independent of time. Next, we introduce the functions R and S defined as

$$R=u_t+c(u)u_x \quad \text{and} \quad S=u_t-c(u)u_x. \tag{1.8}$$

Note that R and S are smooth by assumption. Using (1.8) we can express the energy in (1.7) as

$$\frac{1}{4}\int_{\mathbb{R}}\big(R^2+c(u)\rho^2+S^2+c(u)\sigma^2\big)\,dx. \tag{1.9}$$

From (1.3) we obtain

$$\begin{cases} (R^2+c(u)\rho^2)_t-\Big(c(u)(R^2+c(u)\rho^2)\Big)_x=\dfrac{c'(u)}{2c(u)}(R^2S-RS^2)+\dfrac{c'(u)}{2}(\rho^2S-\sigma^2R),\\ (S^2+c(u)\sigma^2)_t+\Big(c(u)(S^2+c(u)\sigma^2)\Big)_x=-\dfrac{c'(u)}{2c(u)}(R^2S-RS^2)-\dfrac{c'(u)}{2}(\rho^2S-\sigma^2R), \end{cases} \tag{1.10}$$

which yields

$$\begin{cases} \left(\frac{1}{c(u)}\big(R^2+c(u)\rho^2\big)\right)_t-(R^2+c(u)\rho^2)_x=-\frac{c'(u)}{2c^2(u)}(R^2S+RS^2)-\frac{c'(u)}{2c(u)}(\rho^2S+\sigma^2R),\\ \left(\frac{1}{c(u)}\big(S^2+c(u)\sigma^2\big)\right)_t+(S^2+c(u)\sigma^2)_x=-\frac{c'(u)}{2c^2(u)}(R^2S+RS^2)-\frac{c'(u)}{2c(u)}(\rho^2S+\sigma^2R). \end{cases} \tag{1.11}$$

Combining (1.10) and (1.11), we finally obtain

$$\begin{cases} \Big(R^2+c(u)\rho^2+S^2+c(u)\sigma^2\Big)_t-\Big(c(u)\big(R^2+c(u)\rho^2-S^2-c(u)\sigma^2\big)\Big)_x=0,\\ \left(\frac{1}{c(u)}\big(R^2+c(u)\rho^2-S^2-c(u)\sigma^2\big)\right)_t-\Big(R^2+c(u)\rho^2+S^2+c(u)\sigma^2\Big)_x=0. \end{cases} \tag{1.12}$$

In the view of (1.10), we interpret $R^2+c(u)\rho^2$ and $S^2+c(u)\sigma^2$ as the left and right traveling part of the energy density, respectively. Moreover, the right-hand sides of the two equations in (1.10) are equal up to the sign, which means that the right and the left part can interact with each other. That is, energy can swap back and forth between the two parts, while the total energy remains unchanged because of (1.12).

Following [12], we add to the initial data, in order to allow for energy concentration, two positive Radon measures μ_0 and ν_0, such that the absolutely continuous parts equal the classical energy densities, i.e., $(\mu_0)_{\text{ac}}=\frac{1}{4}(R_0^2+c(u_0)\rho_0^2)\,dx$ and $(\nu_0)_{\text{ac}}=\frac{1}{4}(S_0^2+c(u_0)\sigma_0^2)\,dx$. The singular parts of the measures on the other hand contain information about the concentration of energy. We denote the set of initial data by $\mathcal{D}$, which consists of tuples $(u,R,S,\rho,\sigma,\mu,\nu)$.

The construction of a semigroup of weak, global, conservative solutions of (1.3) follows to a large extent the procedure for the NVW equation in [12] and is presented in Sects. 1–6. We state the results in Sects. 1–6 without proof, since the proofs are slight modifications of the ones in [12]. The details can be found in [15].

Our main results, which are contained in Sect. 7, are the following. We consider smooth initial data u_0, R_0, S_0, ρ_0, and σ_0, and absolutely continuous measures μ_0 and ν_0 on a finite interval $[x_l,x_r]$. If ρ_0 and σ_0 are strictly positive on this interval, then for every time $t\in\big[0,\frac{1}{2\kappa}(x_r-x_l)\big]$, the solutions $\rho(t,x)$ and $\sigma(t,x)$ will also be strictly positive for all $x\in[x_l+\kappa t,x_r-\kappa t]$. That is, the strict positivity of ρ_0 and σ_0 is preserved. This has a regularizing effect on the solution at time $t\in\big[0,\frac{1}{2\kappa}(x_r-x_l)\big]$ in the sense that $u(t,x)$, $R(t,x)$, $S(t,x)$, $\rho(t,x)$, and $\sigma(t,x)$ are smooth, and the measures $\mu(t)$ and $\nu(t)$ are absolutely continuous for all $x\in[x_l+\kappa t,x_r-\kappa t]$, see Theorem 7.1. The region where regularity holds is bounded by the backward and forward characteristics as indicated in Fig. 1.

In Theorem 7.9, we consider a sequence of smooth solutions $(u^n,R^n,S^n,\rho^n,\sigma^n,\mu^n,\nu^n)$ with initial data satisfying $u_0^n\to u_0$ in $L^\infty([x_l,x_r])$, $R_0^n\to R_0$, $S_0^n\to S_0$, $\rho_0^n\to 0$, $\sigma_0^n\to 0$ in $L^2([x_l,x_r])$, where u_0, R_0 and S_0 are smooth, and the associated measures μ_0 and ν_0 are absolutely continuous on $[x_l,x_r]$. Then, $u^n(t,\cdot)\to u(t,\cdot)$ in $L^\infty([x_l+\kappa t,x_r-\kappa t])$ for all $t\in\big[0,\frac{1}{2\kappa}(x_r-x_l)\big]$, where u is a solution of the NVW equation with initial data $(u_0,R_0,S_0,\mu_0,\nu_0)$. A central ingredient in the proof is a Gronwall inequality in two variables, see [9].

We point out that these are local results. The main reason for this is that we require the initial data ρ_0 and σ_0 corresponding to the Eqs. (1.3b) and (1.3c) to be bounded from below by a strictly positive constant and to belong to $L^2(\mathbb{R})$, which is not possible globally.

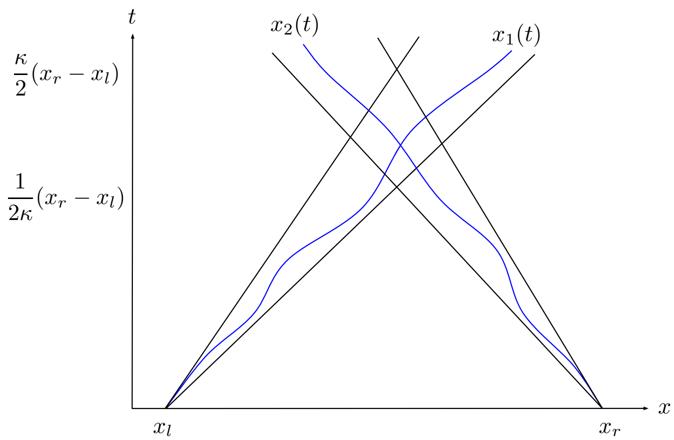

Fig. 1 Characteristics of the NVW equation. The forward characteristic $x_1(t)$ starting from x_l is given by $x_{1,t}(t) = c(u(t, x_1(t)))$, $x_1(0) = x_l$, and the backward characteristic $x_2(t)$ starting from x_r is given by $x_{2,t}(t) = -c(u(t, x_2(t)))$, $x_2(0) = x_r$. Because of (1.5), they intersect at a time t such that $\frac{1}{2\kappa}(x_r - x_l) \leq t \leq \frac{\kappa}{2}(x_r - x_l)$

We hope that further studies of the smooth approximations will be helpful in the understanding of singularities to (1.1).

The motivation for studying (1.3) comes from the two-component Camassa–Holm (2CH) system

$$u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \eta\rho\rho_x = 0, \tag{1.13a}$$

$$\rho_t + (u\rho)_x = 0, \tag{1.13b}$$

where $\kappa \in \mathbb{R}$ and $\eta \in (0, \infty)$ are given numbers. In [11], global, weak, conservative solutions of (1.13) are constructed. It is shown that the solution of (1.13) is regular if initially ρ_0 is strictly positive. Moreover, a sequence of regular solutions, with $\rho_0^n \to 0$, converges in $L^\infty(\mathbb{R})$ to the global, weak conservative solution of the CH equation, which corresponds to $\rho = 0$.

Loosely speaking, since the CH equation has one family of characteristics, see Fig. 2, an extra variable ρ is needed to preserve the positivity of ρ_0 in the characteristic direction. Since the NVW equation has both forward and backward characteristics, we need two extra variables ρ and σ to preserve the positivity of ρ_0 and σ_0 in each characteristic direction. We emphasize that the system (1.3) is constructed in order to have similar properties as (1.13), and is not derived from physical considerations.

We mention that a regularizing system for the Hunter–Saxton equation has been studied in [14].

Next we give a brief outline of the used method.

As for the classical wave equation, (1.3) has two families of characteristics: forward and backward characteristics. The backward characteristics transport the energy described by the measure μ, while the forward characteristics transport the energy described by the measure ν. We interpret the characteristics as particles. At points where the measures are nonsingular there is a finite amount of energy, and there is exactly one forward and one backward characteristic starting from that point. This particle is mapped to one point in the new coordinates (X, Y).

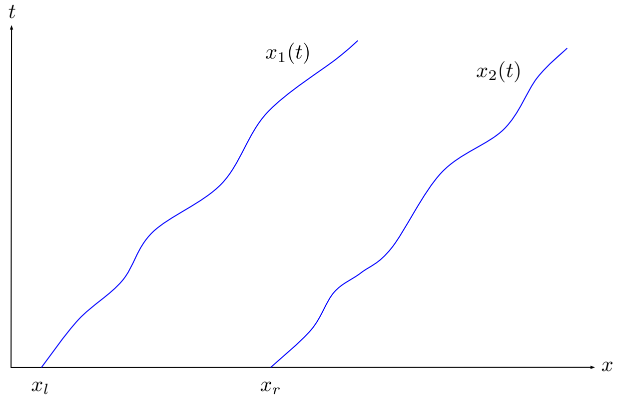

Fig. 2 Characteristics of the CH equation. The characteristic $x_1(t)$ starting from x_l is given by $x_{1,t}(t) = u(t, x_1(t))$, $x_1(0) = x_l$, and the characteristic $x_2(t)$ starting from x_r is given by $x_{2,t}(t) = u(t, x_2(t))$, $x_2(0) = x_r$

At a point where one of the measures is singular and the other is not, there is an infinite amount of energy. Hence, there are infinitely many characteristics corresponding to the singular measure starting from that point, while the nonsingular measure yields one characteristic. This single point is mapped to a horizontal or vertical line in the new coordinates, depending on which measure is singular.

The situation where both measures are singular at a point, means that there is an infinite amount of both backward and forward energy at that point. Infinitely many characteristics of both types start out from that point, and all these particles correspond to a box in the (X, Y)-plane.

The derivation of the system of equations corresponding to (1.3) in the new coordinates is illustrated by assuming that u is smooth, and μ and ν are absolutely continuous. Then, the method of characteristics yields solutions X and Y of the equations

$$X_t - c(u)X_x = 0 \quad \text{and} \quad Y_t + c(u)Y_x = 0. \tag{1.14}$$

The operators acting on X and Y are the two factors $\frac{\partial}{\partial t} - c(u)\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t} + c(u)\frac{\partial}{\partial x}$ corresponding to the highest order derivatives in (1.3a). This defines new coordinates (X, Y). The characteristics for the equations in (1.14) are given by

$$x_t(t) = -c(u(t, x(t))) \quad \text{and} \quad x_t(t) = +c(u(t, x(t))),$$

respectively, for some starting value $x(0) = x_0$. Note that X and Y are constant along characteristics, meaning that particle paths are mapped to straight lines.

Considering the original variables t and x as functions of X and Y, we define $U(X, Y) = u(t(X, Y), x(X, Y))$, $P(X, Y) = \rho(t(X, Y), x(X, Y))$, $p = Px_X$, $Q(X, Y) = \sigma(t(X, Y), x(X, Y))$, and $q = Qx_Y$. We introduce functions J and K, where J corresponds to the energy distribution in the new coordinates. Denoting $Z = (t, x, U, J, K)$, we end up with the identities

$$x_X = c(U)t_X, \quad x_Y = -c(U)t_Y, \tag{1.15a}$$

$$J_X = c(U)K_X, \quad J_Y = -c(U)K_Y, \tag{1.15b}$$

$$2J_X x_X = (c(U)U_X)^2 + c(U)p^2, \quad 2J_Y x_Y = (c(U)U_Y)^2 + c(U)q^2, \tag{1.15c}$$

and a semilinear system of equations

$$Z_{XY} = F(Z)(Z_X, Z_Y), \tag{1.16}$$

where $F(Z)$ is a bilinear and symmetric tensor from $\mathbb{R}^5 \times \mathbb{R}^5$ to $\mathbb{R}^5$.

Moreover, we get two additional equations

$$p_Y = 0 \quad \text{and} \quad q_X = 0, \tag{1.17}$$

which correspond to (1.3b) and (1.3c). From (1.15c) we see that solutions of (1.16) corresponding to (1.1) and (1.3) are not identical. In particular, from (1.15c) we see that the solutions t, x, U, J, K of (1.16) are not independent of the solutions p, q of (1.17). From (1.15) it is clear that the vector Z consists of dependent and independent elements. A fixed point argument is used to prove existence of solutions to the system. This requires a curve $(\mathcal{X}(s), \mathcal{Y}(s))$ parametrized by $s \in \mathbb{R}$ in the (X, Y)-plane that corresponds to the initial time $t = 0$. In the smooth case, the set of points $(X, Y) \in \mathbb{R}^2$ such that $t(X, Y) = 0$ defines this curve, which is strictly monotone. For general initial data this set is the union of strictly monotone curves, horizontal and vertical lines, and boxes. In the case of a box there are in principle infinitely many possible ways of choosing the curve. One has to take this into account when defining initial data in the Lagrangian coordinates.

The initial data in $\mathcal{D}$ is mapped to the Lagrangian variables in $\mathcal{G}_0$ in two steps, see Sect. 3. First we define a map $\mathbf{L}$ from $\mathcal{D}$ to the set $\mathcal{F}$, consisting of elements $\psi = (\psi_1, \psi_2)$ where $\psi_1(X)$ and $\psi_2(Y)$ are six dimensional vectors.

Loosely speaking, the map $\mathbf{L}$ yields the value of Z and its derivatives, and p and q in each characteristic direction, i.e., in the X and the Y direction. Linking the values of ψ_1 and ψ_2 yields the set of points in the (X, Y)-plane where time equals zero. For instance, in the case of initial data where both measures are singular at the same point, this set is a box.

The next map picks one curve $(\mathcal{X}, \mathcal{Y})$ from the set where time equals zero, and sets the value of Z and its derivatives, and p and q on the curve. This map is denoted by $\mathbf{C}$ and maps $\mathcal{F}$ to the set $\mathcal{G}_0$, which is the set of elements $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ corresponding to time equals zero. An element Θ consists of the initial curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ parametrized by $s \in \mathbb{R}$, and yields the value of Z, Z_X, Z_Y, p and q on the curve. This means that $\mathcal{Z}(s) = Z(\mathcal{X}(s), \mathcal{Y}(s))$, $\mathcal{V}(X) = Z_X(X, \mathcal{Y}(\mathcal{X}^{-1}(X)))$, $\mathcal{W}(Y) = Z_Y(\mathcal{X}(\mathcal{Y}^{-1}(Y)), Y)$, $\mathfrak{p}(X) = p(X, \mathcal{Y}(\mathcal{X}^{-1}(X)))$ and $\mathfrak{q}(Y) = q(\mathcal{X}(\mathcal{Y}^{-1}(Y)), Y)$, and we write $\Theta = (Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})$. The functions $\mathcal{Z}$, $\mathcal{V}$, $\mathcal{W}$, $\mathfrak{p}$ and $\mathfrak{q}$ belong, with some modifications, to $L^\infty(\mathbb{R})$. The set $\mathcal{C}$, see Definition 3.2, consists of all curves $(\mathcal{X}, \mathcal{Y})$ such that the functions $\mathcal{X}$ and $\mathcal{Y}$ are continuous, nondecreasing, and have finite distance to the identity. Moreover, the functions satisfy $\mathcal{X} + \mathcal{Y} = 2\,\mathrm{Id}$. In the case of a box, the map $\mathbf{C}$ picks the curve consisting of the left vertical side and the upper horizontal side of the box.

Existence and uniqueness of solutions to (1.16) with initial data Θ follow from a fixed point argument. The solution is first constructed on small rectangular domains Ω in the (X, Y)-plane, where the initial curve $(\mathcal{X}, \mathcal{Y})$ connects the lower left corner with the upper right corner of the rectangle, see Theorem 4.8. Here, a solution basically means that Z, Z_X, Z_Y, p and q are pointwise bounded in the box, and that (1.16) and (1.17) are satisfied almost everywhere in Ω, see Definition 4.5. We consider solutions satisfying some additional properties, i.e., they satisfy the identities in (1.15) and some monotonicity conditions. These properties are contained in Definition 4.7. The set of such solutions is denoted by $\mathcal{H}(\Omega)$. For initial data $\Theta \in \mathcal{G}_0$ we have $\mathcal{V}_2 + \mathcal{V}_4 > 0$ and $\mathcal{W}_2 + \mathcal{W}_4 > 0$ almost everywhere. This property is preserved in the solution and is important in proving that the solution operator from $\mathcal{D}$ to $\mathcal{D}$ is a semigroup.

A pointwise uniform bound on the functions Z, Z_X, Z_Y, p and q in strip like domains containing small rectangles allows us to prove, by an induction argument, existence and uniqueness of solutions in $\mathcal{H}(\Omega)$ on arbitrarily large rectangular domains Ω, see Sect. 4.2.

If a function (Z, p, q) in $\mathcal{H}(\Omega)$ is a solution on any rectangular domain Ω, and there exists a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$, we say that (Z, p, q) is a global solution to (1.16) and (1.17), see Definition 4.13. Here, $\mathcal{G}$ is the analogue of $\mathcal{G}_0$, corresponding to time t different from zero. The set of global solutions is denoted by $\mathcal{H}$. The functions Z, Z_X, Z_Y, p and q are, with some modifications, bounded globally. In particular, the Lagrangian counterpart to the energy is bounded.

In Theorem 4.15 a global solution is constructed by using local solutions in boxes. The procedure is as follows. First we construct solutions on rectangles with diagonal points that lie on the initial curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$. These solutions are then used to construct initial data for adjacent rectangles, and we obtain solutions there as well. Continuing like this one obtains a global solution. We denote the solution map that to any initial data $\Theta \in \mathcal{G}$ yields a unique solution $(Z, p, q) \in \mathcal{H}$ by $\mathbf{S}$.

Having constructed a global solution $(Z, p, q) \in \mathcal{H}$, the goal is to map it back to Eulerian coordinates $\mathcal{D}$ for any time $T > 0$. As addressed before, the points $(X, Y) \in \mathbb{R}^2$ such that $t(X, Y) = T$ may contain boxes. In order to use the sets previously defined for time equal to zero, we shift time to zero, i.e., for $(Z, p, q) \in \mathcal{H}$ we define $(\bar{Z}, \bar{p}, \bar{q}) \in \mathcal{H}$ where $\bar{t}(X, Y) = t(X, Y) - T$. The other elements of $(\bar{Z}, \bar{p}, \bar{q})$ are identical to the ones in (Z, p, q). We call this map $\mathbf{t}_T$. In the case of a box, the curve $(\mathcal{X}, \mathcal{Y})$ corresponding to time T is defined by picking the left vertical side and the upper horizontal side of the box. The element $\Theta \in \mathcal{G}_0$ is then defined as $\Theta = (\bar{Z}, \bar{p}, \bar{q}) \bullet (\mathcal{X}, \mathcal{Y})$, and we denote the map by $\mathbf{E} : \mathcal{H} \to \mathcal{G}_0$. Because of the monotonicity of the function t, the curve corresponding to time T lies below the initial curve. For any $\Theta \in \mathcal{G}_0$ we define a map $\mathbf{D}$ that associates an element $\psi \in \mathcal{F}$. The definition of these maps are contained in Sect. 5.1. The operator $S_T = \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{S} \circ \mathbf{C}$ that for any initial data in $\mathcal{F}$ yields a solution in $\mathcal{F}$ corresponding to time $T > 0$, is a semigroup, see Theorem 5.6.

The remaining step back to Eulerian coordinates is the map $\mathbf{M} : \mathcal{F} \to \mathcal{D}$, which yields an element $(u, R, S, \rho, \sigma, \mu, \nu)(T) \in \mathcal{D}$ at time $T > 0$, see Definition 5.7. Thus, the map $\bar{S}_T = \mathbf{M} \circ S_T \circ \mathbf{L}$ yields an element in $\mathcal{D}$ given any initial data in $\mathcal{D}$. If $\mathbf{L} \circ \mathbf{M} = \mathrm{Id}$, it follows from the semigroup property of S_T that also $\bar{S}_T$ is a semigroup. However, this identity does not hold in general. This is because for any element $\psi \in \mathcal{F}$ we have $x_i + J_i \in G$, where the group G is given by all invertible functions f such that $f - \mathrm{Id}$, $f^{-1} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$ and $(f - \mathrm{Id})' \in L^2(\mathbb{R})$, while the element $\bar{\psi} \in \mathcal{F}$ given by the map $\mathbf{L}$ satisfies $\bar{x}_i + \bar{J}_i = \mathrm{Id}$. Therefore, in general one has $\psi \neq \bar{\psi}$. To overcome this problem, one considers the following approach. Assume that $x_1 + J_1 = f$ and $x_2 + J_2 = g$ where $f, g \in G$, and consider $\tilde{\psi}$ defined by relabeling such that $\tilde{x}_1 = x_1 \circ f^{-1}$, $\tilde{J}_1 = J_1 \circ f^{-1}$, $\tilde{x}_2 = x_2 \circ g^{-1}$ and $\tilde{J}_2 = J_2 \circ g^{-1}$. It then follows that $\tilde{x}_i + \tilde{J}_i = \mathrm{Id}$, $i = 1, 2$. Introduce $\phi = (f^{-1}, g^{-1})$ and $\tilde{\psi} = \psi \cdot \phi$, which defines an action of G^2 on the set $\mathcal{F}$. In particular, the transformation of ψ to $\tilde{\psi}$ defines a projection Π from $\mathcal{F}$ on the set

$$\mathcal{F}_0 = \{\psi = (\psi_1, \psi_2) \in \mathcal{F} \mid x_1 + J_1 = \mathrm{Id} \text{ and } x_2 + J_2 = \mathrm{Id}\},$$

which contains exactly one element of each equivalence class of $\mathcal{F}$ with respect to G^2, see Sect. 5.4. Thus, we have $\tilde{\psi} = \Pi(\psi)$. It turns out that the map $S_T : \mathcal{F} \to \mathcal{F}$ is invariant under the group acting on $\mathcal{F}$, i.e., $S_T(\psi \cdot \phi) = S_T(\psi) \cdot \phi$, where $\phi \in G^2$, since all the maps which S_T is composed of, are invariant under the group action. The action of G^2 on $\mathcal{G}$ and the set of curves $\mathcal{C}$ naturally follows from the definition of the action on $\mathcal{F}$. On the set of curves $\mathcal{C}$, the action corresponds to stretching the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ in the X and Y direction. For the set $\mathcal{H}$, the action is defined such that it commutes with the $\bullet$ operation.

A key result is that the map $\mathbf{M}$ satisfies $\mathbf{M} = \mathbf{M} \circ \Pi$. This implies that to each element in $\mathcal{D}$ there correspond infinitely many elements in $\mathcal{F}$, all belonging to the same equivalence class. The mapping $\mathbf{L} : \mathcal{D} \to \mathcal{F}_0$ on the other hand picks one member of each equivalence class, but we could also pick a different one. Applying the solution operator to all elements belonging to the same equivalence class yields infinitely many solutions in $\mathcal{F}$, which form an equivalence class. Using the mapping $\mathbf{M} : \mathcal{F} \to \mathcal{D}$ on all of these solutions yields the same element in $\mathcal{D}$. Since we get the same solution in the end, we can think of each member of the equivalence class as a different "parametrization" of the initial data in $\mathcal{F}$, which are connected through relabeling. Hence, the map $\bar{S}_T$ is a semigroup. Moreover, the solution produced by the map is a global weak solution of (1.3), see Sect. 6. It is conservative in the sense that for all $T \geq 0$, $\mu(T)(\mathbb{R}) + \nu(T)(\mathbb{R}) = \mu_0(\mathbb{R}) + \nu_0(\mathbb{R})$, where $\mu(T)$ and $\nu(T)$ are the measures at time T, and μ_0 and ν_0 are the initial measures. This is a consequence of the fact that the energy function J in Lagrangian coordinates is such that the limit $\lim_{s\to\pm\infty} J(\mathcal{X}(s), \mathcal{Y}(s))$ is independent of the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$, see Lemma 4.14. Thus, the same limiting values of J are obtained for curves corresponding to different times.

2 Equivalent system

In this section we formally derive a set of equations corresponding to (1.3) in new variables. We assume that u, ρ, and σ are sufficiently smooth and bounded.

For the first equation in (1.14) we find the characteristics

$$t(s, \xi) = s \quad \text{and} \quad x_s(s, \xi) = -c(u(s, x(s, \xi))), \tag{2.1}$$

where we assume that $t(0, \xi) = 0$ and $x(0, \xi) = \xi$ for all $\xi \in \mathbb{R}$. From the last equation in (2.1) we obtain

$$x_\xi(s, \xi) = \exp\left\{ -\int_0^s c'(u(r, x(r, \xi)))u_x(r, x(r, \xi))\, dr \right\}. \tag{2.2}$$

We compute the determinant of the Jacobian corresponding to the map $(s, \xi) \to (t, x)$ and get

$$\det\left(\begin{bmatrix} t_s & t_\xi \\ x_s & x_\xi \end{bmatrix}\right) = t_s x_\xi - t_\xi x_s = x_\xi.$$

Since $0 < x_\xi(s, \xi) < \infty$ we have from the inverse function theorem that the Jacobian corresponding to the map $(t, x) \to (s, \xi)$ satisfies

$$\begin{bmatrix} s_t & s_x \\ \xi_t & \xi_x \end{bmatrix} = \frac{1}{x_\xi} \begin{bmatrix} x_\xi & -t_\xi \\ -x_s & t_s \end{bmatrix}.$$

From (2.1) we get

$$s_t = 1, \quad s_x = 0, \quad \xi_t = -\frac{x_s}{x_\xi}, \quad \xi_x = \frac{1}{x_\xi}, \tag{2.3}$$

so that $s(t, x) = t$ and $\xi_t(t, x) = -x_s(t, \xi(t, x))\xi_x(t, x) = c(u(t, \xi(t, x)))\xi_x(t, x)$.

Furthermore, (1.14) and (2.1) imply that $X(t(s,\xi), x(s,\xi)) = X(0,\xi) = g(\xi)$, for some strictly increasing function $g \in C^1(\mathbb{R})$. Differentiation, combined with (2.1) and (2.3) yields

$$X_t = g'(\xi)\xi_t = -g'(\xi)\frac{x_s}{x_\xi} \quad \text{and} \quad X_x = g'(\xi)\xi_x = g'(\xi)\frac{1}{x_\xi}, \tag{2.4}$$

which implies $0 < X_t < \infty$ and $0 < X_x < \infty$.

Next, we study $Y(t,x)$ with the method of characteristics. We obtain, from (1.14), $\frac{d}{ds}Y(t(s,\xi), x(s,\xi)) = 0$ with the characteristics given by

$$t(s,\xi) = s \quad \text{and} \quad x_s(s,\xi) = c(u(s, x(s,\xi))), \tag{2.5}$$

where we assume that $t(0,\xi) = 0$ and $x(0,\xi) = \xi$ for all $\xi \in \mathbb{R}$. If $Y(0,\xi) = h(\xi)$ for some strictly increasing function $h \in C^1(\mathbb{R})$, then $Y(s, x(s,\xi)) = h(\xi)$. As in the computations above we find

$$Y_t = h'(\xi)\xi_t = -h'(\xi)\frac{x_s}{x_\xi} \quad \text{and} \quad Y_x = h'(\xi)\xi_x = h'(\xi)\frac{1}{x_\xi}, \tag{2.6}$$

so that $-\infty < Y_t < 0$ and $0 < Y_x < \infty$, where

$$x_\xi(s,\xi) = \exp\left\{\int_0^s c'(u(r, x(r,\xi)))u_x(r, x(r,\xi))\, dr\right\}. \tag{2.7}$$

Now we consider the mapping from the (t,x)-plane to the (X,Y)-plane. The determinant of the Jacobian of this map reads

$$d = \det\left(\begin{bmatrix} X_t & X_x \\ Y_t & Y_x \end{bmatrix}\right) = X_tY_x - X_xY_t = 2c(u)X_xY_x = -\frac{2X_tY_t}{c(u)}. \tag{2.8}$$

The inverse function theorem then implies that the Jacobian corresponding to the map $(X,Y) \to (t,x)$ satisfies

$$\begin{bmatrix} t_X & t_Y \\ x_X & x_Y \end{bmatrix} = \frac{1}{d}\begin{bmatrix} Y_x & -X_x \\ -Y_t & X_t \end{bmatrix}.$$

From the above equality many identities can be read off, and we only mention some of them. By using (1.14) and (2.8), we obtain

$$2c(u)t_XX_x = 1, \quad -2c(u)t_YY_x = 1, \quad 2x_XX_x = 1, \quad 2x_YY_x = 1, \tag{2.9}$$

which imply

$$x_X = c(u)t_X \quad \text{and} \quad x_Y = -c(u)t_Y. \tag{2.10}$$

We observe from (2.9) that t_X, t_Y, x_X, and x_Y are nonzero and finite.

Let $U(X,Y) = u(t(X,Y), x(X,Y))$. We insert the derivatives of $u(t,x) = U(X(t,x), Y(t,x))$ in (1.3a) and get

$$\begin{aligned} -\frac{c'(u)}{4}(\rho^2 + \sigma^2) &= 2U_{XY}(X_tY_t - c^2(u)X_xY_x) + U_X(X_{tt} - c^2(u)X_{xx}) \\ &\quad + U_Y(Y_{tt} - c^2(u)Y_{xx}) - c(u)c'(u)(U_X^2X_x^2 + 2U_XU_YX_xY_x + U_Y^2Y_x^2). \end{aligned} \tag{2.11}$$

Due to (1.14) all second order derivatives of U drop out except for the term containing the mixed derivative U_{XY}. We compute the remaining terms. From (1.14) and (2.9) we have

$$R = 2c(u)U_XX_x = c(u)\frac{U_X}{x_X} \quad \text{and} \quad S = -2c(u)U_YY_x = -c(u)\frac{U_Y}{x_Y}. \tag{2.12}$$

By differentiating (1.14) and using (1.8) and (2.9) we obtain

$$X_{tt} - c^2(u)X_{xx} = c'(u)X_x R = \frac{c'(u)R}{2x_X} \quad \text{and} \quad Y_{tt} - c^2(u)Y_{xx} = -c'(u)Y_x S = -\frac{c'(u)S}{2x_Y}.$$

From (1.14) and (2.9) we have

$$X_t Y_t - c^2(u)X_x Y_x = -2c^2(u)X_x Y_x = -\frac{c^2(u)}{2x_X x_Y}.$$

Thus, (2.11) is equivalent to

$$U_{XY} = \frac{c'(u)}{4c^3(u)}\Big((R^2 + c(u)\rho^2)x_X x_Y + (S^2 + c(u)\sigma^2)x_Y x_X\Big) - \frac{c'(u)}{2c(u)}U_X U_Y. \quad (2.13)$$

We introduce

$$J_X = \frac{1}{2}(R^2 + c(u)\rho^2)x_X \quad \text{and} \quad J_Y = \frac{1}{2}(S^2 + c(u)\sigma^2)x_Y, \quad (2.14)$$

which we think of as the left and right traveling part of the energy density in the new variables, respectively. Now (2.13) yields

$$U_{XY} = \frac{c'(u)}{2c^3(U)}(J_X x_Y + J_Y x_X) - \frac{c'(u)}{2c(u)}U_X U_Y.$$

We find it convenient to introduce the function K defined by

$$K_X = \frac{1}{2c(u)}(R^2 + c(u)\rho^2)x_X \quad \text{and} \quad K_Y = -\frac{1}{2c(u)}(S^2 + c(u)\sigma^2)x_Y, \quad (2.15)$$

which satisfies

$$J_X = c(U)K_X \quad \text{and} \quad J_Y = -c(U)K_Y. \quad (2.16)$$

In view of (1.11) and (1.12) we can think of K_X and K_Y as the left and right traveling part of the second conserved quantity $\frac{1}{c(u)}(R^2 + c(u)\rho^2 - S^2 - c(u)\sigma^2)$ in the new coordinates, respectively.

Next, let us derive the equations for t_{XY}, x_{XY}, J_{XY}, and K_{XY}. We have $x_{XY} = x_{YX}$, which by using (2.10) is the same as $(c(U)t_X)_Y = (-c(U)t_Y)_X$. This leads to

$$t_{XY} = -\frac{c'(U)}{2c(U)}(U_Y t_X + U_X t_Y).$$

We find the equation for x_{XY} by using (2.10) in $t_{XY} = t_{YX}$, which yields $(\frac{x_X}{c(U)})_Y = (-\frac{x_Y}{c(U)})_X$ and finally

$$x_{XY} = \frac{c'(U)}{2c(U)}(U_Y x_X + U_X x_Y). \quad (2.17)$$

Using $J_{XY} = J_{YX}$, $K_{XY} = K_{YX}$, and (2.16) we get

$$J_{XY} = \frac{c'(U)}{2c(U)}(U_Y J_X + U_X J_Y)$$

and

$$K_{XY} = -\frac{c'(U)}{2c(U)}(U_Y K_X + U_X K_Y).$$

Let $\rho(t,x) = P(X(t,x), Y(t,x))$. By (1.3b) we get

$$0 = \rho_t - (c(u)\rho)_x = P_X(X_t - c(u)X_x) + P_Y(Y_t - c(u)Y_x) - c'(u)P(U_X X_x + U_Y Y_x).$$

From (1.14) and (2.9) we have

$$c(U)P_Y x_X + \frac{c'(U)}{2}P(U_X x_Y + U_Y x_X) = 0$$

and from (2.17) we see that this is the same as

$$P_Y x_X + P x_{XY} = (P x_X)_Y = 0.$$

We define $p = P x_X$, so that

$$p_Y = 0.$$

Let $\sigma(t,x) = Q(X(t,x), Y(t,x))$. From (1.3c) we have

$$0 = \sigma_t + (c(u)\sigma)_x = Q_X(X_t + c(u)X_x) + Q_Y(Y_t + c(u)Y_x) + c'(u)Q(U_X X_x + U_Y Y_x).$$

Using (1.14) and (2.9) we get

$$c(U)Q_X x_Y + \frac{c'(U)}{2}Q(U_X x_Y + U_Y x_X) = 0$$

and by (2.17) we find

$$Q_X x_Y + Q x_{XY} = (Q x_Y)_X = 0.$$

We define $q = Q x_Y$, so that

$$q_X = 0.$$

By (2.12) and (2.14) we get

$$2J_X x_X = c^2(U)U_X^2 + c(U)p^2 \quad \text{and} \quad 2J_Y x_Y = c^2(U)U_Y^2 + c(U)q^2. \tag{2.18}$$

Finally we end up with the following system of differential equations

$$t_{XY} = -\frac{c'(U)}{2c(U)}(U_Y t_X + U_X t_Y), \tag{2.19a}$$

$$x_{XY} = \frac{c'(U)}{2c(U)}(U_Y x_X + U_X x_Y), \tag{2.19b}$$

$$U_{XY} = \frac{c'(U)}{2c^3(U)}(x_Y J_X + x_X J_Y) - \frac{c'(U)}{2c(U)}U_X U_Y, \tag{2.19c}$$

$$J_{XY} = \frac{c'(U)}{2c(U)}(U_Y J_X + U_X J_Y), \tag{2.19d}$$

$$K_{XY} = -\frac{c'(U)}{2c(U)}(U_Y K_X + U_X K_Y), \tag{2.19e}$$

$$p_Y = 0, \tag{2.19f}$$

$$q_X = 0. \tag{2.19g}$$

We introduce the vector $Z = (t, x, U, J, K)$. The system (2.19a)–(2.19e) then rewrites as

$$Z_{XY} = F(Z)(Z_X, Z_Y), \tag{2.20}$$

where $F(Z)$ is a bilinear and symmetric tensor from $\mathbb{R}^5\times\mathbb{R}^5$ to $\mathbb{R}^5$. Due to the relations (2.10), either one of the equations in (2.19a) and (2.19b) is redundant: one could remove one of them, and the system would remain well-posed, and one retrieves t or x by using (2.10). Similarly, either one of the equations (2.19d) and (2.19e) becomes redundant by (2.16). However, we find it convenient to work with the complete set of variables, i.e., $Z=(t,x,U,J,K)$.

To prove existence of solutions to (2.19), we need a curve $(\mathcal{X},\mathcal{Y})$ in the (X,Y)-plane that corresponds to the initial time, i.e., it consists of all points $(X,Y)\in\mathbb{R}^2$ such that $t(X,Y)=0$. The aim of the next section is to define this curve for general initial data, and to assign values of Z, Z_X, Z_Y, p and q on the curve. Moreover, to solve (2.19), we will require that (2.10), (2.16), (2.18) together with some monotonicity properties hold on the curve, see Definition 4.7.

3 From Eulerian to Lagrangian coordinates

We first define the set $\mathcal{D}$, which consists of possible initial data corresponding to (1.3) in Eulerian coordinates.

Definition 3.1 The set $\mathcal{D}$ consists of the tuples $(u,R,S,\rho,\sigma,\mu,\nu)$ such that $u,R,S,\rho,\sigma\in L^2(\mathbb{R})$,

$$u_x=\frac{1}{2c(u)}(R-S),\quad u_t=\frac{1}{2}(R+S), \tag{3.1}$$

and μ and ν are finite positive Radon measures with

$$\mu_{ac}=\frac{1}{4}(R^2+c(u)\rho^2)\,dx\quad\text{and}\quad \nu_{ac}=\frac{1}{4}(S^2+c(u)\sigma^2)\,dx. \tag{3.2}$$

Note that this definition allows for initial data with concentrated energy.

