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NTNU
Norwegian University of Science and Technology

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Bohr's theorem for general Dirichlet series and different assumptions on frequencies

Master's thesis in mathematics, MLREAL
Supervisor: Karl-Mikael Perfekt
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Abstract

We study general Dirichlet series assuming different conditions on the frequency λ. In particular we consider Dirichlet series belonging to the space $D_\infty^{ext}(\lambda)$ of all somewhere convergent general Dirichlet series which allows a bounded and holomorphic extension to the right half-plane $[\mathrm{Re} > 0]$. We deduce quantitative results for the partial sums of Dirichlet series belonging to $D_\infty^{ext}(\lambda)$, and show that frequencies under certain conditions satisfy Bohr's theorem, namely that the series converges uniformly on the right half-plane.

Sammendrag

Vi studerer generelle Dirichlet-rekker som antar forskjellige antakelser på frekvensen λ. Spesielt betrakter vi Dirichlet-rekker som tilhører rommet $D_\infty^{ext}(\lambda)$ av alle noen steds konvergerende generelle Dirichlet-rekker som tillater en begrenset og holomorf utvidelse til det høyre halvplan $[\mathrm{Re} > 0]$. Vi utleder kvantitative resultater for delsummene av Dirichlet-rekker som tilhører $D_\infty^{ext}(\lambda)$, og viser at frekvenser under visse betingelser tilfresstiller Bohr's teorem, nemlig at rekken konvergerer uniformt på det høyre halvplan.

Preface

This thesis was written from August to December 2022 under the supervision of Karl-Mikael Perfekt, and marks the conclusion of five years as a student at the Department of Mathematical Sciences and at the Teacher Education programme at NTNU.

I would like to thank Karl-Mikael for suggesting an interesting topic, and for all the helpful advice along the way. In addition, a big thanks to my family for always supporting me over the long course of my time as a student.

Jonas Eide
Trondheim, 2022

Contents

Introduction

We begin by giving a brief introduction to the topic at hand, by introducing Dirichlet series and some convergence properties. We then give a short overview of each chapter.

Convergence of Dirichlet series

In this thesis we are concerned with general Dirichlet series, which is an infinite sum on the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

Where a_n is a sequence of complex coefficients, and s is a complex variable on the form $s = \sigma + it$, where σ and t are real variables. $\lambda = (\lambda_n)$ is called the frequency, and is an increasing sequence of real numbers tending to $+\infty$. We observe that by letting $\lambda_n = \log n$, we obtain what is known as ordinary Dirichlet series.

$$\sum_{n=1}^{\infty} a_n n^{-s}$$

A famous example of an ordinary Dirichlet series is that with the coefficients all equal to one, which is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

A Dirichlet series is said to be convergent to $f(s)$, if its sequence $\{S_N f\}$ of partial sums

$$S_N f(s) = \sum_{n=1}^{N} a_n e^{-\lambda_n s}$$

converges to $f(s)$. That is,

$$f(s) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n e^{-\lambda_n s}$$

We are interested in observing for which s a given Dirichlet series converges. Dirichlet series may converge for all, none, or some s. It is well known from complex analysis that power series converges inside disks of the complex plane. Dirichlet series converges on half-planes of the complex plane, where they define holomorphic functions. The half-plane of convergence is solely dependent on σ, the real value of s. For a given Dirichlet series f, the smallest value of σ, such that the series is convergent in all of $[\text{Re } s > \sigma]$, is called the *abscissa of convergence*. The abscissa of convergence shall be denoted by σ_c

$$\sigma_c = \inf\ \left\{\theta \in \mathbb{R}\ :\ f \text{ converges in } [\text{Re } s > \theta]\right\}$$

σ_c need not be finite. If a given Dirichlet series converges everywhere in the complex plane, we set $\sigma_c = -\infty$, and we say that the series is everywhere convergent. On the contrary, if a given Dirichlet series diverges everywhere in the complex plane, we set $\sigma_c = +\infty$, and we say that the series is nowhere convergent. We say that a Dirichlet series whose corresponding abscissa of convergence is not $+\infty$, is a somewhere convergent Dirichlet series. Dirichlet series also define half-planes in which they are, respectively, uniformly convergent, and absolutely convergent. In general, the size of these half-planes differ. We define the abscissa of uniform, and absolute convergence

$$\sigma_u = \inf\ \left\{\theta \in \mathbb{R}\ :\ f \text{ converges uniformly in } [\text{Re } s > \theta]\right\}$$

$$\sigma_a = \inf\ \left\{\theta \in \mathbb{R}\ :\ f \text{ converges absolutely in } [\text{Re } s > \theta]\right\}$$

Harald Bohr was interested in finding the maximal width of the strip for which an ordinary Dirichlet series converges uniformly, but not absolutely.

$$S := \sup\left\{\sigma_a(f) - \sigma_u(f)\ :\ f(s) = \sum_{n=1}^{\infty} a_n n^{-s}\right\}$$

where the supremum is taken over all ordinary Dirichlet series. Bohr was able to show that $S \leq 1/2$ but without being able to produce any examples where this value was obtained. H. F Bohnenblust and E. Hille was able to show some years later, with a nontrivial proof, that in fact $S = 1/2$. We are going to see that boundedness of a function defined by a Dirichlet series and uniform convergence of the series is closely related. We define a fourth abscissa, which defines the half-plane for which a somewhere convergent Dirichlet series is bounded.

$$\sigma_b = \inf\ \left\{\theta \in \mathbb{R}\ :\ f \text{ converges and defines a bounded function } [\text{Re } s > \theta]\right\}$$

Clearly, every σ that defines a half-plane for which a Dirichlet series converges uniformly implies that the limit function is bounded in the same half-plane. For the case of the ordinary Dirichlet series, Bohr showed in [5] that the abscissaes coincide, that is,

$$\sigma_u(f) = \sigma_b(f) \quad , \quad f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

This result is known as *Bohr's theorem*, and an equivalent statement is that for any ordinary Dirichlet series which is somewhere convergent and extends to a bounded and holomorphic function on the half-plane $[\text{Re } s > 0]$, then the Dirichlet series converges uniformly on every smaller half-plane $[\text{Re } s > \varepsilon]$, where $\varepsilon > 0$, and therefore $\sigma_u \leq 0$. We define the space $\mathcal{H}_\infty$ of all somewhere convergent ordinary Dirichlet series with a bounded and holomorphic extension to [Re $s > 0$]. Bohr's theorem extends to the general Dirichlet series. The natural domain of Bohr's theorem is the space $D_\infty^{ext}(\lambda)$ of all somewhere convergent general Dirichlet series which has a bounded and holomorphic extension to [Re $s > 0$]. The frequency λ is said to satisfy Bohr's theorem if every $f \in D_\infty^{ext}(\lambda)$ converges uniformly on [Re $s > \varepsilon$], for every $\varepsilon > 0$. This raises the following question.

For which frequencies λ does Bohr's theorem hold?.

As we just saw, this is the case for $\lambda_n = \log n$. Examples show that any concrete frequency may or may not satisfy Bohr's theorem. We are going to see multiple classes of frequencies for which Bohr's theorem holds, and give their proof with the help of fundamental theorems of general Dirichlet series.

Bohr theorem for the case of the ordinary Dirichlet series, was in 2006, by Balasubramanian, Calado, and H. Queffélec, improved by what can be seen as a quantitative version of Bohr's theorem

$$||S_N f(s)||_\infty \leq C \log N ||f||_\infty$$

This result shows the behavior of the partial sums of a Dirichlet series belonging to $\mathcal{H}_\infty$ is well-controlled. We deduce similar bounds for general Dirichlet series belonging to $\mathcal{D}_\infty^{ext}(\lambda)$ which depends on the restrictions on λ. Bohr proved, in 1913, that frequencies which satisfy the following property

$$\lambda_{n+1} - \lambda_n \geq C e^{-\delta \lambda_n}$$

where C and δ are positive constants, satisfies Bohr's theorem. We refer to this condition as Bohr's condition. Using a generalized version of the Perron-Landau formula

$$A(x) = \frac{1}{2\pi i} \int_{\rho - iT}^{\rho + iT} f(s) \frac{x^s}{s} ds + \mathcal{O}\left[\frac{x^\rho}{T} \sum_{n \geq 1} \frac{|a_n|}{n^\rho |\log(x/n)|} \right]$$

we're able to deduce the following bound for the partial sums where λ satisfies Bohr's condition.

$$||S_N f(s)||_\infty \leq C \lambda_N ||f||_\infty$$

Edmund Landau discovered, in 1921, a different class of frequencies which satisfies Bohr's theorem

$$\lambda_{n+1} - \lambda_n \geq C \exp(-e^{\delta \lambda_n})$$

where C and δ are positive constants. Using a summation method of Riesz means we can similarly to that of Bohr's condition, obtain a quantitative variant for frequencies satisfying Landau's condition. This summation method was first introduced by Marcel Riesz, who extended the domain of definition for a Dirichlet series by introducing what is now known as Riesz means. If f is a general Dirichlet series, then the Riesz means of f of order k is defined as follows

$$R_x^k(f) = \sum_{\lambda_n < x} a_n e^{-\lambda_n s} \left(1 - \frac{\lambda_n}{x}\right)^k$$

We show, with the help of fundamental results inspired by M. Riesz and Hardy's classical monograph [18], that a Dirichlet series which belongs to $\mathcal{D}_\infty^{ext}(\lambda)$ is the uniform limit of its Riesz means of order k on every half-plane [Re $s > \varepsilon$], for all $\varepsilon > 0$. Meaning that, for every $k \geq 0$

$$\lim_{x \to \infty} \sum_{\lambda_n < x} a_n e^{-\lambda_n s} \left(1 - \frac{\lambda_n}{x}\right)^k = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

In [19], Schoolmann gives an estimate of $||S_N f||$ without assuming any condition on λ

$$||S_N f||_\infty \leq C \frac{\Gamma(k+1)}{k} \left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N}\right)^k ||f||_\infty$$

Using this estimate, we're able to deduce the following bound for frequencies under the condition of Landau

$$||S_N f||_\infty \leq C e^{\delta \lambda_N} ||f||_\infty$$

This thesis is mainly inspired by classical works of Bohr, Hardy, and M. Riesz. As well as more recent works of Defant, Schoolmann, H. Queffélec, and M. Queffélec.

Overview of the thesis

Chapter 1. The first chapter is an introduction to classical theorems regarding the convergence domain of Dirichlet series. We give a proof of Abel's partial summation formula, and we prove results regarding maximal distance between the abscissa of simple, uniform, and absolute convergence.

Chapter 2. The second chapter introduces the space $\mathcal{H}_\infty$ of all somewhere convergent ordinary Dirichlet series whose limit function has a bounded and holomorphic extension to $[\mathrm{Re} > 0]$. We show some important properties of series which belongs to this space, as well as proving the Perron-Landau formula. We then use a quantitative result for the partial sums of an ordinary Dirichlet series to prove Bohr's theorem.

Chapter 3. In the third chapter we introduce the space $\mathcal{D}_\infty^{ext}(\lambda)$ of all somewhere convergent general Dirichlet series whose limit function has a bounded and holomorphic extension to $[\mathrm{Re} > 0]$. We study Bohr's theorem for general Dirichlet series belonging to this space, and in particular we prove that a frequency λ which satisfies Bohr's condition, satisfies Bohr's theorem.

Chapter 4. In the fourth chapter we introduce the summation methods by typical means invented by Riesz. We show that a general Dirichlet series belonging to $\mathcal{D}_\infty^{ext}(\lambda)$ is the uniform limit of its Riesz means of order k. And in particular we show that frequencies satisfying Landau's condition, and frequencies which are $\mathbb{Q}$-linearly independent, both satisfies Bohr's theorem.

1 General Theory of ordinary Dirichlet series

We define the space $\mathcal{D}$ of all somewhere convergent ordinary Dirichlet series

$$\mathcal{D} = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \mid \sigma_c(f) < \infty \right\}$$

As stated in the introduction, the domain of convergence for Dirichlet series is defined by half-planes.

Theorem 1.1. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a somewhere convergent ordinary Dirichlet series. Then it converges on the half-plane [Re $s > \sigma_c$] and diverges on the half-plane [Re $s < \sigma_c$].*

Before we give the proof of this statement, we introduce Abel's partial summation formula, which shall be used for multiple purposes throughout the thesis. We introduce the following function for partial summation of Dirichlet coefficients

$$A(x) = \sum_{n \leq x} a_n$$

Lemma 1.2 (Abel's partial summation formula)**.** *Abel's partial summation formula relates a discrete sum to a continuous integral. Let $f(x)$ be a smooth complex-valued function, then*

$$\sum_{x<n\leq y} a_n f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)dt$$

Proof. Since $A(x)$ is defined over the integers, we set $N = \lfloor x \rfloor$ and $M = \lfloor y \rfloor$

$$\begin{aligned}
\sum_{x<n\leq y} a(n)f(n) &= \sum_{N+1}^{M} a(n)f(n) = \sum_{N+1}^{M} [A(n) - A(n-1)]f(n) \\
&= \sum_{N+1}^{M} A(n)f(n) - \sum_{N}^{M-1} A(n)f(n+1) \\
&= A(M)f(M) - A(N)f(N+1) + \sum_{N+1}^{M-1} A(n)[f(n) - f(n+1)]
\end{aligned}$$

Noticing that $\int_n^{n+1} f'(t)dt = f(n+1) - f(n)$, we get that

$$\sum_{x<n\leq y} a(n)f(n) = A(M)f(M) - A(N)f(N+1) - \sum_{N+1}^{M-1} A(n) \int_n^{n+1} f'(t)dt$$

Since $A(t) = A(n)$ over the interval $[n, n+1)$, we can move $A(n)$ inside the integral

$$\sum_{x<n\leq y} a(n)f(n) = A(M)f(M) - A(N)f(N+1) - \sum_{N+1}^{M-1} \int_n^{n+1} A(t)f'(t)dt$$
$$= A(M)f(M) - A(N)f(N+1) - \int_{N+1}^{M} A(t)f'(t)dt$$

Adjusting integral limits back to $[x, y]$

$$\int_x^y A(t)f(t)dt = \left(\int_x^{N+1} + \int_{N+1}^{M} + \int_M^y \right) A(t)f'(t)dt$$

$$-\int_{N+1}^{M} A(t)f'(t)dt = \left(\int_x^{N+1} + \int_M^y - \int_x^y \right) A(t)f'(t)dt$$

Now we use the fact that $A(t) = A(x)$ over $[x, N+1)$, and that $A(t) = A(y)$ over $[M, y]$

$$-\int_{N+1}^{M} A(t)f'(t)dt = A(x)[f(N+1) - f(x)] + A(y)[f(y) - f(M)] - \int_x^y A(t)f'(t)dt$$

which yields that

$$\sum_{x<n\leq y} a_n f(n) = A(M)f(M) - A(N)f(N+1) + A(x)f(N+1)$$
$$- A(x)f(x) + A(y)f(y) - A(y)f(M) - \int_x^y A(t)f'(t)dt$$

finally, since $A(x) = A(N)$, and $A(y) = A(M)$

$$\sum_{x<n\leq y} a_n f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)dt$$

□

We are now ready to state the following theorem, which relates the domain of convergence for Dirichlet series to half-planes. The proof is inspired by Titchmarsh [20, section 9.11], with the help of lemma 1.2.

