Bachelor's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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A Brief Introduction to the Theory of Prolate Spheroidal Wave Functions

Bachelor's thesis in BMAT
Supervisor: Kristian Seip
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Abstract

We go through parts Slepian's article "Some comments on Fourier analysis, uncertainty and modelling" and try to understand the basics of the theory of prolate spheroidal wave functions. We fill in the details where Slepian has been hasty, and try to generally unravel why we care about these objects.

Abstrakt

Vi går gjennom deler av Slepians artikkel "Some comments on Fourier analysis, uncertainty and modelling" og prøver og forstå noen grunnleggende detaljer om prolate sfæriske bølgefunsksjoner. Vi legger til ekstra detaljer der Slepian ikke har vist de åpenbare momentene, og prøver generelt sett å forstå hvorfor vi bryr oss om disse objektene.

This bachelor thesis is dedicated to my first year high-school mathematics teacher, Johan Elon Hake. Without your patience and enthusiasm for teaching, I would have never truly discovered the beautiful world of mathematics. Thank you.

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Contents

1 Introduction

Our tale begins way back in the 60s in the United States, concerning a "Fourier-analytic" question arising in the fields of signal analysis and communication Theory.

The question was related to a certain notion of concentration of interesting functions, called signals, over some interval $[-W, W]$. They used the fact that an $L^2(\mathbb{R})$-function with compact support cannot have a Fourier transform that is also compactly supported. Therefore they were interested in analysing two quantities defined for an $L^2(\mathbb{R})$-function r by

$$\alpha(T) := \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} |r(t)|^2 dt}{\int_{\mathbb{R}} |r(t)|^2 dt},$$

as well as

$$\beta(W) := \frac{\int_{-W}^{W} |\hat{r}(\xi)|^2 d\xi}{\int_{\mathbb{R}} |\hat{r}(\xi)|^2 d\xi}.$$

Slepian & co., who were working on the problem at the time, wanted to find an $L^2(\mathbb{R})$-function with a compactly supported Fourier transform, that maximise this α in some "time slot" $[-T/2, T/2]$. It turned out that by functional analysis a maximising function r of α, must necessarily satisfy the following integral equation:

$$\int_{-W}^{W} \frac{\sin\left[\pi T(\xi' - \xi'')\right]}{\pi(\xi' - \xi'')} \hat{r}(\xi'') d\xi'' = \alpha(T)\hat{r}(\xi'), \text{ for } |\xi'| \leqslant W, \tag{1.1}$$

which we can recognise as a homogeneous Fredholm equation of the second kind. Through clever substitutions, we can go from (1.1) to

$$\int_{-1}^{1} \frac{\sin\left[c(x - y)\right]}{\pi(x - y)} \psi(y) dy = \lambda\psi(x), \text{ for } |x| \leqslant 1. \tag{1.2}$$

Next we deduce some necessary analytic results to help us on our quest. It all boils down to showing that certain integral operators, inspired by (1.2) are compact. This is done so that we can use the well-developed spectral theory for compact, self-adjoint operators on Hilbert spaces. From this we can deduce that (1.2) must have solutions in $L^2(-1, 1)$ only for a discrete set of positive values of λ. The corresponding functions $\psi_1, \psi_2, \ldots$ can be chosen to be real and orthogonal on $(-1, 1)$.

The next thing we do is define the Prolate Spheroidal Wave Functions (PSWFs), and how they are related to the integral equation (1.2). Indeed, we define the PSWFs to be the solutions to the second order differential equation

$$\frac{d}{dx}(1 - x^2)\frac{d\psi}{dx} + (\chi - c^2x^2)\psi = 0. \tag{1.3}$$

This, we recognise as a Sturm-Liouville problem, and we can therefore deduce many interesting properties about them. We also show that we have a remarkable commutation, which allows us to conclude that the ψ's defined in (1.2) are actually the same functions as the PSWFs, but with differing, although related, eigenvalues. The commutation mentioned is the commutation of

$$\begin{aligned} \mathcal{Q}_c \colon L^2[-1,1] &\to L^2[-1,1] \\ \psi &\mapsto \int_{-1}^{1} \frac{\sin[c(x-y)]}{\pi(x-y)} \psi(y) dy, . \end{aligned}$$

the integral operator present in the eigenvalue problem (1.2), and

$$\begin{aligned} L_c \colon L^2[-1,1] &\to L^2[-1,1] \\ \phi &\mapsto -\frac{d}{dx}\left[(1-x^2)\frac{d\phi}{dx}(x)\right] + c^2x^2\phi(x). \end{aligned}$$

To be concrete, what is shown is that for a function ϕ of some regularity, we have that

$$L_c[\mathcal{Q}_c[\phi]](x) = \mathcal{Q}_c[L_c[\phi]](x).$$

The situation described above was apprehended by Henry Pollack, Henry Landau and David Slepian, working at Bell Labs at the time. Slepian described the problem as elegant in two different ways: firstly, because the problem was (seemingly) completely solved. Secondly, because the answer itself was interesting. Indeed, Slepian states: "Usually I struggle for months or years with a problem. If I do "solve" it, it is usually only in part, and the answer itself is rarely interesting. The interest generally lies in the fact that I have proved that this is the answer."[8][pp. 379]

In this thesis, we will take a step-by-step approach to see how Slepian & co. arrived at the definition of the Prolate Spheroidal Wave Functions. The entire thesis will lead up to some extremely elegant results regarding some properties of the PSWFs. The caveat is that I am only providing explicit proofs for two of six of the properties.

2 Preliminaries

I want to make this thesis understandable to as many people as possible, but I don't want to spend too many pages on preliminaries. Thus I have set the reasonably realistic goal of expecting whomever may read this thesis, to have a basic understanding of real & complex analysis (equivalent to courses MA1101, MA1102 and MA2106), as well as introductory functional analysis (equivalent to TMA4145) and Fourier analysis (MA2106). Although I originally wanted to avoid it, I've concluded that I must assume some knowledge of measure and integration theory, equivalent to what one would learn in TMA4225.

2.1 Lebesgue Integration

We begin by way of a reminder from functional analysis.

Definition 2.1 (L^p-spaces)**.** We define the vector space $L^p(\Omega)$, for $1 \leqslant p < \infty$ and $\Omega \subseteq \mathbb{C}$, by

$$L^p(\Omega) := \left\{ f\colon \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)|^p dx < \infty \right\}, \tag{2.1}$$

and for $p = \infty$, we define

$$L^\infty(\Omega) := \left\{ f\colon \Omega \to \mathbb{C} \;\middle|\; \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty \right\}.$$

With this vector space comes the associated norm

$$\|f\|_{L^p(\Omega)} := \left[\int_\Omega |f(x)|^p dx \right]^{\frac{1}{p}},$$

if $p \neq \infty$, and

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|$$

if $p = \infty$.

We say that a function $f\colon \Omega \to \mathbb{C}$ is *integrable* on Ω if its $L^1(\Omega)$-norm is finite. A beautiful result in functional analysis states that these normed spaces $\left(L^p(\Omega), \|\cdot\|_{L^p(\Omega)}\right)$ are Banach spaces (i.e. complete as metric spaces) for all $p \in [1, \infty]$.
In much the same way as we define the Darboux integral as an approximation from below, we would like to do the same for Lebesgue integration with so called *simple functions*.

Definition 2.2 (Simple functions)**.** A function is called *simple* if it only takes finitely many values.

This immediately implies that if $f\colon X \to \mathbb{R}$ is simple, then $f = \sum_{j=1}^{n} c_j \chi_{E_j}$, where $\{c_k\}$ are the distinct non-zero values of f. Here $E_k = f^{-1}(\{c_k\})$. If all the E_k's are intervals, then we say that the function f is a *step-function.*
This next theorem will give us some intuition for the fact that every function can be well-approximated by simple functions in the L^1-norm.

Theorem 2.3. *Suppose $(X, \mathcal{A}, \mu)$ is a measure space, and $f\colon X \to [0,\infty]$ is $\mathcal{A}$-measurable. Then*

$$\int_X f d\mu = \sup \left\{ \sum_{j=1}^{n} c_j \mu(A_j) \middle| A_1, \ldots, A_n \text{ are disjoint sets in } \mathcal{A}, \right.$$
$$c_1, \ldots, c_n \in [0,\infty) \text{ and}$$
$$\left. f(x) \geqslant \sum_{j=1}^{n} c_j \chi_{A_j}(x), \text{ for } x \in X \right\}.$$

Proof. A proof is to be found in [1, pp. 77]. □

Remark. You might have learnt measure theory with Theorem 2.3 as your definition of the Lebesgue integral (provided μ is Lebesgue measure on $\mathbb{R}$ of course). This is totally fine, but in this text it is listed as a theorem rather than a definition, as other equally valid definitions exist (see for instance [1]).

Remark. Sometimes in analysis, it is customary to denote the integral of some function f with respect to some measure μ over some set X, as $\int_X f d\mu$ as is done in Theorem 2.3. In this text, I will always integrate with respect to Lebesgue measure, but I will refrain from specifying this in my notation. Instead, familiar integral notation from single variable calculus will be used; that is, $\int_X f(x)dx$. So whenever you see an integral with an integrand that is not Darboux-integrable, this is meant to be interpreted as a Lebesgue integral.
The reasons why we want to use Lebesgue integration instead of classical Darboux/Riemann integration is threefold. Firstly, we can integrate a larger class of functions; functions with an uncountable amount of discontinuities, like $\chi_{\mathbb{Q}}$, are easily integrated. Secondly, limit-functions of a sequence of measurable functions, are always measurable, and integrable, but limits are not automatically integrable for Riemann integration. Consider, for instance, the following example: let $\{r_j\}_{j=1}^{\infty} := \mathbb{Q} \cap [0,1]$ be all the rational numbers in $[0,1]$. Define $f_k\colon [0,1] \to \mathbb{R}$ by

$$f_k(x) := \begin{cases} 1 & \text{if } x \in \{r_1, \ldots, r_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then each f_k is Riemann-integrable on $[0,1]$ with integral equal to 0, but the limit function f is $\chi_{\mathbb{Q}}$, which is not Riemann-integrable. Lastly, limits are a lot easier to deal with in Lebesgue integration, which can be seen by the next theorem. Indeed a result this strong in the framework of Riemann integration would require much stricter assumptions.

2.2 Integral Theorems

I will make quite liberal use of three particular theorems throughout the entire text. One is about the limit of an integral of a sequence of functions, and the remaining are about when you are allowed to switch up the order of integration.

The first of these results is the *Dominated Convergence Theorem*.

Theorem 2.4 (Dominated Convergence Theorem)**.** *Suppose $(X, \mathcal{A}, \mu)$ is a measure space, $f\colon X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable, and $f_1, f_2, \ldots$ are $\mathcal{A}$-measurable functions from X to $[-\infty, \infty]$ such that*

$$\lim_{k\to\infty} f_k(x) = f(x)$$

for almost every $x \in X$. If there exists an $\mathcal{A}$-measurable function $g\colon X \to [0, \infty]$ such that

$$\int_X g d\mu < \infty \text{ and } |f_k(x)| \leqslant g(x)$$

for every $k \in \mathbb{N}$ and almost every $x \in X$, then

$$\int_X f_k d\mu \to \int_X f d\mu,$$

as $k \to \infty$.

Proof. See [1][pp. 92-93]. □

The next two theorems, which are coincidentally the theorems most used in the entire text, are somewhat technical, so I refer to [1][ch. 5] for a comprehensive treatise.

