Bachelor's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Right Gröbner Basis Theory and Systems of Equations Over Algebras

Bachelor's thesis in Mathematical Sciences
Supervisor: Øyvind Solberg
June 2023

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Sammendrag

Målet med denne bachelor-oppgaven er å gi en metode for å finne løsninger av homogene systemer av lineære ligninger over veialgebraer ved bruk av høyre Gröbnerbasisteori. Den vil introdusere tosidige Gröbnerbasiser og høyre Gröbnerbasiser, for deretter å gå inn i egenskaper disse har som vil være til nytte når vi går over til systemer av ligninger.

Abstract

The goal of this thesis is to provide a method for finding solutions of homogeneous systems of linear equations over path algebras using right Gröbner basis theory. It will introduce two-sided Gröbner bases and right Gröbner bases, to then dive into properties these have that will be of use when we move over to systems of equations.

Contents

1 Introduction

Gröbner bases were first introduced to the public in 1965 by Bruno Buchberger, with similar notions being developed as far back as in 1913, by a russian mathematician Nikolai Grünther. Since then, Gröbner bases have become important in many areas of mathematics and informatics. It has also been extended to include non-commutative settings, by mathematicians such as Edward L. Green. We will here dive deeper into his works.

In this thesis we assume the reader to have some familiarity with basic abstract algebra, quivers and path algebras (see [1]), but, other than that, no particular knowledge of the main topics covered.

Throughout this thesis we are mainly following the article [3] written by Edward L. Green. We will begin by introducing (non-commutative) Gröbner bases for two-sided ideals of K-algebras and move on to right Gröbner bases, which are for right modules. There we will show important results and methods that will build us up to figuring out how to attack the problem of finding solutions of systems of equations over algebras.

2 Gröbner Bases

We start by introducing what a Gröbner basis is, particularly starting with two-sided ones in this section. However, before we do that we first need some prior definitions and results for it to make sense. Here we set R to be a K-algebra, where K is a field. We are looking for a special basis with some order on the basis.

Definition 2.1. Let $\mathcal{B}$ be a K-basis for R. We say $\mathcal{B}$ is a *multiplicative basis* if $\forall b_1, b_2 \in \mathcal{B}$: $b_1 \cdot b_2 \in \mathcal{B} \cup \{0\}$.

Example 2.1.

1. The basis of the polynomial algebra $K[x]$ is $\mathcal{B} = \{1, x, x^2, \dots\}$ which is a multiplicative basis, for if we pick any $x^s, x^t \in \mathcal{B}$ for $s, t \in \mathbb{N}$ we get $x^s \cdot x^t = x^{s+t} \in \mathcal{B}$.
2. The matrix algebra $M(n, K)$ has a basis $\mathcal{B} = \{E_{ij} | 1 \leq i, j \leq n\}$ where E_{ij} is the $n \times n$ matrix with 1 in the (i, j)-th entry. We see that $E_{ij}E_{kl} = \begin{cases} E_{il} \in \mathcal{B} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$ and, hence, is a multiplicative basis.
3. Given any quiver Γ, the path algebra $K\Gamma$ has a basis consisting of all paths of the quiver. Multiplying any paths either gives a new path in the quiver or 0 depending on if they connect or not.

Given a K-algebra with this basis, there is a bijection between the equivalence relations on $\mathcal{B} \cup \{0\}$ and on a special type of ideal called a *2-nomial ideal* that will be useful in knowing when we have a Gröbner basis.

Definition 2.2. Let R be given as above with a multiplicative basis $\mathcal{B}$. An ideal I in R is called a *2-nomial ideal* if it is generated by elements $b - b'$ and b'' where $b, b', b'' \in \mathcal{B}$.

Notice that if I is a 2-nomial ideal with multiplicative basis $\mathcal{B}$, then

$$I = R\mathcal{H}R = \left\{ \sum_{b-b', b'' \in \mathcal{H}} x_{b-b'}(b - b')y_{b-b'} + x_{b''}(b'')y_{b''} : x_{b-b'}, y_{b-b'}, x_{b''}, y_{b''} \in R \right\}$$

where $\mathcal{H}$ is the generating set described in the definition. Since $\forall x \in R$, $x = \sum_{b \in \mathcal{B}} a_b b$ for finite $a_b \in K^*$, we have that each element in I can be written as a finite linear combination of elements in

$$\mathcal{B}\mathcal{H}\mathcal{B} = \{b_1 h b_2 : h \in \mathcal{H} \text{ and } b_1, b_2 \in \mathcal{B}\}$$

with coefficients in K. For each $h \in \mathcal{H}$ we have two cases:

1. $h = b''$: $b_1 b'' \in \mathcal{B} \cup \{0\} \implies (b_1 b'')b_2 \in \mathcal{B} \cup \{0\}$
2. $h = b - b'$: $b_1(b - b')b_2 = b_1 b b_2 - b_1 b' b_2$ where $b_1 b b_2, b_1 b' b_2 \in \mathcal{B} \cup \{0\}$.

Hence, $\mathcal{BHB}$ contains elements of the form $b - b'$ and $\pm b''$. That is, for any $x \in I$:

$$x = \sum_{i=1}^{s} a_i(b_i - b_i') + \sum_{j=1}^{t} c_j b_j''$$

where $b_i, b_i', b_j'' \in \mathcal{B}$, $b_i - b_i', b_j'' \in I$ and $a_i, c_j \in K$ for all i and j, and for some s and t.

Example 2.2. Let $\Gamma\colon 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3$ be a quiver. Then the ideal generated by $\alpha - \beta, \alpha\gamma, \beta\gamma$ is a 2-nomial ideal of the path algebra $K\Gamma$.

We get the following result as was mentioned above.

Theorem 2.1. *There is a bijection between the set of equivalence relations on $\mathcal{B} \cup \{0\}$ and 2-nomial ideals.*

Proof. Let $\sim$ be an equivalence relation on $\mathcal{B} \cup \{0\}$. Define the ideal generated by elements of the form $b - b'$ and b'' if $b \sim b'$ and $b'' \sim 0$, respectively, as $I_\sim$. This ideal is precisely the definition of a 2-nomial ideal. If we have two equivalence relations $\sim$ and $\sim'$ not equal to each other, then $\exists b, b' \in \mathcal{B} \cup \{0\}$, $b \neq b'$ such that $b \sim b'$, but $b \nsim' b'$, meaning that $b - b'$ is a generator of $I_\sim$ whilst not in $I_{\sim'}$. Thus, $I_\sim \neq I_{\sim'}$.

Conversely, if I is a 2-nomial ideal in R define the equivalence relation $\sim_I$ by $b \sim_I b'$ and $b'' \sim_I 0$ when $b - b' \in I$ and $b'' \in I$, respectively, where $b, b', b'' \in \mathcal{B}$. We check that this is an equivalence relation on $\mathcal{B} \cup \{0\}$.

For any $b \in \mathcal{B} \cup \{0\}$ we have $b - b = 0 \in I$, giving us $b \sim_I b$. Let $b - b', b' - b'' \in I$. If $b \sim_I b'$ then $b' \sim_I b$ since $b' - b = -(b - b') \in I$. Lastly, $b \sim_I b' \sim_I b''$, then $b - b'' = (b - b') + (b' - b'')$. Since $b - b', b' - b'' \in I$ it follows that $b - b'' \in I \implies b \sim_I b''$. Thus, it is an equivalence relation and the set of elements is $\mathcal{B} \cup \{0\}$.

Assume $I \neq I'$ as 2-nomial ideals. Then, without loss of generality ($I' \subset I$ or $I \not\subset I'$), there exists $b - b' \in I$ and $b - b' \notin I'$, or $b'' \in I$ and $b'' \notin I'$, or both. If the first case holds, then $b \sim_I b'$ whilst $b \nsim_{I'} b'$. Otherwise, $b'' \sim_I 0$ whilst $b'' \nsim_{I'} 0$. In neither case are the equivalence relations equal.

Hence, we have injections going both ways implying a bijection on sets. □

If we have a 2-nomial ideal we can, then, associate an equivalence relation of $\mathcal{B} \cup \{0\}$ to the ideal.

Definition 2.3. If I is a 2-nomial ideal, the *associated relation to I* is the equivalence relation on $\mathcal{B} \cup \{0\}$ corresponding to I.

We use this to characterize what kind of ideals in a K-algebra with multiplicative basis we need such that the quotient has a multiplicative basis induced from the original basis.

Theorem 2.2. *Let S be a K-algebra with multiplicative basis $\mathcal{B}$. Let $I \subseteq S$ be an ideal and $\pi\colon S \to S/I$ be the canonical surjection. Then $\pi(\mathcal{B}) \setminus \{0\}$ is the multiplicative basis for $S/I \iff I$ is a 2-nomial ideal.*

Proof. (⇒): Suppose $\mathcal{B}' := \pi(\mathcal{B}) \setminus \{0\}$ is a multiplicative basis of S/I. Defining the (equivalence) relation $\sim$ on $\mathcal{B} \cup \{0\}$ by $b \sim b'$ if $\pi(b) = \pi(b')$ we want to show that the 2-nomial ideal, say $I_\sim$, corresponding to $\sim$ (by Theorem 2.1) is I. Since $\pi(b) = \pi(b') \iff \pi(b - b') = 0$ we have $b - b' \in I$ which is also part of the generators of $I_\sim$. For all the generators we have its inclusion in I so $\langle \{b - b' : b \sim b' \text{ for some } b, b' \in \mathcal{B}\} \cup \{b'' : b'' \sim 0 \text{ for some } b'' \in \mathcal{B}\} \rangle = I_\sim \subseteq I$.

To show the other inclusion let $x \in I$. Then $x = \sum_{i=1}^t a_i b_i$ where $b_i \in \mathcal{B}$, $a_i \in K$ and $t \leq |\mathcal{B}|$. Applying the canonical surjection on x we have

$$\pi(x) = \pi(\sum_{i=1}^t a_i b_i) = \sum_{i=1}^t a_i \pi(b_i) = \sum_{\substack{i \\ \pi(b_i) \neq 0}} a_i \pi(b_i) = 0.$$

In the last line we remove the $\pi(b_i)$'s which are zero ($b_i \in I \iff b_i \sim 0 \iff b_i \in I_\sim$). As $\mathcal{B}'$ is K-linear and the a_i's are non-zero, the remaining basis elements must be reccuring. Thus we get that

$$\sum_{\substack{i \\ \pi(b_i) \neq 0}} a_i \pi(b_i) = \sum_{\substack{[b_i] \\ \pi(b_i) \neq 0}} \left(\left(\sum_{\substack{\pi(b_j) = \pi(b_i) \\ j}} a_j \right) \pi(b_i) \right) = 0$$

where $\sum_{b_j \in [b_i]} a_j = 0$, meaning that if we take any a_j, say a_i, from this sum from basis elements in the equivalence class $[b_j]$, we get that $a_i = -\sum_{b_j \in [b_i] \setminus b_i} a_j$. So

$$\begin{aligned} x = \sum_{i=1}^t a_i b_i &= \sum_{\substack{b_i \in I \\ i}} a_i b_i + \sum_{\substack{[b_i] \\ b_i \notin I \\ i}} \left(\left(\sum_{\substack{b_j \in [b_i] \setminus b_i \\ j}} a_j b_j \right) + a_i b_i \right) \\ &= \sum_{\substack{b_i \in I \\ i}} a_i b_i + \sum_{\substack{[b_i] \\ b_i \notin I}} \sum_{\substack{b_j \in [b_i] \setminus b_i \\ j}} a_j (b_j - b_i) \in I_\sim \end{aligned}$$