In Lagrangian coordinates we will admit curves of the following type.

Definition 3.2 We denote by $\mathcal{C}$ the set of curves in the plane $\mathbb{R}^2$ parametrized by $(\mathcal{X}(s),\mathcal{Y}(s))$ with $s\in\mathbb{R}$, such that

$$\mathcal{X}-\mathrm{Id},\ \mathcal{Y}-\mathrm{Id}\in W^{1,\infty}(\mathbb{R}), \tag{3.3a}$$

$$\dot{\mathcal{X}}\geq 0,\ \dot{\mathcal{Y}}\geq 0 \tag{3.3b}$$

with the normalization

$$\frac{1}{2}(\mathcal{X}(s)+\mathcal{Y}(s))=s\quad\text{for all } s\in\mathbb{R}. \tag{3.3c}$$

We set

$$\|(\mathcal{X},\mathcal{Y})\|_{\mathcal{C}}=\|\mathcal{X}-\mathrm{Id}\|_{L^\infty(\mathbb{R})}+\|\mathcal{Y}-\mathrm{Id}\|_{L^\infty(\mathbb{R})}. \tag{3.3d}$$

An essential role throughout the paper is played by G, the group of so-called relabeling functions.

Definition 3.3 The group G is given by all invertible functions f such that

$$f-\mathrm{Id}\ \text{and}\ f^{-1}-\mathrm{Id}\ \text{both belong to}\ W^{1,\infty}(\mathbb{R}), \tag{3.4}$$

and

$$(f-\mathrm{Id})'\in L^2(\mathbb{R}). \tag{3.5}$$

Note that if $f, g \in G$, then also f^{-1}, g^{-1} and $f \circ g$ belong to G. The next step is to map elements from $\mathcal{D}$ to a set $\mathcal{F}$, which is defined as follows.

Definition 3.4 The set $\mathcal{F}$ consists of all functions $\psi = (\psi_1, \psi_2)$ such that

$$\psi_1(X) = (x_1(X), U_1(X), J_1(X), K_1(X), V_1(X), H_1(X))$$

and

$$\psi_2(Y) = (x_2(Y), U_2(Y), J_2(Y), K_2(Y), V_2(Y), H_2(Y))$$

satisfy the following regularity and decay conditions

$$x_1 - \mathrm{Id},\ x_2 - \mathrm{Id},\ J_1,\ J_2,\ K_1,\ K_2 \in W^{1,\infty}(\mathbb{R}), \tag{3.6a}$$
$$x_1' - 1,\ x_2' - 1,\ J_1',\ J_2',\ K_1',\ K_2',\ H_1,\ H_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3.6b}$$
$$U_1,\ U_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3.6c}$$
$$V_1,\ V_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3.6d}$$

and the additional conditions

$$x_1', x_2', J_1', J_2' \geq 0, \tag{3.7a}$$
$$J_1' = c(U_1)K_1', \quad J_2' = -c(U_2)K_2', \tag{3.7b}$$
$$J_1'x_1' = (c(U_1)V_1)^2 + c(U_1)H_1^2, \quad J_2'x_2' = (c(U_2)V_2)^2 + c(U_2)H_2^2, \tag{3.7c}$$
$$x_1 + J_1,\ x_2 + J_2 \in G, \tag{3.7d}$$
$$\lim_{X \to -\infty} J_1(X) = \lim_{Y \to -\infty} J_2(Y) = 0. \tag{3.7e}$$

Moreover, for any curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) \text{ for all } s \in \mathbb{R},$$

we have

$$U_1(\mathcal{X}(s)) = U_2(\mathcal{Y}(s)) \tag{3.8a}$$

for all $s \in \mathbb{R}$ and

$$\frac{d}{ds}U_1(\mathcal{X}(s)) = \frac{d}{ds}U_2(\mathcal{Y}(s)) = V_1(\mathcal{X}(s))\dot{\mathcal{X}}(s) + V_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) \tag{3.8b}$$

for almost all $s \in \mathbb{R}$.

For all our analysis it is vital that $x_i + J_i$ for $i = 1, 2$ are relabeling functions. Next, we define the map from $\mathcal{D}$ to $\mathcal{F}$.

Definition 3.5 Given $(u, R, S, \rho, \sigma, \mu, \nu) \in \mathcal{D}$, we define $\psi_1 = (x_1, U_1, J_1, K_1, V_1, H_1)$ and $\psi_2 = (x_2, U_2, J_2, K_2, V_2, H_2)$ as

$$x_1(X) = \sup\{x \in \mathbb{R} \mid x' + \mu((-\infty, x')) < X \text{ for all } x' < x\}, \tag{3.9a}$$
$$x_2(Y) = \sup\{x \in \mathbb{R} \mid x' + \nu((-\infty, x')) < Y \text{ for all } x' < x\} \tag{3.9b}$$

and

$$J_1(X) = X - x_1(X), \quad J_2(Y) = Y - x_2(Y), \tag{3.9c}$$
$$U_1(X) = u(x_1(X)), \quad U_2(Y) = u(x_2(Y)), \tag{3.9d}$$

$$V_1(X) = x_1'(X)\frac{R(x_1(X))}{2c(U_1(X))}, \quad V_2(Y) = -x_2'(Y)\frac{S(x_2(Y))}{2c(U_2(Y))}, \tag{3.9e}$$

$$K_1(X) = \int_{-\infty}^{X} \frac{J_1'(\bar{X})}{c(U_1(\bar{X}))}\, d\bar{X}, \quad K_2(Y) = -\int_{-\infty}^{Y} \frac{J_2'(\bar{Y})}{c(U_2(\bar{Y}))}\, d\bar{Y}, \tag{3.9f}$$

$$H_1(X) = \frac{1}{2}\rho(x_1(X))x_1'(X), \quad H_2(Y) = \frac{1}{2}\sigma(x_2(Y))x_2'(Y). \tag{3.9g}$$

We denote by $\mathbf{L} : \mathcal{D} \to \mathcal{F}$ the mapping which to any $(u, R, S, \rho, \sigma, \mu, \nu) \in \mathcal{D}$ associates the element $\psi = (\psi_1, \psi_2) \in \mathcal{F}$ as defined above.

As mentioned before, solutions can develop singularities in finite time and energy can concentrate on sets of measure zero. If this is the case one has to put some extra effort into understanding (3.9e) and (3.9g) since they might be of the form $0 \cdot \infty$, when $x_1'(X) = 0$. One has, in the smooth case for $X_1 < X_2$ that

$$\int_{x_1(X_1)}^{x_1(X_2)} \frac{R}{2c(u)}(x)\, dx = \int_{X_1}^{X_2} \frac{R(x_1(\tilde{X}))}{2c(u(x_1(\tilde{X})))} x_1'(\tilde{X})\, d\tilde{X} = \int_{X_1}^{X_2} V_1(\tilde{X})\, d\tilde{X}$$

and

$$\begin{aligned} \left| \int_{x_1(X_1)}^{x_1(X_2)} \frac{R}{2c(u)}(x)\, dx \right| &\le \kappa \sqrt{x_1(X_2) - x_1(X_1)} \sqrt{\mu_{\mathrm{ac}}((x_1(X_1), x_1(X_2))} \\ &\le \kappa \sqrt{x_1(X_2) - x_1(X_1)} \sqrt{J_1(X_2) - J_1(X_1)}. \end{aligned} \tag{3.10}$$

If we now consider the nonsmooth case, (3.10) still holds and the above calculations imply that $V_1(X)$ exists and is bounded. Furthermore, if $x_1'(X) = 0$, we must have that $V_1(X) = 0$.

Given an element in $\mathcal{F}$ we want to define a curve $(\mathcal{X}, \mathcal{Y})$ and the values of ψ on that curve. We define the set $\mathcal{G}$ which consists of curves $(\mathcal{X}, \mathcal{Y})$ and five functions $\mathcal{Z}$, $\mathcal{V}$, $\mathcal{W}$, $\mathfrak{p}$, and $\mathfrak{q}$, next. The idea is that these functions in the smooth case are given through

$$\mathcal{Z}(s) = Z(\mathcal{X}(s), \mathcal{Y}(s))$$

and

$$\begin{aligned} \mathcal{V}(\mathcal{X}(s)) &= Z_X(\mathcal{X}(s), \mathcal{Y}(s)), \quad \mathcal{W}(\mathcal{Y}(s)) = Z_Y(\mathcal{X}(s), \mathcal{Y}(s)), \\ \mathfrak{p}(\mathcal{X}(s)) &= p(\mathcal{X}(s), \mathcal{Y}(s)), \quad \mathfrak{q}(\mathcal{Y}(s)) = q(\mathcal{X}(s), \mathcal{Y}(s)). \end{aligned}$$

Let us explain how the curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ may be defined for regular initial data, and how we assign the value of Z, Z_X, Z_Y, p, and q on the curve. In the derivation in Sect. 2, where we assumed that u_0, R_0, S_0, ρ_0, and σ_0 are smooth and bounded, and the measures μ_0 and ν_0 are absolutely continuous, we found that $0 < t_X < \infty$ and $-\infty < t_Y < 0$. This implies that both $\mathcal{X}(s)$ and $\mathcal{Y}(s)$, which are implicitly given through $t(\mathcal{X}(s), \mathcal{Y}(s)) = 0$, are strictly increasing functions. Indeed, by differentiating $t(\mathcal{X}(s), \mathcal{Y}(s)) = 0$ and using (3.3c) we get $\dot{\mathcal{X}} = -\frac{2t_Y}{t_X - t_Y}$ and $\dot{\mathcal{Y}} = \frac{2t_X}{t_X - t_Y}$, which implies $\dot{\mathcal{X}} > 0$ and $\dot{\mathcal{Y}} > 0$. Thus, in this case $(\mathcal{X}(s), \mathcal{Y}(s))$ is a strictly monotone curve. Next, we use the identities derived in Sect. 2 to assign the value of Z, Z_X, Z_Y, p, and q on the curve $(\mathcal{X}, \mathcal{Y})$. In particular, we set

$$U(\mathcal{X}(s), \mathcal{Y}(s)) = u_0(x(\mathcal{X}(s), \mathcal{Y}(s))).$$

Moreover, to get a set of closed equations, we let

$$2x_X(\mathcal{X}(s), \mathcal{Y}(s)) + J_X(\mathcal{X}(s), \mathcal{Y}(s)) = 1 \quad \text{and} \quad 2x_Y(\mathcal{X}(s), \mathcal{Y}(s)) + J_Y(\mathcal{X}(s), \mathcal{Y}(s)) = 1. \tag{3.11}$$

Following closely the procedure in [12, Section 2] and using (2.14), we define $x(\mathcal{X}(s), \mathcal{Y}(s))$ implicitly as

$$2x(\mathcal{X}(s), \mathcal{Y}(s)) + \frac{1}{4}\int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))} (R_0^2 + c(u_0)\rho_0^2 + S_0^2 + c(u_0)\sigma_0^2)(z)\, dz = 2s. \tag{3.12}$$

Note that the left-hand side is a strictly increasing function with respect to x, so that (3.12) uniquely defines $x(\mathcal{X}(s), \mathcal{Y}(s))$. Furthermore, we have

$$J(\mathcal{X}(s), \mathcal{Y}(s)) = \frac{1}{4}\int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))} (R_0^2 + c(u_0)\rho_0^2 + S_0^2 + c(u_0)\sigma_0^2)(z)\, dz$$

and

$$K(\mathcal{X}(s), \mathcal{Y}(s)) = \frac{1}{4}\int_{-\infty}^{x(\mathcal{X}(s),\mathcal{Y}(s))} \frac{1}{c(u_0)}(R_0^2 + c(u_0)\rho_0^2 - S_0^2 - c(u_0)\sigma_0^2)(z)\, dz$$

Finally using (2.12), (2.14), (2.18), and (3.11), we get the two additional equations

$$p(\mathcal{X}, \mathcal{Y}) = \left(\frac{2\rho_0}{4 + R_0^2 + c(u_0)\rho_0^2}\right)(x(\mathcal{X}, \mathcal{Y}))$$
$$\text{and} \quad q(\mathcal{X}, \mathcal{Y}) = \left(\frac{2\sigma_0}{4 + S_0^2 + c(u_0)\sigma_0^2}\right)(x(\mathcal{X}, \mathcal{Y})).$$

As a motivation for the regularity conditions that are imposed in the definition of the set $\mathcal{G}$, we first note from (3.12) that $x(X, Y)$ is increasing with respect to both its arguments and is therefore unbounded. However, from (3.12) we get

$$|x(\mathcal{X}(s), \mathcal{Y}(s)) - s| \leq \frac{1}{2}(\mu_0(\mathbb{R}) + \nu_0(\mathbb{R})),$$

which belongs to $L^\infty(\mathbb{R})$. Therefore, we require that $\mathcal{Z}_2 - \mathrm{Id}$ belongs to $L^\infty(\mathbb{R})$.

It is convenient to introduce the following notation: to any triplet $(\mathcal{Z}, \mathcal{V}, \mathcal{W})$ of five dimensional vector functions we associate a triplet $(\mathcal{Z}^a, \mathcal{V}^a, \mathcal{W}^a)$ given by

$$\mathcal{Z}_1^a = \mathcal{Z}_1 - \frac{1}{c(0)}(\mathcal{X} - \mathrm{Id}), \quad \mathcal{V}_1^a = \mathcal{V}_1 - \frac{1}{2c(0)}, \quad \mathcal{W}_1^a = \mathcal{W}_1 + \frac{1}{2c(0)}, \tag{3.13a}$$

$$\mathcal{Z}_2^a = \mathcal{Z}_2 - \mathrm{Id}, \quad \mathcal{V}_2^a = \mathcal{V}_2 - \frac{1}{2}, \quad \mathcal{W}_2^a = \mathcal{W}_2 - \frac{1}{2}, \tag{3.13b}$$

$$\mathcal{Z}_i^a = \mathcal{Z}_i, \quad \mathcal{V}_i^a = \mathcal{V}_i, \quad \mathcal{W}_i^a = \mathcal{W}_i \tag{3.13c}$$

for $i \in \{3, 4, 5\}$.

Definition 3.6 The set $\mathcal{G}$ is the set of all elements $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$, which consist of a curve $(\mathcal{X}(s), \mathcal{Y}(s)) \in \mathcal{C}$, three vector-valued functions

$$\mathcal{Z}(s) = (\mathcal{Z}_1(s), \mathcal{Z}_2(s), \mathcal{Z}_3(s), \mathcal{Z}_4(s), \mathcal{Z}_5(s)),$$
$$\mathcal{V}(X) = (\mathcal{V}_1(X), \mathcal{V}_2(X), \mathcal{V}_3(X), \mathcal{V}_4(X), \mathcal{V}_5(X)),$$
$$\mathcal{W}(Y) = (\mathcal{W}_1(Y), \mathcal{W}_2(Y), \mathcal{W}_3(Y), \mathcal{W}_4(Y), \mathcal{W}_5(Y)),$$

and two functions $\mathfrak{p}(X)$ and $\mathfrak{q}(Y)$. We set

$$\|\Theta\|_{\mathcal{G}}^2 = \left\|\mathcal{Z}_3^a\right\|_{L^2(\mathbb{R})}^2 + \left\|\mathcal{V}^a\right\|_{L^2(\mathbb{R})}^2 + \left\|\mathcal{W}^a\right\|_{L^2(\mathbb{R})}^2 + \|\mathfrak{p}\|_{L^2(\mathbb{R})}^2 + \|\mathfrak{q}\|_{L^2(\mathbb{R})}^2 \tag{3.14}$$

and

$$
\begin{aligned}
|||\Theta|||_{\mathcal{G}} &= \|(\mathcal{X},\mathcal{Y})\|_{\mathcal{C}} + \left\|\frac{1}{\mathcal{V}_2+\mathcal{V}_4}\right\|_{L^\infty(\mathbb{R})} \\
&\quad + \left\|\frac{1}{\mathcal{W}_2+\mathcal{W}_4}\right\|_{L^\infty(\mathbb{R})} \\
&\quad + \left\|\mathcal{Z}^a\right\|_{L^\infty(\mathbb{R})} + \left\|\mathcal{V}^a\right\|_{L^\infty(\mathbb{R})} + \left\|\mathcal{W}^a\right\|_{L^\infty(\mathbb{R})} + \|\mathfrak{p}\|_{L^\infty(\mathbb{R})} + \|\mathfrak{q}\|_{L^\infty(\mathbb{R})}. \qquad (3.15)
\end{aligned}
$$

The element Θ belongs to $\mathcal{G}$ if

(i)

$$\|\Theta\|_{\mathcal{G}} < \infty \quad \text{and} \quad |||\Theta|||_{\mathcal{G}} < \infty; \qquad (3.16)$$

(ii)

$$\mathcal{V}_2, \mathcal{W}_2, \mathcal{V}_4, \mathcal{W}_4 \geq 0; \qquad (3.17)$$

(iii) for almost every $s \in \mathbb{R}$, we have

$$\dot{\mathcal{Z}}(s) = \mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s) \qquad (3.18)$$

(iv)

$$\mathcal{V}_2(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_1(\mathcal{X}(s)), \qquad \mathcal{W}_2(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_1(\mathcal{Y}(s)), \qquad (3.19a)$$

$$\mathcal{V}_4(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_5(\mathcal{X}(s)), \qquad \mathcal{W}_4(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_5(\mathcal{Y}(s)), \qquad (3.19b)$$

and

$$2\mathcal{V}_4(\mathcal{X}(s))\mathcal{V}_2(\mathcal{X}(s)) = (c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{p}^2(\mathcal{X}(s)), \qquad (3.19c)$$

$$2\mathcal{W}_4(\mathcal{Y}(s))\mathcal{W}_2(\mathcal{Y}(s)) = (c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{q}^2(\mathcal{Y}(s)), \qquad (3.19d)$$

(v)

$$\lim_{s\to-\infty} \mathcal{Z}_4(s) = 0. \qquad (3.20)$$

We denote by $\mathcal{G}_0$ the subset of $\mathcal{G}$ which parametrize the data at time $t = 0$, that is,

$$\mathcal{G}_0 = \{\Theta \in \mathcal{G} \mid \mathcal{Z}_1 = 0\}.$$

For $\Theta \in \mathcal{G}_0$, we get by using (3.18) and (3.19a), that

$$\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) = \mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s). \qquad (3.21)$$

This implies that

$$\dot{\mathcal{Z}}_2(s) = 2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) = 2\mathcal{W}_2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s). \qquad (3.22)$$

Note that for an element $\Theta \in \mathcal{G}$ we have $\mathcal{V}_2 + \mathcal{V}_4 > 0$ and $\mathcal{W}_2 + \mathcal{W}_4 > 0$ almost everywhere. As we shall see, this property is preserved in the solution and is important in proving that the solution operator from $\mathcal{D}$ to $\mathcal{D}$ is a semigroup.

Definition 3.7 For any $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, we define $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ as

$$\mathcal{X}(s) = \sup\{X \in \mathbb{R} \mid x_1(X') < x_2(2s - X') \text{ for all } X' < X\} \qquad (3.23)$$

and set $\mathcal{Y}(s) = 2s - \mathcal{X}(s)$. We have

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)). \tag{3.24}$$

We define

$$\mathcal{Z}_1(s) = 0, \tag{3.25a}$$
$$\mathcal{Z}_2(s) = x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)), \tag{3.25b}$$
$$\mathcal{Z}_3(s) = U_1(\mathcal{X}(s)) = U_2(\mathcal{Y}(s)), \tag{3.25c}$$
$$\mathcal{Z}_4(s) = J_1(\mathcal{X}(s)) + J_2(\mathcal{Y}(s)), \tag{3.25d}$$
$$\mathcal{Z}_5(s) = K_1(\mathcal{X}(s)) + K_2(\mathcal{Y}(s)) \tag{3.25e}$$

and

$$\mathcal{V}_1(X) = \frac{1}{2c(U_1(X))}x_1'(X), \quad \mathcal{W}_1(Y) = -\frac{1}{2c(U_2(Y))}x_2'(Y), \tag{3.26a}$$
$$\mathcal{V}_2(X) = \frac{1}{2}x_1'(X), \quad \mathcal{W}_2(Y) = \frac{1}{2}x_2'(Y), \tag{3.26b}$$
$$\mathcal{V}_3(X) = V_1(X), \quad \mathcal{W}_3(Y) = V_2(Y), \tag{3.26c}$$
$$\mathcal{V}_4(X) = J_1'(X), \quad \mathcal{W}_4(Y) = J_2'(Y), \tag{3.26d}$$
$$\mathcal{V}_5(X) = K_1'(X), \quad \mathcal{W}_5(Y) = K_2'(Y), \tag{3.26e}$$
$$\mathfrak{p}(X) = H_1(X), \quad \mathfrak{q}(Y) = H_2(Y). \tag{3.26f}$$

Denote by $\mathbf{C} : \mathcal{F} \to \mathcal{G}_\mathbf{0}$ the mapping which to any $\psi \in \mathcal{F}$ associates the element $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}_0$ as defined above.

Observe from (3.9a), (3.9b), (3.23), and (3.25b), that for initial data in $\mathcal{D}$ such that u_0, R_0, S_0, ρ_0, and σ_0 are smooth, and μ_0 and ν_0 are absolutely continuous, we recover the identity (3.12) with $x(\mathcal{X}(s), \mathcal{Y}(s))$ replaced by $\mathcal{Z}_2(s)$.

4 Existence of solutions for the equivalent system

4.1 Existence of short-range solutions

In the following we denote rectangular domains by

$$\Omega = [X_l, X_r] \times [Y_l, Y_r]$$

and we set $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. We define curves in rectangular domains as follows.

Definition 4.1 Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we denote by $\mathcal{C}(\Omega)$ the set of curves in Ω parametrized by $(\mathcal{X}(s), \mathcal{Y}(s))$ with $s \in [s_l, s_r]$ such that $(\mathcal{X}(s_l), \mathcal{Y}(s_l)) = (X_l, Y_l)$, $(\mathcal{X}(s_r), \mathcal{Y}(s_r)) = (X_r, Y_r)$, and

$$\mathcal{X} - \mathrm{Id}, \quad \mathcal{Y} - \mathrm{Id} \in W^{1,\infty}([s_l, s_r]), \tag{4.1a}$$
$$\dot{\mathcal{X}} \geq 0, \quad \dot{\mathcal{Y}} \geq 0, \tag{4.1b}$$
$$\frac{1}{2}(\mathcal{X}(s) + \mathcal{Y}(s)) = s \quad \text{for all } s \in [s_l, s_r]. \tag{4.1c}$$

We set

$$\|(\mathcal{X},\mathcal{Y})\|_{\mathcal{C}(\Omega)} = \|\mathcal{X}-\mathrm{Id}\|_{L^\infty([s_l,s_r])} + \|\mathcal{Y}-\mathrm{Id}\|_{L^\infty([s_l,s_r])}.$$

We introduce the counterpart of $\mathcal{G}$ on bounded domains, which we denote by $\mathcal{G}(\Omega)$.

Definition 4.2 Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, we denote by $\mathcal{G}(\Omega)$ the set of all elements which consist of a curve $(\mathcal{X},\mathcal{Y}) \in \mathcal{C}(\Omega)$, three vector-valued functions $\mathcal{Z}(s)$, $\mathcal{V}(X)$ and $\mathcal{W}(Y)$, and two functions $\mathfrak{p}(X)$ and $\mathfrak{q}(Y)$. We denote $\Theta = (\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})$ and set

$$\|\Theta\|^2_{\mathcal{G}(\Omega)} = \left\|\mathcal{Z}^a_3\right\|^2_{L^2([s_l,s_r])} + \left\|\mathcal{V}^a\right\|^2_{L^2([X_l,X_r])} + \left\|\mathcal{W}^a\right\|^2_{L^2([Y_l,Y_r])} + \|\mathfrak{p}\|^2_{L^2([X_l,X_r])} + \|\mathfrak{q}\|^2_{L^2([Y_l,Y_r])}$$

and

$$\begin{aligned}|||\Theta|||_{\mathcal{G}(\Omega)} = \|(\mathcal{X},\mathcal{Y})\|_{\mathcal{C}(\Omega)} + \left\|\frac{1}{\mathcal{V}_2+\mathcal{V}_4}\right\|_{L^\infty([X_l,X_r])} + \left\|\frac{1}{\mathcal{W}_2+\mathcal{W}_4}\right\|_{L^\infty([Y_l,Y_r])} + \left\|\mathcal{Z}^a\right\|_{L^\infty([s_l,s_r])} \\ + \left\|\mathcal{V}^a\right\|_{L^\infty([X_l,X_r])} + \left\|\mathcal{W}^a\right\|_{L^\infty([Y_l,Y_r])} + \|\mathfrak{p}\|_{L^\infty([X_l,X_r])} + \|\mathfrak{q}\|_{L^\infty([Y_l,Y_r])}.\end{aligned}$$

The element Θ belongs[1] to $\mathcal{G}(\Omega)$, if

(i)

$$|||\Theta|||_{\mathcal{G}(\Omega)} < \infty,$$

(ii)

$$\mathcal{V}_2, \mathcal{W}_2, \mathcal{Z}_4, \mathcal{V}_4, \mathcal{W}_4 \geq 0,$$

(iii) for almost every $s \in \mathbb{R}$, we have

$$\dot{\mathcal{Z}}(s) = \mathcal{V}(\mathcal{X}(s))\dot{\mathcal{X}}(s) + \mathcal{W}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s), \tag{4.2}$$

(iv)

$$\mathcal{V}_2(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_1(\mathcal{X}(s)), \quad \mathcal{W}_2(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_1(\mathcal{Y}(s)), \tag{4.3a}$$

$$\mathcal{V}_4(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_5(\mathcal{X}(s)), \quad \mathcal{W}_4(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_5(\mathcal{Y}(s)) \tag{4.3b}$$

and

$$2\mathcal{V}_4(\mathcal{X}(s))\mathcal{V}_2(\mathcal{X}(s)) = (c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{p}^2(\mathcal{X}(s)), \tag{4.3c}$$

$$2\mathcal{W}_4(\mathcal{Y}(s))\mathcal{W}_2(\mathcal{Y}(s)) = (c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s)))^2 + c(\mathcal{Z}_3(s))\mathfrak{q}^2(\mathcal{Y}(s)). \tag{4.3d}$$

By definition we have for any $(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q}) \in \mathcal{G}(\Omega)$ that the functions $\mathcal{X}$ and $\mathcal{Y}$ are nondecreasing. To any nondecreasing function one can associate its generalized inverse, a concept which is presented in, e.g., [1].

Definition 4.3 Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ and $(\mathcal{X},\mathcal{Y}) \in \mathcal{C}(\Omega)$, we define the generalized inverse of $\mathcal{X}$ and $\mathcal{Y}$ as

$$\begin{aligned}\alpha(X) &= \sup\{s \in [s_l, s_r] \mid \mathcal{X}(s) < X\} \quad \text{for } X \in (X_l, X_r], \\ \beta(Y) &= \sup\{s \in [s_l, s_r] \mid \mathcal{Y}(s) < Y\} \quad \text{for } Y \in (Y_l, Y_r],\end{aligned}$$

respectively. We denote $\mathcal{X}^{-1} = \alpha$ and $\mathcal{Y}^{-1} = \beta$.

[1] Note that condition (i) implies $\|\Theta\|_{\mathcal{G}(\Omega)} < \infty$ because Ω is bounded.

The generalized inverse functions $\mathcal{X}^{-1}$ and $\mathcal{Y}^{-1}$ satisfy the following properties.

Lemma 4.4 *The functions $\mathcal{X}^{-1}$ and $\mathcal{Y}^{-1}$ are lower semicontinuous and nondecreasing. We have*

$$\mathcal{X} \circ \mathcal{X}^{-1} = \mathrm{Id} \quad \textit{and} \quad \mathcal{Y} \circ \mathcal{Y}^{-1} = \mathrm{Id}, \tag{4.4a}$$
$$\mathcal{X}^{-1} \circ \mathcal{X}(s) = s \textit{ for any } s \textit{ such that } \dot{\mathcal{X}}(s) > 0, \tag{4.4b}$$

and

$$\mathcal{Y}^{-1} \circ \mathcal{Y}(s) = s \textit{ for any } s \textit{ such that } \dot{\mathcal{Y}}(s) > 0. \tag{4.4c}$$

Now we define solutions of (2.19) on rectangular domains. Consider the Banach spaces

$$\begin{aligned} &L^\infty_X(\Omega) = L^\infty([Y_l, Y_r], C([X_l, X_r])), \quad L^\infty_Y(\Omega) = L^\infty([X_l, X_r], C([Y_l, Y_r])),\\ &W^{1,\infty}_X(\Omega) = L^\infty([Y_l, Y_r], W^{1,\infty}([X_l, X_r])),\\ &W^{1,\infty}_Y(\Omega) L^\infty([X_l, X_r], W^{1,\infty}([Y_l, Y_r])). \end{aligned}$$

The corresponding norms for $f : \Omega \mapsto \mathbb{R}$ are defined as

$$\begin{aligned} ||f||_{L^\infty_X(\Omega)} &= \operatorname*{ess\,sup}_{Y\in[Y_l,Y_r]} ||f(\cdot, Y)||_{L^\infty([X_l,X_r])}, \quad ||f||_{L^\infty_Y(\Omega)} = \operatorname*{ess\,sup}_{X\in[X_l,X_r]} ||f(X, \cdot)||_{L^\infty([Y_l,Y_r])},\\ ||f||_{W^{1,\infty}_X(\Omega)} &= \operatorname*{ess\,sup}_{Y\in[Y_l,Y_r]} ||f(\cdot, Y)||_{W^{1,\infty}([X_l,X_r])},\\ ||f||_{W^{1,\infty}_Y(\Omega)} &= \operatorname*{ess\,sup}_{X\in[X_l,X_r]} ||f(X, \cdot)||_{W^{1,\infty}([Y_l,Y_r])}. \end{aligned}$$

We introduce the function Z^a, defined as

$$Z^a_1(X, Y) = Z_1(X, Y) - \frac{1}{2c(0)}(X - Y), \tag{4.5a}$$
$$Z^a_2(X, Y) = Z_2(X, Y) - \frac{1}{2}(X + Y), \tag{4.5b}$$
$$Z^a_i(X, Y) = Z_i(X, Y) \quad \text{for } i \in \{3, 4, 5\} \tag{4.5c}$$

in order to conveniently express the decay of Z at infinity in the diagonal direction. Although we are not yet concerned with the behavior at infinity, the notation will be useful when introducing global solutions.

Definition 4.5 We say that (Z, p, q) is a solution of (2.19) in $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ if

(i)

$$\begin{aligned} &Z^a \in [W^{1,\infty}(\Omega)]^5, \quad Z^a_X \in [W^{1,\infty}_Y(\Omega)]^5, \quad Z^a_Y \in [W^{1,\infty}_X(\Omega)]^5,\\ &p \in W^{1,\infty}_Y(\Omega), \quad q \in W^{1,\infty}_X(\Omega), \end{aligned} \tag{4.6}$$

(ii) for almost every $X \in [X_l, X_r]$,

$$(Z_X(X, Y))_Y = F(Z)(Z_X, Z_Y)(X, Y), \tag{4.7}$$

(iii) for almost every $Y \in [Y_l, Y_r]$,

$$(Z_Y(X, Y))_X = F(Z)(Z_X, Z_Y)(X, Y), \tag{4.8}$$

(iv) for almost every $X \in [X_l, X_r]$,

$$p_Y(X, Y) = 0, \tag{4.9}$$

(v) for almost every $Y \in [Y_l, Y_r]$,

$$q_X(X, Y) = 0. \tag{4.10}$$

We say that (Z, p, q) is a global solution of (2.19), if these conditions hold for any rectangular domain Ω.

The following lemma shows that the imposed regularity in Definition 4.5 is necessary to extract relevant data from a curve. Slightly abusing the notation, we denote

$$\mathcal{X}(Y) = \mathcal{X} \circ \mathcal{Y}^{-1}(Y) \quad \text{and} \quad \mathcal{Y}(X) = \mathcal{Y} \circ \mathcal{X}^{-1}(X). \tag{4.11}$$

Lemma 4.6 *Let Ω be a rectangular domain in $\mathbb{R}^2$ and assume that*

$$Z^a \in [W^{1,\infty}(\Omega)]^5, \quad Z^a_X \in [W^{1,\infty}_Y(\Omega)]^5, \quad Z^a_Y \in [W^{1,\infty}_X(\Omega)]^5, \quad p \in W^{1,\infty}_Y(\Omega),$$
$$q \in W^{1,\infty}_X(\Omega).$$

Given a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$, let $(\mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ be defined as

$$\mathcal{Z}(s) = Z(\mathcal{X}(s), \mathcal{Y}(s)) \text{ for all } s \in \mathbb{R}$$

and

$$\mathcal{V}(X) = Z_X(X, \mathcal{Y}(X)), \quad \mathfrak{p}(X) = p(X, \mathcal{Y}(X)) \quad \text{for a.e. } X \in \mathbb{R},$$
$$\mathcal{W}(Y) = Z_Y(\mathcal{X}(Y), Y), \quad \mathfrak{q}(Y) = q(\mathcal{X}(Y), Y) \quad \text{for a.e. } Y \in \mathbb{R},$$

or equivalently

$$\mathcal{V}(\mathcal{X}(s)) = Z_X(\mathcal{X}(s), \mathcal{Y}(s)), \quad \mathfrak{p}(\mathcal{X}(s)) = p(\mathcal{X}(s), \mathcal{Y}(s)) \quad \text{for a.e. } s \in \mathbb{R} \text{ such that } \dot{\mathcal{X}}(s) > 0,$$
$$\mathcal{W}(\mathcal{Y}(s)) = Z_Y(\mathcal{X}(s), \mathcal{Y}(s)), \quad \mathfrak{q}(\mathcal{Y}(s)) = q(\mathcal{X}(s), \mathcal{Y}(s)) \quad \text{for a.e. } s \in \mathbb{R} \text{ such that } \dot{\mathcal{Y}}(s) > 0.$$

Then $\mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q} \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and we denote $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$ by

$$(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}).$$

We now introduce the set $\mathcal{H}(\Omega)$ of all solutions of (2.19) on rectangular domains, which satisfy (2.10), (2.16), (2.18), and some additional constraints.

Definition 4.7 Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, let $\mathcal{H}(\Omega)$ be the set of all solutions (Z, p, q) to (2.19) in the sense of Definition 4.5, which satisfy the following properties

$$x_X = c(U)t_X, \quad x_Y = -c(U)t_Y, \tag{4.12a}$$
$$J_X = c(U)K_X, \quad J_Y = -c(U)K_Y, \tag{4.12b}$$
$$2J_X x_X = (c(U)U_X)^2 + c(U)p^2, \quad 2J_Y x_Y = (c(U)U_Y)^2 + c(U)q^2, \tag{4.12c}$$
$$x_X \geq 0, \quad x_Y \geq 0, \tag{4.12d}$$
$$J_X \geq 0, \quad J_Y \geq 0, \tag{4.12e}$$
$$x_X + J_X > 0, \quad x_Y + J_Y > 0. \tag{4.12f}$$

We have the following short-range existence theorem.

Theorem 4.8 *Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$, then for any $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}(\Omega)$, there exists a unique solution $(Z, p, q) \in \mathcal{H}(\Omega)$ such that*

$$\Theta = (Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}), \tag{4.13}$$

if $s_r - s_l \leq 1/C(|||\Theta|||_{\mathcal{G}(\Omega)})$. Here C denotes an increasing function dependent on Ω, κ, k_1, and k_2.

4.2 Existence of local solutions

We begin with some a priori estimates.

Given a positive constant L, we call domains of the type

$$\{(X, Y) \in \mathbb{R}^2 \mid |Y - X| \leq 2L\}$$

strip domains, which correspond to domains where time is bounded. We have the following a priori estimates for the solution of (2.19).

Lemma 4.9 *Given $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ and $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}(\Omega)$, let $(Z, p, q) \in \mathcal{H}(\Omega)$ be a solution of (2.19) such that $\Theta = (Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})$. Let $\mathcal{E}_0 = ||\mathcal{Z}_4||_{L^\infty([s_l, s_r])} + ||\mathcal{Z}_5||_{L^\infty([s_l, s_r])}$. Then the following statements hold:*

(i) *Boundedness of the energy, that is,*

$$0 \leq J(X, Y) \leq \mathcal{E}_0 \quad \text{for all } (X, Y) \in \Omega \tag{4.14a}$$

and

$$||K||_{L^\infty(\Omega)} \leq (1 + \kappa)\mathcal{E}_0. \tag{4.14b}$$

(ii) *The functions Z, Z_X, Z_Y, p, and q remain uniformly bounded in strip domains which contain Ω, that is, there exists a nondecreasing function $C_1 = C_1(|||\Theta|||_{\mathcal{G}(\Omega)}, L)$ such that for any $L > 0$ and any $(X, Y) \in \Omega$ such that $|X - Y| \leq 2L$, we have*

$$|Z^a(X, Y)| \leq C_1, \quad |Z_X(X, Y)| \leq C_1, \quad |Z_Y(X, Y)| \leq C_1, \tag{4.15a}$$

$$|p(X, Y)| \leq C_1, \quad |q(X, Y)| \leq C_1 \tag{4.15b}$$

and

$$\frac{1}{x_X + J_X}(X, Y) \leq C_1, \quad \frac{1}{x_Y + J_Y}(X, Y) \leq C_1. \tag{4.15c}$$

Condition (ii) *is equivalent to the following:*

(iii) *For any curve $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$, we have*

$$|||(Z, p, q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})|||_{\mathcal{G}(\Omega)} \leq C_1, \tag{4.16}$$

where $C_1 = C_1(||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}, |||\Theta|||_{\mathcal{G}(\Omega)})$ is an increasing function with respect to both its arguments.

We have the following existence and uniqueness result.