Theorem 1.3. *If a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is convergent for some $s_0 = \sigma_0 + it_0$, then it is uniformly convergent throughout the region defined by the inequality $|\arg(s - s_0)| \leq \frac{\pi}{2} - \delta$.*

Proof. Assume that $f(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0}$ is convergent. This implies that $f(s_0)$ has bounded partial sums, i.e that for all $N \geq 1$, $|\sum_{n=1}^{N} a_n n^{-s_0}| \leq M$, for some $M > 0$. For $s \in$

$|\arg(s-s_0)| \le \frac{\pi}{2} - \delta$ we have that $Re\ s > Re\ s_0$. Let $b_n = a_n n^{-s_0}$, and let $f(n) = n^{s_0-s}$. From Abel's partial summation formula we have that

$$\sum_{x<n\le y} a_n n^{-s} = \sum_{x<n\le y} b_n f(n) = B(y)f(y) - B(x)f(x) - \int_x^y B(t)f'(t)dt$$

Using the triangle inequality, and the fact that $|B(x)| \le M$, $\forall x$

$$\begin{aligned}\left|\sum_{x<n\le y} a_n n^{-s}\right| &\le \left|My^{s_0-s}\right| + \left|Mx^{s_0-s}\right| + \left|M(s-s_0)\int_x^y t^{s_0-s-1}dt\right| \\ &\le My^{\sigma_0-\sigma} + Mx^{\sigma_0-\sigma} + M|s-s_0|\left(\frac{y^{\sigma_0-\sigma}-x^{\sigma_0-\sigma}}{\sigma_0-\sigma}\right)\end{aligned}$$

Since $y > x$, and $\sigma > \sigma_0$, we get that

$$My^{\sigma_0-\sigma} + Mx^{\sigma_0-\sigma} \le 2Mx^{\sigma_0-\sigma}$$

as well as

$$|y^{\sigma_0-\sigma} - x^{\sigma_0-\sigma}| = x^{\sigma_0-\sigma} - y^{\sigma_0-\sigma} \le x^{\sigma_0-\sigma} \le 2x^{\sigma_0-\sigma}$$

which results in

$$\begin{aligned}\left|\sum_{x<n\le y} a_n n^{-s}\right| &\le 2Mx^{\sigma_0-\sigma} + 2Mx^{\sigma_0-\sigma}\left(\frac{|s-s_0|}{\sigma-\sigma_0}\right) \\ &= 2Mx^{\sigma_0-\sigma}\left(1 + \frac{|s-s_0|}{\sigma-\sigma_0}\right)\end{aligned}$$

which tends to zero as x tends to infinity, this implies that

$$\left|\sum_{x<n\le y} a_n n^{-s}\right| = |S_y f(s) - S_x f(s)| \to 0$$

the series is Cauchy, and we have convergence for $\sigma > \sigma_0$. Furthermore, if

$$|\arg s - s_0| = \arctan\left(\frac{t-t_0}{\sigma-\sigma_0}\right) \le \frac{1}{2}\pi - \delta$$

then

$$\left(\frac{t-t_0}{\sigma-\sigma_0}\right) \le \tan\left(\frac{1}{2}\pi - \delta\right) = \cot\delta$$

and

$$\frac{|s-s_0|}{\sigma-\sigma_0} = \sqrt{1 + \frac{(t-t_0)^2}{(\sigma-\sigma_0)^2}} \le \sqrt{1+\cot^2\delta} = \csc\delta$$

hence we have that

$$\left| \sum_{x<n\leq y} a_n n^{-s} \right| \leq 2Mx^{\sigma_0-\sigma}(1+\csc\delta)$$

which is independent of s, and we therefore have uniform convergence in the region defined by $|\arg(s-s_0)| \leq \frac{\pi}{2}-\delta$. □

We give the following remark, which can be seen as a consequence of Theorem 1.3

Remark. *The limit function $f : [Re > \sigma_c(D)] \to \mathbb{C}$, given by*

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is holomorphic

Proof. We observe that any compact set in $[Re > \sigma_c(f)]$ can be included within some angular set with vertex in any point of the abscissa $[Re = \sigma_c(f)]$ by taking a wide enough angle. This implies that the series converges uniformly on every compact set in $[Re > \sigma_c(f)]$. By Weierstrass convergence theorem [1, Theorem 1, page 176], we then have that f is holomorphic. □

Theorem 1.3 extends to the general Dirichlet series as well, the proof is fairly similar to that of ordinary Dirichlet series, and can be found in [11, theorem 2.1].

1.1 Abscissa of absolute convergence

A Dirichlet series is said to be absolutely convergent on the half-plane $[\mathrm{Re} > \sigma]$, if the series formed from it by replacing each term by its absolute value is convergent.

$$\sum_{n=1}^{\infty} |a_n n^{-(\sigma+it)}| = \sum_{n=1}^{\infty} |a_n| n^{-\sigma} < \infty \quad \text{for all} \quad s \in [\mathrm{Re} > \sigma]$$

The abscissa of absolute convergence is the smallest value of σ such that the series is absolutely convergent. Same as for the abscissa of simple convergence, we define the abscissa of absolute convergence

$$\sigma_a = \inf \{\theta \in \mathbb{R} \ : \ f \text{ converges absolutely in } [\mathrm{Re} \ > \theta]\}$$

which separates the complex plane into a half-plane of absolute convergence, and a half-plane of where the series does not absolutely converge.

Remark. *If a Dirichlet series f is absolutely convergent, it is convergent, i.e*

$$\sigma_c(f) \leq \sigma_a(f) \tag{1.1}$$

This can be seen easily, since f is absolutely convergent, by a Cauchy criterion, we know that any given $\varepsilon > 0$, there exists an N_0 such that for all N and M each greater than N_0

$$\sum_{n=N+1}^{M} |a_n| n^{-\sigma} < \varepsilon$$

and since we know that the absolute value of a sum never exceeds the sum of the absolute values

$$\left| \sum_{n=N+1}^{M} a_n n^{-s} \right| \leq \sum_{n=N+1}^{M} |a_n| n^{-\sigma} < \varepsilon$$

so the absolute convergence of f implies the convergence of f, as stated.
Lemma 1.4 introduces a maximum for the difference between the abscissa of absolute and simple convergence.

Lemma 1.4. *Let* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$ *be a somewhere convergent ordinary Dirichlet series. Then* $0 \leq \sigma_a(f) - \sigma_c(f) \leq 1$

Proof. Let $\varepsilon > 0$. Convergence of $\sum_{n=1}^{\infty} a_n n^{-(\sigma_c+\varepsilon)}$ implies the following upper bound

$$\sup_n \frac{|a_n|}{n^{\sigma_c+\varepsilon}} < \infty$$

which again implies that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_c+1+\varepsilon}} < \infty$$

We therefore have that $\sigma_a(f) \leq \sigma_c(f) + 1 + \varepsilon$, from which the result follows by letting ε tend to zero.

□

1.2 Abscissa of uniform convergence

A Dirichlet series is said to be uniformly convergent to $f(s)$ on the half-plane $[\text{Re} > \sigma]$, if for an arbitrary positive ε there exists an $N_0(\varepsilon)$, independent of s, such that

$$|S_N f(s) - f(s)| < \varepsilon, \quad \text{for all} \quad N > N_0(\varepsilon), \quad \text{and for all} \quad s \in [\text{Re} > \sigma]$$

In the same manner as we defined σ_c, we also define the *abscissa of uniform convergence*, as the smallest value of σ, such that the series is uniformly convergent in all of σ. The abscissa of uniform convergence shall be denoted by σ_u

$$\sigma_u = \inf \{\theta \in \mathbb{R} \ : \ f \text{ converges uniformly in } [\text{Re } > \theta]\}$$

Example. *For the Riemann zeta function, ζ, the abscissa of simple, uniform, and absolute convergence overlap, that is, $\sigma_c(\zeta) = \sigma_u(\zeta) = \sigma_a(\zeta) = 1$. While for the alternating zeta function, also known as the Dirichlet eta function*

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$$

The abscissa of simple convergence lies at $\sigma_c(\eta) = 0$, while the abscissa of uniform and absolute convergence lies at $\sigma_u(\eta) = \sigma_a(\eta) = 1$.

As we saw in the example of the Dirichlet eta function, the relation $0 \leq \sigma_a - \sigma_c \leq 1$ is optimal. By theorem 1.4 and the fact that $\sigma_c \leq \sigma_u \leq \sigma_a$, we also achieve the following upper bound

$$\sigma_a - \sigma_u \leq 1 \tag{1.2}$$

Harald Bohr was interested in determining the maximal with of the strip, S, where an ordinary Dirichlet series converges uniformly, but not absolutely.

$$S := \sup\{\sigma_a(f) - \sigma_u(f) \ : \ f(s) = \sum_{n=1}^{\infty} a_n n^{-s}\}$$

Bohr found in [5], with a nontrivial proof, that this strip is less than one half. But even though he proved this, he had no examples of Dirichlet series for which this value was attained. In fact, he knew of no Dirichlet series where the abscissa of uniform and absolute convergence differed. In 1914, Toeplitz [21] was able to bound the maximal value of the strip from below, and showed that $S \geq 1/4$. He did this by considering Dirichlet series where $a_n \neq 0$, only when n is the product of two primes, and was able to construct examples for which the strip attained the width of one fourth. Nothing happened on the problem of Bohr for a few decades, until Bohnenblust and Hille [3], in 1931, showed that the strip is in fact equal to one half.

$$S = \frac{1}{2}$$

(1.2) can therefore immediately be improved by the following result.

Theorem 1.5. *For any Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, we have the inequality*

$$\sigma_a \leq \sigma_u + \frac{1}{2}$$

Before we prove theorem 1.5 we need the two following lemmas. The first of which is originally due to Carlson [9], and gives an expression for the square sum of the Dirichlet coefficients for an arbitrary Dirichlet series, we shall give a proof which is due to Defant et al. [10, prop 1.11]. The second lemma gives upper bounds for the abscissa of uniform and absolute convergence, and was originally shown by Cahen in [8].

Lemma 1.6. *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent in [Re > 0], then the following identity holds*

$$\sum_{n=1}^{N} |a_n|^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} a_n n^{it} \right|^2 dt \tag{1.3}$$

Proof. First, lets consider the integral

$$\frac{1}{2T} \int_{-T}^{T} \left(\frac{n}{m}\right)^{it} dt = \frac{1}{2T} \int_{-T}^{T} e^{it \log \frac{n}{m}} dt = \begin{cases} 1 & \text{if } n = m \\ \frac{1}{2iT} \frac{\sin(T \log \frac{n}{m})}{\log \frac{n}{m}} & \text{if } n \neq m \end{cases}$$

When taking the limit as T tends to infinity, we get that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{n}{m}\right)^{it} dt = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} a_n n^{it} \right|^2 dt &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_{n=1}^{N} a_n n^{it} \right) \left(\sum_{m=1}^{N} \overline{a}_m m^{-it} \right) dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{n=1}^{N} \sum_{m=1}^{N} a_n \overline{a}_m \left(\frac{n}{m}\right)^{it} dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{n=1}^{N} |a_n|^2 dt = \sum_{n=1}^{N} |a_n|^2 \end{aligned}$$

□

Lemma 1.7. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Then*

$$\sigma_u(f) \leq \limsup_{N\to\infty} \frac{\log(\sup_{t\in\mathbb{R}} |\sum_{n=1}^{N} a_n n^{-it}|)}{\log N} \tag{1.4}$$

$$\sigma_a(f) \leq \limsup_{N\to\infty} \frac{\log \left(\sum_{n=1}^{N} |a_n| \right)}{\log N} \tag{1.5}$$

Proof. For the proof of (1.4), we follow the lines of [16, theorem 4.2.1], and put

$$a = \limsup_{N\to\infty} \frac{\log(\sup_{t\in\mathbb{R}} |\sum_{n=1}^{N} a_n n^{-it}|)}{\log N}$$

Let $\varepsilon > 0$ and $A_N(t) = \sum_{n=1}^{N} a_n n^{-it}$, such that $A_0(t) = 0$, and $|A_N(t)| \leq C_\varepsilon N^{a+\varepsilon}$. Take $s = \sigma + it$ such that $\sigma \geq a + 2\varepsilon$. We then have that

$$\begin{aligned}\sum_{n=1}^{N} a_n n^{-s} &= \sum_{n=1}^{N} [A_n(t) - A_{n-1}(t)] n^{-\sigma} \\ &= \sum_{n=1}^{N} A_n(t) n^{-\sigma} - \sum_{n=1}^{N} A_{n-1}(t) n^{-\sigma} \\ &= \sum_{n=1}^{N} A_n(t) n^{-\sigma} - \sum_{n=1}^{N-1} A_n(t)(n+1)^{-\sigma} \\ &= A_N(t) N^{-\sigma} + \sum_{n=1}^{N-1} A_n(t)[n^{-\sigma} - (n+1)^{-\sigma}]\end{aligned}$$

The first term tends to zero since

$$|A_N(t) N^{-\sigma}| \leq C_\varepsilon N^{a+\varepsilon} N^{-a-2\varepsilon} = C_\varepsilon N^{-\varepsilon} \xrightarrow{N\to\infty} 0$$

We estimate the telescoping part

$$[n^{-\sigma} - (n+1)^{-\sigma}] = \sigma \int_n^{n+1} t^{-\sigma-1} dt \leq \sigma n^{-\sigma-1}$$

and therefore get that the general term for the series

$$f(s) = \sum_{n=1}^{\infty} A_n(t)[n^{-\sigma} - (n+1)^{-\sigma}] \leq \sum_{n=1}^{\infty} C_\varepsilon n^{a+\varepsilon} \sigma n^{-\sigma-1} \leq C_\varepsilon \sigma \sum_{n=1}^{\infty} \frac{1}{n^{\varepsilon+1}}$$

which converges for all $\varepsilon > 0$. Therefore $\sigma_u(f) \leq a + 2\varepsilon$, and $\sigma_u(f) \leq a$ as ε tends to zero. For the second inequality, we put

$$a = \limsup_{N\to\infty} \frac{\log\left(\sum_{n=1}^{N} |a_n|\right)}{\log N}$$

let $\varepsilon > 0$, then there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\frac{\log\left(\sum_{n=1}^{N} |a_n|\right)}{\log N} \leq a + \varepsilon$$

for all $N \geq N_\varepsilon$. For the finite number of $N < N_\varepsilon$ we let $C_\varepsilon > 0$ be a constant such that

$$\frac{\log\left(\sum_{n=1}^{N} |a_n|\right)}{\log N} \leq a + \varepsilon + \frac{\log C_\varepsilon}{\log N}$$

for all $N \in \mathbb{N}$. This implies that

$$\sum_{n=1}^{N} |a_n| \leq C_\varepsilon N^{a+\varepsilon}$$

We use lemma 1.2, where we let

$$a_n = |a_n| \quad \text{and} \quad f(t) = t^{-\sigma}$$

$$\sum_{n=1}^{N} |a_n| n^{-\sigma} = \frac{A(N)}{N^\sigma} - A(1) - \sigma \int_1^N \frac{A(t)}{t^{\sigma+1}} dt \leq C_\varepsilon \left(\frac{N^{a+\varepsilon}}{N^\sigma} + \sigma \int_1^N \frac{t^{a+\varepsilon}}{t^{\sigma+1}} dt \right)$$

We let $\sigma = a + 2\varepsilon$, and get that

$$\sum_{n=1}^{N} |a_n| n^{-\sigma} \leq C_\varepsilon \left(\frac{1}{N^\varepsilon} + \sigma \int_1^N \frac{1}{t^{\varepsilon+1}} dt \right) = C_\varepsilon \left(\frac{1}{N^\varepsilon} + \frac{\sigma}{\varepsilon} \left(1 - \frac{1}{N^\varepsilon} \right) \right) \leq C_\varepsilon \left(\frac{1}{N^\varepsilon} + \frac{\sigma}{\varepsilon} \right)$$

hence the series is absolutely convergent for $s = \sigma$ which implies that $\sigma_a(f) \leq a + 2\varepsilon$, and furthermore that $\sigma_a(f) \leq a$ as ε tends to zero. □

Proof of Theorem 1.5
Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$, we may assume that $\sigma_a(f) \leq 0$ (if this is not the case, just take $\sigma_0 > \sigma_a(f)$, and consider the translation $\sum_{n=1}^{\infty} a_n n^{-s-\sigma_0}$). We use the Cauchy-Schwarz inequality together with lemma 1.6 to get the following inequality

$$\begin{aligned} \sum_{n=1}^{N} |a_n| &\leq \left(\sum_{n=1}^{N} 1 \right)^{1/2} \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2} \\ &= N^{1/2} \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2} \\ &= N^{1/2} \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} a_n n^{it} \right|^2 dt \right)^{1/2} \\ &\leq N^{1/2} \sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right| \end{aligned}$$

Using the Bohr-Cahen formulas for $\sigma_a(f)$ and $\sigma_u(f)$ obtained in lemma 1.7 we get that

$$\begin{aligned} \sigma_a(f) = \frac{\log \sum_{n=1}^{N} |a_n|}{\log N} &\leq \frac{\log \left(N^{1/2} \sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right| \right)}{\log N} \\ &= \frac{\log N^{1/2}}{\log N} + \frac{\log \left(\sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right| \right)}{\log N} \\ &= \frac{1}{2} + \frac{\log \left(\sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right| \right)}{\log N} = \frac{1}{2} + \sigma_u(f) \end{aligned}$$

□

2 Bohr's theorem

In this chapter we are going to deduce a bound for the partial sums of an ordinary Dirichlet series. Moreover we are going to observe that the partial sums of an ordinary Dirichlet series which is somewhere convergent, and defines bounded functions in [Re > 0], are well-controlled. In proving this we are going to require what is known as the Perron-Landau formula, as well as a result that gives a uniform bound to the coefficients of a Dirichlet series. We start by giving a definition of the space $\mathcal{H}^\infty$, which is the space of all somewhere convergent ordinary Dirichlet series, whose limit function has a holomorphic and bounded extension to the right half-plane [Re > 0].

