Henrik Grüner

The Time-Diversification Controversy

Bridging the Gap between Theory and Practice

Master's thesis in Financial Economics
Supervisor: Prof. Snorre Lindset
June 2023

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Abstract

The time-diversification controversy is a longstanding debate about whether risk decreases with longer investment horizons (time-diversification), as traditionally advised, or whether, as Samuelson (1969) and Bodie (1991) argue, the allocation to risky assets should be either constant or decreasing over time. This thesis delves into this debate by reviewing literature and examining empirical data.

Data from the S&P 500 and US Treasury T-Bills from 1929 to 2023 was used to form portfolios with varying investment horizons. Results indicated that as investment horizon increased, the annualized standard deviation and probabilities of loss and shortfall diminished, while the Sharpe and Sortino ratios grew. The 95%-VaR and 95%-cVaR also reduced to zero after the investment horizon extended beyond 7 and 12.

The data was also used to solve an optimization problem for wealth allocation to risky assets and to calculate the cost of shortfall insurance. Both frameworks suggested a decrease in risk with a longer time horizon when applied to empirical data, contrary to theoretical findings in the literature.

Abstrakt

Kontroversen rundt tidsdiversifisering er en langvarig debatt om hvorvidt lengre investeringshorisont reduserer risiko(tidsdiversifisering), og påfølgende bør allokeringen øke med tidshorisont, eller som Samuelson (1969) og Bodie (1991) argumenterer: allokeringen til risikable eiendeler bør være enten konstant eller avtagende over tid. Denne oppgaven bidrar til debatten gjennomå i) gjennomgå litteraturen rundt opphavet til de motstridende meningene, og ii) Undersøke effekten gjennom empiriske data.

Data fra S&P 500 og US Treasury T-Bills fra 1929 til 2023 ble brukt for å danne porteføljer med varierende investeringshorisonter. Resultatene indikerte at når investeringshorisonten økte, sank det annualiserte standardavviket og sannsynligheten for tap og underprestering (i forhold til risikofri rente), mens Sharpe- og Sortino-forholdene økte. 95%-VaR og 95%-cVaR falt også til null når investeringshorisonten var 7 and 12 eller lengre.

Dataene ble også brukt til å løse et optimeringsproblem for allokering av portefølje til risikable eller risikofrie aktiva m.h.t maksimering av forventet nytte, og for å beregne kostnaden ved forsikring mot at porteføljen underpresterer. Begge analyser indikerte reduksjon i risiko med en lengre tids horisont når de ble anvendt på empiriske data, i motsetning til de teoretiske modellene i litteraturen.

Preface

This report is the culmination of my economic studies and is written for the Department of Economics at the Norwegian University of Science and Technology in Financial Economics.

The topic of this thesis is time diversification, which refers to the belief that risk decreases with the investment horizon. This topic holds personal interest for me, as I have always regarded the principle of time diversification as a self-evident truth that has influenced my own financial decisions. Consequently, I found it intriguing when I encountered Prof. Snorre Lindset's assertion that some academics regard time diversification as a fallacy. If that is indeed the case, then I have fallen victim to this fallacy. Exploring the diverse arguments and different camps surround the time diversification controversy has been an enlightening experience, broadening my perspective on the notion of risk.

Acknowledgements

I wrote this thesis while working full-time as a management consultant. It took a lot of late nights and weekends to complete. Throughout the process, my supervisor, Prof. Snorre Lindset, has provided me with support and valuable guidance. For this, I would like to express my sincerest gratitude. Thank you.

I would also like to thank Mark Kritzman, who provided me with the papers he authored on the subject.

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Chapter 1.

Introduction and motivation

The time-diversification controversy is a longstanding debate among academics and financial practitioners that centers around the relationship between investment horizons and optimal asset allocation. The core of this debate centers around the following question: should the investment horizon affect an investor's portfolio allocation?

Investment advisors and fund managers typically recommend that investors maintain an investment horizon of at least five years when allocating risky assets, such as stocks, to their portfolios. The rationale behind this recommendation is that during longer investment horizons, the fluctuations in securities tend to cancel each other out over time. By adopting a long-term perspective, investors are better positioned to weather periods of market volatility and capitalize on the historically positive long-term performance of equities. Consequently, the five-year minimum investment horizon is often considered a key principle in investing, to the point that it is sometimes referred to as "conventional wisdom" (Bianchi et al., 2016).

According to the Norwegian platform Finansportalen, a Norwegian government-run website created to provide independent and unbiased financial advice for its citizens, a minimum investment horizon of 3-5 years is recommended for stock funds (Finansportalen). Similarly, the majority of wealth management firms in Norway advise investors to hold equity funds for at least five years or longer before selling, including Storebrand, DNB, KLP, Gjensidige, Nordnet, and others (Storebrand; DNB; Gjensidige; Nordnet). KLP even recommends a 10-year horizon for investing in equity funds (KLP).

The basis for the recommendation of a minimum investment horizon of five years is the same for all the aforementioned companies: the risk of losing money decreases as the investment horizon increases. These wealth managers also argue that the allocation towards risky assets should be 100%, gradually decreasing with age to achieve a trade-off between expected returns and acceptable risk level before withdrawal. Implicitly, they put faith in the argument of time-diversification.

Nevertheless, notable academics, such as the late Paul Samuelson, a Nobel laureate economist argue that to maximize the expected utility of any investor, the allocation to risky assets should stay constant irrespective to the investment horizon (Samuelson, 1963, 1969, 1971). Other critics of time-diversification, employ option pricing tools to argue that the allocation to risky assets should actually decrease with investment horizon, as the risk *increases* over the years Bodie (1991, 1995). A behavioral stream, incorporate human behavior and psychological factors, show that the phenomenon of time diversification is consistent with rational thinking. However, they also argue that there are cognitive biases and psychological factors influencing investors' decision-making to such an extent that generalizing how investors perceive risk is not possible (Fisher and Statman, 1999; Olsen and Khaki, 1998; Benartzi and Thaler, 1995b). Consequently, they do not propose a unifying solution.

This is the crux of the time-controversy: multiple distinguished academics reaching the opposite conclusions about a topic which the majority of practitioners agree upon.

1.1. Contributions

This thesis aims to achieve two objectives: 1) provide a comprehensive literature review of the theoretical frameworks within the time diversification controversy, and 2) test if time diversification is present in empirical data.

The first objective is done by showcasing and scrutinizing each school of thought, examining their origins, methodologies, and weaknesses. It is demonstrated that both the expected utility and option pricing approaches rely on unreasonable assumptions regarding the risk attitude of the "typical" investor.

The second objective is done by collecting and analyzing from the S&P 500 and US Treasury T-Bills 1M spanning 1929 to 2023 to generate investment portfolios across various horizons from one to forty years. Then, objective risk measures, such as annualized standard deviation, loss and shortfall probabilities, 95%-VaR, 95%-cVaR, are examined as the investment horizons increases. Furthermore, the risk-adjusted Sharpe and Sortino ratio are found over each investment period. Additionally, the thesis addresses the optimization problem of determining the portion of an investor's wealth to risky or risk-free asset to maximize the investor's expected utility for each time horizon. To examine whether this proportion increases, decreases, or stay constant, hypothesis testing was employed. Lastly, the cost of shortfall insurance through buying a put option was calculated using empirical data for the standard deviation.

Chapter 2.

Investing under uncertainty

The following chapter will provide context and background for the time-diversification controversy.

2.1. Key definitions

2.1.1. Duration

Duration is a measure of the sensitivity of a price of a instrument (e.g. bond or debt instrument) to a change in interest rates. The higher the duration, the more the instrument's price will drop as interest rates rise and vice versa. The interest rate risk is proportional with duration.

2.1.2. Immunization

Immunization is a risk-mitigation strategy that matches the asset and liability duration so portfolio values are protected against interest rate changes. This is often utilized by pension funds e.g., by investing in fixed income securities that match the benefits payable under the plan.

2.2. Performance measures

Consider an investor who buys a stock at price P_{t-1} and sells it at price P_t. The net return, denoted as r_t, can be calculated as:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{2.1}$$

Alternatively, the total returns, denoted as R_t, can be calculated as

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{2.2}$$

Over multiple periods the cumulative returns of the portfolio over the entire period $t = 1, .., T$:

$$R_p = \prod_{t=1}^{T} (1 + R_t). \tag{2.3}$$

2.3. Risk Measures

The concept of risk is a heavily discussed theme in financial modeling, but at its core, it represents the uncertainty surrounding the return of any investment. One of the principal challenges faced by investors, academics, and laymen alike is the challenge of quantifying risk in a meaningful and reliable way. It is conventional to distinguish between *what is being measured*—the risk metric—and *how the risk metric is being measured*—the risk measure. Examples of risk metrics include credit risk, liquidity risk, market risk, operational risk, business risk, etc. Examples of risk measures include standard deviation, Value at Risk, expected shortfall, etc. The landscape of risk metrics, measures, and methods is vast and constitutes an active field within academia. For brevity, the following section will mainly introduce important risk measures.

2.3.1. Coherent risk measure

The class of coherent risk measures was introduced and defined by Artzner et al. (1999) and is a function quantifying risk that satisfies the properties of montonicity, sub-additivy, homogeneity, and translation in-variance. Let X be a random outcome in the space of $\mathcal{L}$, then the function $\varrho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ is defined as a coherent risk measure for $\mathcal{L}$ if it satisfies the following properties (Artzner et al., 1999):

Normalized:

$$\varrho(0) = 0. \tag{2.4}$$

An empty portfolio holds no risk.

Monotonicity:

$$\text{If } Z_1, Z_2 \in \mathcal{L} \text{ and } Z_1 \leq Z_2 \implies \varrho(Z_1) \geq \varrho(Z_2) \tag{2.5}$$

If the portfolio value of Z_2 is better than Z_1, then the risk of Z_2 is lower than the risk of Z_1. Note that better does not mean greater, but rather a higher chance of outperforming the other portfolio/assets. As an example, consider Z_1 to be an in-the-money call option, and Z_2 is also an in-the-money option, but with lower strike price, then the latter option would have lower risk.

Sub-additivity:

$$\text{if } Z_1, Z_2 \in \mathcal{L}, \text{ then } \varrho(Z_1 + Z_2) \leq \varrho(Z_1) + \varrho(Z_2). \tag{2.6}$$

The risk of two portfolios together cannot be worse than adding the two risks separately, otherwise known as the diversification principle: The volatility of any portfolio is lower or equivalent to the combined volatility of its individual components in the portfolio.

Positive homogeneity:

$$\text{If } \alpha > 0 \text{ and } Z \in \mathcal{L}, \text{ then } \varrho(\alpha Z) = \alpha\varrho(Z). \tag{2.7}$$

Risk if linearly proportional to the portfolio size, i.e., if an investor doubles the position size the risk will also be doubled.

Translation invariance:

Consider a deterministic portfolio A with guaranteed return at a and $Z \in \mathcal{L}$, then

$$\varrho Z + A = \varrho Z - a. \tag{2.8}$$

The portfolio A is just adding cash a to your portfolio Z. Translation invariance implies that by adding a sure amount of capital, e.g., cash to a portfolio reduces the risk by the same amount.

2.4. Risk Measures - examples

2.4.1. Standard Deviation

The standard deviation is a statistical metric showing the degree of expected fluctuations from the average value of any given data set. The standard deviation of the expected returns of an investment is the most commonly used measure to quantify the risk associated with that given investment. Assets with higher standard deviations of the expected returns are more volatile, thus riskier, than assets with with a low standard deviation of their returns.

The standard deviation is calculated by taking the square root of the data set's variance. The variance of a data set is the average of the squared differences from the average value of the data set:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2, \tag{2.9}$$

where σ^2 is the variance, n is the number of observations in the data set, x_i, $\text{for} i = [1, 2 \ldots, n]$ is individual data points, and μ is the mean value of the data set. The standard deviation is the square root of the variance:

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2}. \tag{2.10}$$

The square root operation is performed to bring the units of the standard deviation back to the original units of the observations, e.g., for investments, the standard deviation would be measured in returns (percentage of initial investments).

2.4.2. Semi-standard deviation

The standard deviation of the expected returns of any given investment is the most common risk measure within finance. However, it is not without its shortcomings. Standard deviation is symmetric by nature as it does not differentiate between upwards and downwards deviations. Consequently, investments with exceptional returns cannot be distinguished from investments with catastrophic losses by examining the standard deviation of the returns alone. Accordingly, without additional context, the standard deviation cannot be used differentiate between a great and a detrimental investment alone. This limitation could lead to wrongful inclusions of assets in an portfolio.

To alleviate the standard deviation's indifference between upside or downside fluctuations, the semi-standard deviation can be used. The semi-standard deviation only considers the deviation below the mean return, hence it is often referred to as a downside risk measure. It is calculated as:

$$\sigma_d = \sqrt{\frac{1}{n} \sum_{x_i < \mu} (x_i - \mu)^2}, \tag{2.11}$$

where n is the total number of observations with $x_i < \mu$.

2.4.3. Criticism of Standard Deviation as a Risk Measure

Another important clarification of using standard deviation as a risk measure is that many financial models (e.g., Capital Asset Pricing Model (CAPM), Black-Scholes Option Pricing Theory) assumes the expected returns follow a normal distribution. However, this assumption may not hold true of real-world scenarios, where assets often exhibit non-normal distributions, such as fat-tailed or skewed distributions. As the standard deviation quantifies the deviation around the mean, the standard deviation as risk measure might not address the potential for extreme events alone, such as black-swan events. Consequently, by only looking at the standard deviation, the likelihood of extreme events could be underestimated.

During the financial crisis of 2007, the Chief Financial Officer of Goldman Sachs famously commented to the *Financial Times* “We are seeing things that were 25-standard deviations moves, several days in a row” (Haldane, 2009). To put this into perspective, a 25-sigma event would be expected to occur only once every 6×10^{124} lifetimes of the universe(Dowd et al., 2008). This statement brings to light the potential pitfalls of using standard deviation alone as a risk measure. A more plausible explanation than the CFO’s statement being true is that their model was wrong. Likely, the financial models assumed normally distributed returns, whereas the underlying distributions was fat-tailed. Consequently, the use of standard deviation resulted in a significant underestimation of the likelihood of the extreme events that occurred during the financial crisis.

Table 2.1.: Probabilities of High Sigma Events. Table taken from Dowd et al. (2008)

k	Probability in any given day	Expected occurrence
3	0.135%	740.8 days
4	0.00317%	31,559.6 days
5	0.000029%	3,483,046.3 days
10	7.620×10^{-22}%	5.249×10^{20} years
15	3.671×10^{-49}%	1.090×10^{48} years
20	2.754×10^{-87}%	1.453×10^{86} years
25	3.0570×10^{-136}%	1.309×10^{135} years

2.4.4. Value at Risk

Rather than evaluating the deviations of the average, Value at Risk (VaR) is a measure used to estimate the potential loss that could be incurred by an investment with a certain degree of confidence. Consider a portfolio worth $1 million and the investor would like to know what is the maximum loss over the next year with 95% confidence? This could be calculated by using VaR, and if the figure is e.g., $ 100,000, then one would say with 95% confidence that the portfolio will not lose more than $ 100,000 the next year. In general terms, the confidence level is denoted by $\alpha\%$.

The most common ways to calculate the VaR are the historical and the parametric simulation approaches. The historical approach uses the past returns to estimate VaR, with an implicit assumption of future returns to be similar to prior returns. The procedure is to sort the historical returns from worst to best, and determining the returns that correspond to the given confidence level, and calculate the difference between the current portfolio value and the value at the selected percentile.

The parametric approach assumes the returns of the portfolio follow a normal distribution. Under this assumption, the VaR is found by multiplying the portfolio value with the number of standard deviations from the mean corresponding to the given confidence level (i.e., Z-value).

VaR has been criticized for its inability to capture the tail risk of extreme events, as it only considers the maximum loss at the threshold, and not beyond. Further, VaR is not a coherent risk measure as it does not satisfy the sub-additivity property. The sum of the VaRs of two portfolio is not necessarily equal to the VaR of their combined portfolio. An immediate consequense is that VaR might discourage diversification (Artzner et al., 1999).

2.4.5. Conditional Value at Risk

Conditional Value at Risk (cVaR) is a risk measure based on the same concept as VaR, but instead of providing an estimate of the maximum loss at a given level of confidence, cVaR estimates the expected value of the losses that occur beyond the VaR threshold, given that they exceed this threshold. This way, cVaR incorporates the tail risks of extreme events. Additionally, in contrast to the previous defined risk measures, the cVaR is coherent. Whereas VaR is based on a single quantile of the distribution of portfolio returns, the cVaR takes into account the expected value of losses beyond that quantile. As it averages out the worst-case losses, all the different possible states and possibilities are taken into account. Consequently, the diversification effect is captured, and the cVaR is sub-additive, rendering the risk measure as coherent.

cVaR is generally considered to be a superior risk measure over VaR because of the aforementioned properties (Acerbi and Tasche, 2002). Figure 2.1 illustrates how VaR and cVaR with 95% confidence differs in an example with an asset with an expected return of 12% with a standard deviation of 20%.