Lemma 4.10 (Existence and uniqueness on arbitrarily large rectangles) *Given a rectangular domain $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ and $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}(\Omega)$, there exists a unique solution $(Z, p, q) \in \mathcal{H}(\Omega)$ such that*

$$\Theta = (Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}).$$

Lemma 4.11 (A Gronwall lemma for curves) *Let* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ *and assume that* $(Z, p, q) \in \mathcal{H}(\Omega)$ *and* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$. *Then, for any* $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$, *we have*

$$||(Z, p, q) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{G}(\Omega)} \leq C||(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})||_{\mathcal{G}(\Omega)},$$

where $C = C(||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}, |||(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)})$ *is an increasing function with respect to both its arguments.*

Lemma 4.12 (Stability in L^2) *Let* $\Omega = [X_l, X_r] \times [Y_l, Y_r]$ *and assume that* (Z, p, q), $(\tilde{Z}, \tilde{p}, \tilde{q}) \in \mathcal{H}(\Omega)$ *and* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}(\Omega)$. *Then, for any* $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) \in \mathcal{C}(\Omega)$, *we have*

$$||(Z - \tilde{Z}, p - \tilde{p}, q - \tilde{q}) \bullet (\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{G}(\Omega)} \leq D||(Z - \tilde{Z}, p - \tilde{p}, q - \tilde{q}) \bullet (\mathcal{X}, \mathcal{Y})||_{\mathcal{G}(\Omega)},$$

where $D = D(||(\bar{\mathcal{X}}, \bar{\mathcal{Y}})||_{\mathcal{C}(\Omega)}, |||(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)}, |||(\tilde{Z}, \tilde{p}, \tilde{q}) \bullet (\mathcal{X}, \mathcal{Y})|||_{\mathcal{G}(\Omega)})$ *is an increasing function with respect to all its arguments.*

4.3 Existence of global solutions in $\mathcal{H}$

Definition 4.13 (*Global solutions*) Let $\mathcal{H}$ be the set of all functions (Z, p, q) such that

(i) $(Z, p, q) \in \mathcal{H}(\Omega)$ for all rectangular domains Ω;
(ii) there exists a curve $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ such that $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$.

The following lemma shows that condition (ii) does not depend on the particular curve for which it holds. In particular, we can replace this condition by the requirement that $(Z, p, q) \bullet (\mathcal{X}_d, \mathcal{Y}_d) \in \mathcal{G}$ for the diagonal, which is given by $\mathcal{X}_d(s) = \mathcal{Y}_d(s) = s$.

Lemma 4.14 *Given* $(Z, p, q) \in \mathcal{H}$, *we have* $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}$ *for any curve* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$. *Moreover,* $\lim_{s \to \infty} J(\mathcal{X}(s), \mathcal{Y}(s))$ *is independent of the curve* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$.

We have the following global existence theorem.

Theorem 4.15 (Existence and uniqueness of global solutions) *For any initial data* $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}$, *there exists a unique, global solution* $(Z, p, q) \in \mathcal{H}$ *such that* $\Theta = (Z, p, q) \bullet (\mathcal{X}, \mathcal{Y})$. *We denote this solution mapping by*

$$\mathbf{S} : \mathcal{G} \to \mathcal{H}. \tag{4.17}$$

5 From Lagrangian to Eulerian coordinates

5.1 Mapping from $\mathcal{H}$ to $\mathcal{F}$

Given an element (Z, p, q) in $\mathcal{H}$ we now want to map it to an element in the set $\mathcal{G}$ and then further to one in $\mathcal{F}$. For a solution in $\mathcal{H}$ corresponding to time $T > 0$, i.e., $t(X, Y) = T$, we find it convenient to first shift the time to zero so that we can map the solution to an element in $\mathcal{G}_0$ in the next step.

Definition 5.1 Given $T \geq 0$ and $(Z, p, q) \in \mathcal{H}$, we define

$$\bar{t}(X, Y) = t(X, Y) - T \tag{5.1a}$$

and

$$\bar{x}(X, Y) = x(X, Y), \quad \bar{U}(X, Y) = U(X, Y), \tag{5.1b}$$

$$\bar{J}(X, Y) = J(X, Y), \quad \bar{K}(X, Y) = K(X, Y), \tag{5.1c}$$
$$\bar{p}(X, Y) = p(X, Y), \quad \bar{q}(X, Y) = q(X, Y). \tag{5.1d}$$

We denote by $\mathbf{t}_T : \mathcal{H} \to \mathcal{H}$ the mapping which associates to any $(Z, p, q) \in \mathcal{H}$ the element $(\bar{Z}, \bar{p}, \bar{q}) \in \mathcal{H}$. We have

$$\mathbf{t}_{T+T'} = \mathbf{t}_T \circ \mathbf{t}_{T'}. \tag{5.2}$$

Definition 5.2 Given $(Z, p, q) \in \mathcal{H}$, we define

$$\mathcal{X}(s) = \sup\{X \in \mathbb{R} \mid t(X', 2s - X') < 0 \text{ for all } X' < X\} \tag{5.3}$$

and $\mathcal{Y}(s) = 2s - \mathcal{X}(s)$. Then, we have $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}$ and $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}_0$. We denote by $\mathbf{E} : \mathcal{H} \to \mathcal{G}_\mathbf{0}$ the mapping which associates to any $(Z, p, q) \in \mathcal{H}$ the element $(Z, p, q) \bullet (\mathcal{X}, \mathcal{Y}) \in \mathcal{G}_0$.

Definition 5.3 Given $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}_0$, let $\psi_1 = (x_1, U_1, J_1, K_1, V_1, H_1)$ and $\psi_2 = (x_2, U_2, J_2, K_2, V_2, H_2)$ be defined as

$$x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s)) = \mathcal{Z}_2(s), \tag{5.4a}$$
$$U_1(\mathcal{X}(s)) = U_2(\mathcal{Y}(s)) = \mathcal{Z}_3(s), \tag{5.4b}$$
$$J_1(\mathcal{X}(s)) = \int_{-\infty}^{s} \mathcal{V}_4(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)\, d\tau, \quad J_2(\mathcal{Y}(s)) = \int_{-\infty}^{s} \mathcal{W}_4(\mathcal{Y}(\tau))\dot{\mathcal{Y}}(\tau)\, d\tau, \tag{5.4c}$$
$$K_1(\mathcal{X}(s)) = \int_{-\infty}^{s} \mathcal{V}_5(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau)\, d\tau, \quad K_2(\mathcal{Y}(s)) = \int_{-\infty}^{s} \mathcal{W}_5(\mathcal{Y}(\tau))\dot{\mathcal{Y}}(\tau)\, d\tau, \tag{5.4d}$$

and

$$V_1 = \mathcal{V}_3, \quad V_2 = \mathcal{W}_3, \tag{5.4e}$$
$$H_1 = \mathfrak{p}, \quad H_2 = \mathfrak{q}. \tag{5.4f}$$

We denote by $\mathbf{D} : \mathcal{G}_0 \to \mathcal{F}$ the mapping which to any $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}_0$ associates the element $\psi \in \mathcal{F}$ as defined above.

5.2 Semigroup of solutions in $\mathcal{F}$

We define the solution operator on the set $\mathcal{F}$.

Definition 5.4 For any $T \geq 0$, we define the mapping $S_T : \mathcal{F} \to \mathcal{F}$ by

$$S_T = \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{S} \circ \mathbf{C}.$$

In order to show that S_T is a semigroup we need the following result.

Lemma 5.5 *We have*

$$\mathbf{C} \circ \mathbf{D} \circ \mathbf{E} = \mathbf{E}, \quad \mathbf{D} \circ \mathbf{C} = \mathrm{Id}, \quad \mathbf{E} \circ \mathbf{S} \circ \mathbf{C} = \mathbf{C}, \quad \textit{and} \quad \mathbf{S} \circ \mathbf{E} = \mathrm{Id}\,.$$

It follows that $\mathbf{S} \circ \mathbf{C} = (\mathbf{D} \circ \mathbf{E})^{-1}$ *and the sets* $\mathcal{F}$ *and* $\mathcal{H}$ *are in bijection.*

Theorem 5.6 *The mapping* S_T *is a semigroup.*

Proof We have

$$\begin{aligned} S_T \circ S_{T'} &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{S} \circ \mathbf{C} \circ \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_{T'} \circ \mathbf{S} \circ \mathbf{C} \\ &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_T \circ \mathbf{t}_{T'} \circ \mathbf{S} \circ \mathbf{C} \quad \text{by Lemma 5.5} \\ &= \mathbf{D} \circ \mathbf{E} \circ \mathbf{t}_{T+T'} \circ \mathbf{S} \circ \mathbf{C} \quad \text{by (5.2)} \\ &= S_{T+T'}. \end{aligned}$$

□

5.3 Mapping from $\mathcal{F}$ to $\mathcal{D}$

Definition 5.7 Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, we define $(u, R, S, \rho, \sigma, \mu, \nu)$ as[2]

$$u(x) = U_1(X) \quad \text{if } x_1(X) = x \tag{5.5a}$$

or, equivalently,

$$\begin{aligned} u(x) &= U_2(Y) \quad \text{if } x_2(Y) = x, && (5.5\text{b}) \\ R(x)\,dx &= (x_1)_\#(2c(U_1(X))V_1(X)\,dX), && (5.5\text{c}) \\ S(x)\,dx &= (x_2)_\#(-2c(U_2(Y))V_2(Y)\,dY), && (5.5\text{d}) \\ \rho(x)\,dx &= (x_1)_\#(2H_1(X)\,dX), && (5.5\text{e}) \\ \sigma(x)\,dx &= (x_2)_\#(2H_2(Y)\,dY), && (5.5\text{f}) \\ \mu &= (x_1)_\#(J_1'(X)\,dX), && (5.5\text{g}) \\ \nu &= (x_2)_\#(J_2'(Y)\,dY). && (5.5\text{h}) \end{aligned}$$

The relations (5.5c)–(5.5f) are equivalent to

$$\begin{aligned} R(x_1(X))x_1'(X) &= 2c(U_1(X))V_1(X), && (5.6\text{a}) \\ S(x_2(Y))x_2'(Y) &= -2c(U_2(Y))V_2(Y), && (5.6\text{b}) \\ \rho(x_1(X))x_1'(X) &= 2H_1(X), && (5.6\text{c}) \\ \sigma(x_2(Y))x_2'(Y) &= 2H_2(Y), && (5.6\text{d}) \end{aligned}$$

respectively, for almost every X and Y. We denote by $\mathbf{M} : \mathcal{F} \to \mathcal{D}$ the mapping which to any $\psi \in \mathcal{F}$ associates the element $(u, R, S, \rho, \sigma, \mu, \nu) \in \mathcal{D}$ as defined above.

Lemma 5.8 *Given $\psi = (\psi_1, \psi_2) \in \mathcal{F}$, let $(u, R, S, \rho, \sigma, \mu, \nu) = \mathbf{M}(\psi_1, \psi_2)$. Then, for any $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) \in \mathcal{G}_0$ such that $(\psi_1, \psi_2) = \mathbf{D}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q})$, we have*

$$\begin{aligned} u(x) &= \mathcal{Z}_3(s) \quad \text{if } x = \mathcal{Z}_2(s), && (5.7\text{a}) \\ R(x)\,dx &= (\mathcal{Z}_2)_\#(2c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds), && (5.7\text{b}) \\ S(x)\,dx &= (\mathcal{Z}_2)_\#(-2c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds), && (5.7\text{c}) \\ \rho(x)\,dx &= (\mathcal{Z}_2)_\#(2\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds), && (5.7\text{d}) \\ \sigma(x)\,dx &= (\mathcal{Z}_2)_\#(2\mathfrak{q}(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds), && (5.7\text{e}) \\ \mu &= (\mathcal{Z}_2)_\#(\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\,ds), && (5.7\text{f}) \\ \nu &= (\mathcal{Z}_2)_\#(\mathcal{W}_4(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\,ds). && (5.7\text{g}) \end{aligned}$$

[2] The push-forward of a measure λ by a function f is the measure $f_\#\lambda$ defined by $f_\#\lambda(B) = \lambda(f^{-1}(B))$ for Borel sets B.

The relations (5.7b) *and* (5.7d) *are equivalent to*

$$R(\mathcal{Z}_2(s))\mathcal{V}_2(\mathcal{X}(s)) = c(\mathcal{Z}_3(s))\mathcal{V}_3(\mathcal{X}(s)), \tag{5.8a}$$
$$\rho(\mathcal{Z}_2(s))\mathcal{V}_2(\mathcal{X}(s)) = \mathfrak{p}(\mathcal{X}(s)) \tag{5.8b}$$

for any s such that $\dot{\mathcal{X}}(s) > 0$. The relations (5.7c) *and* (5.7e) *are equivalent to*

$$S(\mathcal{Z}_2(s))\mathcal{W}_2(\mathcal{Y}(s)) = -c(\mathcal{Z}_3(s))\mathcal{W}_3(\mathcal{Y}(s)), \tag{5.9a}$$
$$\sigma(\mathcal{Z}_2(s))\mathcal{W}_2(\mathcal{Y}(s)) = \mathfrak{q}(\mathcal{Y}(s)) \tag{5.9b}$$

for any s such that $\dot{\mathcal{Y}}(s) > 0$.

By using the semigroup S_T we can, together with the mappings from $\mathcal{D}$ to $\mathcal{F}$ and vica versa, study the solution in the original set of variables, for given initial data in $\mathcal{D}$.

Lemma 5.9 *Given $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$, let*

$$(u, R, S, \rho, \sigma, \mu, \nu)(T) = \mathbf{M} \circ S_T \circ \mathbf{L}(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$$

and

$$(Z, p, q) = \mathbf{S} \circ \mathbf{C} \circ \mathbf{L}(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0).$$

Then, we have

$$u(t(X, Y), x(X, Y)) = U(X, Y) \tag{5.10}$$

for all $(X, Y) \in \mathbb{R}^2$,

$$R(t(X, Y), x(X, Y))x_X(X, Y) = c(U(X, Y))U_X(X, Y), \tag{5.11a}$$
$$\rho(t(X, Y), x(X, Y))x_X(X, Y) = p(X, Y) \tag{5.11b}$$

for almost every $(X, Y) \in \mathbb{R}^2$ such that $x_X(X, Y) > 0$, and

$$S(t(X, Y), x(X, Y))x_Y(X, Y) = -c(U(X, Y))U_Y(X, Y), \tag{5.12a}$$
$$\sigma(t(X, Y), x(X, Y))x_Y(X, Y) = q(X, Y) \tag{5.12b}$$

for almost every $(X, Y) \in \mathbb{R}^2$ such that $x_Y(X, Y) > 0$. Furthermore, we have

$$u_t = \frac{1}{2}(R + S) \quad \text{and} \quad u_x = \frac{1}{2c(u)}(R - S) \tag{5.13}$$

in the sense of distributions.

5.4 Semigroup of solutions in $\mathcal{D}$

Now we can define a mapping on $\mathcal{D}$, the original set of variables.

Definition 5.10 For any $T > 0$, let $\bar{S}_T : \mathcal{D} \to \mathcal{D}$ be defined as

$$\bar{S}_T = \mathbf{M} \circ S_T \circ \mathbf{L}.$$

Since

$$\bar{S}_T \circ \bar{S}_{T'} = \mathbf{M} \circ S_T \circ \mathbf{L} \circ \mathbf{M} \circ S_{T'} \circ \mathbf{L}$$

it would immediately follow from the semigroup property of S_T that $\bar{S}_T$ is also a semigroup if we had $\mathbf{L} \circ \mathbf{M} = \mathrm{Id}$, but this identity does not hold in general, unless one introduces an equivalence relation based on relabeling. An essential role is played by G, cf. Definition 3.3.

First we define the action of G^2 on the set $\mathcal{F}$.

Definition 5.11 For any $\psi = (\psi_1, \psi_2) \in \mathcal{F}$ and $f, g \in G$, we define $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ as

$$\bar{x}_1(X) = x_1(f(X)), \quad \bar{x}_2(Y) = x_2(g(Y)), \tag{5.14a}$$

$$\bar{U}_1(X) = U_1(f(X)), \quad \bar{U}_2(Y) = U_2(g(Y)), \tag{5.14b}$$

$$\bar{J}_1(X) = J_1(f(X)), \quad \bar{J}_2(Y) = J_2(g(Y)), \tag{5.14c}$$

$$\bar{K}_1(X) = K_1(f(X)), \quad \bar{K}_2(Y) = K_2(g(Y)), \tag{5.14d}$$

$$\bar{V}_1(X) = f'(X)V_1(f(X)), \quad \bar{V}_2(Y) = g'(Y)V_2(g(Y)), \tag{5.14e}$$

$$\bar{H}_1(X) = f'(X)H_1(f(X)), \quad \bar{H}_2(Y) = g'(Y)H_2(g(Y)). \tag{5.14f}$$

The mapping from $\mathcal{F} \times G^2$ to $\mathcal{F}$ given by $\psi \times (f, g) \mapsto \bar{\psi}$ defines an action of the group G^2 on $\mathcal{F}$ and we denote $\bar{\psi} = \psi \cdot (f, g)$.

We refer to [15] for the definition of the action of G^2 on the sets $\mathcal{C}$, $\mathcal{G}$, and $\mathcal{H}$. A key result is the following.

Lemma 5.12 *The mapping S_T is G^2-equivariant, that is, for all $\phi = (f, g) \in G^2$, we have*

$$S_T(\psi \cdot \phi) = S_T(\psi) \cdot \phi$$

for all $\psi \in \mathcal{F}$.

Definition 5.13 We denote by $\mathcal{F}/G^2$ the quotient of $\mathcal{F}$ with respect to the action of the group G^2 on $\mathcal{F}$. More specifically, we define the equivalence relation, $\sim$, on $\mathcal{F}$ as

$$\text{for any } \psi, \bar{\psi} \in \mathcal{F}, \ \psi \sim \bar{\psi} \text{ if there exists } \phi \in G^2 \text{ such that } \bar{\psi} = \psi \cdot \phi.$$

For an element $\psi \in \mathcal{F}$, we denote the equivalence class by

$$[\psi] = \{\bar{\psi} \in \mathcal{F} \mid \bar{\psi} \sim \psi\}.$$

We define the quotient space as

$$\mathcal{F}/G^2 = \{[\psi] \mid \psi \in \mathcal{F}\}.$$

We now define the set $\mathcal{F}_0$, which plays a key role in the following lemma.

Definition 5.14 Let

$$\mathcal{F}_0 = \{\psi = (\psi_1, \psi_2) \in \mathcal{F} \mid x_1 + J_1 = \mathrm{Id} \ \text{and} \ x_2 + J_2 = \mathrm{Id}\}$$

and $\Pi : \mathcal{F} \to \mathcal{F}_0$ be the projection on $\mathcal{F}_0$ given by $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) = \Pi(\psi)$ where $\bar{\psi} \in \mathcal{F}_0$ is defined as follows. Let

$$f(X) = x_1(X) + J_1(X) \quad \text{and} \quad g(Y) = x_2(Y) + J_2(Y) \tag{5.15}$$

and denote $\phi = (f, g) \in G^2$. We set

$$\bar{\psi} = \psi \cdot \phi^{-1}.$$

Lemma 5.15 *The following statements hold:*

(i) *For any ψ and $\bar{\psi}$ in $\mathcal{F}$, we have*

$$\psi \sim \bar{\psi} \quad \textit{if and only if} \quad \Pi(\psi) = \Pi(\bar{\psi}), \tag{5.16a}$$

so that the sets $\mathcal{F}/G^2$ and $\mathcal{F}_0$ are in bijection.

(ii) *We have*

$$\mathbf{M} \circ \Pi = \mathbf{M} \tag{5.16b}$$

and

$$\mathbf{L} \circ \mathbf{M}|_{\mathcal{F}_0} = \mathrm{Id}\,|_{\mathcal{F}_0} \quad \textit{and} \quad \mathbf{M} \circ \mathbf{L} = \mathrm{Id}, \tag{5.16c}$$

so that the sets $\mathcal{D}$, $\mathcal{F}_0$, and $\mathcal{F}/G^2$ are in bijection.

(iii) *We have*

$$\Pi \circ S_T \circ \Pi = \Pi \circ S_T. \tag{5.16d}$$

Note that the first identity in (5.16c) is equivalent to

$$\mathbf{L} \circ \mathbf{M} \circ \Pi = \Pi. \tag{5.17}$$

Now we are finally in position to prove that $\bar{S}_T$ is a semigroup.

Theorem 5.16 *The mapping $\bar{S}_T$ is a semigroup.*

Proof The proof relies on Lemma 5.15 and Theorem 5.6. From Definition 5.10 we have

$$\begin{aligned}
\bar{S}_T \circ \bar{S}_{T'} &= \mathbf{M} \circ S_T \circ \mathbf{L} \circ \mathbf{M} \circ S_{T'} \circ \mathbf{L} \\
&= \mathbf{M} \circ \Pi \circ S_T \circ \mathbf{L} \circ \mathbf{M} \circ \Pi \circ S_{T'} \circ \mathbf{L} \quad \text{by (5.16b)} \\
&= \mathbf{M} \circ \Pi \circ S_T \circ \Pi \circ S_{T'} \circ \mathbf{L} \quad \text{by (5.17)} \\
&= \mathbf{M} \circ \Pi \circ S_T \circ S_{T'} \circ \mathbf{L} \quad \text{by (5.16d).} \\
&= \mathbf{M} \circ S_T \circ S_{T'} \circ \mathbf{L} \quad \text{by (5.16b)} \\
&= \mathbf{M} \circ S_{T+T'} \circ \mathbf{L} \quad \text{by Theorem 5.6} \\
&= \bar{S}_{T+T'}.
\end{aligned}$$

□

6 Existence of weak global conservative solutions

It remains to prove that the solution obtained by using the operator $\bar{S}_T$ is a weak solution of (1.3).

Theorem 6.1 *Let $t > 0$ and $\xi_0 = (u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$. Then $(u, R, S, \rho, \sigma, \mu, \nu)(t) = \bar{S}_t(\xi_0)$ is a weak solution of (1.3), meaning that*

$$\begin{aligned}
&\iint_{(0,\infty)\times\mathbb{R}} \left(\big[\phi_t - c(u)\phi_x\big]R + \big[\phi_t + c(u)\phi_x\big]S + \frac{c'(u)}{c(u)} RS\phi \right) dx\,dt \\
&= \iint_{(0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)} \phi\, d\mu\, dt + \iint_{(0,\infty)\times\mathbb{R}} \frac{2c'(u)}{c(u)} \phi\, d\nu\, dt,
\end{aligned} \tag{6.1a}$$

$$\iint_{(0,\infty)\times\mathbb{R}} \big[\phi_t - c(u)\phi_x\big]\rho\, dx\, dt = 0 \quad \textit{and} \quad \iint_{(0,\infty)\times\mathbb{R}} \big[\phi_t + c(u)\phi_x\big]\sigma\, dx\, dt = 0 \tag{6.1b}$$

for all $\phi = \phi(t, x)$ *in* $C_0^\infty((0, \infty) \times \mathbb{R})$*, where*

$$R = u_t + c(u)u_x \quad \text{and} \quad S = u_t - c(u)u_x \tag{6.1c}$$

in the sense of distributions.

Moreover, the measures μ *and* ν *satisfy the equations*

$$(\mu + \nu)_t - (c(u)(\mu - \nu))_x = 0 \quad \text{and} \quad \left(\frac{1}{c(u)}(\mu - \nu)\right)_t - (\mu + \nu)_x = 0 \tag{6.2}$$

in the sense of distributions.

Note that if the two measures μ and ν are absolutely continuous, (6.2) coincides with (1.12) in the sense of distributions. Moreover, the difference of the sign in front of μ and ν indicates the two opposite traveling directions.

The semigroup of solutions, $\bar{S}_t$, is conservative in the following sense.

Theorem 6.2 *Given* $\xi_0 = (u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$*, let* $(u, R, S, \rho, \sigma, \mu, \nu)(t) = \bar{S}_t(\xi_0)$*. We have:*

(i) *For all* $t \geq 0$*,*

$$\mu(t)(\mathbb{R}) + \nu(t)(\mathbb{R}) = \mu_0(\mathbb{R}) + \nu_0(\mathbb{R}).$$

(ii) *For almost every* $t \geq 0$*, the singular parts of* $\mu(t)$ *and* $\nu(t)$ *are concentrated on the set where* $c'(u) = 0$*.*

Theorem 6.3 (Finite speed of propagation) *For initial data* $\xi_0 = (u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$ *and* $\bar{\xi}_0 = (\bar{u}_0, \bar{R}_0, \bar{S}_0, \bar{\rho}_0, \bar{\sigma}_0, \bar{\mu}_0, \bar{\nu}_0)$ *in* $\mathcal{D}$*, we consider the solutions* $(u, R, S, \rho, \sigma, \mu, \nu)(t) = \bar{S}_t(\xi_0)$ *and* $(\bar{u}, \bar{R}, \bar{S}, \bar{\rho}, \bar{\sigma}, \bar{\mu}, \bar{\nu})(t) = \bar{S}_t(\bar{\xi}_0)$*. Given* $\mathbf{t} > 0$ *and* $\mathbf{x} \in \mathbb{R}$*, if* $\xi_0(x) = \bar{\xi}_0(x)$ *for* $x \in [\mathbf{x} - \kappa\mathbf{t}, \mathbf{x} + \kappa\mathbf{t}]$*, then* $u(\mathbf{t}, \mathbf{x}) = \bar{u}(\mathbf{t}, \mathbf{x})$*.*

7 Regularity of solutions

7.1 Existence of local smooth solutions

Theorem 7.1 *Let* $-\infty < x_l < x_r < \infty$ *and consider* $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$*. Let* $m \in \mathbb{N}$ *and assume that,*

(A1) $u_0 \in L^\infty([x_l, x_r])$*,*
(A2) $R_0, S_0, \rho_0, \sigma_0 \in W^{m-1,\infty}([x_l, x_r])$*,*
(A3) *there are constants* $d > 0$ *and* $e > 0$ *such that* $\rho_0(x) \geq d$ *and* $\sigma_0(x) \geq e$ *for all* $x \in [x_l, x_r]$*,*
(A4) μ_0 *and* ν_0 *are absolutely continuous on* $[x_l, x_r]$*,*
(A5) $c \in C^{m-1}(\mathbb{R})$ *and* $\max_{u\in\mathbb{R}} \left|\frac{d^i}{du^i}c(u)\right| \leq k_i$ *for constants* k_i*,* $i = 3, 4, 5, \ldots, m-1$*.*

For any $\tau \in \left[0, \frac{1}{2\kappa}(x_r - x_l)\right]$ *consider* $(u, R, S, \rho, \sigma, \mu, \nu)(\tau) = \bar{S}_\tau(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$*. Then*

(P1) $u(\tau, \cdot) \in W^{m,\infty}([x_l + \kappa\tau, x_r - \kappa\tau])$*,*
(P2) $R(\tau, \cdot), S(\tau, \cdot), \rho(\tau, \cdot), \sigma(\tau, \cdot) \in W^{m-1,\infty}([x_l + \kappa\tau, x_r - \kappa\tau])$*,*
(P3) *there are constants* $\bar{d} > 0$ *and* $\bar{e} > 0$ *such that* $\rho(\tau, x) \geq \bar{d}$ *and* $\sigma(\tau, x) \geq \bar{e}$ *for all* $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$*,*

(P4) *$\mu(\tau,\cdot)$ and $\nu(\tau,\cdot)$ are absolutely continuous on $[x_l+\kappa\tau, x_r-\kappa\tau]$.*

For $\tau \in \big[-\frac{1}{2\kappa}(x_r-x_l), 0\big]$, the solution satisfies the same properties on the interval $\big[x_l-\kappa\tau, x_r+\kappa\tau\big]$.

Note that since $(u_0)_x = \frac{1}{2c(u_0)}(R_0-S_0)$, it follows from assumptions (A1), (A2) and (A5) that $u_0 \in W^{m,\infty}([x_l,x_r])$.

Specifically, (A4) means that $\mu_0((-\infty,x_l)) = \mu_0((-\infty,x_l])$, $\mu_0((-\infty,x_r)) = \mu_0((-\infty,x_r])$, $\nu_0((-\infty,x_l)) = \nu_0((-\infty,x_l])$, and $\nu_0((-\infty,x_r)) = \nu_0((-\infty,x_r])$.

By (1.5) and (1.6), (A5) holds for $i=0,1,2$.

Proof In the following, we will consider the case $0<\tau\leq \frac{1}{2\kappa}(x_r-x_l)$. The case $-\frac{1}{2\kappa}(x_r-x_l)\leq\tau<0$ can be treated in the same way.

We decompose the proof into three steps.

Step 1. We first consider the case $m=1$.

(i) Consider $(\psi_1,\psi_2)=\mathbf{L}(u_0,R_0,S_0,\rho_0,\sigma_0,\mu_0,\nu_0)$. Since μ_0 is absolutely continuous on $[x_l,x_r]$ we have from (3.9a), $x_1(X)+\mu_0((-\infty,x_1(X)))=X$ for all $x_1(X)\in[x_l,x_r]$. For X_l and X_r satisfying $x_1(X_l)=x_l$ and $x_1(X_r)=x_r$ we have $X_l=x_l+\mu_0((-\infty,x_l))$ and $X_r=x_r+\mu_0((-\infty,x_r))$. Therefore, since x_1 is nondecreasing, we get

$$x_1(X)+\mu_0((-\infty,x_1(X)))=X \tag{7.1}$$

for all $X\in[X_l,X_r]$. Similarly we find by using (3.9b), $x_2(Y)+\nu_0((-\infty,x_2(Y)))=Y$ for all $Y\in[Y_l,Y_r]$, where $x_2(Y_l)=x_l$, $x_2(Y_r)=x_r$, $Y_l=x_l+\nu_0((-\infty,x_l))$, and $Y_r=x_r+\nu_0((-\infty,x_r))$. We define $\Omega=[X_l,X_r]\times[Y_l,Y_r]$. From now on we only consider $(X,Y)\in\Omega$. Rewriting (7.1) yields

$$x_1(X)+\mu_0((-\infty,x_l))+\frac{1}{4}\int_{x_l}^{x_1(X)}(R_0^2+c(u_0)\rho_0^2)(x)\,dx=X.$$

Differentiating implies that

$$x_1'(X)=\frac{4}{(R_0^2+c(u_0)\rho_0^2)\circ x_1(X)+4}. \tag{7.2}$$

Since $R_0,\rho_0\in L^\infty([x_l,x_r])$, we get the lower bound

$$x_1'(X)\geq\frac{4}{||R_0||^2_{L^\infty([x_l,x_r])}+\kappa||\rho_0||^2_{L^\infty([x_l,x_r])}+4}=:d_1>0, \tag{7.3}$$

and since $\rho_0(x)\geq d$, we find the upper bound

$$x_1'(X)\leq\frac{4\kappa}{d^2+4\kappa}. \tag{7.4}$$

Similarly, we find

$$0<e_1:=\frac{4}{||S_0||^2_{L^\infty([x_l,x_r])}+\kappa||\sigma_0||^2_{L^\infty([x_l,x_r])}+4}\leq x_2'(Y)\leq\frac{4\kappa}{e^2+4\kappa}. \tag{7.5}$$

(ii) Let $\Theta=(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})=\mathbf{C}(\psi_1,\psi_2)$. By (3.24) we have $x_1(\mathcal{X}(s))=x_2(\mathcal{Y}(s))$, which after differentiating and using $\mathcal{X}(s)+\mathcal{Y}(s)=2s$, yields

$$\dot{\mathcal{X}}(s)=\frac{2x_2'(\mathcal{Y}(s))}{x_1'(\mathcal{X}(s))+x_2'(\mathcal{Y}(s))}\quad\text{and}\quad\dot{\mathcal{Y}}(s)=\frac{2x_1'(\mathcal{X}(s))}{x_1'(\mathcal{X}(s))+x_2'(\mathcal{Y}(s))}. \tag{7.6}$$

This implies, by (7.3)–(7.5), that

$$\dot{\mathcal{X}}(s) \geq 2e_1\left(\frac{4\kappa}{d^2+4\kappa}+\frac{4\kappa}{e^2+4\kappa}\right)^{-1} \text{ and } \dot{\mathcal{Y}}(s) \geq 2d_1\left(\frac{4\kappa}{d^2+4\kappa}+\frac{4\kappa}{e^2+4\kappa}\right)^{-1} \tag{7.7}$$

for all s such that $(\mathcal{X}(s), \mathcal{Y}(s)) \in \Omega$, that is, for all values of s which satisfy $X_l \leq \mathcal{X}(s) \leq X_r$ and $Y_l \leq \mathcal{Y}(s) \leq Y_r$. Using this together with the identity $\mathcal{X}(s)+\mathcal{Y}(s) = 2s$, we find that (7.7) is valid for all $s \in [s_l, s_r]$, where $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. Hence, $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ are strictly increasing functions on $[s_l, s_r]$.

(iii) Consider $(Z, p, q) = \mathbf{S}(\Theta)$. First we prove that

$$(x_X + J_X)(X, Y) \leq C(\mathcal{V}_2 + \mathcal{V}_4)(X) \tag{7.8}$$

for all $(X, Y) \in \Omega$, where C depends on κ, k_1, $|||\Theta|||_{\mathcal{G}(\Omega)}$, $X_r - X_l$, and $Y_r - Y_l$. From (4.12c), we have $2J_X x_X \geq (c(U)U_X)^2$ and $2J_Y x_Y \geq (c(U)U_Y)^2$. By Young's inequality, we get $|U_X| \leq \frac{\kappa}{\sqrt{2}}(x_X + J_X)$ and $|U_Y| \leq \frac{\kappa}{\sqrt{2}}(x_Y + J_Y)$. This implies, after using (2.19b) and (2.19d), that

$$\begin{aligned} 0 < (x_X + J_X)(X, Y) &\leq (x_X + J_X)(X, \mathcal{Y}(X)) \\ &\quad + \left|\int_{\mathcal{Y}(X)}^{Y} \frac{k_1\kappa^2}{\sqrt{2}}(x_X + J_X)(x_Y + J_Y)(X, \bar{Y})\, d\bar{Y}\right|. \end{aligned}$$

By Gronwall's inequality,

$$(x_X + J_X)(X, Y) \leq (x_X + J_X)(X, \mathcal{Y}(X)) \exp\left\{\frac{k_1\kappa^2}{\sqrt{2}}\left|\int_{\mathcal{Y}(X)}^{Y}(x_Y + J_Y)(X, \tilde{Y})\, d\tilde{Y}\right|\right\}.$$

Since x and J are nondecreasing with respect to both variables, we have

$$\begin{aligned} \left|\int_{\mathcal{Y}(X)}^{Y}(x_Y + J_Y)(X, \tilde{Y})\, d\tilde{Y}\right| &\leq \left|x(X_r, Y_r) - x(X_l, Y_l)\right| + \left|J(X_r, Y_r) - J(X_l, Y_l)\right| \\ &= \left|\mathcal{Z}_2^a(s_r) - \mathcal{Z}_2^a(s_l) + s_r - s_l\right| + \left|\mathcal{Z}_4^a(s_r) - \mathcal{Z}_4^a(s_l)\right| \\ &\leq 2|||\Theta|||_{\mathcal{G}(\Omega)} + \frac{1}{2}(X_r - X_l) + \frac{1}{2}(Y_r - Y_l), \end{aligned}$$

where we used $s_l = \frac{1}{2}(X_l + Y_l)$, $s_r = \frac{1}{2}(X_r + Y_r)$, and the definition of $|||\Theta|||_{\mathcal{G}(\Omega)}$. Defining $C = \exp\left\{\frac{k_1\kappa^2}{\sqrt{2}}(2|||\Theta|||_{\mathcal{G}(\Omega)} + \frac{1}{2}(X_r - X_l) + \frac{1}{2}(Y_r - Y_l))\right\}$ completes the proof of (7.8). Note that C is strictly positive.

We have $\mathcal{V}_2 + \mathcal{V}_4 = \frac{1}{2}x_1' + J_1'$ and since $J_1' = 1 - x_1'$, $x_1' \geq 0$ and $J_1' \geq 0$, this implies that $\frac{1}{2} \leq \mathcal{V}_2 + \mathcal{V}_4 \leq 1$. Thus, we get

$$(x_X + J_X)(X, Y) \leq C. \tag{7.9}$$

By (3.26f), (3.9g), (A3) and (7.3) we obtain

$$p(X, \mathcal{Y}(X)) = \mathfrak{p}(X) = H_1(X) = \frac{1}{2}\rho_0(x_1(X))x_1'(X) \geq \frac{1}{2}dd_1 =: d_2. \tag{7.10}$$

Similarly we obtain

$$q(\mathcal{X}(Y), Y) = \mathfrak{q}(Y) = H_2(Y) = \frac{1}{2}\sigma_0(x_2(Y))x_2'(Y) \geq \frac{1}{2}ee_1 =: e_2.$$

Then, since $p_Y = 0$, we get for all $(X, Y) \in \Omega$,

$$\begin{aligned} d_2^2 &\le p^2(X, \mathcal{Y}(X)) = p^2(X, Y) \le \left(\frac{1}{c(U)}((c(U)U_X)^2 + c(U)p^2)\right)(X, Y) \\ &\le \kappa((c(U)U_X)^2 + c(U)p^2)(X, Y) = 2\kappa(J_X x_X)(X, Y) \le 2\kappa(J_X + x_X)x_X(X, Y) \\ &\le 2\kappa C x_X(X, Y), \end{aligned}$$

where we used (4.12c) and (7.9). Hence,

$$x_X(X, Y) \ge \frac{d_2^2}{2\kappa C}. \tag{7.11}$$

Using that $q_X = 0$, we find in the same way that

$$x_Y(X, Y) \ge \frac{e_2^2}{2\kappa \tilde{C}}, \tag{7.12}$$

where $\tilde{C}$ is strictly positive and depends on $|||\Theta|||_{\mathcal{G}(\Omega)}$, $X_r - X_l$, and $Y_r - Y_l$. From (4.12a), we then get

$$t_X(X, Y) \ge \frac{d_2^2}{2\kappa^2 C} \quad \text{and} \quad t_Y(X, Y) \le -\frac{e_2^2}{2\kappa^2 \tilde{C}}. \tag{7.13}$$

(iv) Let $0 < \tau \le \frac{1}{2\kappa}(x_r - x_l)$ and consider $\Theta(\tau) = \mathbf{E} \circ \mathbf{t}_\tau(Z, p, q)$.

We claim that $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ lies below the curve[3] $(\mathcal{X}(s), \mathcal{Y}(s))$. For any $\bar{s}$ and s such that $\mathcal{X}(\tau, \bar{s}) = \mathcal{X}(s)$ we have $t(\mathcal{X}(s), \mathcal{Y}(s)) = 0 < \tau = t(\mathcal{X}(\tau, \bar{s}), \mathcal{Y}(\tau, \bar{s}))$, so that by (4.12a) and (4.12d),

$$\mathcal{Y}(\tau, \bar{s}) < \mathcal{Y}(s), \tag{7.14}$$

which proves the claim.