Theorem 2.5 (Tonelli's Theorem)**.** *Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Suppose $f\colon X \times Y \to [0, \infty]$ is $\mathcal{S} \otimes \mathcal{T}$-measurable. Then*

$$x \mapsto \int_Y f(x, y) d\nu(y) \text{ is an } \mathcal{S}\text{-measurable function on } X,$$

$$y \mapsto \int_X f(x, y) d\mu(x) \text{ is a } \mathcal{T}\text{-measurable function on } Y,$$

and

$$\int_{X\times Y} f d(\mu \times \nu) = \int_X \int_Y f(x, y) d\nu(y) d\mu(x) = \int_Y \int_X f(x, y) d\mu(x) d\nu(y).$$

Proof. A proof is to be found in [1][pp. 129-130]. □

Theorem 2.6 (Fubini's Theorem)**.** *Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Suppose $f\colon X \times Y \to [-\infty, \infty]$ is $\mathcal{S} \otimes \mathcal{T}$-measurable and $\int_{X\times Y} |f| d(\mu \times \nu) < \infty$. Then*

$$\int_Y |f(x, y)| d\nu(y) < \infty \text{ for almost every } x \in X$$

and

$$\int_X |f(x,y)|d\mu(x) < \infty \text{ for almost every } y \in Y.$$

Furthermore

$$x \mapsto \int_Y f(x,y)d\nu(y) \text{ is an } \mathcal{S}\text{-measurable function on } X,$$

$$y \mapsto \int_X f(x,y)d\mu(x) \text{ is a } \mathcal{T}\text{-measurable function on } Y,$$

and

$$\int_{X\times Y} fd(\mu\times\nu) = \int_X \int_Y f(x,y)d\nu(y)d\mu(x) = \int_Y \int_X f(x,y)d\mu(x)d\nu(y).$$

Proof. A proof is to be found in [1][pp. 132-133]. □

Remark. It is sometimes customary to merge Fubini and Tonelli into one "super theorem" called Fubini-Tonelli's theorem, such that we are allowed to switch order of integration if either non-negativity or absolute integrability is present. This is the approach I will take. Therefore, whenever I switch order of integration it is due to this Fubini-Tonelli theorem.

2.3 Fourier analysis & Complex analysis

I will give a short repetition of how we define the Fourier transform on the real line.

Definition 2.7 (Fourier Transform)**.** If $f \in L^1(\mathbb{R})$, we define its *Fourier transform* $\hat{f}$ for $\xi \in \mathbb{R}$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi}dx. \tag{2.2}$$

Remark. The Fourier transform, as defined in Definition 2.7, is well defined for $f \in L^1(\mathbb{R})$. To see why, note that

$$\begin{aligned} |\hat{f}(\xi)| &= \left|\int_{\mathbb{R}} f(x)e^{-2\pi ix\xi}dx\right| \\ &\leqslant \int_{\mathbb{R}} \left|f(x)e^{-2\pi ix\xi}\right|dx \\ &= \int_{\mathbb{R}} |f(x)|dx \\ &= \|f\|_{L^1(\mathbb{R})}. \end{aligned}$$

Remark. I have been told by my algebraist friends that there is sometimes confusion as to what fg means for functions f and g. Indeed fg often denotes composition of functions. Throughout the text, I will without exception refer to fg as the product of f and g defined as the function $fg\colon x \mapsto f(x)g(x)$. The notation $f \circ g$ will define composition given some appropriate domain and co-domain for g and f.

Next we will show that the zeros of an holomorphic function are, in a certain sense, well behaved. Recall that we denote the vector space of holomorphic functions on a region $\Omega \subseteq \mathbb{C}$ by $H(\Omega)$.

Theorem 2.8 (Isolated Zeros Theorem)**.** *Let f be holomorphic in a region $\Omega \subseteq \mathbb{C}$ such that $f|_\Omega \not\equiv 0$. If f has a zero $z_0 \in \Omega$ then there exists a $\delta > 0$ such that $f \in H(\mathbb{D}(z_0, \delta))$, and it has no other zeros there.*

Proof. See [7][pp. 30] □

Thus, holomorphic functions cannot have "too many" zeros in small enough disks.

2.4 Functional analysis

We care deeply about operators in functional analysis, and one way to measure their "size" is through their *operator norm*.

Definition 2.9 (Operator norm)**.** Let $T\colon X \to Y$ be a linear operator between normed spaces. Then we define the *operator norm* of T to be

$$\|T\| = \sup_{\|u\|\leqslant 1} \|T[u]\|.$$

for $u \in X$.

Theorem 2.10. *Let $(x_i)_{i\in I}$ be an orthonormal basis for an Hilbert space $\mathcal{H}$. Then, for every $x \in \mathcal{H}$, we have*

$$\|x\|^2 = \sum_{i\in I} |\langle x_i, x\rangle|^2.$$

Proof. A proof is to be found in [6][pp. 187]. □

Definition 2.11. We say that a sequence of linear operators (T_n) between normed spaces X and Y converges in the operator norm to an operator $T\colon X \to Y$ if

$$\lim_{n\to\infty} \|T - T_n\| = 0.$$

3 Preludium

3.1 Signal analysis

We will motivate the discussion of the problem given in the introduction by looking at some chief uses of Fourier analysis in signal analysis. Like any analyst, we are interested in functions, and functions of principal interest to us are so-called *signals*. This leads to our first definition.

Definition 3.1 (Signals & Energy)**.** We define a *signal* $r\colon \mathbb{R} \to \mathbb{R}$, as an element in the Lebesgue space $L^2(\mathbb{R})$, equipped with its usual norm. We also define the *energy*, E of a signal r, as the square of its L^2-norm,

$$E[r] := \|r\|_{L^2}^2 = \int_{\mathbb{R}} |r(t)|^2 dx.$$

Indeed we see that E defines a functional

$$\begin{aligned} E\colon L^2(\mathbb{R}) &\to \mathbb{R}_{\geqslant 0} \\ r &\mapsto \|r\|_{L^2}^2. \end{aligned}$$

It is sometimes customary for engineers to call $L^2(\mathbb{R})$ the *signal space*. To further the discussion, we need to define the *Fourier transform* of a signal r. Since $\mu(\mathbb{R}) = +\infty$, we can not without justification say that r is integrable, and thus we can not in good conscience use Definition 2.7; we have that $L^2(\Omega) \subset L^1(\Omega)$ only if $\mu(\Omega) < +\infty$. This follows from the Cauchy-Schwartz inequality. Indeed, if $f \in L^2(\Omega)$, we have

$$\begin{aligned} \int_\Omega |f(x)| dx &= \int_\Omega |f(x)| \cdot 1 dx \\ &\leqslant \left(\int_\Omega |f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega 1^2 dx \right)^{\frac{1}{2}} \\ &= \sqrt{\mu(\Omega)} \|f\|_{L^2(\Omega)} \\ &< \infty, \end{aligned}$$

if $\mu(\Omega) < \infty$. Thus we need to modify our usual definition somewhat. Before we do this modification of sorts, however, we need the famous *Plancherel's theorem* to show that $f \mapsto \hat{f}$ preserves $L^2(\mathbb{R})$ norms on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

Theorem 3.2 (Plancherel's Theorem)**.** *Suppose $r \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\|r\|_{L^2} = \|\hat{r}\|_{L^2}$.*

Proof. See [9, pp. 143-144] for proof. □

Our modified definition will hinge on a density result, which is proven below.

Theorem 3.3. *$L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.*

Proof. Let f be an arbitrary element of $L^2(\mathbb{R})$, and define a sequence (f_j) by

$$f_j(x) := \begin{cases} f(x) & \text{if } |x| < j, \\ 0 & \text{if } |x| \geqslant j. \end{cases}$$

Then each entry of our sequence $(f_j)_{j\in\mathbb{N}}$ is obviously square-integrable and by the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \|f_j\|_{L^1(\mathbb{R})} &= \int_{|x|<j} |f(x)|dx \\ &\leqslant \left(\int_{|x|<j} |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{|x|<j} 1^2 dx\right)^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

so we know that $f_j \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $j \in \mathbb{N}$ (in simpler terms, it is both integrable and square-integrable), and quite trivially we know that $f_j \to f$ pointwise. Since $|f(x)-f_j(x)| \leqslant |f(x)|$, we know by the Dominated convergence theorem that

$$\int_{\mathbb{R}} |f(x) - f_j(x)|^2 dx \to 0.$$

with dominating function $|f|^2$. This we recognise as equivalent to the fact that $\lim_{j\to\infty} \|f - f_j\|^2_{L^2(\mathbb{R})} = 0$, so obviously $\lim_{j\to\infty} \|f - f_j\|_{L^2(\mathbb{R})} = 0$, which completes the proof. $\square$

Because of Theorem 3.2 one can show that the map $r \mapsto \hat{r}$ uniquely extends to a bounded linear map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Indeed, we have an even more general result.

Theorem 3.4. *Suppose U is a subspace of a normed vector space V. Suppose also that W is a Banach space and that $S\colon U \to W$ is a bounded linear map. Then there exists a unique continuous function $T\colon \overline{U} \to W$ such that $T|_U = S$.*

Proof. See [1][pp. 171]. $\square$

The reason why Theorem 3.4 is of interest to us is that we now know that we can *uniquely* extend the Fourier transform to $L^2(\mathbb{R})$. Indeed, let $U = L^1(\mathbb{R}) \cap L^2(\mathbb{R}), V = L^1(\mathbb{R})$ and $W = L^2(\mathbb{R})$.

Definition 3.5 (Fourier transform on $L^2(\mathbb{R})$)**.** The *Fourier transform*, $\mathcal{F}$, on $L^2(\mathbb{R})$, is the bounded operator on $L^2(\mathbb{R})$ such that $\mathcal{F}[r] = \hat{r}$ for all $r \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, where $\hat{f}$ is the Fourier transform of an $L^1(\mathbb{R})$ function f.

Although the above definition distinguishes between the notation for the Fourier transform of an L^1-function ($\hat{r}$) and an L^2-function ($\mathcal{F}[r]$), we will only use the first notation for both. Rigorously, we would not define the Fourier transform on $L^2(\mathbb{R})$ as the usual integral (because we do not know if r is integrable), but as the limit of the Fourier transform of sequences in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$

that converges to r in the $L^2(\mathbb{R})$-norm.
We now give a technical, but important definition for the functions that we will later use. It will also serve as an alternative method for defining the Fourier transform of an $L^2(\mathbb{R})$-function.

Definition 3.6 (Schwartz space $\mathcal{S}(\mathbb{R})$)**.** The *Schwartz space* is defined as

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \sup_{x\in\mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty, (k,\ell) \in \mathbb{N}_0^2 \right\}.$$

In other words, $\mathcal{S}(\mathbb{R})$ consists of all smooth functions f, in the sense of infinite continuous differentiability, such that $f^{(n)}$ are rapidly decreasing for all $n \in \mathbb{N}_0$, where $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0,1,2,\dots\}$. Note that $f^{(0)} = f$. We can formulate the Schwartz space in a slightly more functional analytic way. Indeed if we define the (k,ℓ)-norm of a smooth function f to be

$$\|f\|_{(k,\ell)} := \sup_{x\in\mathbb{R}} |x|^k |f^{(\ell)}(x)|,$$

then the Schwartz space $\mathcal{S}(\mathbb{R})$ is the space of smooth functions f such that $\|f\|_{(k,\ell)} < \infty$, for all $(k,\ell) \in \mathbb{N}_0^2$. By the continuity of polynomials, and of the function f, one can also reformulate the definition of the Schwartz space to be all smooth functions f, such that $|x|^k|f^{(\ell)}(x)|$ vanishes at infinity. Let us give some examples and non-examples to showcase how the definition of a Schwartz function works with some functions that we all know.