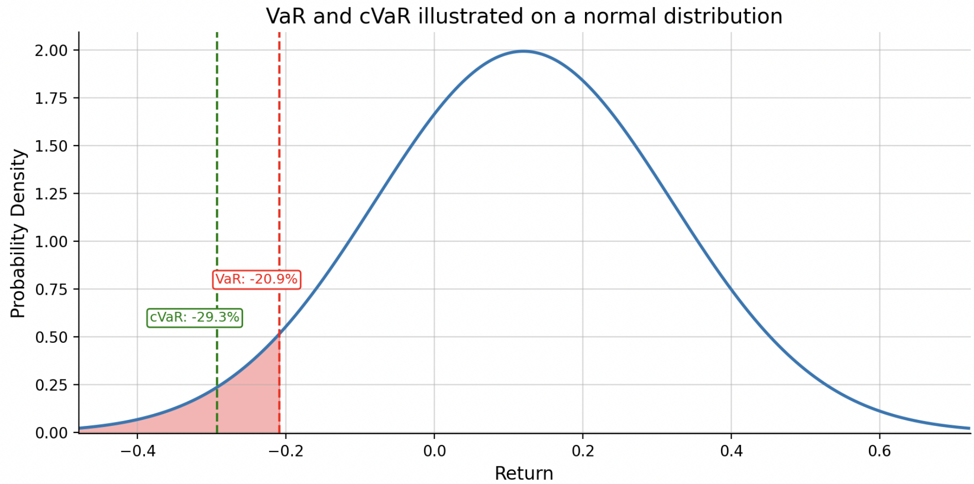

Figure 2.1.: Probability density function of a normal distribution with a mean of 12% and a standard deviation of 20%, along with the VaR and cVaR at a 95% confidence level. The VaR (red dashed line) is the maximum expected loss that could occur with a 5% probability. The cVaR (green dashed line) is the expected loss if the loss exceeds the VaR. The area shaded in red represents the probability of losses beyond the VaR.

2.4.6. Shortfall risk

Shortfall risk[1] is defined as the risk that the portfolio returns do not exceed a predefined minimum return (benchmark). Let R_p denote the expected return of a portfolio with volatility σ_p and R_b represent the minimum acceptable return (benchmark). Assuming normally distributed returns, the shortfall risk is calculated by finding the z-score and using the cumulative distribution function (CDF) for the standard normal distribution, denoted as $\Phi(z)$:

$$z = \frac{R_p - R_b}{\sigma_p}, \text{ then } P(\text{shortfall}) = \Phi(z). \tag{2.12}$$

[1] Note that the term Expected shortfall is commonly used synonymous with Conditional Value at Risk. This thesis, however, defines shortfall risk in alignment with Bodie (1991).

Consider an asset with normally distributed returns, an expected return of 12%, and a standard deviation of 20%. Furthermore, let the benchmark R_b be the risk-free rate of 5%. The z-score is then calculated by $z = \frac{0.12-0.05}{0.20} = -0.35$. By using the standard normal CDF, the shortfall is calculated as $P(\text{shortfall}) = \Phi(-0.35) \sim 36.32\%$.

Figure 2.2 illustrates the distribution of the asset's returns, with the shaded red area indicating all scenarios in which the asset underperforms relative to the risk-free rate.

Figure 2.2.: The probability density function for an asset's expected return following a normal distribution with mean of 12% with a standard deviation of 20%. The red vertical line indicates the risk-free rate of 5%, and the shaded regions are all scenarios where the asset underperforms compared to the risk-free rate. The shortfall risk of 36.32% is the probability of the asset's return falling below the threshold of 5%.

Shortfall risk is a fitting addition upon examining time-diversification, particularly with the risk-free rate as a benchmark. As the investment horizon increases, the probability of incurring a loss on initial investments diminishes. Consequently, it is more interesting to change the focus from potential losses to the returns that could have been achieved risk-free instead. Implicitly, "conventional wisdom", suggests that shortfall risk should decrease as the investment horizon lengthens.

2.5. Risk-Adjusted Performance Measures

2.5.1. Sharpe Ratio

Using returns alone to measure the performance of the portfolio is not sufficient, as it fails to consider the risk taken to attain said returns. Given the possibility of leverage, more return can always be earned by taking more risk. To alleviate the one-dimensional view on portfolio performance, Sharpe (1966) introduced the Sharpe ratio, which measures the risk-adjusted performance of the portfolio. The Sharpe ratio is calculated as follows:

$$\text{Sharpe Ratio} = \frac{R_p - R_f}{\sigma_p}, \tag{2.13}$$

where R_p and R_f is the expected portfolio return and risk-free rate and σ is the portfolio's standard deviation. The Sharpe ratio provides a single metric that combines both risk and return. A higher Sharpe ratio indicates a better risk-adjusted performance, as it suggests that the portfolio achieved higher returns relative to its level of risk. However, the Sharpe ratio suffers from the standard deviation's inability to distinguish between deviations to the upside from the downside. As such, the Sharpe ratio could actually be increased by removing the largest positive returns in a distribution. It is a particularly poor performance metric for negatively skewed distributions.

2.5.2. Sortino Ratio

Sortino and Price (1994) introduces the Sorting Ratio, which instead of using the standard deviation to adjust the performance for risk, it uses the semi-standard deviation. This change solves the problems the Sharpe ratio has when evaluating skewed distributions. It can be calculated as

$$\text{Sortino Ratio} = \frac{R_p - R_f}{\sigma_d}. \tag{2.14}$$

where R_p and R_f is the expected portfolio return and risk-free rate and σ_d is the portfolio's semi-standard deviation. As investor's are more interested in downside risk, Sortino is a more effective tool. Furthermore, it alleviates the troubles Sharpe ratio has with positively-skewed portfolios.

2.6. Modelling returns

2.6.1. Random Walk

The random walk hypothesis asserts that successive price changes are independent and identically distributed, hence impossible to predict the future prices based on historical data, hence the name. Let P_t be an asset's price at time t. Then by assuming random walk,

$$P_{t+1} = P_t + \varepsilon, \quad \text{where } \varepsilon \sim \text{i.i.d}(0, \sigma^2) \tag{2.15}$$

where ε is an independent, identically distributed random variable with zero mean with σ^2 variance. Figure 2.3 illustrates 100 simulations of random walks over time with for normally distributed asset with a 12% expected return and 20% volatility.

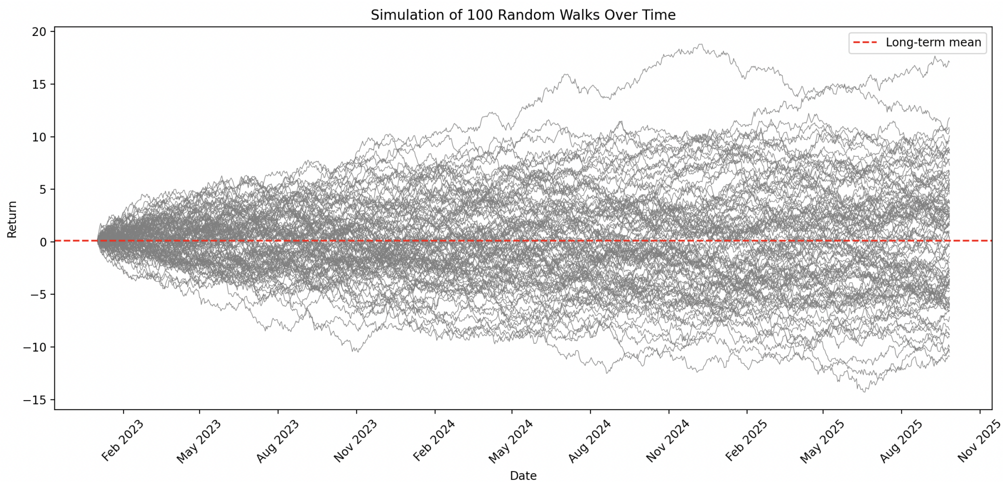

Figure 2.3.: The random walk hypothesis illustrated for 100 scenarios for a normally distributed asset with a 12% expected return and 20% volatility.

2.6.2. Mean Reversion

Mean reversion, or reversion to the mean, predicts that asset price volatility and historical returns will eventually revert to the long-run mean of the data set. The concept assumes that if the return of the asset significantly strays from the long-term mean, it is expected to regress back to the long-term mean. Consider an asset with return R, then the process is given by the Ornstein-Uhlenbeck process:

$$dR_t = \alpha(\mu - R_t)dt + \sigma dW_t \tag{2.16}$$

Where α is the mean reversion rate, μ is the long-term mean, σ is the standard deviation of the asset's returns, and dW_t denotes a standard Wiener process (Durrett, 2019). Mean reversion theory implies that the optimal allocations towards risk-free and risky asset should be dynamically adjusted over time. If the returns of an assets has been consistently higher than the long-term mean, a follower of mean reversion theory would adjust the allocation down, and vice versa for returns lower than the long-time mean. Thus, mean reversion theory suggests the allocation to risky and risk-free should be dynamically managed by investors reallocating the portfolio based on deviations from the long-term mean.

Figure 2.4 illustrates how an asset with 12% long-term mean with 20% volatility is modeled as a mean-reverting time series. Figure 2.5 illustrates the difference between the mean-reversion and random walk hypotheses.

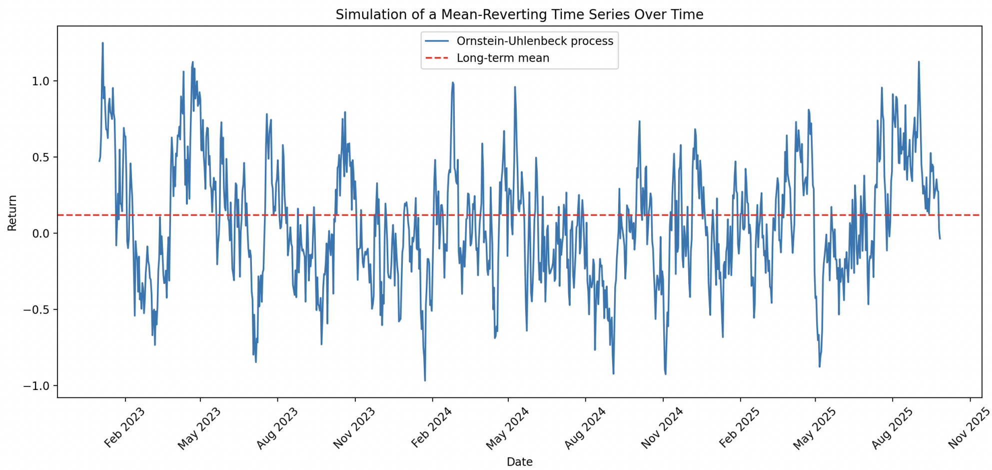

Figure 2.4.: Illustration of mean-reversion in a time period of 1000 days. The return is modeled as a mean-reverting time series with a long-term mean of 12% and volatility of 20%, with a mean reversion rate of 0.15. Note that the values oscillate over and under the long-term mean.

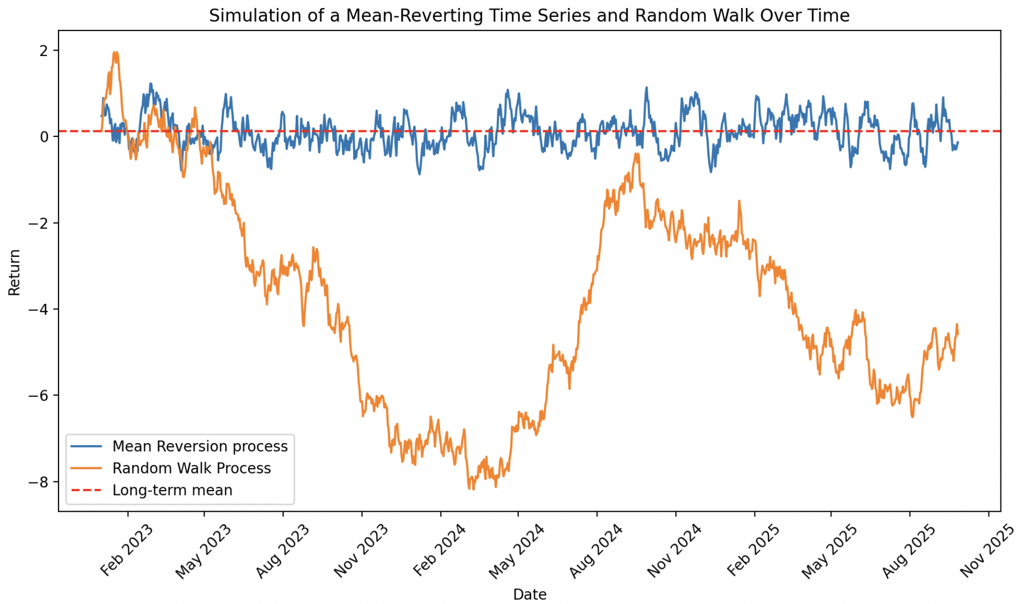

Figure 2.5.: Comparing random walk and mean reversion hypotheses for one simulation with a a normally distributed asset with expected return of 12% and volatility of 20%. Note how the mean reverting time series follows the mean, whereas the random walk does not follow any pattern, hence the name.

2.7. Modern Portfolio Theory

Portfolio diversification is the financial counterpart to not putting all your eggs in one basket. Rather than investing all your funds (eggs) into a single stock (basket), diversification refers to spreading out the funds into multiple assets across sectors and countries (baskets). As the risk of any portfolio stems from either company specific risk or systematic market risk, the goal of diversification is to eliminate any company specific risk. However, the systematic risk inherent to the financial system still remains. (Markowitz, 1952) seminal modern portfolio theory was the first formal treatment of the benefits of portfolio diversification. The paper showed that by constructing a portfolio of multiple assets, it is possible to achieve the same or higher return with lower risk than any of the individual stocks alone - given that the stocks are not perfectly correlated. Formally, a portfolio p, is said to be mean-variance efficient if it yields a higher return for a given level of risk than any other portfolio q constructed by the same assets.

$$E[R_p] > E[R_q] \quad \text{where } \sigma_p^2 = \sigma_q^2 \tag{2.17}$$

or with a given level of return

$$\sigma_p^2 < \sigma_q^2 \quad \text{where } E[R_p] = E[R_q] \tag{2.18}$$

where $E[R_x]$ and σ_x^2 is the expected return and variance of portfolio x respectively.

The return and allocations of the optimal portfolio p consisting of n assets is equal to

$$E[R_p] = w^T R \tag{2.19}$$

$$\sigma_p^2 = w^T \Sigma w \tag{2.20}$$

where $w^T \in [1, n]$ and $w \in [n, 1]$ are the row and column vectors of the portfolio weights of n assets, $R \in [1, n]$ is the expected returns for the n assets, and $\Sigma \in [n, n]$ is the covariance matrix between the assets. For the steps to calculate the the optimal portfolio, see (Markowitz, 1952).

Even though the modern portfolio theory put forth by Markowitz remains one of the iconic theories in finance, taught at every finance study, it is a simplification of real world investing relying on several assumptions criticized in the literature. MTP assumes that the returns from assets are distributed normally, rational and risk-averse investors, all investors has the same information, that the risk and returns of each asset is known, and MTP assumes single-period investing. Each of these assumption are interesting and worth a takedown, however due to brevity, this thesis will concentrate on the single-period assumption. For a comprehensive discussion on MTP the interested reader is directed to (Elton and Gruber, 1997).

2.8. Black-Scholes Option Pricing

The Black-Scholes model introduced by Black and Scholes (1973), is a mathematical model used for pricing options. Options are financial derivatives that give buyers the right, but not the obligation, to buy or sell an underlying asset at an agreed-upon price and date. The agreed-upon price is referred to as the strike price. The Black-Scholes calculates the theoretical price of European call and put options, which can only be exercised at expiration. The Black-Scholes pricing equations for European call options $C(S, t)$ and European put options $P(S, t)$ are:

$$C(S, t) = S_t N(d_1) - X e^{-r_f t} N(d_2) \tag{2.21}$$

$$P(S, t) = X e^{-r_f t} N(-d_2) - S_t N(-d_1), \tag{2.22}$$

where d_1 and d_2 are defined as:

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{X}\right) + \left(r_f + \frac{\sigma^2}{2}\right)(T-t)\right], \tag{2.23}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{2.24}$$

S_t is the spot price of the asset, X is the strike price, T is the expiry date, t is the current time, r_f is the risk-free rate, σ is the volatility of the asset's returns, and $N(\cdot)$ is the standard normal CDF.The model is widely used, although often with some adjustments, by options market participants.Bodie et al. (2009). There are a lot of assumptions for the model to be accurate, which will not be discussed here. The interested reader is guided to Black and Scholes (1973) or Bodie et al. (2009).

2.9. Expected utility theory

Traditional finance models posits that an investor should always choose the investment with the highest expected value. If the asset price is lower than the expected asset price, the net expected value is positive and the investor should buy asset. However, these models have been criticized for several reasons. First, they implicitly assume that investors are indifferent to the risk associated with each investment. Consider a decision-maker faced with two choices: Entering a free lottery with a 50%–50% chance of winning \$100, and a sure payoff of \$50. The expected value of both choices are \$50, however, the choice will depend on the risk appetite of the decision maker. To illustrate that individuals consider more than just the expected value of choice during decision is perfectly shown in the St. Petersburg Paradox posed by Nicholas Bernoulli in 1728.