Definition 1. *We define $\mathcal{H}^\infty$ to be the space of all somewhere convergent ordinary Dirichlet series, which allow a bounded and holomorphic extension to $\mathbb{C}_0$.*

$$\mathcal{H}^\infty = \Big\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \mid f \textit{ converges and defines a bounded function in [Re } > 0] \Big\}$$

The norm on $\mathcal{H}^\infty$ is defined as the standard supremum-norm

$$||f||_\infty = \sup_{\text{Re } s > 0} |f(s)|$$

We shall see that $|| \cdot ||_\infty$ if fact defines a norm on $\mathcal{H}^\infty$ after the proof of theorem 2.2.

2.1 The Fourier-Bohr formulas

The classical Cauchy formula for calculating the coefficients of a Taylor series can be extended to Dirichlet series. The coefficients of a Taylor series can be obtained by differentiating the function f at a point a.

$$a_n = \frac{f^{(n)}(a)}{n!}$$

We can estimate the coefficients of a Dirichlet series in $\mathcal{H}^\infty$ by the Fourier-Bohr formulas defined below, the definition holds in fact for all ordinary Dirichlet series which are somewhere convergent.

Lemma 2.1 (Fourier-Bohr formulas)**.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$. Then for every $\rho > \sigma_a(f)$, and $n \in \mathbb{N}$*

$$a_n = \lim_{T\to\infty} \frac{1}{2iT} \int_{\rho - iT}^{\rho + iT} f(s) n^s ds \tag{2.1}$$

$$a_n n^{-\rho} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\rho + it) n^{it} dt \tag{2.2}$$

Proof. Fix $T > 0$, and let $\rho > \sigma_a(f)$. We then have uniform convergence of f on the strip $\{\rho + it \ : -T \leq t \leq T\}$. Consider the integral

$$\begin{aligned}\frac{1}{2iT}\int_{\rho-iT}^{\rho+iT} f(s)m^s ds &= \frac{1}{2T}\int_{-T}^{T} f(\rho+it)m^{\rho+it}dt \\ &= \frac{1}{2T}\int_{-T}^{T}\left[\sum_{n=1}^{\infty} a_n n^{-\rho-it}\right]m^{\rho+it}dt \\ &= \frac{m^\rho}{2T}\sum_{n=1}^{\infty}\frac{a_n}{n^\rho}\int_{-T}^{T}\left(\frac{m}{n}\right)^{it}dt \\ &= a_m + \frac{m^\rho}{2T}\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{a_n}{n^\rho}\int_{-T}^{T}\left(\frac{m}{n}\right)^{it}dt \\ &= a_m + m^\rho\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{a_n}{n^\rho}\left[\frac{\sin(T\log(\frac{m}{n}))}{T\log(\frac{m}{n})}\right]\end{aligned}$$

We have the following inequality

$$\left|m^\rho\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{a_n}{n^\rho}\left[\frac{\sin(T\log(\frac{m}{n}))}{T\log(\frac{m}{n})}\right]\right| \leq m^\rho\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{|a_n|}{n^\rho}\frac{1}{T|\log(\frac{m}{m+1})|} \leq \frac{C}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^\rho}$$

Where C is some constant. Since $\sum_{n=1}^{\infty}|a_n|n^{-\rho}$ is convergent, this term tends to zero as $T \longrightarrow \infty$, which leaves us with the identity

$$a_n = \lim_{T\to\infty}\frac{1}{2iT}\int_{\rho-iT}^{\rho+iT} f(s)n^s ds$$

Multiplying by $n^{-\rho}$ on both sides, and using $ds = i\ dt$, yields the second identity

$$a_n n^{-\rho} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\rho+it)n^{it}dt$$

□

We estimate the coefficients of a Dirichlet series in $\mathcal{H}^\infty$.

Theorem 2.2 (16, theorem 6.1.1)**.** *If* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty$, *then* $\sup_{n\in\mathbb{N}}|a_n| \leq ||f||_\infty$, $\forall f \in \mathcal{H}^\infty$, *and* $\sigma_a(f) \leq 1$.

Proof. Since $f \in \mathcal{H}^\infty$, we can choose $\rho > \sigma_a(f)$, which means that $\sum_{n=1}^{\infty}|a_n|n^{-\rho} < \infty$ Denote by Γ_ε, the rectangle with corners $\varepsilon - iT$, $\varepsilon + iT$, $\rho - iT$, and $\rho + iT$, where $0 < \varepsilon < \rho$.

By the Cauchy integral theorem for rectangles we have

$$\begin{aligned} 0 &= \lim_{T\to\infty} \frac{1}{2iT} \int_{\Gamma_\varepsilon} f(s) n^s ds \\ &= \lim_{T\to\infty} \frac{1}{2iT} \left(\int_{\varepsilon+iT}^{\varepsilon-iT} + \int_{\varepsilon-iT}^{\rho-iT} + \int_{\rho-iT}^{\rho+iT} + \int_{\rho+iT}^{\varepsilon+iT} \right) f(s) n^s ds \end{aligned}$$

Using (2.1), we get that

$$a_n = \lim_{T\to\infty} \frac{1}{2iT} \left(\int_{\varepsilon-iT}^{\varepsilon+iT} + \int_{\varepsilon+iT}^{\rho+iT} - \int_{\varepsilon-iT}^{\rho-iT} \right) f(s) n^s ds$$

For the second and third integral we have

$$\lim_{T\to\infty} \left| \frac{1}{2iT} \int_{\varepsilon+iT}^{\rho+iT} f(s) n^s ds \right| \leq \lim_{T\to\infty} \frac{\rho n^\rho ||f||_\infty}{2T} = 0$$

and

$$\lim_{T\to\infty} \left| \frac{1}{2iT} \int_{\varepsilon-iT}^{\rho-iT} f(s) n^s ds \right| \leq \lim_{T\to\infty} \frac{\rho n^\rho ||f||_\infty}{2T} = 0$$

and for the first integral

$$\lim_{T\to\infty} \left| \frac{1}{2iT} \int_{\varepsilon-iT}^{\varepsilon+iT} f(s) n^s ds \right| \leq n^\varepsilon ||f||_\infty$$

by lemma 2.1 it follows that $|a_n| \leq ||f||_{\mathcal{H}_\infty}$ as ε tends to zero. Moreover, having bounded coefficients implies that the Dirichlet series is absolutely convergent for [Re $s > 1 + \varepsilon$], for all $\varepsilon > 0$, and $\sigma_a(f) \leq 1$. □

We note that by theorem 2.2 it follows that the coefficients of a Dirichlet series are all equal to zero provided that $||f||_\infty = 0$, and $\mathcal{H}_\infty$ is indeed a normed space.

2.2 The Perron-Landau formula

We will now prove an important result regarding estimating the summation function $A(x) = \sum_{n\leq x} a_n$ of a Dirichlet series. Because the function $f(s)$ is holomorphic in the half-plane of absolute convergence, this can be done by looking at the behaviour of $f(s)$ on some line Re $s = \rho$, which lies in this plane.

Lemma 2.3 (Elementary Perron-Landau formulas)**.** *(14, p. 342) Let $a > 0$, and $T \to +\infty$, then for $0 < y < 1$:*

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{y^s}{s} ds = \mathcal{O}(y^a/T|\log y|) \tag{2.3}$$

and for $y > 1$:

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}ds = 1+\mathcal{O}(y^a/T\log y) \tag{2.4}$$

where the $\mathcal{O}$'s are absolute

Proof. Let $0 < y < 1$, $c > a$, and define by Γ_a the rectangle with corners $c-iT$, $c+iT$, $a+iT$, and $a-iT$. Then, by Cauchy's theorem for rectangles

$$\int_{\Gamma_a}\frac{y^s}{s}ds = 0$$

Using the triangle inequality

$$\left|\int_{a-iT}^{a+iT}\frac{y^s}{s}ds\right| \leq \left|\int_{a+iT}^{c+iT}\frac{y^s}{s}ds\right| + \left|\int_{c+iT}^{c-iT}\frac{y^s}{s}ds\right| + \left|\int_{c-iT}^{a-iT}\frac{y^s}{s}ds\right|$$

We bound the integrals

$$\left|\int_{a+iT}^{c+iT}\frac{y^s}{s}ds\right| \leq \int_a^c\frac{|y^{u+iT}|}{|u+iT|}du = \int_a^c\frac{y^u}{\sqrt{u^2+T^2}}du \leq \frac{1}{T}\int_a^c y^u du = \frac{y^c-y^a}{T\log y}$$

$$\left|\int_{a-iT}^{c-iT}\frac{y^s}{s}ds\right| \leq \int_a^c\frac{|y^{u-iT}|}{|u-iT|}du = \int_a^c\frac{y^u}{\sqrt{u^2+T^2}}du \leq \frac{1}{T}\int_a^c y^u du = \frac{y^c-y^a}{T\log y}$$

$$\left|\int_{c+iT}^{c-iT}\frac{y^s}{s}ds\right| \leq \int_{-T}^{T}\frac{|y^{c+it}|}{|c+it|}dt = y^c\int_{-T}^{T}\frac{1}{\sqrt{c^2+t^2}}dt = -y^c\log\left(\frac{-T+\sqrt{c^2+T^2}}{T+\sqrt{c^2+T^2}}\right)$$

This then yields that

$$\left|\int_{a-iT}^{a+iT}\frac{y^s}{s}ds\right| \leq \frac{2(y^c-y^a)}{T\log y} - y^c\log\left(\frac{-T+\sqrt{c^2+T^2}}{T+\sqrt{c^2+T^2}}\right)$$

Which tends to $2y^a/T\log y$ when c tends to infinity. We therefore have that

$$\left|\int_{a-iT}^{a+iT}\frac{y^s}{s}ds\right| \leq \frac{2y^a}{T|\log y|}$$

and therefore

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}ds\right| \leq \frac{y^a}{\pi T|\log y|}$$

For $0 < y < 1$, this results in

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}ds = \mathcal{O}(y^a/T|\log y|)$$

Now, let $y > 1$, and consider the rectangle Γ_{a^-} with corners $-c-iT$, $-c+iT$, $a+iT$, and $a-iT$. The point $s = 0$, which is a simple pole, lies within the rectangle, and we therefore have, by the residue theorem, that

$$\int_{\Gamma_{a^-}}\frac{y^s}{s}ds = 2\pi i$$

Using the triangle inequality

$$\left|\int_{a-iT}^{a+iT}\frac{y^s}{s}ds - 2\pi i\right| \le \left|\int_{a+iT}^{-c+iT}\frac{y^s}{s}ds\right| + \left|\int_{-c+iT}^{-c-iT}\frac{y^s}{s}ds\right| + \left|\int_{-c-iT}^{a-iT}\frac{y^s}{s}ds\right|$$

We bound the integrals as before, but this time we achieve the bounds

$$\left|\int_{-c-iT}^{a-iT}\frac{y^s}{s}ds\right| \le \frac{y^a - y^{-c}}{T\log y}$$

$$\left|\int_{a+iT}^{-c+iT}\frac{y^s}{s}ds\right| \le \frac{y^a - y^{-c}}{T\log y}$$

$$\left|\int_{-c+iT}^{-c-iT}\frac{y^s}{s}ds\right| \le y^{-c}\log\left(\frac{-T+\sqrt{c^2+T^2}}{T+\sqrt{c^2+T^2}}\right)$$

and thus, when c tends to infinity, we have for $y > 1$

$$\left|\int_{a-iT}^{a+iT}\frac{y^s}{s}ds - 2\pi i\right| \le \frac{2y^a}{T|\log y|}$$

And therefore

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}ds - 1\right| \le \frac{y^a}{\pi T|\log y|}$$

which this results in

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{y^s}{s}ds = 1 + \mathcal{O}(y^a/T\log y)$$

□

Using the elementary Perron-Landau formulas, we are now ready to define what is known as the Perron-Landau formula, for which we prove following the lines of [16, Theorem 4.2.3]

Theorem 2.4 (Perron-Landau formula)**.** *Let* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *be a somewhere convergent Dirichlet series. Let* $\rho > \max(0, \sigma_a)$*,* $T \geq 1$ *and let* $x \geq 1$*, not an integer. Then:*

$$A(x) = \frac{1}{2\pi i}\int_{\rho-iT}^{\rho+iT} f(s)\frac{x^s}{s}ds + \mathcal{O}\left[\frac{x^\rho}{T}\sum_{n\geq 1}\frac{|a_n|}{n^\rho|\log(x/n)|}\right] \tag{2.5}$$

Proof. Consider the integral

$$\frac{1}{2\pi i}\int_{\rho-iT}^{\rho+iT} f(s)\frac{x^s}{s}ds$$

By lemma 2.3, and using the substitution $y = x/n$, we get that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\rho-iT}^{\rho+iT} f(s)\frac{x^s}{s}ds &= \frac{1}{2\pi i}\int_{\rho-iT}^{\rho+iT}\sum_{n=1}^{\infty} a_n\frac{(x/n)^s}{s}ds \\
&= \sum_{n\leq x} a_n\left[1 + \mathcal{O}\left(\frac{(x/n)^\rho}{T\log(x/n)}\right)\right] + \sum_{n>x} a_n\cdot\mathcal{O}\left(\frac{(x/n)^\rho}{T|\log(x/n)|}\right) \\
&= A(x) + \left[\sum_{n=1}^{\infty} a_n\cdot\mathcal{O}\left(\frac{(x/n)^\rho}{T|\log(x/n)|}\right)\right] \\
&= A(x) + \mathcal{O}\left[\frac{x^\rho}{T}\sum_{n\geq 1}\frac{|a_n|}{n^\rho|\log(x/n)|}\right]
\end{aligned}$$

Rearranging the equation we end up with the Perron-Landau formula, which concludes the proof. □

2.3 Control of partial sums

We are now ready, by the help of the results deduced, to show that the partial sums of an ordinary Dirichlet series is well-controlled. The result was originally proved by Balasubramanian, Calado and Queffélec in [2].

Lemma 2.5. *There is a constant* $C > 0$ *such that for all* $f \in \mathcal{H}_\infty$ *and for all* $x \geq 2$

$$||S_x f||_\infty \leq C\log x||f||_\infty \tag{2.6}$$

Proof. We have from the Perron-Landau formula

$$A(x) = \frac{1}{2\pi i}\int_{\rho-iT}^{\rho+iT} f(s)\frac{x^s}{s}ds + \mathcal{O}\left[\frac{x^\rho}{T}\sum_{n\geq 1}\frac{|a_n|}{n^\rho|\log(x/n)|}\right]$$

Assume that $x = N + \frac{1}{2}$, where $N \in \mathbb{N}^*$.