Thus, $I = I_\sim$

(⇐): Assuming I is a 2-nomial ideal we want to show $\mathcal{B}'$ is a multiplicative basis, that is, being multiplicative and a K-basis of S/I. Let $\pi(b), \pi(b') \in \mathcal{B}'$ for $b, b' \in \mathcal{B}$. As $\mathcal{B}$ is multiplicative, $bb' \in \mathcal{B} \cup \{0\}$, we get $\pi(b) \cdot \pi(b') = \pi(bb') \in \pi(\mathcal{B} \cup \{0\}) = \mathcal{B}' \cup \{0\}$. To show that $\mathcal{B}'$ is a K-basis, we only need to see that its elements are linearly independent, as π is surjective. Suppose $\sum_{i=1}^n \alpha_i \pi(b_i) = 0 \implies \sum_{i=1}^n \alpha_i b_i \in I$, assuming $0 \neq \pi(b_i) \neq \pi(b_j) \neq 0$ for $i \neq j$, $1 \leq i, j \leq n$. As I is a 2-nomial ideal ¤

$$\sum_{i=1}^n \alpha_i b_i = \sum_{j=1}^s \beta_j (b_j - b_j') + \sum_{k=1}^t \gamma_k b_k''$$

where the b_i's are some of the b_j, b_j' and b_k''. If there was some $\beta_j(b_j - b_j')$ then, since $0 = \pi(\beta_j(b_j - b_j')) = \beta_j(\pi(b_j) - \pi(b_j'))$, we get $\pi(b_j) = \pi(b_j')$. But since all such elements were distinct $b_j = b'_j$. Similarly, if there was a $\gamma_k b_k$ in the sum, this would imply we had a $b_k \in I$, but all b_j are such that $\pi(b_j) \neq 0$. Thus, $\sum_{i=1}^{n} \alpha_i b_i = 0 \iff \alpha_1 = \cdots = \alpha_n = 0$.

□

From this, by having the ideal be a 2-nomial ideal we are quotienting out, we are, in some sense, preserving the multiplicative basis structure. This important result will be applied later on to show that this ideal is also one of the necessary conditions for the quotient of a path algebra to have our special multiplicative basis, namely a Gröbner basis.

Next we introduce an ordering on the basis as it gives more structure to work with, and is part of the basis we are working towards.

Definition 2.4. The pair $(\mathcal{B}, >)$ is an *ordered multiplicative basis* of R if $\mathcal{B}$ is a multiplicative basis and $>$ is an *admissible order* on $\mathcal{B}$. That is, the following properties hold:

A1. $>$ is a well-ordering on $\mathcal{B}$

A2. $\forall b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_1 b_3 > b_2 b_3$, when $b_1 b_3 \neq 0$ and $b_2 b_3 \neq 0$

A3. $\forall b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_3 b_1 > b_3 b_2$, when $b_3 b_1 \neq 0$ and $b_3 b_2 \neq 0$

A4. $\forall b_1, b_2, b_3, b_4 \in \mathcal{B}$, if $b_1 = b_2 b_3 b_4$ then $b_1 \geq b_3$

We will set R to have now an ordered multiplicative basis $(\mathcal{B}, >)$ continuing this chapter (still being a K-algebra), in addition, a multiplicative identity, 1, not necessarily in $\mathcal{B}$. The identity is then of the form $1 = \sum_{i=1}^{n} \alpha_i v_i$ with $0 \neq \alpha_i \in K$ and distinct $v_i \in \mathcal{B}$. Now we give some important results for these basis elements which give us the identity.

Theorem 2.3. *Let $\Gamma_0 := \{v_1, v_2, ..., v_n\}$ where $1 = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_i \in K^*$. Then*

1. *Γ_0 is a set of orthogonal idempotents and $\alpha_i = 1$ for all $i = 1, 2, ..., n$.*
2. *If $b \in \mathcal{B}$ then $\exists! i, j \in \{1, .., n\}$ such that $v_i b = b$ and $b v_j = b$. We denote these by $o(b) = v_i$ and $t(b) = v_j$. In addition, if $k \neq i$ then $v_k b = 0$, and if $k \neq j$ then $b v_k = 0$.*
3. *If $o(b)$ or $t(b)$ is v_i, then $b \geq v_i$.*
4. *If $b \in \mathcal{B} \setminus \Gamma_0$ then $b^2 \neq b$.*
5. *The elements of Γ_0 are primitive, that is, they cannot be written as a sum of two orthogonal nonzero idempotents.*

Proof. 1. Pick some $v_i, v_j \in \Gamma_0$ where $v_i \cdot v_j \neq 0$. By the multiplicative structure of the basis we have that $v_i \cdot v_j = b \in \mathcal{B}$. Looking at v_i first we have $v_i = v_i \cdot 1 = \sum_{c=1}^{n} \alpha_c v_i v_c$. Assume $b \neq v_i$. Since v_i is a basis element and we have a recurrence of b for when $c = j$, we must have an $s \neq j$ such that $v_i v_s = b$ also. As our basis is well-ordered

(A1.) we have two possible cases. Assume $v_j > v_s$. Then by A3. $b = v_i v_j > v_i v_s = b$, which is impossible, so $v_i v_j = 0$ or $v_i v_s = 0 \implies b = 0$. Doing the same arguments we get $b = 0$ for the other case. Thus, $b = v_i = v_i v_j \in \Gamma_0$. Now, we also have $v_j = 1 \cdot v_j = \sum_{c=1}^{n} \alpha_c v_c v_j$. We get the same result assuming $b \neq v_j$ using A2.. So, $v_i = v_i v_j = v_j$.

By the above we have $v_i = \sum_{j=1}^{n} \alpha_j v_i v_j = \alpha_i v_i v_i = \alpha_i v_i \implies \alpha_i = 1$ for any $v_i \in \Gamma_0$, meaning that we can rewrite $1 = \sum_{i=1}^{n} v_i$.

2. Take some $b \in \mathcal{B}$. Then $b = b \cdot 1 = \sum_{i=1} b v_i$. If there is some $b \neq b_1 = b v_j \neq 0$ then by the same argument as in the first proof we get that $b = b v_j$. If there was another $v_k \neq v_j$ such that $b = b v_k$, then $b = b v_k = (b v_j) v_k = 0$. So the j is unique. Similarly, following the same arguments, considering for the case $b = 1 \cdot b$ we get that $b = v_i b$ for some $v_i \in \Gamma_0$ and for any other $v_k \neq v_i$ that $v_k b = 0$.

3. Let $b \in \mathcal{B}$. from (2.) we have some v_i such that $b = b v_i = b v_i v_i$, as v_i is idempotent. By A4. this implies $b \geq v_i$.

4. Let $b \in \mathcal{B} \setminus \Gamma_0$. As $\mathcal{B}$ is multiplicative $b \cdot b \in \mathcal{B} \cup \{0\}$. If $b^2 = 0$ then it is OK. So assume $b^2 \neq 0$. We have $b > o(b)$ from (3.) and by the assumption $b \notin \Gamma_0$. Multiplying by b from the right, we get $b^2 > o(b)b = b$. Hence, $b^2 \neq b$.

5. Suppose $x + y = v_i \in \Gamma_0$ for some $x, y \in R$ nonzero orthogonal idempotents. There are some $\alpha_s, \beta_j \in K$ and $b_j \in \mathcal{B} \setminus \Gamma_0$ such that $x = \sum_k \alpha_k v_k + \sum_j \beta_j b_j$. Since $o(x + y) = o(v_i) = v_i = t(v_i) = t(x + y)$ we have that $\alpha_k = 0$ for $k \neq i$, and $\beta_j = 0$ if $o(b_j) \neq v_i$ or $t(b_j) \neq v_i$. So, after rewriting, we have $y + \alpha_i v_i + \sum_j \beta_j b_j = v_i$. As they are orthogonal we have

$$0 = xy = \left(\alpha_i v_i + \sum_j \beta_j b_j\right)\left((1-\alpha_i)v_i - \sum_j \beta_j b_j\right)$$
$$= \alpha_i(1-\alpha_i)v_i + (1-2\alpha_i)\sum_j \beta_j b_j - \left(\sum_j \beta_j b_j\right)^2$$

By (3.), for any $b_m, b_n \in \mathcal{B} \setminus \Gamma_0$ from the sum in x such that $b_m b_n \neq 0$ we have that $b_m, b_n > v_i$. Multiplying from the left by b_m we get, by A3.,

$$b_m b_n > b_m v_i = b_m > v_i. \tag{1}$$

This means that $\alpha_i(1-\alpha_i) = 0$. So $\alpha_i \in \{0, 1\}$. Since $\mathcal{B}$ is ordered we can choose the smallest b in the sum in x such that $\beta \neq 0$. Then from xy we can see b occuring, which can only be cancelled out by some $b_m b_n$, if possible. However, since $b \leq b_m$ and by (1) this never happens if the product has b in it. Otherwise, we have $b < b_n, b_m$ by minimality, but this means $b_m > b > v_i \implies b_m b_n > b b_n > b_n > b$. Hence, $\beta = 0$. Continuing choosing the next smallest element we end up with all $\beta_j = 0$. Thus, we get $x = \alpha v_i$ and $y = (1-\alpha)v_i$ where $y = 0$ if $\alpha = 1$ or $x = 0$ if $\alpha = 0$, contradicting our assumption that both of them are nonzero. □

Next we want to find a subset of basis elements which generate the the entire multiplicative basis $\mathcal{B}$. Thinking of Γ_0 as the vertex set of a quiver, we know that the set of arrows in a quiver constructs all possible paths. We will see that such a subset with Γ_0 does the same in this case as well. Define Γ_1 to be the set of all elements in $\mathcal{B} \setminus \Gamma_0$ that cannot be written as a multiple of two other elements in this set, that is, $\Gamma_1 := \{b \in \mathcal{B} \setminus \Gamma_0 \mid \neg\exists b_i, b_j \in \mathcal{B} \setminus \Gamma_0 \text{ such that } b_i b_j = b\}$. These are called the *product indecomposable* elements in $\mathcal{B} \setminus \Gamma_0$.

Theorem 2.4. *Let R be a K-algebra with ordered multiplicative basis* $(\mathcal{B}, >)$*. If* Γ_0 *and* Γ_1 *are defined as above, then* $(\Gamma_0 \cup \Gamma_1) := \{\prod_{finite} b \in R^* : b \in \Gamma_0 \cup \Gamma_1\} = \mathcal{B}$.

Proof. We want to show that any $b \in \mathcal{B}$ is a product of elements in $\Gamma_0 \cup \Gamma_1$. Take $b \in \mathcal{B} \setminus \Gamma_0 \cup \Gamma_1$, for otherwise we are done. Then $b = b_1 b_2$ for $b_1, b_2 \in \mathcal{B} \setminus \Gamma_0$. $b = b_1 b_2 t(b_2) = o(b_1) b_1 b_2$ so by A4. $b \geq b_1$ and $b \geq b_2$. If $b = b_1$ then since $b_2 > t(b)$ we have by A3. $b = b b_2 > b t(b) = b$. Thus $b > b_1$. Similarly, by A2. we get $b > b_2$. If any of the elements are not in Γ_1 we can again split them into two components both being less than the original, as we did above. By A1. there is a least element making this process of splitting elements not in Γ_1 stop. Thus we will end up at some point with $b = b_{i_1} b_{i_2} \cdots b_{i_h}$ where each b_{i_j} not being able to be divided further, which is exactly being in the set Γ_1. □

With the result above we can associate a graph to $\mathcal{B}$ by letting Γ_0 be the set of vertices and Γ_1 the set of arrows, b, going from the vertices $o(b)$ to $t(b)$. This directed graph we will use to see what form R must be in when its basis is equipped with an admissible order. Before that we define some notions we will use in the proceeding chapter.