The St. Petersburg paradox involves a hypothetical game where a fair coin is tossed in a casino setting. The initial stake is \$1 and doubles every time a tail appears[2]. The game ends when the first head appears, and the player wins the current stake. Thus, the player wins \$1 if a head appears on the first toss, \$2 if it appears on the second toss, \$4 on the third, and so on. The payoff is therefore $\$2^{k-1}$ for k consecutive tails followed by a head. What is the fair price to enter the game? By examining the expected value, the probability of each payoff can be calculated as

[2]Different versions of the game exists, most prominently another where the initial stake is 2\$. The difference does not affect the outlined findings

$$\begin{aligned} E &= \frac{1}{2} \cdot \$1 + \frac{1}{4} \cdot \$2 + \frac{1}{8} \cdot \$4 + \frac{1}{16} \cdot \$16 + \cdots \\ &= \$\frac{1}{2} + \$\frac{1}{2} + \$\frac{1}{2} + \$\frac{1}{2} + \cdots \\ &= \$\infty, \end{aligned}$$

therefore, the gamble should be preferred to any finite sure gain. However, in reality, few individuals would be willing to pay significant amounts to enter the game, hence the paradox. A resolution was proposed by Daniel Bernoulli, a cousin of Nicholas Bernoulli. Bernoulli (1954) suggest that individuals maximize their expected utility of the outcomes, rather than the expected monetary value. They argue that a gain of \$200 does not necessarily represent double the utility of a gain of \$100, highlighting the concept of diminishing marginal utility of money. To capture this expected utility, Bernoulli proposed a model:

$$U(p) = \sum u(x_k) p_k, \tag{2.25}$$

where p_k is the probability of outcome x_k and function u represents the utility assigned to each outcome. Hence, U is the expected utility. Bernoulli suggested using a logarithmic function as it exhibits diminishing marginal utility. Then, the expected utility is calculated as:

$$U = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^k \ln(2^{k-1}), \tag{2.26}$$

which is indeed finite. Bernoulli took this one step further and incorporated the wealth level of the individual to provide a good criterion for real individuals behavior. To find what any individual would pay to enter the gamble, consider a sure gain ζ which gives the same utility as the gamble. Then, by comparing the individuals utility with the certain gain against the expected incremental utility from the payoffs from the gamble multiplied the probability of the given state, then one can find the maximal price the individual is willing to pay. Let W be the individuals wealth, then

$$U(W + \zeta) = \frac{1}{2} u(W + 1) + \frac{1}{4} u(W + 2) + \frac{1}{8} u(W + 4) + \cdots . \tag{2.27}$$

By assuming the utility function is the logarithmic function $u(x) = ln(x)$ and the wealth is \$ 50 000, the individual's certainty equivalent is only $\sim$ \$9 (Machina, 1987), even though the gamble has an infinite expected value. Figure 2.6 illustrates how the certainty equivalent changes with the wealth of the individual.

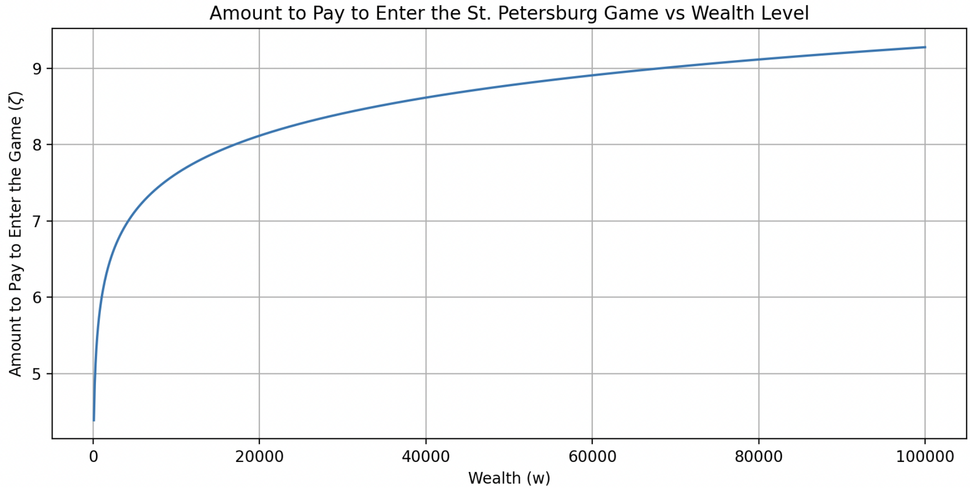

Figure 2.6.: The amount one would be willing to pay to enter the St. Petersburg game, assuming a logarithmic utility function, varies as a function of the individual's wealth. This is a far cry from the game's theoretical expected value of infinity.

2.10. Neumann-Morgenstern Utility Function

Neumann et al. (1944) formally proved that maximizing utility is a rational decision criterion. Specifically, authors John von Neumann and Oskar Morgenstern established five basic axioms. When these conditions are met, it is possible to assign numerical utilities to outcomes. These utilities, when combined with their associated probabilities, can be used to calculate an expected utility. Greater expected utility correspond to a higher preference.

To set up the five axioms, consider an individual agent faced with options called *lotteries*. The lotteries are scenarios where each outcome happens with a given probability. Note that the outcomes are mutually exclusive and all probabilites sum up to one. A lottery with many outcomes are denoted L

$$L = \sum p_k A_k, \tag{2.28}$$

with $\sum p_i = 1$. If lottery L_1 is preferred over lottery L_2, the notation is $L_1 \succ L_2$. Indifference is denoted by $\sim$. The five axioms are

1. **Completeness**: For every L_1 and L_2, either $L_1 \succeq L_2$ or $L_2 \succeq L_1$ or both

(indifference). Completeness means that any individual can either prefer one lottery to the other, or be indifferent between them.

2. **Transitivity**: For every L_1,L_2, and L_3, with $L_1 \succeq L_2$ and $L_2 \succeq L_3$, then $L_1 \succeq L_3$. Individual's preferences are consistent.
3. **Continuity**: For every L_1,L_2, and L_3, with $L_1 \succeq L_2 \succeq L_3$ then L_2 is equally preferred to $pL_1 + (1-p)L_3$ for some $p \in [0,1]$.
4. **Independence**: If $L_1 \succeq L_2$, then for all L_3 and $p \in [0,1]$, $pL_1+(1-p)L_3 \succ pL_2 + (1-p)L_3$. That is, a mix of L_1 and L_3 with probability p is preferred to a mix of L_2 and L_3 with the same probability.
5. **Non-satiation**: If L_1 and L_2 are two lotteries with the same probabilities, but L_1 always has equal or greater outcomes than L_2, then $L_1 \succeq L_2$.

The above axioms are sufficient to guarantee that there exists a utility index such that the ordering of the lotteries by their expected utilities fully coincides with the individual's actual preferences (Schoemaker, 1982). It is important to note that it is not the expected values themselves that are important, but rather the ranking they represent out of the choices available. Further, the absolute values of any utility function $U(x)$ are not interesting, as the origin and unit could be arbitrarily selected. Instead, is the relative value of an expected utility when compared to other expected utilities that hold importance.

2.11. Neumann Morgenstern expected utility theorem

The von Neumann—Morgenstern utility theorems, show that, under the satisfaction of the five theorems of rational behavior outlined above, a decision maker faced with probabilistic outcomes of different choices, will behave as he or shew is maximizing the expected value of some function (utility function) defined over the potential outcomes at some specified point in the future (Neumann et al., 1944).

As long as the five axioms hold for an individual's preferences, a utility function may be used to weigh the different outcomes of scenarios. This could then be used to deduct how the individual will choose during uncertainty. Consider a risk-averse individual with \$10 000 in savings is given the option to gamble it for a 30% chance of a \$100 000 gain, and 70% chance of receiving zero. The von Neumann-Morgenstern model posits that the individual will only take the bet if the expected utility of the bet is higher than the expected utility of his or her savings. The expected utility of taking the bet can be modelled as

$$\text{EU(taking the bet)} = 0.3\text{u}(\$100\ 000) + 0.7\text{u}(\$0),$$

and hence, the individual will only take the bet under the following condition

$$EU(\text{taking the bet}) \succ u(\$20000). \tag{2.29}$$

2.12. Risk aversion

An important concept in Expected utility theory is an individuals attitude towards risk. Recall the introductory bet with a 50%–50% chance for payoffs of \$100 or \$0 dollars. Assume that there is a certainty equivalent such that an individual is indifferent to the certain payoff or the gamble with regards to expected utility. If the certainty equivalent is lower (or equal) than the expected value of \$50, the individual is said to be risk averse. A concave utility function captures risk aversion of the individual. A person with the same certainty equivalent is said to be risk neutral, whereas a person with a higher certainty equivalent is said to be risk loving.

A concave function is used to represent a risk averse individual utility. The higher the curvature on the function $u(x)$, the more risk averse the individual. On the other hand, a risk loving individual's utility is represented by a convex function. A risk neutral will have a linear function. See Figure 2.7 for an illustration of the utility functions of a risk averse, risk neutral, and risk loving individual.

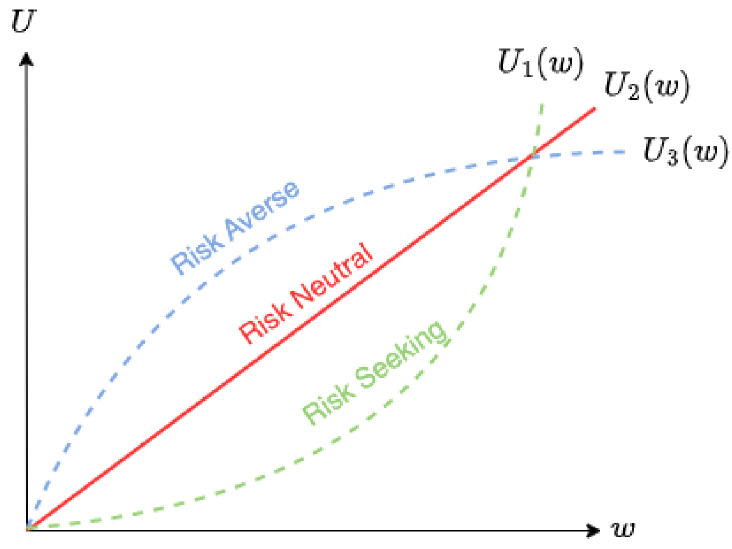

Figure 2.7.: Illustrations of the curvature of utility function for 1) Risk seeking investor (U_1 highlighted in green). 2) Risk neutral investor (U_2 highlighted in red), 2) Risk averse investor (U_3 highlighted in blue). Most individuals exhibit risk averse behavior, hence most portfolio theory is developed under the assumption of risk aversion.

The expected utility framework will be used in this thesis to examine the utility of different portfolio allocations. It is generally understood that risk-loving investors prefer portfolios dominated by risky assets, and they may exclude risk-free assets entirely due to their preference for higher returns. However, for the purpose of this thesis, and reflecting the reality that many investors exhibit risk-averse behavior, the following sections will primarily focus on the portfolio allocation strategies for risk-averse individuals.

2.13. Utility functions

The logarithmic function is merely one example of a concave function that could be used to represent risk aversion. There are numerous other utility functions that serve the same purpose. The second derivative of a function can be used to evaluate its curvature; if the second derivative is negative for all values, then the function is concave. However, the second derivative alone is not a sufficient measure of risk aversion. A measure that remains invariant to linear (affine) transformations, which scale, shift, skew, or rotate an object, is more robust.

Pratt (1964) and Arrow (1971) proposed a measure, fittingly named Arrow-Pratt measure of absolute risk aversion (ARA) that would remain consistent after an affine transformation of the utility function:

$$R_A(w) = -\frac{u''(w)}{u'(w)}, \tag{2.30}$$

where $u'(w)$ and $u''(w)$ are the first and second derivatives with respect to w. As an example, consider the logarithmic function with a shift, then $u(w) = \alpha + \beta \ln(w)$:

$$u'(w) = \frac{\beta}{w}, \quad u''(w) = -\frac{1}{w^2} \Rightarrow R_A(w) = \frac{1}{w} \tag{2.31}$$

As the second derivative is negative, the sign of the function changes. Therefore, a larger ARA value signifies a higher level of risk aversion. Notice how the Arrow-Pratt ARA measure is inversely proportional to wealth, i.e., lower wealth corresponds to greater risk aversion.

It is reasonable to assume that the wealth level of the individual affects the risk aversion, hence Pratt (1964) and Arrow (1971) further proposed the measure Relative risk aversion (RRA):

$$R_R(w) = -w\frac{u''(w)}{u'(w)}. \tag{2.32}$$

The RRA is simply ARA multiplied by the wealth level.

2.14. Constant Absolute Risk Aversion (CARA)

Individuals with CARA utility exhibit the same level of risk aversion regardless of their wealth. For example, consider a gamble with a 50% chance of either gaining \$200 or losing \$100. An individual with CARA utility who rejects this bet at low wealth levels would consistently reject the bet no matter their wealth level, even if they were to gain a billion dollars. One notable utility function that exhibits constant absolute risk aversion is the negative exponential utility function:

$$u(w) = -e^{-\eta w}, \quad u'(w) = \eta e^{-\eta w}, \quad u''(w) = -\eta^2 e^{-\eta w} \tag{2.33}$$

where $u(w)$ represents the utility of an individual with wealth w and a positive risk aversion parameter η. Then $R_A = \eta$, which remains constant for all wealth levels.

2.15. Constant Relative Risk Aversion (CRRA)

Returning to the 50% bet example with a $200 gain and $100 loss, an individual with CARA utility consistently refuses this bet, irrespective of their wealth. However, under the CRRA model, this person might accept the bet if their wealth were greater. This is because their risk aversion is based on the proportion of wealth at stake, not the absolute amount. The only utility function exhibiting CRRA is the isoelastic function,

$$u(w) = \begin{cases} \frac{1}{1-\eta} w^{1-\eta} & \text{if } \eta > 0, \eta \neq 1 \\ \ln w & \text{if } \eta = 1, \end{cases}$$

where $u(w)$ represents the individual's utility given a wealth of w, and η is the risk aversion parameter. A higher η indicates a more risk-averse individual. The marginal utility is always positive and diminishing:

$$u'(w) = w^{-\eta}, \qquad u''(w) = -\eta w^{-\eta-1} < 0. \tag{2.34}$$

The isoelastic function exhibits constant relative risk aversion (CRRA),

$$R_R(w) = -\eta. \tag{2.35}$$

2.16. Decreasing Absolute Risk Aversion (DARA)

An example of this utility function is the one used by Bernoulli in the origins of expected utility theory, i.e., the logarithmic function,

$$u(w) = \ln(w), \quad u'(w) = -\frac{1}{w}, \quad u''(w) = -\frac{1}{w^2}. \tag{2.36}$$

The ARA value is thus $R_A = \frac{1}{w}$, which implies that the risk aversion decreases when the wealth increases. Returning to the 50%-50% bet with $200 gain and $100 loss: an individual with a logarithmic utility function would evaluate the bet in terms of the proportional change in their wealth, rather than the absolute change. As the wealth grows, the individual becomes less concerned about the absolute loss of $100. Table 2.2 presents a collection of utility functions and their properties. Increasing risk aversion implies that the risk aversion of an individual would increase with wealth.

Table 2.2.: Examples of utility functions with their exhibited properties. Note that the quadratic utility varies depending on the parameters, however, it typically exhibit decreasing absolute risk aversion.

Utility Function	**Expression**	**ARA**	**ARA**	**RRA**	**RRA**
Logarithmic	$u(w) = \ln w$	$1/w$	Decreasing	1	Constant
Exponential (CARA)	$u(w) = -e^{-\eta w}$	η	Constant	η/w	Decreasing
Isoelastic (CRRA)	$u(w) = \frac{1}{1-\eta} w^{1-\eta}$	η/w	Decreasing	η	Constant
Quadratic	$u(w) = aw - \frac{1}{2}bw^2$	b/w	Decreasing	Varies	Varies
Square Root	$u(w) = \sqrt{w}$	$1/(4w)$	Decreasing	1/2	Constant
Inverse Power	$u(w) = -1/w^\eta$	η/w	Decreasing	η	Increasing

Chapter 3.

Time Diversification

The debate around time diversification is about the relationship between investment risk and investment horizon. That is, does the length of the investment horizon affect the risk of an investment? Beyond the premise itself, the debaters do not agree on much. Despite a diversity of different schools of thought, theoretical frameworks, and quantitative methods, there is no unified conclusion with collective endorsement.

The thesis will give a comprehensive review over the existing literature and respective methods used to examine the time-diversification controversy, and why it remains a puzzle. The first section introduces the mean-variance utility framework and how it predicts, counter-intuitively, that the allocation towards risky assets declines exponentially with the investment horizon.

3.1. Practitioner's view

Before going into the economic theories, it is fitting to start with what Thorley (1995) denotes as the practitioners risk measure. The starting point is to evaluate the effect of investment horizon on expected return and risk is to compare two portfolios. Assume an initial wealth W_0. After t time periods with a risk-free rate of r_f, the value of a portfolio with full allocation to the risk-free asset will be:

$$W_t^f = W_0 e^{(r_f t)} \tag{3.1}$$

Further assuming a normally distributed and continuous expected return with a mean of $E[R_m]$ and standard deviation of σ, then the value of the risky portfolio W_t^r after t time periods is:

$$W_t^r = W_0 e^{(E[r_m]t+\sigma\sqrt{t}z)} \tag{3.2}$$

where z is a random variable following the standard normal distribution. Rearrange and apply the natural logarithm:

$$\ln\left[\frac{E[W_t^r]}{W_0}\right] = \left(E[r_m]t + \sigma\sqrt{t}z\right). \tag{3.3}$$

Then, one can infer that the probability of the risky portfolio under-performing the risk-free portfolio at period t is:

$$\text{Prob}(W_t^r < W_t^f) = 1 - \phi\left(\frac{E[r_m] - r_f}{\sigma}\sqrt{t}\right), \tag{3.4}$$

where ϕ denotes the standard normal cumulative density function (CDF), $E[r_m]$ is the expected return of the risky asset, r_f is the return of the risk-free asset, σ is the standard deviation of the returns of the risky asset, and t is the investment horizon. The fraction inside the parenthesis is proportional with the square root of the time horizon. Thus, as the investment horizon goes to infinity, the value of the standard normal CDF will approach 1, and the probability of shortfall approach zero:

$$\begin{aligned}\lim_{t\to\infty} \text{Prob}(W_t^r < W_t^f) &= \lim_{t\to\infty}\left(1 - \phi\left(\frac{E[r_m] - r_f}{\sigma}\sqrt{t}\right)\right)\\ &= 1 - \lim_{t\to\infty}\phi\left(\frac{E[r_m] - r_f}{\sigma}\sqrt{t}\right)\\ &= 1 - 1\\ &= 0\%.\end{aligned} \tag{3.5}$$

Hence, when the investment horizon increases, the shortfall risk approaches zero. Furthermore, the lowest value the fraction inside the parenthesis can take is zero (either from no market premium or time period zero, both are not relevant cases). Thus, as the investment horizon goes to zero, the value of the standard normal CDF will approach 0, and the probability of shortfall approach 50%:

$$
\begin{aligned}
\lim_{t\to 0} \text{Prob}(W_t^r < W_t^f) &= \lim_{t\to 0}\left(1-\phi\left(\frac{E[r_m]-r_f}{\sigma}\sqrt{t}\right)\right) \\
&= 1 - \lim_{t\to 0}\phi 0 \\
&= 1-\frac{1}{2} \\
&= 50\%.
\end{aligned} \tag{3.6}
$$

Effectively, the domain of shortfall risk is reduced between $[0.5, 0]$[1]. To evaluate the monetarily effect the investment horizon has in this context, consider an individual with a fortune of \$1 million planning to save until retirement. Suppose the individual's only two alternatives are either a) market portfolio with expected returns of 12% with a 20% volatility, or b) a 5% risk-free rate in 10-year T-Bills. The utility or risk attitude of the individual is not taken into account, and the money is locked during the investment period. Table 3.1 illustrates the expected return of the strategy, and the shortfall risk with varying investment horizons.