If $n > x$

$$\left|\log\frac{x}{n}\right| = \log\frac{n}{x} \geq \log\frac{N+1}{N+\frac{1}{2}} \geq \frac{1}{4(N+1/2)}$$

and if $n < x$

$$\log\frac{x}{n} \geq \log\frac{N+1}{N} \geq \frac{1}{4N}$$

Which means that $|\log x/n|^{-1} = \mathcal{O}(x)$
Since $f \in \mathcal{H}_\infty$ we know that $\sigma_a(f) \leq 1$, and we can therefore let $\rho = 2$. By the fact that $|a_n| \leq ||f||_\infty$, we observe that

$$\frac{x^2}{T}\sum_{n\geq 1}\frac{|a_n|}{n^2|\log(x/n)|} \leq \frac{x^3}{T}\sum_{n\geq 1}\frac{|a_n|}{n^2} = C||f||_\infty$$

for some $C > 0$. Now, let $0 < \varepsilon < 2$ and denote by Γ_ε, the rectangle with corners $\varepsilon - iT$, $\varepsilon + iT$, $2 - iT$, and $2 + iT$. By the Cauchy integral theorem for rectangles we have

$$\begin{aligned} 0 &= \int_{\Gamma_\varepsilon} f(s)\frac{x^s}{s}ds \\ &= \left(\int_{\varepsilon+iT}^{\varepsilon-iT} + \int_{\varepsilon-iT}^{2-iT} + \int_{2-iT}^{2+iT} + \int_{2+iT}^{\varepsilon+iT}\right) f(s)\frac{x^s}{s}ds \end{aligned}$$

which yields

$$\begin{aligned} \int_{2-iT}^{2+iT} f(s)\frac{x^s}{s}ds &= \left(\int_{\varepsilon-iT}^{\varepsilon+iT} + \int_{\varepsilon+iT}^{2+iT} + \int_{2-iT}^{\varepsilon-iT}\right) f(s)\frac{x^s}{s}ds \\ &= \int_{\varepsilon-iT}^{\varepsilon+iT} f(s)\frac{x^s}{s}ds + \int_\varepsilon^2 f(u+iT)\frac{x^{u+iT}}{u+iT}du - \int_\varepsilon^2 f(u-iT)\frac{x^{u-iT}}{u-iT}du \end{aligned}$$

If we let $T = x^3$

$$\begin{aligned} \left|\int_\varepsilon^2 f(u+iT)\frac{x^{u+iT}}{u+iT}du\right| &\leq \left|\frac{x^2}{T}||f||_\infty\right| = \frac{||f||_\infty}{x} \\ \left|\int_\varepsilon^2 f(u-iT)\frac{x^{u-iT}}{u-iT}du\right| &\leq \left|\frac{x^2}{T}||f||_\infty\right| = \frac{||f||_\infty}{x} \end{aligned}$$

For the remaining integral we use the substitution $s = \varepsilon + it$

$$\begin{aligned} \int_{\varepsilon-iT}^{\varepsilon+iT} f(s)\frac{x^s}{s}ds = \int_{-T}^{T} f(\varepsilon+it)\frac{x^{\varepsilon+it}}{\varepsilon+it}\, i\, dt &\leq x^\varepsilon||f||_\infty \int_{-T}^{T}\frac{1}{\sqrt{\varepsilon^2+t^2}}dt \\ &= 2x^\varepsilon||f||_\infty\int_0^{T/\varepsilon}\frac{1}{\sqrt{u^2+1}}du \\ &\leq 2x^\varepsilon||f||_\infty\left(1 + \int_1^{T/\varepsilon}\frac{1}{u}du\right) \leq x^\varepsilon||f||_\infty\log(T/\varepsilon) \end{aligned}$$

Since ε is an arbitrary constant, we can let $\varepsilon = 1/\log x$, and we obtain

$$|A(x)| \leq ||f||_\infty \log(x^3 \log x) \leq C||f||_\infty \log x$$

Finally, fix some $s_0 \in \mathbb{C}_0$, and define $f_{s_0}(s) = \sum_{n=1}^{\infty} a_n n^{-s_0} n^{-s} = f(s+s_0)$, we have that

$$\left| S_x f(s) \right| \leq C \log x ||f_{s_0}||_\infty = C \log x \sup_{Re\ s_0 > Re\ s} |f(s_0)| \leq C \log x ||f||_\infty$$

$$||S_x f||_\infty \leq C \log x ||f||_\infty$$

□

2.4 Bohr's theorem for ordinary Dirichlet series

We are going to see that boundedness and uniform convergence of a Dirichlet series is closely related. We know that, clearly, if a Dirichlet series is uniformly convergent for some σ, then it is also bounded, which shows the inequality

$$\sigma_b \leq \sigma_u$$

Bohr proved for ordinary Dirichlet series which are somewhere convergent, that the abscissa of boundedness, and the abscissa of uniform convergence, coincide.

$$\sigma_b = \sigma_u$$

This deep result is known a consequence of what is known as Bohr's theorem, which we are now ready to state in detail. It was originally proved by Bohr in [5], but we shall prove it by the use of lemma 2.5.

Theorem 2.6. *(Bohr's Theorem). Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty$. Then f converges uniformly in [Re $> \varepsilon$] for all $\varepsilon > 0$.*

Proof. Let $f \in \mathcal{H}^\infty$ and let $\varepsilon > 0$. Then f converging uniformly in [Re $> \varepsilon$] is equivalent to $\sum_{n=1}^{\infty} a_n n^{-s-\varepsilon}$ converging uniformly in [Re > 0] for all $\varepsilon > 0$. Let $S_n(s) = \sum_{j=1}^{n} a_j j^{-s}$, with $S_0(s) = 0$. We then have the Abel transformation

$$\begin{aligned}\sum_{n=1}^{N} a_n n^{-s-\varepsilon} &= \sum_{n=1}^{N}[(S_n(s) - S_{n-1}(s)]n^{-\varepsilon} \\ &= S_N(s)N^{-\varepsilon} + \sum_{n=1}^{N-1} S_n(s)[n^{-\varepsilon} - (n+1)^{-\varepsilon}] \\ &\leq \frac{C\log N||f||_\infty}{N^\varepsilon} + \sum_{n=1}^{N-1} \frac{C_\varepsilon \log n||f||_\infty}{n^{\varepsilon+1}}\end{aligned}$$

The last step uses lemma 2.5, and the fact that

$$n^{-\varepsilon} - (n+1)^{-\varepsilon} = \varepsilon \int_n^{n+1} \frac{1}{t^{\varepsilon+1}} dt \leq \frac{\varepsilon}{n^{\varepsilon+1}}$$

This shows that $\sum_{n=1}^{\infty} a_n n^{-s-\varepsilon}$ is uniformly convergent in [Re > 0] since the bounds are independent of s, and hence that $\sigma_u(f) \leq 0$. □

Corollary 2.6.1. *For every ordinary Dirichlet series* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$,

$$\sigma_b(f) = \sigma_u(f)$$

Proof. We first note that if a Dirichlet series is nowhere convergent,then

$$\sigma_b = \sigma_u = \infty$$

We then assume that the Dirichlet series is somewhere convergent. As stated previously, we obiviously have that if a Dirichlet series is unformly convergent, then it is bounded. What remains to show is then that

$$\sigma_u(f) \leq \sigma_b(f)$$

Assume that $f(s)$ converges and defines a bounded function on [Re $s > s_0$], then the translated Dirichlet series

$$f_{s_0}(s) = \sum_{n=1}^{\infty} a_n n^{-(s+s_0)}$$

converges and defines a bounded function on [Re $s > 0$]. By Bohr's theorem this series therefore is uniformly convergent on [Re $s > \varepsilon$] for every $\varepsilon > 0$. This implies that $f(s)$ is uniformly convergent on [Re $s > s_0 + \varepsilon$] for every $\varepsilon > 0$, and we obtain the desired inequality. □

3 Bohr's condition

Bohr's theorem also extends to the general Dirichlet series, and we are now going to generalize the results from the previous chapter, where we considered the space $\mathcal{H}^\infty$ of all somewhere convergent ordinary Dirichlet series whose limit function has a holomorphic and bounded extension to [Re > 0]. In this chapter we are going to consider the space $\mathcal{D}_\infty^{ext}(\lambda)$ of all somewhere convergent general Dirichlet series whose limit function has a holomorphic and bounded extension to the half-plane [Re > 0].

Definition 2. ***The*** $\mathcal{D}_\infty^{ext}(\lambda)$***-space***

$$\mathcal{D}_\infty^{ext}(\lambda) = \left\{ f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \mid \sigma_c(f) < \infty \text{ , } f \textit{ has an holomorphic and bounded extension to } \mathbb{C}_0 \right\}$$

The norm on $\mathcal{D}_\infty^{ext}(\lambda)$ is defined as the standard supremum-norm

$$||f||_\infty = \sup_{\text{Re } s > 0} |f(s)|$$

Similarly to the case of $\mathcal{H}_\infty$, we shall after lemma 3.5 see that $|| \cdot ||_\infty$ indeed defines a norm on $\mathcal{D}_\infty^{ext}(\lambda)$. We say that a frequency λ satisfies Bohr's theorem if every $f(s) \in \mathcal{D}_\infty^{ext}(\lambda)$ converges uniformly in [Re > 0]. Bohr showed, in [5], that Bohr's theorem holds for a class of λ's satisfying a condition that shall be referred to as Bohr's condition:

Definition 3. *(Bohr's condition) We say that a frequency λ satisfies Bohr's condition (BC) if, there exists $\delta > 0$, $C > 0$ such that for all $n \in \mathbb{N}$:*

$$\lambda_{n+1} - \lambda_n \geq C e^{-\delta \lambda_n} \tag{3.1}$$

We start by showing a Bohr-Cahen type formula for the abscissa of uniform convergence equivalent to that of lemma 1.7, but for general Dirichlet series. The proof follows the same line as that for the ordinary Dirichlet series.

Lemma 3.1. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Then*

$$\sigma_u(f) \leq \limsup_{N \to \infty} \frac{\log(\sup_{t \in \mathbb{R}} |\sum_{n=1}^{N} a_n e^{-\lambda_n it}|)}{\lambda_N} \tag{3.2}$$

Proof. We put

$$a = \limsup_{N \to \infty} \frac{\log(\sup_{t \in \mathbb{R}} |\sum_{n=1}^{N} a_n e^{-\lambda_n it}|)}{\lambda_N}$$

Let $\varepsilon > 0$ and $A_N(t) = \sum_{n=1}^{N} a_n e^{-\lambda_n it}$, such that $A_0(t) = 0$, and $|A_N(t)| \leq C_\varepsilon e^{\lambda_N (a+\varepsilon)}$. Take $s = \sigma + it$ such that $\sigma \geq a + 2\varepsilon$

We then have that

$$\begin{aligned}\sum_{n=1}^{N} a_n e^{-\lambda_n s} &= \sum_{n=1}^{N} [A_n(t) - A_{n-1}(t)] e^{-\lambda_n \sigma} \\ &= \sum_{n=1}^{N} A_n(t) e^{-\lambda_n \sigma} - \sum_{n=1}^{N} A_{n-1}(t) e^{-\lambda_n \sigma} \\ &= \sum_{n=1}^{N} A_n(t) e^{-\lambda_n \sigma} - \sum_{n=1}^{N-1} A_n(t) e^{-\lambda_{n+1} \sigma} \\ &= A_N(t) e^{-\lambda_N \sigma} + \sum_{n=1}^{N-1} A_n(t) [e^{-\lambda_n \sigma} - e^{-\lambda_{n+1} \sigma}]\end{aligned}$$

The first term tends to zero since

$$|A_N(t) e^{-\lambda_N \sigma}| \leq C_\varepsilon e^{\lambda_N (a+\varepsilon)} e^{-\lambda_N (a+2\varepsilon)} = C_\varepsilon e^{-\lambda_N \varepsilon} \xrightarrow{N \to \infty} 0$$

We estimate the telescoping part

$$e^{-\lambda_n \sigma} - e^{-\lambda_{n+1} \sigma} = \sigma \int_{\lambda_n}^{\lambda n+1} e^{-t\sigma} dt \leq \sigma(\lambda_{n+1} - \lambda_n) e^{-\lambda_n \sigma}$$

and therefore get that the general term for the series

$$f(s) = \sum_{n=1}^{\infty} A_n(t) [e^{-\lambda_n \sigma} - e^{-\lambda_{n+1} \sigma}] \leq \sum_{n=1}^{\infty} C_\varepsilon \sigma (\lambda_{n+1} - \lambda_n) e^{\lambda_n (a+\varepsilon)} e^{-\lambda_n \sigma} \leq C_\varepsilon \sigma \sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{e^{\lambda_n \varepsilon}}$$

which converges for all $\varepsilon > 0$. Therefore $\sigma_u(f) \leq a + 2\varepsilon$, and furthermore that $\sigma_u(f) \leq a$ as ε tends to zero. □

3.1 Determination of somewhere absolute convergence

In lemma 1.4 from chapter 1, we saw that the distance between the abscissa of simple and absolute convergence for ordinary Dirichlet series is at most one. This implies that if an ordinary Dirichlet series is somewhere convergent, then it is also somewhere absolutely convergent. This property also obviously implies, by definition, that all the defined abscissas neatly lies inside a strip of width one for any somewhere convergent ordinary Dirichlet series. This fact is not necessarily true for general Dirichlet series. We may have Dirichlet series where the distance between the abscissa of simple and absolute convergence is not even finite. In our proof for lemma 2.5 we relied on the fact that the Dirichlet series in question had a finite abscissa of absolute convergence. In the following lemma we determine an upper bound for the distance between the abscissa of simple and absolute convergence for a general Dirichlet series, and we shall later see that if a frequency λ satisfies Bohr's condition, then this distance is finite. And since we're considering Dirichlet series which belong to the $\mathcal{D}^{\infty}_{ext}(\lambda)$-space of somewhere convergent general Dirichlet series, we're can then confirm that we have a finite abscissa of absolute convergence, and we're able to apply similar techniques as that in lemma 2.5 to proving such a frequency satisfies Bohr's theorem.

Lemma 3.2 (18, Theorem 9)**.** *Let* $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$*. Then*

$$L(\lambda) := \limsup_{n\to\infty} \frac{\log n}{\lambda_n} \geq \sigma_a(f) - \sigma_c(f)$$

Proof. Assume that $\sigma_c(f) > 0$. The truth of the lemma is obviously independent of this restriction since by a simple translation argument, we have that if

$$f_{s_0}(s) = \sum_{n=1}^{\infty} = a_n e^{\lambda_n(s+s_0)}$$

then

$$\sigma_a(f) - \sigma_c(f) = \sigma_a(f_{s_0}) - \sigma_c(f_{s_0})$$

Given $\delta > 0$, we can choose $n_0 \in \mathbb{N}$ so that, for $n > n_0$:

$$|A(n)| < e^{(\sigma_c+\delta)\lambda_n}$$

and

$$|a_n| = |A(n) - A(n-1)| < |A(n)| + |A(n-1)| < 2e^{(\sigma_c+\delta)\lambda_n} < e^{(\sigma_c+2\delta)\lambda_n}$$

where the last inequality uses the fact that $2 < e^{\delta\lambda_n}$, which is obviously possible. Furthermore, if n is sufficiently large in comparison to n_0

$$\sum_{1}^{n} |a_i| < \sum_{1}^{n_0} |a_i| + ne^{(\sigma_c+2\delta)\lambda_n} < ne^{(\sigma_0+3\delta)\lambda_n}$$

and finally

$$\sigma_a \leq \frac{\log \sum_1^n |a_i|}{\lambda_n} < \frac{\log n}{\lambda_n} + \sigma_c + 3\delta$$

$$\sigma_a - \sigma_c \leq \limsup_{n\to\infty} \frac{\log n}{\lambda_n}$$

□

An immediate consequence of this lemma is that if λ is a frequency such that $L(\lambda) = 0$, then the abscissa of simple, and absolute convergence coincide, which again leads to the implication that $\sigma_b(f) = \sigma_u(f)$, or equivalently, λ satisfies Bohr's theorem. We also note that if $\lambda_n = \log n$, which is that for ordinary Dirichlet series, then $L(\lambda) = 1$, which is the result from lemma 1.4.