Definition 2.5. Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$.

1. The *tip* of $x = \sum_{i=1}^{n} \alpha_i b_i \in R$, where $\alpha_i \neq 0$, denoted $\textsc{Tip}(x)$, is defined as $\textsc{Tip}(x) := b_i \geq b_j \; \forall j \in \{1, 2, ..., n\}$, ie. the largest basis element occuring in x.

2. The *coefficient of the tip* is defined as $\textsc{CTip}(x) := \alpha_j$

3. The *tip* of I, a subset of R, is defined by a set of the tips of all non-zero elements in I,
$$\textsc{Tip}(I) := \{\textsc{Tip}(x) \in \mathcal{B} : x \in I \setminus \{0\}\} \subseteq \mathcal{B}.$$

4. $\textsc{NonTip}(I) := \mathcal{B} \setminus \textsc{Tip}(I)$.

Note that for an ideal I by the definition of $\textsc{Tip}(I)$ we have that $\textsc{Tip}(I) = (\textsc{Tip}(I)) \cap \mathcal{B}$ where $(\textsc{Tip}(I))$ is the ideal generated by the tip of I.

The next definition defines the main topic of this chapter.

Definition 2.6. Let R be a K-algebra with an ordered multiplicative basis $(\mathcal{B}, >)$. A subset $\mathcal{G} \subset I$ is a *Gröbner basis of I with respect to* $>$ if $(\textsc{Tip}(\mathcal{G})) = (\textsc{Tip}(I))$.

Following from the definitions of tips, we have an important general result showing the use of $\textsc{NonTip}(*)$ that will be used for proving the existence of Gröbner bases.

Theorem 2.5. *Let V be a vector space over K with basis $\mathcal{B}$ having a well-ordering $>$ and W be a subspace of V. Then*

$$V = W \oplus \operatorname{Span}(\textsc{NonTip}(W))$$

Proof. We first show that $V = W + \operatorname{Span}(\textsc{NonTip}(W))$. Obviously we have $V \supseteq W + \operatorname{Span}(\textsc{NonTip}(W))$. So, suppose for a contradiction that $U := V \setminus (W + \operatorname{Span}(\textsc{NonTip}(W))) \neq \emptyset$. Let $v \in U$ be chosen such that $\textsc{Tip}(v) = b$ is minimal for all elements in U. Then $\textsc{Tip}(v - \textsc{CTip}(v)\, b) < \textsc{Tip}(v)$.

Suppose $b \in \textsc{NonTip}(W)$. Then, since v was chosen such that it had the minimal tip, we have $v - \alpha b \in W + \operatorname{Span}(\textsc{NonTip}(W))$ so $v - \alpha b = w + n_w \implies$

$$v = w + (n_w + \alpha b) \in W + \operatorname{Span}(\textsc{NonTip}(W))$$

where $w \in W$ and $n_w \in \operatorname{Span}(\textsc{NonTip}(W))$.

Otherwise, if $b \in \textsc{Tip}(W)$ then $\exists w \in W$ such that $\textsc{Tip}(w) = b$. Then, by the minimality as above, we have some $w' \in W$ and $n_{w'} \in \operatorname{Span}(\textsc{NonTip}(W))$ such that $v - \frac{\textsc{CTip}(v)}{\textsc{CTip}(w)} w = w' + n_{w'}$ giving us again

$$v = \left(w' + \frac{\textsc{CTip}(v)}{\textsc{CTip}(w)} w\right) + n_{w'} \in W + \operatorname{Span}(\textsc{NonTip}(W)),$$

a contradiction, so $U = \emptyset$ resulting in $V = W + \operatorname{Span}(\textsc{NonTip}(W))$.

Finally, let $x \in W^*$. Then $\textsc{Tip}(x) \in \textsc{Tip}(W)$. If also $x \in \operatorname{Span}(\textsc{NonTip}(W))$ then $\textsc{Tip}(x) \in \textsc{NonTip}(W)$. Hence, $W \cap \operatorname{Span}(\textsc{NonTip}(W)) = \{0\}$. □

By the result above, we have that for any $v \in V$ it can uniquely be written as $v = w_v + n_v$ where $w_v \in W$ and $n_v \in \operatorname{Span}(\textsc{NonTip}(W))$. We define this for K-algebras.

Definition 2.7. Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$ and I an ideal of R. As vector spaces, let $r \in R$. Then, $r = i_r + n_r$ for unique $i_r \in I$ and $n_r \in \operatorname{Span}(\textsc{NonTip}(I))$. We call $\operatorname{Norm}(r) := n_r = r - i_r$ the *normal form* of r.

Notice that for $x, y \in R$ we have $(i_x + \operatorname{Norm}(x)) + (i_y + \operatorname{Norm}(y)) = x + y = i_{x+y} + \operatorname{Norm}(x + y)$. Thus, $\operatorname{Norm}(x + y) = \operatorname{Norm}(x) + \operatorname{Norm}(y)$.

Similarly, one can show that $\operatorname{Norm}(xy) = \operatorname{Norm}(x)\operatorname{Norm}(y)$. This will be useful at the end of our journey.

Continuing with our search, we now show that these Gröbner bases indeed exist.

Theorem 2.6. *Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$ and I an ideal of R. Then there exists a Gröbner basis of I.*

Proof. We want to construct a subset $\mathcal{G}$ of I such that $(\textsc{Tip}(I)) = (\textsc{Tip}(\mathcal{G}))$. From Theorem 2.5 we know that for any $b \in \textsc{Tip}(I) \subset R$ that $b = i_b + \operatorname{Norm}(b)$ for some $i_b \in I$ and $\operatorname{Norm}(b) \in \operatorname{Span}(\textsc{NonTip}(I))$. Let, then, $\mathcal{G} := \{b - \operatorname{Norm}(b) : b \in \textsc{Tip}(I)\} \subset I$, where for each $b \in \textsc{Tip}(I)$ we have $\textsc{Tip}(b - \operatorname{Norm}(b)) = b$. Thus $\textsc{Tip}(\mathcal{G}) = \textsc{Tip}(I) \implies (\textsc{Tip}(\mathcal{G})) = (\textsc{Tip}(I))$. □

Hence, we can say that if a K-algebra has an ordered multiplicative basis, then it has a *Gröbner basis theory*. Next gives a simple example of a Gröbner basis in a path algebra.

Example 2.3. Using Example 2.2 we can define an admissible ordering of the basis as such: $\beta\gamma > \alpha\gamma > \gamma > \beta > \alpha > v_3 > v_2 > v_1$. We also have the 2-nomial ideal $I = \{a_1(\alpha - \beta) + a_2\alpha\gamma + a_3\beta\gamma : a_1, a_2, a_3 \in K\}$. The tip of this ideal is $\text{TIP}(I) = \{\beta, \alpha\gamma, \beta\gamma\}$. Picking the subset $\mathcal{G} = \{\alpha - \beta, \alpha\gamma\} \subset I$ we get that

$$\begin{aligned}(\text{TIP}(\mathcal{G})) &= \{r_1\beta r_2 + r_3\alpha\gamma r_4 : r_1, r_2, r_3, r_4 \in K\Gamma\} \\ &= \{a_1\beta + a_2\alpha\gamma + a_3\beta\gamma : a_1, a_2, a_3 \in K\} \\ &= (\text{TIP}(I)).\end{aligned}$$

We state the main theorem of this chapter.

Theorem 2.7. *Let R be a K-algebra with an ordered multiplicative basis $(\mathcal{B}, >)$. Let $\Gamma = \Gamma_0 \cup \Gamma_1$ be the graph associated to $\mathcal{B}$. Then there exists a 2-nomial ideal I of the path algebra $K\Gamma$ such that*

$$K\Gamma/I \cong R.$$

Proof. Define a function $g : \Gamma \to R$ sending all vertices and arrows of Γ to the corresponding in R as its associated to $\mathcal{B}$. Then it follows that

1. $\sum_{v\in\Gamma_0} g(v) = \sum_{v\in\Gamma_0} v = 1$,
2. for any $u, v \in \Gamma$ such that $uv = 0$, then $g(u)g(v) = uv = 0$ in R,
3. for any $v \in \Gamma_0$, $g(v) = v = v^2 = g(v)^2$ and
4. for any $a \in \Gamma_1$, $g(o(a))g(a) = o(a)a = a = at(a) = g(a)g(t(a))$

By Lemma 1.2 in [4] g extends uniquely to a K-algebra homomorphism $f : K\Gamma \to R$ where $f(v) := g(v)$ for all $v \in \Gamma_0$ and for any path $a_1 \cdots a_n$ of the graph, $f(a_1 \cdots a_n) := g(a_1) \cdots g(a_n)$. From Theorem 2.4 f hits all basis elements of R and so it is also onto. Since $\ker(f)$ an ideal of $K\Gamma$, we then have by the isomorphism theorem a canonical surjection $\pi : K\Gamma \to K\Gamma/\ker(f) \cong \text{im}(f) = R$. By Theorem 2.2, as $\pi(\Gamma) = \mathcal{B} \cup \{0\}$, $\ker(f)$ is a 2-nomial ideal. □

Thus, we have that R has a Gröbner basis theory associated to $(\mathcal{B}, >)$ if it is equal to, up to isomorphism, a path algebra quotiented by a 2-nomial ideal. The big question, however, which remains unanswered, is under which circumstances for the 2-nomial ideal it holds the other way. That is, what would be the necessary and sufficient conditions for a 2-nomial ideal I of a path algebra $K\Gamma$ such that $K\Gamma/I$ has a Gröbner basis theory?

3 Right Gröbner Bases

In this section we will learn about right Gröbner bases in an algebra of Gröbner basis theory. We will see that specific projective right modules have right Gröbner basis theory and, at the end, see how one can find a right Gröbner basis of intersections of submodules of a projective right module, that will come in handy in the last section.

Before all that, let us build up the knowledge to understand the theory fully.

For now, set R to be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$ where Γ_0 is the set of orthogonal idempotents of $\mathcal{B}$, where $\sum_{v \in \Gamma_0} v = 1$. We look at the right R-modules, where we particularly have the requirement $m \cdot 1 = m$ for all m in the module.

Similarly to earlier we introduce a basis with an ordering for right modules.

Definition 3.1. Let M be a right R-module with K-basis $\mathcal{M}$. A subset $\mathcal{M}$ of M is a *coherent* (K-) basis of M if $\forall m \in \mathcal{M}, \forall b \in \mathcal{B}$, $mb \in \mathcal{M} \cup \{0\}$

We have an immediate result from this.