With an investment horizon of only one year, the risky portfolio is, not surprisingly, expected to outperform the risk-free portfolio. However, there is still a 36.2% chance of shortfall, where the expected value of those portfolios being \$ 0.9 million. For a five-year investment horizon, the expected value of the risky portfolio is \$2 million, whereas the risk free is \$1.3 million, with am expected shortfall risk of 21.6%. This is still a considerable shortfall risk after five years, however, the mean of the underperforming portfolio is \$1 million, hence, no expected loss of the initial investment.

For an investor with a 40-year investment horizon, the expected value of the risky portfolio is \$ 272.6 million, whereas the risk-free is \$7.4 million, with a shortfall risk of 1.3%. For a fifty-year investment horizon, the expected value of the risky portfolio is a hefty $\sim$ \$1.1 billion, whereas the risk-free is \$12.2 million, a 90-fold difference. The lowest 10% of the risky portfolios has an expected value of \$65.1, whereas the top 10% has an expected value of \$ 2.46 billion. Furthermore, the shortfall risk is reduced to 0.7%, with the mean of those underperforming portfolios being \$8.3.

[1]Note that if shortfall risk were to be over 50%, this would require the market's *expected return to be lower than the risk-free rate.* In such a scenario, investing in risky assets would be nonsensical.

Table 3.1.: The table showcases the growth of a $1 million portfolio invested in either a risk-free asset with a 5% return or a risky portfolio with an expected return of 12% and 20% volatility. Furthermore, the table presents expected shortfall risk and the expected value of those underperforming portfolios. It demonstrates shortfall risk decreases with investment horizon length, and even after 5 years, the underperforming portfolios incurs no expected loss. For a 50-year investment horizon, the difference between a risky portfolio and a risk-free portfolio is an expected 90-fold difference. Furthermore, the shortfall risk is 0.7%, with the expected value of those portfolio at $8.3 million, whereas the risk-free portfolio returns $12.2 million.

		Risky portfolio value ($1m)			Underperforming	
Horizon	Risk-free portfolio value ($1m)	Mean	10th Percentile	90th Percentile	Shortfall	Mean ($1m)
1	1.1	1.2	0.9	1.5	36.2%	0.9
5	1.3	2.0	1.0	3.2	21.6%	1.0
10	1.6	4.1	1.5	7.5	13.3%	1.2
20	2.7	16.5	3.5	34.7	5.8%	2.0
30	4.5	66.8	9.0	149.5	2.8%	3.1
40	7.4	272.6	24.2	619.8	1.3%	5.1
50	12.2	1097.7	65.1	2460.7	0.7%	8.3

Figure 3.1 illustrates how the expected shortfall risk varies by investment horizon. The trend that shortfall risk decreases

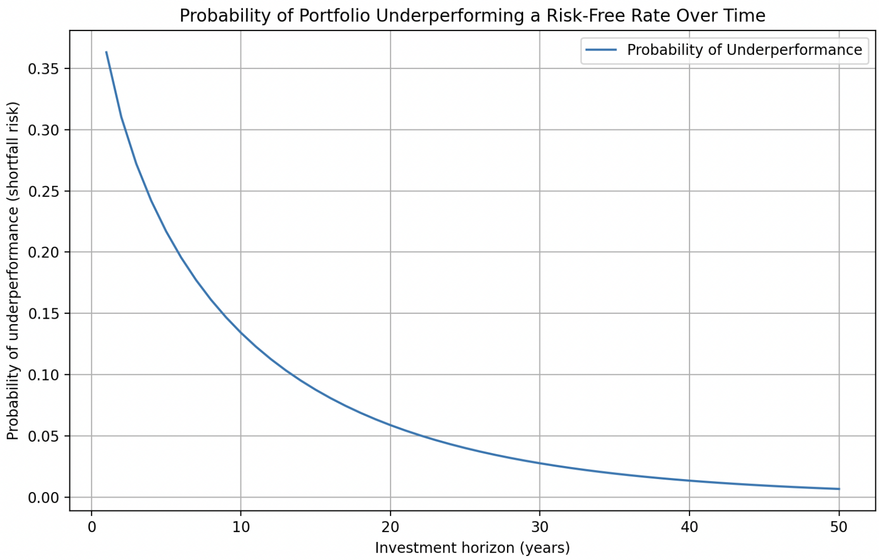

Figure 3.1.: The shortfall risk over varying investment horizons for a risky portfolio with normally distributed expected returns of 12% with 20% volatility and risk-free rate of 5% as benchmark. It is clear that the shortfall risk decreases as the investment horizon increases.

The example highlights the traditional way of thinking when it comes to time-diversification. The results show two clear trends on how the length of the investment horizons affect the expected return and shortfall risk. Firstly, the longer investment horizon, the lower the risk of both incurring a loss on initial investment and shortfall. Even though the standard deviation of the risky portfolio is high, the chance of incurring a loss on the portfolio is virtually zero, hence one could claim, that time diversifies risk. Secondly, the expected value of the risky portfolio and the value of the risk-free portfolio diverges as the investment horizon increases. Specifically, for a 50-year horizon, the top 10% of risky portfolios show an expected return 90 times greater than the risk-free alternative, while the bottom 10% still show a five-fold increase.

How come then, prominent academics disregard the notion of time-diversification? Putting aside the unrealistic assumptions of constant return rates, normally distributed returns, lack of taxes, etc,. The reason is the exclusion of the investors risk attitude and expected utility of these choices. The model operates under the assumption of fixed investor behavior, not accounting for the expected utility of investors throughout the period. Benartzi and Thaler (1995b) show investors exhibit some degree risk averse behavior and the point where most academics argue about time diversification is what the optimal mix of assets within a portfolio when maximizing the investors utility.

Rather than evaluating shortfall risk and expected returns alone, the risk attitude and expected utility of the investors needs to be incorporated.

3.2. Mean Variance Utility Optimization

Consider a utility function that increases with the expected return of the portfolio $E[r_p]$, but decreases with the variance porportional to the investors risk aversion quantified as A.

$$U = E[r_p] - \frac{1}{2}A\sigma^2(r_p) \tag{3.7}$$

The investor has two investment possibilities a risk-free asset r_f or a risky asset denoted as r_m. If the investor invest α in the risky asset, then the combined return will be the weighted average:

$$E[r_p] = \alpha E[r_m] + (1-\alpha)r_f.$$

Then the utility can be expressed as

$$U = \alpha E[r_m] + (1-\alpha)r_f - \frac{1}{2}A\sigma^2(\alpha E[r_m]), \tag{3.8}$$

rearranged to

$$U = \alpha E[r_m] + (1-\alpha)r_f - \frac{1}{2}A\alpha^2\sigma^2(r_m). \tag{3.9}$$

To maximize the expected utility, the derivative is found:

$$\frac{dU}{d\alpha} = E[r_m] - r_f - \alpha A\sigma^2(r_m) \tag{3.10}$$

Solving for α finds the optimal allocation in the risky fund to maximize utility:

$$\alpha^* = \frac{E[r_m] - r_f}{A\,\sigma^2(r_m)}. \tag{3.11}$$

The expression is inversely proportional with the investors risk aversion, i.e., the higher the risk aversion, the lower the fraction of wealth is to be invested in the risky asset. Furthermore, the fraction of wealth is proportional with the market premium, and inversely proportional to the variance. However, how does the investment horizon affect the optimal allocation to risky asset, based on this setup?

As a rule of thumb, some researches approximate that the expected mean and variance of stochastic risky returns increase proportionally with the investment horizon (Thorley, 1995). Then, by ignoring any compounding effects, after two periods, the fraction will be reduced by 25%. Assuming the rule of thumb holds true, the optimal fraction will decrease exponentially when the investment horizon increases. However, by assuming a log-normal distribution, the effect time horizon has on the optimal fraction α could be modeled directly. Denote the mean and variance of the risky asset over time as $E[r_m]_t$ and $Var(r_m)_t$, respectively. Assuming log-normality, the mean and variance over time can be modeled as:

$$E[r_m]_t = e^{(\mu+\frac{1}{2}\sigma^2)t}, \tag{3.12}$$

$$V_t = e^{(2\mu+\sigma^2)t}\left(e^{\sigma^2 t} - 1\right). \tag{3.13}$$

Then, by substituting into Eq. 3.11, the time-dependent optimal fraction is found:

$$\alpha^* = \frac{e^{(E[r_m]+\frac{1}{2}\sigma^2)t} - e^{r_f t}}{Ae^{(2E[r_m]+\sigma^2)t}\left(e^{\sigma^2 t} - 1\right)}. \tag{3.14}$$

At time zero, the optimal fraction of wealth that should be allocated to risky assets is

$$\lim_{t\to 0} \alpha^* = \frac{\mu + \frac{1}{2}\sigma^2 - r_f}{A\sigma^2}. \tag{3.15}$$

By examining the expression's value when the investment horizon approaches infinity, the optimal fraction of wealth allocated to risky asset in order to maximize the expected utility is zero.

$$\lim_{t\to\infty} \alpha^* = 0 \tag{3.16}$$

The optimal fraction of wealth allocated to risky assets declines monotonically from (see Appendix A for proof):

$$\lim_{t\to 0} \alpha^* = \frac{\mu + \frac{1}{2}\sigma^2 - r_f}{A\sigma^2} \quad \text{to} \quad \lim_{t\to\infty} \alpha^* = 0. \tag{3.17}$$

Hence, the mean-variance utility framework predicts that investors with longer investment horizons will invest less to risky assets, and at very long horizons, none. For a numerical example, assume the risky asset has an expected return of 12% with 20% volatility, whereas the risk-free asset returns 5%. Figure 3.2 illustrates how the optimal allocation to risky assets decreases exponentially with the investment horizon length.

Mean-variance utility results are deemed unreasonable when considering variable investment horizons. As noted by Thorley (1995), the framework, while effective for fixed time horizons, fails to produce reasonable outcomes over varying horizons. For instance, it predicts that an investor with a 40-year horizon and a risk aversion coefficient of $A = 1$ would allocate zero percent of their portfolio to risky assets, which is an unrealistic outcome.

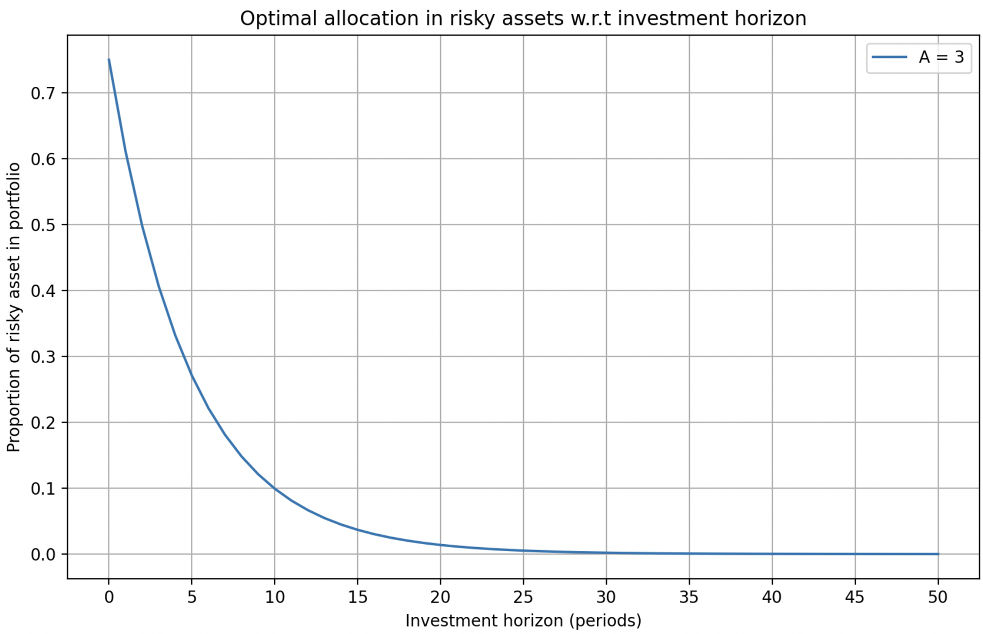

Figure 3.2.: The optimal allocation allocated to risky assets assuming an expected market return of 12% with 20% volatility and risk-free rate of 5%. The risk aversion coefficient A is set to 3.

3.3. Expected Utility Framework

Time diversification was first examined in an expected utility framework in Samuelson (1969). In his paper, Samuelson showed in a mathematical framework that the allocation of a portfolio towards risky assets maximizing expected utility is fixed over the portfolio life and is only determined by the investor's inherent risk tolerance. Samuelson argues that the allocation of risky assets is equal for the investors prime and nearing the end of his life, and that the “chance of recouping“ losses is not relevant. In Samuelson (1969) the following theorem is formulated:

> “The optimal portfolio decision is independent of wealth at each stage and independent of all consumption-saving decisions, leading to a constant w^*.”

3.3.1. Method

The proof is based on the following procedure: Let an individual with utility function U with consumption $C(t)$ as input, and risk aversion coefficient as ρ maximize the expected utility over his or hers lifetime

$$\int_0^T e^{-\rho t} U[C(t)]dt \tag{3.18}$$

subject to initial wealth W_0 which can be invested with an exogenously given rate of return; or subject to the constraint

$$C(t) = r_f W(t) - \dot{W}(t) \tag{3.19}$$

i.e., the accumulation of wealth subtracted the consumption of wealth at each year. By assuming no bequest at death (i.e., terminal wealth $W_T = 0$) and inserting Eq. 3.19 into Eq. 3.18

$$J = \underset{\{W(t)\}}{\text{Max}} \int_0^T e^{-\rho t} U[r_f W(t) - \dot{W}(t)]dt \tag{3.20}$$

For simplicity, the problem is converted to discrete time optimization problem:

$$\underset{\{W(t)\}}{\text{Max}} \sum_{t=0}^{T} (1+\rho)^{-t} U[C_t] \tag{3.21}$$

Subject to:

$$C_t = W_t - \frac{W_{t+1}}{1+r_f} \tag{3.22}$$

or alternatively

$$\underset{\{W(t)\}}{\text{Max}} \sum_{t=0}^{T} (1+\rho)^{-t} U\left[W_t - \frac{W_{t+1}}{1+r_f}\right] \tag{3.23}$$

Introducing a risky asset

Up until this point the only asset to invest in has been a risk free asset with return r_f. Consider a stochastic asset in which one invest \$1 and after one period t get a return of $\$Z_t$, where Z_t is a random variable following the probability distribution

$$Prob\{Z_t \leq z\} = P(z), \qquad z \geq 0 \tag{3.24}$$

and thus $Z_{t+1} - 1$ is the percentage yield of every outcome. Further assume independence over the yields at different times such that

$$P(z_0, z_1, \ldots, z_T) = P(z_0)P(z_1)\ldots P(z_T) \tag{3.25}$$

The expected value of the risky assets must exceed the expected value of the safe asset.

The optimization problem

At each period, what is the optimal fraction w_t invested in the risky asset, and consequently $1 - w_t$ in the risk-free asset? With the newly introduced risky asset Eq. 3.22 can be rewritten to

$$C_t = W_t - \frac{W_{t+1}}{(1 - w_t)(1 + r_f) + w_t Z_t} \tag{3.26}$$

Which yields the stochastic counterpart to Eq. 3.21

$$\underset{\{C_t, W_t\}}{\text{Max}} \sum_{t=0}^{T} (1 + \rho)^{-t} U[C_t] \tag{3.27}$$

subject to

$$C_t = W_t - \frac{W_{t+1}}{(1 - w_t)(1 + r_f) + w_t Z_t} \tag{3.28}$$

where W_0 is given, and $W_{T+1} = 0$ by assuming no bequest at death.

Solution of the problem

The problem is solved recursively by focusing on the last period $T - 1$, and then inserting the result into period $T - 1$ all the way to the initial starting point. By following the recursive procedure, Samuelson show the optimal decision rules for consumption-saving and portfolio selection as:

$$C_t^* = f[W_t; Z_{t+1}, \ldots, Z_0] = f_{T-t}[W_T] \quad \text{if the Z's are independently distributed.} \tag{3.29}$$

and

$$w_t^* = g[W_t; Z_{t+1}, \ldots, Z_0] = g_{T-t}[W_T] \quad \text{if the Z's are independently distributed.} \tag{3.30}$$

Which is not dependent on time! In other words, Samuelson argues that in order to maximize an investors expected utility, the proportion invested in risky assets should remain constant irrespective of investment horizon.

3.3.2. Criticism

The expected utility stream tend to observe risk indirectly. The proponents of this stream does not define a risk in any particular way (e.g., standard deviation, VaR) and examine it over various investment horizons. Instead, scholars within the expected utility stream make several assumptions about the behavior, preferences, and risk attitude of an investor, and then, adhering to the aforementioned assumptions solves for the optimal mix of assets. This method allows the researches to infer how the risk changes as the investment horizon is altered. Consequently, the scholars within the expected utility stream assign the risk aversion specification of the investor by selecting a utility function. This approach is said to be normative, because it generalizes how the investor perceive risk, and then observe the relationship of interest.