Lemma 3.3 (6, lemma 3)**.** *Let λ be a sequence of non negative real numbers tending to $+\infty$, then*

$$L(\lambda) = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} = \sigma_c\left(\sum e^{-\lambda_n s}\right)$$

Before we prove lemma 3.6, we give the following result to show what if a frequency satisfies Bohr's condition, we have a finite abscissa of absolute convergence. The result is due to Bohr [6, lemma 4]

Lemma 3.4. *Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \in \mathcal{D}_{ext}^{\infty}(\lambda)$ and λ a frequency which satisfies (BC), then $L(\lambda) < \infty$. And in particular f has a finite abscissa of absolute convergence.*

Proof. Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ and assume that λ satisfies Bohr's condition. By Lemma 3.3 it is sufficient to show that the series $\sum_{n=1}^{\infty} e^{-\lambda_n s}$ converges for some $\sigma_c < \infty$. Since λ satisfies Bohr's condition we have that for some $C, \delta > 0$

$$\lambda_{n+1} - \lambda_n \geq C e^{-\lambda_n \delta}$$

Let $N(\varepsilon) = N$ such that for all $n \geq N$, and every $\varepsilon > 0$ we have that

$$\lambda_{n+1} - \lambda_n \geq e^{-\lambda_n(\delta + \frac{\varepsilon}{2})}$$

It follows that for $n \geq N$

$$\begin{aligned} e^{-\lambda_{n+1}(\delta+\varepsilon)} &= e^{-\lambda_{n+1}\frac{\varepsilon}{2}} e^{-\lambda_{n+1}(\delta+\frac{\varepsilon}{2})} \\ &\leq e^{-\lambda_{n+1}\frac{\varepsilon}{2}} e^{-\lambda_{n}(\delta+\frac{\varepsilon}{2})} \\ &\leq e^{-\lambda_{n+1}\frac{\varepsilon}{2}} (\lambda_{n+1} - \lambda_n) \\ &\leq \int_{\lambda_n}^{\lambda_{n+1}} e^{-\frac{\varepsilon}{2}x} dx \end{aligned}$$

and

$$\sum_{n=1}^{\infty} e^{-\lambda_{n+1}(\delta+\varepsilon)} = \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} e^{-\frac{\varepsilon}{2}x} dx = \int_{\lambda_1}^{\infty} e^{-\frac{\varepsilon}{2}x} dx$$

The right hand side is convergent for all $\varepsilon > 0$, and we therefore have that

$$\sigma_a(f) - \sigma_c(f) \leq L(\lambda) = \sigma_c\left(\sum_{n=1}^{\infty} e^{-\lambda_n s}\right) < \infty$$

□

3.2 The Fourier-Bohr formulas for general Dirichlet series

Lemma 3.5 (Fourier-Bohr formulas for general Dirichlet series)**.** *Let* $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \in \mathcal{D}_{\infty}^{ext}(\lambda)$. *Then for every* $\rho > \sigma_a(f)$, *and* $n \in \mathbb{N}$

$$a_n = \lim_{T\to\infty} \frac{1}{2T} \int_{\rho-iT}^{\rho+iT} f(\rho+it)e^{\lambda_n(\rho+it)}dt \tag{3.3}$$

$$a_n e^{-\lambda_n \rho} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\rho+it)e^{\lambda_n it}dt \tag{3.4}$$

for all $\sigma > 0$. *In particular,* $\sup_{n\in\mathbb{N}} |a_n| \leq ||f||_{\infty}$.

Proof. Fix $T > 0$, since $f \in \mathcal{D}_{\infty}^{ext}(\lambda)$, we know that $\sigma_a(f) < \infty$. Let $\rho > \sigma_a(f)$. We then have uniform convergence of f on the strip $\{\rho + it \ : -T \leq t \leq T\}$. Consider the integral

$$\begin{aligned}
\frac{1}{2iT}\int_{\rho-iT}^{\rho+iT} f(s)e^{\lambda_m s}ds &= \frac{1}{2T}\int_{-T}^{T} f(\rho+it)e^{\lambda_m(\rho+it)}dt \\
&= \frac{1}{2T}\int_{-T}^{T}\left[\sum_{n=1}^{\infty} a_n e^{-\lambda_n(\rho-it)}\right]e^{\lambda_m(\rho+it)}dt \\
&= \frac{e^{\lambda_m\rho}}{2T}\sum_{n=1}^{\infty}\frac{a_n}{e^{\lambda_n\rho}}\int_{-T}^{T}\left(\frac{e^{\lambda_m}}{e^{\lambda_n}}\right)^{it}dt \\
&= a_m + \frac{e^{\lambda_m\rho}}{2T}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{a_n}{e^{\lambda_n\rho}}\int_{-T}^{T}\left(\frac{e^{\lambda_m}}{e^{\lambda_n}}\right)^{it}dt \\
&= a_m + e^{\lambda_m\rho}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{a_n}{e^{\lambda_n\rho}}\left[\frac{\sin(T(\lambda_m-\lambda_n))}{T(\lambda_m-\lambda_n)}\right]
\end{aligned}$$

We have the following inequality

$$\left|e^{\lambda_m\rho}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{a_n}{e^{\lambda_n\rho}}\left[\frac{\sin(T(\lambda_m-\lambda_n))}{T(\lambda_m-\lambda_n)}\right]\right| \leq e^{\lambda_m\rho}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{|a_n|}{e^{\lambda_n\rho}}\frac{1}{|T(\lambda_m-\lambda_{m+1})|} \leq \frac{C}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{e^{\lambda_n\rho}}$$

Where C is some constant. Since $\sum_{n=1}^{\infty} |a_n|e^{-\lambda_n\rho}$ is convergent, this term tends to zero as $T \longrightarrow \infty$, which leaves us with the identity

$$a_n = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\rho+it)e^{\lambda_n(\rho+it)}dt$$

Multiplying by $e^{-\rho\lambda_n}$ on both sides, yields the second identity

$$a_n e^{-\lambda_n\rho} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\rho+it)e^{\lambda_n it}dt$$

□

We make the remark that the previous lemma confirms that $||.||_\infty$ in fact is a norm. This follows from the observation that if $||f||_\infty = 0$, then the coefficients $|a_n|$ all vanish, and we get the implication

$$||f||_\infty = 0 \quad \text{if and only if} \quad f(s) = 0$$

3.3 Bohr's theorem under Bohr's condition

We are now ready to prove the main result of this chapter. Namely that a frequency satisfying Bohr's condition (3.1) satisfies Bohr's theorem. The proof follows similar lines as that of lemma 2.5.

Lemma 3.6. *Let λ be a frequency satisfying Bohr's condition. Then λ satisfies Bohr's theorem.*

Proof. Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \in \mathcal{D}_\infty^{ext}(\lambda)$, and assume λ satisfies Bohr's condition. We then need to show that $f(s)$ converges uniformly in [Re >0]. We first generalize the Perron-Landau formula for general Dirichlet series

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\rho-iT}^{\rho+iT} f(s)\frac{x^s}{s}ds &= \frac{1}{2\pi i}\sum_{n=1}^{\infty} a_n \int_{\rho-iT}^{\rho+iT}\left(xe^{-\lambda_n}\right)^s \frac{1}{s}ds \\
&= \sum_{\lambda_n<\log x} a_n\left[1+\mathcal{O}\left(\left(xe^{-\lambda_n}\right)^\rho \frac{1}{T\log(xe^{-\lambda_n})}\right)\right] \\
&+ \sum_{\lambda_n>\log x} a_n\left[\mathcal{O}\left(\left(xe^{-\lambda_n}\right)^\rho \frac{1}{T\log|xe^{-\lambda_n}|}\right)\right] \\
&= \sum_{\lambda_n<\log x} a_n + \mathcal{O}\left[\frac{x^\rho}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{e^{\rho\lambda_n}\log(xe^{-\lambda_n})}\right]
\end{aligned}$$

Let $\log x = \frac{\lambda_{N+1}+\lambda_N}{2}$. If $xe^{-\lambda_n} < 1$:

$$|\log xe^{-\lambda_n}| = |\log x - \lambda_n| = \left|\frac{\lambda_{N+1}+\lambda_N}{2} - \lambda_n\right| \geq \left|\frac{\lambda_{N+1}+\lambda_N}{2} - \lambda_N\right| = \frac{\lambda_{N+1}-\lambda_N}{2}$$

and if $xe^{-\lambda_n} > 1$:

$$\log xe^{-\lambda_n} \geq \frac{\lambda_{N+1}-\lambda_N}{2}$$

Since λ satisfies Bohr's condition we have that

$$\lambda_{n+1} - \lambda_n \geq Ce^{-\delta\lambda_n}$$

Which shows that $(\log xe^{-\lambda_n})^{-1} = \mathcal{O}(e^{\delta\lambda_N})$
Since λ satisfies (BC), we know by lemma 3.4 that $\sigma_a(f) < \infty$, and we can therefore let $\rho > \sigma_a(f)$. By the fact that $|a_n| \leq ||f||_\infty$, we observe that

$$\frac{x^\rho}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{e^{\rho\lambda_n}\log(xe^{-\lambda_n})} \leq \frac{Cx^\rho}{T}\sum_{n\geq 1}\frac{|a_n|}{e^{\rho\lambda_n}} = C'||f||_\infty$$

for some $C' > 0$. Now, let $0 < \varepsilon < 2$ and denote by Γ_ε, the rectangle with corners $\varepsilon - iT$, $\varepsilon + iT$, $\rho - iT$, and $\rho + iT$. By the Cauchy integral theorem for rectangles we have

$$\begin{aligned} 0 &= \int_{\Gamma_\varepsilon} f(s)\frac{x^s}{s}ds \\ &= \left(\int_{\varepsilon+iT}^{\varepsilon-iT} + \int_{\varepsilon-iT}^{\rho-iT} + \int_{\rho-iT}^{\rho+iT} + \int_{\rho+iT}^{\varepsilon+iT}\right) f(s)\frac{x^s}{s}ds \end{aligned}$$

which yields

$$\begin{aligned} \int_{\rho-iT}^{\rho+iT} f(s)\frac{x^s}{s}ds &= \left(\int_{\varepsilon-iT}^{\varepsilon+iT} + \int_{\varepsilon+iT}^{\rho+iT} + \int_{\rho-iT}^{\varepsilon-iT}\right) f(s)\frac{x^s}{s}ds \\ &= \int_{\varepsilon-iT}^{\varepsilon+iT} f(s)\frac{x^s}{s}ds + \int_\varepsilon^\rho f(u+iT)\frac{x^{u+iT}}{u+iT}du - \int_\varepsilon^\rho f(u-iT)\frac{x^{u-iT}}{u-iT}du \end{aligned}$$

If we let $T = x^{\rho+1}$, we have for the second and third integral

$$\left|\int_\varepsilon^\rho f(u+iT)\frac{x^{u+iT}}{u+iT}du\right| \leq \left|\frac{x^\rho}{T}||f||_\infty\right| = \frac{||f||_\infty}{x}$$

$$\left|\int_\varepsilon^\rho f(u-iT)\frac{x^{u-iT}}{u-iT}du\right| \leq \left|\frac{x^\rho}{T}||f||_\infty\right| = \frac{||f||_\infty}{x}$$

For the remaining integral we use the substitution $s = \varepsilon + it$

$$\begin{aligned} \int_{\varepsilon-iT}^{\varepsilon+iT} f(s)\frac{x^s}{s}ds &= \int_{-T}^{T} f(\varepsilon+it)\frac{x^{\varepsilon+it}}{\varepsilon+it}\, i\,dt \leq x^\varepsilon||f||_\infty\int_{-T}^{T}\frac{1}{\sqrt{\varepsilon^2+t^2}}dt \\ &= 2x^\varepsilon||f||_\infty\int_0^{T/\varepsilon}\frac{1}{\sqrt{u^2+1}}du \leq 2x^\varepsilon||f||_\infty\left(1+\int_1^{T/\varepsilon}\frac{1}{u}du\right) \leq x^\varepsilon||f||_\infty\log(T/\varepsilon) \end{aligned}$$

Since ε is an arbitrary constant, we can let $\varepsilon = Te^{-\lambda_N}$, and we obtain

$$\left|\sum_{\lambda_n<\log x} a_n\right| \leq C\lambda_N||f||_\infty \tag{3.5}$$

Finally, fix some $s_0 \in \mathbb{C}_0$, and define $f_{s_0}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n(s+s_0)} = f(s+s_0)$, we have that

$$\left|S_N f(s)\right| \leq C\lambda_N ||f_{s_0}||_\infty = C\lambda_N \sup_{Re\ s_0 > Re\ s} |f(s_0)| \leq C\lambda_N ||f||_\infty$$

$$||S_N f||_\infty \leq C\lambda_N ||f||_\infty$$

It remains to show that we have uniform convergence in [Re $s > \varepsilon$], for all $\varepsilon > 0$. We use our result from (3.5) together with lemma 3.1

$$\sigma_u(f) \leq \limsup_{N\to\infty} \frac{\log ||S_N f||_\infty}{\lambda_N} \leq \limsup_{N\to\infty} \frac{\log C\lambda_N ||f||_\infty}{\lambda_N} = 0$$

and λ satisfies Bohr's theorem. □

Landau later found, in [15], a weaker class of frequencies which satisfies Bohr's theorem. In the following chapter we shall give the definition of this condition, and give a proof that frequencies satisfying this condition satisfies Bohr's theorem.

4 Riesz summability of general Dirichlet series

We have, until now, only been concerned with convergent Dirichlet series. The way of defining the sum of series which is not convergent, but oscillating, was generalized by Cesàro, in which the sum of the series is defined as the limit of the arithmetic mean of its first n partial sums, since a series may not have a partial sum that converges to a given values, meanwhile the series arithmetic means may converge. We say that a series is Cesàro summable to the sum σ if

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{1}{n+1} \sum_{n=0}^{\infty} s_n = \sigma$$

where

$$s_n = a_0 + a_1 + \cdots + a_n$$

Cesàros method of summation has shown to be useful in the study of, among other things, theory of Fourier analysis. Joseph Fourier stated that almost any real valued function could be represented by a trigonometric series on the form

$$\sum_{n\in\mathbb{Z}} a_n e^{inx}$$

It was believed for some time, that the Fourier series for some function f will converge to the value of $f(x)$, at all points of continuity of the function. However, in [12, p. 572], D.B Reymond disproved this, when he showed that additional conditions were required for the Fourier series to converge to its associated function. This weakened the reliability of the theory of Fourier series. L. Fejer in [13, p. 51] reedemed the situation by showing that the Fourier series of a continuous function is always summable by the method of arithmetic means. That is, the expression

$$\sigma_N = \frac{1}{N+1} \sum_{0}^{N} S_n(x)$$

tends to $f(x)$ with x tending to infinity. This result is known as Fejér's theorem, and the trigonometric polynomials $\sigma_N(x)$ are often called the Fejér means of f.

Applications of Cesàro summability was shown to be useful in the study of Dirichlet series as well, since the domain of convergence may be extended further than for that of the series itself. A function which is continuous may not necessarily converge to its associated Dirichlet series. Riesz and Bohr showed independently that the arithmetic means formed in Cesàro's manner from an ordinary Dirichlet series may have domains of convergence more extensive than that of the series itself [18, p. 20]. To understand what the concept of summability does for a Dirichlet series, let us again consider the example of the Dirichlet eta function.

We know from earlier that this function converges for [Re $s > 0$], and diverges for all [Re $s < 0$]. However, at the point $s = 0$, the series reduces to

$$\sum_{n=1}^{N}(-1)^{n+1}$$

the value of this series is either 0 or 1, depending on if N is even or odd. Therefore the series is not convergent. It is however, Cesàro summable to the sum $\frac{1}{2}$.

4.1 Riesz means

Riesz worked on the problem of extending the domain of definiton for a Dirichlet series by substituting summability in the place of convergence as the criterion. He found that the arithmetic means shown above are not very well adapted to the study of general Dirichlet series as certain other means formed in different manners, which led to the following generalization to that of Cesàro's means, which is known as Riesz means. [22, p.27]. For $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ a Dirichlet series, Riesz defined the summation method

$$S_N^1(f) = \lambda_N \sum_{n=1}^{N} a_n e^{-\lambda_n s} - \sum_{n=1}^{N} \lambda_n a_n e^{-\lambda_n s} = \sum_{n=1}^{N} a_n e^{-\lambda_n s}(\lambda_N - \lambda_n) \tag{4.1}$$

and more generally

$$S_N^k(f) = \sum_{n=1}^{N} a_n e^{-\lambda_n s}(\lambda_N - \lambda_n)^k \tag{4.2}$$

where k is known as the order of summability. In [17], Riesz modified his method of summation by introducting a continuous parameter $x > 0$, and express the sum as

$$S_x^k(f) = \sum_{\lambda_n < x} a_n e^{-\lambda_n s}(x - \lambda_n)^k = k\int_0^x S_t(f)(x-t)^{k-1}dt \tag{4.3}$$

where

$$S_t(f) = S_t^0(f) = \sum_{\lambda_n < t} a_n e^{-\lambda_n s} \tag{4.4}$$

and

$$R_x^k(f) = \frac{S_x^k(f)}{x^k} = \sum_{\lambda_n < x} a_n e^{-\lambda_n s}\left(1 - \frac{\lambda_n}{x}\right)^k = \frac{k}{x^k}\int_0^x S_t(f)(x-t)^{k-1}dt \tag{4.5}$$

are said to be the Riesz means of f of order k. If $\lim_{x\to\infty} R_x^k(f) = f(s)$, we say that $\sum a_n e^{-\lambda_n s}$ is (λ, k)-Riesz summable to f. This leads us to the following proposition, which states that, given any order $k > 0$, then on every half-plane [Re $> \varepsilon$], $\varepsilon > 0$, the limit function of a Dirichlet series $f \in D_\infty^{ext}(\lambda)$ is the uniform limit of its Riesz means of order k.