Lemma 3.1. *If $\mathcal{M}$ is a coherent basis of M then $\forall m \in \mathcal{M}, \exists! v \in \Gamma_0$ such that $mv = m$.*

Proof. Pick any $m \in \mathcal{M}$. We have that $m = m \cdot 1 = \sum_{v \in \Gamma_0} mv$. For each v we have $mv \in \mathcal{M} \cup \{0\}$. If there was some v' such that $mv' = m' \neq m$ then you would need cancellation, but since every coefficient is 1 we conclude that $mv' = 0$ for each v' except the ones that give $mv' = m$, as this has to occur to satisfy $m = m \cdot 1$. It has to also be unique, for otherwise suppose v' and v'' are such elements. Then $m = mv' = (mv'')v' = m(v''v') = 0$. □

Generally we call elements $x \in M \setminus \{0\} =: M^*$ that satisfy $xv = x$ for some $v \in \Gamma_0$ *left uniform*. Note also that, since $\sum_{m \in \mathcal{M}} \alpha_m m \in M$ where finite $\alpha_m \in K^*$, every element in M is a sum of left uniform elements.

Next, we define an ordering for the coherent basis.

Definition 3.2. Let M be a right R-module as before. We say that $(\mathcal{M}, \succ)$ is an *ordered basis of M* if $\mathcal{M}$ is a coherent K-basis of M and $\succ$ is a *right admissible order on $\mathcal{M}$*, that is the following properties hold:

M1. $\succ$ is a well-order.

M2. $\forall m_1, m_2 \in \mathcal{M}$, $\forall b \in \mathcal{B}$, if $m_1 \succ m_2$ and $m_1 b \neq 0 \neq m_2 b$ then $m_1 b \succ m_2 b$.

M3. $\forall m \in \mathcal{M}$, $\forall b_1, b_2 \in \mathcal{B}$, if $b_1 > b_2$ and $mb_1 \neq 0 \neq mb_2$ then $mb_1 \succ mb_2$.

Lemma 3.2. *Let $(\mathcal{M}, \succ)$ be an ordered basis of M. Let $m \in \mathcal{M}$ and $b \in \mathcal{B} \setminus \Gamma_0$ and suppose $mb \neq 0$. Then $mb \succ m$.*

Proof. By the previous result m is left uniform for some $v \in \Gamma_0$. Since $0 \neq mb = m(vb)$ we have that $o(b) = v$. We know that $b > v$ so by M3., $mb \succ mv = m$. □

Continuing forward, we set M to be a right R-module with ordered basis $(\mathcal{M}, \succ)$. We can now define the tips as we did similarly previously.

Definition 3.3. Let $(\mathcal{M}, \succ)$ be an ordered basis of M. Let $x = \sum_{i=1}^{r} \alpha_i m_i$, where $\alpha_i \in K^*$ and distinct $m_i \in \mathcal{M}$. Let X be a subset of M. We define the following:

1. The tip of x is defined as $\textsc{Tip}^*(x) := m_j \succeq m_i$, $\forall i \in \{1, 2, \ldots, r\}$,
2. Denote the coefficient of the tip of x as $\textsc{CTip}^*(x) := \alpha_j$,
3. Denote the set of all tips in X as $\textsc{Tip}^*(X) := \{\textsc{Tip}^*(x) \in \mathcal{M} : x \in X \setminus \{0\}\}$,
4. $\textsc{NonTip}^*(X) := \mathcal{M} \setminus \textsc{Tip}^*(X)$.

With this we can define the notion of a right Gröbner basis for right modules.

Definition 3.4. Let M be a right module of R with ordered basis $(\mathcal{M}, \succ)$. Let N be a right submodule of M. A set $\mathcal{G}$ is said to be a *right Gröbner basis of N with respect to* $\succ$ if $\mathcal{G} \subset N$ and $\langle \textsc{Tip}^*(\mathcal{G})\rangle_R = \langle \textsc{Tip}^*(N)\rangle_R$ where $\langle * \rangle_R$ denotes the right submodule of M generated by $*$.

Using the result from before (Theorem 2.5) we can again use the same arguments as in Theorem 2.6 to show the existence of right Gröbner bases. We, thus, say that M has a *right Gröbner basis theory with respect to* $\succ$ if $(\mathcal{M}, \succ)$ is an ordered basis of M. Note that, as vector spaces, $M/N \cong \mathrm{Span}(\textsc{NonTip}^*(N))$ for a right submodule N of M and we say the normal form of $m \in M$ with respect to $\succ$ to be $\mathrm{Norm}(m)$, where $m = n_m + \mathrm{Norm}(m) \in N \oplus \mathrm{Span}(\textsc{NonTip}^*(N))$, similar to as was the case in the previous chapter.

The next theorem will show us basic properties that follow from the definition which is similar to the situation when we were dealing with ideals.

Theorem 3.3. *Let M be a right R-module with ordered basis* $(\mathcal{M}, \succ)$*, N a submodule of M and* $\mathcal{G}$ *a right Gröbner basis for N with respect to* $\succ$*. Then* $\mathcal{G}$ *generates N as a right submodule.*

Proof. Assume $\mathcal{G}$ does not generate N. Let $z \in N$ such that $\textsc{Tip}^*(z)$ is minimal and such that $z \notin \langle \mathcal{G} \rangle_R$. By hypothesis there exists $g \in \mathcal{G}$ such that $\textsc{Tip}^*(g)\, b = \textsc{Tip}^*(z)$ for some $b \in \mathcal{B}$. However, then

$$\textsc{Tip}^*\left(z - \frac{\textsc{CTip}^*(z)}{\textsc{CTip}^*(g)} g\, b\right) < \textsc{Tip}^*(z).$$

Hence $z - \frac{\textsc{CTip}^*(z)}{\textsc{CTip}^*(g)} g\, b \in \langle \mathcal{G} \rangle_R \implies z \in \langle \mathcal{G} \rangle_R$. A contradiction. □

Right Gröbner bases can have certain properties that we will next define and show when we have these.

Definition 3.5. Let M be a right R-module with ordered basis $(\mathcal{M}, \succ)$. Let $m, m' \in \mathcal{M}$. We say m *(properly) left divides* m' if $m' = m b$ (and $m' \neq m$) for some $b \in \mathcal{B}$.

Definition 3.6. Let N be a right submodule of M and let $\mathcal{G}$ be a right Gröbner basis of N with respect to $\succ$.

1. We say $\mathcal{G}$ is *reduced* if for each $g = \sum_{i=1}^{r} \alpha_i m_i \in \mathcal{G}$, where $\alpha_i \in K^*$ and $m_i \in \mathcal{M}$ distinct, there is no $g' \in \mathcal{G} \setminus \{g\}$ such that $\textsc{Tip}^*(g')$ left divides m_i for any $1 \le i \le r$, and $\mathrm{C}\textsc{Tip}^*(g) = 1$

2. $\mathcal{G}$ is *tip-reduced* if for every $g, g' \in \mathcal{G}$ where $\textsc{Tip}^*(g)$ left divides $\textsc{Tip}^*(g')$, then $g = g'$.

We see that the reduced right Gröbner basis is also tip-reduced from the definition.

We want to show that these type of Gröbner bases indeed exist.

Proposition 3.4. *Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$. Let M be a right R-module with ordered basis $(\mathcal{M}, \succ)$ and N a right submodule of M. Then there exists a tip-reduced right Gröbner basis for N, particularly a reduced one that is unique with respect to $\succ$. In addition, there exists a tip-reduced (left) uniform right Gröbner basis.*

Proof. As we mentioned, it is enough to show the existence and uniqueness of a reduced right Gröbner basis for N.

Let

$$\mathcal{T} = \{t \in \textsc{Tip}^*(N) : \text{no tip } t' \in \textsc{Tip}^*(N) \text{ properly left divides } t\}$$

(if t' left divides t, then $t' = t$). We then consider the set $\mathcal{G} := \{t - \mathrm{Norm}(t) : t \in \mathcal{T}\}$. To verify it is a right Gröbner basis, we can check that $\langle \textsc{Tip}^*(\mathcal{T}) = \mathcal{T} \rangle_R = \langle \textsc{Tip}^*(N) \rangle_R$, since $\mathcal{G}$ has all the tips as $\mathcal{T}$ has by construction. Assume $t \in \textsc{Tip}^*(N)$ and $t \notin \mathcal{T}$. This implies t is properly left divided by some $t' \in \textsc{Tip}^*(N)$. We assume $t' \in \mathcal{T}$, for otherwise we repeat the process. Then there is some $b \in \mathcal{B}$ such that $t'b = t \in \langle \mathcal{T} \rangle_R$. Hence, it is a right Gröbner basis. Also, for each element $g = t - \mathrm{Norm}(t) \in \mathcal{G}$ we have that $\mathrm{C}\textsc{Tip}^*(g) = 1$, and its tip is only left divided by the same tip, which is unique in the set, thus also reduced.

Suppose now $\mathcal{H}$ is another reduced right Gröbner basis of N. Let $h \in \mathcal{H}$. Then $\exists g \in \mathcal{G}$ such that $\textsc{Tip}^*(g)$ left divides $\textsc{Tip}^*(h)$. Then, also $\exists h' \in \mathcal{H}$ such that $\textsc{Tip}^*(h')$ left divides $\textsc{Tip}^*(g)$, so $\textsc{Tip}^*(h')$ left divides $\textsc{Tip}^*(h)$, hence $\textsc{Tip}^*(g) = \textsc{Tip}^*(h') = \textsc{Tip}^*(h)$. Thus we get that $h - g \in \mathrm{Span}(\textsc{NonTip}^*(N))$. But

$$h - g \in N \implies h - g = 0 \implies \mathcal{H} \subseteq \mathcal{G}.$$

Doing the same procedure on $\mathcal{G}$, we see that $\mathcal{G} = \mathcal{H}$, and hence we have showed uniqueness.

From above all we are left to show is that we can make the tip-reduced right Gröbner basis uniform out of a tip-reduced one, say $\mathcal{G}$. From Lemma 3.1 we have that for each $g \in \mathcal{G}$ there exists a unique $v \in \Gamma_0$ such that $\textsc{Tip}^*(g)v = \textsc{Tip}^*(g)$, and $gv \in N$. So let

$$\mathcal{G}' = \{gv \in N : g \in \mathcal{G},\, v \in \Gamma_0 \text{ such that } \textsc{Tip}^*(g)v = \textsc{Tip}^*(g)\}.$$

Hence, $\mathcal{G}'$ is a tip-reduced uniform right Gröbner basis. ☐

We can write R as $\coprod_{v\in\Gamma_0} vR$, and since R is a right projective R-module each submodule constructed by summing an arbitrary of the summands is also right projective, for if we have a subset $\mathcal{G} \subseteq \Gamma_0$, then assuming there are two right modules N, M and an epimorphism $f : N \to M$ and a homomorphism $g\colon \coprod_{v\in\mathcal{G}} vR \to M$, then we have the inclusion $\iota\colon \coprod_{v\in\mathcal{G}} vR \hookrightarrow R$ and the projection $\pi\colon R \to \coprod_{v\in\mathcal{G}} vR$. Since R is right projective and there is a homomorphism $g \circ \pi\colon R \to M$ there exists a $h\colon R \to N$. We get $f \circ h = g \circ \pi \implies f \circ h \circ \iota = g \circ \pi \circ \iota = g \circ \mathrm{id} = g$, hence $\coprod_{v\in\mathcal{G}} vR$ is a right projective R-module. We want to construct an ordered basis for this.

Theorem 3.5. *Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$. Let I be an index set and $V : I \to \Gamma_0$. Then the right projective module $P := \coprod_{i\in I} V(i)R$ has a ordered basis $(\mathcal{P}, \succ)$.*

Proof. First we have to show that for each $v \in \Gamma_0$, $(v\mathcal{B}, >)$, where $v\mathcal{B} := \{b \in \mathcal{B} : vb = b\}$, is an ordered basis for vR and $>$ is the admissible order on $\mathcal{B}$ restricted to $v\mathcal{B}$. Let $b, b' \in v\mathcal{B}$. Then $bb' \in \mathcal{B} \cup \{0\}$. If $bb' = 0$ then OK. Assume $0 \neq bb' = b''$. Since $b = vb$ we have that $b'' = vbb' = vb''$. Hence, $v\mathcal{B}$ is a coherent basis. The right admissible order follows from $\mathcal{B}$. Hence, $(v\mathcal{B}, >)$ is an ordered basis.