Example. If f is given by $f\colon x \mapsto e^{-x^2}$, a Gaußian function, then $f \in \mathcal{S}(\mathbb{R})$. To see why, let us have a look at the derivatives of f.

$$\begin{aligned}
&f(x) = e^{-x^2},\\
&f'(x) = -2xf(x),\\
&f''(x) = (4x^2-1)f(x),\\
&f'''(x) = (12x - 8x^3)f(x),\\
&\dots
\end{aligned}$$

From this we postulate that $x^k \frac{d^\ell}{dx^\ell} f(x) = p(x)f(x)$, where $p \in \mathcal{P}_{k+\ell}(\mathbb{R})$ is a $k+\ell$-th degree polynomial. l'Hôpital's rule then tells us that pf must be bounded on the real line for any polynomial p (x^k is a polynomial), and the conclusion follows.

Example. If g is given by $g\colon x \mapsto e^{-|x|}$, then $g \notin \mathcal{S}(\mathbb{R})$, because $g \notin C^\infty(\mathbb{R})$.

Example. If h is given by $h\colon x \mapsto \frac{1}{1+x^2}$, then $h \notin \mathcal{S}(\mathbb{R})$, as h doesn't decay fast enough. Indeed, let $k = 3$ and $\ell = 0$. Then $\|f\|_{(3,0)} = \sup_{x\in\mathbb{R}} \frac{x^3}{1+x^2} = \infty$.

From this definition there are some immediately desirable qualities we observe apply to Schwartz functions. Indeed, if $f, g \in \mathcal{S}(\mathbb{R})$, then the product $fg \in \mathcal{S}(\mathbb{R})$. One can also show that the Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an isomorphism. Last but not least, every Schwartz function is uniformly continuous (even Lipschitz) as their derivative is bounded. This follows by considering $\|f\|_{(0,1)} = |f'(x)| < \infty$.

A well-known fact is that $\overline{\mathcal{S}(\mathbb{R})} = L^p(\mathbb{R})$ for all $1 \leqslant p < \infty$. In particular, we have the result for $p \in \{1, 2\}$, so we can "naïvely" write the Fourier transform of signals as an integral. This is because of Theorem 3.4. One must never make the mistake of believing that the usual definition of the Fourier transform of an $L^1(\mathbb{R})$-function is valid for signals. We always define the Fourier transform as a sequence of Fourier transforms of functions from either $\mathcal{S}(\mathbb{R})$ or from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We can choose either one due to the previous density results.

We will, with our newfound knowledge, write

$$\hat{r}(\xi) = \int_{\mathbb{R}} r(t)e^{-2\pi i\xi t}dt,$$

as the Fourier transform of a signal r. Note that these functions are themselves square-integrable, and therefore elements of $L^2(\mathbb{R})$. Engineers typically call the Fourier transform of a signal r its *amplitude spectrum.* We relate the signal to its amplitude spectrum with the inverse Fourier transform, given by:

$$r(t) = \int_{\mathbb{R}} \hat{r}(\xi)e^{2\pi i\xi t}d\xi. \tag{3.1}$$

The *sinusoid of frequency* ξ is the $e^{2\pi i\xi t}$ term in the integrand of (3.1). It has *amplitude* given by $|\hat{r}(\xi)|$ and *phase* given by $\arg(\hat{r}(\xi)) \in (-\pi, \pi]$. We will often like to think of our signals as being influenced or modified by some external force, maybe from some concoction The Engineer has built. Perhaps naively, we model the change as some nice operator applied to our signals $\{r_j\}_j$ giving rise to *outputs* or *responses* $\{s_j\}_j$, and one model that might be of interest to us might be the following: if, for $M\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $s_j(t) = M[r_j](t)$, then

$$as_1(t) + bs_2(t) = M[ar_1 + br_2](t), \tag{3.2}$$

which we recognise as linearity and

$$s_1(t - T) = M[r_1](t - T), \tag{3.3}$$

which is called *translation invariance* of M. (3.3) & (3.4) are meant to hold for all real numbers T, complex numbers a, b and all signal inputs r_1, r_2. We call this family of operators $\mathcal{L}_T(L^2(\mathbb{R}))$ the family of linear, translation-invariant operators acting on $L^2(\mathbb{R})$.

We realise that we can easily calculate the response s_ξ of a sinusoidal $e^{2\pi i\xi t}$ to be

$$s_\xi(t) = M[e^{2\pi i\xi t}],$$

which tells us that

$$s_\xi(t+T) = M[e^{2\pi i\xi(t+T)}] = e^{2\pi i\xi T} M[e^{2\pi i\xi t}] = e^{2\pi i\xi T} s_\xi(t). \tag{3.4}$$

Letting $t \searrow 0$ in (3.4), and realising that the identity holds for all real T (in particular $T = t$), we get

$$s_\xi(t) = e^{2\pi i\xi t} s_\xi(0),$$

which gives us

$$s_\xi(t) = M[e^{2\pi i\xi t}] = Y_M e^{2\pi i\xi t}, \tag{3.5}$$

where we let $Y_M(\xi) := s_\xi(0)$. Thus we have shown that the response to the sinusoidal is a sinusoid with similar frequency ξ, but with different amplitude. We call the function $Y_M(\xi)$ the *transfer function* of M. In total, from (3.5) we get the following theorem.

Theorem 3.7. *Let M be a linear, translation-invariant operator on $L^2(\mathbb{R})$. Then the eigenvectors of M are of the form $e^{2\pi i\xi t}$, $\xi \in \mathbb{R}$. Furthermore, they have eigenvalues $Y_M(\xi)$.*

Before we can move on to our next result we will need the following lemma.

Lemma 3.8. *Let $M\colon L^2(\Omega) \to L^2(\Omega)$ be a linear, translation-invariant operator as defined in (2.3) & (2.4) on a non-empty, connected, open interval $\Omega \subseteq \mathbb{R}$. Then, there exists a function $h \in L^2(\Omega)$ such that*

$$M[f] = f * h \tag{3.6}$$

for every $f \in L^2(\Omega)$, where the convolution is defined as

$$(f * h)(x) := \int_\Omega f(y)h(y-x)dy = \int_\Omega f(y-x)h(y)dy.$$

As every signal r can be expanded in terms of sinusoidals in the spirit of (3.1), we can calculate the response of $M \in \mathcal{L}_T(L^2(\mathbb{R}))$ applied to an arbitrary signal r to see that

$$\begin{aligned} s(t) = M[r](t) &= M\left[\int_\mathbb{R} \hat{r}(\xi)e^{2\pi i\xi t}d\xi\right] \\ &= \int_\mathbb{R} \hat{r}(\xi)M[e^{2\pi i\xi t}]d\xi \\ &= \int_\mathbb{R} \hat{r}(\xi)Y_M(\xi)e^{2\pi i\xi t}d\xi, \end{aligned}$$

where we must somehow justify the fact that we have pulled M inside the integral. Indeed we observe that, by Lemma 3.8, we have

$$\begin{aligned}
M[r](t) &= (r * h)(t) \\
&= \int_{\mathbb{R}} r(x)h(t-x)dx \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \hat{r}(\xi)e^{2\pi i\xi x}d\xi \right] h(t-x)dx \\
&= \int_{\mathbb{R}} \hat{r}(\xi) \left[\int_{\mathbb{R}} h(t-x)e^{2\pi i\xi x}dx \right] d\xi \\
&= \int_{\mathbb{R}} \hat{r}(\xi)[h * e^{2\pi i\xi t}]d\xi \\
&= \int_{\mathbb{R}} \hat{r}(\xi)M[e^{2\pi i\xi t}]d\xi \\
&= \int_{\mathbb{R}} \hat{r}(\xi)Y_M(\xi)e^{2\pi i\xi t}d\xi,
\end{aligned}$$

where we used the Fubini-Tonelli theorem to interchange order of integration. Therefore, we can say that the amplitude spectrum of $M[r]$, is

$$S(\xi) := Y_M(\xi)\hat{r}(\xi). \tag{3.7}$$

This makes more sense if we compare the inverse Fourier transform formula in (3.1). The amplitude spectrum of r is the function $\hat{r}$ such that

$$r(t) = \int_{\mathbb{R}} \hat{r}(\xi)e^{2\pi i\xi t}d\xi;$$

since we just showed that

$$M[r](t) = \int_{\mathbb{R}} \hat{r}(\xi)Y_M(\xi)e^{2\pi i\xi t}d\xi,$$

which is an inverse Fourier transform, we get that $S(\xi)$ is the amplitude spectrum of $M[r]$ by comparison with (3.1).

3.2 Limited signals

In real life, we often do not see sinusoids of arbitrarily high frequency without attenuation, i.e. the decay of the amplitude of the signal; indeed, the transfer functions Y_M often tend to zero as ξ increases. It follows from (3.7) that the amplitude spectra of the responses to signals of finite energy are negligibly small. For those interested in the real world, Slepian says:

"Examination of the most natural classes of input signals shows that they too have amplitude spectra of finite support. For example, Fourier analysis of recorded male speech gives an amplitude spectrum that is zero for frequencies higher than 8000 hertz (= cycles/second). Conventional orchestral music has no frequencies higher than $20,000$ hertz, while the output of a television camera (vidicon) has an amplitude spectrum vanishing for $|\xi| > 2 \cdot 10^6$ hertz." [8][pp. 3]

These observations lead us to consider the following definitions regarding when the Fourier transform of a signal vanishes, as well as when the signal itself vanishes.

Definition 3.9 (Bandlimited and Timelimited signals)**.** Let $r \in L^2(\mathbb{R})$ be a signal. Then r is called *bandlimited* if its Fourier transform vanishes for $|\xi| > W$ (i.e. is compactly supported on $[-W, W] \subset \mathbb{R}$), where $W > 0$ is some real number. The space of these functions is denoted by $B_W = \{r \in L^2(\mathbb{R}) \mid \operatorname{supp}(\hat{r}) \subset [-W, W]\}$. Likewise, we define a signal to be *timelimited* if it vanishes whenever $|t| > T/2$. We similarly denote the space of these functions as $\mathcal{T}_T = \{r \in L^2(\mathbb{R}) \mid \operatorname{supp}(r) \subset [-T/2, T/2]\}$, with $T > 0$ some real number.

Definition 3.10 (Bandwidth)**.** The number W in Definition 3.9 is called the *bandwidth* of a bandlimited signal $r \in B_W$.

The space $B_W \subset L^2(\mathbb{R})$ is called the *Paley-Wiener space.*

A beautiful theorem from complex analysis is the so called *Paley-Wiener theorem.* It is also exactly what we need!

Theorem 3.11 (Paley-Wiener theorem)**.** *A function f is bandlimited to $[-W, W]$ if and only if*

$$f(t) = \int_{-W}^{W} g(\omega) e^{-i\omega t} d\omega,$$

for some $g \in L^2(-W, W)$ and if and only if f is an entire function of exponential type that is square-integrable on $\mathbb{R}$, i.e. f is an entire function with

$$|f(z)| \leqslant \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| \exp(W|y|),$$

and

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty,$$

for $z = x + iy \in \mathbb{C}$.

Proof. A proof can be found in [5]. □

Notice that

$$r(t) = \int_{(-W,W)} \hat{r}(\xi) e^{2\pi i \xi t} d\xi$$

is a finite Fourier transform. By the Paley-Wiener theorem it is an entire function. It is also smooth. It has no singularity in the complex plane, it is infinitely differentiable everywhere, and has a power series about every point with an infinite radius of convergence.

As a corollary of the Paley-Wiener theorem, it follows that any non-trivial bandlimited signal cannot vanish on any interval on the t-axis. Indeed if we found this to be the case, then it would follow that all its power series coefficients would be zero at some interior point of the interval, meaning that it would be identically zero everywhere.