Proposition 1 (19, Proposition 3.4)**.** *Let* $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \in \mathcal{D}_{\infty}^{ext}(\lambda)$ *with extension* $\tilde{f}$*. Then for all* $k > 0$ *the Dirichlet polynomials*

$$R_x^k(f) = \sum_{\lambda_n < x} a_n e^{-\lambda_n s}\left(1 - \frac{\lambda_n}{x}\right)^k$$

converge uniformly to $\tilde{f}$ *on [Re* $> \varepsilon$*] as* $x \to \infty$ *for all* $\varepsilon > 0$*.*

Before we prove this statement, we first deduce expressions for, respectively, increasing and decreasing the order of a summatory function $S_x^k(f)$.

Lemma 4.1 (18, Lemma 6)**.** *Let* f *be a Dirichlet series. If* $k > 0$*,* $\mu > 0$*, then*

$$S_x^{k+\mu}(f) = \frac{\Gamma(k+\mu+1)}{\Gamma(k+1)\Gamma(\mu)} \int_0^x S_u^k(f)(x-u)^{\mu-1} du$$

and if $k > 0$*,* $\mu < 1$*,* $\mu \leq k$*, then*

$$S_x^{k-\mu}(f) = \frac{\Gamma(k-\mu+1)}{\Gamma(k+1)\Gamma(1-\mu)} \int_0^x \frac{d}{du} S_u^k(f)(x-u)^{-\mu} du$$

Proof. For the first identity we use the expression for $S_\lambda^k(u)$ given in 4.3

$$\begin{aligned}\int_0^x S_u^k(f)(x-u)^{\mu-1} du &= \int_0^x k\left(\int_0^u S_t(f)(u-t)^{k-1} dt\right)(x-u)^{\mu-1} du \\ &= k\int_0^x S_t(f) \int_t^x (u-t)^{k-1}(x-u)^{\mu-1} du dt\end{aligned}$$

We use the expression deduced in A.6, and get that

$$\begin{aligned}k\int_0^x S_t(f) \int_t^x (u-t)^{k-1}(x-u)^{\mu-1} du dt &= k\frac{\Gamma(k)\Gamma(\mu)}{\Gamma(k+\mu)} \int_0^x S_t(f)(x-t)^{k+\mu-1} dt \\ &= \frac{k}{k+\mu} \frac{\Gamma(k)\Gamma(\mu)}{\Gamma(k+\mu)} S_t^{k+\mu}(f) \\ &= \frac{\Gamma(k+1)\Gamma(\mu)}{\Gamma(k+\mu+1)} S_x^{k+\mu}(f)\end{aligned}$$

by rearranging the formula we get the desired result.
For the second part of the lemma, we use the following identity, which is a consequence of [11, Lemma 4.1]

$$\frac{d}{dx} S_x^{k-\mu+1}(f) = (k-\mu+1) S_x^{k-\mu}(f)$$

Since $-\mu+1>0$, we can use the first part of the lemma to get that

$$S_x^{k-\mu+1}(f)=\frac{\Gamma(k-\mu+2)}{\Gamma(k+1)\Gamma(1-\mu)}\int_0^x S_u^k(f)(x-u)^{-\mu}du$$

which yields that

$$\begin{aligned} S_x^{k-\mu}(f) &= \frac{1}{k-\mu+1}\frac{\Gamma(k-\mu+2)}{\Gamma(k+1)\Gamma(1-\mu)}\frac{d}{dx}\int_0^x S_u^k(f)(x-u)^{-\mu}du \\ &= \frac{\Gamma(k-\mu+1)}{\Gamma(k+1)\Gamma(1-\mu)}\frac{d}{dx}\int_0^x S_u^k(f)(x-u)^{-\mu}du \end{aligned}$$

For the integral we use integration by parts

$$\begin{aligned} \int_0^x S_u^k(f)(x-u)^{-\mu}du &= \left[\frac{S_u^k(f)(x-u)^{1-\mu}}{\mu-1}\right]_0^x + \frac{1}{1-\mu}\int_0^x \frac{d}{du}S_u^k(f)(x-u)^{1-\mu}du \\ &= \frac{1}{1-\mu}\int_0^x \frac{d}{du}S_u^k(f)(x-u)^{1-\mu}du \end{aligned}$$

and finally we get that

$$\begin{aligned} S_x^{k-\mu}(f) &= \frac{\Gamma(k-\mu+1)}{\Gamma(k+1)\Gamma(1-\mu)}\frac{d}{dx}\frac{1}{1-\mu}\int_0^x \frac{d}{du}S_u^k(f)(x-u)^{1-\mu}du \\ &= \frac{\Gamma(k-\mu+1)}{\Gamma(k+1)\Gamma(1-\mu)}\int_0^x \frac{d}{du}S_u^k(f)(x-u)^{-\mu}du \end{aligned}$$

and we are done. □

We define the following abscissa, which characterizes the largest possible half-plane on which f is uniformly (λ, k)-Riesz summable.

$$\sigma_u^k(f)=\inf\left\{\theta\in\mathbb{R} : f \text{ uniformly } (\lambda,k)\text{-Riesz summable in } [\mathrm{Re}>\theta]\right\}$$

The upper bound for this abscissa can be determined for $0<k\le 1$ by a Bohr-Cahen type formula, which is deduced in [19, lemma 3.8]

$$\sigma_u^k \le \limsup_{x\to\infty}\frac{\log(||R_x^k(f)||_\infty)}{x} \tag{4.6}$$

Theorem 4.2. *If the series is summable $(\lambda.k)$, where $k > 0$, and $\rho > Re\ s_0$,*

$$\sum_{\lambda_n < x} a_n e^{-\lambda_n s_0}(x-\lambda_n)^k = \frac{\Gamma(k+1)}{2\pi i}\int_{\rho - i\infty}^{\rho+i\infty} \frac{f(s)}{(s-s_0)^{k+1}} e^{x(s-s_0)} ds$$

Proof. Let $\rho > 0$. We consider the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{e^s}{s^{k+1}} ds$$

where γ is the path shown in Figure 1. We know by Cauchy's integral theorem that

$$\frac{1}{2\pi i}\int_\gamma \frac{e^s}{s^{k+1}} ds = 0$$

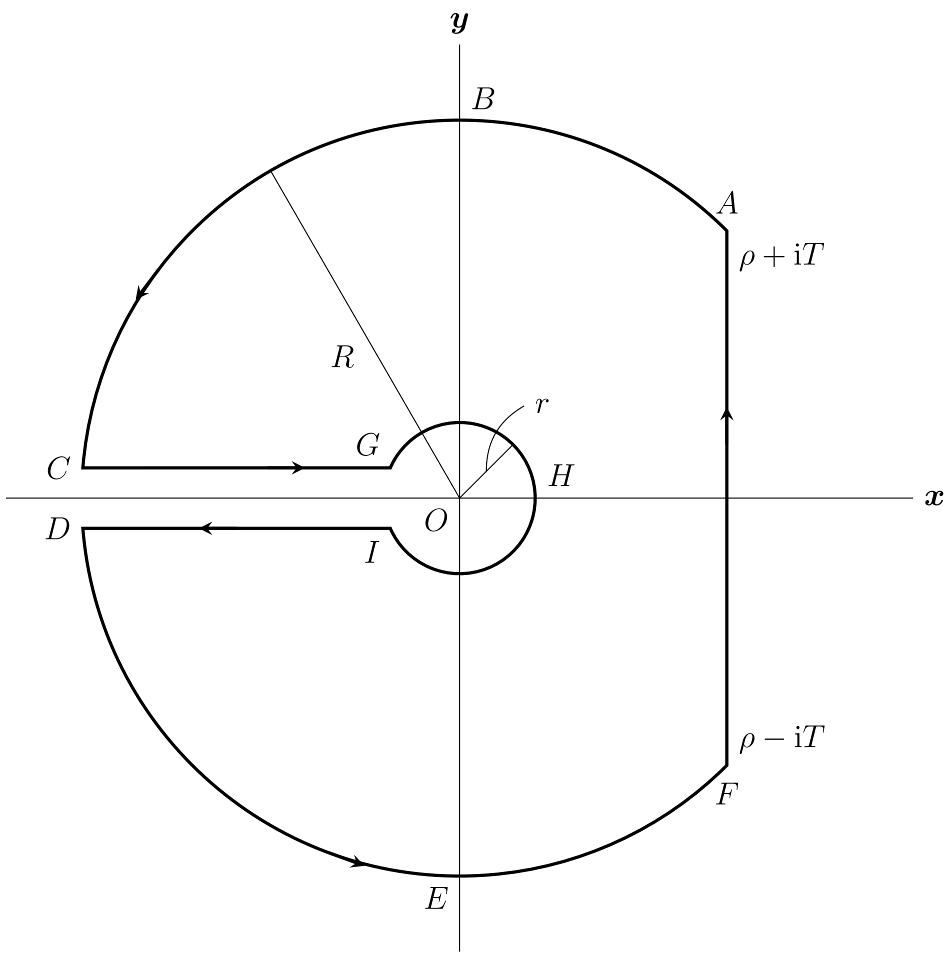

Figure 1: The path γ

We let $r \to 0$, $T \to \infty$, and notice that R tends to $+\infty$ with T. We start by considering the circular paths connecting the points ABC and DEF. Let $s = Re^{i\theta}$, and

$$I_1 = \int_{BC} \frac{e^s}{s^{k+1}} ds = \int_{\pi/2}^{\pi} \frac{e^{R(\cos\theta + i\sin\theta)}}{(Re^{i\theta})^{k+1}} Rie^{i\theta} d\theta$$

For R large and $k > -1$ we have that

$$\left| \frac{1}{(Re^{i\theta})^{k+1}} \right| < \varepsilon$$

for some $\varepsilon > 0$, and therefore

$$|I_1| < \varepsilon R \int_{\pi/2}^{\pi} e^{R\cos\theta} d\theta = \varepsilon R \int_0^{\pi/2} e^{-R\sin\alpha} d\alpha$$

where $\theta = \alpha + \pi/2$. We use the inequality obtained in (A.1), and so

$$|I_1| < \varepsilon R \int_0^{\pi/2} e^{-\frac{R2\alpha}{\pi}} d\alpha = \frac{\varepsilon\pi}{2}(1 - e^{-R})$$

We therefore get that, for $k > -1$: when $R \to \infty$, $\varepsilon \to 0$, and so $I_1 \to 0$. In evaluating the integral along DE we obtain the same type bound. We may replace the arc AB by a line from B perpendicular to the line AF. Thus, we let $s = x + iy$

$$I_2 = \int_{AB} \frac{e^s}{s^{k+1}} ds = -\int_0^{\rho} \frac{e^{x+iy}}{s^{k+1}} ds$$

$$|I_2| \le e^{\rho} \int_0^{\rho} \frac{1}{|x+iy|^{k+1}} dx \le \rho e^{\rho} \varepsilon$$

which, for $k > -1$, goes to zero as $R \to \infty$ and $\varepsilon \to 0$. A similar result is obtained for the arc EF. What we're now left with is, for $k > -1$:

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{\rho - iT}^{\rho + iT} \frac{e^s}{s^{k+1}} ds = -\frac{1}{2\pi i} \left(\int_{CG} + \int_{GHI} + \int_{ID} \right) \frac{e^s}{s^{k+1}} ds$$

For the integral around the smaller circle GHI, let $s = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$$-\frac{1}{2\pi i} \int_{GHI} \frac{e^s}{s^{k+1}} ds = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{r(\cos\theta + i\sin\theta)}}{(re^{i\theta})^{k+1}} ire^{i\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{r(\cos\theta + i\sin\theta) - i\theta k} r^{-k} d\theta$$

which for $k < 0$, tends to zero as $r \to 0$. For the path CG, we let $s = te^{i\pi}$

$$I_3 = -\frac{1}{2\pi i} \int_{CG} \frac{e^s}{s^{k+1}} ds = \frac{e^{-\pi i k}}{2\pi i} \int_0^{\infty} \frac{e^{-t}}{t^{k+1}} dt$$

For the path DI, we let $s = te^{-i\pi}$

$$I_4 = -\frac{1}{2\pi i} \int_{DI} \frac{e^s}{s^{k+1}} ds = \frac{-e^{\pi i k}}{2\pi i} \int_0^{\infty} \frac{e^{-t}}{t^{k+1}} dt$$

We sum I_3 and I_4

$$I_3 + I_4 = \frac{\sin(-k\pi)}{\pi} \int_0^\infty e^{-t} t^{-k-1} dt = -\frac{\sin(k\pi)}{\pi} \Gamma(-k)$$

we use Eulers reflection formula given in lemma A.1, and get that, finally

$$\frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \frac{e^s}{s^{k+1}} ds = \frac{\Gamma(-k)}{\Gamma(-k)\Gamma(1+k)} = \frac{1}{\Gamma(k+1)} \tag{4.7}$$

For which our result is only valid for $-1 < k < 0$. We consider the improper integral, and apply integration by parts

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{\rho+it}}{(\rho+it)^{k+1}} dt &= \lim_{T\to\infty} \frac{1}{2\pi} \left(\left[\frac{e^{\rho+it}}{i(\rho+it)^{k+1}} \right]_{-T}^{T} + (k+1) \int_{-T}^{T} \frac{e^{\rho+it}}{(\rho+it)^{k+2}} dt \right) \\
&= \lim_{T\to\infty} \frac{k+1}{2\pi} \int_{-T}^{T} \frac{e^{\rho+it}}{(\rho+it)^{k+2}} dt = \frac{1}{\Gamma(k+1)}
\end{aligned}$$

We substitute back in $s = \rho + it$, divide both sides by $k+1$, and use (A.3)

$$\frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \frac{e^s}{s^{k+2}} ds = \frac{1}{\Gamma(k+2)}$$

Which is valid for $-1 < k < 0$, and if we let $\tilde{k} = k+1$, then we get the same expression as (4.7), which is now valid for $0 < \tilde{k} < 1$. Hence by an induction argument, and the fact that $1/\Gamma(k+1)$ is analytic for all k, (4.7) can be extended and is valid for $k > -1$.
We wish to replace s by xs. We let $\omega = xs$, and $\eta = x\rho$, with $x > 0$ and consider

$$\frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} \frac{e^\omega}{\omega^{k+1}} d\omega = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \frac{e^{xs}}{(xs)^{k+1}} x ds = \frac{x^{-k}}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \frac{e^{xs}}{s^{k+1}} ds = \frac{1}{\Gamma(k+1)}$$

multiplying both sides by x^k gives

$$\frac{1}{2\pi i} \int_{\rho - iT}^{\rho + iT} \frac{e^{xs}}{s^{k+1}} ds = \frac{x^k}{\Gamma(k+1)}$$

and for $x \leq 0$:

$$\frac{1}{2\pi i} \int_{\rho - iT}^{\rho + iT} \frac{e^{xs}}{s^{k+1}} ds = 0$$

and we therefore get that

$$\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{e^{xs}}{s^{k+1}}ds=\begin{cases}\frac{x^k}{\Gamma(k+1)} & \text{if } x>0\\ 0 & \text{if } x\leq 0\end{cases}$$

The next part of the proof follows similar lines to that of the Perron-Landau formula given in theorem 2.4. We consider the following integral, where $f(s)=\sum_{n=1}^{\infty}a_ne^{-\lambda_n s}$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s)\frac{e^{xs}}{s^{k+1}}ds=\sum_{n=1}^{\infty}a_n\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{s(x-\lambda_n)}}{s^{k+1}}ds=\sum_{\lambda_n\leq x}a_n\frac{(x-\lambda_n)^k}{\Gamma(k+1)}$$

multiplying both sides by $\Gamma(k+1)$ gives the identity

$$\sum_{\lambda_n\leq x}a_n(x-\lambda_n)^k=\frac{\Gamma(k+1)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f(s)}{s^{k+1}}e^{xs}ds$$

And more generally we have, for $\rho > Re\ s_0$

$$\sum_{\lambda_n<x}a_ne^{-\lambda_n s_0}(x-\lambda_n)^k=\frac{\Gamma(k+1)}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{f(s)}{(s-s_0)^{k+1}}e^{x(s-s_0)}ds$$