Now we want to make an ordered basis for P. For each $i \in I$ let

$$\mathcal{P}_i = \{x \in P : x_j = 0 \text{ for } i \neq j \in I \text{ and } x_i \in V(i)\mathcal{B}\}.$$

We have a basis for P: $\mathcal{P} := \bigcup_{i\in I} \mathcal{P}_i$. We need an admissible order $\succ$ on $\mathcal{P}$. Have some well-ordering $>_I$ on I and for $x_1, x_2 \in \mathcal{P}$, we define $x_1 \succ x_2$ if $b_1 > b_2$ where b_1 is the non-zero entry of x_1 and b_2 is the non-zero entry of x_2, or, if the entries are equal, the non-zero entry of x_1 occurs in the ith component and the non-zero entry of x_2 occurs in the jth component and $i >_I j$.

We want to verify that $(\mathcal{P}, \succ)$ is an ordered basis of P. Let $x \in \mathcal{P}$ and $b \in \mathcal{B}$. Then there is some $j \in I$ such that $x \in V(j)\mathcal{B}$, and since this is a coherent basis as shown previously, so is $\mathcal{P}$.

Now we wish to show $\succ$ is a right admissible order on $\mathcal{P}$. Since $>$ is a well-ordering, there is a minimal element in $\mathcal{B}$, say, b'. Since $>_I$ is a well-ordering, there is a minimal element in

$$\{i \in I : \exists b \in V(i)\mathcal{B},\ b = b'\} \subseteq I,$$

say, k. We see that for any other element in $\mathcal{P}$ by the definition of the ordering of $\succ$ any element must have to have at least the same non-zero entry, and since the index is chosen such that it is minimal, it cannot be any other. Hence, $\succ$ is well-ordered.

Let $x_1, x_2 \in \mathcal{P}$ and suppose $x_1 \succ x_2$. Let $b \in \mathcal{B}$. We check both cases:

1. $x_1 > x_2$ as elements in $\mathcal{B}$: Then, since $>$ is an admissible order on $\mathcal{B}$ we have $x_1 b > x_2 b$ if $x_1 b \neq 0 \neq x_2 b$, so $x_1 b \succ x_2 b$.

2. $i >_I j$ where $V(i)x_1 = x_1$ and $V(j)x_2 = x_2$: Then, since $x_1 = x_2$ as elements in $\mathcal{B}$, we have $x_1 b = x_2 b$ keeping the equality, whilst also index of non-zero entries are not changed, as $v\mathcal{B}$ is an ordered basis for all $v \in \Gamma_0$, unless the products are 0. Hence, $x_1 b \succ x_2 b$ in both cases.

Finally we have to check that M3. holds. Let $x \in \mathcal{P}$. Let $b_1, b_2 \in \mathcal{B}$ and suppose $b_1 > b_2$. x is in some $v\mathcal{B} \subseteq \mathcal{B}$ for $v \in \Gamma_0$, so we have that $xb_1 > xb_2$ as elements in $\mathcal{B}$, particularly in $v\mathcal{B}$, as long as the products are not zero. Hence, $xb_1 \succ xb_2$. $\square$

Suppose now $R = K\Gamma$, a path algebra, and $\mathcal{B}$ is a set of finite paths in R with an admissible order $>$. Since R is now a path algebra we have that for each $p_1, p_2 \in \mathcal{B}$,

$$p_1p_2 = \begin{cases} p \in \mathcal{B} & \text{if } t(p_1) = o(p_2) \\ 0 & \text{otherwise.} \end{cases}$$

Suppose we have some left and right uniform $x, y \in R$, respectively, so that for some $v \in \Gamma_0$, $xv = x = \sum_{i=1}^{r_1} \alpha_i p_i$ and $vy = y = \sum_{j=1}^{r_2} \beta_j q_j$ for $p_i v, vq_j \in \mathcal{B}$ and $\alpha_i, \beta_j \in K^*$. Then all the basis elements occuring in xy are in $\{p_iq_j \in \mathcal{B} : 1 \leq i \leq r_1 \text{ and } 1 \leq j \leq r_2\}$. Notice that $\text{Tip}^*(x) \succ p_i \implies \text{Tip}^*(x)q_j \succ p_iq_j$ for all i, except i such that $p_i = \text{Tip}^*(x)$, and j, since $>$ is an admissible order and none of the compositions are zero. Similarly, $\text{Tip}^*(y) = \text{Tip}(y) > q_j \implies p_i\text{Tip}^*(y) \succ p_iq_j$. Thus, we have

$$\begin{aligned} \text{Tip}^*(x)\text{Tip}^*(y) &\succ p_i\text{Tip}^*(y) \succ p_iq_j \\ \text{Tip}^*(x)\text{Tip}^*(y) &\succ \text{Tip}^*(x)q_j \succ p_iq_j \end{aligned}$$

Hence, $\text{Tip}^*(xy) = \text{Tip}^*(x)\text{Tip}^*(y)$. Next we want to show that we have more right projective R-modules.

Lemma 3.6. *Let $x \in R^*$ left uniform, with $xv = x$ for some $v \in \Gamma_0$. Then $vR \cong xR$ and, hence, xR is a right projecive R-module.*

Proof. Define the homomorphism $f : vR \to xR$ by $vr \mapsto x(vr) = xr$ which is clearly onto. We want to show that it is one-to-one. Suppose $f(vr) = xr = 0$ and assume $vr \neq 0$. Then $\text{Tip}^*(vr) = v\text{Tip}^*(r) \neq 0$ and $\text{Tip}^*(xr) = 0$. Since $xv = x \in R^*$, $\text{Tip}^*(x)v \neq 0$, but then $0 \neq \text{Tip}^*(x)\text{Tip}^*(vr) = \text{Tip}^*(xr)$. A contradiction. So $vr = 0$, and hence f is an isomorphism of right modules. $\square$

Hence, we get a fundamental result from this which is used throughout.

Theorem 3.7. *Let $P = \coprod_{i \in I} V(i)R$ be a projective right R-module with ordered basis $(\mathcal{P}, \succ)$. If $\mathcal{G}$ is a uniform tip-reduced subset of P, then the right submodule generated by $\mathcal{G}$ is the right projective module $\coprod_{g \in \mathcal{G}} gR$.*

In addition, if Q has a finitely generating set, then every uniform tip-reduced right Gröbner basis is finite.

Proof. Let Q be the right submodule generated by $\mathcal{G}$. To show that the sum is direct, we have to show that if $x = \sum_{g \in \mathcal{G}} gr_g = 0$ then necessarily $r_g = 0$ for all $g \in \mathcal{G}$. Suppose there is some $g' \in \mathcal{G}$ such that $r_{g'} \neq 0$. Then for some $v \in \Gamma_0$, $r_{g'}v \neq 0$, so $xv \neq 0$. We replace x such that $xv = x$. Since $\mathcal{G}$ is uniform there is some $v_g \in \Gamma_0$ for each g such that $gv_g = g$, hence $gr_g = gv_gr_g$ and we can assume $v_gr_g = r_g$. From these assumptions we have that $\text{Tip}^*\left(gr_g\right) = \text{Tip}^*(g)\text{Tip}^*\left(r_g\right)$. Pick the $g_0 \in \mathcal{G}$ such that $\text{Tip}^*\left(g_0r_{g_0}\right)$ is maximal in the sum. We have that $\text{Tip}^*(g_0) = V(j)p_0 = p_0$

for some $j \in I$ and $p_0 \in \mathcal{B}$. Since $x = 0$ and it is maximal, there must be another $g \in \mathcal{G}$ for cancellation to occur, where $\textsc{Tip}^*(g) = V(j)p = p \in \mathcal{B}$. So $\textsc{Tip}^*\left(gr_g\right) = \textsc{Tip}^*\left(g_0 r_{g_0}\right)$ and $g \neq g_0$. However,

$$p\textsc{Tip}^*\left(r_g\right) = \textsc{Tip}^*(g)\textsc{Tip}^*\left(r_g\right) = \textsc{Tip}^*(g_0)\textsc{Tip}^*\left(r_{g_0}\right) = p_0\textsc{Tip}^*\left(r_{g_0}\right)$$

So either $p = p_0 q$ or $pq = p_0$ for some $q \in \mathcal{B}$. In both cases the tips left divide each other, contradicting the assumption that $\mathcal{G}$ is tip-reduced. From Lemma 3.6 we have for each $g \in \mathcal{G}$, gR is projective, by the assumption that they are left uniform, and so the result follows.

Finally, if Q is finitely generated, and, for a contradiction, Q has a uniform tip-reduced right Gröbner basis $\mathcal{G}$ that is infinite, then by what we have shown earlier, $Q = \coprod_{g\in\mathcal{G}} gR$ is not finitely generated. A contradiction. So $\mathcal{G}$ is finite. □

We use this to also show a stronger result in the case of path algebras.

Theorem 3.8. *Let $R = K\Gamma$ be a path algebra. Let $P = \coprod_{i\in I} V(i)R$ for some index set I with ordered basis $(\mathcal{P}, \succ)$ and where Q is a right submodule of P. Then there is a tip-reduced uniform right Gröbner basis of Q. For every tip-reduced uniform right Gröbner basis $\mathcal{G}$ of Q:*

$$Q = \coprod_{g\in\mathcal{G}} gR.$$

Furthermore, for every uniform tip-reduced generating set, G, of Q, G is a right Gröbner basis with respect to $\succ$.

Proof. From Proposition 3.4 the existence follows.

Suppose now $\mathcal{G}$ is a uniform tip-reduced right Gröbner basis of Q. By Theorem 3.7, $\langle\mathcal{G}\rangle_R = \coprod_{g\in\mathcal{G}} gR$. Furthermore, by Theorem 3.3, $\langle\mathcal{G}\rangle_R = Q$. Hence $\coprod_{g\in\mathcal{G}} gR = Q$.

We now prove the last claim. Again, by Theorem 3.7, we have that $Q = \coprod_{g\in G} gR$. Let $x = \sum_{g\in G} gr_g \in Q^*$ where finite $r_g \in R^*$. Let $\textsc{Tip}^*\left(g_0 r_{g_0}\right)$ be the largest tip from the gr_g occuring in x. Suppose $\textsc{Tip}^*(x) \neq \textsc{Tip}^*\left(g_0 r_{g_0}\right)$. Then there has to be some $\textsc{Tip}^*\left(gr_g\right)$ cancelling out with $\textsc{Tip}^*\left(g_0 r_{g_0}\right)$, so we have that $\textsc{Tip}^*(g)\textsc{Tip}^*\left(r_g\right) = \textsc{Tip}^*(g_0)\textsc{Tip}^*\left(r_{g_0}\right)$. So either $\textsc{Tip}^*(g) = \textsc{Tip}^*(g_0)q$ or $\textsc{Tip}^*(g)q = \textsc{Tip}^*(g_0)$ for some $q \in \mathcal{B}$, which contradicts the assumption that G is tip-reduced. Hence, G is a right Gröbner basis. □

Next, we provide an algorithm that will give us a uniform tip-reduced right Gröbner basis for submodules that are finitely generated using what we have learned. This will be useful later on, when we are working with intersections of submodules.