The normativity of the expected utility framework, including the sensitivity to the selection of utility function represent the key weakness. The very existence of behavioral economics as a field contend to the idea that the populations of investors are highly heterogeneous and the assumptions in classical economics is seldom realistic for everyone. The critique of the normative approach has motivated both the option pricing and behavioral stream in the literature.

Samuelson's findings of constant allocation towards risky assets irrespective of investment horizon rely on three core assumptions:

1. The investor exhibits constant relative risk aversion (CRRA), which means that they maintain the same percentage exposure to risky assets regardless of changes in their wealth.
2. The investment returns are independent and identically distributed, which means they follow a random walk.
3. The investor's wealth is solely determined by the returns on the portfolio.

However, Merton (1969), a page later [2] generalized Samuelson's findings in continuous time, but tested the results against the alternative specification of risk aversion by constant absolute risk aversion (CARA). The conclusion was that rather than the percentage of the portfolio remains constant in risky asset's irrespective of investment horizon, the absolute dollar wealth does. The implication is that the fraction of wealth invested in risky assets decrease as the wealth level increase. Merton (1969) does point out that the utility function is not plausible, however, it highlighted how sensitive the framework is to the framework's assumptions.

Kritzman and Rich (1998b) work extended Samuelson's work by examining the differences in outcome when assumption one and two changes. Five utility specifications are employed, each with returns modeled as a random walk, mean reversion process, or mean aversion. Table 3.3 illustrates just how sensitive the expected utility framework is to the Samuelson's first two assumptions. The only utility function that leads to the result that the optimal mix of assets should be constant is the logarithmic function. For the power and square root utility, the optimal allocation is predicted should either increase, decrease, or stay constant based on how the returns are represented. It is interesting to note that only three combinations predicts that the allocation to risky assets should increase as the investment horizon lengthens. Effectively, by altering the assumptions, any desired result could be achieved with the expected utility framework.

[2] Merton (1969) was, in fact, in the same edition of the Review of Economics and Statistics as Samuelson (1969), just a page later. Credits to Bianchi et al. (2016) for pointing it out.

Figure 3.3.: By utilizing different utility function and assumptions about the risky assets return, one can achieve all the different results with the expected utility framework. The table shows how the allocation to risky assets varies with different utility functions, risk aversion, and the asset return process. Table is taken from Kritzman and Rich (1998b)

			Impact of Time on Allocation to Risky Assets		
Utility Function	**Absolute risk aversion**	**Relative Risk Aversion**	**Random Walk**	**Mean Reversion**	**Mean Aversion**
Logarithmic Utility U = log(W)	Decreasing	Constant	Hold Constant	Hold Constant	Hold Constant
Square root Utility U = W$^{1/2}$	Decreasing	Constant	Hold Constant	Increase	Decrease
Power Utility U = -1/ W	Decreasing	Constant	Hold Constant	Increase	Decrease
Quadratic Utility U = 25W+0.1W^2	Decreasing	Increasing	Decrease	Decrease	Decrease
Combination Utility U = 1/W + log(W)	Decreasing	Decreasing	Increase	Decrease	Increase

Further, as pointed out in Bianchi et al. (2016), even the most prominent scholars within the expected utility theory school concede that their general findings that disprove of time diversification may not hold with alternative utility specifications (Samuelson, 1969, 1971, 1989, 1990, 1994; Kritzman, 1994, 2002)

Moreover, Samuelson (1989) admits that if the investor is anxious not to fall below a subsistence level of terminal wealth, such that it is the utility of terminal wealth minus the subsistence level, it is correct that the investor will reallocate to a lower fraction of his or her wealth to risky assets when nearing the retirement age. Milevsky (1999) extends this reasoning by applying the framework with a discontinuous utility function, which supports Samuelson (1969) findings of constant allocation irrespective of investment horizon.

Moreover, the results from the expected utility framework is not merely sensitive to the utility specification, but also the modeling of returns. Unfortunately, the views on which process asset's returns follow are also mixed. Several authors, Lo (1999); Fama and French (1988, 1989), contrary to Samuelson (1969) random walk assumption, argues that the serial correlation between assets are not zero in the short term. Shi (2000) argues that this caused the irrational exuberance of the the dot-com boom. Negative serial correlations is equal to mean reversion, and positive serial correlations is equal to mean aversion.

Hence, it is clear that the time diversification controversy *within* the expected utility framework is more of a controversy about the assumptions made of the investor and return processes, than the fact whether time diversifies risk. Even though the framework is prevalent and long-lived, there are various reasons why it does not solve the puzzle. Bianchi et al. (2016) points out three main reasons: 1) there is no consensus regarding a generalization for the typical investor and the most fitting utility specification, 2) The findings are not robust to various levels of risk tolerance and different utility specifications. Any result from the framework could be countered with other assumptions, 3) the normative approach of selecting a utility function and then analyzing the relationship between risk and time.

3.4. Option Pricing Framework

Bodie (1991) was the first to diverge from the expected utility framework and set out to 'investigate the implications of option pricing theory for investment policy of defined benefit pension plans'. The implications are found by considering a defined benefit pension plan, where the sponsor guarantees benefit to plan participants.

The paper defines shortfall risk as the probability that the returns of an investment portfolio is less than the risk-free rate in the same period. Then, Bodie (1991) equates risk with the cost of insuring against said shortfall. By implicitly equating risk with the cost of shortfall insurance, Bodie (1991) argues that the allocation to risky asset's should decrease with investment horizon, or in his own words:

> "If the objective of pension asset management is to minimize the cost of providing guaranteed benefits, then the longer the time horizon, the *lower* the proportion of assets that should be invested in stocks."

3.4.1. Method

Consider a defined benefit pension plan, where the sponsor, hereafter referred to as pension fund, guarantees benefits to plan participants. If the plan is over funded, the exceeding amount goes to the participants, whereas if the plan is under-funded, the pension fund absorbs the down-side risk. To ensure that the pension fund is a position to pay out all the benefits, Bodie highlights two strategies: a) immunization: to invest in risk-free bonds to match the liabilities and assets of the fund; or b) to invest in equities. Two evaluate the strategies, consider an example in which a fund has an obligation to pay \$100 N years from now. Let the continuously compounded risk-free rate $r_f = 8\%$, and the annual rate of return in risky assets be 16% with a standard deviation of 20%.

Alternative a) - Immunization

The fund can immunize the liability by investing in a zero coupon bond maturing in N years. Let $N = 20$, then the cost of a immunization would be equal to the net present value of such a bond, i.e.,

$$\text{Cost} = \$100e^{-r_f * n} = \$100 * e^{-0.08*20} = \$20.18 \tag{3.31}$$

The result would be that the pension plan is fully funded, and no additional funds will be required to cover any under-funding (nor will there be a surplus).

Alternative b) - Invest in risky assets

The other strategy is that the pension fund invests in risky assets instead, e.g., equity. The pension fund's asset would be the net present value of the liabilities, \$20.18. If the fund is exposed to shortfall risk, it would need to cover the difference between the terminal value and the liability of \$100, hence a guard against shortfall risk is needed. The fund could invest in a put option with an exercises price of \$100 expiring in N years (terminal date), ensuring the liability is fully funded. However, the cost of such a put will vary based on the value of N, i.e., investment horizon.

Bodie (1995) shows that when insuring against the risk-free rate, the Black-Scholes formula simplifies to:

$$P/S = N(d_1) - N(d_2), \tag{3.32}$$

$$d_1 = \frac{\sigma\sqrt{T}}{2}, \tag{3.33}$$

$$d_2 = -\frac{\sigma\sqrt{T}}{2}. \tag{3.34}$$

Table 3.2 demonstrates how the cost increases with investment horizon. For pensions due in one year, the cost is $\sim$ 8% of the pension's present value, whereas for pensions due in 10 years, it's 25%. For a very long investment horizon, i.e., 50-year, the cost of protecting against shortfall risk exceeds half of the benefit's current value. Bodie (1991) asserts that because the cost of the put option, alternatively the insurance against shortfall, increases with investment horizon, *risk* must also increase with time, reaching the conclusion that allocation to risky assets should decrease with time.

Table 3.2.: The cost of insuring against shortfall risk as a function of maturity. The table is taken from Bodie (1991) and s derived using the Black-Scholes option pricing formula with $\sigma = 0.2$. The option is exactly at the money, such that the present value of the exercise price equals the current market value of the underlying portfolio.

N (Years)	Value of a put option as a percentage of present value of benefits (%)
1	7.98
5	17.72
10	24.84
20	34.54
30	41.63
50	52.08

Bodie (1995) expanded further on the option pricing stream by testing the findings on two asset return models: the random walk assumption and mean reversion, and reaches the same conclusion as Bodie (1991) that the optimal allocation to risky stocks decreases with investment horizon, strengthening the framework. Figure 3.4 illustrates how how the cost of shortfall insurance monotonically increases over very long investment horizons.

To summarize, Bodie (1991, 1995) findings diverge from from Samuelson (1969) – arguing that allocation should be constant irrespective of investment horizon – and is in direct opposition of 'conventional wizdom' – arguing that the allocation towards risky asset should increase with time, highlighting why it is dubbed the time diversification *controversy*.

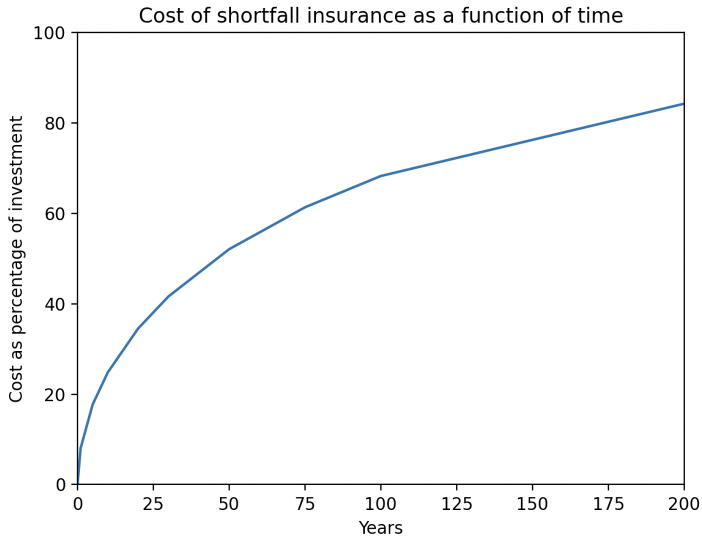

Figure 3.4.: The cost of insuring a portfolio against shortfall as a function of time. The table was derived using a simplified Black-Scholes formula with $\sigma = 20\%$ a year. The insurance cost is independent of risk-free rate. The figure and results are adopted from Bodie (1995).

3.4.2. Criticism

Merrill and Thorley (1996) argues that option pricing theory has an advantage over expected utility theory because the results derived are 'independent of any specific model of investor utility or risk aversion'. The authors attempts to examine how other strategies within option pricing framework reveals about time diversification. To do this Merrill and Thorley (1996) considers Protected Equity Notes (PEN) and Collars. A PEN is composed of a risk-free asset and a call option. The risk-free asset ensures a minimum guaranteed return, and the call option allows the investor to participate in any potential equity market returns above the guaranteed rate. Merrill and Thorley (1996) show that the proportion needed to invest in the risk-free asset to guarantee a certain return decreases with time due to the compounding effect, hence the higher proportion could be allocated to the call option, i.e., the risky asset. For the Collar strategy, Merrill and Thorley (1996) demonstrates that the cap rate (the maximum rate of return) incrases with investment horizon, for any floor less than the risk-free rate. Both of these strategy implies that their measurement of risk reduces over time, hence time diversifies risk. In the authors own words

> "Longer time horizons reduce the cost of risk elimination and, by implication, risk itself. Specifically, the cost of insuring a minimum rate

> of return on an equity investment is shown to decrease with the investment horizon."

Again, as with the expected utility framework, there are disagreements within the same school of though.

Furthermore Kritzman and Rich (1998a) argues that the conclusion of Bodie (1995) is misleading. The argument is that option valuation does not reveal that stocks become riskier with time, rather it is the increase in the stock's cumulative volatility that raise the value of an option. Then, by equating risk with the cost of insuring against shortfall risk, the result is tautological. The option value increases because as the horizon lengthens, the risky asset's return dispersion causes the option value to increase with time. Table 3.3 demonstrates this fact. Let the risk-free rate be 5%. Then, consider an asset at the price of \$100. A five-year put option with strike price of $\$128.40 = 100e^{0.05*5}$ with an annual volatility of 20% and cumulative volatility of $44.72\% = 20\%\sqrt{5}$ is equal to a one-year put option with strike price at 105.13 written on an otherwise identical asset with annual standard deviation of 44.72%.

Table 3.3.: The Cost of Insurance as a Function of Time

Investment Horizon	Strike Price	Annual Deviation	Standard Deviation	Put Value
1 Year	\$105.13	20.00%	20.00%	\$7.97
5 Years	\$128.40	20.00%	44.72%	\$17.69
10 Years	\$164.87	20.00%	63.25%	\$24.82
20 Years	\$271.83	20.00%	89.44%	\$34.53
Investment Horizon	**Strike Price**	**Annual Deviation**	**Standard Deviation**	**Put Value**
1 Year	\$105.13	20.00%	20.00%	\$7.97
1 Year	\$105.13	44.72%	44.72%	\$17.69
1 Year	\$105.13	63.25%	63.25%	\$24.82
1 Year	\$105.13	89.44%	89.44%	\$34.53

Another criticism of the option pricing framework is the one-sided focus on risk, rather than the trade-off between reward and risk. Bodie (1991, 1995) does not take into account the good outcomes, as it merely equates the cost of insuring the minimum level of liabilities, putting no weight on any expected returns exceeding those of the risk-free rate. One could argue that the time-diversification controversy is about the relationship between risk and investment horizon alone. However, why would one invest in the stock market, if not for the higher expected returns? Regardless of the latter criticism, it is clear that the option pricing framework offer no resolution ot the time diversification controversy, with two opposite conclusions.

A final major criticism of the option pricing framework is that it assumes a constant standard deviation over time. Bianchi et al. (2016) argues that the argument fails when the constant standard deviation in the pricing model is replaced with a non-standard constant deviation.

3.5. Behavioral Economics

The streams of expected utility and option pricing have been characterized by frameworks with strong theoretical foundations and rigorous mathematical approaches. The stream of behavioral economics, however, does not share these traits. Instead, this stream focusing instead on how human psychology influences the findings within the time diversification controversy. As demonstrated, the results within the expected utility and option pricing framework are highly contingent on the assumptions made about the risk itself, and the hypothetical investor's behavior and risk attitude. It is therefore fitting to scrutinize the accuracy of these assumptions and examine how investors *actually* perceive risk. Consequently, while behavioral economics does not provide an alternative analytical framework, it illuminates the weaknesses of the existing solutions. The principal conclusion drawn from this stream suggests that the other frameworks fails to frame risk properly, and has a too narrow focus upon evaluating the time diversification controversy.

Olsen and Khaki (1998) argues that the lack of closure on time diversification results from the profession's inability to agree on a common definition of risk. They then expand this discussion by integrating concepts of risk and rationality from the perspective of behavioral economics. Given these considerations, they assert that time diversification is both theoretically and empirically consistent.

They dismiss the expected utility framework outright: 'the dismissal of time diversification on positive grounds cannot be justified by appeal to the traditional discounted SEU model.' To substantiate this claim, Olsen and Khaki (1998) underline that empirical evidence suggests investors don't evaluate probabilities and outcomes in a multiplicative manner, which is a premise of the expected utility framework. Instead, Olsen and Khaki (1998) find that investors perceive risk as a positive function of both the probability and size of loss, but place substantially more weight on the potential magnitude of the loss than the likelihood of it. As such, investors could be seen as loss-averse, rather than risk-averse. This behavior implies that loss-averse investors may be inclined to hold risky investments over longer periods.

Olsen and Khaki (1998) presents three main arguments in favor of time diversification. The first is made by highlighting various studies within behavioral economics that demonstrate investors' willingness to accept risk increases with the delay in realizing the outcomes—a principle consistent with time diversification. The second point they make is that investors discount potential future losses at higher rates than potential gains, indicating diminishing concern with risk as the investment horizon lengthens. The final argument highlights the common behavior of investors, such as de-emphasizing low-probability events, indicating a level of comfort with risk over time, and setting minimum return goals, implying a degree of risk tolerance compatible with time diversification. However, Olsen and Khaki (1998) emphasize the importance of not disregarding investors' tendency to underestimate 'low-probability negative outcomes,' warning that such oversight could lead to financial catastrophe.

Fisher and Statman (1999) in their paper 'A behavioral framework for time diversification' continue the critique of the expected utility and option theory streams. They incorporate insights from human behavior and cognitive biases to better understand investor decisions. While they concede the mathematical robustness of the expected utility framework, they argue that the assumptions that necessitates its findings, do not accurately reflect the reality of investor behavior and decision-making. Fisher and Statman (1999) challenges the three key assumptions of the expected utility framework: 1) the investor exhibit constant relative risk aversion, 2) stock returns follow a random walk, and 3) investor's wealth is solely determined by the returns of the portfolio. They challenge the first assumption, arguing that the utility of behavioral investors cannot be described as a function of wealth, and that investors are not consistently risk averse.[3] While the remaining two assumptions are not elaborated upon, the authors do state, 'Samuelson's mathematics are right, but his assumptions are wrong.'