Corollary 3.12. *For B_W and $\mathcal{T}_T$ as defined in Definition 3.9, we have*

$$B_W \cap \mathcal{T}_T = \{0\}.$$

Proof. By the Paley-Wiener theorem, if $\hat{f}$ has compact support, then f is an entire function. Therefore, if f also has compact support, it is necessarily zero on a set with an accumulation point. By the isolated zeros theorem, $f \equiv 0$, and since $\mathcal{F}$ is an injection, we must also have that $\hat{f} \equiv 0$. □

In addition to sending smooth, continuous signals, like speech or music, we also sometimes send short pulses of information such as the dots and dashes used in telegraphy. We have shown that a signal which posses a time-support of finite measure cannot be bandlimited, and it must contain sinusoids of (arbitrarily) large ξ. Simultaneously, we also know that bandlimited signals cannot possibly have time-support of finite measure; they must "go on forever". Almost paradoxical!

The mathematical description of such an observation is encoded in the *uncertainty principle* of Fourier analysis. It says that a signal and its amplitude spectrum cannot be simultaneously "localised".

Theorem 3.13 (Heisenberg's Uncertainty principle)**.** *Suppose $\psi \in \mathcal{S}(\mathbb{R})$, such that $E[\psi] = 1$. Then*

$$\left(\int_{\mathbb{R}} x^2 |\psi(x)|^2 dx\right)\left(\int_{\mathbb{R}} \xi^2 |\hat{\psi}(\xi)|^2 d\xi\right) \geqslant \frac{1}{16\pi^2}.$$

Proof. A proof can be found in [9, pp. 158-159]. □

3.3 Concentration

The standard notion of variance from statistics, appearing in Theorem 3.13, is actually of minor use to us, so we decide to introduce a more helpful notion of concentration for a signal r. Indeed, let

$$\alpha(T) := \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} |r(t)|^2 dt}{\int_{\mathbb{R}} |r(t)|^2 dt} \tag{3.8}$$

be a measure of the concentration of the signal. Similarly, we can define

$$\beta(W) := \frac{\int_{-W}^{W} |\hat{r}(\xi)|^2 d\xi}{\int_{\mathbb{R}} |\hat{r}(\xi)|^2 d\xi}. \tag{3.9}$$

If we had that r was a timelimited signal on $(-T/2, T/2)$, then $\alpha(T)$ would attain its greatest value, which is 1 (as the integral in the denominator must always be greater or equal to the one in the numerator). But, we know that a nontrivial bandlimited signal cannot be particularly timelimited. So, a natural question arises: how large can $\alpha(T)$ become for $r \in B_W$?

Recalling that $|z|^2 = z\overline{z}$ and that we can relate r to $\hat{r}$ via a finite Fourier transform (see (3.1)), together with just a dash of Plancherel's (Theorem 3.2), we get that (3.8) becomes

$$\begin{aligned}\alpha(T) &= \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} |r(t)|^2 dt}{\int_{\mathbb{R}} |r(t)|^2 dt} = \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} r(t)\overline{r(t)} dt}{\int_{\mathbb{R}} |\hat{r}(\xi)|^2 d\xi} \\ &= \frac{\int_{-T/2}^{T/2} \left(\int_{-W}^{W} e^{2\pi i \xi'' t} \hat{r}(\xi'') d\xi''\right) \left(\int_{-W}^{W} e^{-2\pi i \xi' t} \overline{\hat{r}(\xi')} d\xi'\right) dt}{\int_{-W}^{W} |\hat{r}(\xi)|^2 d\xi}. \qquad (3.10)\end{aligned}$$

How do we continue our surgery on this fraction? For sake of brevity, let $\mathcal{W} := (-W, W)^2 \subset \mathbb{R}^2$, and $dA := d\xi'' d\xi'$. If we set aside our attention to just the numerator of (3.10), we have

$$\begin{aligned}&\int_{-T/2}^{T/2} \left(\int_{-W}^{W} e^{2\pi i \xi'' t} \hat{r}(\xi'') d\xi''\right) \left(\int_{-W}^{W} e^{-2\pi i \xi' t} \overline{\hat{r}(\xi')} d\xi'\right) dt \\ &= \int_{-T/2}^{T/2} \left(\iint_{\mathcal{W}} \hat{r}(\xi'') \overline{\hat{r}(\xi')} e^{2\pi i (\xi'' - \xi') t} dA\right) dt \\ &= \iint_{\mathcal{W}} \hat{r}(\xi'') \overline{\hat{r}(\xi')} \left(\int_{-T/2}^{T/2} e^{2\pi i (\xi'' - \xi') t} dt\right) dA, \qquad (3.11)\end{aligned}$$

where we in the last step of the above calculation have used Fubini-Tonelli's theorem for double integrals. For maximum clarity, let us deal with the exponential integral in (3.11) separately. Indeed we have

$$\begin{aligned}\int_{-T/2}^{T/2} e^{2\pi i (\xi'' - \xi') t} dt &= \frac{1}{2\pi i (\xi'' - \xi')} e^{2\pi i (\xi'' - \xi') t} \Big|_{-T/2}^{T/2} \\ &= \frac{1}{2\pi i (\xi'' - \xi')} \left[e^{\frac{2\pi i (\xi'' - \xi') T}{2}} - e^{\frac{-2\pi i (\xi'' - \xi') T}{2}}\right] \\ &= \frac{\sin[\pi T (\xi'' - \xi')]}{\pi (\xi'' - \xi')}.\end{aligned}$$

Altogether, this gives us

$$\alpha(T) = \frac{\int_{-W}^{W} \left[\int_{-W}^{W} \frac{\sin\left[\pi T (\xi' - \xi'')\right]}{\pi (\xi' - \xi'')} \hat{r}(\xi'') \overline{\hat{r}(\xi')} d\xi''\right] d\xi'}{\int_{-W}^{W} \hat{r}(\xi') \overline{\hat{r}(\xi')} d\xi'}. \qquad (3.12)$$

Now we regard $\hat{r}$ as an arbitrary element of $L^2(-W, W)$. Recall that we are working over the Hilbert space $L^2(-W, W)$, which means that

$$\langle f, g\rangle_{L^2(-W,W)} = \int_{-W}^{W} f(x)g(x)dx.$$

Let

$$k(x, y) := \begin{cases} \frac{\sin[\pi T(x-y)]}{\pi(x-y)} & \text{if } x \neq y, \\ T & \text{if } x = y \end{cases} \tag{3.13}$$

be the sin(...)-term of the integrand of the integral in (3.12). We also let

$$\mathcal{T}[\hat{r}](\xi') := \int_{-W}^{W} \frac{\sin\left[\pi T(\xi' - \xi'')\right]}{\pi(\xi' - \xi'')} \hat{r}(\xi'')d\xi''. \tag{3.14}$$

We see that our maximising problem boils down to finding

$$\max_{\hat{r}\in L^2} \frac{\langle \mathcal{T}[\hat{r}], \hat{r}\rangle_{L^2}}{\|\hat{r}\|_{L^2}^2},$$

or maximising the so-called Rayleigh quotient $\alpha(\hat{r})$, as given in (3.12). Notice that by an appropriate scaling, we can assume $\|\hat{r}\|_{L^2} = 1$.
Thus the Rayleigh quotient reduces to

$$\alpha(\hat{r}) = \langle \mathcal{T}[\hat{r}], \hat{r}\rangle, \tag{3.15}$$

and we would the like to find $\max_{\|\hat{r}\|_{L^2}=1} \langle \mathcal{T}[\hat{r}], \hat{r}\rangle$.

It is from this point on-wards obvious that we are working over the Hilbert space L^2, and therefore no confusion arises in from now on neglecting to indicate which inner product/norm it is we are working with.
One can also show that

$$\mathcal{V}\colon \psi \mapsto \int_{\mathcal{K}} \frac{\sin(x-y)}{x-y}\psi(y)dy \tag{3.16}$$

is a positive definite operator between $L^2(\mathcal{K})$ and $L^2(\mathcal{K})$ for any compact $\mathcal{K} \subset \mathbb{R}$.

Lemma 3.14. *The operator $\mathcal{V}$ given by*

$$\begin{aligned} \mathcal{V}\colon L^2([-1,1]) &\to L^2([-1,1]) \\ \psi &\mapsto \int_{-1}^{1} \frac{\sin(x-y)}{x-y}\psi(y)dy \end{aligned}$$

is a positive definite operator.

Proof. Stated without proof in [8]. $\square$

Integral operators with kernels that are elements of $L^2(\Omega)$ are called *Hilbert-Schmidt operators*. Indeed, we note that $\Omega = \mathcal{W} = (-W, W)^2$ is Lebesgue-finite, so we know that $\|k\|_{L^2(\mathcal{W})} < +\infty$.
Thus the operator $\mathcal{T}$ as defined in (3.14) is an Hilbert-Schmidt operator. Note also that our kernel k, as defined in (3.13) is symmetric (i.e. $k(x, y) = k(y, x)$). We have some results on operators with symmetric kernels.

Theorem 3.15. *For all integral operators T with symmetric kernel k, we have*

$$\langle T[f], g\rangle = \langle f, T[g]\rangle \tag{3.17}$$

Proof. Let $k(x, y)$ be symmetric such that $k(x, y) = k(y, x)$, for all $(x, y) \in \mathbb{R}^2$. Let $f, g \in L^2(\mathbb{R})$. Then

$$\begin{aligned}
\langle T[f], g\rangle &= \int_{\mathbb{R}} T[f](x)g(x)dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} k(x, y)f(y)dy \right) g(x)dx \\
&= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} k(x, y)g(x)dx \right) dy \\
&= \int_{\mathbb{R}} f(y)T[g](y)dy \\
&= \langle f, T[g]\rangle.
\end{aligned}$$

The result follows by Fubini-Tonelli. □

Thus, integral operators with symmetric kernels are self-adjoint. We can easily show that self-adjoint operators have real eigenvalues.

Theorem 3.16. *If $S\colon X \to Y$ is self-adjoint, then all its eigenvalues are real. Eigenvectors corresponding to different eigenvalues are pairwise orthogonal.*

Proof. Without loss of generality, we can assume that for an eigenvector x, $\|x\| = 1$. Then we have

$$\begin{aligned}
\lambda = \lambda\langle x, x\rangle &= \langle \lambda x, x\rangle = \langle Sx, x\rangle = \langle x, Sx\rangle \\
&= \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle = \overline{\lambda}.
\end{aligned}$$

We conclude that λ is real. As to the orthogonality, we observe that

$$\langle Sx_1, x_2\rangle = \langle x_1, Sx_2\rangle,$$

for eigenvectors $x_1 \neq x_2$, which implies that

$$\langle \lambda_1 x_1, x_2\rangle = \langle x_1, \lambda_2 x_2\rangle.$$

This is equivalent to

$$(\lambda_1 - \overline{\lambda_2})\langle x_1, x_2\rangle = 0.$$

We conclude that $\langle x_i, x_j\rangle = \delta_{ij}$, for all i, j. □

We now give an important definition regarding a certain quality of operators.

Definition 3.17 (Compact operator)**.** An operator T on a Hilbert space $\mathcal{H}$ is called *compact* if, for every bounded sequence $(f_j)_j$ in $\mathcal{H}$, the sequence $(Tf_j)_j$ has a convergent sub-sequence.
We denote the family of compact operators on $\mathcal{H}$ by $\mathcal{C}(\mathcal{H})$.

Since the spectral theory for compact operators is a well-developed one, we ideally want to show that our operator $\mathcal{V}$, or equally $\mathcal{T}$, as defined in (3.16) and (3.14) respectively, is a compact operator. In actuality, we will show an even more general result, stating that all Hilbert-Schmidt integral operators are compact.