Algorithm 1 Construction of Uniform Tip-Reduced Right Gröbner Bases For Finitely Generated Submodules Q of P

Input: Finite uniform set $\mathcal{H} = \{h_1, \ldots, h_r\} \subset P$
Output: Uniform tip-reduced right Gröbner basis $\mathcal{H}$
$\mathcal{T}_{\mathcal{H}} \leftarrow \emptyset$
while $|\mathcal{T}_{\mathcal{H}}| \neq |\mathcal{H}|$ **do**
    $\mathcal{T}_{\mathcal{H}} \leftarrow \emptyset$
    for all $h \in \mathcal{H}$ **do** ▷ Tip-reduce $\mathcal{H}$.
        $\mathcal{T}_{\mathcal{H}} \leftarrow \mathcal{T}_{\mathcal{H}} \cup \{h\}$
        for all $h' \in \mathcal{H} \setminus \{h\}$ **do**
            if $\text{TIP}^*(h')$ left divides $\text{TIP}^*(h)$ **then**
                $\mathcal{T}_{\mathcal{H}} \leftarrow \mathcal{T}_{\mathcal{H}} \setminus \{h\}$
            end if
        end for
    end for
    $Q' \leftarrow \coprod_{h \in \mathcal{T}_{\mathcal{H}}} hR$
    for all $h \in \mathcal{T}_{\mathcal{H}}$ **do**
        $n \leftarrow \text{Norm}(h) \in \text{Span}(\text{NONTIP}^*(Q'))$ ▷ Use Theorem 3.7. Reduce tips using $\mathcal{H}$, to get $\text{Norm}(h)$, which remains uniform.
        if $n \neq 0$ **then**
            $\mathcal{H} \leftarrow \mathcal{H} \cup \{n\}$
        end if
    end for
end while
return $\mathcal{H}$

We now go back to the case of two-sided ideals, where we wish to find a right Gröbner basis of these ideals as right ideals.

Definition 3.7. Let $p, p_1, p_2 \in \mathcal{B}$ If $p = p_1 p_2$, then we say p_1 is a *prefix* of p, and a *proper prefix* if $p_2 \notin \Gamma_0$.

Remember in the case of right modules, M, we had left uniformity on $m \in M$ if there was some $v \in \Gamma_0$ such that $mv = m$. For two-sided ideals, we say that an element $g \in I$ is *strongly uniform* if it has some $v_1, v_2 \in \Gamma_0$ such that $v_1 g v_2 = g$.

In the theorem that follows, we show how we can construct a right Gröbner basis from an ideal.

Theorem 3.9. *Let I be a two-sided ideal of $R = K\Gamma$ with ordered multiplicative basis $(\mathcal{B}, >)$ with paths in Γ and suppose $\mathcal{G}$ is a reduced strongly uniform Gröbner basis of I. Let*

$$\mathcal{X} = \{pg \in I : p \in \text{NONTIP}(I), g \in \mathcal{G} \text{ and no proper prefix } p_1 \in \text{TIP}(I) \text{ of } \text{TIP}(pg)\}.$$

Then, as right ideals, $\mathcal{X}$ is a tip-reduced uniform right Gröbner basis of I and

$$I = \coprod_{x \in \mathcal{X}} xR.$$

Proof. Let $pg, p'g' \in \mathcal{X}$. Then, by construction, if $\textsc{Tip}(pg) = \textsc{Tip}(p'g')\,b$ for some $b \in \mathcal{B}$, then $b \in \Gamma_0$. g' is strongly uniform, so

$$p'\textsc{Tip}\left(g'\right) = \textsc{Tip}\left(p'g'\right) = \textsc{Tip}(pg) = p\textsc{Tip}(g),$$

for otherwise $\textsc{Tip}(p'g')\,b = 0$. Then, either $\textsc{Tip}(g') = q\textsc{Tip}(g)$ or $q\textsc{Tip}(g') = \textsc{Tip}(g)$. However, $\mathcal{G}$ is reduced. A contradiction. So $g' = g$, and so $p' = p$. Thus, $\mathcal{X}$ is tip-reduced.

We also have that $\mathcal{X} \subset I$ as $\mathcal{G} \subset I$, as well as left uniform as $\mathcal{G}$ is both left and right uniform. We want to show for each $x \in I$, x is also in the right ideal generated by $\mathcal{X}$. Assume to the contrary that there are some elements in I, but not in the right ideal. Call the set S. Then there is some $\textsc{Tip}(x)$ minimal element in $\textsc{Tip}(S)$, and so a $g \in \mathcal{G}$ such that $\textsc{Tip}(x) = p\textsc{Tip}(g)\,q$ for some $p, q \in \mathcal{B}$. Choose such that p is minimal. Then, as $\mathcal{G}$ is a reduced Gröbner basis for I, $p\textsc{Tip}(g)$ has no proper prefix in $\textsc{Tip}(I)$, so $p \in \textsc{NonTip}(I)$. Thus $pg \in \mathcal{X}$. In the right ideal generated by $\mathcal{X}$ we, thus, have pgq. Since r had the smallest tip out of the elements not in the right ideal and

$$r' = r - \frac{\textsc{CTip}(r)}{\textsc{CTip}(g)}pgq$$

has a tip smaller than r, we arrive at a conclusion that r' and hence r is in the right ideal. A contradiction. Therefore, $\mathcal{X}$ is a right Gröbner basis of I.

Finally, by Theorem 3.7, it immediately follows that $I = \coprod_{x \in \mathcal{X}} xR$.

□

This will show its usefulness in what will soon follow.

We want to end this section with introducing the elimination theory in the noncommutative case of path algebras and finally the problem of generating intersections between two right submodules of a right projective module.

We start off by introducing a special admissible ordering, as the usual lex ordering on $\mathcal{B}$ (basis of paths) is not a well ordering in path algebras, and then extend to a right admissible ordering.

Definition 3.8. Let $R = K\Gamma$, a path algebra, and $\mathcal{B}$ be the basis of paths.

1. Let $>_c$ be defined as the *commutative lex order* on $\mathcal{B}$ as such: Identify all vertices as 1 and let all the arrows commute, viewing the paths this way as commutative monomials in a commutative polynomial ring, where the arrows are the commutative variables.

2. Define $>_{nc}$ to be a *noncommutative lex order* on $\mathcal{B}$, where the order of the vertices and arrows of Γ such that all vertices are less than the arrows and for $p, q \in \mathcal{B}$ then

$$p >_{nc} q \iff \begin{cases} p >_c q, \text{ or} \\ p =_c q \text{ and } p >_l q \end{cases}$$

   where $>_l$ is the left lexicographical order using the fixed order on arrows and vertices and $>_c$ as mentioned above with the same fixed order.

We show that this is an admissible ordering.

Proposition 3.10. $>_{nc}$ *is an admissible ordering on* $\mathcal{B}$.

Proof. We have a minimal element in $\mathcal{B}$, that is the smallest vertex. Now suppose $b_1, b_2, b_3 \in \mathcal{B}$ such that $b_1 >_{nc} b_2$. We wish to show that $b_1 b_3 >_{nc} b_2 b_3$ whenever $b_1 b_3 \neq 0 \neq b_2 b_3$, and $b_3 b_1 >_{nc} b_3 b_2$ whenever $b_3 b_1 \neq 0 \neq b_3 b_2$. We have two cases.

1. $b_1 >_c b_2$: Multiplying by b_3 on left or right side does not matter in the case we are assuming the arrows commute and adding extra and same variables to b_1 and b_2 does not affect the commutative inequality. Hence,

$$b_1 b_3 >_c b_2 b_3 \implies b_1 b_3 >_{nc} b_2 b_3$$
$$b_3 b_1 >_c b_3 b_2 \implies b_3 b_1 >_{nc} b_3 b_2$$

2. $b_1 =_c b_2$ and $b_1 >_l b_2$: Multiplying from the right by b_3 clearly does not affect the first left inequality of arrows/vertices for b_1 and b_2. Multiplying from the left, we just have the same terms for the first arrows and vertices of b_3 until we again encounter the left-most inequality of arrows/vertices. Hence,

$$b_1 b_3 =_c b_2 b_3 \text{ and } b_1 b_3 >_l b_2 b_3 \implies b_1 b_3 >_{nc} b_2 b_3$$
$$b_3 b_1 =_c b_3 b_2 \text{ and } b_3 b_1 >_l b_3 b_2 \implies b_3 b_1 >_{nc} b_3 b_2.$$

Lastly, we have to check for $b, b_1, b_2, b_3 \in \mathcal{B}$ that if $b =_{nc} b_1 b_2 b_3$ then $b \geq_{nc} b_2$. Since $b =_c b_1 b_2 b_3$ we have $b >_c b_2 \implies b >_{nc} b_2$ whenever b_1 or b_3 are in $\mathcal{B} \setminus \Gamma_0$. So suppose both $b_1, b_3 \in \Gamma_0$. Then $b =_l b_1 b_2 b_3 =_l b_2 \implies b =_{nc} b_2$, and otherwise 0 if $o(b_2) \neq b_1$ and $t(b_2) \neq b_3$.

Thus $>_{nc}$ is an admissible ordering on $\mathcal{B}$. □

Now we can extend $>_{nc}$ to a right admissible order $\succ$ on the basis of $P = \coprod_{i \in I} V(i)R$ exactly as we did in the proof of Theorem 3.5. Thus we have an ordered basis $(\mathcal{P}, \succ)$ for P.

For the elimination theory that follows, we define what it means to eliminate arrows.

Definition 3.9. Let Γ be a quiver, $R = K\Gamma$ a path algebra and $\mathcal{B}$ the basis of paths. Let $a \in \Gamma_1$. We define the following.

1. Denote Γ_a as the quiver where $(\Gamma_a)_0 := \Gamma_0$ and $(\Gamma_a)_1 := \Gamma_1 \setminus \{a\}$. Then $K\Gamma_a$ is viewed as a subalgebra of $K\Gamma$. If S is a subset of $K\Gamma$, then $(S)_a := S \cap K\Gamma_a$.

2. Denote $P_a := \coprod_{i \in I} (V(i)K\Gamma)_a$. If S is a subset of P, then $(S)_a := S \cap P_a$.

3. Let $(>_{nc})_a$ be the restriction of $>_{nc}$ to the paths in Γ_a, $\mathcal{B}_a$, which is an non-commutative lex order on $K\Gamma_a$ and an admissible order on $\mathcal{B}_a$.

4. Let $\succ_a$ be the extension of $(>_{nc})_a$ to P_a, as we did for $>_{nc}$.

Recursively, we can define this for a subset of arrows we wish to remove. Suppose $U = a_1, \dots, a_n \subseteq \Gamma_1$. Then, define $\Gamma_U := (\Gamma_{U \setminus \{a_1\}})_{a_1}$.

What follows is known as The Elimination Theorem, showing that the Gröbner basis for a right projective module, with an eliminated arrow as defined above, is same, but where the elements with the arrow are removed. Furthermore it will preserve uniformity and remain reduced. Notice there is more to the result.