We prove that there exist $\bar{s}_{\min}$ and $\bar{s}_{\max}$ satisfying $s_l < \bar{s}_{\min} \le \bar{s}_{\max} < s_r$ such that $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ belongs to Ω for all $s \in [\bar{s}_{\min}, \bar{s}_{\max}]$. First we need an estimate. Since $c(u) \le \kappa$, $t_X(X, Y) > 0$, and $t_Y(X, Y) < 0$, we have

$$\begin{aligned} \frac{1}{2\kappa}(x_r - x_l) &= \frac{1}{2\kappa}(x(X_r, Y_r) - x(X_r, Y_l) + x(X_r, Y_l) - x(X_l, Y_l)) \\ &= \frac{1}{2\kappa}\left(-\int_{Y_l}^{Y_r} c(U(X_r, Y))t_Y(X_r, Y)\,dY + \int_{X_l}^{X_r} c(U(X, Y_l))t_X(X, Y_l)\,dX\right) \\ &\le \frac{1}{2}\left(-\int_{Y_l}^{Y_r} t_Y(X_r, Y)\,dY + \int_{X_l}^{X_r} t_X(X, Y_l)\,dX\right) = t(X_r, Y_l), \end{aligned} \tag{7.15}$$

where we used (4.12a). Consider $\bar{s}_{\max}$ such that $\mathcal{X}(\tau, \bar{s}_{\max}) = X_r$. Note that since $\mathcal{X}(\tau, \bar{s}_{\max}) = \mathcal{X}(s_r) = X_r$, we get from (7.14), $\mathcal{Y}(\tau, \bar{s}_{\max}) < \mathcal{Y}(s_r) = Y_r$. From (7.15) we get $t(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) = \tau \le \frac{1}{2\kappa}(x_r - x_l) \le t(X_r, Y_l)$, so that $\mathcal{Y}(\tau, \bar{s}_{\max}) \ge Y_l$. In particular, we have

$$Y_l \le \mathcal{Y}(\tau, \bar{s}_{\max}) < Y_r. \tag{7.16}$$

Since $\mathcal{X}(\tau, \cdot)$ and $\mathcal{Y}(\tau, \cdot)$ are nondecreasing and continuous, there exists $\bar{s}_{\min}$ such that $\bar{s}_{\min} \le \bar{s}_{\max}$, $\mathcal{Y}(\tau, \bar{s}_{\min}) = Y_l$ and $\mathcal{X}(\tau, \bar{s}_{\min}) \le X_r$. Since $t(X_l, Y_l) = 0 < \tau = t(\mathcal{X}(\tau, \bar{s}_{\min}), Y_l)$

[3] Note that this is also true outside Ω.

we find that $\mathcal{X}(\tau, \bar{s}_{\min}) > X_l$ and we have

$$X_l < \mathcal{X}(\tau, \bar{s}_{\min}) \leq X_r. \tag{7.17}$$

For any $s \in [\bar{s}_{\min}, \bar{s}_{\max}]$ we get from (7.16) and (7.17), since $\mathcal{X}(\tau, \cdot)$ and $\mathcal{Y}(\tau, \cdot)$ are nondecreasing, that

$$\begin{aligned} &X_l < \mathcal{X}(\tau, \bar{s}_{\min}) \leq \mathcal{X}(\tau, s) \leq \mathcal{X}(\tau, \bar{s}_{\max}) = X_r \quad \text{and} \\ &Y_l = \mathcal{Y}(\tau, \bar{s}_{\min}) \leq \mathcal{Y}(\tau, s) \leq \mathcal{Y}(\tau, \bar{s}_{\max}) < Y_r. \end{aligned}$$

In other words, the curve $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ lies in Ω for all $s \in [\bar{s}_{\min}, \bar{s}_{\max}]$, and hence all the estimates obtained in (iii) are valid along this curve. Observe that $s_l = \frac{1}{2}(X_l + Y_l) < \frac{1}{2}\big(\mathcal{X}(\tau, \bar{s}_{\min}) + \mathcal{Y}(\tau, \bar{s}_{\min})\big) = \bar{s}_{\min}$ and $s_r = \frac{1}{2}(X_r + Y_r) > \frac{1}{2}\big(\mathcal{X}(\tau, \bar{s}_{\max}) + \mathcal{Y}(\tau, \bar{s}_{\max})\big) = \bar{s}_{\max}$, which implies $s_l < \bar{s}_{\min} \leq \bar{s}_{\max} < s_r$.

By differentiating $t(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s)) = \tau$ and using that $\dot{\mathcal{X}}(\tau, s) + \dot{\mathcal{Y}}(\tau, s) = 2$, where $\cdot$ denotes the derivative with respect to s, we obtain

$$\dot{\mathcal{X}}(\tau, s) = \frac{-2t_Y(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))}{t_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s)) - t_Y(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))}. \tag{7.18}$$

By (4.5), we have $|t_X(X, Y)| \leq ||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{\kappa}{2}$ and $|t_Y(X, Y)| \leq ||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{\kappa}{2}$ for all $(X, Y) \in \Omega$. Using this and (7.13) in (7.18), we find

$$\dot{\mathcal{X}}(\tau, s) \geq \frac{e_2^2\kappa^{-2}\tilde{C}^{-1}}{||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + ||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \kappa} =: \alpha_1 \tag{7.19}$$

for all $s \in [\bar{s}_{\min}, \bar{s}_{\max}]$. Similarly, one proves that

$$\dot{\mathcal{Y}}(\tau, s) \geq \alpha_2 \tag{7.20}$$

for some positive constant α_2 that depends on d_2, C, κ, $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$, and $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$.

Next, we prove that there exist $\bar{s}_1$ and $\bar{s}_2$ satisfying $\bar{s}_{\min} \leq \bar{s}_1 \leq \bar{s}_2 \leq \bar{s}_{\max}$ such that

$$x(\mathcal{X}(\tau, \bar{s}_1), \mathcal{Y}(\tau, \bar{s}_1)) = x_l + \kappa\tau \quad \text{and} \quad x(\mathcal{X}(\tau, \bar{s}_2), \mathcal{Y}(\tau, \bar{s}_2)) = x_r - \kappa\tau. \tag{7.21}$$

Observe that since x is increasing with respect to both variables this will imply

$$x_l + \kappa\tau \leq x(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s)) \leq x_r - \kappa\tau$$

for all $s \in [\bar{s}_1, \bar{s}_2]$ and in particular, $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ belongs to Ω for all $s \in [\bar{s}_1, \bar{s}_2]$, see Figs. 3 and 4.

By (4.12a), (4.12d), and (1.5), we have

$$x(\mathcal{X}(\tau, \bar{s}_{\min}), Y_l) \leq x(X_l, Y_l) + \kappa \int_{X_l}^{\mathcal{X}(\tau, \bar{s}_{\min})} t_X(X, Y_l)\, dX = x_l + \kappa\tau.$$

Using the lower bound on c, we end up with

$$x_l + \frac{1}{\kappa}\tau \leq x(\mathcal{X}(\tau, \bar{s}_{\min}), Y_l) \leq x_l + \kappa\tau. \tag{7.22}$$

Following the same lines, one obtains

$$x_r - \kappa\tau \leq x(X_r, \mathcal{Y}(\tau, \bar{s}_{\max})) \leq x_r - \frac{1}{\kappa}\tau. \tag{7.23}$$

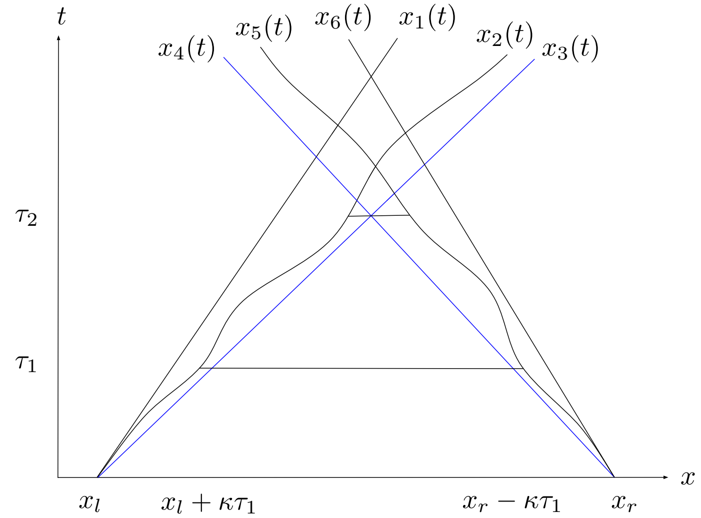

Fig. 3 The region bounded by the characteristics $x_2(t)$ (forward) and $x_5(t)$ (backward) starting from x_l and x_r, respectively, at $t = 0$. Here, $0 < \tau_1 < \tau_2 < \frac{1}{2\kappa}(x_r - x_l)$. The remaining functions are given by $x_1(t) = x_l + \frac{t}{\kappa}$, $x_3(t) = x_l + \kappa t$, $x_4(t) = x_r - \kappa t$ and $x_6(t) = x_r - \frac{t}{\kappa}$. We have $x_3(\tau_1) = x_l + \kappa\tau_1 = x(\mathcal{X}(\tau_1, \bar{s}_1), \mathcal{Y}(\tau_1, \bar{s}_1))$ and $x_4(\tau_1) = x_r - \kappa\tau_1 = x(\mathcal{X}(\tau_1, \bar{s}_2), \mathcal{Y}(\tau_1, \bar{s}_2))$

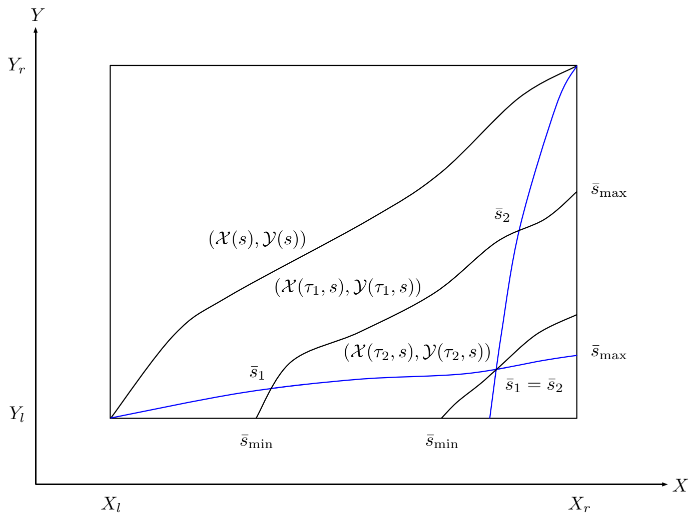

Fig. 4 The region from Fig. 3 in Lagrangian coordinates. The curves $(\mathcal{X}(s), \mathcal{Y}(s))$, $(\mathcal{X}(\tau_1, s), \mathcal{Y}(\tau_1, s))$ and $(\mathcal{X}(\tau_2, s), \mathcal{Y}(\tau_2, s))$ correspond to $t = 0$, $t = \tau_1$ and $t = \tau_2$, respectively

Since $0 < \tau \leq \frac{1}{2\kappa}(x_r - x_l)$ we have $x_l + \kappa\tau = x_l + 2\kappa\tau - \kappa\tau \leq x_l + x_r - x_l - \kappa\tau = x_r - \kappa\tau$, which implies, since x is continuous with respect to both variables, that there exist $\bar{s}_1$ and $\bar{s}_2$ such that $\bar{s}_1 \leq \bar{s}_2$ and

$$x(\mathcal{X}(\tau, \bar{s}_1), \mathcal{Y}(\tau, \bar{s}_1)) = x_l + \kappa\tau \quad \text{and} \quad x(\mathcal{X}(\tau, \bar{s}_2), \mathcal{Y}(\tau, \bar{s}_2)) = x_r - \kappa\tau. \tag{7.24}$$

From (7.22), (7.24), and the fact that x increases along the curve $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$, we have $\mathcal{X}(\tau, \bar{s}_{\min}) \leq \mathcal{X}(\tau, \bar{s}_1)$ which implies $\bar{s}_{\min} \leq \bar{s}_1$. By (7.23) and (7.24) we get $\mathcal{Y}(\tau, \bar{s}_2) \leq \mathcal{Y}(\tau, \bar{s}_{\max})$ and $\bar{s}_2 \leq \bar{s}_{\max}$. This concludes the proof of (7.21).

We prove (P1). From (5.7a) we have $u(\tau, x) = \mathcal{Z}_3(\tau, s)$ if $x = \mathcal{Z}_2(\tau, s)$. Since the function $\mathcal{Z}_2(\tau, s)$ is nondecreasing, the smallest and biggest value it can attain for $s \in [\bar{s}_1, \bar{s}_2]$ are

$$\mathcal{Z}_2(\tau, \bar{s}_1) = x(\mathcal{X}(\tau, \bar{s}_1), \mathcal{Y}(\tau, \bar{s}_1)) = x_l + \kappa\tau \quad \text{and} \quad \mathcal{Z}_2(\tau, \bar{s}_2) = x(\mathcal{X}(\tau, \bar{s}_2), \mathcal{Y}(\tau, \bar{s}_2)) = x_r - \kappa\tau,$$

respectively. The function $\mathcal{Z}_2(\tau, \cdot)$ is in fact strictly increasing for $s \in [\bar{s}_1, \bar{s}_2]$, as we now show. We differentiate the relation $\mathcal{Z}_2(\tau, s) = x(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ and get

$$\dot{\mathcal{Z}}_2(\tau, s) = x_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(\tau, s) + x_Y(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{Y}}(\tau, s).$$

From (7.11), (7.12), (7.19) and (7.20) we have

$$\dot{\mathcal{Z}}_2(\tau, s) \geq (2\kappa)^{-1}(\alpha_1 d_2^2 C^{-1} + \alpha_2 e_2^2 \tilde{C}^{-1}) > 0. \tag{7.25}$$

Hence, the mapping $s \mapsto \mathcal{Z}_2(\tau, s)$ is strictly increasing for $s \in [\bar{s}_1, \bar{s}_2]$ and therefore invertible on $[\bar{s}_1, \bar{s}_2]$. For any $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$ we get

$$u(\tau, x) = \mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x)), \tag{7.26}$$

where $\mathcal{Z}_2^{-1}(\tau, x)$ denotes the inverse of the function $s \mapsto \mathcal{Z}_2(\tau, s)$. Since

$$|\mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x))| = |U(\mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau, x)), \mathcal{Y}(\tau, \mathcal{Z}_2^{-1}(\tau, x)))| \leq ||U||_{L^\infty(\Omega)}$$

we have

$$u(\tau, \cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau]). \tag{7.27}$$

Next, we differentiate (7.26) and get

$$u_x(\tau, x) = \dot{\mathcal{Z}}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x))\frac{d}{dx}\mathcal{Z}_2^{-1}(\tau, x) = \frac{\dot{\mathcal{Z}}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x))}{\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau, x))}. \tag{7.28}$$

We have

$$\begin{aligned} |\dot{\mathcal{Z}}_3(\tau, s)| &= |U_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{X}}(\tau, s) + U_Y(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))\dot{\mathcal{Y}}(\tau, s)| \\ &\leq 2||U_X||_{W_Y^{1,\infty}(\Omega)} + 2||U_Y||_{W_X^{1,\infty}(\Omega)}, \end{aligned}$$

so that $\dot{\mathcal{Z}}_3(\tau, \cdot) \in L^\infty([\bar{s}_1, \bar{s}_2])$. By (7.25) we end up with

$$|u_x(\tau, x)| \leq \frac{2\kappa||\dot{\mathcal{Z}}_3(\tau, \cdot)||_{L^\infty([\bar{s}_1, \bar{s}_2])}}{\alpha_1 d_2^2 C^{-1} + \alpha_2 e_2^2 \tilde{C}^{-1}},$$

which implies that

$$u_x(\tau, \cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau]). \tag{7.29}$$

From (7.27) and (7.29) we conclude that (P1) holds.

We prove (P2). From (5.8a) and (5.8b), we have

$$R(\tau, \mathcal{Z}_2(\tau, s))\mathcal{V}_2(\tau, \mathcal{X}(\tau, s)) = c(\mathcal{Z}_3(\tau, s))\mathcal{V}_3(\tau, \mathcal{X}(\tau, s))$$

and

$$\rho(\tau, \mathcal{Z}_2(\tau, s))\mathcal{V}_2(\tau, \mathcal{X}(\tau, s)) = \mathfrak{p}(\tau, \mathcal{X}(\tau, s))$$

for all $s \in [\bar{s}_1, \bar{s}_2]$. We multiply these equations by $2\dot{\mathcal{X}}(\tau, s)$ and use (3.22) to get

$$R(\tau, \mathcal{Z}_2(\tau, s))\dot{\mathcal{Z}}_2(\tau, s) = 2c(\mathcal{Z}_3(\tau, s))\mathcal{V}_3(\tau, \mathcal{X}(\tau, s))\dot{\mathcal{X}}(\tau, s)$$

and

$$\rho(\tau, \mathcal{Z}_2(\tau, s))\dot{\mathcal{Z}}_2(\tau, s) = 2\mathfrak{p}(\tau, \mathcal{X}(\tau, s))\dot{\mathcal{X}}(\tau, s),$$

which yields

$$R(\tau, x) = \frac{2c(\mathcal{Z}_3(\tau, \mathcal{Z}_2^{-1}(\tau, x)))\mathcal{V}_3(\tau, \mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau, x)))\dot{\mathcal{X}}(\tau, \mathcal{Z}_2^{-1}(\tau, x))}{\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau, x))} \tag{7.30}$$

and

$$\rho(\tau, x) = \frac{2\mathfrak{p}(\tau, \mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau, x)))\dot{\mathcal{X}}(\tau, \mathcal{Z}_2^{-1}(\tau, x))}{\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau, x))} \tag{7.31}$$

for all $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$. Since $\mathcal{V}_3(\tau, \mathcal{X}(\tau, s)) = U_X(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$, we have $|\mathcal{V}_3(\tau, \mathcal{X}(\tau, s))| \leq ||U_X||_{W_Y^{1,\infty}(\Omega)}$ or in other words $\mathcal{V}_3(\tau, \mathcal{X}(\tau, \cdot)) \in L^\infty([\bar{s}_1, \bar{s}_2])$, and since $\mathfrak{p}(\tau, \mathcal{X}(\tau, s)) = p(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s))$ it follows that $\mathfrak{p}(\tau, \mathcal{X}(\tau, \cdot)) \in L^\infty([\bar{s}_1, \bar{s}_2])$. Using (7.25) in (7.30) and (7.31) we obtain

$$|R(\tau, x)| \leq \frac{8\kappa^2 ||\mathcal{V}_3(\tau, \mathcal{X}(\tau, \cdot))||_{L^\infty([\bar{s}_1, \bar{s}_2])}}{\alpha_1 d_2^2 C^{-1} + \alpha_2 e_2^2 \tilde{C}^{-1}} \quad \text{and} \quad |\rho(\tau, x)| \leq \frac{8\kappa ||\mathfrak{p}(\tau, \mathcal{X}(\tau, \cdot))||_{L^\infty([\bar{s}_1, \bar{s}_2])}}{\alpha_1 d_2^2 C^{-1} + \alpha_2 e_2^2 \tilde{C}^{-1}}$$

for all $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$. Therefore $R(\tau, \cdot), \rho(\tau, \cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$. In a similar way one shows that $S(\tau, \cdot), \sigma(\tau, \cdot) \in L^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$ and we have proved (P2).

We prove (P3). By inserting

$$\dot{\mathcal{Z}}_2(\tau, \mathcal{Z}_2^{-1}(\tau, x)) = 2x_X(\mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau, x)), \mathcal{Y}(\tau, \mathcal{Z}_2^{-1}(\tau, x)))\dot{\mathcal{X}}(\tau, \mathcal{Z}_2^{-1}(\tau, x))$$

into (7.31) we get

$$\rho(\tau, x) = \frac{\mathfrak{p}(\tau, \mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau, x)))}{x_X(\mathcal{X}(\tau, \mathcal{Z}_2^{-1}(\tau, x)), \mathcal{Y}(\tau, \mathcal{Z}_2^{-1}(\tau, x)))}$$

for all $s \in [\bar{s}_1, \bar{s}_2]$. Since $p_Y = 0$ we get from (7.10),

$$\mathfrak{p}(\tau, X) = p(X, \mathcal{Y}(\tau, \mathcal{X}^{-1}(\tau, X))) = p(X, \mathcal{Y}(X)) \geq d_2$$

for all $X \in [X_l, X_r]$. Recalling (7.9) and that $J_X(X, Y) \geq 0$ we get $\rho(\tau, x) \geq d_2 C^{-1}$ for all $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$. Similarly we find $\sigma(\tau, x) \geq e_2 \tilde{C}^{-1}$ for all $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$. This concludes the proof of (P3).

We prove (P4). Let $M \subset [x_l + \kappa\tau, x_r - \kappa\tau]$ be a Borel set. By an argument as in [12, Proof of Lemma 11], we have

$$\mu_{\text{sing}}(\tau, M) = \int_{\mathcal{Z}_2^{-1}(\tau, M) \cap A^c} \mathcal{V}_4(\tau, \mathcal{X}(\tau, s))\dot{\mathcal{X}}(\tau, s)\, ds,$$

where $A = \{s \in \mathbb{R} \mid \mathcal{V}_2(\tau, \mathcal{X}(\tau, s)) > 0\}$. We have $\text{meas}(A^c) = 0$, which implies that $\mu_{\text{sing}}(\tau, M) \leq 2|||\Theta(\tau)|||_{\mathcal{G}(\Omega)} \, \text{meas}(A^c) = 0$. This proves that $\mu(\tau)$ is absolutely continuous on $[x_l + \kappa\tau, x_r - \kappa\tau]$. Similarly, one shows that $\nu(\tau)$ is absolutely continuous on $[x_l + \kappa\tau, x_r - \kappa\tau]$.

Step 2. Assume that $m = 2$. By Step 1, (P3) and (P4) hold. Moreover, (P1) and (P2) hold for $m = 1$. It remains to prove that $u_{xx}(\tau, \cdot), R_x(\tau, \cdot), S_x(\tau, \cdot), \rho_x(\tau, \cdot), \sigma_x(\tau, \cdot) \in L^\infty([x_l +$

$\kappa\tau, x_r - \kappa\tau]$). In order to do so, we have to show that $Z_{XX} = (t_{XX}, x_{XX}, U_{XX}, J_{XX}, K_{XX})$ and $Z_{YY} = (t_{YY}, x_{YY}, U_{YY}, J_{YY}, K_{YY})$ exist and are bounded. We first consider Z_{XX}, which is the unique solution of the semilinear system $Z_{XXY}(X, Y) = f(X, Y, Z_{XX})$, where f is Lipschitz continuous with respect to the Z_{XX} variable. This can be seen by differentiating the governing equations (2.19). For instance, we have

$$\begin{aligned} x_{XXY} &= \frac{c'(U)}{2c(U)}(x_X U_{XY} - U_X x_{XY}) + \frac{c''(U)}{2c(U)} U_X (U_X x_Y + U_Y x_X) \\ &\quad + \frac{c'(U)}{2c(U)}(x_Y U_{XX} + U_Y x_{XX}) \end{aligned} \tag{7.32}$$

and

$$|x_{XX}(X, Y)| \leq |x_{XX}(X, \mathcal{Y}(X))| + \int_Y^{\mathcal{Y}(X)} |x_{XXY}(X, \tilde{Y})|\, d\tilde{Y}, \tag{7.33}$$

if we assume without loss of generality that $Y \leq \mathcal{Y}(X)$. The other case is similar.

Let us find a bound on x_{XX} at time $\tau = 0$. We differentiate (7.2) and get

$$x_1''(X) = -\frac{1}{4} x_1'(X)^3 (2R_0 R_{0x} + c'(u_0) u_{0x} \rho_0^2 + 2c(u_0)\rho_0 \rho_{0x}) \circ x_1(X),$$

which, by (7.4), implies that

$$\begin{aligned} |x_1''(X)| &\leq \frac{1}{4}\left(\frac{4\kappa}{d^2 + 4\kappa}\right)^3 (2||R_0||_{L^\infty([x_l, x_r])} ||(R_0)_x||_{L^\infty([x_l, x_r])} \\ &\quad + k_1 ||(u_0)_x||_{L^\infty([x_l, x_r])} ||\rho_0||^2_{L^\infty([x_l, x_r])} + 2\kappa ||\rho_0||_{L^\infty([x_l, x_r])} ||(\rho_0)_x||_{L^\infty([x_l, x_r])}) \end{aligned} \tag{7.34}$$

and we conclude that $x_1'' \in L^\infty([X_l, X_r])$.

By (3.26b), we have $x_X(X, \mathcal{Y}(X)) = \mathcal{V}_2(X) = \frac{1}{2} x_1'(X)$. We differentiate and get

$$x_{XX}(X, \mathcal{Y}(X)) + x_{XY}(X, \mathcal{Y}(X)) \left(\frac{\dot{\mathcal{Y}}}{\dot{\mathcal{X}}}\right) \circ \mathcal{X}^{-1}(X) = \frac{1}{2} x_1''(X),$$

so that by (7.19),

$$\begin{aligned} |x_{XX}(X, \mathcal{Y}(X))| &\leq \frac{1}{2}|x_1''(X)| + |x_{XY}(X, \mathcal{Y}(X))| \left(\frac{\dot{\mathcal{Y}}}{\dot{\mathcal{X}}}\right) \circ \mathcal{X}^{-1}(X) \\ &\leq \frac{1}{2}||x_1''||_{L^\infty([X_l, X_r])} + \frac{k_1 \kappa}{\alpha_1}\left(|||\Theta|||^2_{\mathcal{G}(\Omega)} + \frac{1}{2}|||\Theta|||_{\mathcal{G}(\Omega)}\right) \end{aligned}$$

and $x_{XX}(\cdot, \mathcal{Y}(\cdot)) \in L^\infty([X_l, X_r])$. Here we used the estimate

$$\begin{aligned} |x_{XY}(\mathcal{X}(s), \mathcal{Y}(s))| &= \left|\frac{c'(\mathcal{Z}_3(s))}{2c(\mathcal{Z}_3(s))}(\mathcal{V}_3(\mathcal{X}(s))\mathcal{W}_2(\mathcal{Y}(s)) + \mathcal{W}_3(\mathcal{Y}(s))\mathcal{V}_2(\mathcal{X}(s)))\right| \quad \text{by (2.19)} \\ &\leq \frac{1}{2} k_1 \kappa \Bigg(||\mathcal{V}_3^a||_{L^\infty([X_l, X_r])} \left(||\mathcal{W}_2^a||_{L^\infty([Y_l, Y_r])} + \frac{1}{2}\right) \\ &\quad + ||\mathcal{W}_3^a||_{L^\infty([Y_l, Y_r])} \left(||\mathcal{V}_2^a||_{L^\infty([X_l, X_r])} + \frac{1}{2}\right)\Bigg) \\ &\leq \frac{1}{2} k_1 \kappa \left(|||\Theta|||^2_{\mathcal{G}(\Omega)} + \frac{1}{2}|||\Theta|||_{\mathcal{G}(\Omega)}\right). \end{aligned}$$

We estimate x_{XXY}. Since $|Z_X^a| \leq ||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$ and $|Z_Y^a| \leq ||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, we get from (2.19) that

$$|Z_{XY}| \leq \eta, \tag{7.35}$$

where η depends on $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$, $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, κ, and k_1. We obtain from (7.32),

$$\begin{aligned}
|x_{XXY}| \leq\; & k_1\kappa\left(||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{1}{2}\right)\eta \\
& + k_2\kappa||Z_X^a||_{W_Y^{1,\infty}(\Omega)}\left(||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{1}{2}\right)\left(||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{1}{2}\right) \\
& + \frac{1}{2}k_1\kappa\left(||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{1}{2}\right)(|U_{XX}| + |x_{XX}|).
\end{aligned} \tag{7.36}$$

We insert (7.36) into (7.33) and get

$$\begin{aligned}
|x_{XX}(X,Y)| \leq\; & ||x_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])} + \Bigg[k_1\kappa\left(||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{1}{2}\right)\eta \\
& + k_2\kappa||Z_X^a||_{W_Y^{1,\infty}(\Omega)}\left(||Z_X^a||_{W_Y^{1,\infty}(\Omega)} + \frac{1}{2}\right)\left(||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{1}{2}\right)\Bigg]|Y_r - Y_l| \\
& + \int_Y^{\mathcal{Y}(X)} \frac{1}{2}k_1\kappa\left(||Z_Y^a||_{W_X^{1,\infty}(\Omega)} + \frac{1}{2}\right)(|U_{XX}| + |x_{XX}|)(X,\tilde{Y})\,d\tilde{Y}.
\end{aligned}$$

Following the same lines for the other components of Z_{XX}, we obtain

$$\begin{aligned}
&(|t_{XX}| + |x_{XX}| + |U_{XX}| + |J_{XX}| + |K_{XX}|)(X,Y) \\
&\quad \leq ||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])} + C_1|Y_r - Y_l| \\
&\qquad + \int_Y^{\mathcal{Y}(X)} C_2(|t_{XX}| + |x_{XX}| + |U_{XX}| + |J_{XX}| + |K_{XX}|)(X,\tilde{Y})\,d\tilde{Y},
\end{aligned}$$

where C_1 and C_2 depend on $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$, $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, κ, k_1, and k_2. By Gronwall's lemma, we obtain

$$\begin{aligned}
&(|t_{XX}| + |x_{XX}| + |U_{XX}| + |J_{XX}| + |K_{XX}|)(X,Y) \\
&\quad \leq (||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])} + C_1|Y_r - Y_l|)e^{C_2|Y-\mathcal{Y}(X)|}.
\end{aligned} \tag{7.37}$$

A similar procedure yields

$$\begin{aligned}
&(|t_{YY}| + |x_{YY}| + |U_{YY}| + |J_{YY}| + |K_{YY}|)(X,Y) \\
&\quad \leq (||Z_{YY}(\mathcal{X}(\cdot),\cdot)||_{L^\infty([Y_l,Y_r])} + \tilde{C}_1|X_r - X_l|)e^{\tilde{C}_2|X-\mathcal{X}(Y)|},
\end{aligned} \tag{7.38}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ depend on $||Z_X^a||_{W_Y^{1,\infty}(\Omega)}$, $||Z_Y^a||_{W_X^{1,\infty}(\Omega)}$, κ, k_1, and k_2.

We also need estimates for $\ddot{\mathcal{X}}(\tau,s)$ and $\ddot{\mathcal{Y}}(\tau,s)$. By (3.21), we have $x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s) = x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)$. We differentiate and get

$$\begin{aligned}
&x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)^2 + x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{X}}(\tau,s) \\
&\quad = x_{YY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)^2 + x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{Y}}(\tau,s).
\end{aligned} \tag{7.39}$$

Since $\mathcal{X}(\tau,s)+\mathcal{Y}(\tau,s)=2s$, we have $\ddot{\mathcal{Y}}(\tau,s)=-\ddot{\mathcal{X}}(\tau,s)$, so that

$$\ddot{\mathcal{X}}(\tau,s)=\frac{x_{YY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)^2-x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)^2}{x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))+x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))}.$$

By (7.11), (7.12), (7.37), and (7.38), we find

$$\begin{aligned}|\ddot{\mathcal{X}}(\tau,s)|&\leq 4\bigg(\frac{|x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))|+|x_{YY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))|}{x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))+x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))}\bigg)\\&\leq\frac{8\kappa}{d_2^2C^{-1}+e_2^2\tilde{C}^{-1}}\Big((||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+C_1|Y_r-Y_l|)e^{C_2|Y-\mathcal{Y}(X)|}\\&\quad+(||Z_{YY}(\mathcal{X}(\cdot),\cdot)||_{L^\infty([Y_l,Y_r])}+\tilde{C}_1|X_r-X_l|)e^{\tilde{C}_2|X-\mathcal{X}(Y)|}\Big).\end{aligned}\tag{7.40}$$

Thus, we conclude that $\ddot{\mathcal{X}}(\tau,\cdot)\in L^\infty([\bar{s}_1,\bar{s}_2])$.

We differentiate $\mathcal{Z}_2(\tau,s)=x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ twice and get, by (7.39), that

$$\begin{aligned}\ddot{\mathcal{Z}}_2(\tau,s)&=2x_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)^2+2x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\ddot{\mathcal{X}}(\tau,s)\\&\quad+2x_{XY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)\dot{\mathcal{Y}}(\tau,s).\end{aligned}$$

Using (7.35) and (7.37), we end up with

$$\begin{aligned}|\ddot{\mathcal{Z}}_2(\tau,s)|&\leq 8(||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])}+C_1|Y_r-Y_l|)e^{C_2|Y-\mathcal{Y}(X)|}\\&\quad+2\bigg(||Z_X^a||_{W_Y^{1,\infty}(\Omega)}+\frac{1}{2}\bigg)||\ddot{\mathcal{X}}(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}+8\eta\end{aligned}$$

and $\ddot{\mathcal{Z}}_2(\tau,\cdot)\in L^\infty([\bar{s}_1,\bar{s}_2])$.

Following the same lines we prove that $\ddot{\mathcal{Z}}_3(\tau,\cdot)\in L^\infty([\bar{s}_1,\bar{s}_2])$.

We compute u_{xx} from (7.28) and get

$$u_{xx}(\tau,x)=\frac{\ddot{\mathcal{Z}}_3(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))^2}-\frac{\dot{\mathcal{Z}}_3(\tau,\mathcal{Z}_2^{-1}(\tau,x))\ddot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))^3}.$$

By (7.25) we obtain

$$|u_{xx}(\tau,x)|\leq\frac{4\kappa^2||\ddot{\mathcal{Z}}_3(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\big(\alpha_1d_2^2C^{-1}+\alpha_2e_2^2\tilde{C}^{-1}\big)^2}+\frac{8\kappa^3||\dot{\mathcal{Z}}_3(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{Z}}_2(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\big(\alpha_1d_2^2C^{-1}+\alpha_2e_2^2\tilde{C}^{-1}\big)^3},$$

and we conclude that $u_{xx}(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$.

By differentiating (7.30) we get

$$\begin{aligned}R_x(\tau,x)&=\frac{2\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\big[\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]^2}\big[c'(\mathcal{Z}_3(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{Z}}_3(\tau,\mathcal{Z}_2^{-1}(\tau,x))\mathcal{V}_3(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\\&\quad+c(\mathcal{Z}_3(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{V}}_3(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\big]\\&\quad+\frac{2c(\mathcal{Z}_3(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\mathcal{V}_3(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\ddot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\big[\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]^2}\\&\quad-\frac{2c(\mathcal{Z}_3(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\mathcal{V}_3(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))\ddot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\big[\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]^3},\end{aligned}$$

where we denote $\dot{\mathcal{V}}_3(\tau,\mathcal{X}(\tau,s))=\frac{d}{ds}\mathcal{V}_3(\tau,\mathcal{X}(\tau,s))$.

Since $\mathcal{V}_3(\tau,\mathcal{X}(\tau,s))=U_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$, we have

$$\dot{\mathcal{V}}_3(\tau,\mathcal{X}(\tau,s))=U_{XX}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)+U_{XY}(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s).$$

From (7.35) and (7.37) we obtain

$$|\dot{\mathcal{V}}_3(\tau,\mathcal{X}(\tau,s))| \le 2\big[(||Z_{XX}(\cdot,\mathcal{Y}(\cdot))||_{L^\infty([X_l,X_r])} + C_1|Y_r - Y_l|)e^{C_2|Y-\mathcal{Y}(X)|} + \eta\big].$$

Therefore we have $\dot{\mathcal{V}}_3(\tau,\mathcal{X}(\tau,\cdot)) \in L^\infty([\bar{s}_1,\bar{s}_2])$. By (7.25) we get

$$\begin{aligned}|R_x(\tau,x)| \le{}& \frac{8\kappa^2}{\big(\alpha_1 d_2^2 C^{-1} + \alpha_2 e_2^2 \tilde{C}^{-1}\big)^2}\\ &\times \Big(2k_1||\dot{\mathcal{Z}}_3(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\mathcal{V}_3(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}\\ &+ 2\kappa||\dot{\mathcal{V}}_3(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}\\ &+ \kappa||\mathcal{V}_3(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{X}}(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}\Big)\\ &+ \frac{32\kappa^4||\mathcal{V}_3(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{Z}}_2(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\big(\alpha_1 d_2^2 C^{-1} + \alpha_2 e_2^2 \tilde{C}^{-1}\big)^3},\end{aligned}$$

which implies that $R_x(\tau,\cdot) \in L^\infty([x_l+\kappa\tau, x_r-\kappa\tau])$.

We differentiate (7.31) and get

$$\begin{aligned}\rho_x(\tau,x) ={}& \frac{2\big[\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))^2 + \mathfrak{p}(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\ddot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]}{\big[\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]^2}\\ &- \frac{2\mathfrak{p}(\tau,\mathcal{X}(\tau,\mathcal{Z}_2^{-1}(\tau,x)))\dot{\mathcal{X}}(\tau,\mathcal{Z}_2^{-1}(\tau,x))\ddot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))}{\big[\dot{\mathcal{Z}}_2(\tau,\mathcal{Z}_2^{-1}(\tau,x))\big]^3},\end{aligned}$$

where we denote $\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,s)) = \frac{d}{ds}\mathfrak{p}(\tau,\mathcal{X}(\tau,s))$. Since $\mathfrak{p}(\tau,\mathcal{X}(\tau,s)) = p(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ and $p_Y(X,Y) = 0$ we have

$$\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,s)) = p_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s).$$

Furthermore, by (3.26f) and (3.9g), we have

$$p(X,Y) = p(X,\mathcal{Y}(X)) = \mathfrak{p}(X) = H_1(X) = \frac{1}{2}\rho_0(x_1(X))x_1'(X). \tag{7.41}$$

We differentiate and get, since $p_Y(X,Y) = 0$,

$$p_X(X,Y) = p_X(X,\mathcal{Y}(X)) = \frac{1}{2}(\rho_0)_x(x_1(X))x_1'(X)^2 + \frac{1}{2}\rho_0(x_1(X))x_1''(X).$$

Using (7.4), this leads to the estimate

$$|p_X(X,Y)| \le \frac{1}{2}||(\rho_0)_x||_{L^\infty([x_l,x_r])}\left(\frac{4\kappa}{d^2+4\kappa}\right)^2 + \frac{1}{2}||\rho_0||_{L^\infty([x_l,x_r])}||x_1''||_{L^\infty([X_l,X_r])},$$

which implies that

$$|\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,s))| \le ||(\rho_0)_x||_{L^\infty([x_l,x_r])}\left(\frac{4\kappa}{d^2+4\kappa}\right)^2 + ||\rho_0||_{L^\infty([x_l,x_r])}||x_1''||_{L^\infty([X_l,X_r])}.$$

Recalling (7.34), it follows that $\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,\cdot)) \in L^\infty([\bar{s}_1,\bar{s}_2])$. Using (7.25), we end up with

$$|\rho_x(\tau,x)| \le \frac{8\kappa^2\big(4||\dot{\mathfrak{p}}(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])} + ||\mathfrak{p}(\tau,\mathcal{X}(\tau,\cdot))||_{L^\infty([\bar{s}_1,\bar{s}_2])}||\ddot{\mathcal{X}}(\tau,\cdot)||_{L^\infty([\bar{s}_1,\bar{s}_2])}\big)}{\big(\alpha_1 d_2^2 C^{-1} + \alpha_2 e_2^2 \tilde{C}^{-1}\big)^2}$$

$$+\frac{32\kappa^3\|\mathfrak{p}(\tau,\mathcal{X}(\tau,\cdot))\|_{L^\infty([\bar{s}_1,\bar{s}_2])}\|\ddot{\mathcal{Z}}_2(\tau,\cdot)\|_{L^\infty([\bar{s}_1,\bar{s}_2])}}{\left(\alpha_1 d_2^2 C^{-1}+\alpha_2 e_2^2\tilde{C}^{-1}\right)^3}$$

and we conclude that $\rho_x(\tau,\cdot)\in L^\infty([x_l+\kappa\tau, x_r-\kappa\tau])$.