Fisher and Statman (1999) continue their scrutinizing of the expected utility stream further by bringing up an unstated assumption of the framework—that investors correctly assess the probabilities of losses. Fisher and Statman (1999) bluntly states outright 'they do not'. they suggest that cognitive biases cause investors to treat small probabilities as equivalent to zero probabilities and cite the behavioral economics research by Tversky and Kahneman (1992), which argues that 'very small probabilities can be either greatly over-weighted or neglected altogether'. On the other hand, Fisher and Statman (1999) also points to what Benartzi and Thaler (1995a) dubbed 'myopic loss aversion', which proposes that the more frequently an investor evaluates their portfolio, the higher the perceived probability of observing a loss becomes, thus enhancing susceptibility to loss aversion. This effect leads investors to be fooled thinking the probability of losses is higher than it is.

[3]This observation motivated Daniel Kahneman and Amos Tversky to develop prospect theory, an alternative to expected utility theory. For a deeper dive into this theory, readers are directed to Kahneman and Tversky (2013)

Fisher and Statman (1999) goes on to detail the many cognitive biases and behavioral factors that influence the investor's decision-making processes. For instance, recency bias, where investors tend to incorrectly extrapolate recent events in the stock market. This is exemplified by an investor that initially planned for a 40-year horizon, only to experience loss the first three years, causing the investor to reconsider, and the horizon ended after three years. They further cite Clarke and Statman (1998), which finds that investors become optimistic after increases in stock prices and vice versa. Moreover, Fisher and Statman (1999) argue that investors struggling with self-control may use the stay-the-course rules of time diversification to stop themselves from cashing in. These rules reduce regret over paper losses as the possibility of future recovery remains.

Without stating it, Fisher and Statman (1999) indirectly shows a disinterest in whether the time-diversification debate can be resolved at all. The sum of the cognitive biases makes it difficult, if not impossible, to generalize investors' behavior, and as such, it needs to be evaluated on the basis of each investor. The classical view of the financial choices of investors is that it should be a trade-off between risk and return. However, as Fisher and Statman (1999) points out, the concept of risk is too narrow. Therefore, they propose a shift from this perspective, asserting that 'time diversification debate teaches us that the box of risk and expected returns, tidy as it seems, is inadequate for a description of financial choices. We should abandon that box and turn to the box of many choice factors'.

Fisher and Statman (1999) offers the following conclusion:

> The time diversification debate teaches us little about the relationship between risk and the investment horizon, but it teaches us much about the many factors that affect financial choices.

While the behavioral stream does argue that time diversification is consistent both theoretically and empirically, given the understanding of rationality and risk from a behavioral perspective, it does not resolve the debate. Instead of seeking a universal solution, it stresses the influence of cognitive biases and human psychological factors of investors influencing their decision making. As such, it can be inferred that there is no universal solution to the time diversification controversy, as it is dependent on each individual choice factors.

Chapter 4.

Data and Methodology

4.1. Problem Statement

The problem statement can be formulated as a two-part question: Is the empirical data consistent with time diversification, i.e., risk decreases with investment horizon? If so, what is the minimum investment horizon before investing in stocks?

The upcoming chapter will introduce the data and methodologies used to investigate the phenomenon of time diversification. The chapter will also discuss potential sources of error that might have influenced the obtained results. Python was employed for the calculations, and the code is accessable in my GitHub (https://github.com/henrikgruner/FinansMaster).

4.2. Data set

This thesis has a quantitative approach where the data material is fetched from multiple sources and gathered into a single source with data-processing in Python. The data needs to be a representative sample consisting of consisting of several market sectors, industries, and asset classes. Further, the time horizon needs to be extensive enough to be able to capture several life-time allocations (up until 40 years). Moreover, ideally, the asset classes should be denominated in the same currency to eliminate any currency risk.

These limitations restrict the domain of potential data sets severely. To be able to satisfy the aforementioned criteria, the S&P 500 is used as a proxy for market data. It is possible to find open-source data sets for both of these from 1928 to 2023, enabling the analysis of longer investment horizons. The S&P 500 data set is obtained from Yahoo Finance, which provides historical daily price data for the S&P 500 Index.

The S&P 500 Index is a widely recognized benchmark of U.S. stock market performance that includes 500 leading companies and captures approximately 80% coverage of available market capitalization. The data set includes the daily open and closing price, however, it does not include the reinvesting of dividends. To address the limitation of the S&P 500 data in capturing the impact of dividends, Shiller (2015) dividend data is used. The data set includes the 12-month rolling average of dividends. By combining the S&P 500 with the dividend data the total returns can be calculated to get the full picture of investing in risky assets.

A risk-free rate does not exists. All existing investments carry some kind of risk. Given the absence of a truly risk-free assets, the U.S government backed treasury bills (T-bills) are often used as a proxy. T-bills are considered nearly free of default risk because they are fully backed by the U.S. government. This thesis has used the T-bills 1M data from French as a proxy for the risk-free rate. This data set provides daily data on the yields of 1-month T-bills

All the figures are nominal and total returns assuming reinvested dividends. The frequency of the data is monthly. Using daily data would induce a computationally heavy load. Furthermore, as the dividend data is monthly, interpolation the dividend data to daily would introduce artificial noise in the data. However, monthly data from January, 1929 to April 2023 provides 1132 observations of returns.

4.3. Calculating returns

The monthly returns for both risky and risk-free assets were calculated by:

$$r_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}}, \tag{4.1}$$

Where r_t, P_t, and D_t represent the net return, price, and dividend at time t. D_t is zero for the risk-free asset. To calculate the cumulative returns (CR) over a specific period:

$$CR = \prod_{t=1}^{T}(1 + r_t) - 1 \tag{4.2}$$

To find the annualized returns (AR):

$$AR = (1 + CR)^{\frac{1}{T}} - 1 \tag{4.3}$$

4.4. Descriptive Statistics

Table 4.1 illustrates the descriptive statistics of S&P 500 with and without dividends reinvested, and T-bills 1M. The mean monthly return for the S&P 500 with reinvested dividends is 0.86%, whereas the minimum is -26.47% in one month. This was, not surprisingly, during the great depression from October to November 1929. On the other hand, the maximum return is 51.35%, from July to August. Moreover, the standard deviation of the returns are 4.48%, which aligns with expectations. For the T-bills 1M, the mean monthly returns are 0.27%, which converts to an annual rate of 3.3%, also aligning with expectations.

Table 4.1.: Descriptive Statistics over the S&P 500 returns and T-bills 1M from 1929 to 2023.

Variable	**Count**	**Mean**	**Std**	**Min**	**25%**	**50%**	**75%**	**Max**
S&P 500	1132	0.56%	4.48%	-26.47%	-1.34%	0.95%	2.93%	50.30%
Incl. dividends	1132	0.86%	4.49%	-26.19%	-1.08%	1.22%	3.21%	51.35 %
T-bills 1M	1132	0.27%	0.25%	-0.07%	0.03%	0.21%	0.42%	1.35%

To evaluate the difference in returns of the assets, we begin by comparing the cumulative returns of risky and risk-free assets over time. Figures 4.1 offer an initial perspective on this comparison, illustrating the historical performance of S&P 500 (risky asset) and T-bills 1M (risk-free asset) between the end of 1928 and 2023. Figure 4.1 demonstrates that, over the 95-year period, risky assets, have consistently outperformed the risk-free asset. In fact, the S&P 500, when reinvesting dividends, surpassed the T-bills 1M by almost 107-fold, highlighting the potential benefits of investing in risky assets for long-term growth. By reinvesting the dividends the risky-portfolio has a cumulative return of a staggering 767 809%, which is 9.87% annualized.

Table 4.2 summarizes the annualized and cumulative returns of the S&P 500 with and without dividends, in addition to the risk-free T-bills 1M. It is clear that the market by far outperforms the risk-free rate.

Table 4.2.: Summary of annualized returns and cumulative returns after investing in S&P 500 with and without dividends or T-bills 1M from 1928 to 2023.

	Return Metrics	
Asset	**Annualized Return**	**Cumulative Return**
T-bills 1M	3.24%	1 964.94%
S&P 500	5.95%	23 225.08%
S&P 500 with Dividends	9.87%	767 709.12%

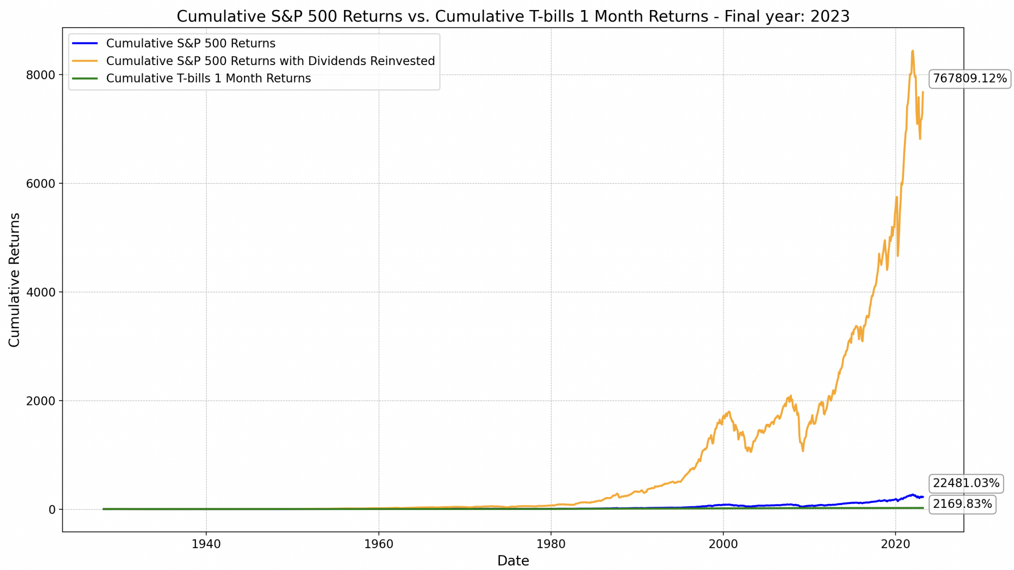

Figure 4.1.: The figure demonstrates the cumulative return of the S%P 500 with and without reinvested dividends, and T-bills 1M from 1929 to 2023.

4.5. Simulating the market portfolio

The rich and long data set allows for generation of multiple portfolios. To perform the analysis of time-diversification, I constructed portfolios with investment horizon ranging from one-year to 40-years. This resulted in 620 portfolios, as the latest had to start 40 years before 2023. In addition to the portfolios ranging from one-year to forty-years, portfolios ranging from one-month to 30-years horizon with a stride of one month was also calculated. This allowed for 773 observations per horizon.

4.6. Calculations

4.6.1. Returns

To calculate the mean cumulative return (MCR) and mean annualized return (MAR), the cumulative return (CR) and annualized return (AR) is calculated for all the portfolios, then averaged:

$$MCR_H = \frac{1}{N}\sum_{i=1}^{N} CR_i \tag{4.4}$$

$$MAR_H = \frac{1}{N}\sum_{i=1}^{N} AR_i. \tag{4.5}$$

where N is the total number of portfolios, CR_i is the cumulative return of the i-th portfolio, and AR_i is the annualized return of the i-th portfolio. The standard deviations of cumulative returns (SDRC) and the standard deviation of the annualized returns (SDAR) are calculated by the same procedure. Calculate the annualized and cumulative standard deviation first, then take the averages.

$$SDCR_H = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(CR_i - MCR_H)^2} \tag{4.6}$$

$$SDAR_H = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(AR_i - MAR_H)^2} \tag{4.7}$$

where CR_i and AR_i are as defined earlier, and MCR_H and MAR_H are the mean cumulative return and mean annualized return for the given investment horizon H.

4.7. Option Pricing Experiment

The portfolio data was also used to evaluate how Bodie (1991, 1995) theoretical shortfall insurance cost faces the realities of non-constant standard deviation.

$$P/S = N(d_1) - N(d_2), \tag{4.8}$$

$$d_1 = \frac{\sigma\sqrt{T}}{2}, \tag{4.9}$$

$$d_2 = -\frac{\sigma\sqrt{T}}{2}. \tag{4.10}$$

The standard deviation used is the average annualized standard deviation for the relevant time horizon.

4.8. Optimal Proportion of Wealth Allocated to Risky Asset

The optimization problem of Strong and Taylor (2001) is replicated, as the original implementation suffered from limited observations, leaning on the bootstrapping technique. The optimization problem is find the allocation to risky or risk-free asset's that maximized the investor's utility, formulated as

$$\max_{w} \frac{1}{T} \sum_{t=1}^{T} U\left(\sum_{i=1}^{s} w_i(1 + r_{i,t})\right)$$

To find the optimal allocation of risky asset, the following formula based on isoelastic utility is used

$$w_i = \frac{r_i - r_{i,f}}{\sigma_i^2}, \text{if} \quad \eta \neq 1 \tag{4.11}$$

$$w_i = \frac{r_i - r_{rf}}{\eta \cdot \sigma_i^2}, \quad \text{otherwise} \tag{4.12}$$

where w is the optimal allocation to the risky asset, r_i and r_{if} are the risky and risk-free returns respectively, and σ_i is the standard deviation of the risky returns and i is the i-th portfolio. The average optimal weight is then calculated for each horizon.

$$\overline{w} = \frac{1}{N} \sum_{i=1}^{N} w_i \tag{4.13}$$

where $\overline{w}$ is the average allocation to the risky asset, w_i is the allocation for the i-th portfolio, and N is the total number of portfolios. The allocations are bounded to maximum 100%.

The horizon invariance test from Strong and Taylor (2001) is carried out with the null hypothesis:

- **H0** The optimal allocation w^* is equal for one-month horizon as for 5 years.

- **H1** The optimal allocation w^* is *not* equal for one-month horizon as for 5 years.

The null hypothesis is tested with a student-t test. If the null is rejected, there is evidence that risk, in the eyes of the expected utility framework, decrease over investment horizon.

Chapter 5.

Results

This chapter will present the results from the empirical analysis. This includes a preliminary analysis of the portfolio distributions, descriptive statistics of the average annualized and cumulative returns, and an analysis of various risk measures ranging from standard deviation and shortfall risk, to more sophisticated risk measures such as Value at Risk and conditional Value at Risk. Risk-adjusted measures will also be evaluated in relation to the investment horizon. Equipped with empirical data from the market, the option pricing framework will be assessed using the empirical standard deviation as a function of time. Furthermore, the optimal allocation to risky assets in light of the investor's expected utility will be tested. This will be followed by a discussion regarding time diversification and the potential existence of a minimum investment horizon.

5.1. Return and Risk Measures

5.1.1. Standard Deviation

Table 5.1 shows the annualized returns and standard deviance as a function that investment horizon for both the S&P 500 and the risk-free asset. For both assets, the annualized returns is within a narrow area, as aligns with expectations. For one-year horizon portfolios, the expected standard deviation is 22.84%. After 40-years, however, the annualized standard deviation is under 1%, whereas the annualized return is almost 11%. It is clear that the expected standard deviation decreases as investment horizon lengthens.

Table 5.1.: The Annualized returns of the risk-free rate and the risky asset, including the annualized standard deviation. It is clear that the annualized standard deviations decreases with horizon, hence, at least one objective measure of risk decreases as time lengthens.

	Risk-Free	S&P 500	
Horizon	**Annualized Return**	**Annualized Return**	**Annualized σ**
1	3.24%	10.81%	22.84%
5	3.45%	10.37%	8.02%
10	3.73%	10.72%	5.35%
15	3.96%	11.23%	4.19%
20	4.19%	11.56%	3.34%
30	4.46%	11.23%	1.28%
40	4.52%	10.92%	0.96%

5.1.2. Distribution of Outcomes

Table 5.2 shows the distributions of both the cumulative and annualized returns over time. From here, there are some interesting insights. The most notable is the high spread of the returns with one-year investment horizon. Even though the median is 10.8% annualized, the worst performing portfolio experienced a catastrophic loss of -62.2%, whereas the best performing has a whopping return of 140.3% annualized[1]. It is also interesting to note that the 25th percentile is positive after only five years, however the 1st percentile is -48% and -21.9% for investment horizons of 5- and 10-years, respectively, meaning the risk is still present. After 20 years, the minimum cumulative return is 50%, which is positive, hence there is none of the 620 portfolios that incurs a loss on initial investment after 20 years. The minimum return of 50.4% is still below the expected risk-free rate of 156.3% with the same horizon. After 30 years, the worst performing portfolio still outperforms the risk-free rate by 2.5 times. For the annualized returns, the pattern is, not surprisingly, the same. The returns seem to be normally distributed.

[1]The minimum stems from the great depression in 1929, whereas the maximum stems from the rally in 1932.

Table 5.2.: The distribution percentiles of the cumulative and annualized returns of the portfolios, as a function of time. It is clear that as time increases, the risk of incurring a loss on initial investment diminishes, as with shortfall risk. At a 15-year investment horizon, the worst 1st performing portfolios is still expected to have a positive return.

Cumulative Returns (%)								
Horizon	r_f	**Min**	**1st**	**25th**	**50th**	**75th**	**90th**	**Max**
1	3.2	-62.2	-42.9	-3.3	10.8	24.3	69.8	140.3
5	19.7	-61.1	-48.0	33.1	72.2	108.7	232.2	319.8
10	50.4	-33.3	-21.9	97.2	205.5	328.3	514.4	580.5
15	94.7	-4.8	11.2	208.7	466.6	697.5	1136.3	1283.0
20	156.3	50.4	75.6	404.8	940.5	1341.7	2464.9	2607.4
30	331.3	808.0	998.1	1801.1	2483.5	3130.2	4961.3	5413.5
40	561.0	1958.2	2313.0	4881.4	6578.0	8023.9	11275.4	14003.7
Annualized Returns (%)								
Horizon	r_f	**Min**	**1st**	**25th**	**50th**	**75th**	**90th**	**Max**
1	3.2	-62.2	-42.9	-3.3	10.8	24.3	69.8	140.3
5	3.4	-17.2	-12.3	5.9	10.4	15.8	27.1	33.2
10	3.7	-4.0	-2.4	7.0	10.7	15.7	19.9	21.1
15	4.0	-0.3	0.7	7.8	11.2	14.8	18.3	19.1
20	4.2	2.1	2.9	8.4	11.6	14.3	17.6	17.9
30	4.5	7.6	8.3	10.3	11.2	12.3	14.0	14.3
40	4.5	7.9	8.3	10.3	10.9	11.6	12.6	13.2

Table 5.3 demonstrates the confidence intervals for the annualized and cumulative returns. For a one-year horizon, the annualized returns has a lower 95% confidence interval of 9.05% and a upper 95% confidence interval of 12.56%. These are very narrow, as a testament to the large number of observations.