Lemma 3.18. *If $(\phi_i(x))$ is an orthonormal basis for $L^2(\Omega)$, then $(\phi_i(x)\phi_j(y))$ is a basis for $L^2(\Omega \times \Omega)$.*

Proof. A method for proving the lemma is found in [6][pp. 262]. □

Lemma 3.19. *Let (T_n) be a sequence of compact, linear operators from X to Y. Suppose that $T_n \to T$ in the operator norm. Then T is compact.*

Proof. A proof is to be found in [6][pp. 261]. □

Lemma 3.20. *Let $T\colon X \to Y$ be a linear operator between normed spaces such that $\dim T(X)$ is finite. Then T is compact.*

Proof. Since T has finite rank, $T(X)$ is a finite-dimensional normed space. Thus, as normed spaces, we have that $T(X) \cong \mathbb{R}^n$ where $n = \dim T(X)$. Then, for any bounded sequence (x_n) in X, the sequence $(T[x_n])$ is bounded in $T(X)$, so by Bolzano-Weierstraß, the sequence must contain a convergent sub-sequence. Thus T is compact. □

Theorem 3.21. *Let the kernel $k\colon \Omega \times \Omega \to \mathbb{R}$ be $L^2(\Omega \times \Omega)$. Then the integral operator K given by*

$$K[u](x) := \int_\Omega k(x,y)u(y)dy$$

is compact.

Proof. By Lemma 3.18, we can construct a basis for $L^2(\Omega \times \Omega)$ out of an orthonormal basis for $L^2(\Omega)$. We can then expand k using this basis by

$$k(x,y) = \sum_{i,j=1}^{\infty} k_{ij}\phi_i(x)\phi_j(y),$$

where the sum converges to k in the $L^2(\Omega \times \Omega)$-norm, and

$$k_{ij} := \iint_{\Omega^2} k(x,y)\phi_i(x)\phi_j(y)dxdy.$$

Furthermore by Theorem (2.10) we have

$$\iint_{\Omega^2} |k(x,y)|^2 dxdy = \sum_{i,j=1}^{\infty} |k_{ij}|^2. \tag{3.18}$$

We can now define an operator K_n and a kernel k_n by

$$k_n(x,y) := \sum_{i,j=1}^{n} k_{ij}\phi_i(x)\phi_j(y),$$

and

$$K_n[u](x) := \int_\Omega k_n(x,y)u(y)dy.$$

We say that k_n and K_n are a *separable kernel* and a *separable operator* respectively. It is seen that the operator K_n is both bounded and has a finite-dimensional image, so it has finite rank, and is therefore compact by Lemma (3.20). Recall that for an operator T, we define (or show that your favourite definition is equivalent to)

$$\|T\| = \sup_{\|u\|\leqslant 1} \|T[u]\|.$$

Thus we can show that

$$\begin{aligned}
\|K - K_n\|^2 &= \sup_{\|u\|\leqslant 1} \|(K - K_n)[u]\|^2 \\
&= \sup_{\|u\|\leqslant 1} \int_\Omega \left[\int_\Omega [k(x,y) - k_n(x,y)]u(y)dy\right] dx \\
&\leqslant \sup_{\|u\|\leqslant 1} \int_\Omega \left[\int_\Omega |k(x,y) - k_n(x,y)|^2 dy\right] \left[\int_\Omega |u(y)|^2 dy\right] dx \qquad (3.19)\\
&= \sup_{\|u\|\leqslant 1} \int_\Omega \left[\int_\Omega |k(x,y) - k_n(x,y)|^2 dy\right] \|u\|^2 dx \\
&= \iint_{\Omega^2} |k(x,y) - k_n(x,y)|^2 dxdy,
\end{aligned}$$

where the inequality in (3.19) is by virtue of Hölder's inequality, so that

$$\|K - K_n\|^2 \leqslant \iint_{\Omega^2} |k(x,y) - k_n(x,y)|^2 dxdy.$$

By (3.18) we have that

$$\begin{aligned}
0 \leqslant \lim_{n\to\infty} \|K - K_n\| &\leqslant \lim_{n\to\infty} \iint_{\Omega^2} |k(x,y) - k_n(x,y)|^2 dxdy \qquad (3.20)\\
&= \lim_{n\to\infty} \left[\sum_{i,j=1}^{\infty} |k_{ij}|^2 - \sum_{i,j=1}^{n} |k_{ij}|^2\right] \qquad (3.21)\\
&= \lim_{n\to\infty} \sum_{i,j=n+1}^{\infty} |k_{ij}|^2 = 0. \qquad (3.22)
\end{aligned}$$

Thus K_n converges to K in the operator norm, which shows that K is compact, by Lemma 3.19. □

We can now show that our maximising problem is really an eigenvalue problem.

Theorem 3.22. *Let T be a bounded, linear and self-adjoint operator on an Hilbert space $\mathcal{H}$. Then*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Proof. We begin by letting $m := \sup_{\|x\|=1} |\langle Tx, x\rangle|$. Then $|\langle Tx, x\rangle| \leqslant m$ for all $x \in \mathcal{H}$ with $\|x\| = 1$. By the Cauchy-Schwartz inequality, we have

$$|\langle Tx, x\rangle| \leqslant \|Tx\|\|x\| = \|Tx\| \leqslant \|T\|.$$

Hence $m \leqslant \|T\|$. To prove inequality in the opposite direction we let $x, y \in \mathcal{H}$, which implies that $\langle T(x \pm y), x \pm y\rangle = \langle Tx, x\rangle + 2\mathrm{Re}\langle Tx, y\rangle + \langle Ty, y\rangle$. Thus

$$\begin{aligned} 4\mathrm{Re}\langle Tx, y\rangle &= \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\ &\leqslant |\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle| \\ &\leqslant m\left(|x+y\|^2 + \|x-y\|^2\right) \\ &= 2m(\|x\|^2 + \|y\|^2), \end{aligned} \tag{3.23}$$

where (3.23) is due to the Parallelogram law. For every complex number z, we have that $z = |z|\exp(i\theta)$, thus $\langle Tx, y\rangle = |\langle Tx, y\rangle|\exp(i\theta)$, for real θ. If we let $x \mapsto x\exp(i\theta)$, then we arrive at

$$4|\langle Tx, y\rangle| \leqslant 2m(\|x\|^2 + \|y\|^2). \tag{3.24}$$

Lettnig $y = \frac{\|x|}{\|Tx\|}Tx$ in (3.24), we get

$$\|Tx\| \leqslant m\|x\|.$$

We conclude that $m = \|T\|$. □

Theorem 3.23. *Let T be a bounded, linear and self-adjoint operator on an Hilbert space $\mathcal{H}$. Then the following holds:*

i. Let $\lambda := \inf_{|x\|=1} \langle Tx, x\rangle$. If there exists an $x_0 \in \mathcal{H}$ such that $\|x_0\| = 1$ and $\lambda = \langle Tx_0, x_0\rangle$, then λ is an eigenvalue of T with corresponding eigenvector x_0.

ii. Let $\mu := \sup_{|x\|=1} \langle Tx, x\rangle$. If there exists an $x_1 \in \mathcal{H}$ such that $\|x_1\| = 1$ and $\mu = \langle Tx_1, x_1\rangle$, then μ is an eigenvalue of T with corresponding eigenvector x_1.

Proof. We see by the definition of λ that

$$\langle T(x_0 + \alpha v), x_0 + \alpha v\rangle \geqslant \lambda\langle x_0 + \alpha v, x_0 + \alpha v\rangle, \tag{3.25}$$

for every $\alpha \in \mathbb{C}$ and $v \in \mathcal{H}$. Let us first expand the left-hand side of (3.25). This becomes

$$\begin{aligned}\langle T(x_0 + \alpha v), x_0 + \alpha v\rangle &= \langle Tx_0, x_0\rangle + \overline{\alpha}\langle Tx_0, v\rangle + \alpha\langle Tv, x_0\rangle + |\alpha|^2\langle Tv, v\rangle \\ &= \langle Tx_0, x_0\rangle + \overline{\alpha}\langle Tx_0, v\rangle + \alpha\overline{\langle Tx_0, v\rangle} + |\alpha|^2\langle Tv, v\rangle \\ &= \langle Tx_0, x_0\rangle + \overline{\alpha}\langle Tx_0, v\rangle + \overline{\overline{\alpha}\langle Tx_0, v\rangle} + |\alpha|^2\langle Tv, v\rangle \\ &= \langle Tx_0, x_0\rangle + 2\mathrm{Re}[\alpha\langle Tx_0, v\rangle] + |\alpha|^2\langle Tv, v\rangle. \end{aligned} \tag{3.26}$$

The right-hand side of (3.25 becomes)

$$\begin{aligned}\lambda\langle x_0 + \alpha v, x_0 + \alpha v\rangle &= \lambda\left[\langle x_0, x_0\rangle + \overline{\alpha}\langle x_0, v\rangle + \alpha\langle v, x_0\rangle + |\alpha|^2\langle v, v\rangle\right] \\ &= \lambda\left[\langle x_0, x_0\rangle + \overline{\alpha}\langle x_0, v\rangle + \alpha\overline{\langle x_0, v\rangle} + |\alpha|^2\langle v, v\rangle\right] \\ &= \lambda\left[\langle x_0, x_0\rangle + \overline{\alpha}\langle x_0, v\rangle + \overline{\overline{\alpha}\langle x_0, v\rangle} + |\alpha|^2\langle v, v\rangle\right] \\ &= \lambda\langle x_0, x_0\rangle + 2\mathrm{Re}\left[\alpha\lambda\langle x_0, v\rangle\right] + \lambda|\alpha|^2\langle v, v\rangle. \end{aligned} \tag{3.27}$$

Combining (3.25), (3.27) and (3.26), we get

$$\begin{aligned}&\langle Tx_0, x_0\rangle + 2\mathrm{Re}[\alpha\langle Tx_0, v\rangle] + \langle Tv, v\rangle \\ &\geqslant \lambda\langle x_0, x_0\rangle + 2\mathrm{Re}\left[\alpha\lambda\langle x_0, v\rangle\right] + \lambda|\alpha|^2\langle v, v\rangle. \end{aligned} \tag{3.28}$$

Let us now substitute $\lambda = \langle Tx_0, x_0\rangle$ and $\alpha = r\overline{\langle v, (T - \lambda I)x_0\rangle}$ for $r \in \mathbb{R}$. It follows from a lengthy, but not particularly insightful calculation that (3.28) implies that $\alpha = 0$, which implies that

$$r\overline{\langle v, (T - \lambda I)x_0\rangle} = 0$$

which by Theorem 3.22 implies that $(T - \lambda I)x_0 = 0$. This is equivalent to $Tx_0 = \lambda x_0$. The proof for *ii.* is identical if we replace T by $-T$. □

Theorem 3.24. *If T is a linear, bounded, compact and self-adjoint operator on an Hilbert space $\mathcal{H}$, then at least one of the numbers $\|T\|$ or $-\|T\|$ is an eigenvalue of T.*

Proof. By Theorem 3.23, we can find a sequence $(x_n)_{n\in I} \subseteq \mathcal{H}$, such that $\|x_n\| = 1$ for every $n \in I$ such that $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda$, where $\lambda \in \{\pm\|T\|\}$. Then

$$\begin{aligned}0 \leqslant \|Tx_n - \lambda x_n\|^2 &= \|Tx_n\|^2 + \lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \\ &\leqslant 2\lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \to 0 \text{ as } n \to \infty.\end{aligned}$$

Since T is compact, there exists a sub-sequence $(Tx_{n_j}) \subset (Tx_n)$ that converges to some $y \in \mathcal{H}$. Thus $Tx_{n_j} - \lambda x_{n_j} \to 0$ as $n \to \infty$, but this means that $x_n \to \frac{1}{\lambda}y$. Hence $y = \lim_{n\to\infty} Tx_{n_j} = \frac{1}{\lambda}Ty$. Thus $Ty = \lambda y$, and λ is an eigenvalue. □

Corollary 3.25. *If T is a bounded, linear, compact and self-adjoint operator on an Hilbert space $\mathcal{H}$, then* $\max_{\|x\|=1} |\langle Tx, x\rangle| = \|T\|$.