In a similar way one shows that $S_x(\tau,\cdot),\sigma_x(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$.

Step 3. Assume that the result holds for $m=n$, that is, if $u_0\in L^\infty([x_l,x_r])$ and $R_0,S_0,\rho_0,\sigma_0\in W^{n-1,\infty}([x_l,x_r])$, then $u(\tau,\cdot)\in W^{n,\infty}([x_l+\kappa\tau,x_r-\kappa\tau])$, $R(\tau,\cdot)$, $S(\tau,\cdot)$, $\rho(\tau,\cdot)$, $\sigma(\tau,\cdot)\in W^{n-1,\infty}([x_l+\kappa\tau,x_r-\kappa\tau])$ and (P3) and (P4) hold. We show by induction that the result also holds for $m=n+1$. Note that it is enough to show that if R_0,S_0,ρ_0,σ_0 belong to $W^{n,\infty}([x_l,x_r])$, then $u(\tau,\cdot)\in W^{n+1,\infty}([x_l+\kappa\tau,x_r-\kappa\tau])$ and $R(\tau,\cdot),S(\tau,\cdot),\rho(\tau,\cdot),\sigma(\tau,\cdot)\in W^{n,\infty}([x_l+\kappa\tau,x_r-\kappa\tau])$.

Since the result holds for $m=n$ we get, following closely the argument used in Step 2 to derive (7.37) and (7.38), that

$$\frac{\partial^{\alpha+\beta}}{\partial X^\alpha\partial Y^\beta}Z\in[L^\infty(\Omega)]^5,\quad \alpha,\beta=0,1,\dots,n,\quad \alpha+\beta\le n. \tag{7.42}$$

Since $R_0,S_0,\rho_0,\sigma_0\in W^{n,\infty}([x_l,x_r])$ we get by Definitions 3.5 and 3.7 that $\frac{\partial^{\alpha+\beta}}{\partial X^\alpha\partial Y^\beta}Z$, $\alpha,\beta=0,1,\dots,n+1,\alpha+\beta=n+1$ is bounded on the curve $(\mathcal{X}(s),\mathcal{Y}(s))$, $s\in[s_l,s_r]$.

Since the governing Eq. (2.19) is semilinear, there exists a unique solution $\frac{\partial^{n+1}}{\partial X^\alpha\partial Y^\beta}Z$, $\alpha,\beta=0,1,\dots,n+1,\alpha+\beta=n+1$ in Ω of the system

$$\begin{aligned}\frac{\partial}{\partial Y}&\sum_{\substack{\alpha,\beta=0,1,\dots,n+1\\ \alpha+\beta=n+1}}\frac{\partial^{n+1}}{\partial X^\alpha\partial Y^\beta}(t+x+U+J+K)\\ &=f+\sum_{\substack{\alpha,\beta=0,1,\dots,n+1\\ \alpha+\beta=n+1}}\left\langle g_{\alpha,\beta},\frac{\partial^{n+1}}{\partial X^\alpha\partial Y^\beta}Z\right\rangle,\end{aligned}\tag{7.43}$$

where f and $g_{\alpha,\beta}$ depend on derivatives up to order n and $g_{\alpha,\beta}$ denotes $n+1$ five dimensional vectors. By (7.42), the functions f and $g_{\alpha,\beta}$ are bounded. To clarify the notation, let us compute (7.43) for $n=2$. We have

$$\begin{aligned}\frac{\partial}{\partial Y}&\sum_{\substack{\alpha,\beta=0,1,2,3\\ \alpha+\beta=3}}\frac{\partial^3}{\partial X^\alpha\partial Y^\beta}(t+x+U+J+K)\\ &=f+\langle g_{3,0},Z_{XXX}\rangle+\langle g_{2,1},Z_{XXY}\rangle+\langle g_{1,2},Z_{XYY}\rangle+\langle g_{0,3},Z_{YYY}\rangle.\end{aligned}$$

By Gronwall's lemma, we obtain $\frac{\partial^{n+1}}{\partial X^\alpha\partial Y^\beta}Z\in[L^\infty(\Omega)]^5$, $\alpha,\beta=0,1,\dots,n+1$, $\alpha+\beta=n+1$. This implies, since $x_X(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{X}}(\tau,s)=x_Y(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))\dot{\mathcal{Y}}(\tau,s)$, $\mathcal{Z}_2(\tau,s)=x(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$, and $\mathcal{Z}_3(\tau,s)=U(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$, that $\frac{d^{n+1}}{ds^{n+1}}\mathcal{X}(\tau,\cdot)$, $\frac{d^{n+1}}{ds^{n+1}}\mathcal{Y}(\tau,\cdot)$, $\frac{d^{n+1}}{ds^{n+1}}\mathcal{Z}_2(\tau,\cdot)$, $\frac{d^{n+1}}{ds^{n+1}}\mathcal{Z}_3(\tau,\cdot)\in L^\infty([s_l,s_r])$. It then follows from (7.25) and (7.26) that $\frac{\partial^{n+1}}{\partial x^{n+1}}u(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$, and from (7.25) and (7.30) we obtain $\frac{\partial^n}{\partial x^n}R(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$.

By (7.41), $\frac{\partial^k}{\partial X^k}p(\cdot,\mathcal{Y}(\cdot))$, $k=0,1,\dots,n$, is bounded on the curve $(\mathcal{X}(s),\mathcal{Y}(s))$, $s\in[s_l,s_r]$. Since $\frac{\partial^k}{\partial X^k}p(X,\mathcal{Y}(X))=\frac{\partial^k}{\partial X^k}p(X,Y)$, we get from (7.25) and (7.31), $\frac{\partial^n}{\partial x^n}\rho(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$.

Similarly, one proves that $\frac{\partial^n}{\partial x^n}S(\tau,\cdot)$, $\frac{\partial^n}{\partial x^n}\sigma(\tau,\cdot)\in L^\infty([x_l+\kappa\tau,x_r-\kappa\tau])$. □

Remark 7.2 A close inspection of the proof of (7.21) reveals that to any $0 \leq \tau \leq \frac{1}{2\kappa}(x_r - x_l)$ there exist $\bar{s}_1$ and $\bar{s}_2$ such that $s_l < \bar{s}_1 \leq \bar{s}_2 < s_r$, $(\mathcal{X}(\tau, s), \mathcal{Y}(\tau, s)) \in \Omega$ for all $s \in [\bar{s}_1, \bar{s}_2]$ and

$$x(\mathcal{X}(\tau, \bar{s}_1), \mathcal{Y}(\tau, \bar{s}_1)) = x_l + \kappa\tau \quad \text{and} \quad x(\mathcal{X}(\tau, \bar{s}_2), \mathcal{Y}(\tau, \bar{s}_2)) = x_r - \kappa\tau.$$

In particular, this result also holds in the case $\rho = 0 = \sigma$.

Remark 7.3 In [10] it has been highlighted that singularities appear, when either $R = u_t + c(u)u_x = c(u)\frac{U_X}{x_X}$ or $S = u_t - c(u)ux = -c(u)\frac{U_Y}{x_Y}$ become unbounded. As a closer look at the above proof reveals, both x_X and x_Y remain strictly positive in the studied region, cf. (7.11) and (7.12), while U_X and U_Y remain bounded. Thus neither R nor S can become unbounded in the studied region and hence no singularities form.

From Theorem 7.1 we obtain the following result.

Corollary 7.4 *Let* $-\infty < x_l < x_r < \infty$ *and consider* $(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0) \in \mathcal{D}$. *Assume that*

(A1') $u_0, R_0, S_0, \rho_0, \sigma_0 \in C^\infty([x_l, x_r])$,
(A2') *there are constants* $d > 0$ *and* $e > 0$ *such that* $\rho_0(x) \geq d$ *and* $\sigma_0(x) \geq e$ *for all* $x \in [x_l, x_r]$,
(A3') μ_0 *and* ν_0 *are absolutely continuous on* $[x_l, x_r]$,
(A4') $c \in C^\infty(\mathbb{R})$ *and* $c^{(m)} \in L^\infty(\mathbb{R})$ *for* $m = 3, 4, 5, \ldots$.

For any $\tau \in \left[0, \frac{1}{2\kappa}(x_r - x_l)\right]$ *consider* $(u, R, S, \rho, \sigma, \mu, \nu)(\tau) = \bar{S}_\tau(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$. *Then*

(P1') $u(\tau, \cdot), R(\tau, \cdot), S(\tau, \cdot), \rho(\tau, \cdot), \sigma(\tau, \cdot) \in C^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$,
(P2') *there are constants* $\bar{d} > 0$ *and* $\bar{e} > 0$ *such that* $\rho(\tau, x) \geq \bar{d}$ *and* $\sigma(\tau, x) \geq \bar{e}$ *for all* $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$,
(P3') $\mu(\tau, \cdot)$ *and* $\nu(\tau, \cdot)$ *are absolutely continuous on* $[x_l + \kappa\tau, x_r - \kappa\tau]$.

For $\tau \in \left[-\frac{1}{2\kappa}(x_r - x_l), 0\right]$, *the solution satisfies the same properties on the interval* $\left[x_l - \kappa\tau, x_r + \kappa\tau\right]$.

7.2 Convergence results

In this section we state and prove convergence results needed to prove our main result, which is presented in Sect. 7.3. Our main theorem is a local result, and strictly speaking we only require local convergence of the Lagrangian variables to prove it. However, some of the steps can be done globally, see Lemmas 7.5 and 7.7, and we choose to present them here as it is more challenging than the local case. In the proof of our main result, Theorem 7.9, we will point out whenever the local convergence differs from Lemmas 7.5 and 7.7.

Lemma 7.5 *Let* $(u, R, S, \rho, \sigma, \mu, \nu)$ *and* $(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n)$ *belong to* $\mathcal{D}$, *and assume that* μ, ν, μ^n, *and* ν^n *are absolutely continuous. Consider* $\psi_1 = (x_1, U_1, J_1, K_1, V_1, H_1)$, $\psi_2 = (x_2, U_2, J_2, K_2, V_2, H_2)$, $\psi_1^n = (x_1^n, U_1^n, J_1^n, K_1^n, V_1^n, H_1^n)$, *and* $\psi_2^n = (x_2^n, U_2^n, J_2^n, K_2^n, V_2^n, H_2^n)$ *defined by* $(\psi_1, \psi_2) = \mathbf{L}(u, R, S, \rho, \sigma, \mu, \nu)$ *and* $(\psi_1^n, \psi_2^n) = \mathbf{L}(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n)$. *Assume that*

$$u^n \to u, \quad R^n \to R, \quad S^n \to S, \quad \rho^n \to \rho, \quad \text{and} \quad \sigma^n \to \sigma \quad \text{in} \quad L^2(\mathbb{R}).$$

Then, x_i and x_i^n are strictly increasing, and

$$\begin{aligned}
&x_i^n \to x_i, \quad (x_i^n)^{-1} \to x_i^{-1}, \quad U_i^n \to U_i, \quad J_i^n \to J_i, \quad K_i^n \to K_i \quad \text{in} \quad L^\infty(\mathbb{R}),\\
&\quad U_i^n \to U_i, \quad V_i^n \to V_i, \quad H_i^n \to H_i \quad \text{in} \quad L^2(\mathbb{R}),\\
&(x_i^n)' \to x_i', \quad (J_i^n)' \to J_i', \quad (K_i^n)' \to K_i' \quad \text{in} \quad L^1(\mathbb{R})
\end{aligned}$$

for $i = 1, 2$.

We mention that the functions which converge in $L^1(\mathbb{R})$ also converge in $L^2(\mathbb{R})$, because convergence in $L^1(\mathbb{R})$ and (uniform) boundedness in $L^\infty(\mathbb{R})$ imply convergence in $L^2(\mathbb{R})$.

Proof We start by proving that $u^n \to u$ in $L^\infty(\mathbb{R})$. By (3.1) and the Cauchy Schwarz inequality, we have

$$\begin{aligned}
(u(x) - u^n(x))^2 &= \int_{-\infty}^x (u - u^n)\left(\frac{1}{c(u)}(R - S) - \frac{1}{c(u^n)}(R^n - S^n)\right)(y)\, dy\\
&\le \int_{\mathbb{R}} \kappa |u - u^n|(|R| + |R^n| + |S| + |S^n|)(y)\, dy\\
&\le \kappa ||u - u^n||_{L^2(\mathbb{R})}(2||R||_{L^2(\mathbb{R})} + ||R - R^n||_{L^2(\mathbb{R})}\\
&\quad + 2||S||_{L^2(\mathbb{R})} + ||S - S^n||_{L^2(\mathbb{R})}),
\end{aligned}$$

which implies, since $R^n \to R$, $S^n \to S$, and $u^n \to u$ in $L^2(\mathbb{R})$, that $u^n \to u$ in $L^\infty(\mathbb{R})$.

Let $\psi = (\psi_1, \psi_2) = \mathbf{L}(u, R, S, \rho, \sigma, \mu, \nu)$ and $\psi^n = (\psi_1^n, \psi_2^n) = \mathbf{L}(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n)$. We will only prove the results for the components of ψ_1, the proof for ψ_2 can be done in a similar way.

Let $f = \frac{1}{4}(R^2 + c(u)\rho^2)$ and $f^n = \frac{1}{4}((R^n)^2 + c(u^n)(\rho^n)^2)$. From (3.2) we have $\mu((-\infty, x)) = \int_{-\infty}^x f(z)\, dz$ and $\mu^n((-\infty, x)) = \int_{-\infty}^x f^n(z)\, dz$, since μ and μ^n are absolutely continuous measures. Consider the functions $F(x) = x + \int_{-\infty}^x f(z)\, dz$ and $F^n(x) = x + \int_{-\infty}^x f^n(z)\, dz$, which are strictly increasing and continuous. By (3.9a), we have that $F(x_1(X)) = X$ and $F^n(x_1^n(X)) = X$ for all $X \in \mathbb{R}$, which implies that x_1 and x_1^n are strictly increasing and continuous. Since $x_1^{-1} = F$ and $(x_1^n)^{-1} = F^n$, the inverses x_1^{-1} and $(x_1^n)^{-1}$ exist and are strictly increasing and continuous. We prove that $(x_1^n)^{-1} \to x_1^{-1}$ in $L^\infty(\mathbb{R})$. Since $|x_1^{-1}(x) - (x_1^n)^{-1}(x)| \le ||f - f^n||_{L^1(\mathbb{R})}$ we have to show that $f^n \to f$ in $L^1(\mathbb{R})$. By using the estimate

$$\left|\frac{1}{c(u)} - \frac{1}{c(u^n)}\right| = \left|\int_{u^n}^u -\frac{c'(w)}{c^2(w)}\, dw\right| \le \kappa^2 k_1 |u - u^n| \tag{7.44}$$

and the Cauchy–Schwarz inequality, we find

$$\begin{aligned}
||f - f^n||_{L^1(\mathbb{R})} &= \frac{1}{4}\int_{\mathbb{R}} |R^2 - (R^n)^2 + \rho^2(c(u) - c(u^n)) + c(u^n)(\rho^2 - (\rho^n)^2)|(x)\, dx\\
&\le \frac{1}{4}\int_{\mathbb{R}} \big(|R^2 - (R^n)^2| + k_1\rho^2|u - u^n| + \kappa|\rho^2 - (\rho^n)^2|\big)(x)\, dx\\
&\le \frac{1}{4}\big(2||R||_{L^2(\mathbb{R})} + ||R - R^n||_{L^2(\mathbb{R})}\big)||R - R^n||_{L^2(\mathbb{R})}\\
&\quad + \frac{k_1}{4}||\rho||^2_{L^2(\mathbb{R})}||u - u^n||_{L^\infty(\mathbb{R})}\\
&\quad + \frac{\kappa}{4}\big(2||\rho||_{L^2(\mathbb{R})} + ||\rho - \rho^n||_{L^2(\mathbb{R})}\big)||\rho - \rho^n||_{L^2(\mathbb{R})}.
\end{aligned}$$

Since $R^n \to R$ and $\rho^n \to \rho$ in $L^2(\mathbb{R})$, and $u^n \to u$ in $L^\infty(\mathbb{R})$, it follows that $f^n \to f$ in $L^1(\mathbb{R})$. Therefore $(x_1^n)^{-1} \to x_1^{-1}$ in $L^\infty(\mathbb{R})$.

We prove that $x_1^n \to x_1$ in $L^\infty(\mathbb{R})$. By direct calculations, we get for $x_1^n(X) \le x_1(X)$, that

$$\begin{aligned} x_1(X) - x_1^n(X) &= -\int_{-\infty}^{x_1(X)} (f(x) - f^n(x))\,dx - \int_{x_1^n(X)}^{x_1(X)} f^n(x)\,dx \\ &\le -\int_{-\infty}^{x_1(X)} (f(x) - f^n(x))\,dx, \end{aligned}$$

since $f^n \ge 0$. Treating the case $x_1(X) \le x_1^n(X)$ in a similar way, we end up with $|x_1(X) - x_1^n(X)| \le ||f - f^n||_{L^1(\mathbb{R})}$, which implies, since $f^n \to f$ in $L^1(\mathbb{R})$, that $x_1^n \to x_1$ in $L^\infty(\mathbb{R})$. Using (3.9c) we get $J_1^n \to J_1$ in $L^\infty(\mathbb{R})$.

We prove that $(x_1^n)' \to x_1'$ in $L^1(\mathbb{R})$. As in (7.2) we get $(x_1^n)' = \frac{1}{f^n \circ x_1^n + 1}$ and $x_1' = \frac{1}{f \circ x_1 + 1}$, so that

$$(x_1^n)' - x_1' = (f \circ x_1 - f^n \circ x_1^n)(x_1^n)' x_1'. \tag{7.45}$$

For every $\varepsilon > 0$, there exists a function l in $C_c(\mathbb{R})$ such that $||f - l||_{L^1(\mathbb{R})} \le \varepsilon$. Here $C_c(\mathbb{R})$ denotes the space of continuous functions with compact support. Applying the triangle inequality to (7.45) yields

$$\begin{aligned} &\int_{\mathbb{R}} |x_1'(X) - (x_1^n)'(X)|\,dX \\ &\le \int_{\mathbb{R}} |f \circ x_1(X) - l \circ x_1(X)| x_1'(X)\,dX + \int_{\mathbb{R}} |l \circ x_1(X) - l \circ x_1^n(X)|\,dX \\ &\quad + \int_{\mathbb{R}} |l \circ x_1^n(X) - f \circ x_1^n(X)| (x_1^n)'(X)\,dX \\ &\quad + \int_{\mathbb{R}} |f \circ x_1^n(X) - f^n \circ x_1^n(X)| (x_1^n)'(X)\,dX \\ &= 2||f - l||_{L^1(\mathbb{R})} + ||l \circ x_1 - l \circ x_1^n||_{L^1(\mathbb{R})} + ||f - f^n||_{L^1(\mathbb{R})}, \end{aligned} \tag{7.46}$$

by a change of variables. Furthermore we used that $0 \le x_1' \le 1$ and $0 \le (x_1^n)' \le 1$. It remains to show that $\lim_{n\to\infty} ||l \circ x_1 - l \circ x_1^n||_{L^1(\mathbb{R})} = 0$. We have $l \circ x_1^n \to l \circ x_1$ pointwise almost everywhere. In order to use the dominated convergence theorem we have to show that $l \circ x_1^n$ can be uniformly bounded by a function which belongs to $L^1(\mathbb{R})$. We prove a slightly more general result, which will be used many times throughout the text.

Lemma 7.6 *Assume that $g \in C_c(\mathbb{R})$, and that h and h_n satisfy $h - \mathrm{Id}$, $h_n - \mathrm{Id} \in L^\infty(\mathbb{R})$, and $h_n \to h$ in $L^\infty(\mathbb{R})$. Then there exists a constant $0 < K < \infty$ that is independent of n such that*

$$|g \circ h_n| \le ||g||_{L^\infty(\mathbb{R})} \chi_{[-K,K]}, \tag{7.47}$$

where $\chi_{[-K,K]}$ denotes the indicator function of the interval $[-K, K]$.

Proof Since g has compact support there is a constant $0 < k < \infty$ such that $\mathrm{supp}(g) \subset [-k, k]$. Writing $h_n(x) = h_n(x) - h(x) + h(x) - x + x$, we get $\{x \mid h_n(x) \in [-k, k]\} \subset [-K_n, K_n]$ where $K_n = k + ||h - h_n||_{L^\infty(\mathbb{R})} + ||h - \mathrm{Id}\,||_{L^\infty(\mathbb{R})}$. Since $h_n \to h$ in $L^\infty(\mathbb{R})$ we can find a constant M such that $||h - h_n||_{L^\infty(\mathbb{R})} \le M$ for all n. If we set $K = k + M + ||h - \mathrm{Id}\,||_{L^\infty(\mathbb{R})}$, we get

$$|g \circ h_n(x)| \le |g \circ h_n(x)| \chi_{[-K,K]} \le ||g||_{L^\infty(\mathbb{R})} \chi_{[-K,K]},$$

which proves (7.47). □

From (7.47) we conclude that $l \circ x_1^n$ can be uniformly bounded by an $L^1(\mathbb{R})$ function. By the dominated convergence theorem we obtain $\lim_{n\to\infty} ||l \circ x_1 - l \circ x_1^n||_{L^1(\mathbb{R})} = 0$. We conclude that the right-hand side of (7.46) can be made arbitrarily small, so that $(x_1^n)' \to x_1'$ in $L^1(\mathbb{R})$ and as an immediate consequence $(J_1^n)' \to J_1'$ in $L^1(\mathbb{R})$ by (3.9c).

We show that $U_1^n \to U_1$ in $L^\infty(\mathbb{R})$. By (3.9d) and the Cauchy–Schwarz inequality we get

$$\begin{aligned}
\big|U_1(X) - U_1^n(X)\big| &\le \big|u(x_1(X)) - u(x_1^n(X))\big| + \big|u(x_1^n(X)) - u^n(x_1^n(X))\big| \\
&\le \left| \int_{x_1^n(X)}^{x_1(X)} u_x(x)\,dx \right| + \big|u(x_1^n(X)) - u^n(x_1^n(X))\big| \\
&\le ||u_x||_{L^2(\mathbb{R})} ||x_1 - x_1^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}} + ||u - u^n||_{L^\infty(\mathbb{R})}.
\end{aligned}$$

From (3.1) we have $||u_x||_{L^2(\mathbb{R})} \le \frac{\kappa}{2}(||R||_{L^2(\mathbb{R})} + ||S||_{L^2(\mathbb{R})})$, and since $x_1^n \to x_1$ and $u^n \to u$ in $L^\infty(\mathbb{R})$, we conclude that $U_1^n \to U_1$ in $L^\infty(\mathbb{R})$.

Let us prove that $U_1^n \to U_1$ in $L^2(\mathbb{R})$. Since $u \in H^1(\mathbb{R})$ there is for every $\varepsilon > 0$ a function $\eta \in C_c(\mathbb{R})$ such that $||u - \eta||_{L^2(\mathbb{R})} \le \varepsilon$ and $||u - \eta||_{L^\infty(\mathbb{R})} \le \varepsilon$. We have

$$\begin{aligned}
||U_1 - U_1^n||_{L^2(\mathbb{R})} &\le ||u \circ x_1 - \eta \circ x_1||_{L^2(\mathbb{R})} + ||\eta \circ x_1 - \eta \circ x_1^n||_{L^2(\mathbb{R})} \\
&\quad + ||\eta \circ x_1^n - u \circ x_1^n||_{L^2(\mathbb{R})} + ||u \circ x_1^n - u^n \circ x_1^n||_{L^2(\mathbb{R})}.
\end{aligned} \tag{7.48}$$

Let $D_1 = \big\{X \in \mathbb{R} \mid x_1'(X) < \frac{1}{2}\big\}$ and $D_2 = \big\{Y \in \mathbb{R} \mid x_2'(Y) < \frac{1}{2}\big\}$. Observe that $\mathrm{meas}(D_1) \le \int_\mathbb{R} 2J_1'(X)\,dX$ and $\mathrm{meas}(D_2) \le \int_\mathbb{R} 2J_2'(Y)\,dY$, since $J_i' = 1 - x_i'$, $i = 1, 2$. Since $\lim_{X\to-\infty} J_1(X) = 0 = \lim_{Y\to-\infty} J_2(Y)$, we get $\mathrm{meas}(D_1) \le 2||J_1||_{L^\infty(\mathbb{R})}$ and $\mathrm{meas}(D_2) \le 2||J_2||_{L^\infty(\mathbb{R})}$. In a similar way we find that the measures of the sets $D_1^n = \big\{X \in \mathbb{R} \mid (x_1^n)'(X) < \frac{1}{2}\big\}$ and $D_2^n = \big\{Y \in \mathbb{R} \mid (x_2^n)'(Y) < \frac{1}{2}\big\}$ have the bounds $\mathrm{meas}(D_1^n) \le 2||J_1^n||_{L^\infty(\mathbb{R})}$ and $\mathrm{meas}(D_2^n) \le 2||J_2^n||_{L^\infty(\mathbb{R})}$. Since $J_i^n \to J_i$ in $L^\infty(\mathbb{R})$, we can find constants E_i that are independent of n such that $\mathrm{meas}(D_1^n) \le E_1$ and $\mathrm{meas}(D_2^n) \le E_2$ for all n. We have

$$\begin{aligned}
&||u \circ x_1 - \eta \circ x_1||_{L^2(\mathbb{R})}^2 \\
&\quad = \int_{D_1} \big(u \circ x_1(X) - \eta \circ x_1(X)\big)^2\,dX + \int_{D_1^c} \big(u \circ x_1(X) - \eta \circ x_1(X)\big)^2\,dX \\
&\quad \le \mathrm{meas}(D_1)||u - \eta||_{L^\infty(\mathbb{R})}^2 + 2\int_{D_1^c} \big(u \circ x_1(X) - \eta \circ x_1(X)\big)^2 x_1'(X)\,dX \\
&\quad \le \mathrm{meas}(D_1)||u - \eta||_{L^\infty(\mathbb{R})}^2 + 2||u - \eta||_{L^2(\mathbb{R})}^2
\end{aligned} \tag{7.49}$$

by a change of variables. Similarly,

$$||\eta \circ x_1^n - u \circ x_1^n||_{L^2(\mathbb{R})}^2 \le E_1||u - \eta||_{L^\infty(\mathbb{R})}^2 + 2||u - \eta||_{L^2(\mathbb{R})}^2 \tag{7.50}$$

and

$$||u \circ x_1^n - u^n \circ x_1^n||_{L^2(\mathbb{R})}^2 \le E_1||u - u^n||_{L^\infty(\mathbb{R})}^2 + 2||u - u^n||_{L^2(\mathbb{R})}^2.$$

Since $u_n \to u$ in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we get that

$$\lim_{n\to\infty} ||u \circ x_1^n - u^n \circ x_1^n||_{L^2(\mathbb{R})} = 0. \tag{7.51}$$

For the remaining term in (7.48) we use the dominated convergence theorem. We have $\eta \circ x_1^n \to \eta \circ x_1$ pointwise almost everywhere, and by Lemma 7.6 we find that $\eta \circ x_1^n$ can be uniformly bounded by an $L^2(\mathbb{R})$ function. By the dominated convergence theorem we get

$$\lim_{n\to\infty} ||\eta \circ x_1 - \eta \circ x_1^n||_{L^2(\mathbb{R})} = 0. \tag{7.52}$$

From (7.51) and (7.52), and since the right-hand sides of (7.49) and (7.50) can be made arbitrarily small, we conclude that $U_1^n \to U_1$ in $L^2(\mathbb{R})$.

We prove that $K_1^n \to K_1$ in $L^\infty(\mathbb{R})$. By (3.9f) we get

$$\begin{aligned} K_1(X) - K_1^n(X) &= \int_{-\infty}^{X} J_1'(\bar{X})\Big[\frac{1}{c(U_1(\bar{X}))} - \frac{1}{c(U_1^n(\bar{X}))}\Big]\, d\bar{X} \\ &\quad + \int_{-\infty}^{X} \frac{1}{c(U_1^n(\bar{X}))}\big[J_1'(\bar{X}) - (J_1^n)'(\bar{X})\big]\, d\bar{X}. \end{aligned} \tag{7.53}$$

For the first term on the right-hand side we use the Cauchy–Schwarz inequality and get

$$\begin{aligned} \Big|\int_{-\infty}^{X} J_1'(\bar{X})\Big[\frac{1}{c(U_1(\bar{X}))} - \frac{1}{c(U_1^n(\bar{X}))}\Big]\, d\bar{X}\Big| &\le ||J_1'||_{L^2(\mathbb{R})}\Big|\int_{-\infty}^{X}\Big(\int_{U_1^n(\bar{X})}^{U_1(\bar{X})} -\frac{c'(U)}{c^2(U)}\, dU\Big)^2 d\bar{X}\Big|^{\frac{1}{2}} \\ &\le k_1\kappa^2 ||J_1'||_{L^2(\mathbb{R})} ||U_1 - U_1^n||_{L^2(\mathbb{R})}, \end{aligned}$$

and for the second term we have

$$\Big|\int_{-\infty}^{X} \frac{1}{c(U_1^n(\bar{X}))}\big[J_1'(\bar{X}) - (J_1^n)'(\bar{X})\big]\, d\bar{X}\Big| \le \kappa ||J_1' - (J_1^n)'||_{L^1(\mathbb{R})},$$

which implies that $K_1^n \to K_1$ in $L^\infty(\mathbb{R})$.

The above proof in fact also shows that $(K_1^n)' \to K_1'$ in $L^1(\mathbb{R})$, since

$$\int_{\mathbb{R}} \big|K_1'(X) - (K_1^n)'(X)\big|\, dX = \int_{\mathbb{R}} \Big|\frac{J_1'(X)}{c(U_1(X))} - \frac{(J_1^n)'(X)}{c(U_1^n(X))}\Big|\, dX.$$

We prove that $H_1^n \to H_1$ in $L^2(\mathbb{R})$. Since ρ and ρ^n belong to $L^2(\mathbb{R})$ there exist for every $\varepsilon > 0$ functions ϕ and ϕ^n in $C_c(\mathbb{R})$ such that $||\rho - \phi||_{L^2(\mathbb{R})} \le \varepsilon$ and $||\rho^n - \phi^n||_{L^2(\mathbb{R})} \le \varepsilon$. Since $\rho^n \to \rho$ in $L^2(\mathbb{R})$ we can for every $\varepsilon > 0$ choose n so large that $||\rho - \rho^n||_{L^2(\mathbb{R})} \le \varepsilon$. This implies that for large n we have $||\phi - \phi^n||_{L^2(\mathbb{R})} \le 3\varepsilon$. From (3.9g) we have

$$\begin{aligned} H_1 - H_1^n &= \frac{1}{2}x_1'(\rho \circ x_1 - \phi \circ x_1) + \frac{1}{2}((\phi \circ x_1)x_1' - (\phi^n \circ x_1^n)(x_1^n)') \\ &\quad + \frac{1}{2}(x_1^n)'(\phi^n \circ x_1^n - \rho^n \circ x_1^n), \end{aligned}$$

so that

$$\begin{aligned} ||H_1 - H_1^n||_{L^2(\mathbb{R})} &\le \frac{1}{2}||x_1'(\rho \circ x_1 - \phi \circ x_1)||_{L^2(\mathbb{R})} + \frac{1}{2}||(\phi \circ x_1)x_1' - (\phi^n \circ x_1^n)(x_1^n)'||_{L^2(\mathbb{R})} \\ &\quad + \frac{1}{2}||(x_1^n)'(\phi^n \circ x_1^n - \rho^n \circ x_1^n)||_{L^2(\mathbb{R})}. \end{aligned} \tag{7.54}$$

Since $0 \le x_1' \le 1$ we get for the first term on the right-hand side by a change of variables,

$$||x_1'(\rho \circ x_1 - \phi \circ x_1)||_{L^2(\mathbb{R})}^2 \le \int_{\mathbb{R}} x_1'(X)(\rho(x_1(X)) - \phi(x_1(X)))^2\, dX = ||\rho - \phi||_{L^2(\mathbb{R})}^2. \tag{7.55}$$

Similarly we get for the third term,

$$||(x_1^n)'(\phi^n \circ x_1^n - \rho^n \circ x_1^n)||_{L^2(\mathbb{R})} \leq ||\rho^n - \phi^n||_{L^2(\mathbb{R})}. \tag{7.56}$$

For the second term we have

$$\begin{aligned}
&||(\phi \circ x_1)x_1' - (\phi^n \circ x_1^n)(x_1^n)'||_{L^2(\mathbb{R})} \\
&\quad \leq ||(\phi \circ x_1)(x_1' - (x_1^n)')||_{L^2(\mathbb{R})} + ||(x_1^n)'(\phi \circ x_1 - \phi \circ x_1^n)||_{L^2(\mathbb{R})} \\
&\qquad + ||(x_1^n)'(\phi \circ x_1^n - \phi^n \circ x_1^n)||_{L^2(\mathbb{R})} \\
&\quad \leq \sqrt{2}||\phi||_{L^\infty(\mathbb{R})}||x_1' - (x_1^n)'||_{L^1(\mathbb{R})}^{\frac{1}{2}} + ||\phi \circ x_1 - \phi \circ x_1^n||_{L^2(\mathbb{R})} + ||\phi - \phi^n||_{L^2(\mathbb{R})},
\end{aligned} \tag{7.57}$$

where we used that $0 \leq x_1' \leq 1$, $0 \leq (x_1^n)' \leq 1$, and a change of variables for the last term on the right-hand side. We have $\phi \circ x_1^n \to \phi \circ x_1$ pointwise almost everywhere, and by Lemma 7.6 we get that $\phi \circ x_1^n$ can be uniformly bounded by an $L^2(\mathbb{R})$ function. By the dominated convergence theorem we have $\lim_{n\to\infty} ||\phi \circ x_1 - \phi \circ x_1^n||_{L^2(\mathbb{R})} = 0$.

Using the estimates (7.55)–(7.57) in (7.54) we observe that all terms on the right-hand side can be made arbitrarily small, which implies that $H_1^n \to H_1$ in $L^2(\mathbb{R})$.

We can prove in more or less the same way that $V_1^n \to V_1$ in $L^2(\mathbb{R})$, where we also have to use that $U_1^n \to U_1$ in $L^\infty(\mathbb{R})$ and the boundedness of c and c'. □

Lemma 7.7 *Let (ψ_1, ψ_2) and (ψ_1^n, ψ_2^n) belong to $\mathcal{F}$, where $\psi_i = (x_i, U_i, J_i, K_i, V_i, H_i)$ and $\psi_i^n = (x_i^n, U_i^n, J_i^n, K_i^n, V_i^n, H_i^n)$, $i = 1, 2$. Assume that x_i and x_i^n are strictly increasing, and that $x_i + J_i = \mathrm{Id}$, and $x_i^n + J_i^n = \mathrm{Id}$. Consider $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) = \mathbf{C}(\psi_1, \psi_2)$ and $(\mathcal{X}^n, \mathcal{Y}^n, \mathcal{Z}^n, \mathcal{V}^n, \mathcal{W}^n, \mathfrak{p}^n, \mathfrak{q}^n) = \mathbf{C}(\psi_1^n, \psi_2^n)$. Assume*

$$\begin{aligned}
&x_i^n \to x_i, \quad (x_i^n)^{-1} \to x_i^{-1}, \quad U_i^n \to U_i, \quad J_i^n \to J_i, \quad K_i^n \to K_i \quad in \quad L^\infty(\mathbb{R}), \\
&U_i^n \to U_i, \quad V_i^n \to V_i, \quad H_i^n \to H_i \quad in \quad L^2(\mathbb{R}), \\
&(x_i^n)' \to x_i', \quad (J_i^n)' \to J_i', \quad (K_i^n)' \to K_i' \quad in \quad L^2(\mathbb{R})
\end{aligned}$$

for $i = 1, 2$. Then $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}^n$, and $\mathcal{Y}^n$ are strictly increasing, and

$$\begin{aligned}
&\mathcal{X}^n \to \mathcal{X}, \quad \mathcal{Y}^n \to \mathcal{Y}, \quad \mathcal{Z}_i^n \to \mathcal{Z}_i \quad in \quad L^\infty(\mathbb{R}), \\
&\mathcal{Z}_3^n \to \mathcal{Z}_3, \quad \mathcal{V}_i^n \to \mathcal{V}_i, \quad \mathcal{W}_i^n \to \mathcal{W}_i, \quad \mathfrak{p}^n \to \mathfrak{p}, \quad \mathfrak{q}^n \to \mathfrak{q} \quad in \quad L^2(\mathbb{R})
\end{aligned}$$

for $i = 1, \dots, 5$.