Table 5.3.: The confidence interval for annualized and cumulative returns. The confidence interval is narrow.

	Annualized Returns		Cumulative Returns	
	Lower 95% CI	Upper 95% CI	Lower 95% CI	Upper 95% CI
1	9.05	12.56	9.05	12.56
5	9.75	10.98	67.64	76.73
10	10.31	11.13	195.04	215.87
15	10.91	11.56	444.22	489.02
20	11.30	11.81	896.86	984.14
30	11.14	11.33	2413.45	2553.52
40	10.85	11.00	6414.22	6741.70

Figure 5.1 compares the expected cumulative returns, including the worst performing portfolio for every horizon, against the risk-free rate. The best performing portfolio has a total return of 14000%, 25 times the risk-free rate. The percentiles are also including, to show the full picture. At a 25 year investment horizon, every portfolio outperforms the risk-free rate, at which point, there has not been one incident in history (at least for the S&P 500), a market portfolio has been outperformed by the risk-free rate.

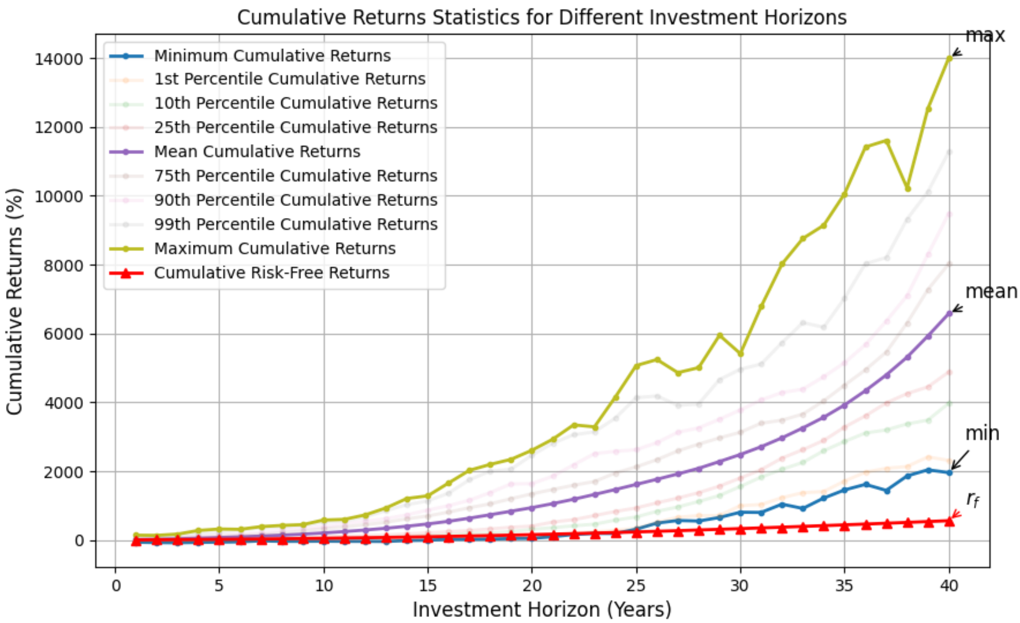

Figure 5.1.: The expected cumulative return of risky and risk-free asset.

Figure 5.2 compares the expected annualized return of the market portfolios with the risk-free rate. It is clear that as the investment horizon increases, the expected returns become normalized and closer to the mean. The superb expected returns (>25% annualized) does not occur at investment horizons longer than 8 years. Further, we see that the 10th percentile still has expected loss up till an investment horizon of 5 years.

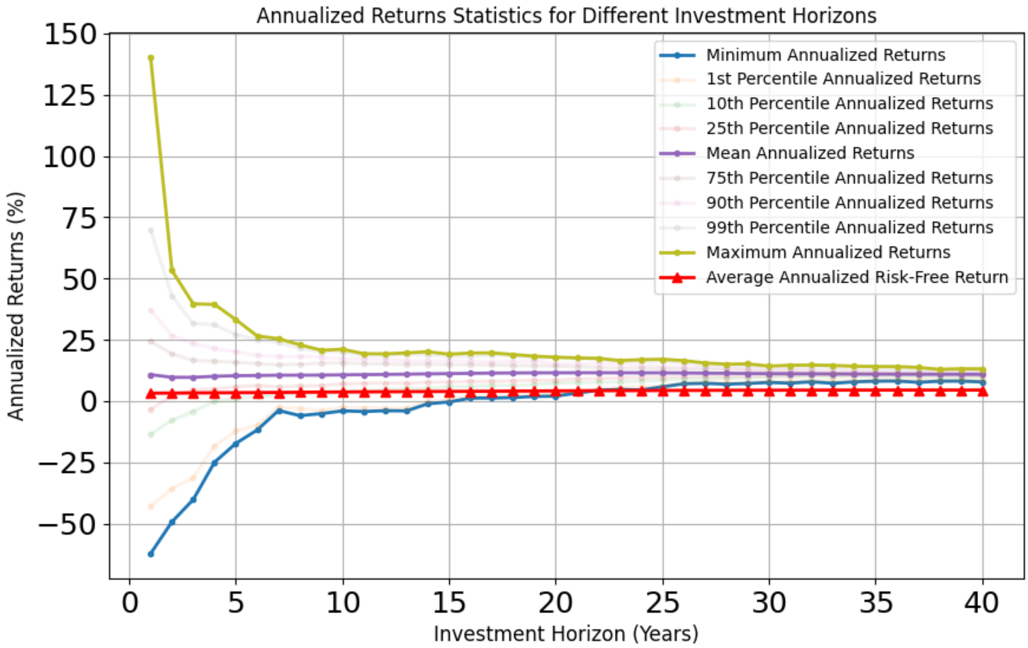

Figure 5.2.: The annualized returns for the risky and risk-free asset.

Figure 5.3 provides a close-up for the expected performance with of the 1st percentile portfolios, including the worst performing at the differnt investment horizons. Based on historical performance of the S&P 500, to not inucrr a loss, investor's should have at least 15 year investment horizon. To be certain of excess return over the risk-free rate, the investment horizon should be 22 years.

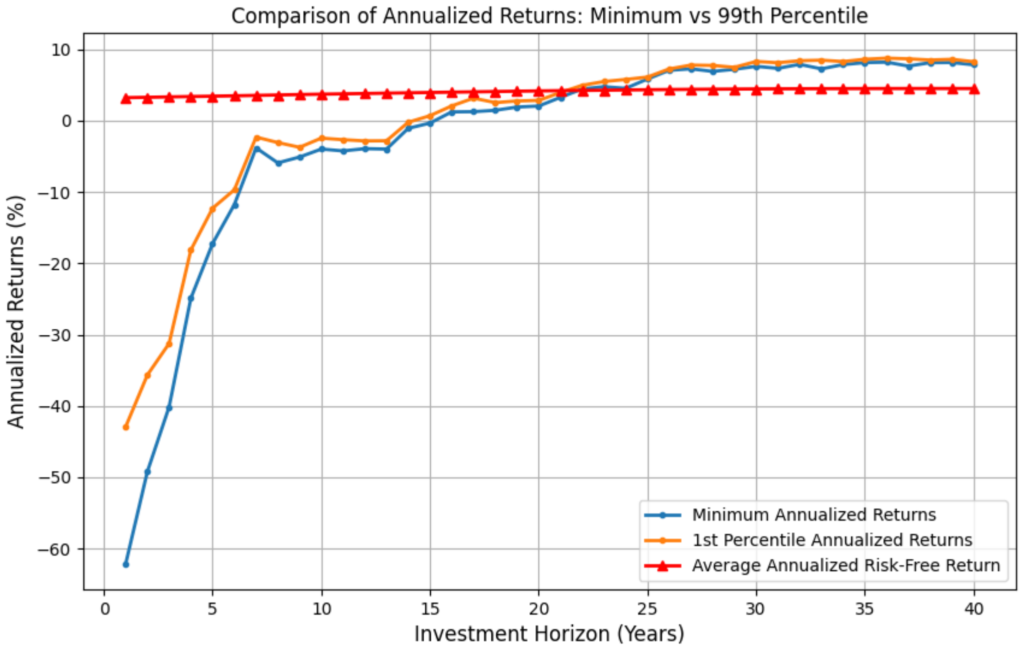

Figure 5.3.: Comparing the expected annualized return of the 1st percentile and worst performing portfolio to the risk-free rate. For the 1st percentile to have an expected value over the risk-free rate, the investment horizon should be 22 years. However, all investment horizon expected consistent outperforming of the risk-free rate.

5.1.3. Expected Probability of Loss or Shortfall risk

The second risk metric after standard deviation is to examine is the probability of shortfall. A very risk averse investor might be worse of by the thought of losing parts of the initial investments than gaining a two-fold return. Figure 5.4 shows the expected probability of loss on initial investment or shortfall with the risk-free rate as bench-mark. The trend is clear, the probability of both loss and shortfall decreases when the investment length increases. With a 20-year investment horizon, the expected probability of loss is zero. To eliminate any expected probability of shortfall risk, the investor should have an investment horizon of 27.

In line with the investment advisors and wealth management firms recommendations for investment horizon, it is clear that the probability of incurring a loss on the initial investment decreases as the investment horizon is extended. This finding is contrary to the logic that insuring against shortfall risk should increase over time, as the option theory stream predicts. Based on the findings as of yet time does diversify the risk of shortfall or loss.

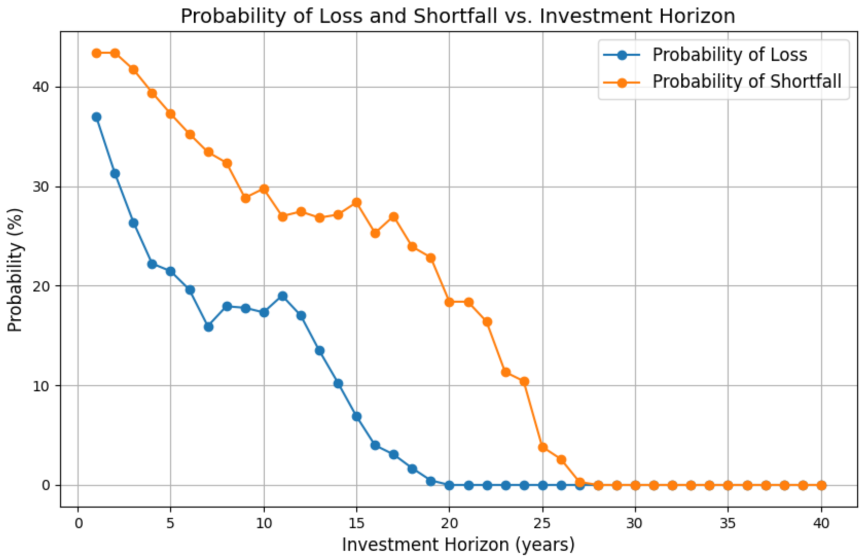

Figure 5.4.: The expected probability of loss or shortfall steadily decreases when the investment horizon increases, but is nevertheless present until 20 and 26 years for loss and shortfall respectively.

5.1.4. The Value at risk

Examining the Value at Risk (VaR) as a function of investment horizon provides insight into whether a portfolio's risk diminishes as the investment period extends. It has the property of incorporating not only the probability of loss, but also the magnitude. As has been discusses in Section 3.5, there is evidence of investors putting greater weight on the magnitude of the loss tan the probability of said loss. Figures 5.5 and 5.6 illustrates the 95% and 99% Value at Risk (VaR) and Conditional VaR (cVaR).

A clear pattern emerges in both figures, demonstrating a decline in the expected 95%-VaR and 95%-cVaR as the investment horizon lengthens. The most striking result is that the 95%-VaR turns positive after only 7 years, meaning that with a 95% confidence level, the expected losses will not exceed a positive figure. The 95%-cVaR tells a similar story, after 12 years, the 95%-cVaR turns positive. This means that the average of the most unfavorable 5% of outcomes is, in fact, positive.

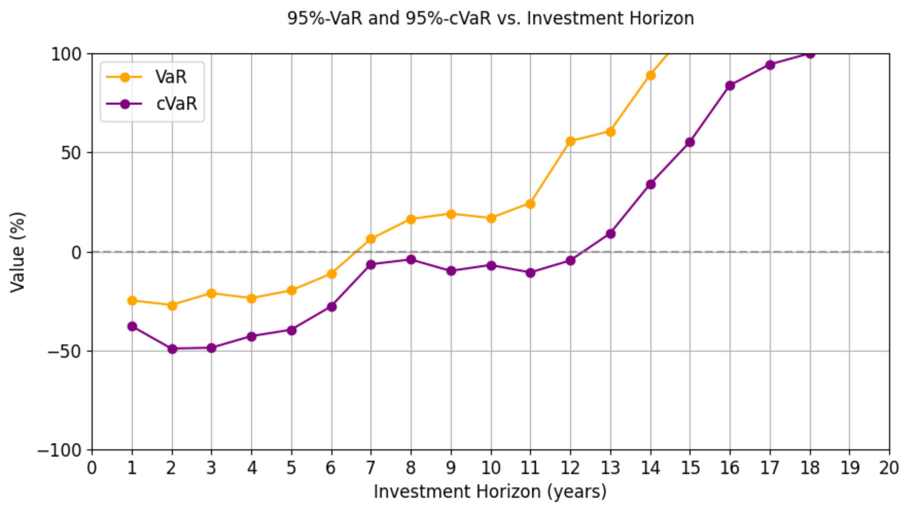

Figure 5.5.: The 95%-VaR and 95%-cVaR as a function of investment horizon. It is clear that both the 95%-VaR and 95%-cVaR decreases as the investment horizon increases. The results indicate that after 12 years, all curves are above the zero-line, indicating that there is no value at risk at a 95% confidence level. After which, the 95%-VaR and 95%-cVaR is actually positive, meaning that the investor's "Value at risk" is simply gaining less than the other scenarios.

The 99%-VaR and 99%-cVaR follow a similar trend, but with a longer required investment horizon before the expected 99%-VaR and 99%-cVaR turns positive. A positive 99%-VaR is achieved after 14 years, and 15 years for the 99%-cVaR. That means that the worst 1% of outcomes of the simulated scenarios will still have a expected positive net gain to the investor's portfolio.

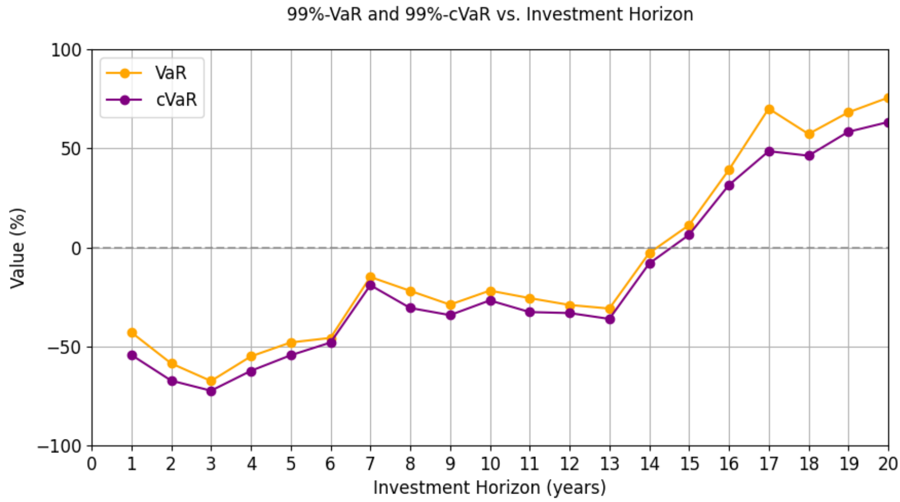

Figure 5.6.: The 99%-VaR and 99%-cVaR as a function of investment horizon. It is clear that both the 99%-VaR and 99%-cVaR decreases as the investment horizon increases. The results indicate that after 15 years, all curves are above the zero-line, indicating that there is no value at risk at a 99% confidence level. After which, the 99%-VaR and 99%-cVaR is actually positive, meaning that the investor's "Value at risk" is simply gaining less than the other scenarios.

The results point in favor for time-diversification, as both the expected 95%- and 99%-VaR and cVaR steadily decline as the investment horizon increases. It does also align with the conventional wisdom of having a time-horizon of at least 5 years before investing in equity funds, even though there is still a significant decrease in expected 95%-VaR and 95%-cVar with 6 and 7 year investment horizon.

5.2. Risk-Adjusted Performance

Until now, the focus has been on annualized and cumulative returns, the distribution of portfolios, and various return measures. However, analyzing the risk-adjusted performance will provide more insights into the validity of time-diversification. Risk-adjusted performance measures provide a rewards-for-risk ratio, which the option pricing stream is accused of neglecting. The natural risk-adjusted performance would be linear. If the measures exceed a linear relationship, an increase in investment horizon should yield higher expected returns than the indicated risk level.

5.2.1. Sharpe Ratio

The Expected Sharpe Ratio is calculated as the expected excess return divided by the standard deviation. It is computed using annualized values. Figure 5.7 illustrates the expected Sharpe ratio for different investment horizons. The trend is evident: as the investment horizon lengthens, the ratio is expected to increase. The peak is reached at an investment horizon of 35 years, after which it slightly declines. To maximize the expected Sharpe ratio, an investor should aim for an investment horizon of 35 years. This finding aligns with the concept of time diversification, as a longer investment period can potentially reduce risk and subsequently increase the Sharpe ratio, all else being equal.