Applying these results to $\mathcal{T}$, defined in (3.14), we see that a maximising $\hat{r} \in L^2(-W, W)$, must satisfy the eigenvalue problem

$$\int_{-W}^{W} \frac{\sin\left[\pi T(\xi' - \xi'')\right]}{\pi(\xi' - \xi'')} \hat{r}(\xi'')d\xi'' = \alpha(T)\hat{r}(\xi'), \text{ for } |\xi'| \leqslant W$$

or

$$\mathcal{T}[\hat{r}](\xi') = \alpha\hat{r}(\xi'). \tag{3.29}$$

We can see that (3.29) belongs to a specific family of integral equations which we define below.

Definition 3.26 (Homogeneous Fredholm equation of the 2^{nd} kind)**.** An equation of the form

$$\phi(t) = \lambda \int_{\Omega} K(t, s)\phi(s)ds \tag{3.30}$$

is called a Homogeneous Fredholm equation of the 2^{nd} kind, with kernel K, with ϕ some sufficiently smooth function and Ω some domain.

Another definition we will recognise is the following type of integral equation.

Definition 3.27 (Covolutional Volterra Equations)**.** A convolutional Volterra equation of the second kind is an expression of the form

$$\varphi(x) = \int_{a}^{x} K(x - t)\varphi(t)dt + \sigma(x) \tag{3.31}$$

where $a, b \in \mathbb{R}$ are such that $a < b$, and the functions $\sigma, K\colon [a, b] \to \mathbb{C}$ are elements of $L^2[a, b]$, and $\varphi\colon [a, b] \to \mathbb{C}$ is some function to be found.

For this type of equation, we have the following uniqueness and regularity result.

Theorem 3.28. *The equation (3.31) always has a unique solution φ on $[a, b]$, and if $K, \sigma \in C^k[a, b]$, then the solution φ is also k times continuously differentiable.*

Proof. See [4][pp. 7] □

Now we do some clean-up. Indeed, let us define the following:

$$y := \frac{\xi''}{W}, \ x := \frac{\xi'}{W}, \ \psi(y) := \hat{r}(Wy), \ \lambda := \alpha(T), \ c := \pi WT. \tag{3.32}$$

This gives us

$$|x| = \left|\frac{\xi'}{W}\right| \leqslant 1, \tag{3.33}$$

as well as

$$d\xi'' = d(Wy) = W dy, \tag{3.34}$$

so that

$$\xi'' \in (-W, W) \text{ which implies } y \in (-1, 1). \tag{3.35}$$

Combining (3.27)-(3.30), we ultimately get that (3.29) becomes

$$\int_{-1}^{1} \frac{\sin\left[c(x-y)\right]}{\pi(x-y)}\psi(y)dy = \lambda\psi(x), \text{ for } |x| \leqslant 1, \tag{3.36}$$

which is only depending on $c \in \mathbb{R}$. We interpret (3.36) as an eigenvalue problem, where ψ is interpreted as an eigenfunction of an integral operator, with corresponding eigenvalue λ on $[-1, 1]$.

4 Prolate spheroidal wave functions

In the style of all these operators, we give one final reformulation and clean-up of our original champion $\mathcal{T}$, as defined in (3.14), which we will use in our all our future proofs and definitions. We define

$$\begin{aligned} \mathcal{Q}_c \colon L^2[-1,1] &\to L^2[-1,1] \\ \psi &\mapsto \int_{-1}^{1} \frac{\sin[c(x-y)]}{\pi(x-y)} \psi(y) dy. \end{aligned} \tag{4.1}$$

The eigenfunctions $(\psi_n)_n$ of the operator $\mathcal{Q}_c$ as defined in (4.1), are real and complete in $L^2[-1,1]$, where we mean complete in the following sense:

Definition 4.1 (Mean-square convergence)**.** A sum of functions $\sum_{j=1}^{\infty} g_j$, where $(g_j)_j$ is a sequence of $L^1(\mathbb{R})$ functions, is convergent *in the mean-square*, and its sum is the $L^1(\mathbb{R})$-function g, if

$$\lim_{J\to\infty} \left\| g - \sum_{j=1}^{J} g_j \right\|_{L^2(\mathbb{R})} = 0. \tag{4.2}$$

Definition 4.2 (Completeness)**.** An orthonormal sequence $(\phi_j)_j$ is said to be *complete* if

$$f = \sum_{j=1}^{\infty} \langle f, \phi_j \rangle \phi_j, \tag{4.3}$$

in the mean-square, for every $f \in L^1(\mathbb{R})$.

The completeness of the eigenfunctions is a result proven with classical Sturm-Liouville theory, and will be done in a while. As noted, our kernel K, as defined in (3.13) is symmetric (i.e. $K(x,y) = K(y,x)$). So $\mathcal{Q}_c$ is self-adjoint. Since our operator $\mathcal{Q}_c$ acts on the space $L^2(-1,1)$, the eigenvectors also lie there. We also have the following result on the eigenvalues of the operator.

Theorem 4.3 (Spectral Theorem for compact self-adjoint operators)**.** *Let $T \in \mathcal{C}(L^2[-1,1])$ be self-adjoint. Then there exists a system of orthonormal functions $(\psi_j)_j$ consisting of eigenfunctions of T, with corresponding eigenvalues $(\lambda_j)_j$ such that, if there are infinitely many eigenvalues, then*

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \dots$$

with

$$\lim_{j\to\infty} \lambda_j = 0.$$

Proof. We will prove the theorem in two steps.

Step 1: Construction of eigenvectors.

We use our previous result on these operators , namely Theorem 3.24, to construct eigenvectors and eigenvalues.
First we let $H_1 := L^2[-1,1]$, and $T_1 := T$. By Theorem 3.24, there exists and eigenvalue-eigenvector pair (λ_1, ψ_1) such that $\|\psi_1\| = |\lambda_1| = \|T_1\|$. Since $\text{span}\{\psi_1\}$ is a closed subspace of H_1, we know by the projection theorem that $H_1 = \text{span}\{\psi_1\} \oplus \text{span}\{\psi_1\}^\perp$. Let $H_2 := \text{span}\{\psi_1\}$; H_2 is a closed subspace of H_1, and we have that $T(H_2) \subseteq H_2$.
Now, let $T_2 := T_1|_{H_2}$. Then T_2 is compact and self-adjoint in $\mathcal{B}(H_2)$, the space of bounded operators on H_2.

If it so happened that $T_2 \equiv 0$ then we are practically done, so let us suppose that T_2 isn't the zero operator. Then, by the same argument used to construct our first eigenvalue-eigenvector tuple, there exists a pair (λ_2, ψ_2) such that $|\lambda_2| = \|T_2\|$, and $\|\psi_2\| = 1$. Since T_2 is a restriction of T_1, we obviously have that $|\lambda_2| = \|T_2\| \leqslant \|T_1\| = |\lambda_1|$; we also have that $\langle \psi_1, \psi_2 \rangle = 0$ by Theorem (2.14).

Now we unsurprisingly let $H_3 := \text{span}\{\psi_1, \psi_2\}^\perp$. Then, as before, we have that $H_3 \subseteq H_2$ and $T(H_3) \subseteq H_3$. The operator $T_3 := T_1|_{H_3}$ is compact and self-adjoint. Thus we know there exists a eigenvalue-eigenvector tuple (λ_3, ψ_3) such that $\|\psi_3\| = 1$. Since $|\lambda_3| = \|T_3\|$, we have that $|\lambda_3| \leqslant |\lambda_2| \leqslant |\lambda_1|$.

Continuing in this fashion we either end up with $T_n \equiv 0$ or we get our desired chain: $|\lambda_{i+1}| \leqslant |\lambda_i|$ for all $i \in \mathbb{N}$.

Step 2: Limit of (λ_n).

Suppose for sake of contradiction that $\lambda \not\to 0$ as $n \to \infty$. Then there exists an $\varepsilon > 0$ such that $|\lambda_n| \geqslant \varepsilon$ for infinitely many n. if we let $n \neq m$, then

$$\begin{aligned}
&\|T\psi_n - T\psi_m\|^2 = \|\lambda_n\psi_n - \lambda_m\psi_m\|^2 \\
&= \langle \lambda_n\psi_n, \lambda_n\psi_n \rangle - \langle \lambda_n\psi_n, \lambda_m\psi_m \rangle \\
&- \langle \lambda_m\psi_m, \lambda_n\psi_n \rangle + \langle \lambda_m\psi_m, \lambda_m\psi_m \rangle \\
&= |\lambda_n|^2 + |\lambda_m|^2 > \varepsilon^2.
\end{aligned}$$

Thus $(T[\phi_n])$ has no convergent sub-sequence, which contradicts the assumption that T is compact. □

Since the maximising problem that led to (3.36) only needs to hold on the closed unit ball, we extend the domain for the eigenfunctions such that

$$\psi_j(x) := \frac{1}{\lambda_j} \int_{-1}^{1} \frac{\sin[c(x-y)]}{\pi(x-y)} \psi_j(y) dy, \text{ for } |x| > 1.$$

Now, to simplify our calculations, let us divert our attention to the following operator, given by

$$\begin{aligned} F_c\colon L^2[-1,1] &\to L^2[-1,1] \\ \psi &\mapsto \int_{-1}^{1} \psi(t)e^{icxt}dt. \end{aligned} \tag{4.4}$$

Now we come to the real meat of the problem. Indeed, we actually have what Slepian essentially describes as a miracle. Our most essential object of study is the differential operator given by

$$\begin{aligned} L_c\colon L^2[-1,1] &\to L^2[-1,1] \\ \phi &\mapsto -\frac{d}{dx}\left[(1-x^2)\frac{d\phi}{dx}(x)\right] + c^2x^2\phi(x). \end{aligned} \tag{4.5}$$

we will slowly build our way towards the final result. It is this operator that gives rise to the PSWFs. Indeed the PSWFs are the *eigenfunctions* of the operator L_c as defined in (4.5).

Theorem 4.4. *Suppose $c > 0$ is a real number, and that L_c, F_c are defined as in (4.5) and (4.4) respectively. Suppose additionally, that $\psi \in C^2([-1,1],\mathbb{C})$. Then*

$$L_c[F_c[\psi]](x) = F_c[L_c[\psi]](x). \tag{4.6}$$

for all $x \in [-1,1]$.

Proof. Leibniz' rule gifts us with

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}e^{icxt}\right) - c^2x^2e^{icxt} = \frac{d}{dt}\left((1-t^2)\frac{d}{dt}e^{icxt}\right) - c^2t^2e^{icxt} \tag{4.7}$$

which follows from direct computations. Combining (4.7) with the definitions given in (4.5) and (4.4), we get that

$$\begin{aligned} L_c[F_c[\psi]](x) &= L_c\left[\int_{-1}^{1}\psi(t)e^{icxt}(x)dt\right](x) \\ &= \int_{-1}^{1}\psi(t)L_c[e^{icxt}](x)dt \\ &= -\int_{-1}^{1}\psi(t)\left[\frac{d}{dx}\left((1-x^2)\frac{d}{dx}e^{icxt}\right) - c^2x^2e^{icxt}\right]dt \\ &= -\int_{-1}^{1}\psi(t)\left[\frac{d}{dt}\left((1-t^2)\frac{d}{dt}e^{icxt}\right) - c^2t^2e^{icxt}\right]dt, \end{aligned}$$

where we pull the operator into the integral by virtue of the Dominated convergence theorem, as a derivative is really a limit. Performing integration by parts twice, we get thus

$$
\begin{aligned}
&\int_{-1}^{1} \psi(t)\frac{d}{dt}\left((1-t^2)\frac{d}{dt}e^{icxt}\right)dt \\
&= \psi(t)(1-t^2)\frac{d}{dt}e^{icxt}\Big|_{-1}^{1} - \int_{-1}^{1} \psi'(t)(1-t^2)\frac{d}{dt}e^{icxt}dt \\
&= -\psi'(t)(1-t^2)\frac{d}{dt}e^{icxt}\Big|_{-1}^{1} + \int_{-1}^{1} \frac{d}{dt}(\psi'(t)(1-t^2))e^{icxt}dt \\
&= \int_{-1}^{1} \frac{d}{dt}(\psi'(t)(1-t^2))e^{icxt}dt.
\end{aligned}
$$

where we first let

$$u = \psi(t),\ dv = \frac{d}{dt}\left[(1-t^2)\frac{d}{dt}e^{icxt}\right],\ \text{giving } du = \psi'(t),\ v = (1-t^2)\frac{d}{dt}e^{icxt}$$

and then

$$\tilde{u} = \psi'(t)(1-t^2),\ d\tilde{v} = \frac{d}{dt}e^{icxt},\ \text{ gives } d\tilde{u} = \frac{d}{dt}[\psi'(t)(1-t^2)],\ \tilde{v} = e^{icxt}.$$

We then see that

$$L_c[F_c[\psi]](x) = -\int_{-1}^{1}\left[\frac{d}{dt}(\psi'(t)(1-t^2)) - c^2t^2\psi(t)\right]e^{icxt}dt = F_c[L_c[\psi]](x).$$

□

Next, we will show a result about the structure of the adjoint operator of F_c, which we call F_c^*.