□

Proof of Proposition 1
We first show that $R_x^k(f)$ is convergent. Let $x>0$ and $\varepsilon=\frac{1}{x}$. We fix some $s_0=\sigma_0+it_0$, $\sigma_0>0$. $c=\sigma_0+\varepsilon$. Applying theorem 4.2, we get that

$$\begin{aligned}\sup_{\sigma_0>0}\left|\sum_{\lambda_n<x}a_ne^{-\lambda_n s_0}\left(1-\frac{\lambda_n}{x}\right)^k\right| &\leq \left|\frac{\Gamma(k+1)}{2\pi i}\frac{1}{x^k}\int_{c-i\infty}^{c+i\infty}\frac{f(s)}{(s-s_0)^{k+1}}e^{x(s-s_0)}ds\right|\\ &\leq ||f||_\infty\frac{\Gamma(k+1)}{\pi}\frac{e^{x\varepsilon}}{x^k}\int_0^\infty\frac{1}{|\varepsilon+it|^{k+1}}dt\\ &\leq ||f||_\infty\frac{e}{\pi}\frac{\Gamma(k+1)}{k} \qquad (4.8)\end{aligned}$$

Hence we have that $R_x^k(f)$ is convergent, and by (4.6) we have that $f(s)$ is (λ,k)-Riesz summable for $0<k\leq 1$.
Now, let $k>1$, and $k=k'+l$, where $l\in\mathbb{N}$, and $0<k\leq 1$. By lemma 4.1 we have that

$$R_x^k(f)=\frac{S_x^{k'+l}(f)}{x^k}=\frac{\Gamma(k+1)}{\Gamma(k'+1)\Gamma(l)}\frac{1}{x^k}\int_0^x\left(\sum_{\lambda_n<u}a_ne^{\lambda_n s}(u-\lambda_n)^{k'}\right)(x-u)^{l-1}du$$

We use the definition of the Beta function (A.4), and the relation between the Beta and the Gamma function (A.5) to see that

$$\frac{\Gamma(k'+1)\Gamma(l)}{\Gamma(k+1)} = \beta(k'+1,l) = \frac{1}{x^k}\int_0^x u^{k'}(x-u)^{l-1}du := \frac{1}{C}$$

Furthermore, we again use lemma 4.1, and consider

$$\begin{aligned}
R_x^k(f) - f(s) &= R_x^k(f) - f(s)\frac{C}{x^k}\int_0^x u^{k'}(x-u)^{l-1}du \\
&= \frac{C}{x^k}\int_0^x \Bigg(\sum_{\lambda_n<u} a_n e^{\lambda_n s}(u-\lambda_n)^{k'}\Bigg)(x-u)^{l-1}du - f(s)\frac{C}{x^k}\int_0^x u^{k'}(x-u)^{l-1}du \\
&= \frac{C}{x^k}\int_0^x u^{k'}(x-u)^{l-1}\Big(R_\lambda^{k'}(u) - f(s)\Big)du
\end{aligned}$$

We take the absolute value, and separate the integral at some finite point x_0

$$\begin{aligned}
|R_x^k(f) - f(s)| &= \left|\frac{C}{x^k}\left(\int_0^{x_0} + \int_{x_0}^x\right)u^{k'}(x-u)^{l-1}\Big(R_\lambda^{k'}(u) - f(s)\Big)du\right| \\
&\leq \sup_{y\geq 0}||R_y^{k'}(f) - f(s)||_\infty \frac{C}{x^k}\int_0^{x_0} u^{k'}(x-u)^{l-1}du + \varepsilon\frac{C}{x^k}\int_{x_0}^x u^{k'}(x-u)^{l-1}du \\
&\leq ||f||_\infty \frac{C}{x^k}x_0 \sup_{u\in[0,x_0]} u^{k'}(x-u)^{l-1} + \varepsilon\frac{C}{x^k}(x-x_0)\sup_{u\in[x_0,x]} u^{k'}(x-u)^{l-1} \\
&\leq ||f||_\infty \frac{C}{x^k}x_0 x^{k-1} + \varepsilon\frac{C}{x^k}x^k = C\Bigg(||f||_\infty\frac{x_0}{x} + \varepsilon\Bigg)
\end{aligned}$$

Which finally shows that

$$\lim_{x\to\infty} R_\lambda^k(x) = f(s)$$

□

4.2 Boundedness of partial sums

We introduce the following lemma, which gives a bound of the partial sums for a general Dirichlet series

Lemma 4.3 (19, Lemma 3.5). *If $f(s) = \sum_{n=1}^\infty a_n e^{-\lambda_n s}$, and $0 < k \leq 1$, we have for all $N \in \mathbb{N}$*

$$\left|\sum_{n=1}^N a_n\right| \leq 3\left(\frac{1}{\lambda_{N+1}-\lambda_N}\right)^k \sup_{0\leq x\leq\lambda_{N+1}}\left|\sum_{\lambda_n<x} a_n(x-\lambda_n)^k\right| \tag{4.9}$$

Before we give the proof, we need the following lemma

Lemma 4.4 (18, Lemma 7)**.** *If a_n is real for all n, $0 \leq x \leq \omega$, and $0 < k \leq 1$, then*

$$\left| k \int_0^x \sum_{n=1}^{\infty} a_n (\omega - t)^{k-1} dt \right| \leq \sup_{0 \leq t \leq x} |A_t^k(f)| \tag{4.10}$$

where $A_t^k(f) = \sum_{\lambda_n < t} a_n (t - \lambda_n)^k$

Proof of lemma 4.3.

$$\begin{aligned} \left| \sum_{n=1}^{N} a_n \right| (\lambda_{N+1} - \lambda_N)^k &= \left| \sum_{n=1}^{N} a_n \right| k \int_{\lambda_N}^{\lambda_{N+1}} (\lambda_{N+1} - t)^{k-1} dt \\ &= \left| k \int_{\lambda_N}^{\lambda_{N+1}} A_t(f) (\lambda_{N+1} - t)^{k-1} dt \right| \\ &= \left| k \left(\int_0^{\lambda_{N+1}} - \int_0^{\lambda_N} \right) A_t(f) (\lambda_{N+1} - t)^{k-1} dt \right| \\ &\leq \left| \sum_{n=1}^{N} a_n (\lambda_{n+1} - \lambda_n)^k \right| + \left| k \int_0^{\lambda_N} A_t(f) (\lambda_{N+1} - t)^{k-1} dt \right| \\ &\leq \sup_{0 \leq x \leq \lambda_{N+1}} \left| \sum_{\lambda_n \leq x} a_n (x - \lambda_n)^k \right| + \left| k \int_0^{\lambda_N} A_t(f) (\lambda_{N+1} - t)^{k-1} dt \right| \end{aligned}$$

the fourth step uses (4.3) and the triangle inequality. We bound the integral as follows

$$\left| k \int_0^{\lambda_N} A_t(f) (\lambda_{N+1} - t)^{k-1} dt \right| \leq$$
$$\left| k \int_0^{\lambda_N} \left(\sum_{\lambda_n < t} Re\ a_n \right) (\lambda_{N+1} - t)^{k-1} dt \right| + \left| k \int_0^{\lambda_N} \left(\sum_{\lambda_n < t} Im\ a_n \right) (\lambda_{N+1} - t)^{k-1} dt \right|$$

Since $|\text{Re } a_n| \leq |a_n|$ and $|\text{Im } a_n| \leq |a_n|$, for all n, and using (4.10), we get that

$$\begin{aligned} \left| k \int_0^{\lambda_N} A_t(f) (\lambda_{N+1} - t)^{k-1} dt \right| &\leq 2 \sup_{0 \leq x \leq \lambda_{N+1}} |A_x^k(f)| \\ &= 2 \sup_{0 \leq x \leq \lambda_{N+1}} \left| \sum_{\lambda_n < x} a_n (x - \lambda_n)^k \right| \end{aligned}$$

which finally yields that

$$\left| \sum_{n=1}^{N} a_n \right| \leq 3 \left(\frac{1}{\lambda_{N+1} - \lambda_N} \right)^k \sup_{0 \leq x \leq \lambda_{N+1}} \left| \sum_{\lambda_n \leq x} a_n (x - \lambda_n)^k \right|$$

□

Lemma 4.5 (19, theorem 3.2)**.** *For all $0 < k \leq 1$, $N \in \mathbb{N}$ and $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \in \mathcal{D}_{\infty}^{ext}(\lambda)$ we have*

$$\sup_{[Re>0]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| \leq C \frac{\Gamma(k+1)}{k} \left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^k ||f||_\infty$$

where $C > 0$ is a universal constant and Γ denotes the Gamma function.

Proof. For $0 < k \leq 1$ we have the following inequality when $\lambda_N \leq x \leq \lambda_{N+1}$

$$(x - \lambda_N)^k \leq \lambda_{N+1}^k \left(1 - \frac{\lambda_N}{x}\right)^k$$

Furthermore we get that

$$\sup_{0 \leq x \leq \lambda_{N+1}} \left| \sum_{\lambda_n < x} a_n e^{-\lambda_n s} (x - \lambda_n)^k \right| \leq \lambda_{N+1}^k \sup_{0 \leq x \leq \lambda_{N+1}} \left| \sum_{\lambda_n < x} a_n e^{-\lambda_n s} \left(1 - \frac{\lambda_n}{x}\right)^k \right|$$
$$\leq \lambda_{N+1}^k \frac{e}{\pi} \frac{\Gamma(k+1)}{k} ||f||_\infty$$

Where the last step uses the bound obtained in (4.8). This result together with lemma 4.3 gives us that

$$\sup_{[Re>0]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| \leq 3 \frac{e}{\pi} \frac{\Gamma(k+1)}{k} \left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^k ||f||_\infty$$

□

4.3 Bohr's theorem under Landau's condition

We are now ready to give the definition of Landau's condition.

Definition 4. *(Landau's condition) We say that a frequency λ satisfies Landaus's condition (LC) if, for all $\delta > 0$, there exists $C > 0$, so that for all $n \in \mathbb{N}$:*

$$\lambda_{n+1} - \lambda_n \geq C \exp(-e^{\delta \lambda_n})$$

Having established a bound for the norm of the partial sums of a given Dirichlet series, we use this together with our bound for the abscissa of uniform convergence given in (3.1) to prove that the condition of Landau satisfies Bohr's theorem.

Lemma 4.6. *Let λ be a frequency satisfying (LC). Then λ satisfies Bohr's theorem.*

Proof. Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ be a Dirichlet series, and assume w.l.o.g that $\lambda_{n+1} - \lambda_n \leq 1$ for all n. If $\lambda_{n+1} - \lambda_n > 1$ for some n, we define a new frequency λ^2, by the method described in [19, page 17], which satisfies (LC) and $\lambda_{n+1} - \lambda_n \leq 1$. Let $\delta > 0$, and set $k_N = e^{-\delta\lambda_N}$. By lemma 4.5 we have that

$$||S_N f||_\infty \leq C \frac{\Gamma(k_N+1)}{k_N} \left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^{k_N} ||f||_\infty$$

Since λ satisfies (LC), we have that

$$C \exp(-k_N^{-1}) \leq \lambda_{N+1} - \lambda_N \leq 1$$

for some $C > 0$. We have the following bounds

$$\Gamma(k_N+1) \leq 1$$

$$\left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^{k_N} \leq \left(\frac{\lambda_{N+1}}{C \exp(-k_N{}^{-1})} \right)^{k_N} = \left(\frac{1}{C} \lambda_{N+1} \exp\left(e^{\delta\lambda_N}\right) \right)^{k_N} = \frac{1}{C^{k_N}} \lambda_{N+1}^{k_N} e$$

$$\sup \lambda_{N+1}^{k_N} = \lim_{\delta \to 0} (\lambda_N + 1)^{e^{-\delta\lambda_N}} = \lambda_N + 1$$

$$\sup \frac{1}{C^{k_N}} = \max\left(\frac{1}{C}, 1\right) = K$$

for some constant K. We therefore have the bound

$$||S_N f||_\infty \leq C \frac{K}{k_N} ||f||_\infty = C_1 e^{\delta\lambda_N} ||f||_\infty \tag{4.11}$$

where C_1 is some positive constant. It remains to show that we have uniform convergence in [Re $s > \varepsilon$], for all $\varepsilon > 0$. We use our result from (4.11) together with lemma 3.1

$$\begin{aligned}
\sigma_u(f) &\leq \limsup_{N\to\infty} \frac{\log ||S_N f||_\infty}{\lambda_N} \\
&\leq \limsup_{N\to\infty} \frac{\log C_1 e^{\delta\lambda_N} ||f||_\infty}{\lambda_N} \\
&= \limsup_{N\to\infty} \frac{\delta\lambda_N + \log C_1 ||f||_\infty}{\lambda_N} \\
&= \delta + \limsup_{N\to\infty} \frac{\log C_1 ||f||_\infty}{\lambda_N} = \delta
\end{aligned}$$

Letting δ tend to zero we get that

$$\sigma_u(f) \leq 0$$

and λ satisfies Bohr's theorem.

□

4.4 Kronecker's approximation theorem

We give another application to that of Riesz-means of first kind, which is that λ's which are $\mathbb{Q}$-linearly independent satisfies Bohr's theorem. We first give the definition of $\mathbb{Q}$-linear independence.

Definition 5. *($\mathbb{Q}$-linearly independent) We say that a frequency $\lambda = (\lambda_n)$ is a $\mathbb{Q}$-linearly independent frequency if for all rational sequences $q = (q_n)$*

$$\sum q_n \lambda_n = 0$$

implies that $q_n = 0$ for all n.

Bohr showed in [4] that for general Dirichlet series where λ is a $\mathbb{Q}$-linearly independent frequency satisfies the identity

$$\sigma_b^{ext} = \sigma_a \tag{4.12}$$

where

$$\sigma_b^{ext} = \inf \left\{ \theta \in \mathbb{R} \;:\; f \text{ allows a holomorphic and bounded extension to } [\mathrm{Re} > \theta] \right\}$$

and since we have the general relation $\sigma_b^{ext} \leq \sigma_b \leq \sigma_u \leq \sigma_a$, we have that λ satisfies Bohr's theorem. Before we show that Bohr's theorem holds for linearly independent frequencies, we give a famous result shown originally by Kronecker, but we shall give a proof which is due to Bohr and Jessen. [7]

Lemma 4.7. *(Kronecker's approximation theorem). Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real, linearly-independent numbers, and let $\mu_1, \mu_2, \ldots, \mu_N$ be a sequence of real numbers. Then there exists $t \in \mathbb{R}$ and a sequence $g_1, g_2, \ldots, g_N$ of integers such that for all $\varepsilon > 0$, and for all $n = 1, 2, \ldots, N$*

$$|t\lambda_n - \mu_n - g_n| < \varepsilon$$

Proof. The lemma is equivalent to the statement that the complex number

$$e^{2\pi i(\lambda_n t - \mu_n)}$$

differ by less than ε from $e^0 = 1$, for all $n = 1, 2, \ldots, N$. We define the function f as

$$f(t) = 1 + \sum_{n=1}^{N} e^{2\pi i(\lambda_i t - \mu_i)} \tag{4.13}$$

and define it's upper bound by Γ

$$\sup_{t\in\mathbb{R}} |f(t)| = \Gamma$$

we immediately notice that $\Gamma \leq N + 1$. We need to show that $\Gamma \geq N + 1$.
The Fejér kernel is defined as follows

$$F_N(t) = \frac{1}{N}\sum_{k=0}^{N-1} D_k(x) = 1 + \frac{n-1}{n}(e^{-it} + e^{it}) + \dots$$

Where $D_k(x)$ denotes the k'th order Dirichlet kernel. The Fejér kernel has the property that it is always non-negative, as well as for all $t \in \mathbb{R}$ the mean

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F_N(t)dt = 1$$

We define the composed kernel

$$K_n(t) = \prod_{n=1}^{N} F_N(2\pi(\lambda_n t - \mu_n))$$

Since λ is linearly independent, we obtain by multiplying out

$$K_n(t) = 1 + \frac{n-1}{n}(e^{-2\pi i(\lambda_1 t - \mu_1)} + e^{-2\pi i(\lambda_2 t - \mu_2)} + \dots + e^{-2\pi i(\lambda_N t - \mu_N)}) + R(t)$$

where $R(t)$ is a trigonometric polynomial whose exponents are all different from $0, -2\imath\lambda_1, \dots, -2\pi\lambda_N$.
Hence we have that

$$F(t)K_n(t) = 1 + \frac{n-1}{n}N + S(t)$$

where $S(t)$ is a trigonometric polynomial whose exponents are all different from zero. Same as for the Fejér kernel, the composed kernel has the property that it is always non-negative, as well as for all $t \in \mathbb{R}$ the mean