Proof Since x_i and x_i^n are continuous and strictly increasing for $i = 1, 2$, we have by (3.23) that for every $s \in \mathbb{R}$ there exist *unique* points $\mathcal{X}(s)$ and $\mathcal{X}^n(s)$ such that $x_1(\mathcal{X}(s)) = x_2(2s - \mathcal{X}(s))$ and $x_1^n(\mathcal{X}^n(s)) = x_2^n(2s - \mathcal{X}^n(s))$. Moreover, $\mathcal{X}$ and $\mathcal{X}^n$ are strictly increasing and continuous. It follows from (3.3c) that $\mathcal{Y}$ and $\mathcal{Y}^n$ are strictly increasing and continuous. Thus, the inverse functions $\mathcal{X}^{-1}$, $\mathcal{Y}^{-1}$, $(\mathcal{X}^n)^{-1}$ and $(\mathcal{Y}^n)^{-1}$ exist, and they are continuous and strictly increasing.

We prove that $\mathcal{X}^n \to \mathcal{X}$ in $L^\infty(\mathbb{R})$. To begin with we show that $J_i^n \circ (x_i^n)^{-1} \to J_i \circ x_i^{-1}$ in $L^\infty(\mathbb{R})$. Write

$$J_i \circ x_i^{-1}(X) - J_i^n \circ (x_i^n)^{-1}(X) = \int_{(x_i^n)^{-1}(X)}^{x_i^{-1}(X)} J_i'(Z)\, dZ + J_i \circ (x_i^n)^{-1}(X) - J_i^n \circ (x_i^n)^{-1}(X)$$

to get

$$|J_i \circ x_i^{-1}(X) - J_i^n \circ (x_i^n)^{-1}(X)| \leq ||x_i^{-1} - (x_i^n)^{-1}||_{L^\infty(\mathbb{R})} + ||J_i - J_i^n||_{L^\infty(\mathbb{R})}.$$

Here, we used that $0 \leq J_i' \leq 1$, where the upper bound comes from the identity $x_i' + J_i' = 1$, and $x_i' \geq 0$. Using the convergence assumptions implies that $J_i^n \circ (x_i^n)^{-1} \to J_i \circ x_i^{-1}$ in $L^\infty(\mathbb{R})$.

We insert $X = \mathcal{X}(s)$ in $x_1(X) + J_1(X) = X$ and get

$$\mathcal{X}(s) = x_1 \circ \mathcal{X}(s) + J_1 \circ \mathcal{X}(s) = x_1 \circ \mathcal{X}(s) + J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X}(s). \tag{7.58}$$

Similarly we get

$$\mathcal{Y}(s) = x_2 \circ \mathcal{Y}(s) + J_2 \circ \mathcal{Y}(s) = x_1 \circ \mathcal{X}(s) + J_2 \circ x_2^{-1} \circ x_1 \circ \mathcal{X}(s), \tag{7.59}$$

where we used that $x_1(\mathcal{X}(s)) = x_2(\mathcal{Y}(s))$. The expressions for $\mathcal{X}^n$ and $\mathcal{Y}^n$ are defined in a similar way. By (3.3c), $\mathcal{X}(s) + \mathcal{Y}(s) - \mathcal{X}^n(s) - \mathcal{Y}^n(s) = 0$, which combined with the above expressions yields

$$\begin{aligned}
&2\big(x_1 \circ \mathcal{X}(s) - x_1^n \circ \mathcal{X}^n(s)\big) + J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X}(s) - J_1^n \circ (x_1^n)^{-1} \circ x_1^n \circ \mathcal{X}^n(s) \\
&\quad + J_2 \circ x_2^{-1} \circ x_1 \circ \mathcal{X}(s) - J_2^n \circ (x_2^n)^{-1} \circ x_1^n \circ \mathcal{X}^n(s) = 0.
\end{aligned}$$

Since $J_i^n \circ (x_i^n)^{-1}$ is increasing we get

$$\begin{aligned}
&2\big|x_1 \circ \mathcal{X}(s) - x_1^n \circ \mathcal{X}^n(s)\big| + |J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X}(s) - J_1 \circ x_1^{-1} \circ x_1^n \circ \mathcal{X}^n(s)| \\
&\quad + |J_2 \circ x_2^{-1} \circ x_1 \circ \mathcal{X}(s) - J_2 \circ x_2^{-1} \circ x_1^n \circ \mathcal{X}^n(s)| \\
&\quad \leq |J_1^n \circ (x_1^n)^{-1} \circ x_1^n \circ \mathcal{X}^n(s) - J_1 \circ x_1^{-1} \circ x_1^n \circ \mathcal{X}^n(s)| \\
&\quad + |J_2^n \circ (x_2^n)^{-1} \circ x_1^n \circ \mathcal{X}^n(s) - J_2 \circ x_2^{-1} \circ x_1^n \circ \mathcal{X}^n(s)|.
\end{aligned}$$

Therefore

$$\begin{aligned}
||x_1 \circ \mathcal{X} - x_1^n \circ \mathcal{X}^n||_{L^\infty(\mathbb{R})} &\leq \frac{1}{2}||J_1^n \circ (x_1^n)^{-1} - J_1 \circ x_1^{-1}||_{L^\infty(\mathbb{R})} \\
&\quad + \frac{1}{2}||J_2^n \circ (x_2^n)^{-1} - J_2 \circ x_2^{-1}||_{L^\infty(\mathbb{R})}, \\
||J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X} - J_1 \circ x_1^{-1} \circ x_1^n \circ \mathcal{X}^n||_{L^\infty(\mathbb{R})} &\leq ||J_1^n \circ (x_1^n)^{-1} - J_1 \circ x_1^{-1}||_{L^\infty(\mathbb{R})} \\
&\quad + ||J_2^n \circ (x_2^n)^{-1} - J_2 \circ x_2^{-1}||_{L^\infty(\mathbb{R})},
\end{aligned}$$

and

$$\begin{aligned}
||J_1 \circ \mathcal{X} - J_1^n \circ \mathcal{X}^n||_{L^\infty(\mathbb{R})} &= ||J_1 \circ x_1^{-1} \circ x_1 \circ \mathcal{X} - J_1^n \circ (x_1^n)^{-1} \circ x_1^n \circ \mathcal{X}^n||_{L^\infty(\mathbb{R})} \\
&\leq 2||J_1^n \circ (x_1^n)^{-1} - J_1 \circ x_1^{-1}||_{L^\infty(\mathbb{R})} \\
&\quad + ||J_2^n \circ (x_2^n)^{-1} - J_2 \circ x_2^{-1}||_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Thus, we showed that $x_1^n \circ \mathcal{X}^n \to x_1 \circ \mathcal{X}$ and $J_1^n \circ \mathcal{X}^n \to J_1 \circ \mathcal{X}$ in $L^\infty(\mathbb{R})$ since $J_i^n \circ (x_i^n)^{-1} \to J_i \circ x_i^{-1}$ in $L^\infty(\mathbb{R})$. Recallling (3.25b) and (3.25d), we thus have $\mathcal{Z}_2^n \to \mathcal{Z}_2$ and $\mathcal{Z}_4^n \to \mathcal{Z}_4$ in $L^\infty(\mathbb{R})$. Furthermore from (7.58), (3.3c), and (7.59) it immediately follows that $\mathcal{X}^n \to \mathcal{X}$ and $\mathcal{Y}^n \to \mathcal{Y}$ in $L^\infty(\mathbb{R})$.

We show that $\mathcal{Z}_5^n \to \mathcal{Z}_5$ in $L^\infty(\mathbb{R})$. By (3.25e) we have

$$\begin{aligned}
\mathcal{Z}_5(s) - \mathcal{Z}_5^n(s) &= K_1(\mathcal{X}(s)) - K_1(\mathcal{X}^n(s)) + K_1(\mathcal{X}^n(s)) - K_1^n(\mathcal{X}^n(s)) \\
&\quad + K_2(\mathcal{Y}(s)) - K_2(\mathcal{Y}^n(s)) + K_2(\mathcal{Y}^n(s)) - K_2^n(\mathcal{Y}^n(s))
\end{aligned}$$

and for the first line we get

$$\begin{aligned}&\left|K_1(\mathcal{X}(s)) - K_1(\mathcal{X}^n(s)) + K_1(\mathcal{X}^n(s)) - K_1^n(\mathcal{X}^n(s))\right| \\ &\quad \leq ||K_1'||_{L^\infty(\mathbb{R})}||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})} + ||K_1 - K_1^n||_{L^\infty(\mathbb{R})}.\end{aligned}$$

A similar estimate for the second line yields $\mathcal{Z}_5^n \to \mathcal{Z}_5$ in $L^\infty(\mathbb{R})$.

By (3.25c) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}|\mathcal{Z}_3(s) - \mathcal{Z}_3^n(s)| &\leq \left| \int_{\mathcal{X}^n(s)}^{\mathcal{X}(s)} U_1'(X)\, dX \right| + |U_1(\mathcal{X}^n(s)) - U_1^n(\mathcal{X}^n(s))| \\ &\leq ||U_1'||_{L^2(\mathbb{R})}||\mathcal{X} - \mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}} + ||U_1 - U_1^n||_{L^\infty(\mathbb{R})},\end{aligned}$$

which shows that $\mathcal{Z}_3^n \to \mathcal{Z}_3$ in $L^\infty(\mathbb{R})$.

We prove that $\mathcal{Z}_3^n \to \mathcal{Z}_3$ in $L^2(\mathbb{R})$. We have

$$||\mathcal{Z}_3 - \mathcal{Z}_3^n||_{L^2(\mathbb{R})}^2 = ||\mathcal{Z}_3||_{L^2(\mathbb{R})}^2 - 2\langle \mathcal{Z}_3, \mathcal{Z}_3^n \rangle + ||\mathcal{Z}_3^n||_{L^2(\mathbb{R})}^2, \tag{7.60}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\mathbb{R})$. Since $\dot{\mathcal{X}} + \dot{\mathcal{Y}} = 2$ we get from (3.25c) and a change of variables,

$$\begin{aligned}||\mathcal{Z}_3||_{L^2(\mathbb{R})}^2 &= \frac{1}{2}\int_{\mathbb{R}} U_1^2(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds + \frac{1}{2}\int_{\mathbb{R}} U_2^2(\mathcal{Y}(s))\dot{\mathcal{Y}}(s)\, ds \\ &= \frac{1}{2}||U_1||_{L^2(\mathbb{R})}^2 + \frac{1}{2}||U_2||_{L^2(\mathbb{R})}^2,\end{aligned} \tag{7.61}$$

and similarly

$$||\mathcal{Z}_3^n||_{L^2(\mathbb{R})}^2 = \frac{1}{2}||U_1^n||_{L^2(\mathbb{R})}^2 + \frac{1}{2}||U_2^n||_{L^2(\mathbb{R})}^2. \tag{7.62}$$

Using that $\dot{\mathcal{X}}^n + \dot{\mathcal{Y}}^n = 2$ we have

$$2\langle \mathcal{Z}_3, \mathcal{Z}_3^n \rangle = \int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\, ds + \int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{Y}}^n(s)\, ds.$$

Since $\mathcal{Z}_3 \in L^2(\mathbb{R})$ there exists for every $\varepsilon > 0$ a function $\phi \in C_c^\infty(\mathbb{R})$ such that $||\mathcal{Z}_3 - \phi||_{L^2(\mathbb{R})} \leq \varepsilon$. Write

$$\begin{aligned}&\int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\, ds \\ &\quad = \int_{\mathbb{R}} \big[\mathcal{Z}_3(s) - \phi(s)\big]\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\, ds + \int_{\mathbb{R}} \phi(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\, ds - \int_{\mathbb{R}} \phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\, ds \\ &\qquad + \int_{\mathbb{R}} \mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\big[\phi(s) - \mathcal{Z}_3(s)\big]\, ds + \int_{\mathbb{R}} \mathcal{Z}_3^2(s)\dot{\mathcal{X}}(s)\, ds = T_1^n + \int_{\mathbb{R}} \mathcal{Z}_3^2(s)\dot{\mathcal{X}}(s)\, ds.\end{aligned}$$

By a change of variables we get

$$\int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\, ds = T_1^n + ||U_1||_{L^2(\mathbb{R})}^2 \tag{7.63}$$

and in a similar way we obtain

$$\int_{\mathbb{R}} \mathcal{Z}_3(s)\mathcal{Z}_3^n(s)\dot{\mathcal{Y}}^n(s)\, ds = T_2^n + ||U_2||_{L^2(\mathbb{R})}^2, \tag{7.64}$$

where T_2^n is equal to T_1^n with $\mathcal{X}(s)$ and $\mathcal{X}^n(s)$ replaced by $\mathcal{Y}(s)$ and $\mathcal{Y}^n(s)$, respectively.

Using (7.61)–(7.64) in (7.60) we get

$$||\mathcal{Z}_3 - \mathcal{Z}_3^n||^2_{L^2(\mathbb{R})} = \frac{1}{2}\Big(||U_1^n||^2_{L^2(\mathbb{R})} - ||U_1||^2_{L^2(\mathbb{R})} + ||U_2^n||^2_{L^2(\mathbb{R})} - ||U_2||^2_{L^2(\mathbb{R})}\Big) - T_1^n - T_2^n. \tag{7.65}$$

The strong convergence $U_i^n \to U_i$ in $L^2(\mathbb{R})$ implies that $||U_i^n||_{L^2(\mathbb{R})} \to ||U_i||_{L^2(\mathbb{R})}$ for $i = 1, 2$. Thus, it remains to show that $T_i^n \to 0$ for $i = 1, 2$.

Using (3.25c) and $0 \leq \dot{\mathcal{X}}^n \leq 2$ we get by the Cauchy–Schwarz inequality and a change of variables,

$$\left|\int_{\mathbb{R}} \mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\big[\phi(s) - \mathcal{Z}_3(s)\big]\, ds\right| \leq \sqrt{2}||U_1||_{L^2(\mathbb{R})}||\mathcal{Z}_3 - \phi||_{L^2(\mathbb{R})}, \tag{7.66}$$

and similarly

$$\left|\int_{\mathbb{R}} \big[\mathcal{Z}_3(s) - \phi(s)\big]\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\, ds\right| \leq \sqrt{2}\Big(||U_1^n - U_1||_{L^2(\mathbb{R})} + ||U_1||_{L^2(\mathbb{R})}\Big)||\mathcal{Z}_3 - \phi||_{L^2(\mathbb{R})}. \tag{7.67}$$

Since ϕ has compact support, there exists $k > 0$ such that $\operatorname{supp}(\phi) \subset [-k, k]$. Integration by parts yields

$$\int_{\mathbb{R}} \phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\, ds = \int_{-k}^{k} \phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\, ds = -\int_{-k}^{k} \phi'(s)\int_{-k}^{s} \mathcal{Z}_3(t)\dot{\mathcal{X}}(t)\, dt\, ds,$$

where the first term in the integration by parts equals zero because ϕ has compact support and the second integral is finite, since

$$\left|\int_{-k}^{s} \mathcal{Z}_3(t)\dot{\mathcal{X}}(t)\, dt\right| = \left|\int_{\mathcal{X}(-k)}^{\mathcal{X}(s)} U_1(X)\, dX\right| \leq \big(2||\mathcal{X} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2k\big)^{\frac{1}{2}}||U_1||_{L^2(\mathbb{R})}$$

for $s \in [-k, k]$. We have

$$\int_{\mathbb{R}} \phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\, ds = -\int_{-k}^{k} \phi'(s)\int_{\mathcal{X}(-k)}^{\mathcal{X}(s)} U_1(X)\, dX\, ds$$

and similarly we obtain

$$\int_{\mathbb{R}} \phi(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\, ds = -\int_{-k}^{k} \phi'(s)\int_{\mathcal{X}^n(-k)}^{\mathcal{X}^n(s)} U_1^n(X)\, dX\, ds.$$

By the Cauchy–Schwarz inequality we get

$$\begin{aligned}
&\left|\int_{\mathcal{X}(-k)}^{\mathcal{X}(s)} U_1(X)\, dX - \int_{\mathcal{X}^n(-k)}^{\mathcal{X}^n(s)} U_1^n(X)\, dX\right| \\
&\quad \leq \left|\int_{\mathcal{X}(-k)}^{\mathcal{X}(s)} \big[U_1(X) - U_1^n(X)\big]\, dX\right| + \left|\int_{\mathcal{X}^n(-k)}^{\mathcal{X}(-k)} U_1^n(X)\, dX\right| + \left|\int_{\mathcal{X}^n(s)}^{\mathcal{X}(s)} U_1^n(X)\, dX\right| \\
&\quad \leq ||U_1 - U_1^n||_{L^2(\mathbb{R})}\big|\mathcal{X}(k) - \mathcal{X}(-k)\big|^{\frac{1}{2}} + 2||U_1^n||_{L^2(\mathbb{R})}||\mathcal{X} - \mathcal{X}^n||^{\frac{1}{2}}_{L^\infty(\mathbb{R})},
\end{aligned} \tag{7.68}$$

which implies that

$$\begin{aligned}&\left|\int_{\mathbb{R}}\phi(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds-\int_{\mathbb{R}}\phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\,ds\right|\\&\quad\le 2k||\phi'||_{L^\infty(\mathbb{R})}\Big[||U_1-U_1^n||_{L^2(\mathbb{R})}\big(2||\mathcal{X}-\mathrm{Id}\,||_{L^\infty(\mathbb{R})}+2k\big)^{\frac{1}{2}}\\&\qquad+2\big(||U_1^n-U_1||_{L^2(\mathbb{R})}+||U_1||_{L^2(\mathbb{R})}\big)||\mathcal{X}-\mathcal{X}^n||^{\frac{1}{2}}_{L^\infty(\mathbb{R})}\Big].\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}}\phi(s)\mathcal{Z}_3^n(s)\dot{\mathcal{X}}^n(s)\,ds-\int_{\mathbb{R}}\phi(s)\mathcal{Z}_3(s)\dot{\mathcal{X}}(s)\,ds\right|=0. \tag{7.69}$$

From (7.66), (7.67) and (7.69) we conclude that $T_1^n\to 0$ as $n\to\infty$. In a similar way we can prove that $T_2^n\to 0$. This implies by (7.65) that $\mathcal{Z}_3^n\to\mathcal{Z}_3$ in $L^2(\mathbb{R})$.

Using (3.26a) we get

$$||\mathcal{V}_1-\mathcal{V}_1^n||_{L^2(\mathbb{R})}\le\left|\left|x_1'\left(\frac{1}{2c(U_1)}-\frac{1}{2c(U_1^n)}\right)\right|\right|_{L^2(\mathbb{R})}+\left|\left|\frac{1}{2c(U_1^n)}\left(x_1'-(x_1^n)'\right)\right|\right|_{L^2(\mathbb{R})},$$

and recalling (7.44), we see that $\mathcal{V}_1^n\to\mathcal{V}_1$ in $L^2(\mathbb{R})$.

From (3.26b)–(3.26f) and the assumptions we immediately get that $\mathcal{V}_i^n\to\mathcal{V}_i, i=2,\dots,5$ and $\mathfrak{p}^n\to\mathfrak{p}$ in $L^2(\mathbb{R})$.

The corresponding results for $\mathcal{W}$ and $\mathfrak{q}$ can be proved in a similar way. □

Theorem 7.9 deals with convergence of the elements u, ρ, and σ for $(u,R,S,\rho,\sigma,\mu,\nu)$ in $\mathcal{D}$, see (P1”) and (P2”) therein. The following result indicates the type of convergence we have to assume for elements in $\mathcal{G}_0$ in order to get weak-star convergence for the remaining elements in $\mathcal{D}$.

Lemma 7.8 *Let* $\Theta=(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{V},\mathcal{W},\mathfrak{p},\mathfrak{q})$ *and* $\Theta^n=(\mathcal{X}^n,\mathcal{Y}^n,\mathcal{Z}^n,\mathcal{V}^n,\mathcal{W}^n,\mathfrak{p}^n,\mathfrak{q}^n)$ *belong to* $\mathcal{G}_0$. *Consider* $(u,R,S,\rho,\sigma,\mu,\nu)=\mathbf{M}\circ\mathbf{D}(\Theta)$ *and* $(u^n,R^n,S^n,\rho^n,\sigma^n,\mu^n,\nu^n)=\mathbf{M}\circ\mathbf{D}(\Theta^n)$. *Assume that*

$$\begin{aligned}&\mathcal{X}^n\to\mathcal{X},\quad\mathcal{Y}^n\to\mathcal{Y},\quad\mathcal{Z}_i^n\to\mathcal{Z}_i\quad in\quad L^\infty(\mathbb{R}),\\&\mathcal{Z}_3^n\to\mathcal{Z}_3,\quad\mathcal{V}_i^n\to\mathcal{V}_i,\quad\mathcal{W}_i^n\to\mathcal{W}_i,\quad\mathfrak{p}^n\to\mathfrak{p},\quad\mathfrak{q}^n\to\mathfrak{q}\quad in\quad L^2(\mathbb{R}),\end{aligned}$$

for $i=1,\dots,5$. *Then*

$$u^n\to u\quad in\quad L^\infty(\mathbb{R})$$

and

$$R^n\overset{*}{\rightharpoonup}R,\quad S^n\overset{*}{\rightharpoonup}S,\quad\rho^n\overset{*}{\rightharpoonup}\rho,\quad\sigma^n\overset{*}{\rightharpoonup}\sigma,\quad\mu^n\overset{*}{\rightharpoonup}\mu,\quad and\quad\nu^n\overset{*}{\rightharpoonup}\nu.$$

Observe that there are no assumptions on the monotonicity of $(\mathcal{X},\mathcal{Y})$ and $(\mathcal{X}^n,\mathcal{Y}^n)$. This means that as functions of s they are nondecreasing, but not necessarily strictly increasing.

From these results it follows immediately that $u_x^n\overset{*}{\rightharpoonup}u_x$ due to (3.1).

Proof We will use Lemma 5.8. For any x, there exist s and s^n, which are not necessarily unique, such that $x=\mathcal{Z}_2(s)$ and $x=\mathcal{Z}_2^n(s^n)$. By (5.7a), we have $u(x)=\mathcal{Z}_3(s)$ and $u^n(x)=\mathcal{Z}_3^n(s^n)$. We have

$$u(x)-u^n(x)=\mathcal{Z}_3(s)-\mathcal{Z}_3^n(s^n)=\mathcal{Z}_3(s)-\mathcal{Z}_3(s^n)+\mathcal{Z}_3(s^n)-\mathcal{Z}_3^n(s^n). \tag{7.70}$$

We estimate the difference $\mathcal{Z}_3(s) - \mathcal{Z}_3(s^n)$. We assume that $s^n \leq s$, the other case can be treated similar. We have

$$\begin{aligned}
|\mathcal{Z}_3(s) - \mathcal{Z}_3(s^n)| &= \left| \int_{s^n}^{s} \dot{\mathcal{Z}}_3(\bar{s})\, d\bar{s} \right| = \left| \int_{s^n}^{s} (\mathcal{V}_3(\mathcal{X}(\bar{s}))\dot{\mathcal{X}}(\bar{s}) + \mathcal{W}_3(\mathcal{Y}(\bar{s}))\dot{\mathcal{Y}}(\bar{s}))\, d\bar{s} \right| \\
&\leq \left(\int_{s^n}^{s} \dot{\mathcal{X}}(\bar{s})\, d\bar{s} \right)^{\frac{1}{2}} \left(\int_{s^n}^{s} \mathcal{V}_3^2(\mathcal{X}(\bar{s}))\dot{\mathcal{X}}(\bar{s})\, d\bar{s} \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{s^n}^{s} \dot{\mathcal{Y}}(\bar{s})\, d\bar{s} \right)^{\frac{1}{2}} \left(\int_{s^n}^{s} \mathcal{W}_3^2(\mathcal{Y}(\bar{s}))\dot{\mathcal{Y}}(\bar{s})\, d\bar{s} \right)^{\frac{1}{2}}
\end{aligned} \tag{7.71}$$

by the Cauchy–Schwarz inequality. From (3.19c) and (3.22) we get

$$\begin{aligned}
\int_{s^n}^{s} \mathcal{V}_3^2(\mathcal{X}(\bar{s}))\dot{\mathcal{X}}(\bar{s})\, d\bar{s} &\leq \int_{s^n}^{s} \frac{2\mathcal{V}_2(\mathcal{X}(\bar{s}))\mathcal{V}_4(\mathcal{X}(\bar{s}))}{c^2(\mathcal{Z}_3(\bar{s}))} \dot{\mathcal{X}}(\bar{s})\, d\bar{s} \\
&\leq \kappa^2 ||\mathcal{V}_4^a||_{L^\infty(\mathbb{R})} \int_{s^n}^{s} \dot{\mathcal{Z}}_2(\bar{s})\, d\bar{s} \leq \kappa^2 ||\mathcal{V}_4^a||_{L^\infty(\mathbb{R})} ||\mathcal{Z}_2^n - \mathcal{Z}_2||_{L^\infty(\mathbb{R})},
\end{aligned} \tag{7.72}$$

where we used that $\mathcal{Z}_2(s) = \mathcal{Z}_2^n(s^n)$. In a similar way, we obtain

$$\int_{s^n}^{s} \mathcal{W}_3^2(\mathcal{Y}(\bar{s}))\dot{\mathcal{Y}}(\bar{s})\, d\bar{s} \leq \kappa^2 ||\mathcal{W}_4^a||_{L^\infty(\mathbb{R})} ||\mathcal{Z}_2^n - \mathcal{Z}_2||_{L^\infty(\mathbb{R})}. \tag{7.73}$$

Using $\mathcal{Z}_2(s) = \mathcal{Z}_2^n(s^n)$ once more we get

$$\begin{aligned}
\int_{s^n}^{s} \dot{\mathcal{X}}(\bar{s})\, d\bar{s} &= \big(\mathcal{X}(s) - s\big) - \big(\mathcal{X}(s^n) - s^n\big) - \big(\mathcal{Z}_2(s) - s\big) \\
&\quad + \big(\mathcal{Z}_2^n(s^n) - \mathcal{Z}_2(s^n)\big) + \big(\mathcal{Z}_2(s^n) - s^n\big) \\
&\leq 2||\mathcal{X} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2||\mathcal{Z}_2 - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + ||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{7.74}$$

Similarly, we get

$$\int_{s^n}^{s} \dot{\mathcal{Y}}(\bar{s})\, d\bar{s} \leq 2||\mathcal{Y} - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + 2||\mathcal{Z}_2 - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} + ||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty(\mathbb{R})}. \tag{7.75}$$

Combining (7.72)–(7.75) in (7.71) and using that $\mathcal{Z}_2^n \to \mathcal{Z}_2$ in $L^\infty(\mathbb{R})$ we find that $\mathcal{Z}_3(s^n) \to \mathcal{Z}_3(s)$. Using this and that $\mathcal{Z}_3^n \to \mathcal{Z}_3$ in $L^\infty(\mathbb{R})$ in (7.70) we conclude that $u^n \to u$ in $L^\infty(\mathbb{R})$.

We prove that $\mu^n \overset{*}{\rightharpoonup} \mu$, that is,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi(x)\, d\mu^n = \int_{\mathbb{R}} \phi(x)\, d\mu$$

for all $\phi \in C_0(\mathbb{R})$. Here $C_0(\mathbb{R})$ is the space of continuous functions that vanish at infinity. Since $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ it suffices to consider test functions ϕ in $C_c^\infty(\mathbb{R})$. By (5.7f) we have

$$\int_{\mathbb{R}} \phi(x)\, d\mu = \int_{\mathbb{R}} \phi(\mathcal{Z}_2(s))\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s)\, ds. \tag{7.76}$$

By (3.18) and (3.19b) we have $2\mathcal{V}_4(\mathcal{X}(s))\dot{\mathcal{X}}(s) = \dot{\mathcal{Z}}_4(s) + c(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_5(s)$. Inserting this in (7.76) and using integration by parts yields

$$\begin{aligned}\int_{\mathbb{R}} \phi(x)\, d\mu &= \frac{1}{2}\int_{\mathbb{R}} \big[\phi(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_4(s) + \phi(\mathcal{Z}_2(s))c(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_5(s)\big]\, ds \\ &= -\frac{1}{2}\int_{\mathbb{R}} \phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\mathcal{Z}_4(s)\, ds \\ &\quad -\frac{1}{2}\int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s)) + \phi(\mathcal{Z}_2(s))c'(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_3(s)\big]\mathcal{Z}_5(s)\, ds.\end{aligned}$$

Combining this with the corresponding expression for $\int_{\mathbb{R}} \phi(x)\, d\mu^n$ yields

$$\begin{aligned}&\int_{\mathbb{R}} \phi(x)\, d\mu - \int_{\mathbb{R}} \phi(x)\, d\mu^n \\ &= -\frac{1}{2}\bigg(\int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\mathcal{Z}_4(s) - \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)\mathcal{Z}_4^n(s)\big]\, ds \\ &\quad + \int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s) - \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3^n(s))\mathcal{Z}_5^n(s)\big]\, ds \\ &\quad + \int_{\mathbb{R}} \big[\phi(\mathcal{Z}_2(s))c'(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_3(s)\mathcal{Z}_5(s) - \phi(\mathcal{Z}_2^n(s))c'(\mathcal{Z}_3^n(s))\dot{\mathcal{Z}}_3^n(s)\mathcal{Z}_5^n(s)\big]\, ds\bigg).\end{aligned} \tag{7.77}$$

The three integrals on the right-hand side of (7.77) can be treated in more or less the same way, and we only consider the second one. We have

$$\begin{aligned}&\int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s) - \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3^n(s))\mathcal{Z}_5^n(s)\big]\, ds \\ &= \int_{\mathbb{R}} \dot{\mathcal{Z}}_2(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s)\big[\phi'(\mathcal{Z}_2(s)) - \phi'(\mathcal{Z}_2^n(s))\big]\, ds \quad (I_1^n) \\ &\quad + \int_{\mathbb{R}} \phi'(\mathcal{Z}_2^n(s))c(\mathcal{Z}_3(s))\mathcal{Z}_5(s)\big[\dot{\mathcal{Z}}_2(s) - \dot{\mathcal{Z}}_2^n(s)\big]\, ds \quad (I_2^n) \\ &\quad + \int_{\mathbb{R}} \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3(s))\big[\mathcal{Z}_5(s) - \mathcal{Z}_5^n(s)\big]\, ds \quad (I_3^n) \\ &\quad + \int_{\mathbb{R}} \phi'(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)\mathcal{Z}_5^n(s)\big[c(\mathcal{Z}_3(s)) - c(\mathcal{Z}_3^n(s))\big]\, ds \quad (I_4^n).\end{aligned} \tag{7.78}$$

We have

$$\big|I_1^n\big| \leq 4\kappa\bigg(||\mathcal{V}_2^a||_{L^\infty(\mathbb{R})} + \frac{1}{2}\bigg)||\mathcal{Z}_5||_{L^\infty(\mathbb{R})}||\phi' \circ \mathcal{Z}_2 - \phi' \circ \mathcal{Z}_2^n||_{L^1(\mathbb{R})} \tag{7.79}$$

since

$$0 \leq \dot{\mathcal{Z}}_2(s) = 2\mathcal{V}_2(\mathcal{X}(s))\dot{\mathcal{X}}(s) \leq 4\mathcal{V}_2(\mathcal{X}(s)) = 4\bigg(\mathcal{V}_2^a(\mathcal{X}(s)) + \frac{1}{2}\bigg).$$

We have $\phi' \circ \mathcal{Z}_2^n \to \phi' \circ \mathcal{Z}_2$ pointwise almost everywhere, and by Lemma 7.6 we find that $\phi' \circ \mathcal{Z}_2^n$ can be uniformly bounded by an $L^1(\mathbb{R})$ function. By the dominated convergence theorem we get $\lim_{n\to\infty} ||\phi' \circ \mathcal{Z}_2 - \phi' \circ \mathcal{Z}_2^n||_{L^1(\mathbb{R})} = 0$, and (7.79) implies $\lim_{n\to\infty} I_1^n = 0$.

Integration by parts yields

$$\begin{aligned}I_2^n = -\int_{\mathbb{R}} &\big[\phi''(\mathcal{Z}_2^n(s))\dot{\mathcal{Z}}_2^n(s)c(\mathcal{Z}_3(s))\mathcal{Z}_5(s) + \phi'(\mathcal{Z}_2^n(s))c'(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_3(s)\mathcal{Z}_5(s) \\ &+ \phi'(\mathcal{Z}_2^n(s))c(\mathcal{Z}_3(s))\dot{\mathcal{Z}}_5(s)\big]\big[\mathcal{Z}_2(s) - \mathcal{Z}_2^n(s)\big]\, ds,\end{aligned}$$

which implies

$$\begin{aligned} |I_2^n| \leq \Bigg(& \kappa ||\mathcal{Z}_5||_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \big|\phi''(\mathcal{Z}_2^n(s))\big| \dot{\mathcal{Z}}_2^n(s)\, ds \\ & + \Big[2k_1 ||\mathcal{Z}_5||_{L^\infty(\mathbb{R})} \big(||\mathcal{V}_3^a||_{L^\infty(\mathbb{R})} + ||\mathcal{W}_3^a||_{L^\infty(\mathbb{R})} \big) \\ & + 2\kappa \big(||\mathcal{V}_5^a||_{L^\infty(\mathbb{R})} + ||\mathcal{W}_5^a||_{L^\infty(\mathbb{R})} \big) \Big] \int_{\mathbb{R}} \big|\phi'(\mathcal{Z}_2^n(s))\big|\, ds \Bigg) ||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty(\mathbb{R})}. \end{aligned}$$

By a change of variables and an estimate as in the proof of Lemma 7.6 we get

$$\begin{aligned} |I_2^n| \leq \Bigg(& \kappa ||\mathcal{Z}_5||_{L^\infty(\mathbb{R})} ||\phi''||_{L^\infty(\mathbb{R})} \mathrm{meas}\big(\mathrm{supp}(\phi'')\big) \\ & + \Big[2k_1 ||\mathcal{Z}_5||_{L^\infty(\mathbb{R})} \big(||\mathcal{V}_3^a||_{L^\infty(\mathbb{R})} + ||\mathcal{W}_3^a||_{L^\infty(\mathbb{R})} \big) \\ & + 2\kappa \big(||\mathcal{V}_5^a||_{L^\infty(\mathbb{R})} + ||\mathcal{W}_5^a||_{L^\infty(\mathbb{R})} \big) \Big] \tilde{C} ||\phi'||_{L^\infty(\mathbb{R})} \Bigg) ||\mathcal{Z}_2 - \mathcal{Z}_2^n||_{L^\infty(\mathbb{R})} \end{aligned}$$

for a constant $\tilde{C}$ that is independent of n. Since $\mathcal{Z}_2^n \to \mathcal{Z}_2$ in $L^\infty(\mathbb{R})$ we get $\lim_{n\to\infty} I_2^n = 0$.

Following the same lines as above we find $\lim_{n\to\infty} I_3^n = 0 = \lim_{n\to\infty} I_4^n$.

We return to (7.78) and obtain

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} \big[\phi'(\mathcal{Z}_2(s)) \dot{\mathcal{Z}}_2(s) c(\mathcal{Z}_3(s)) \mathcal{Z}_5(s) - \phi'(\mathcal{Z}_2^n(s)) \dot{\mathcal{Z}}_2^n(s) c(\mathcal{Z}_3^n(s)) \mathcal{Z}_5^n(s) \big]\, ds \right| = 0. \tag{7.80}$$

By using (7.80) in (7.77) we get

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} \phi(x)\, d\mu - \int_{\mathbb{R}} \phi(x)\, d\mu^n \right| = 0 \tag{7.81}$$

for all $\phi \in C_c^\infty(\mathbb{R})$, and we conclude that $\mu^n \overset{*}{\rightharpoonup} \mu$. Similarly we prove that $\nu^n \overset{*}{\rightharpoonup} \nu$.

Next we show that $\rho^n \overset{*}{\rightharpoonup} \rho$, that is,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \rho^n(x)\phi(x)\, dx = \int_{\mathbb{R}} \rho(x)\phi(x)\, dx \tag{7.82}$$

for all $\phi \in L^2(\mathbb{R})$. Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it suffices to consider test functions ϕ in $C_c^\infty(\mathbb{R})$.

Consider a test function ϕ in $C_c^\infty(\mathbb{R})$. By (5.7d), we get

$$\begin{aligned} \int_{\mathbb{R}} \phi(x)\big[\rho(x) - \rho^n(x)\big]\, dx = 2 & \int_{\mathbb{R}} \phi(\mathcal{Z}_2(s)) \big[\mathfrak{p}(\mathcal{X}(s)) \dot{\mathcal{X}}(s) - \mathfrak{p}^n(\mathcal{X}^n(s)) \dot{\mathcal{X}}^n(s) \big]\, ds \quad (A_1^n) \\ & + 2 \int_{\mathbb{R}} \mathfrak{p}^n(\mathcal{X}^n(s)) \dot{\mathcal{X}}^n(s) \big[\phi(\mathcal{Z}_2(s)) - \phi(\mathcal{Z}_2^n(s)) \big]\, ds \quad (A_2^n). \end{aligned} \tag{7.83}$$

Since ϕ has compact support, there exists $k > 0$ such that $\mathrm{supp}(\phi) \subset [-k, k]$. Thus, we only integrate over s such that $-k \leq \mathcal{Z}_2(s) \leq k$ in A_1^n. Since the quantity $\mathcal{Z}_2 - \mathrm{Id}$ belongs to $L^\infty(\mathbb{R})$, the region we integrate over is contained in $-k - ||\mathcal{Z}_2 - \mathrm{Id}\,||_{L^\infty(\mathbb{R})} \leq s \leq k + ||\mathcal{Z}_2 - \mathrm{Id}\,||_{L^\infty(\mathbb{R})}$, or written more compactly, $-M \leq s \leq M$ with $M = k + ||\mathcal{Z}_2 - \mathrm{Id}\,||_{L^\infty(\mathbb{R})}$.