Theorem 3.11. *Let Γ be a quiver with a noncommutative lex ordering $>_{nc}$ on the paths of Γ. Let $a \in \Gamma_1$ be the maximal arrow in Γ with respect to $>_{nc}$. Let $P = \coprod_{i \in I} V(i)K\Gamma$ with ordered basis $(\mathcal{P}, \succ)$. If $\mathcal{G}$ is a uniform right Gröbner basis for P with respect to $\succ$, then $\mathcal{G}_a$ is a uniform right Gröbner basis for P_a with respect to $(\succ)_a$.*

Furthermore, if $U = \{a_1, \dots, a_n\} \subset \Gamma_1$ where

$$a_1 >_{nc} a_2 >_{nc} \cdots >_{nc} a_n >_{nc} a$$

for all $a \in (\Gamma_1)_U$, then $\mathcal{G}_U$ is a uniform right Gröbner basis for P_U with respect to $(\succ)_U$

Proof. We want to show that $\mathcal{G}_a = \mathcal{G} \cap P_a$ is a uniform right Gröbner basis for P_a with respect to $\succ_a$. Since $\mathcal{G}$ is already uniform, so is $\mathcal{G}_a$.

Let $z \in P_a \subset P \implies \exists g \in \mathcal{G}$ such that $\text{TIP}^*(g)\, b = \text{TIP}^*(z)$ for some $b \in \mathcal{B}$. If $g \in P_a$, then $g \in \mathcal{G}_a$, and this would mean the right modules generated by the tips of P_a and $\mathcal{G}_a$ are equal. We have that $\text{TIP}^*(g) = \text{TIP}^*(z) \in P_a$. Then there is some $i \in I$ and $p \in (V(i)\mathcal{B})_a$ such that $\text{TIP}^*(g)$ has p in the ith component and 0 everywhere else. Now for any other basis element, q, that has p_q at the i_qth component and 0 everywhere else, occuring in z we have $\text{TIP}^*(z) \succ q$, and so $p \geq_{nc} p_q$. Hence by the definition of $>_{nc}$ a cannot occur in q, so $q \in P_a$. Thus $g \in P_a$. Thus, $\mathcal{G}_a$ is a uniform right Gröbner basis.

To show the final claim, we remove a_1 which is maximal for all arrows, and we have a right uniform right Gröbner basis as above. Next, we remove a_2 which is in the current maximal arrow of Γ_a. Continuing the recursion this way we arrive at the result. □

The next construction, that will be defined, will be useful for finding a generating set of the intersection of two (right) ideals.

Definition 3.10. Let Γ be a quiver.

1. Define $\Gamma[T]$ to be the quiver such that

$$\Gamma[T]_0 = \Gamma_0$$
$$\Gamma[T]_1 = \Gamma_1 \cup \{(T_v : v \to v) : v \in \Gamma_0\}.$$

   We view $K\Gamma$ as a subalgebra of $K\Gamma[T]$.

2. If $P = \coprod_{i \in I} V(i)K\Gamma$ for $V : I \to \Gamma_0$, denote $P[T] := \coprod_{i \in I} V(i)K\Gamma[T]$.

3. Let $\mathcal{B}[T]$ be a basis of paths in $K\Gamma[T]$. Let $(\mathcal{P}, \succ)$ be an ordered basis of P, with noncommutative lex ordering $>_{nc}$ on $\mathcal{B}$. Then, extend $>_{nc}$ to $\mathcal{B}[T]$ by fixing a ordering to T_v for $v \in \Gamma_0$ where $T_v >_{nc} a$ for all $a \in \Gamma_1$.

4. Let $T := \sum_{v\in\Gamma_0} T_v$.

By the definition of T we see that for a $p \in \mathcal{B}$, then $Tp = T_{o(p)}p$ and $(1-T)p = p - T_{o(p)}p$. For any $x \in P$, we have the ith component is $\sum_{j=1}^{r} \alpha_j p_j$ where $\alpha_j \in K^*$ and $p_j \in V(i)\mathcal{B}$. So for the case of $Tx \in P[T]$ we define that the ith component is $\sum_{j=1}^{r} \alpha_j T p_j = \sum_{j=1}^{r} \alpha_j T_{V(i)} p_j$. Similarly, we define for $(1-T)x \in P[T]$.

As usual, for a right submodule, Q, of P, we denote TQ for the right submodule of $P[T]$ where it consists of $\{Tx \in P[T] : x \in Q\}$, and similarly for $(1-T)Q$.

Next we show a result where we can express the intersection of two right submodules of P using the construction above.

Theorem 3.12. *Let $R = K\Gamma$ be a path algebra. Let the right projective R-module be $P = \coprod_{i\in I} V(i)R$ with ordered basis $(\mathcal{P}, \succ)$, where Q_1 and Q_2 are right submodules of P. Then*

$$Q_1 \cap Q_2 = (TQ_1 + (1-T)Q_2) \cap P.$$

Proof. Suppose $h \in Q_1 \cap Q_2$. Then $h \in P$ and $h = Th + (1-T)h \in TQ_1 + (1-T)Q_2$. So $Q_1 \cap Q_2 \subseteq (TQ_1 + (1-T)Q_2) \cap P$.

Now let $h \in (TQ_1 + (1-T)Q_2) \cap P$. Then there is some $q_1 \in Q_1$ and $q_2 \in Q_2$ such that $h = Tq_1 + (1-T)q_2 = T(q_1 - q_2) + q_2$. Since $h \in P$ we cannot have any loops from T in h, so $T(q_1 - q_2) = 0$, and since q_1 and q_2 must then be equal in every component they are non-zero at, $q_1 = q_2$. Thus $q_1 = q_2 = h \in Q_1 \cap Q_2$.

Thus, we arrive at the result $Q_1 \cap Q_2 = (TQ_1 + (1-T)Q_2) \cap P$. ☐

From this result, we can find a generating set from the intersection, given we have Q_1 and Q_2's uniform tip-reduced Gröbner bases $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Assume these are finite.

From earlier, we have right submodules $TQ_1, (1-T)Q_2 \subseteq P[T]$ and their uniform tip-reduced generating sets $T\mathcal{G}_1$ and $(1-T)\mathcal{G}_2$, respectively. For the right submodule $TQ_1 + (1-T)Q_2$ we then have a generating set

$$G_{1+2} = \{Tg + (1-T)f : g \in \mathcal{G}_1 \text{ and } f \in \mathcal{G}_2\}.$$

Making the set uniform by applying fv for every $f \in G_{1+2}$, and every $v \in \Gamma_0$, we can use Algorithm 1 to make an uniform tip-reduced Gröbner basis, $\mathcal{G}_{1+2}$, for $TQ_1 + (1-T)Q_2$ for the extended ordering on $\mathcal{P}$ to $\mathcal{P}[T]$. Then we can, by Theorem 3.11, go back to the original right module, by removing all the loops, as they are defined to be greater than any other arrow in Γ_1, to get $(\mathcal{G}_{1+2})_{\{T_v : v\in\Gamma_0\}}$ that is a uniform tip-reduced Gröbner basis for

$$(TQ_1 + (1-T)Q_2)_{\{T_v : v\in\Gamma_0\}} = (TQ_1 + (1-T)Q_2) \cap P = Q_1 \cap Q_2.$$

4 Systems of Equations Over Algebras

Let $R = K\Gamma$ be a path algebra with ordered multiplicative basis $(\mathcal{B}, >)$ where $\mathcal{B}$ is the set of paths in Γ. Let $\Lambda = K\Gamma/I$ where I is an ideal of $K\Gamma$, and $A \in \Lambda^{n\times m}$:

$$A = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,m} \\ \vdots & \ddots & \vdots \\ \lambda_{n,1} & \cdots & \lambda_{n,m} \end{pmatrix} \tag{2}$$

where $\lambda_{i,j} \in \Lambda$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Our aim for this section is finding solutions, generators for solutions and, finally, an algorithm for finding such a generating set of the solutions to some homogeneous systems of linear equations with coefficients in Λ, that is finding all $x \in \Lambda^m$ such that

$$Ax = 0. \tag{3}$$

However, first we show that we can transform any linear system to a linear system such that the entries are strongly uniform (that is we have for each entry $\lambda_{i,j}$ some $v, u \in \Gamma_0$ such that $v\lambda_{i,j}u = \lambda_{i,j}$), which also preserves all the solutions in the original system.

Proposition 4.1. *Let Λ as above and $A \in \Lambda^{n\times m}$. Then we can associate a new matrix $A' \in \Lambda^{n'\times m'}$ such that there exist $V : \{1,\ldots,n'\} \to \Gamma_0$ and $W : \{1,\ldots,m'\} \to \Gamma_0$ such that for each entry $\lambda'_{i,j}$ in A' we have $V(i)\lambda'_{i,j}W(j) = \lambda'_{i,j}$, and there is a bijection between the solutions of $Ax = 0$ for $x \in \Lambda^m$ and $A'x' = 0$ for $x' \in \coprod_{j=1}^{m'} W(j)\Lambda$.*

Proof. Suppose $\Gamma_0 = \{v_1,\ldots,v_r\}$. Let $R(\Gamma_0) := \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix}$. First, we can start by replacing the i-th row $\begin{pmatrix} \lambda_{i,1} & \cdots & \lambda_{i,m} \end{pmatrix}$ in A with $\begin{pmatrix} \lambda_{i,1}v_1 & \cdots & \lambda_{i,1}v_r & \cdots & \lambda_{i,m}v_r \end{pmatrix}$ for $1 \leq i \leq n$ which we get from multiplying each row with

$$I_R := \begin{pmatrix} R(\Gamma_0) & & 0 \\ & \ddots & \\ 0 & & R(\Gamma_0) \end{pmatrix}$$

from the right side. So our new matrix is AI_R. Next, we replace each column, $1 \leq j \leq rm$, in the AI_R with $\begin{pmatrix} v_1\lambda_{1,s}v_t & \cdots & v_r\lambda_{1,s}v_t & \cdots & v_r\lambda_{n,s}v_t \end{pmatrix}^T$, where $s = \lceil \frac{j}{r} \rceil$ and $t = j - r(s-1)$, by multiplying from the left with

$$I_L := \begin{pmatrix} R(\Gamma_0)^T & & 0 \\ & \ddots & \\ 0 & & R(\Gamma_0)^T \end{pmatrix}.$$

The final matrix is then $A' := I_LAI_R \in \Lambda^{rn\times rm}$. We see that for each entry in A' we have left and right uniformity. We can define $W : \{1,\ldots,m\} \times \{1,\ldots,r\} \to \Gamma_0$ and

$V:\{1,\dots,n\}\times\{1,\dots,r\}\to\Gamma_0$ by $W(j,l)=v_l$ and $V(k,l')=v_{l'}$ respectively, where $1\le j\le m$, $1\le k\le n$, and $1\le l,l'\le r$.

To see that there is a bijection between the solutions of $Ax=0$ and $A'x'=0$ we notice that for each solution $\begin{pmatrix}x_1 & x_2 & \cdots & x_m\end{pmatrix}^T$ of the first problem we have a solution $\begin{pmatrix}v_1x_1 & v_2x_1 & \cdots & v_rx_1 & \cdots & v_rx_m\end{pmatrix}^T$ of the 2nd problem. Similarly, for each solution $\begin{pmatrix}x_{1,1} & x_{1,2} & \cdots & x_{1,r} & \cdots & x_{m,r}\end{pmatrix}^T$ of $A'x'=0$ we have a solution $\begin{pmatrix}v_1x_{1,1}+v_2x_{1,2}+\cdots+v_rx_{1,r} & \cdots & v_1x_{m,1}+\cdots+v_rx_{m,r}\end{pmatrix}^T$ for $Ax=0$. □

We can then assume that each matrix A has this property to continue. We denote $f_j\in\coprod_{i=1}^n V(i)R$ for the j-th column of A, and notice it is left uniform ($f_jW(j)=f_j$). Particularly, in this section, we are interested in the solutions for matrices A where the direct sum $\coprod_{j=1}^m f_jR$ holds. We have seen from earlier that this holds if the set of f_j's is uniform and tip-reduced.