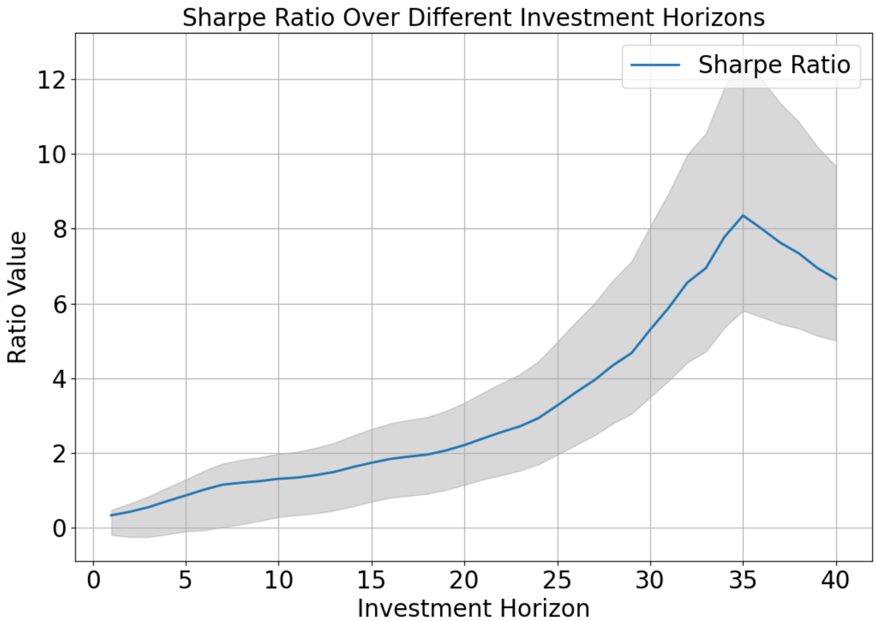

Figure 5.7.: The Expected Sharpe ratio as a function of investment horizon. The shaded area illustrates the variability. The expected Sharpe increases for each year the investment, until the investment horizon is 35 years. Afterwards, for each year the investment horizon increases, the expected Sharpe ratio decreases.

5.2.2. Sortino Ratio

The Expected Sortino Ratio is calculated as the expected excess return divided by the semi-standard deviation. Figure 5.8 has the same pattern as the Sharpe ratio, which aligns with time diversification.

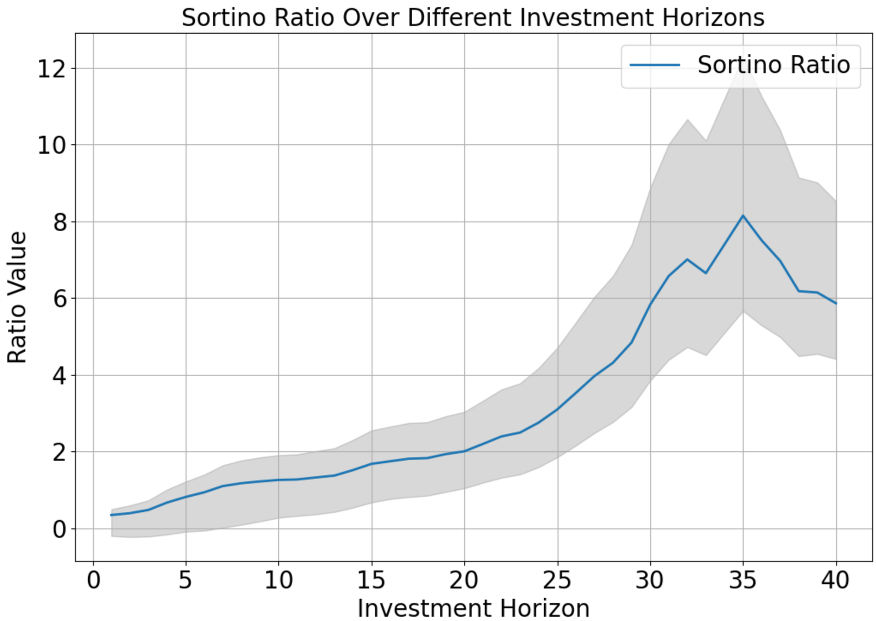

Figure 5.8.: Sortino Ratio as a Function of Investment Horizon

5.3. Calculating Shortfall Insurance with Empirical Data

The shortfall insurance cost as a function of investment horizon illustrated in Section 3.4 is based on the criticized assumption of constant standard deviation. Bianchi et al. (2016) argues that the argument fails when the constant standard deviation in the pricing model is replaced with a non-standard constant deviation. Henceforth, I have calculated the shortfall insurance cost as a function of investment horizon with the empirical annual standard deviation. Table 5.4 demonstrates the results. When accounting for the declining expected annual standard deviation over time, the insurance cost actually decreases. This finding is in starch contrast to Bodie (1991, 1995), which equates risk with the cost of shortfall insurance. Consequently, by Bodie (1995)'s own measure of risk, there are signs that time diversification exists, as the cost does decrease from around 9% to 2.42%.

Table 5.4.: The calculated shortfall insurance cost calculated with both constant standard deviation, and then with the non-constant empirical standard deviation. When the empirical standard deviation is used, the The shortfall insurance cost *decreases* as the investment horizon increases. By Bodie (1995) own measure, time diversfication is real.

	Constant σ		Empirical σ	
Horizon	σ	**Insurance Cost**	σ	**Insurance Cost**
1	20%	7.97%	22.84%	9.09%
5	20%	17.69%	8.02%	7.14%
10	20%	24.82%	5.35%	6.74%
15	20%	30.15%	4.19%	6.47%
20	20%	34.53%	3.34%	5.95%
30	20%	41.61%	1.28%	2.80%
40	20%	47.29%	0.96%	2.42%

Figure 5.9 illustrates the difference between an of shortfall insurance cost calculated with an assumed constant standard deviation and the expected annual standard deviation.

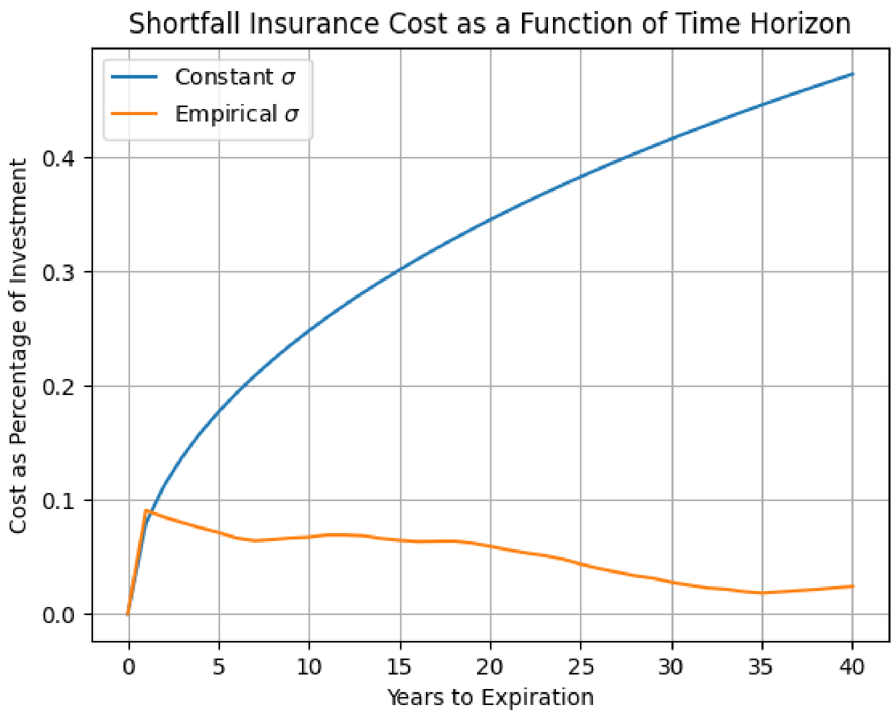

Figure 5.9.: Comparison between the shortfall insurance cost with constant and non-constant empirical standard deviation. Rather than increasing with time, the insurance cost actually decreases.

5.4. Optimal Allocations to Maximize Expected Utility

For the following section, the model with monthly portfolios is applied. The model with yearly investment horizons was not granular enough for the purpose, and limited the insights. Table 5.5 presents the estimated optimal proportion's of wealth for an investor with isoelastic utility, where η is the risk coefficient. There is a clear trend that the higher risk aversion, the longer investment horizon is required before the entirely of the investors portfolio is to be invested in risky assets. For example, for investors with $\eta = 2$, the proportion of wealth allocated to risky assets is 53.18% when a 6-month investment horizon is used, rising to 100% at the 18-months investment horizon. For a more risk averse investor, e.g., $\eta = 8$, a 42-month horizon is required. All optimal weights for investment horizons above 48-months are 100% to risky assets.

Table 5.5.: The table illustrates the optimal proportions of portfolio invested in risky asset's based on an individual with CRRA utility function (isoelastic). 'M' is months. For an investor with $\eta = 1$, i.e., logarithmic utility, the optimal proportion to maximize the expected utility is 1.45%. However, when the investment horizon is 6 months, the proportion is 100%. For an more risk averse investor, e.g., someone with $\eta = 6$ does not have 100% before 42 months. The rightmost values are the p-values for the null hypothesis of the proportion of wealth in risky assets is equal for a 1 month portfolio as for a portfolio with 5-year or 10-year investment horizon. The p-values are all zero, hence the null hypothesis is rejected.

	Investment Horizon in Months									p-value	
η	1M	6M	12M	18M	24M	30M	36M	42M	48M	5-year	10-year
1	1.45%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	0.00	0.00
2	0.73%	53.18%	82.99%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	0.00	0.00
4	0.36%	26.59%	41.50%	59.75%	82.83%	100.00%	100.00%	100.00%	100.00%	0.00	0.00
6	0.24%	17.73%	27.66%	39.84%	55.22%	73.68%	92.55%	100.00%	100.00%	0.00	0.00
8	0.18%	13.29%	20.75%	29.88%	41.42%	55.26%	69.41%	86.81%	100.00%	0.00	0.00

The statistical significance of the difference between optimal proportion allocated to risky asset with an investment horizon of 1-month versus five or ten years is carried out by the horizon invariance test. Recall the null hypothesis: The allocation w^* is equal. It is clear that for all risk aversions, the null can be rejected, as the p-value is zero. This implies that the proportion to risky asset is significantly higher for a five-year horizon than one month, and consequently, strengthens the claim of time diversification. Figure 5.10 illustrates how the optimal allocation to risky asset changes with the investment horizon length. Note how they all increase with time, i.e., the expected utility of holding risky asset's increase with time.

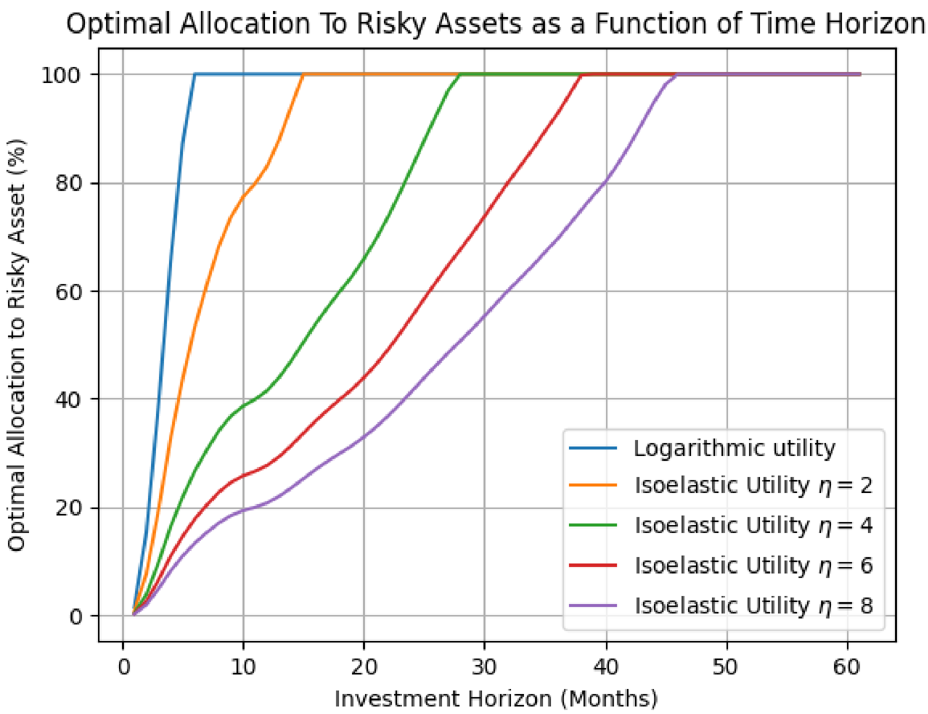

Figure 5.10.: The optimal proportion of wealth allocated to risky asset over investment horizons for an investor with isoelastic utility.

The result is likely a result of the decreasing annualized standard deviations over time. Investors with isoelastic utility has CRRA, i.e., constant relative risk aversion. This implies that the investor will allocate the same relative proportion of his or her wealth to a risky bet, no matter the size of the investor's fortune. However, when the risk of the bet (in this case the annualized standard deviation), the investor may take on more of the bet. Consider an investor with $\eta = 6$, according to Figure 5.10, if this investor has a horizon of only 12-months, the proportion of wealth allocated to risky assets is only 27.66%. However, for a 24-month horizon, the proportion has increased to 55.22%. From Table 5.1, we see that when the horizon increases, the expected annualized returns is almost constant, whereas the expected annualized standard deviation decreases—the risk decreases, whereas the returns stay the same.

5.5. Discussion

The findings supports time diversification. Every objective risk measure has decreased as the investment horizon has lengthened from 1-year to forty-year investment horizon, the expected annualized standard deviation decreased from 22.84% to 0.96%. Furthermore, with 20-years investment horizon and above, no portfolios incurred a loss. For a 27-year investment horizon and above, no portfolio experienced shortfall. The 95%-VaR and 95%-cVaR were reduced to zero after 7 and 12 years, respectively, from their initial levels of approximately 25% and 40%. For the 99%-VaR and 99%-cVaR, the investment horizon are 14 and 15 year. The risk-adjusted performance measures such as the expected Sharpe and Sortino ratios increased with the investment horizon. By equating risk with objective measures, there is strong evidence of time diversification.

The empirical data was further utilized to solve the optimization problem of determining the proportion of wealth allocated to risky assets, maximizing the investor's expected utility for each time horizon. It was found, significantly, that the proportion to risky assets steadily as a function of investment horizon, for investors with isoelastic utilities. Even for investors with a risk coefficient of $\eta = 8$, after the investment horizon is longer than 4 years, the optimal allocation is 100% risky assets. There is strong evidence of time diversification.

Additionally, the empirical data was used to calculate the cost of shortfall insurance. The findings indicated that when using empirical data, both frameworks predicted a *decrease* in risk with the time horizon, contradicting their theoretical findings. By Bodie (1991, 1995) measure of risk, i.e., cost of insuring against shortfall, time diversification is real.

However, there may be errors in the data, such as serial correlations etc. Future works include: testing for mean reversion, other data sets, more granular data. Finding minimum time horizon, such that a 95% confidence interval could be constructed with no expected loss.

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Appendix A.

Proof of limit

Given the assumptions that $E[r_m] > r_f$ and $\sigma > 0$, consider the equation:

$$\alpha^* = \frac{e^{(E[r_m]+\frac{1}{2}\sigma^2)t} - e^{r_f t}}{Ae^{(2E[r_m]+\sigma^2)t}\left(e^{\sigma^2 t} - 1\right)} \tag{A.1}$$

As t approaches infinity, and given that $E[r_m] > r_f$ and $\sigma > 0$, the term $e^{r_f t}$ in the numerator will be dwarfed by $e^{(E[r_m]+\frac{1}{2}\sigma^2)t}$, effectively becoming zero. Similarly, in the denominator, the term -1 in $\left(e^{\sigma^2 t} - 1\right)$ will also be negligible compared to $e^{\sigma^2 t}$. So the equation simplifies to:

$$\alpha^* = \frac{e^{(E[r_m]+\frac{1}{2}\sigma^2)t}}{Ae^{(2E[r_m]+\sigma^2)t}e^{\sigma^2 t}} \tag{A.2}$$

Simplifying further by combining terms with the same base:

$$\alpha^* = \frac{1}{Ae^{(2E[r_m]+\frac{1}{2}\sigma^2)t}} \tag{A.3}$$

As t approaches infinity, it is clear that the fraction will approach zero:

$$\lim_{t\to\infty} \alpha^* = 0. \tag{A.4}$$

What is the value when time approaches zero? Let's start with the original expression again

$$\alpha^* = \frac{e^{(E[r_m]+\frac{1}{2}\sigma^2)t} - e^{r_f t}}{Ae^{(2E[r_m]+\sigma^2)t}\left(e^{\sigma^2 t} - 1\right)}. \tag{A.5}$$

As t approaches zero, the exponentials in the equation approach 1, except for the term $e^{\sigma^2 t}$ in the denominator, which approaches $e^0 = 1$. Hence, the equation simplifies to:

$$\alpha^* = \frac{1 + (E[r_m] + \frac{1}{2}\sigma^2)t - (1 + r_f t)}{A(1 + (2E[r_m] + \sigma^2)t)(1 - 1)}. \tag{A.6}$$

This further simplifies to:

$$\alpha^* = \frac{(E[r_m] + \frac{1}{2}\sigma^2 - r_f)t}{A\sigma^2 t}. \tag{A.7}$$

The t in the numerator and the denominator cancel out, and we have:

$$\lim_{t \to 0} \alpha^* = \frac{E[r_m] + \frac{1}{2}\sigma^2 - r_f}{A\sigma^2}. \tag{A.8}$$

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