Integration by parts yields

$$\begin{aligned}
A_1^n &= 2\int_{-M}^{M} \phi(\mathcal{Z}_2(s))\big[\mathfrak{p}(\mathcal{X}(s))\dot{\mathcal{X}}(s) - \mathfrak{p}^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big]\,ds \\
&= 2\bigg[\phi(\mathcal{Z}_2(s))\int_{-M}^{s} \big[\mathfrak{p}(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) - \mathfrak{p}^n(\mathcal{X}^n(\tau))\dot{\mathcal{X}}^n(\tau)\big]\,d\tau\bigg]_{s=-M}^{s=M} \\
&\quad - 2\int_{-M}^{M} \phi'(\mathcal{Z}_2(s))\dot{\mathcal{Z}}_2(s)\bigg(\int_{-M}^{s} \big[\mathfrak{p}(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) - \mathfrak{p}^n(\mathcal{X}^n(\tau))\dot{\mathcal{X}}^n(\tau)\big]\,d\tau\bigg)\,ds. \quad (7.84)
\end{aligned}$$

By a change of variables we have

$$\int_{-M}^{s} \big[\mathfrak{p}(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) - \mathfrak{p}^n(\mathcal{X}^n(\tau))\dot{\mathcal{X}}^n(\tau)\big]\,d\tau = \int_{\mathcal{X}(-M)}^{\mathcal{X}(s)} \mathfrak{p}(X)\,dX - \int_{\mathcal{X}^n(-M)}^{\mathcal{X}^n(s)} \mathfrak{p}^n(X)\,dX.$$

Using an estimate as in (7.68) yields

$$\begin{aligned}
&\bigg|\int_{-M}^{s} \big[\mathfrak{p}(\mathcal{X}(\tau))\dot{\mathcal{X}}(\tau) - \mathfrak{p}^n(\mathcal{X}^n(\tau))\dot{\mathcal{X}}^n(\tau)\big]\,d\tau\bigg| \\
&\quad\le ||\mathfrak{p}-\mathfrak{p}^n||_{L^2(\mathbb{R})}\big(2||\mathcal{X}-\mathrm{Id}\,||_{L^\infty(\mathbb{R})}+2M\big)^{\frac{1}{2}} \\
&\qquad + 2\big(||\mathfrak{p}^n-\mathfrak{p}||_{L^2(\mathbb{R})}+||\mathfrak{p}||_{L^2(\mathbb{R})}\big)||\mathcal{X}-\mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}}.
\end{aligned}$$

This implies that the first term on the right-hand side of (7.84) equals zero. Moreover, we get

$$\begin{aligned}
|A_1^n| \le 2\Big[&||\mathfrak{p}-\mathfrak{p}^n||_{L^2(\mathbb{R})}\big(2||\mathcal{X}-\mathrm{Id}\,||_{L^\infty(\mathbb{R})}+2M\big)^{\frac{1}{2}} \\
&+ 2\big(||\mathfrak{p}^n-\mathfrak{p}||_{L^2(\mathbb{R})}+||\mathfrak{p}||_{L^2(\mathbb{R})}\big)||\mathcal{X}-\mathcal{X}^n||_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\Big]||\phi'||_{L^\infty(\mathbb{R})}\mathrm{meas}\big(\mathrm{supp}(\phi')\big),
\end{aligned}$$

where we used that

$$\begin{aligned}
\int_{-M}^{M} |\phi'(\mathcal{Z}_2(s))|\dot{\mathcal{Z}}_2(s)\,ds &\le \int_{\mathbb{R}} |\phi'(\mathcal{Z}_2(s))|\dot{\mathcal{Z}}_2(s)\,ds = \int_{\mathbb{R}} |\phi'(x)|\,dx \\
&\le ||\phi'||_{L^\infty(\mathbb{R})}\mathrm{meas}\big(\mathrm{supp}(\phi')\big).
\end{aligned}$$

Since $\mathcal{X}^n \to \mathcal{X}$ in $L^\infty(\mathbb{R})$ and $\mathfrak{p}^n \to \mathfrak{p}$ in $L^2(\mathbb{R})$ we find that $\lim_{n\to\infty} A_1^n = 0$.

The second integral in (7.83) is estimated as follows. Using the Cauchy–Schwarz inequality and that $0 \le \dot{\mathcal{X}}^n \le 2$ we obtain

$$\begin{aligned}
|A_2^n| &\le 2\bigg(\int_{\mathbb{R}} \big(\mathfrak{p}^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big)^2\,ds\bigg)^{\frac{1}{2}}||\phi\circ\mathcal{Z}_2-\phi\circ\mathcal{Z}_2^n||_{L^2(\mathbb{R})} \\
&\le 2\sqrt{2}||\mathfrak{p}^n||_{L^2(\mathbb{R})}||\phi\circ\mathcal{Z}_2-\phi\circ\mathcal{Z}_2^n||_{L^2(\mathbb{R})} \\
&\le 2\sqrt{2}\big(||\mathfrak{p}^n-\mathfrak{p}||_{L^2(\mathbb{R})}+||\mathfrak{p}||_{L^2(\mathbb{R})}\big)||\phi\circ\mathcal{Z}_2-\phi\circ\mathcal{Z}_2^n||_{L^2(\mathbb{R})}
\end{aligned}$$

by a change of variables. We have $\phi\circ\mathcal{Z}_2^n \to \phi\circ\mathcal{Z}_2$ pointwise almost everywhere, and by Lemma 7.6 we get that $\phi\circ\mathcal{Z}_2^n$ can be uniformly bounded by an $L^2(\mathbb{R})$ function. Thus, the dominated convergence theorem implies that $\lim_{n\to\infty}||\phi\circ\mathcal{Z}_2-\phi\circ\mathcal{Z}_2^n||_{L^2(\mathbb{R})} = 0$. Since also $\mathfrak{p}^n \to \mathfrak{p}$ in $L^2(\mathbb{R})$ we find that $\lim_{n\to\infty} A_2^n = 0$.

We return to (7.83), and conclude that (7.82) holds. In a similar way one shows that $\sigma^n \overset{*}{\rightharpoonup} \sigma$.

To show that $R^n \overset{*}{\rightharpoonup} R$, we consider a test function ϕ in $C_c^\infty(\mathbb{R})$, use (5.7b), and write

$$\begin{aligned}\int_{\mathbb{R}} \phi(x)\big[R(x) - R^n(x)\big]\,dx &= 2\int_{\mathbb{R}} \phi(\mathcal{Z}_2(s))c(\mathcal{Z}_3(s))\big[\mathcal{V}_3(\mathcal{X}(s))\dot{\mathcal{X}}(s) - \mathcal{V}_3^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big]\,ds\\ &\quad + 2\int_{\mathbb{R}} c(\mathcal{Z}_3(s))\mathcal{V}_3^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big[\phi(\mathcal{Z}_2(s)) - \phi(\mathcal{Z}_2^n(s))\big]\,ds\\ &\quad + 2\int_{\mathbb{R}} \phi(\mathcal{Z}_2^n(s))\mathcal{V}_3^n(\mathcal{X}^n(s))\dot{\mathcal{X}}^n(s)\big[c(\mathcal{Z}_3(s)) - c(\mathcal{Z}_3^n(s))\big]\,ds,\end{aligned}$$

where the terms on the right-hand side can be treated more or less as demonstrated earlier. In a similar way one shows that $S^n \overset{*}{\rightharpoonup} S$. □

7.3 Approximation by local smooth solutions

Theorem 7.9 *Let $-\infty < x_l < x_r < \infty$. Consider $\xi_0 = (u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$ and $\xi_0^n = (u_0^n, R_0^n, S_0^n, \rho_0^n, \sigma_0^n, \mu_0^n, \nu_0^n)$ in $\mathcal{D}$. Assume that for all $n \in \mathbb{N}$,*

(A1”) *$u_0, R_0, S_0, \rho_0, \sigma_0, u_0^n, R_0^n, S_0^n, \rho_0^n, \sigma_0^n \in C^\infty([x_l, x_r])$,*
(A2”) *$\rho_0(x) = 0$ and $\sigma_0(x) = 0$ for all $x \in [x_l, x_r]$,*
(A3”) *there are constants $d_n > 0$ and $e_n > 0$ such that $\rho_0^n(x) \geq d_n$ and $\sigma_0^n(x) \geq e_n$ for all $x \in [x_l, x_r]$,*
(A4”) *$u_0^n \to u_0$ in $L^\infty([x_l, x_r])$, $R_0^n \to R_0$, $S_0^n \to S_0$, $\rho_0^n \to \rho_0$, and $\sigma_0^n \to \sigma_0$ in $L^2([x_l, x_r])$,*
(A5”) *μ_0, ν_0, μ_0^n, and ν_0^n are absolutely continuous on $[x_l, x_r]$,*
(A6”) *$\mu_0((-\infty, x_l)) = \mu_0^n((-\infty, x_l))$, $\mu_0((-\infty, x_r,)) = \mu_0^n((-\infty, x_r))$, $\nu_0((-\infty, x_l)) = \nu_0^n((-\infty, x_l))$, and $\nu_0((-\infty, x_r,)) = \nu_0^n((-\infty, x_r))$,*
(A7”) *$c \in C^\infty(\mathbb{R})$ and $c^{(m)} \in L^\infty(\mathbb{R})$ for $m = 3, 4, 5, \ldots$.*

For any $\tau \in \big[0, \frac{1}{2\kappa}(x_r - x_l)\big]$ consider $(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n)(\tau) = \bar{S}_\tau(\xi_0^n)$ and $(u, R, S, \rho, \sigma, \mu, \nu)(\tau) = \bar{S}_\tau(\xi_0)$. Then we have

(P1”) *$u^n(\tau, \cdot) \to u(\tau, \cdot)$ in $L^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$,*
(P2”) *$\rho^n(\tau, \cdot) \to 0$ and $\sigma^n(\tau, \cdot) \to 0$ in $L^1([x_l + \kappa\tau, x_r - \kappa\tau])$.*

The same conclusion holds on the interval $\big[x_l - \kappa\tau, x_r + \kappa\tau\big]$ in the case where $\tau \in \big[-\frac{1}{2\kappa}(x_r - x_l), 0\big]$.

Observe that we assume in (A4”) $u_0^n \to u_0$ in $L^\infty([x_l, x_r])$, in contrast to the $L^2(\mathbb{R})$ convergence in the global case, see Lemma 7.5. This is because functions in $H^1(\mathbb{R})$ tend to 0 as $x \to \pm\infty$, while here we have no assumptions on the values of u_0 and u_0^n at the endpoints of $[x_l, x_r]$. Since $[x_l, x_r]$ is a bounded interval, the convergence in $L^\infty([x_l, x_r])$ implies that $u_0^n \to u_0$ in $L^2([x_l, x_r])$.

By the assumptions (A5”) and (A6”) we mean that $\mu_0([x_l, x_r]) = \mu_0^n([x_l, x_r])$ and $\nu_0([x_l, x_r]) = \nu_0^n([x_l, x_r])$ for all n.

Note that the approximating sequence in Theorem 7.9 is locally smooth. Indeed, by Corollary 7.4 we have $u^n(\tau, \cdot), R^n(\tau, \cdot), S^n(\tau, \cdot), \rho^n(\tau, \cdot), \sigma^n(\tau, \cdot) \in C^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$. Furthermore, there are constants $\bar{d}_n > 0$ and $\bar{e}_n > 0$ such that $\rho^n(\tau, x) \geq \bar{d}_n$ and $\sigma^n(\tau, x) \geq \bar{e}_n$ for all $x \in [x_l + \kappa\tau, x_r - \kappa\tau]$, and $\mu^n(\tau, \cdot)$ and $\nu^n(\tau, \cdot)$ are absolutely continuous on $[x_l + \kappa\tau, x_r - \kappa\tau]$ for all n. However, the limit solution does not in general satisfy these properties, which is illustrated in an example in [15].

In addition, an inspection of the proof shows, that the above theorem actually states that locally, i.e., on $\{x \in [x_l + \kappa\tau, x_r - \kappa\tau] \mid \tau \in [0, \frac{1}{2\kappa}(x_r - x_l)]\}$, the conservative solution

$(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n)(\tau) = \bar{S}_\tau(\xi_0^n)$ to (1.3) converges to the conservative solution $(u^n, R^n, S^n, \rho^n, \sigma^n, \mu^n, \nu^n)(\tau) = \bar{S}_\tau(\xi_0^n)$ of the nonlinear variational wave equation (1.1), since $\mathfrak{p}^n \to 0$ and $\mathfrak{q}^n \to 0$.

Proof We will only consider the case $0 < \tau \leq \frac{1}{2\kappa}(x_r - x_l)$. The case $-\frac{1}{2\kappa}(x_r - x_l) \leq \tau < 0$ can be treated in the same way.

We split the proof into four steps.

Step 1. Set $(\psi_1, \psi_2) = \mathbf{L}(u_0, R_0, S_0, \rho_0, \sigma_0, \mu_0, \nu_0)$ and $(\psi_1^n, \psi_2^n) = \mathbf{L}(u_0^n, R_0^n, S_0^n, \rho_0^n, \sigma_0^n, \mu_0^n, \nu_0^n)$, where $\psi_1 = (x_1, U_1, J_1, K_1, V_1, H_1)$, $\psi_2 = (x_2, U_2, J_2, K_2, V_2, H_2)$, $\psi_1^n = (x_1^n, U_1^n, J_1^n, K_1^n, V_1^n, H_1^n)$, and $\psi_2^n = (x_2^n, U_2^n, J_2^n, K_2^n, V_2^n, H_2^n)$. Let us find out what kind of region the interval $[x_l, x_r]$ corresponds to in Lagrangian coordinates (X, Y). Since the measures are assumed to be absolutely continuous we get from (3.9a), $x_1(X) + \mu_0((-\infty, x_1(X))) = X$ for all $x_1(X) \in [x_l, x_r]$. We show which range of the X-variable this corresponds to. If $\hat{X}$ is such that $x_1(\hat{X}) = x_l$ then $x_l + \mu_0((-\infty, x_l)) = \hat{X}$. We also have $x_1^n(X) + \mu_0^n((-\infty, x_1^n(X))) = X$ for all $x_1^n(X) \in [x_l, x_r]$. If $\check{X}$ is such that $x_1^n(\check{X}) = x_l$ we get $x_l + \mu_0^n((-\infty, x_l)) = \check{X}$. Using (A6”) we obtain $\hat{X} = \check{X}$ and we denote $X_l = \hat{X} = \check{X}$. If $\bar{X}$ and $\tilde{X}$ are such that $x_1(\bar{X}) = x_1^n(\tilde{X}) = x_r$ then $x_r + \mu_0((-\infty, x_r)) = \bar{X}$ and $x_r + \mu_0^n((-\infty, x_r)) = \tilde{X}$, and by (A6”) we get $\bar{X} = \tilde{X}$ which we denote $X_r = \bar{X} = \tilde{X}$. We have $X_l \leq X_r$ since $X_l = x_l + \mu_0((-\infty, x_l)) \leq x_r + \mu_0((-\infty, x_r)) = X_r$. In other words, we can define the interval $[X_l, X_r]$ using either the measure μ_0 or μ_0^n. Observe that this is a consequence of the assumptions (A5”) and (A6”).

In a similar way we find that $x_2(Y) + \nu_0((-\infty, x_2(Y))) = Y$ and $x_2^n(Y) + \nu_0^n((-\infty, x_2^n(Y))) = Y$ for all $Y \in [Y_l, Y_r]$ where $Y_l = x_l + \nu_0((-\infty, x_l))$ and $Y_r = x_r + \nu_0((-\infty, x_r))$. Let $\Omega = [X_l, X_r] \times [Y_l, Y_r]$.

Following closely the proof of Lemma 7.5, we obtain that x_1, x_1^n, x_2, and x_2^n are strictly increasing for $X \in [X_l, X_r]$ and $Y \in [Y_l, Y_r]$, respectively, and

$$
\begin{aligned}
&x_1^n \to x_1, \quad (x_1^n)^{-1} \to x_1^{-1}, \quad U_1^n \to U_1, \quad J_1^n \to J_1 \quad \text{in } L^\infty([X_l, X_r]),\\
&x_2^n \to x_2, \quad (x_2^n)^{-1} \to x_2^{-1}, \quad U_2^n \to U_2, \quad J_2^n \to J_2 \quad \text{in } L^\infty([Y_l, Y_r]),\\
&V_1^n \to V_1, \quad H_1^n \to 0, \quad (x_1^n)' \to x_1', \quad (J_1^n)' \to J_1', \quad (K_1^n)' \to K_1' \quad \text{in } L^2([X_l, X_r]),\\
&V_2^n \to V_2, \quad H_2^n \to 0, \quad (x_2^n)' \to x_2', \quad (J_2^n)' \to J_2', \quad (K_2^n)' \to K_2' \quad \text{in } L^2([Y_l, Y_r]).
\end{aligned}
$$

Note that (3.9f) does not imply $K_1^n \to K_1$ in $L^\infty([X_l, X_r])$ and $K_2^n \to K_2$ in $L^\infty([Y_l, Y_r])$. However, we have $K_1^n - K_1^n(X_l) \to K_1 - K_1(X_l)$ in $L^\infty([X_l, X_r])$ and $K_2^n - K_2^n(Y_l) \to K_2 - K_2(Y_l)$ in $L^\infty([Y_l, Y_r])$. The proof of this closely follows the investigation of (7.53) in Lemma 7.5.

Step 2. Let $\Theta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{V}, \mathcal{W}, \mathfrak{p}, \mathfrak{q}) = \mathbf{C}(\psi_1, \psi_2)$ and $\Theta^n = (\mathcal{X}^n, \mathcal{Y}^n, \mathcal{Z}^n, \mathcal{V}^n, \mathcal{W}^n, \mathfrak{p}^n, \mathfrak{q}^n) = \mathbf{C}(\psi_1^n, \psi_2^n)$. From (3.23) and since both x_1 and x_2 are strictly increasing on $[X_l, X_r]$ and $[Y_l, Y_r]$, respectively, the relevant values of s are those satisfying $\mathcal{X}(s) \in [X_l, X_r]$ and $\mathcal{Y}(s) \in [Y_l, Y_r]$. In other words, since $x_1(X_l) = x_2(Y_l) = x_l$ and $x_1(X_r) = x_2(Y_r) = x_r$, we have $\frac{1}{2}(X_l + Y_l) \leq s \leq \frac{1}{2}(X_r + Y_r)$ and we let $s_l = \frac{1}{2}(X_l + Y_l)$ and $s_r = \frac{1}{2}(X_r + Y_r)$. The same conclusion holds for $(\mathcal{X}^n(s), \mathcal{Y}^n(s))$. In particular, one has

$$
(\mathcal{X}(s_l), \mathcal{Y}(s_l)) = (X_l, Y_l) = (\mathcal{X}^n(s_l), \mathcal{Y}^n(s_l)), \; (\mathcal{X}(s_r), \mathcal{Y}(s_r)) = (X_r, Y_r) = (\mathcal{X}^n(s_r), \mathcal{Y}^n(s_r)), \tag{7.85}
$$

i.e., the lower left and upper right corner of Ω.

Following closely the proof of Lemma 7.7, we obtain that $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}^n$, and $\mathcal{Y}^n$ are strictly increasing on $[s_l, s_r]$, and

$$\begin{aligned}&\mathcal{X}^n \to \mathcal{X}, \quad \mathcal{Y}^n \to \mathcal{Y}, \quad \mathcal{Z}_j^n \to \mathcal{Z}_j \quad \text{in } L^\infty([s_l, s_r]),\\ &\mathcal{V}_i^n \to \mathcal{V}_i, \quad \mathfrak{p}^n \to 0 \quad \text{in } L^2([X_l, X_r]),\\ &\mathcal{W}_i^n \to \mathcal{W}_i, \quad \mathfrak{q}^n \to 0 \quad \text{in } L^2([Y_l, Y_r])\end{aligned}$$

for $i = 1, \ldots, 5$, $j = 1, \ldots, 4$. Using (A2”), (A3”), and (7.3), we get from (3.26f) and (3.9g), that for all $X \in [X_l, X_r]$, $\mathfrak{p}(X) = 0$ and $\mathfrak{p}^n(X) \geq k_n > 0$ for some constant k_n.

Since $K_1^n - K_1^n(X_l) \to K_1 - K_1(X_l)$ in $L^\infty([X_l, X_r])$ and $K_2^n - K_2^n(Y_l) \to K_2 - K_2(Y_l)$ in $L^\infty([Y_l, Y_r])$ we get from (3.25e) and (7.85) that $\mathcal{Z}_5^n - \mathcal{Z}_5^n(s_l) \to \mathcal{Z}_5 - \mathcal{Z}_5(s_l)$ in $L^\infty([s_l, s_r])$.

Step 3. Consider $(Z, p, q) = \mathbf{S}(\Theta)$ and $(Z^n, p^n, q^n) = \mathbf{S}(\Theta^n)$. We prove a Gronwall type estimate. We claim that for all (X, Y) in Ω,

$$\begin{aligned}&\Big[\big[Z_3 - Z_3^n\big]^2 + \sum_{i=1}^5 \Big(\big[Z_{i,X} - Z_{i,X}^n\big]^2 + \big[Z_{i,Y} - Z_{i,Y}^n\big]^2\Big)\Big](X, Y)\\ &\leq K\Big\{||U_1 - U_1^n||^2_{L^\infty([X_l, X_r])} + \sum_{j=1}^5 \Big(\big[\mathcal{V}_j(X) - \mathcal{V}_j^n(X)\big]^2 + \big[\mathcal{W}_j(Y) - \mathcal{W}_j^n(Y)\big]^2\Big)\\ &\quad + \big[\mathcal{Y} \circ \mathcal{X}^{-1}(X) - \mathcal{Y}^n \circ (\mathcal{X}^n)^{-1}(X)\big]^2 + \big[\mathcal{X} \circ \mathcal{Y}^{-1}(Y) - \mathcal{X}^n \circ (\mathcal{Y}^n)^{-1}(Y)\big]^2\Big\},\end{aligned} \tag{7.86}$$

where K depends on κ, k_1, k_2, $|||\Theta|||_{\mathcal{G}(\Omega)}$ and the size of Ω.

Let $(X, Y) \in \Omega$. Subtracting the equations

$$t_X(X, Y) = t_X(X, \mathcal{Y}(X)) + \int_{\mathcal{Y}(X)}^Y t_{XY}(X, \tilde{Y})\, d\tilde{Y}$$

and

$$t_X^n(X, Y) = t_X^n(X, \mathcal{Y}^n(X)) + \int_{\mathcal{Y}^n(X)}^Y t_{XY}^n(X, \tilde{Y})\, d\tilde{Y} \tag{7.87}$$

yields, using $Z_X(X, \mathcal{Y}(X)) = \mathcal{V}(X)$ and $Z_X^n(X, \mathcal{Y}^n(X)) = \mathcal{V}^n(X)$,

$$\begin{aligned}t_X(X, Y) - t_X^n(X, Y) &= \mathcal{V}_1(X) - \mathcal{V}_1^n(X) - \int_{\mathcal{Y}^n(X)}^{\mathcal{Y}(X)} t_{XY}^n(X, \tilde{Y})\, d\tilde{Y}\\ &\quad + \int_{\mathcal{Y}(X)}^Y (t_{XY}(X, \tilde{Y}) - t_{XY}^n(X, \tilde{Y}))\, d\tilde{Y}.\end{aligned} \tag{7.88}$$

Using (2.19a) we get

$$\begin{aligned}t_{XY} - t_{XY}^n &= -\frac{c'(U)}{2c(U)}(U_X t_Y + U_Y t_X) + \frac{c'(U^n)}{2c(U^n)}(U_X^n t_Y^n + U_Y^n t_X^n)\\ &= -\frac{c'(U)}{2c(U)}\Big(U_X(t_Y - t_Y^n) + t_Y^n(U_X - U_X^n) + U_Y(t_X - t_X^n) + t_X^n(U_Y - U_Y^n)\Big)\\ &\quad - \frac{1}{2}(U_X^n t_Y^n + U_Y^n t_X^n)\int_{U^n}^U \left(\frac{c''(V)}{c(V)} - \frac{c'(V)^2}{c(V)^2}\right) dV.\end{aligned} \tag{7.89}$$

Next we need a pointwise uniform bound on the components of Z_X^n and Z_Y^n. By (2.19a) we get

$$|t_{XY}^n| \leq \frac{1}{2}k_1\kappa\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)\Big(|t_Y^n|+|x_Y^n|+|U_Y^n|+|J_Y^n|+|K_Y^n|\Big),$$

and by doing the same kind of estimate for the other components we get

$$\begin{aligned}&\Big(|t_{XY}^n|+|x_{XY}^n|+|U_{XY}^n|+|J_{XY}^n|+|K_{XY}^n|\Big)\\&\quad\leq B_1\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)\Big(|t_Y^n|+|x_Y^n|+|U_Y^n|+|J_Y^n|+|K_Y^n|\Big)\end{aligned}\tag{7.90}$$

for a constant B_1 that only depends on κ and k_1. By (7.87) and the corresponding expressions for x_X^n, U_X^n, J_X^n and K_X^n we obtain

$$\begin{aligned}&\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)(X,Y)\\&\quad\leq\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)(X,\mathcal{Y}^n(X))\\&\qquad+\Bigg|\int_{\mathcal{Y}^n(X)}^{Y}B_1\Big\{\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)\\&\qquad\times\Big(|t_Y^n|+|x_Y^n|+|U_Y^n|+|J_Y^n|+|K_Y^n|\Big)\Big\}(X,\tilde{Y})\,d\tilde{Y}\Bigg|,\end{aligned}\tag{7.91}$$

where we used (7.90), and by Gronwall's inequality,

$$\begin{aligned}&\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)(X,Y)\\&\quad\leq\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)(X,\mathcal{Y}^n(X))\\&\qquad\times\exp\Bigg\{B_1\Bigg|\int_{\mathcal{Y}^n(X)}^{Y}\Big(|t_Y^n|+|x_Y^n|+|U_Y^n|+|J_Y^n|+|K_Y^n|\Big)(X,\tilde{Y})\,d\tilde{Y}\Bigg|\Bigg\}.\end{aligned}$$

By (4.12a), (4.12b), (4.12d), and (4.12e) we get

$$\begin{aligned}|t_Y^n|+|x_Y^n|+|U_Y^n|+|J_Y^n|+|K_Y^n|&=\frac{1}{c(U^n)}x_Y^n+x_Y^n+|U_Y^n|+J_Y^n+\frac{1}{c(U^n)}J_Y^n\\&\leq(1+\kappa)(x_Y^n+J_Y^n)+|U_Y^n|.\end{aligned}\tag{7.92}$$

From (4.12c) we have $2J_Y^n x_Y^n \geq (c(U^n)U_Y^n)^2$, and by Young's inequality, $|U_Y^n| \leq \frac{\kappa}{\sqrt{2}}(J_Y^n + x_Y^n)$, which by (7.92) implies

$$|t_Y^n|+|x_Y^n|+|U_Y^n|+|J_Y^n|+|K_Y^n|\leq\Bigg[1+\Big(1+\frac{1}{\sqrt{2}}\Big)\kappa\Bigg](x_Y^n+J_Y^n).$$

Using this in (7.91) we obtain

$$\begin{aligned}&\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)(X,Y)\\&\quad\leq\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)(X,\mathcal{Y}^n(X))\exp\Bigg\{B_2\Bigg|\int_{\mathcal{Y}^n(X)}^{Y}(x_Y^n+J_Y^n)(X,\tilde{Y})\,d\tilde{Y}\Bigg|\Bigg\}\end{aligned}\tag{7.93}$$

for a new constant B_2 that again depends on κ and k_1. Since x^n and J^n are nondecreasing with respect to both variables, we have

$$\left|\int_{\mathcal{Y}^n(X)}^{Y}(x_Y^n+J_Y^n)(X,\tilde{Y})\,d\tilde{Y}\right| \leq \left|x^n(X_r,Y_r)-x^n(X_l,Y_l)\right|+\left|J^n(X_r,Y_r)-J^n(X_l,Y_l)\right|$$
$$=\left|\mathcal{Z}_2^n(s_r)-\mathcal{Z}_2^n(s_l)\right|+\left|\mathcal{Z}_4^n(s_r)-\mathcal{Z}_4^n(s_l)\right|.$$

Since

$$\mathcal{Z}_2^n(s_r)-\mathcal{Z}_2^n(s_l)=\mathcal{Z}_2^n(s_r)-\mathcal{Z}_2(s_r)+\mathcal{Z}_2^a(s_r)+s_r-\mathcal{Z}_2^n(s_l)+\mathcal{Z}_2(s_l)-\mathcal{Z}_2^a(s_l)-s_l$$

and

$$\mathcal{Z}_4^n(s_r)-\mathcal{Z}_4^n(s_l)=\mathcal{Z}_4^n(s_r)-\mathcal{Z}_4(s_r)+\mathcal{Z}_4^a(s_r)-\mathcal{Z}_4^n(s_l)+\mathcal{Z}_4(s_l)-\mathcal{Z}_4^a(s_l)$$

we end up with

$$\left|\int_{\mathcal{Y}^n(X)}^{Y}(x_Y^n+J_Y^n)(X,\tilde{Y})\,d\tilde{Y}\right| \leq 2||\mathcal{Z}_2-\mathcal{Z}_2^n||_{L^\infty([s_l,s_r])}+2||\mathcal{Z}_2^a||_{L^\infty([s_l,s_r])}+s_r-s_l$$
$$+2||\mathcal{Z}_4-\mathcal{Z}_4^n||_{L^\infty([s_l,s_r])}+2||\mathcal{Z}_4^a||_{L^\infty([s_l,s_r])}.$$

The convergence $\mathcal{Z}_i^n\to\mathcal{Z}_i$ in $L^\infty([s_l,s_r])$ implies that for every $\varepsilon>0$ there exists N such that for all $n\geq N$, we have $||\mathcal{Z}_i-\mathcal{Z}_i^n||_{L^\infty([s_l,s_r])}\leq\varepsilon$, $i=2,4$. Hence,

$$\left|\int_{\mathcal{Y}^n(X)}^{Y}(x_Y^n+J_Y^n)(X,\tilde{Y})\,d\tilde{Y}\right| \leq 4\varepsilon+2||\mathcal{Z}_2^a||_{L^\infty([s_l,s_r])}+2||\mathcal{Z}_4^a||_{L^\infty([s_l,s_r])}+s_r-s_l. \tag{7.94}$$

We use (7.94) and $Z_{i,X}^n(X,\mathcal{Y}^n(X))=\mathcal{V}_i^n(X)$ in (7.93) and obtain

$$\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)(X,Y)\leq B_3\sum_{i=1}^{5}\left|\mathcal{V}_i^n(X)\right| \tag{7.95}$$

where B_3 only depends on κ, k_1 and $|||\Theta|||_{\mathcal{G}(\Omega)}$. Since μ_0^n is absolutely continuous in $[x_l,x_r]$ we obtain as in (7.2) that $0\leq(x_1^n)'(X)\leq 1$ for all $X\in[X_l,X_r]$. Using (3.26a), (3.26b), (3.26d), (3.26e), (3.9c) and (3.9f) we get $0\leq\mathcal{V}_1^n(X)\leq\frac{1}{2}\kappa$, $0\leq\mathcal{V}_2^n(X)\leq\frac{1}{2}$, $0\leq\mathcal{V}_4^n(X)\leq 1$, $0\leq\mathcal{V}_5^n(X)\leq\kappa$ for all $X\in[X_l,X_r]$. From (3.26c) and (3.7c) we have $|\mathcal{V}_3^n(X)|\leq\frac{1}{c(U_1^n(X))}\sqrt{(J_1^n)'(X)(x_1^n)'(X)}\leq\kappa$. Inserting these estimates in (7.95) we get

$$\Big(|t_X^n|+|x_X^n|+|U_X^n|+|J_X^n|+|K_X^n|\Big)(X,Y)\leq B_4\quad\text{for all }(X,Y)\in\Omega \tag{7.96}$$

for a new constant B_4, which only depends on κ, k_1, and $|||\Theta|||_{\mathcal{G}(\Omega)}$. Similarly we can show that there exist constants B_5, B_6, and B_7, which only depend on κ, k_1, and $|||\Theta|||_{\mathcal{G}(\Omega)}$, such that for all $(X,Y)\in\Omega$,

$$\Big(|t_Y^n|+|x_Y^n|+|U_Y^n|+|J_Y^n|+|K_Y^n|\Big)(X,Y)\leq B_5, \tag{7.97}$$

$$\Big(|t_X|+|x_X|+|U_X|+|J_X|+|K_X|\Big)(X,Y)\leq B_6, \tag{7.98}$$

and

$$\Big(|t_Y|+|x_Y|+|U_Y|+|J_Y|+|K_Y|\Big)(X,Y)\leq B_7. \tag{7.99}$$

From (7.90) we get for all $(X, Y) \in \Omega$ that

$$\Big(|t^n_{XY}| + |x^n_{XY}| + |U^n_{XY}| + |J^n_{XY}| + |K^n_{XY}|\Big)(X, Y) \le D, \tag{7.100}$$

for a constant D that depends on κ, k_1, and $|||\Theta|||_{\mathcal{G}(\Omega)}$. From (7.96)–(7.99) we get in (7.89), for all $(X, Y) \in \Omega$ that

$$\begin{aligned} &\Big|t_{XY} - t^n_{XY}\Big|(X, Y) \\ &\quad \le C_1\Big(|U - U^n| + |t_X - t^n_X| + |U_X - U^n_X| + |t_Y - t^n_Y| + |U_Y - U^n_Y|\Big)(X, Y), \end{aligned} \tag{7.101}$$

where C_1 depends on κ, k_1, k_2, and $|||\Theta|||_{\mathcal{G}(\Omega)}$.

Recall the notation in (4.11). Using the estimates (7.101) and (7.100) in (7.88) we get

$$\begin{aligned} &|t_X(X, Y) - t^n_X(X, Y)| \\ &\quad \le |\mathcal{V}_1(X) - \mathcal{V}^n_1(X)| + D|\mathcal{Y} \circ \mathcal{X}^{-1}(X) - \mathcal{Y}^n \circ (\mathcal{X}^n)^{-1}(X)| \\ &\qquad + C_1\Big|\int_{\mathcal{Y}(X)}^{Y} \big(|U - U^n| + |t_X - t^n_X| + |U_X - U^n_X| + |t_Y - t^n_Y| + |U_Y - U^n_Y|\big)(X, \tilde{Y})\, d\tilde{Y}\Big|. \end{aligned}$$

To write the estimates more compactly, we denote $Z = (Z_1, Z_2, Z_3, Z_4, Z_5) = (t, x, U, J, K)$, and similar for Z^n. By the same procedure as above, we obtain

$$\begin{aligned} |Z_{j,X}(X, Y) - Z^n_{j,X}(X, Y)| &\le |\mathcal{V}_j(X) - \mathcal{V}^n_j(X)| + D|\mathcal{Y} \circ \mathcal{X}^{-1}(X) - \mathcal{Y}^n \circ (\mathcal{X}^n)^{-1}(X)| \\ &\quad + C_j\Big|\int_{\mathcal{Y}(X)}^{Y} \Big[|Z_3 - Z^n_3| + \sum_{i=1}^{5}\big(|Z_{i,X} - Z^n_{i,X}| + |Z_{i,Y} - Z^n_{i,Y}|\big)\Big](X, \tilde{Y})\, d\tilde{Y}\Big|, \end{aligned} \tag{7.102}$$

for $j = 1, 2, 3, 4, 5$, where C_j depends on κ, k_1, k_2 and $|||\Theta|||_{\mathcal{G}(\Omega)}$, and D depends on κ, k_1 and $|||\Theta|||_{\mathcal{G}(\Omega)}$.

Following the same lines as above we find estimates for the partial derivatives with respect to Y. We obtain for $j = 1, 2, 3, 4, 5$,

$$\begin{aligned} &|Z_{j,Y}(X, Y) - Z^n_{j,Y}(X, Y)| \\ &\quad \le |\mathcal{W}_j(Y) - \mathcal{W}^n_j(Y)| + D|\mathcal{X} \circ \mathcal{Y}^{-1}(Y) - \mathcal{X}^n \circ (\mathcal{Y}^n)^{-1}(Y)| \\ &\qquad + C_j\Big|\int_{X}^{\mathcal{X}(Y)} \Big[|Z_3 - Z^n_3| + \sum_{i=1}^{5}\big(|Z_{i,X} - Z^n_{i,X}| + |Z_{i,Y} - Z^n_{i,Y}|\big)\Big](\tilde{X}, Y)\, d\tilde{X}\Big|. \end{aligned} \tag{7.103}$$

We have

$$\begin{aligned} U(X, Y) - U^n(X, Y) = U(X, \mathcal{Y}(X)) - U^n(X, \mathcal{Y}^n(X)) - \int_{\mathcal{Y}^n(X)}^{\mathcal{Y}(X)} U^n_Y(X, \tilde{Y})\, d\tilde{Y} \\ + \int_{\mathcal{Y}(X)}^{Y} (U_Y(X, \tilde{Y}) - U^n_Y(X, \tilde{Y}))\, d\tilde{Y}. \end{aligned}$$

To any X in $[X_l, X_r]$, there exist unique s and s^n in $[s_l, s_r]$ such that $X = \mathcal{X}(s)$ and $X = \mathcal{X}^n(s^n)$ and we can write

$$\begin{aligned} U(X, \mathcal{Y}(X)) - U^n(X, \mathcal{Y}^n(X)) &= U(\mathcal{X}(s), \mathcal{Y}(s)) - U^n(\mathcal{X}^n(s^n), \mathcal{Y}^n(s^n)) = \mathcal{Z}_3(s) - \mathcal{Z}^n_3(s^n) \\ &= U_1(\mathcal{X}(s)) - U^n_1(\mathcal{X}^n(s^n)) = U_1(X) - U^n_1(X). \end{aligned}$$

This implies, using the alternative notation,

$$\begin{aligned}&|Z_3(X,Y)-Z_3^n(X,Y)|\\&\quad\leq ||U_1-U_1^n||_{L^\infty([X_l,X_r])}+B_5|\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)|\\&\qquad+\left|\int_{\mathcal{Y}(X)}^{Y}\left[|Z_3-Z_3^n|+\sum_{i=1}^{5}\left(|Z_{i,X}-Z_{i,X}^n|+|Z_{i,Y}-Z_{i,Y}^n|\right)\right](X,\tilde{Y})\,d\tilde{Y}\right|,\end{aligned}\tag{7.104}$$

where we used (7.97).