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} K_n(t)dt = 1$$

$$1 + \frac{n-1}{n}N = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F(t)K_n(t)dt \leq \max_{t\in\mathbb{R}} |F(t)| \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} K_n(t)dt = \Gamma$$

and finally

$$1 + N = \lim_{n\to\infty}\left(1 + \frac{n-1}{n}N\right) \leq \Gamma$$

□

We also need the following lemma, which is due to Bohr [4, Theorem 2b]

Lemma 4.8. *Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where λ is a $\mathbb{Q}$-linearly independent frequency. Let $\sigma_0 > \sigma_a(f)$, then for every $\varepsilon > 0$, there exists $t_0 \in \mathbb{R}$ such that*

$$\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n \sigma_0} - \left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n(\sigma_0 + it_0)} \right| < \varepsilon$$

Proof. Let $q_n = |a_n| e^{-\lambda_n \sigma_0}$, and since $f(s)$ is absolutely convergent for σ_0, we can set $\sum_{n=1}^{\infty} q_n = R$. We define a domain D as follows;
If for some q_i, $i = 1, 2, \dots$, we have that

$$q_i > \sum_{n \neq i} q_n = R - q_i$$

we let

$$D := \left\{ z \in \mathbb{C} : r \leq |z| \leq R \right\}$$

where $r = q_i - \sum_{n \neq i} q_n = 2q_i - R$
Else, we let

$$D := \left\{ z \in \mathbb{C} : |z| \leq R \right\}$$

We can see that $|f(s_0 + it_0)|$ takes values in D, since if we have case 1,

$$r = q_i - \sum_{n \neq i} q_n \leq \left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n(\sigma_0 + it_0)} \right| \leq \sum_{n=1}^{\infty} q_n = R$$

and if case 2,

$$\left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n(s_0 + it_0)} \right| \leq R$$

Need to show that $\sum_{n=1}^{\infty} q_n$ is dense in D. For any $z \in D$, $\delta > 0$, we need to show existence of some $T \in \mathbb{R}$ such that

$$|f(\sigma_0 + iT) - z| < \delta$$

Pick $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} q_n < \delta$$

we now use a result which we shall not prove, but whose details can be found in [4, Lemma 1]. The result states that there exists a sequence $\phi_1, \phi_2, \dots \in \mathbb{R}$ such that the series

$$\sum_{n=1}^{\infty} q_n e^{i\phi_n}$$

is dense in D. We therefore set this series equal to z. So for any $t \in \mathbb{R}$, it follows that

$$
\begin{aligned}
|f(\sigma_0 + it) - z| &= \left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n(\sigma_0+it_0)} - \sum_{n=1}^{\infty} q_n e^{i\phi_n} \right| \leq \left| \sum_{n=1}^{\infty} |a_n| e^{i\alpha_n} e^{-\lambda_n(\sigma_0+it_0)} \right| \\
&\leq \left| \sum_{n=1}^{N} |a_n| e^{-\lambda_n\sigma_0} \left(e^{i(\alpha_n - \lambda_n t)} - e^{i\phi_n} \right) \right| + \left| \sum_{n=N+1}^{\infty} |a_n| e^{-\lambda_n\sigma_0} \left(e^{i(\alpha_n - \lambda_n t)} - e^{i\phi_n} \right) \right| \\
&\leq \left| \sum_{n=1}^{N} |a_n| e^{-\lambda_n\sigma_0} e^{i(\alpha_n - \lambda_n t)} \right| + \left| \sum_{n=1}^{N} |a_n| e^{-\lambda_n\sigma_0} e^{i\phi_n} \right| + \frac{2\delta}{3} \\
&\leq \sum_{n=1}^{N} |a_n| e^{-\lambda_n\sigma_0} \left| e^{i(\alpha_n - \lambda_n t)} \right| \left| 1 - e^{i(\phi_n + \lambda_n t - \alpha_n)} \right| + \frac{2\delta}{3} \\
&= \sum_{n=1}^{N} |a_n| e^{-\lambda_n\sigma_0} \left| 1 - \exp\left(2\pi i \left(\frac{\lambda_n t}{2\pi} - \frac{\alpha_n - \phi_n}{2\pi} \right) \right) \right| + \frac{2\delta}{3}
\end{aligned}
$$

finally, by lemma 4.7, we know that there exists $g_1, g_2, \cdots \in \mathbb{N}$, and $t \in \mathbb{Z}$, such that for all $\varepsilon > 0$

$$
\left| \frac{\lambda_n t}{2\pi} - \frac{\alpha_n - \phi_n}{2\pi} - g_n \right| < \varepsilon
$$

which means that the exponent is arbitrarily close to an integer for all n, and we finally get that

$$
\sum_{n=1}^{N} |a_n| e^{-\lambda_n\sigma_0} \left| 1 - \exp\left(2\pi i \left(\frac{\lambda_n t}{2\pi} - \frac{\alpha_n - \phi_n}{2\pi} \right) \right) \right| < \frac{\delta}{3}
$$

and

$$
|f(\sigma_0 + iT) - z| < \delta
$$

□

4.5 Bohr's theorem under linearly-independent frequencies

We are now ready to prove that for $\mathbb{Q}$-linearly independent frequencies, we have the identity from (4.12)

Lemma 4.9 (4, Theorem 6)**.** *Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where λ is a $\mathbb{Q}$-linearly independent frequency. Then $f(s)$ is absolutely convergent for $\sigma > \sigma_b^{ext}(f)$*

Proof. Assume that f is somewhere absolutely convergent. Let $\beta = \sigma_b^{ext}(f)$. We then have that for [Re $s > \beta$]:

$$|f(s)| < K$$

for some real K. Let $\delta > 0$, we want to prove that $f(s)$ is absolutely convergent for [Re $s \geq \beta + \delta$]. Let $s_0 = \beta + \delta + it$
By Cauchy's integral formula for the n'th derivative

$$f'(\beta + \delta + it) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - (\beta + \delta + it))^2} ds$$

where C is the circle with center $\beta+\delta+it$, and radius δ. We use the substitution $s = s_0 + \delta e^{i\theta}$

$$|f'(\beta + \delta + it)| \leq \frac{K}{2\pi} \int_0^{2\pi} \frac{1}{\delta^2} \delta d\theta = \frac{K}{\delta} = K'$$

Now, let $\gamma > \beta + \delta$, where $\gamma > \sigma_a(f)$. Let $T \in \mathbb{R}$, $x > 1$, and $s_0 = \beta + \delta + iT$. We have by lemma 4.2 for $k = 1$:

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{f(s)}{(s - s_0)^2} e^{x(s - s_0)} ds = \sum_{\lambda_n < x} a_n e^{-\lambda_n s_0} (x - \lambda_n)$$

define by Γ the rectangle with corners $\beta - iV$, $\beta + iV$, $\gamma + iV$, and $\gamma - iV$. We set

$$F(s) = \frac{f(s)}{(s - s_0)^2} e^{x(s - s_0)}$$

and use the limit formula for higher-order poles

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma F(s) ds &= \text{Res } (F(s), s_0) = \lim_{s \to s_0} \frac{d}{ds} (s - s_0)^2 F(s) \\ &= \lim_{s \to s_0} f'(s) e^{x(s - s_0)} + x f(s) e^{x(s - s_0)} = f'(s_0) + x f(s_0) \end{aligned}$$

We separate the integral

$$\frac{1}{2\pi i} \int_{\gamma - iV}^{\gamma + iV} F(s) ds = \frac{1}{2\pi i} \left(\int_{\gamma - iV}^{\beta - iV} + \int_{\beta - iV}^{\beta + iV} + \int_{\beta + iV}^{\gamma + iV} \right) F(s) ds + f'(s_0) + x f(s_0) \tag{4.14}$$

We have the following bounds for the integrals parallell to the real axis

$$\left| \int_{\gamma - iV}^{\beta - iV} F(s) ds \right| \leq \left| \int_{\gamma - iV}^{\beta - iV} \frac{K}{(t - T)^2} e^{x(\gamma - (\beta + \delta))} dt \right| \leq \left| \int_{\gamma - iV}^{\beta - iV} \frac{K}{(t - T)^2} dt \right|$$

which tends to zero as V tends to infinity, we get the same bound for the integral between $\beta + iV$ and $\gamma + iV$. For the remaining integral

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\beta-iV}^{\beta+iV} F(s)ds\right| &= \left|\frac{1}{2\pi}\int_{-V}^{V}\frac{f(\beta+it)}{(\beta+it-(\beta+\delta+iT))^2}e^{x(\beta+it-(\beta+\delta+iT))}dt\right| \\ &\leq \frac{Ke^{-x\delta}}{2\pi}\int_{-V}^{V}\frac{1}{\delta^2+(t-T)^2}dt = \frac{Ke^{-x\delta}}{2\pi}\int_{-V-T}^{V-T}\frac{1}{\delta^2+u^2}du \\ &\leq K_1 e^{-x\delta} < K_1\end{aligned}$$

By (4.14) and 4.2 we're left with

$$\begin{aligned}\left|\sum_{\lambda_n<x} a_n e^{\lambda_n s_0}(x-\lambda_n)\right| &= \frac{1}{2\pi i}\int_{\beta-iV}^{\beta+iV} F(s)ds + f'(s_0) + xf(s) \\ &< K_1 + K' + xK < x(K_1 + K' + K) = xK_2\end{aligned}$$

where K_2 is independent of T and x. By lemma 4.8 we have that

$$\sum_{n=1}^{\infty}|a_n|e^{-\lambda_n(\beta+\delta)}(x-\lambda_n) - \left|\sum_{n=1}^{\infty} a_n e^{-\lambda_n(\beta+\delta+it_0)}(x-\lambda_n)\right| < \varepsilon$$

and for $t_0 = T$

$$\sum_{n=1}^{\infty}|a_n|e^{-\lambda_n(\beta+\delta)}(x-\lambda_n) < \varepsilon + xK_2 < x(\varepsilon + K_2) = xK_3$$

furthermore

$$\sum_{\lambda_n<\frac{x}{2}}|a_n|e^{-\lambda_n(\beta+\delta)}\left(\frac{x}{2}\right) \leq \sum_{\lambda_n<\frac{x}{2}}|a_n|e^{-\lambda_n(\beta+\delta)}(x-\lambda_n) \leq \sum_{n=1}^{\infty}|a_n|e^{-\lambda_n(\beta+\delta)}(x-\lambda_n) \leq xK_3$$

$$\sum_{\lambda_n<\frac{x}{2}}|a_n|e^{-\lambda_n(\beta+\delta)} \leq 2K_3 = K_4$$

for all $x > 1$. Since K_4 is independent of x, we can let x tend to infinity, and we finally get that

$$\sum_{n=1}^{\infty}|a_n|e^{-\lambda_n(\beta+\delta)} \leq K_4 < \infty$$

and hence $f(s)$ is absolutely convergent for all s such that [Re $s > \sigma_b^{ext}(f)$], and moreover, λ satisfies Bohr's theorem. □

A Appendix

Proposition 2. *If $\theta \in [0, \pi/2]$, then*

$$\sin\theta \geq \frac{2\theta}{\pi} \tag{A.1}$$

Proof. Let $\theta \in [0, \pi/2]$ and $y = \sin\theta/\theta$, then

$$\frac{dy}{d\theta} = \frac{\theta\cos\theta - \sin\theta}{\theta^2}$$

and

$$\frac{d}{d\theta}(\theta\cos\theta - \sin\theta) = -\theta\sin\theta \leq 0$$

Thus we have that $\theta\cos\theta - \sin\theta$ decreases from zero when θ increases. So $dy/d\theta < 0$ when $\theta \in [0, \pi/2]$, hence y decreases in this range, and since $\lim_{\theta\to 0} y = 1 > 2/\pi$, the result follows.

□

A.1 The Gamma function

Definition 6. *(The Gamma function) Let $s = \sigma + it$. We define the gamma function for $\sigma > 0$ as*

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt \tag{A.2}$$

Proposition 3. *Let s be a complex variable not equal to a nonnegative integer, then the following identity holds*

$$s\Gamma(s) = \Gamma(s+1) \tag{A.3}$$

Proof. Using integration by parts, one sees that

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty e^{-t}t^s dt = \Big[-e^{-t}t^s\Big]_0^\infty + \int_0^\infty se^{-t}t^{s-1}dt \\ &= \lim_{t\to\infty}[-e^{-t}t^s] + s\int_0^\infty e^{-t}t^{s-1}dt = s\int_0^\infty e^{-t}t^{s-1}dt = s\Gamma(s)\end{aligned}$$

□

Definition 7. *(The Beta function) Let s_1, s_2 be complex variables. We define the beta function for $Re(s_1), Re(s_2) > 0$ as*

$$\beta(s_1, s_2) = \int_0^1 t^{s_1-1}(1-t)^{s_2-1}dt \tag{A.4}$$

Remark. *The beta function is symmetric, meaning that*

$$\beta(s_1, s_2) = \beta(s_2, s_1)$$

for all inputs s_1 and s_2.

Proposition 4. *Let s_1 and s_2 be complex variables. For $Re(s_1), Re(s_2) > 0$ the following identity holds*

$$\beta(s_1, s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1 + s_2)} \tag{A.5}$$

Proof.

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) &= \int_{u=0}^{\infty} e^{-u}u^{s_1-1}du \int_{v=0}^{\infty} e^{-v}v^{s_2-1}dv \\ &= \int_{v=0}^{\infty}\int_{u=0}^{\infty} e^{-u-v}u^{s_1-1}v^{s_2-1}dudv\end{aligned}$$

Using the change of variables $u = kt$ and $v = k(1-t)$, one sees that

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) &= \int_{s=0}^{\infty}\int_{t=0}^{1} e^{-k}(kt)^{s_1-1}(k(1-t))^{s_2-1}kdtdk \\ &= \int_{k=0}^{\infty} e^{-k}k^{s_1+s_2-1}ds \int_{t=0}^{1} t^{s_1-1}(1-t)^{s_2-1}dt = \Gamma(s_1+s_2)\beta(s_1, s_2)\end{aligned}$$

Dividing both sides by $\Gamma(s_1 + s_2)$ gives the desired result.

□

Remark. *We consider the beta function where we shift the integral limits by x, and substitute $x + 1$ by k to get the following expression*

$$\begin{aligned}\beta(s_1, s_2) &= \int_0^1 t^{s_1-1}(1-t)^{s_2-1}dt = \int_x^{x+1} (t-x)^{s_1-1}(1-t+x)^{s_2-1}dt \\ &= \int_x^k (t-x)^{s_1-1}(k-t)^{s_2-1}dt = (k-x)^{s_1+s_2-1}\beta(s_1, s_2)\end{aligned}$$

and therefore

$$\int_x^k (t-x)^{s_1-1}(k-t)^{s_2-1}dt = (k-x)^{s_1+s_2-1}\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} \tag{A.6}$$

Lemma A.1. *(Euler's reflection formula) Let $k \in \mathbb{C}/\mathbb{Z}$, then*

$$\Gamma(k)\Gamma(1-k) = \frac{\pi}{\sin \pi k} \tag{A.7}$$

Proof. We use the Weierstrass definition of the gamma function, which is valid for all complex numbers k except for the non-positive integers.

$$\Gamma(k) = \frac{e^{-\gamma k}}{k}\prod_{n\geq 1}\left(1 + \frac{k}{n}\right)^{-1} e^{k/n} \tag{A.8}$$

where

$$\gamma = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \log N\right)$$

denotes the Euler-Mascheroni constant. Since $\Gamma(1-k) = -k\Gamma(-k)$ we consider

$$-k\Gamma(-k) = e^{\gamma k}\prod_{n\geq 1}\left(1-\frac{k}{n}\right)^{-1} e^{-k/n} \tag{A.9}$$

Multiplying (A.8) and (A.9) gives

$$\Gamma(k)\Gamma(1-k) = \frac{1}{k}\prod_{n\geq 1}\left(1-\frac{k^2}{n^2}\right)^{-1}$$

Euler's product formula for sine is given by

$$\sin k = k\prod_{n\geq 1}\left(1-\frac{k^2}{\pi^2 n^2}\right) \tag{A.10}$$

We replace k by πk, and divide both sides by π

$$\frac{\sin \pi k}{\pi} = k\prod_{n\geq 1}\left(1-\frac{\pi^2 k^2}{\pi^2 n^2}\right)$$

cancelling the π's and taking the inverse on both sides yields the desired result.

$$\frac{\pi}{\sin \pi k} = \frac{1}{k}\prod_{n\geq 1}\left(1-\frac{k^2}{n^2}\right)^{-1} = \Gamma(k)\Gamma(1-k)$$

□

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