Theorem 4.5. *Suppose $c > 0$ is a real number, and that the integral operator $F_c\colon L^2[-1,1] \to L^2[-1,1]$ is given by (4.4). Suppose further that $F_c^*\colon L^2[-1,1] \to L^2[-1,1]$ is the adjoint operator of F_c such that*

$$\int_{-1}^{1} F_c[\phi](x)\overline{\psi(x)}dx = \int_{-1}^{1} \phi(x)\overline{F_c^*[\psi](x)}dx$$

for any two functions $\phi, \psi \in L^2([-1,1],\mathbb{C})$. Then

$$F_c^*[\phi](x) = \int_{-1}^{1} \phi(t)e^{-icxt}dt$$

for every $\phi \in L^2[-1,1]$.

Proof. Follows from simple calculations. Indeed we see that

$$\begin{aligned}
&\int_{-1}^{1} F_c[\phi](x)\overline{\psi(x)}dx \\
&= \int_{-1}^{1} \left[\int_{-1}^{1} \phi(t)e^{icxt}dt\right] \overline{\psi(x)}dx \\
&= \iint_{[-1,1]^2} \phi(t)e^{icxt}\overline{\psi(x)}dxdt \\
&= \iint_{[-1,1]^2} \phi(t)\overline{e^{-icxt}\psi(x)}dxdt \\
&= \int_{-1}^{1} \phi(t) \left[\int_{-1}^{1} \overline{\psi(x)e^{-icxt}dx}\right] dt \\
&= \int_{-1}^{1} \phi(x)\overline{F_c^*[\psi](x)}dx
\end{aligned}$$

where everything converges nicely due to the L^2-regularity. The result follows. □

We immediately prove the following result.

Corollary 4.6. *Suppose $\mathcal{Q}_c \colon L^2[-1,1] \to L^2[-1,1]$ is defined as in (4.1). Furthermore, suppose $\phi \in L^2[-1,1]$. Then*

$$F_c[F_c^*[\phi]](x) = \frac{2\pi}{c}\mathcal{Q}_c[\phi](x) = F_c^*[F_c[\phi]](x)$$

for every real $x \in [-1,1]$.

Proof. From direct calculations, we get

$$\begin{aligned}
F_c[F_c^*[\phi]](x) &= \int_{-1}^{1} F_c^*[\phi](t)e^{icxt}dt \\
&= \int_{-1}^{1} \phi(s) \int_{-1}^{1} e^{ict(x-s)}dtds. \\
&= \frac{2}{c}\int_{-1}^{1} \phi(s)\frac{\sin[c(x-s)]}{x-s}ds \\
&= \frac{2\pi}{c}\mathcal{Q}_c[\phi](x).
\end{aligned}$$

The proof for the second equality is identical. □

Following in the same fashion, we prove yet another result on commutation.

Theorem 4.7. *Suppose that $c > 0$ is a real number, and that L_c, F_c^* are defined as in (4.5) and Theorem 3.5. Suppose as well, that $\phi \in C^2([-1,1],\mathbb{C})$. Then*

$$L_c[F_c^*[\phi]](x) = F_c^*[L_c[\phi]](x). \tag{4.8}$$

Proof. The proof is indeed nearly identical to that in Theorem 3.4 and is therefore omitted. □

Now, we have arrived at the meat of the problem. Indeed we can show that the operators $\mathcal{Q}_c, L_c$ commute.

Theorem 4.8. *Suppose that $c > 0$ is a real number, and that L_c and $\mathcal{Q}_c$ are defined as in (4.5) and (4.1). Suppose also that $\phi \in C^2([-1,1], \mathbb{C})$. Then*

$$L_c[\mathcal{Q}_c[\phi]](x) = \mathcal{Q}_c[L_c[\phi]](x) \tag{4.9}$$

for all $x \in [-1, 1]$.

Proof. Combining Theorem 4.4, Theorem 4.5, Corollary 4.6 & Theorem 4.7 we get

$$\begin{aligned} L_c[\mathcal{Q}_c[\phi]](x) &= \frac{c}{2\pi} L_c[F_c F_c^*[\phi]](x) = \frac{c}{2\pi} F_c[L_c F_c^*[\phi]](x) \\ &= \frac{c}{2\pi} F_c[F_c^* L_c \phi](x) = \mathcal{Q}_c[L_c[\phi]](x), \end{aligned}$$

as required. □

The proof of every theorem and corollary ranging from Theorem 4.4 to Theorem 4.8 is due to [4][pp. 10-12].

From this commuting relationship, we can deduce many useful properties of the ψ_n and λ_n. Most of the results below follow from standard theory for ODEs, so we will have to rely on more general theory.

Definition 4.9 (Sturm-Liouville problem)**.** A *Sturm-Liouville problem* is a second order ODE of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = -\lambda\omega(x)y, \tag{4.10}$$

for given functions p, q, ω of some given regularity, and an unknown function $y = y(x)$, together with an unknown constant λ, and some boundary condition imposed on y.

The ω is to be understood and interpreted as a density or weight function for the problem. As with most of differential equation theory, we want to somehow impose functional analytic methods, which often involves writing the equation in operator form $Df = \lambda f$, where D is some differential operator. In our context, we call this D the *Sturm-Liouville operator*.For our sake, we can ignore the weight ω by setting it equal to unity; then we can define $D[f](x) := -\frac{d}{dx}\left[p(x)\frac{df}{dx}\right] + q(x)f$. One can show multiple results on the properties of this kind of operator. One that stands out to us is the following result:

Theorem 4.10. *The operator*

$$\begin{aligned} D\colon L^2(I) &\to L^2(I) \\ f &\mapsto -\frac{d}{dx}\left[p(x)\frac{df}{dx}\right] + q(x)f \end{aligned} \tag{4.11}$$

is self-adjoint for any compact interval $I \subset \mathbb{R}$.

Proof. Proof follows from integration by parts twice, where the boundary terms vanish by virtue of the given boundary conditions. $\square$

Since D is self-adjoint, its eigenvalues are purely real, and also eigenfunctions are pairwise orthogonal We observe that our linear differential operator L_c as defined in (4.5) can be analysed under the microscope of Sturm-Liouville theory. Indeed if one is clever in their choice of p, q and ω, then we can see that the eigenvalue problem for the operator L_c is indeed a Sturm-Liouville problem! Let us now see some examples of how to reduce ODEs to Sturm-Liouville problems.

Example (The Bessel equation)**.** The equation

$$x^2y''(x) + xy' + (x^2 - \nu^2)y(x) = 0$$

is called the *Bessel equation*, and can easily be reduced to a Sturm-Liouville problem by multiplying the equation through by $\frac{1}{x}$. Then, after some surgery is done, we are left with

$$[xy'(x)]' + \left(x - \frac{\nu^2}{x}\right)y(x) = 0,$$

which we readily recongise as a Sturm-Liouville problem (given some boundary conditions of course).

Example (Prolate spheroidal wave functions)**.** Our prestigious L_c as defined in (4.5) is also a Sturm-Liouville operator. Indeed we can let $p(x) := 1 - x^2$ and $q(x) := c^2x^2$, we reach our desirable conclusion.

We can now prove the completeness of the prolate spheroidal wave functions. Indeed, the following result holds for every set of orthonormal sequence of eigenfunctions of a Sturm-Liouville operator corresponding to a Sturm-Liouville eigenvalue problem with some boundary conditions on an interval $[a, b]$ with $-\infty < a < b < \infty$.

Lemma 4.11 (C^2-approximation)**.** *Let* $f \in C^2[a, b]$. *Then*

$$f = \sum_{n=1}^{\infty} \langle f, \psi_n \rangle \psi_n,$$

where the convergence is uniform on $[a, b]$.

Proof. Omitted, as it includes the use of Green's functions, which is beyond the relevancy for this thesis. □

Theorem 4.12. *The sequence $(\psi_j)_{j=1}^{\infty}$ consisting of the prolate spheroidal wave functions, is complete in $L^2[-1,1]$.*

Proof. It is enough to show that an integrable (L^1) function can be arbitrarily well approximated in mean-square by some linear combination of ψ_n's. Let f be integrable on $[-1,1]$. Then we can assume without loss of generality that f is real-valued on $[-1,1]$, and we must necessarily have that f is bounded on $[-1,1]$; we let $M := \sup\{|f(x)| \mid x \in [-1,1]\}$. We are now going to approximate f in mean-square by a C^2-function.
Let $\varepsilon > 0$. By Theorem (2.3), we may approximate f arbitrarily well by a step-function $s = s(x)$ in the L^1-norm. Thus

$$\int_{-1}^{1} |f(x) - s(x)|dx < \frac{\varepsilon^2}{18M}.$$

This gives

$$\begin{aligned}
\|f - s\|_{L^2}^2 &= \int_{-1}^{1} |f(x) - s(x)|^2 dx \\
&= \int_{-1}^{1} |f(x) - s(x)||f(x) - s(x)|dx \\
&\leqslant \int_{-1}^{1} (|f(x)| + |s(x)|)|f(x) - s(x)|dx \\
&\leqslant 2M \int_{-1}^{1} |f(x) - s(x)|dx < \frac{\varepsilon^2}{9},
\end{aligned}$$

and thus $\|f - s\|_{L^2} < \frac{\varepsilon}{3}$.
The next thing we do is approximate s by a C^2-function g in mean-square, such that $\|s - g\|_{L^2} < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}
\|f - g\|_{L^2} &= \|f - s + s - g\|_{L^2} \\
&\leqslant \|f - s\|_{L^2} + \|s - g\|_{L^2} \\
&< \frac{2\varepsilon}{3}.
\end{aligned}$$

By Lemma 4.11, we can approximate g such that

$$g(x) = \sum_{n=1}^{\infty} \langle g, \psi_n \rangle \psi_n(x),$$

where the convergence of the series is uniform on $[-1,1]$. Therefore we know there exists an $N \in \mathbb{N}$ such that

$$\sup_{x \in [a,b]} \left| g(x) - \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n(x) \right| < \frac{\varepsilon}{3\sqrt{2}}.$$

Then we have that

$$\begin{aligned}
\left\| g - \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n \right\|_{L^2} &= \sqrt{\int_{-1}^{1} \left| g(x) - \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n(x) \right|^2 dx} \\
&\leqslant \sqrt{2 \sup_{x \in [-1,1]} \left| g(x) - \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n(x) \right|^2} \\
&= \sqrt{2 \left(\sup_{x \in [-1,1]} \left| g(x) - \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n(x) \right| \right)^2} \\
&< \sqrt{2 \frac{\varepsilon^2}{18}} \\
&= \frac{\varepsilon}{3},
\end{aligned}$$

such that

$$\left\| g - \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n \right\|_{L^2} < \frac{\varepsilon}{3}. \tag{4.12}$$

Therefore

$$\left\| f - \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n \right\|_{L^2} \leqslant \| f - g \|_{L^2} + \left\| g - \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n \right\|_{L^2} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

We also have these intriguing facts about the PSWFs.