Applying Hölder's inequality to (7.102)–(7.104) yields for all $(X,Y)\in\Omega$,

$$\begin{aligned}&\left[[Z_3-Z_3^n]^2+\sum_{i=1}^{5}\left([Z_{i,X}-Z_{i,X}^n]^2+[Z_{i,Y}-Z_{i,Y}^n]^2\right)\right](X,Y)\\&\quad\leq 3||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}+3(B_5^2+5D^2)\left[\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)\right]^2\\&\qquad+15D^2\left[\mathcal{X}\circ\mathcal{Y}^{-1}(Y)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)\right]^2+3\sum_{j=1}^{5}\left[\mathcal{V}_j(X)-\mathcal{V}_j^n(X)\right]^2\\&\qquad+3\sum_{j=1}^{5}\left[\mathcal{W}_j(Y)-\mathcal{W}_j^n(Y)\right]^2\\&\qquad+33(1+C)|Y-\mathcal{Y}(X)|\left|\int_{\mathcal{Y}(X)}^{Y}\left[[Z_3-Z_3^n]^2+\sum_{i=1}^{5}\left([Z_{i,X}-Z_{i,X}^n]^2\right.\right.\right.\\&\qquad\qquad\left.\left.\left.+[Z_{i,Y}-Z_{i,Y}^n]^2\right)\right](X,\tilde{Y})\,d\tilde{Y}\right|\\&\qquad+33C|\mathcal{X}(Y)-X|\left|\int_{X}^{\mathcal{X}(Y)}\left[[Z_3-Z_3^n]^2+\sum_{i=1}^{5}\left([Z_{i,X}-Z_{i,X}^n]^2\right.\right.\right.\\&\qquad\qquad\left.\left.\left.+[Z_{i,Y}-Z_{i,Y}^n]^2\right)\right](\tilde{X},Y)\,d\tilde{X}\right|,\end{aligned}\tag{7.105}$$

where we introduced $C=\sum_{j=1}^{5}C_j^2$.

At this point we need a Gronwall inequality, which can be found in [9], see Theorem 1, case 2, at the beginning of the chapter "Gronwall inequalities in higher dimensions". For a proof, we refer to [15].

Lemma 7.10 *Consider a nonnegative function $u(x,y)$ in the region $x\geq 0$ and $y\geq 0$. Assume that*

$$u(x,y)\leq c+a\int_0^x u(r,y)\,dr+b\int_0^y u(x,s)\,ds,$$

where a, b, and c are nonnegative constants. Then we have

$$u(x,y)\leq ce^{2ax+2by+abxy}.$$

Using Lemma 7.10 we get from (7.105),

$$\begin{aligned}
&\Big[\big[Z_3 - Z_3^n\big]^2 + \sum_{i=1}^{5}\Big(\big[Z_{i,X} - Z_{i,X}^n\big]^2 + \big[Z_{i,Y} - Z_{i,Y}^n\big]^2\Big)\Big](X,Y) \\
&\quad \le \Big\{3||U_1 - U_1^n||^2_{L^\infty([X_l,X_r])} + 3(B_5^2 + 5D^2)\big[\mathcal{Y}\circ\mathcal{X}^{-1}(X) - \mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)\big]^2 \\
&\qquad + 15D^2\big[\mathcal{X}\circ\mathcal{Y}^{-1}(Y) - \mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(Y)\big]^2 \\
&\qquad + 3\sum_{j=1}^{5}\big[\mathcal{V}_j(X) - \mathcal{V}_j^n(X)\big]^2 + 3\sum_{j=1}^{5}\big[\mathcal{W}_j(Y) - \mathcal{W}_j^n(Y)\big]^2\Big\} \\
&\qquad\quad \times \exp\Big\{66C\big[\mathcal{X}(Y) - X\big]^2 + 66(1+C)\big[Y - \mathcal{Y}(X)\big]^2 \\
&\qquad\qquad + 33^2C(1+C)\big[\mathcal{X}(Y) - X\big]^2\big[Y - \mathcal{Y}(X)\big]^2\Big\}.
\end{aligned} \tag{7.106}$$

Since all the differences appearing in the exponential function are bounded by either $X_r - X_l$ or $Y_r - Y_l$ we can find a new constant K which depends on $\kappa, k_1, k_2, |||\Theta|||_{\mathcal{G}(\Omega)}$, and the size of Ω such that (7.86) holds. Thus, we have proved the claim.

From (7.86) we obtain an estimate for the difference $Z_i(X, 2s - X) - Z_i^n(X, 2s - X)$ for $i = 1, 2, 3, 4$. We have

$$\begin{aligned}
&Z_i(X,2s-X) - Z_i^n(X,2s-X) \\
&\quad = \mathcal{Z}_i(s) - \mathcal{Z}_i^n(s) + \int_{\mathcal{X}(s)}^{X}\big[(Z_{i,X} - Z_{i,X}^n) - (Z_{i,Y} - Z_{i,Y}^n)\big](\xi, 2s-\xi)\,d\xi \\
&\qquad - \int_{\mathcal{X}^n(s)}^{\mathcal{X}(s)}(Z_{i,X}^n - Z_{i,Y}^n)(\xi, 2s-\xi)\,d\xi,
\end{aligned}$$

which implies by (7.96), (7.97), and the Cauchy–Schwarz inequality, that

$$\begin{aligned}
&\big|Z_i(X,2s-X) - Z_i^n(X,2s-X)\big| \\
&\quad \le ||\mathcal{Z}_i - \mathcal{Z}_i^n||_{L^\infty([s_l,s_r])} + (B_4+B_5)||\mathcal{X} - \mathcal{X}^n||_{L^\infty([s_l,s_r])} + \big[X - \mathcal{X}(s)\big]^{\frac{1}{2}} \\
&\qquad \times \Big\{\Big|\int_{\mathcal{X}(s)}^{X}(Z_{i,X} - Z_{i,X}^n)^2(\xi,2s-\xi)\,d\xi\Big|^{\frac{1}{2}} + \Big|\int_{\mathcal{X}(s)}^{X}(Z_{i,Y} - Z_{i,Y}^n)^2(\xi,2s-\xi)\,d\xi\Big|^{\frac{1}{2}}\Big\} \\
&\quad \le ||\mathcal{Z}_i - \mathcal{Z}_i^n||_{L^\infty([s_l,s_r])} + (B_4+B_5)||\mathcal{X} - \mathcal{X}^n||_{L^\infty([s_l,s_r])} + 2(X_r - X_l)^{\frac{1}{2}} \\
&\qquad \times \Big|\int_{\mathcal{X}(s)}^{X}\Big[\big[Z_3 - Z_3^n\big]^2 + \sum_{i=1}^{5}\Big(\big[Z_{i,X} - Z_{i,X}^n\big]^2 + \big[Z_{i,Y} - Z_{i,Y}^n\big]^2\Big)\Big](\xi,2s-\xi)\,d\xi\Big|^{\frac{1}{2}}.
\end{aligned} \tag{7.107}$$

From (7.86), we have

$$\Big[\big[Z_3 - Z_3^n\big]^2 + \sum_{i=1}^{5}\Big(\big[Z_{i,X} - Z_{i,X}^n\big]^2 + \big[Z_{i,Y} - Z_{i,Y}^n\big]^2\Big)\Big](\xi, 2s-\xi)$$

$$
\begin{aligned}
&\leq K\Big\{||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}+\sum_{j=1}^5\Big(\big[\mathcal{V}_j(\xi)-\mathcal{V}_j^n(\xi)\big]^2+\big[\mathcal{W}_j(2s-\xi)-\mathcal{W}_j^n(2s-\xi)\big]^2\Big)\\
&\quad+\big[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\big]^2+\big[\mathcal{X}\circ\mathcal{Y}^{-1}(2s-\xi)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(2s-\xi)\big]^2\Big\}.
\end{aligned}
$$

Integration and a change of variables leads to

$$
\begin{aligned}
&\left|\int_{\mathcal{X}(s)}^X\Big[\big[Z_3-Z_3^n\big]^2+\sum_{i=1}^5\Big(\big[Z_{i,X}-Z_{i,X}^n\big]^2+\big[Z_{i,Y}-Z_{i,Y}^n\big]^2\Big)\Big](\xi,2s-\xi)\,d\xi\right|\\
&\quad\leq K\Big\{|X-\mathcal{X}(s)|||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}+\sum_{j=1}^5\Big(\Big|\int_{\mathcal{X}(s)}^X\big[\mathcal{V}_j(\xi)-\mathcal{V}_j^n(\xi)\big]^2\,d\xi\Big|\\
&\qquad+\Big|\int_{2s-X}^{\mathcal{Y}(s)}\big[\mathcal{W}_j(\xi)-\mathcal{W}_j^n(\xi)\big]^2\,d\xi\Big|\Big)+\Big|\int_{\mathcal{X}(s)}^X\big[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\big]^2\,d\xi\Big|\\
&\qquad+\Big|\int_{2s-X}^{\mathcal{Y}(s)}\big[\mathcal{X}\circ\mathcal{Y}^{-1}(\xi)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(\xi)\big]^2\,d\xi\Big|\Big\}\\
&\quad\leq K\Big\{(X_r-X_l)||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}\\
&\quad+\sum_{j=1}^5\Big(||\mathcal{V}_j-\mathcal{V}_j^n||^2_{L^2([X_l,X_r])}+||\mathcal{W}_j-\mathcal{W}_j^n||^2_{L^2([Y_l,Y_r])}\Big)\\
&\qquad+\int_{X_l}^{X_r}\big[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\big]^2\,d\xi\\
&\quad+\int_{Y_l}^{Y_r}\big[\mathcal{X}\circ\mathcal{Y}^{-1}(\xi)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(\xi)\big]^2\,d\xi\Big\}.
\end{aligned}
\tag{7.108}
$$

From (3.3c) we have $X+\mathcal{Y}\circ\mathcal{X}^{-1}(X)=2\mathcal{X}^{-1}(X)$ and $X+\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)=2(\mathcal{X}^n)^{-1}(X)$, so that

$$
\mathcal{Y}\circ\mathcal{X}^{-1}(X)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(X)=2(\mathcal{X}^{-1}(X)-(\mathcal{X}^n)^{-1}(X)). \tag{7.109}
$$

This leads to

$$
\begin{aligned}
\int_{X_l}^{X_r}\big[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\big]^2\,d\xi&=4\int_{X_l}^{X_r}\big[\mathcal{X}^{-1}(\xi)-(\mathcal{X}^n)^{-1}(\xi)\big]^2\,d\xi\\
&\leq 4(s_r-s_l)\int_{X_l}^{X_r}|\mathcal{X}^{-1}(\xi)-(\mathcal{X}^n)^{-1}(\xi)|\,d\xi.
\end{aligned}
\tag{7.110}
$$

To estimate the above integral, we need to introduce another sequence of curves on $[s_l,s_r]$ given by $(\hat{\mathcal{X}}^n(s),\hat{\mathcal{Y}}^n(s))=(\max\{\mathcal{X}(s),\mathcal{X}^n(s)\},\min\{\mathcal{Y}(s),\mathcal{Y}^n(s)\})$. This sequence satisfies that both $(\mathcal{X},\mathcal{Y})$ and $(\mathcal{X}^n,\mathcal{Y}^n)$ lie above or are equal to $(\hat{\mathcal{X}}^n,\hat{\mathcal{Y}}^n)$. This implies that $\mathcal{X}^{-1}(X)\geq(\hat{\mathcal{X}}^n)^{-1}(X)$ and $(\mathcal{X}^n)^{-1}(X)\geq(\hat{\mathcal{X}}^n)^{-1}(X)$ for all X in $[X_l,X_r]$, and that

$$
\int_{X_l}^{X_r}|\mathcal{X}^{-1}(\xi)-(\mathcal{X}^n)^{-1}(\xi)|\,d\xi=\int_{X_l}^{X_r}(\mathcal{X}^{-1}(\xi)-(\hat{\mathcal{X}}^n)^{-1}(\xi))\,d\xi
$$

$$+\int_{X_l}^{X_r}((\mathcal{X}^n)^{-1}(\xi)-(\hat{\mathcal{X}}^n)^{-1}(\xi))\,d\xi. \tag{7.111}$$

By a change of variables and integration by parts we get

$$\int_{X_l}^{X_r}\mathcal{X}^{-1}(\xi)\,d\xi=\int_{\mathcal{X}^{-1}(X_l)}^{\mathcal{X}^{-1}(X_r)}s\dot{\mathcal{X}}(s)\,ds=X_r\mathcal{X}^{-1}(X_r)-X_l\mathcal{X}^{-1}(X_l)-\int_{\mathcal{X}^{-1}(X_l)}^{\mathcal{X}^{-1}(X_r)}\mathcal{X}(s)\,ds,$$

and similarly we find

$$\int_{X_l}^{X_r}(\hat{\mathcal{X}}^n)^{-1}(\xi)\,d\xi=X_r(\hat{\mathcal{X}}^n)^{-1}(X_r)-X_l(\hat{\mathcal{X}}^n)^{-1}(X_l)-\int_{(\hat{\mathcal{X}}^n)^{-1}(X_l)}^{(\hat{\mathcal{X}}^n)^{-1}(X_r)}\hat{\mathcal{X}}^n(s)\,ds.$$

Note that $\mathcal{X}^{-1}(X_l)=s_l=(\hat{\mathcal{X}}^n)^{-1}(X_l)$ and $\mathcal{X}^{-1}(X_r)=s_r=(\hat{\mathcal{X}}^n)^{-1}(X_r)$. Therefore,

$$\int_{X_l}^{X_r}|\mathcal{X}^{-1}(\xi)-(\hat{\mathcal{X}}^n)^{-1}(\xi)|\,d\xi=\int_{s_l}^{s_r}(\hat{\mathcal{X}}^n(s)-\mathcal{X}(s))\,ds.$$

We obtain in a similar way,

$$\int_{X_l}^{X_r}|(\mathcal{X}^n)^{-1}(\xi)-(\hat{\mathcal{X}}^n)^{-1}(\xi)|\,d\xi=\int_{s_l}^{s_r}(\hat{\mathcal{X}}^n(s)-\mathcal{X}^n(s))\,ds.$$

Combining (7.111) and the above estimates we end up with

$$\begin{aligned}\int_{X_l}^{X_r}|\mathcal{X}^{-1}(\xi)-(\mathcal{X}^n)^{-1}(\xi)|\,d\xi&=\int_{s_l}^{s_r}(\hat{\mathcal{X}}^n(s)-\mathcal{X}(s))\,ds+\int_{s_l}^{s_r}(\hat{\mathcal{X}}^n(s)-\mathcal{X}^n(s))\,ds\\&=\int_{s_l}^{s_r}|\mathcal{X}^n(s)-\mathcal{X}(s)|\,ds,\end{aligned}$$

which inserted in (7.110) yields

$$\int_{X_l}^{X_r}\left[\mathcal{Y}\circ\mathcal{X}^{-1}(\xi)-\mathcal{Y}^n\circ(\mathcal{X}^n)^{-1}(\xi)\right]^2d\xi\leq 4(s_r-s_l)^2||\mathcal{X}-\mathcal{X}^n||_{L^\infty([s_l,s_r])}. \tag{7.112}$$

By similar calculations we find

$$\int_{Y_l}^{Y_r}\left[\mathcal{X}\circ\mathcal{Y}^{-1}(\xi)-\mathcal{X}^n\circ(\mathcal{Y}^n)^{-1}(\xi)\right]^2d\xi\leq 4(s_r-s_l)^2||\mathcal{Y}-\mathcal{Y}^n||_{L^\infty([s_l,s_r])}. \tag{7.113}$$

Returning to (7.108) we now get

$$\begin{aligned}&\left|\int_{\mathcal{X}(s)}^{X}\left[[Z_3-Z_3^n]^2+\sum_{i=1}^{5}\left([Z_{i,X}-Z_{i,X}^n]^2+[Z_{i,Y}-Z_{i,Y}^n]^2\right)\right](\xi,2s-\xi)\,d\xi\right|\\&\leq K\Bigg\{(X_r-X_l)||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}\\&\quad+\sum_{j=1}^{5}\left(||\mathcal{V}_j-\mathcal{V}_j^n||^2_{L^2([X_l,X_r])}+||\mathcal{W}_j-\mathcal{W}_j^n||^2_{L^2([Y_l,Y_r])}\right)\\&\quad+4(s_r-s_l)^2\left(||\mathcal{X}-\mathcal{X}^n||_{L^\infty([s_l,s_r])}+||\mathcal{Y}-\mathcal{Y}^n||_{L^\infty([s_l,s_r])}\right)\Bigg\},\end{aligned}$$

which we insert in (7.107) and get

$$\begin{aligned}
&\big|Z_i(X,2s-X)-Z_i^n(X,2s-X)\big|\\
&\quad\le ||\mathcal{Z}_i-\mathcal{Z}_i^n||_{L^\infty([s_l,s_r])}+(B_4+B_5)||\mathcal{X}-\mathcal{X}^n||_{L^\infty([s_l,s_r])}+2\sqrt{K(X_r-X_l)}\\
&\qquad\times\Bigg\{(X_r-X_l)||U_1-U_1^n||^2_{L^\infty([X_l,X_r])}\\
&\qquad+\sum_{j=1}^{5}\Big(||\mathcal{V}_j-\mathcal{V}_j^n||^2_{L^2([X_l,X_r])}+||\mathcal{W}_j-\mathcal{W}_j^n||^2_{L^2([Y_l,Y_r])}\Big)\\
&\qquad+4(s_r-s_l)^2\Big(||\mathcal{X}-\mathcal{X}^n||_{L^\infty([s_l,s_r])}+||\mathcal{Y}-\mathcal{Y}^n||_{L^\infty([s_l,s_r])}\Big)\Bigg\}^{\frac{1}{2}}
\end{aligned}\tag{7.114}$$

for $i=1,\dots,4$. A similar inequality is valid for

$$|(Z_5(X,2s-X)-Z_5(X_l,2s_l-X_l))-(Z_5^n(X,2s-X)-Z_5^n(X_l,2s_l-X_l))\big|,$$

the only difference from (7.114) being that we get $||(\mathcal{Z}_5-\mathcal{Z}_5(s_l))-(\mathcal{Z}_5^n-\mathcal{Z}_5^n(s_l))||_{L^\infty([s_l,s_r])}$ on the right-hand side.

Step 4. Let $0<\tau\le\frac{1}{2\kappa}(x_r-x_l)$ and consider $\Theta(\tau)=\mathbf{E}\circ\mathbf{t}_\tau(Z,p,q)$, $\Theta^n(\tau)=\mathbf{E}\circ\mathbf{t}_\tau(Z^n,p^n,q^n)$, $(u,R,S,\rho,\sigma,\mu,\nu)(\tau)=\mathbf{M}\circ\mathbf{D}(\Theta(\tau))$ and $(u^n,R^n,S^n,\rho^n,\sigma^n,\mu^n,\nu^n)(\tau)=\mathbf{M}\circ\mathbf{D}(\Theta^n(\tau))$.

We prove (P1”).

Recall Remark 7.2 and consider $z\in[x_l+\kappa\tau,x_r-\kappa\tau]$. Then there exist $s\in[\bar{s}_1,\bar{s}_2]$ and $s^n\in[\bar{s}_1^n,\bar{s}_2^n]$ such that $z=\mathcal{Z}_2(\tau,s)=\mathcal{Z}_2^n(\tau,s^n)$. Using (5.7a) we obtain

$$\begin{aligned}
u(\tau,z)-u^n(\tau,z)&=\mathcal{Z}_3(\tau,s)-\mathcal{Z}_3^n(\tau,s^n)\\
&=U(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))-U^n(\mathcal{X}^n(\tau,s^n),\mathcal{Y}^n(\tau,s^n)).
\end{aligned}\tag{7.115}$$

Now we can have several different scenarios depending on the order of the points s and s^n, and the intervals $[\bar{s}_1,\bar{s}_2]$ and $[\bar{s}_1^n,\bar{s}_2^n]$. We only show one of the challenging cases, the others can be treated in a similar way. Assume that $s\le\bar{s}_1^n\le\bar{s}_{\max}$. Here $[\bar{s}_{\min},\bar{s}_{\max}]$ denotes the maximal interval such that $(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))$ belongs to Ω for all $s\in[\bar{s}_{\min},\bar{s}_{\max}]$. Observe that in this case the point $(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n))$ is in Ω, but the point $(\mathcal{X}^n(\tau,s),\mathcal{Y}^n(\tau,s))$ may be outside Ω. So when estimating (7.115) we have to carefully choose points on the curve so that we do not end up outside Ω, see Fig. 5.

Write

$$\begin{aligned}
&U(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))-U^n(\mathcal{X}^n(\tau,s^n),\mathcal{Y}^n(\tau,s^n))\\
&\quad=U(\mathcal{X}(\tau,s),\mathcal{Y}(\tau,s))-U(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n))\quad(A_1^n)\\
&\qquad+U(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n))-U(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))\quad(A_2^n)\\
&\qquad+U(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))-U^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))\quad(A_3^n)\\
&\qquad+U^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))-U^n(\mathcal{X}^n(\tau,s^n),\mathcal{Y}^n(\tau,s^n))\quad(A_4^n).
\end{aligned}$$

By (3.18) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}
|A_1^n|&=|\mathcal{Z}_3(\tau,\bar{s}_1^n)-\mathcal{Z}_3(\tau,s)|=\left|\int_s^{\bar{s}_1^n}\dot{\mathcal{Z}}_3(\tau,r)\,dr\right|\\
&\le\left(\int_s^{\bar{s}_1^n}\dot{\mathcal{X}}(\tau,r)\,dr\right)^{\frac{1}{2}}\left(\int_s^{\bar{s}_1^n}\mathcal{V}_3^2(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)\,dr\right)^{\frac{1}{2}}
\end{aligned}$$

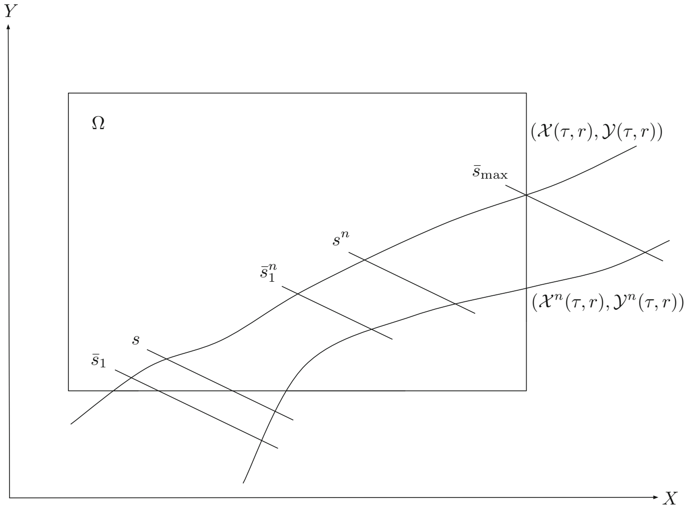

Fig. 5 An example of the situation described in Step 4. Note that the point $(\mathcal{X}^n(\tau, s), \mathcal{Y}^n(\tau, s))$ lies outside Ω

$$+\left(\int_s^{\bar{s}_1^n} \dot{\mathcal{Y}}(\tau, r)\,dr\right)^{\frac{1}{2}}\left(\int_s^{\bar{s}_1^n} \mathcal{W}_3^2(\tau, \mathcal{Y}(\tau, r))\dot{\mathcal{Y}}(\tau, r)\,dr\right)^{\frac{1}{2}}. \tag{7.116}$$

From (3.19c) and (3.22) we get

$$\begin{aligned}0 &\le \int_s^{\bar{s}_1^n} \mathcal{V}_3^2(\tau, \mathcal{X}(\tau, r))\dot{\mathcal{X}}(\tau, r)\,dr \le \int_s^{\bar{s}_1^n} \frac{2\mathcal{V}_4(\tau, \mathcal{X}(\tau, r))\mathcal{V}_2(\tau, \mathcal{X}(\tau, r))}{c^2(\mathcal{Z}_3(\tau, r))}\dot{\mathcal{X}}(\tau, r)\,dr\\ &\le \kappa^2 B_6 \int_s^{\bar{s}_1^n} \dot{\mathcal{Z}}_2(\tau, r)\,dr \le \kappa^2 B_6\big[\mathcal{Z}_2(\tau, \bar{s}_1^n) - \mathcal{Z}_2^n(\tau, \bar{s}_1^n)\big],\end{aligned} \tag{7.117}$$

where we used that $\mathcal{Z}_2(\tau, s) = \mathcal{Z}_2^n(\tau, s^n)$ and $\mathcal{Z}_2^n(\tau, \bar{s}_1^n) \le \mathcal{Z}_2^n(\tau, s^n)$ since $\bar{s}_1^n \le s^n$ and $\mathcal{Z}_2^n(\tau, \cdot)$ is nondecreasing. We also used that since $(\mathcal{X}(\tau, r), \mathcal{Y}(\tau, r)) \in \Omega$ for $r \in [s, \bar{s}_1^n]$ we can use the estimate in (7.98) to obtain $|\mathcal{V}_4(\tau, \mathcal{X}(\tau, r))| = |J_X(\mathcal{X}(\tau, r), \mathcal{Y}(\tau, r))| \le B_6$. We have

$$\begin{aligned}\mathcal{Z}_2(\tau, \bar{s}_1^n) - \mathcal{Z}_2^n(\tau, \bar{s}_1^n) &= x(\mathcal{X}(\tau, \bar{s}_1^n), \mathcal{Y}(\tau, \bar{s}_1^n)) - x^n(\mathcal{X}^n(\tau, \bar{s}_1^n), \mathcal{Y}^n(\tau, \bar{s}_1^n))\\ &= x(\mathcal{X}(\tau, \bar{s}_1^n), \mathcal{Y}(\tau, \bar{s}_1^n)) - x(\mathcal{X}^n(\tau, \bar{s}_1^n), \mathcal{Y}^n(\tau, \bar{s}_1^n))\\ &\quad + x(\mathcal{X}^n(\tau, \bar{s}_1^n), \mathcal{Y}^n(\tau, \bar{s}_1^n)) - x^n(\mathcal{X}^n(\tau, \bar{s}_1^n), \mathcal{Y}^n(\tau, \bar{s}_1^n)).\end{aligned} \tag{7.118}$$

By (3.3c), (4.12a), and since $t_X \ge 0$ and $t_Y \le 0$ we get

$$\begin{aligned}&|x(\mathcal{X}(\tau, \bar{s}_1^n), \mathcal{Y}(\tau, \bar{s}_1^n)) - x(\mathcal{X}^n(\tau, \bar{s}_1^n), \mathcal{Y}^n(\tau, \bar{s}_1^n))|\\ &= \left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} [c(U)(t_X + t_Y)](X, 2\bar{s}_1^n - X)\,dX\right|\\ &\le \kappa \left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (t_X - t_Y)(X, 2\bar{s}_1^n - X)\,dX\right|\\ &= \kappa|t(\mathcal{X}(\tau, \bar{s}_1^n), \mathcal{Y}(\tau, \bar{s}_1^n)) - t(\mathcal{X}^n(\tau, \bar{s}_1^n), \mathcal{Y}^n(\tau, \bar{s}_1^n))|.\end{aligned}$$

Since $t(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) = \tau = t^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))$ we have

$$\begin{aligned}&t(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - t(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))\\&\quad = t^n(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) - t(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)).\end{aligned} \tag{7.119}$$

Therefore,

$$|x(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - x(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))| \leq \kappa|t-t^n|(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))$$

and (7.118) yields

$$0 \leq \mathcal{Z}_2(\tau,\bar{s}_1^n) - \mathcal{Z}_2^n(\tau,\bar{s}_1^n) \leq \big(|x-x^n| + \kappa|t-t^n|\big)(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)). \tag{7.120}$$

We insert this in (7.117) and get

$$0 \leq \int_s^{\bar{s}_1^n} \mathcal{V}_3^2(\tau,\mathcal{X}(\tau,r))\dot{\mathcal{X}}(\tau,r)\,dr \leq \kappa^2 B_6\big(|x-x^n| + \kappa|t-t^n|\big)(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)).$$

Similarly we get

$$0 \leq \int_s^{\bar{s}_1^n} \mathcal{W}_3^2(\tau,\mathcal{Y}(\tau,r))\dot{\mathcal{Y}}(\tau,r)\,dr \leq \kappa^2 B_7\big(|x-x^n| + \kappa|t-t^n|\big)(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)).$$

We have

$$\int_s^{\bar{s}_1^n} \dot{\mathcal{X}}(\tau,r)\,dr = \mathcal{X}(\tau,\bar{s}_1^n) - \mathcal{X}(\tau,s) \leq X_r - X_l \quad \text{and} \quad \int_s^{\bar{s}_1^n} \dot{\mathcal{Y}}(\tau,r)\,dr \leq Y_r - Y_l.$$

Using these estimates in (7.116) gives

$$|A_1^n| \leq v_1\big(|x-x^n| + \kappa|t-t^n|\big)^{\frac{1}{2}}(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n)) \tag{7.121}$$

for a constant v_1 that is independent of n.

By (4.12c) we have the estimates $|U_X| \leq \frac{\sqrt{2J_X x_X}}{c(U)}$ and $|U_Y| \leq \frac{\sqrt{2J_Y x_Y}}{c(U)}$, which leads to

$$\begin{aligned}|A_2^n| &\leq |U(\mathcal{X}(\tau,\bar{s}_1^n),\mathcal{Y}(\tau,\bar{s}_1^n)) - U(\mathcal{X}^n(\tau,\bar{s}_1^n),\mathcal{Y}^n(\tau,\bar{s}_1^n))|\\
&\leq \sqrt{2}\kappa\left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} \Big(\sqrt{J_X x_X} + \sqrt{J_Y x_Y}\Big)(X, 2\bar{s}_1^n - X)\,dX\right|\\
&\leq \sqrt{2}\kappa\left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} J_X(X,2\bar{s}_1^n - X)\,dX\right|^{\frac{1}{2}}\left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} x_X(X,2\bar{s}_1^n - X)\,dX\right|^{\frac{1}{2}}\\
&\quad + \sqrt{2}\kappa\left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} J_Y(X,2\bar{s}_1^n - X)\,dX\right|^{\frac{1}{2}}\left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} x_Y(X,2\bar{s}_1^n - X)\,dX\right|^{\frac{1}{2}}\\
&\leq 2\sqrt{2}\kappa\left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (J_X+J_Y)(X,2\bar{s}_1^n - X)\,dX\right|^{\frac{1}{2}}\left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (x_X+x_Y)(X,2\bar{s}_1^n - X)\,dX\right|^{\frac{1}{2}}\end{aligned}$$

since $x_Y \geq 0$ and $J_Y \geq 0$. From (4.12a) we get

$$\left|\int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (x_X + x_Y)(X,2\bar{s}_1^n - X)\,dX\right|$$

$$\leq \kappa \left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (t_X - t_Y)(X, 2\bar{s}_1^n - X)\, dX \right|$$
$$= \kappa |t(\mathcal{X}(\tau,\bar{s}_1^n), \mathcal{Y}(\tau,\bar{s}_1^n)) - t(\mathcal{X}^n(\tau,\bar{s}_1^n), \mathcal{Y}^n(\tau,\bar{s}_1^n))| = \kappa |t - t^n|(\mathcal{X}^n(\tau,\bar{s}_1^n), \mathcal{Y}^n(\tau,\bar{s}_1^n)),$$

where the last equality follows from (7.119). Using (7.98) and (7.99) yields

$$\left| \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}(\tau,\bar{s}_1^n)} (J_X + J_Y)(X, 2\bar{s}_1^n - X)\, dX \right| \leq (B_6 + B_7)|\mathcal{X}(\tau,\bar{s}_1^n) - \mathcal{X}^n(\tau,\bar{s}_1^n)|$$
$$\leq (B_6 + B_7)(X_r - X_l).$$

Now we get

$$|A_2^n| \leq v_2 |t - t^n|^{\frac{1}{2}} (\mathcal{X}^n(\tau,\bar{s}_1^n), \mathcal{Y}^n(\tau,\bar{s}_1^n)) \tag{7.122}$$

for a constant v_2 which is independent of n.

By estimating as we did for A_1^n we get

$$|A_4^n| \leq v_3 \big(|x - x^n| + \kappa |t - t^n|\big)^{\frac{1}{2}} (\mathcal{X}^n(\tau,\bar{s}_1^n), \mathcal{Y}^n(\tau,\bar{s}_1^n)) \tag{7.123}$$

for a constant v_3 that is independent of n.

Using (7.121), (7.122) and (7.123) in (7.115) yields

$$|u(\tau, z) - u^n(\tau, z)|$$
$$\leq \Big[(v_1 + v_3)\big(|x - x^n| + \kappa|t - t^n|\big)^{\frac{1}{2}} + v_2 |t - t^n|^{\frac{1}{2}} + |U - U^n| \Big] (\mathcal{X}^n(\tau,\bar{s}_1^n), \mathcal{Y}^n(\tau,\bar{s}_1^n))$$

and we conclude, after using (7.114), that $u^n(\tau,\cdot) \to u(\tau,\cdot)$ in $L^\infty([x_l + \kappa\tau, x_r - \kappa\tau])$.

Since $p_Y = p_Y^n = 0$ we have $\mathfrak{p}(\tau, X) = p(X, \mathcal{Y}(\tau, \mathcal{X}^{-1}(\tau, X))) = p(X, \mathcal{Y}(\mathcal{X}^{-1}(X))) = \mathfrak{p}(X)$, and similarly $\mathfrak{p}^n(\tau, X) = \mathfrak{p}^n(X)$. From Step 2 we get $\mathfrak{p}(\tau, X) = 0$ and $\mathfrak{p}^n(\tau, X) \geq k_n > 0$ for all $X \in [X_l, X_r]$. Then by (5.7d) we obtain

$$\int_{x_l+\kappa\tau}^{z} \rho(\tau, y)\, dy = \int_{\bar{s}_1}^{s} 2\mathfrak{p}(\tau, \mathcal{X}(\tau, r))\dot{\mathcal{X}}(\tau, r)\, dr = 0,$$

so that by a change of variables,

$$\left| \int_{x_l+\kappa\tau}^{z} (\rho^n - \rho)(\tau, y)\, dy \right| = \int_{\mathcal{X}^n(\tau,\bar{s}_1^n)}^{\mathcal{X}^n(\tau,s^n)} 2\mathfrak{p}^n(X)\, dX \leq 2\sqrt{X_r - X_l}\, ||\mathfrak{p}^n||_{L^2([X_l,X_r])}.$$

Since $\mathfrak{p}^n \to 0$ in $L^2([X_l, X_r])$ this implies that

$$\lim_{n\to\infty} \int_{x_l+\kappa\tau}^{z} \rho^n(\tau, y)\, dy = \int_{x_l+\kappa\tau}^{z} \rho(\tau, y)\, dy = 0. \tag{7.124}$$

As mentioned in the comment before the proof, there is for any n a constant $\bar{d}_n > 0$ such that $\rho^n(\tau, z) \geq \bar{d}_n$ for all $z \in [x_l + \kappa\tau, x_r - \kappa\tau]$. Since $\rho^n(\tau,\cdot)$ is positive, (7.124) is the same as $\rho^n(\tau,\cdot) \to 0$ in $L^1([x_l + \kappa\tau, x_r - \kappa\tau])$.

Similarly we obtain $\sigma^n(\tau,\cdot) \to 0$ in $L^1([x_l + \kappa\tau, x_r - \kappa\tau])$. This concludes the proof of (P2”). ◻

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Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

References

1. Brenier, Y.: L^2 formulation of multidimensional scalar conservation laws. Arch. Ration. Mech. Anal. **193**, 1–19 (2009)
2. Bressan, A.: Uniqueness of conservative solutions for nonlinear wave equations via characteristics. Bull. Braz. Math. Soc. (N.S.) **47**, 157–169 (2016)
3. Bressan, A., Chen, G.: Generic regularity of conservative solutions to a nonlinear wave equation. Ann. Inst. H. Poincaré Anal. Non Linéaire **34**, 335–354 (2017)
4. Bressan, A., Chen, G.: Lipschitz metrics for a class of nonlinear wave equations. Arch. Ration. Mech. Anal. **226**, 1303–1343 (2017)
5. Bressan, A., Chen, G., Zhang, Q.: Unique conservative solutions to a variational wave equation. Arch. Ration. Mech. Anal. **217**, 1069–1101 (2015)
6. Bressan, A., Huang, T.: Representation of dissipative solutions to a nonlinear variational wave equation. Commun. Math. Sci. **14**, 31–53 (2016)
7. Bressan, A., Huang, T., Yu, F.: Structurally stable singularities for a nonlinear wave equation. Bull. Inst. Math. Acad. Sin. (N.S.) **10**, 449–478 (2015)
8. Bressan, A., Zheng, Y.: Conservative solutions to a nonlinear variational wave equation. Commun. Math. Phys. **266**, 471–497 (2006)
9. Fink, A.M., Mitrinović, D.S., Pečarić, J.E.: Inequalities Involving Functions and Their Integrals and Derivatives. Mathematics and Its Applications (East European Series), vol. 53. Kluwer Academic Publishers Group, Dordrecht (1991)
10. Glassey, R.T., Hunter, J.K., Zheng, Y.: Singularities of a variational wave equation. J. Differ. Equ. **129**, 49–78 (1996)
11. Grunert, K., Holden, H., Raynaud, X.: Global solutions for the two-component Camassa–Holm system. Commun. Partial Differ. Equ. **37**, 2245–2271 (2012)
12. Holden, H., Raynaud, X.: Global semigroup of conservative solutions of the nonlinear variational wave equation. Arch. Ration. Mech. Anal. **201**, 871–964 (2011)
13. Hunter, J.K., Saxton, R.: Dynamics of director fields. SIAM J. Appl. Math. **51**, 1498–1521 (1991)
14. Nordli, A.: A Lipschitz metric for conservative solutions of the two-component Hunter–Saxton system. Methods Appl. Anal. **23**, 215–232 (2016)
15. Reigstad, A.: A regularized system for the nonlinear variational wave equation. Ph.D. thesis, NTNU Norwegian University of Science and Technology (2021)

16. Saxton, R.A.: Dynamic instability of the liquid crystal director. In: Lindquist, W.B. (ed.) Current Progress in Hyperbolic Systems: Riemann Problems and Computations (Brunswick, ME, 1988), Contemp. Math., vol. 100, pp. 325–330. Amer. Math. Soc., Providence (1989)
17. Zhang, P., Zheng, Y.: Weak solutions to a nonlinear variational wave equation. Arch. Ration. Mech. Anal. **166**, 303–319 (2003)
18. Zhang, P., Zheng, Y.: Weak solutions to a nonlinear variational wave equation with general data. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 207–226 (2005)
19. Zhang, P., Zheng, Y.: On the global weak solutions to a variational wave equation. In: Dafermos, C.M., Feireisl, E. (eds.) Handbook of Differential Equations. Evolutionary Equations, vol. 2, pp. 561–648. Elsevier, Amsterdam (2005)