Let M be the set of solutions for the system, that is $M=\{x\in\coprod_{j=1}^m W(j)\Lambda : Ax=0\}$. We wish to find a generating set for M.

Theorem 4.2. *Let $R=K\Gamma$ be a path algebra with ordered multiplicative basis $(\mathcal{B},>)$ of paths where $>$ is a noncommutative lex order on $\mathcal{B}$. Let $P:=\coprod_{i=1}^n V(i)R$ projective right R-module with an ordered basis $(\mathcal{P},\succ)$. Let $\mathcal{G}$ be a tip-reduced uniform right Gröbner basis for $\coprod_{j=1}^m f_jR\cap\coprod_{i=1}^n V(i)I\subseteq P$, so for each $g\in\mathcal{G}$, $g=\sum_{j=1}^m f_ja_{g,j}$ for some $a_{g,j}\in R$. Then*

$$\left\{\begin{pmatrix}\mathrm{Norm}\left(a_{g,1}\right) & \mathrm{Norm}\left(a_{g,2}\right) & \cdots & \mathrm{Norm}\left(a_{g,m}\right)\end{pmatrix}^T\in\coprod_{j=1}^m W(j)\Lambda : g\in\mathcal{G}\right\}$$

is a generating set for M.

Proof. We want to show any $x=(x_1,x_2,\dots,x_m)\in M$ can be written as a linear combination of elements in the set above. We have that $\sum_{j=1}^m f_jx_j=0$ in $\coprod_{i=1}^n V(i)\Lambda$. In $\coprod_{i=1}^n V(i)R$ we have that $\sum_{j=1}^m f_jx_j\in\coprod_{i=1}^n V(i)I$, so the sum lies in $\coprod_{j=1}^m f_jR\cap\coprod_{i=1}^n V(i)I$, which is generated by $\mathcal{G}$ as right R-module. Hence,

$$\sum_{j=1}^m f_jx_j=\sum_{g\in G} gs_g$$

for some $s_g\in R$ (finite $s_g\neq 0$). As we have for each $g\in\mathcal{G}$, $g=\sum_{j=1}^m f_ja_{g,j}$ we get

$$\sum_{j=1}^m f_jx_j=\sum_{g\in G}\left(\sum_{j=1}^m f_ja_{g,j}\right)s_g=\sum_{j=1}^m f_j\sum_{g\in\mathcal{G}} a_{g,j}s_g.$$

By assumption we have $\coprod_{j=1}^m f_jR$, so for $1\le j\le m$, $x_j=\sum_{g\in\mathcal{G}} a_{g,j}s_g$. Since $x_j\in W(j)\Lambda$,

$$x_j=\mathrm{Norm}\left(\sum_{g\in\mathcal{G}} a_{g,j}s_g\right)=\sum_{g\in\mathcal{G}}\mathrm{Norm}\left(a_{g,j}\right)\mathrm{Norm}\left(s_g\right)$$

where $\text{Norm}\left(s_g\right) \in \Lambda$. Altogether, we have

$$X = \sum_{g \in \mathcal{G}} \begin{pmatrix} \text{Norm}\left(a_{g,1}\right) \\ \text{Norm}\left(a_{g,2}\right) \\ \vdots \\ \text{Norm}\left(a_{g,m}\right) \end{pmatrix} \text{Norm}\left(s_g\right).$$

□

Before continuing forward, we note an important result about the relation of the dimensionality of quotients of K-algebras with ordered multiplicative bases and Gröbner bases of the kernels of the quotient (ideal) with respect to $>$.

Theorem 4.3. *Let S be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$ and assume $\mathcal{B} \cup \{0\}$ is a finitely generated semigroup with 0. Let I be an ideal of S such that S/I has finite dimension over K. Then there exists a finite Gröbner basis for I with respect to $>$.*

Proof. Note that $\text{Span}(\text{NonTip}(I)) = S/I$ as vector spaces, so $\text{NonTip}(I)$ has $\dim_k(S/I)$ elements. Let $\mathcal{T} = \{t \in \text{Tip}(I) : \text{there do not exist } b, b' \in \mathcal{B} \setminus \Gamma_0 \text{ such that } t = bb'\}$. As we have seen earlier we can define $\mathcal{G} = \{t - \text{Norm}(t) \in I : t \in \mathcal{T}\}$ which is a (reduced) Gröbner basis for I.

We want to show that $\mathcal{T}$ is finite. Particularly, we want to show that $\mathcal{T} \subseteq \{nb \in \mathcal{B} : n \in \text{NonTip}(I) \cup B\} \cup B$, where B is the finite set of generators of $\mathcal{B}$. Let $t \in \mathcal{T} \setminus \Gamma_0$. Then, by construction, every proper factor of t must be in $\text{NonTip}(I)$. So $t = o(t)b_1 \cdots b_r$. If $b_r \in B \setminus \{0\}$, then $o(t)b_1 \cdots b_{r-1} \in \text{NonTip}(I)$ or in Γ_0. If in Γ_0, then $t = b_r \in B$. Else, if the factor is in $\text{NonTip}(I)$, then $t \in \{nb \in \mathcal{B} : n \in \text{NonTip}(I) \cup B\}$. Thus, $\mathcal{T}$ is a finite set, and the result follows. □

We see that this result holds for path algebras, where Γ is a finite graph, as $\mathcal{B}$ is finitely generated by its vertices and arrows.

The final result comes now showing when we can have a finite uniform tip-reduced right Gröbner basis for a intersection of the submodules of P, with some additional assumptions.

Theorem 4.4. *Let $R = K\Gamma$ be a path algebra with ordered multiplicative basis $(\mathcal{B}, >)$ of paths where $>$ is a noncommutative lex order on $\mathcal{B}$. Let $P := \coprod_{i=1}^{n} V(i)R$ with an ordered basis $(\mathcal{P}, \succ)$. Let $f_j \in P$, $1 \leq j \leq m$, be a tip-reduced set of uniform elements. Let I be an ideal such that $\Lambda := K\Gamma/I$ is finite dimensional over K and assume $P/\coprod_{j=1}^{m} f_j I$ is finite dimensional. Then there is a finite uniform tip-reduced right Gröbner basis of $\coprod_{j=1}^{m} f_j R \cap \coprod_{i=1}^{n} V(i)I$ in P with respect to $\succ$.*

Proof. Note that $Q = \coprod_{j=1}^{m} f_j R \cap \coprod_{i=1}^{n} V(i)I$ contains the right submodule $Z = \coprod_{j=1}^{m} f_j I$. We have that Q/Z is a right submodule of P/Z, which is assumed to be finite dimensional. Thus, Q/Z is finite dimensional. Let the K-basis of Q/Z be B. For each $b \in B$ choose a uniform $b' \in Q$ such that $b' + Z = b$. Then let $B' = \{b' \in Q : b \in B\}$.

By Theorem 4.3, since Λ is finite dimensional, the ideal I has a finite reduced Gröbner basis and $\textsc{NonTip}(I)$ has $\dim_K(\Lambda)$ elements. Then using Theorem 3.9 we have a finite uniform tip-reduced right Gröbner basis for I as right ideals. Let this set be $\mathcal{G}_I$. Then we have a finite uniform right generating set for Z, call it $G = \{f_j h : 1 \leq j \leq m \text{ and } h \in \mathcal{G}_I\}$. Then $G \cup B'$ is a finite uniform generating set for Q. Finally, by tip-reducing, we get the result. □

It can be shown that $\dim_K(P/Z) < \infty \iff \dim_K(P/\coprod_{j=1}^{m} f_j R) < \infty$, so either of those assumptions of the previous theorem will do it.

From the results we have gotten so far, we can construct an algorithm that will give us the generating set for solutions of $Ax = 0$ where the set of columns of A is a uniform tip-reduced set. We end this section by partially including an algorithm for these generators.

Algorithm 2 Generating Set of Solutions for System of Equations

Find a reduced Gröbner basis, $\mathcal{F}$, for I. $K\Gamma/I$ finite dimensional $\Longrightarrow$ exists a finite algorithm to do this. By Theorem 9.2 (9.1) there is a finite reduced Gröbner basis for I. See [2] for an algorithm.

Find $\textsc{NonTip}(I)$. There is a finite algorithm to find this finite set. $\mathcal{B}$ is finite, and $\textsc{NonTip}(I) = \mathcal{B} \setminus \textsc{Tip}(I)$. We have for all $x \in \textsc{Tip}(I)$, $\exists g \in \mathcal{G}$ such that $x = b_1 \textsc{Tip}(g)\, b_2$. Hence, an algorithm can be:

$H \leftarrow \emptyset$
for $g \in \mathcal{G}$ **do**
  for $b_1 \in \mathcal{B}$ **do**
    for $b_2 \in \mathcal{B}$ **do**
      $H \leftarrow \{b_1 \textsc{Tip}(g)\, b_2\} \cup H$
    end for
  end for
end for
$\textsc{Tip}(I) \leftarrow H$
$\textsc{NonTip}(I) \leftarrow \mathcal{B} \setminus \textsc{Tip}(I)$

Find the uniform reduced right Gröbner basis for I. Use Theorem 3.9.

$\mathcal{G} \leftarrow \emptyset$
for $p \in \textsc{NonTip}(I)$ **do**
  for $g \in \mathcal{G}$ **do**
    $\mathcal{G} \leftarrow \mathcal{G} \cup \{pg\}$
    for $p_1 \in \textsc{Tip}(I)$ **do**
      if p_1 is a proper prefix of $\textsc{Tip}(pg)$ **then**
        $\mathcal{G} \leftarrow \mathcal{G} \setminus \{pg\}$
      end if
    end for
  end for
end for

Set up matrix A and f_j's where $\coprod_{j=1}^{m} f_j K\Gamma$.

Find uniform tip-reduced right Gröbner basis for $\coprod_{j=1}^{m} f_j K\Gamma \cap \coprod_{i=1}^{n} V(i)I$. By using the assumptions of Theorem 9.5 this basis is finite. We have two right submodules of P, and they have uniform tip-reduced Gröbner bases $\{f_j : 1 \leq j \leq m\}$ for $\coprod_{j=1}^{m} f_j K\Gamma$ and $\mathcal{G}$ for $\coprod_{i=1}^{n} V(i)I$. Using the information from the end of chapter 3, using the elimination and intersection methods we can get a finite uniform tip-reduced right Gröbner basis.

Use Theorem 4.2 to get the generating set of solutions using this right Gröbner basis.

References

[1] Maurice Auslander, Idun Reiten, and Sverre O. Smalo. *Examples of algebras and modules*, page 49–99. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[2] Edward L. Green. Noncommutative gröbner bases, and projective resolutions. In P. Dräxler, C. M. Ringel, and G. O. Michler, editors, *Computational Methods for Representations of Groups and Algebras*, pages 29–60, Basel, 1999. Birkhäuser Basel.

[3] Edward L. Green. Multiplicative bases, gröbner bases, and right gröbner bases. *Journal of Symbolic Computation*, 29(4):601–623, 2000.

[4] Øystein Ingmar Skartsæterhagen. Quivers and admissible relations of tensor products and trivial extensions. Master's thesis, Norwegian University of Science and Technology, 2011.

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