Corollary 4.13. *For the ψ_n and λ_n as defined in (3.29) the following holds:*

i. $\lambda_0 > \lambda_1 > \lambda_2 > \dots$ *with* $\lim_{n \to \infty} \lambda_n = 0$;

ii. For every $c > 0$, there exists a strictly increasing unbounded sequence of positive numbers $\chi_0 < \chi_1 < \dots$ such that for every $n \in \mathbb{N}_0$, the differential equation

$$(1 - x^2)\psi''(x) - 2x\psi'(x) + (\chi_n - c^2 x^2)\psi(x) = 0$$

has a continuous solution on $[-1, 1]$.

iii. ψ_n *is even or odd with* n;

iv. $\psi_n(x)$ *has exactly* n *zeros in* $(-1, 1)$;

v. $\psi_n(x)$ *is asymptotically equal to* $k_n \frac{\sin(cx)}{x}$, *as* $n \to \infty$;

vi. $\int_{-1}^{1} \psi_n(t) e^{2\pi i x t} dt = \alpha_n \psi_n(2\pi x / n)$ *for* $x \in \mathbb{R}$, *where k_n and α_n are independent of x.*

Before we give proofs, we will have to delve some more into some relevant ODE theory from [2][pp. 321-340].

Notice that the last equation tells us the peculiar fact that the Fourier transform of ψ_n restricted to $|t| < 1$ has the same form as ψ_n modulo some scaling factor. Note that we denote the eigenvalues of the operator L_c, as defined in (4.5), by (χ_n).

Lemma 4.14. *Let $u(t) \not\equiv 0$ be a real-valued solution to the equation*

$$(p(t)u')' + q(t)u = 0, \tag{4.13}$$

on $[a, b]$, where $p(t) > 0$ and q is real-valued, and both are continuous. Let u have exactly n zeros $t_1 < t_2 < \cdots < t_n$ on $(a, b]$. Let φ be a continuous function defined by

$$\varphi(t) := \arctan\left(\frac{u(t)}{p(t)u'(t)}\right). \tag{4.14}$$

and $\varphi(a) \in [0, \pi)$. Then $\varphi(t_k) = k\pi$ and $\varphi(t) > k\pi$ for $t \in (t_k, b]$ and $k \in \mathbb{N}$.

Proof. Note that at the t-values where $u = 0$, i.e. where $\varphi \equiv 0 \mod \pi$, (4.14) implies that $\varphi' = \frac{1}{p} > 0$. Thus, φ is an increasing function in the neighbourhoods of all points such that $\varphi(t) = j\pi$ for some integer j. This implies that if $\sigma \in [a, b]$ and $\varphi(\sigma) \geqslant j\pi$, then $\varphi(t) > j\pi$ for $t \in (\sigma, b]$. Also, if $\varphi(\sigma) \leqslant j\pi$, then $\varphi(t) < j\pi$ for $t \in [a, \sigma)$. This proves the assertion. □

In light of equation (4.13), we define the equations

$$(p_j(t)u')' + q_j(t)u = 0, \; j = 1, 2, \tag{4.15}$$

where p_j and q_j are real-valued and continuous on some interval $J \subset \mathbb{R}$. Suppose further that

$$p_1(t) \geqslant p_2(t) > 0 \text{ and } q_1(t) \leqslant q_2(t) \tag{4.16}$$

on J. We call equation (4.15) with $j = 2$, a *Sturm majorant* for (4.15) with $j = 1$ on J, and we call the opposite case a *Sturm minorant* on J. If the inequalities in (4.16) are strict, then we have a *strict Sturm majorant* and a *strict Sturm minorant*.

Theorem 4.15 (Sturm's First Comparison Theorem)**.** *Let the functions p_j, q_j in (4.15) be point-wise continuous on an interval $J = [a, b]$ and let (4.15) with $j = 2$ be a Sturm majorant for (4.15) with $j = 1$. Let $u = u_1(t) \not\equiv 0$ be a solution of (4.15) with $j = 1$, and let u_1 have exactly $n (\in \mathbb{N})$ zeros $t = t_1 < t_2 < \cdots < t_n$ on $t \in (a, b]$. Let $u = u_2 \not\equiv 0$ be a solution of (4.15) with $j = 2$ satisfying*

$$\frac{p_1(t)u_1'(t)}{u_1(t)} \geqslant \frac{p_2(t)u_2'(t)}{u_2(t)} \tag{4.17}$$

at $t = t_0$. Then u_2 has at least n zeros on $(a, t_n]$. Furthermore u_2 has at least n zeros on (a, t_n) if either the inequality in (4.17) holds at $t = t_0$ or (4.15) with $j = 2$ is a strict Sturm majorant for (4.15) with $j = 1$ on $[t_0, t_n]$.

Proof. Unfortunately quite long, but insightful nonetheless. A proof is to be found in [2][pp. 334-335]. □

Now we are ready to prove *ii.* & *iv.* in Corollary 3.13.

Proof of ii. & iv. Let $u = u(t, \lambda)$ be a solution of (4.13) (or (4.10) with $\omega(x) \equiv 1$). For a fixed λ, define a continuous function $\varphi(t, \lambda)$ of t on $[a, b]$ by

$$\varphi(t, \lambda) := \arctan\left(\frac{u(t, \lambda)}{p(t)u'(t, \lambda)}\right). \tag{4.18}$$

It is a simple task to show that (4.18) has a continuous derivative given by

$$\varphi' = \frac{1}{p(t)} \cos^2(\varphi) + [q(t) + \lambda] \sin^2(\varphi). \tag{4.19}$$

With a bit of work, the details of which we will exclude, one can show that φ is continuous in $(t, \lambda) \in [a, b] \times \mathbb{R}$. Next we see that by Lemma 3.14, $\varphi(\cdot, \lambda)$ is an increasing function of λ. Note that

$$\varphi(b, \lambda) \to \infty \text{ as } \lambda \to \infty. \tag{4.20}$$

To see this, let us introduce a new independent variable through $ds = dt/p(t)$ and $s(a) = 0$; thus (4.13) becomes

$$\ddot{u} + p(t)[q(t) + \lambda]u = 0,\ t = t(s),\ \dot{u} = \frac{du}{ds}. \tag{4.21}$$

If $M > 0$ is a real number, λ can be chosen so large that $p(t)[q(t) + \lambda] \geqslant M^2$ for any $t \in [a, b]$, since we always assume $p(t) > 0$. We now apply Sturm's Comparison Theorem, to the systems (4.21) and

$$\ddot{u} + M^2 u = 0;$$

this tells us that if n is an arbitrary natural number, and $M > 0$ is sufficiently big, then a non-trivial solution of (4.21) will have at least n zeros on the curious interval $\left[0, \int_a^b dt/p(t)\right]$. In other words we must have that $\varphi(b, \lambda) \geqslant n$ if $\lambda > 0$ is sufficiently large by Lemma 4.14. Similarly, we show that

$$\varphi(b, \lambda) \to 0 \text{ as } \lambda \to -\infty. \tag{4.22}$$

We know by Lemma 4.14 that $\varphi(b, \lambda) \geqslant 0$. Let $-\lambda > 0$ be large enough so that $p(t)[q(t) + \lambda] \leqslant -M^2 < 0$. Consider a solution $u = u(s) \not\equiv 0$ to the equation

$$\ddot{u} - M^2 u = 0. \tag{4.23}$$

Define the analogue to (4.18) by

$$\psi(s, M) := \arctan\left(\frac{u(s)}{\dot{u}(s)}\right). \tag{4.24}$$

Any solution to (4.23) must necessarily be of the form $u(s) = Ae^{Ms} + Be^{-Ms}$, where $\{A, B\} \subset \mathbb{R}$. It follows that for any fixed $s > 0$, we have

$$\lim_{M \to \infty} \frac{u(s)}{\dot{u}(s)} = 0. \tag{4.25}$$

Hence, $\psi(b_0, M) \to 0$ as $M \to \infty$, where $b_0 := \int_a^b dt/p(t)$. By Sturm's Comparison Theorem, we must have that $\varphi(b, \lambda) \leqslant \psi(b_0, M)$. This proves (4.22). Because of the fact that φ is a monotonically increasing function of λ, and the limiting relations (4.20) and (4.22), we must have that there exists some set of numbers $\lambda_0, \lambda_1, \ldots$ such that

$$\varphi(b, \lambda_n) = \beta + n\pi \text{ for } n \in \mathbb{N}_0,$$

where we suppose $\beta \in (0, \pi]$. Furthermore $\varphi(b, \lambda) \not\equiv \beta \mod \pi$ unless $\lambda = \lambda_n$. This concludes our proof. □

Since we have that every eigenfunction ψ_j is defined on $\mathbb{R}$, we can show that they are indeed orthogonal here. Let us normalise the eigenfunctions to have unit energy. Then we have

$$\int_{\mathbb{R}} \psi_n(x)\psi_m(x)dx = \delta_{mn}$$

which implies that

$$\int_{-1}^{1} \psi_n(x)\psi_m(x)dx = \lambda_n \delta_{mn} \tag{4.26}$$

where

$$\delta_{mn} := \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n \end{cases}$$

is the Kronecker delta. This tells us that we can expand every pointwise continuous function f defined on all of $\mathbb{R}$ by

$$f(t) = \sum_{n=0}^{\infty} \gamma_n \psi_n(t),$$

with

$$\gamma_n = \int_{\mathbb{R}} f(t)\psi_n(t)dt;$$

of course this obeys Parseval's identity:

$$\sum_{n=0}^{\infty} |\gamma_n|^2 = \int_{\mathbb{R}} |f(t)|^2 dt.$$

For a detailed proof of (4.26), see [2][pp. 340].

5 Conclusion

So, are we happy? The anticlimactic answer is "I suppose".

The properties stated in Corollary 4.13 are elegant, but finding proofs for these have turned out to be more difficult than I would've imagined. Nonetheless, I am quite happy with what I have been able to show in the semester I have spent chasing this project. If I can allow myself to be somewhat tongue in cheek, I must say that experts in this field of research really try their hardest to obscurify the precise details of results. That being said, I get how tiresome it must get if every article included a large portion dedicated to proofs of elementary results. I must at some point have accepted that my experience with the field is too ephemeral. At least this is my experience spending a whole semester trying to piece together the glossed-over details of [8]. I suppose it is this that has evolved into this thesis: trying to understand how in the world these results "follow from simple calculations". By talking with my fellow students, I don't think I am alone in this frustration.

All in all I am somewhat proud that I managed to show most of Slepian's derivations and calculations myself, with almost all of it being excluded from [8]. With the help of [4], I have actually managed to fully show the commutation relationship between L_c & $\mathcal{Q}_c$, which from my understanding is really what Slepian in [8] wants to relay as the most important detail.

Although this bachelor thesis is (hopefully) done, I would love to maybe revisit PSWFs in the future. There are plenty of analytic detail and information about PSFWs that I know exist, and I would love to get a stronger grasp on the theory at some point. Another point worth mentioning is the relevancy of the PSWFs. It is my understanding that PSWFs are still used in modern numerical and analytic research, although to what extent I am unsure. A quick glance at [3] seems to tell the story of continued interest; it should be noted that the article works with the theory in a discrete framework as well as the continuous one.

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