Choice of representations in combinatorial problems

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This paper is based on classroom sessions where Norwegian 9-year-old (Grade 4) children work on combinatorial problems. The classroom sessions are part of a four-year long research project where the topic of multiplicative structures was central. I will investigate to what extent the pupils recognise the combinatorial problems as multiplicative and identify possible connections between the semiotic representations chosen in the solutions and in the formulation of the problems.

Keywords: Combinatorics, multiplicative structures, register, semiotic representations.

Introduction

This paper is based on data from the project Language Use and Development in the Mathematics Classroom (LaUDiM)—a four-year collaboration project between researchers at the Norwegian University of Science and Technology and two primary schools. In this project, a central theme was multiplicative structures, and a recurring question was to investigate connections between the formulation of a problem, the pupils' choice of semiotic representations to solve the problem, and to what extent they recognised a given situation as multiplicative.

In this paper, I study pupils in Grade 4 (nine-year-olds) working with two combinatorial problems (see Figures 1 and 2). The problems were presented with no previous instruction that could give an indication about what mathematical knowledge and techniques that would be helpful for solving the problems. Pupils worked in pairs, and work in selected pairs as well as in whole-class sessions were video recorded. I pose the following research question: *In what ways can the context of a situation be seen to influence the choice of registers in the solution, and how can the chosen registers provide evidence about the extent to which the situations are perceived as multiplicative?*

Theoretical framework

The concept of *register* is used in somewhat different ways by different authors. Duval uses the term register to mean a semiotic representation system (e.g., natural language, symbolic systems, graphics) and emphasises that "[c]hanging representation register is the threshold of mathematical comprehension for learners at each stage of the curriculum" (Duval, 2006, p. 128). I follow Duval's usage of the term, which is also in accordance with the usage by Prediger and Wessel (2013) in their model concerning changing and relating registers. This model entails a transition between different registers - a concrete representational register, a graphical representational register, different verbal registers and a symbolic-algebraic or symbolic-numeric register (Prediger & Wessel, 2013, Fig. 1, p. 437). This resembles the process described by O'Halloran when she writes that language is used to introduce and describe a mathematical problem, later to visualise the problem, and then the problem is solved using mathematical symbolism (O'Halloran, 2005, p. 94).

Before discussing the classroom situations, I will define what is meant by *a multiplicative structure,* or *a multiplicative situation*. Steffe defines a multiplicative situation as a counting situation where "it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit" (Steffe, 1994, p. 19). This is the basis for my discussion of multiplicative structures. There are several different classifications of multiplicative structures to be found in the literature (see e.g., Greer, 1992). I will rely on the classification given by Vergnaud who splits multiplicative structures in three classes: *Isomorphy of Measures*, *Product of Measures*, and *Multiple Proportions* (Vergnaud, 1983, p. 128). The latter will not be discussed here. *Isomorphy of Measures* is defined as a structure involving a direct proportion between two measure spaces, M_1 and M_2 (Vergnaud, 1983, p. 129). Situations like equivalent groups and multiplicative comparison (Greer, 1992) fall into this category. Further, *Product of Measures* is defined as a structure involving a mapping from a product of two measure spaces into a third measure space, $M_1 \times M_2 \rightarrow M_3$ (Vergnaud, 1983, p. 134). Combinatorial problems (Cartesian products) and rectangular area problems fall into this category. Rectangular *array* problems (Fosnot & Dolk, 2001), to find the total number of items laid out in a row-column pattern with a certain number of items in each row and each column, may look similar to area problems but they actually belong to the class *Isomorphy of Measures*, with M_1 = [number of rows (columns)] and M_2 = [number of items in each row (column)]. Unlike many other Isomorphy of Measures-problems, these are *symmetric* (Rønning, 2012). In combinatorial problems, the measure space M_3 may not be initially present. M_3 is where the counting unit is situated, and that this space may initially be unknown, represents a challenge when solving such problems. This is also connected to the phenomenon that the counting unit is of indefinite quantity and that there is not always a clear strategy to determine when the problem is actually solved (English, 1991; Shin & Steffe, 2009).

The tasks given to the pupils

The first task given to the pupils is presented, in its simplest version, in Figure 1, the second task in Figure 2. The tasks were given on two different days of the same week. No instructions were given on how to solve the tasks, and what strategies that might be helpful could also not be inferred from what the class had worked with immediately before the sessions where these tasks were presented.

> How many different gingerbread biscuits can we make if we have cutters in these four shapes ☆ ◇ ○ ♡ and we have white, green and red icing?

Figure 1: Task 1

> Ms. Hall has 3 pairs of trousers and 5 sweaters. The trousers are in the colours blue, black, and grey. The sweaters are in the colours blue, red, black, green and purple. She will use one pair of trousers and one sweater each day, and she will combine different pairs of trousers with different sweaters. How many days in a row can Ms. Hall wear different outfits?

Figure 2: Task 2 (Ms. Hall is the teacher in the class)

After agreeing on a solution, each pupil in the pair was asked to produce a written account of the method used. Both tasks induce a mapping $M_1 \times M_2 \rightarrow M_3$, with M_1 containing shapes in Task 1 and trousers in Task 2. M_2 contains colours in Task 1 and sweaters in Task 2. In Task 1, M_3 contains coloured shapes (biscuits). It could be argued that since M_3 in Task 1 contains coloured shapes, the measure space M_3 is not really new, it is a variation of M_1. A more precise representation of the mapping in Task 1 could therefore be $M_1 \times M_2 \to M_1^*$, where M_1^* denotes *coloured shapes*. In Task 2 one may think of a mapping $M_1 \times M_2 \to M_3 \to M_4$, where M_3 contains outfits and M_4 contains days of the week. M_3 and M_4 are isomorphic, so this transition would be expected not to be challenging.

In the language of Steffe (1994), one can say that it makes sense to distribute the composite unit from M_1 over the elements of M_2, or the other way around, which means that the situation is symmetric (Rønning, 2012). Both problems can be seen as a matrix product **c** of two vectors, **a** and **b**, where **a** and **b** represent the composite units from M_1 and M_2, respectively, and $\mathbf{c} \in M_3$ (see Figure 3).

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix} = \begin{bmatrix} a_1 b_1 & \dots & a_1 b_m \\ \vdots & & \vdots \\ a_n b_1 & \dots & a_n b_m \end{bmatrix}.$$

Figure 3: Matrix structure of a combinatorial problem

Each element, $a_i b_j$, of the product matrix **c** represents one possible combination (composition). This representation shows that the dimension of M_3 equals the product of the dimensions of M_1 and M_2.

The discussion above shows that although the problems in the two tasks are computationally equivalent, the mappings between measure spaces are somewhat different.

Design and method

The project LaUDiM was based on interventions consisting of two-three classroom sessions dealing with the same mathematical topic, preceded by planning meetings where teachers and researchers met. Between and after the sessions, reflection sessions were held. The design of the classroom sessions was based on the Theory of Didactical Situations (Brousseau, 1997).

The classroom sessions contained whole-class activities and activities where pupils worked in pairs with given tasks. For each session, the work of two pairs was video-recorded, as were all whole-class activities. Attempts were made to choose pairs to be recorded differently for each session, so that the pupils should not feel that only a few participated in the project. The pairs were determined by the teacher, based on her expectations of who would collaborate and communicate well. The school from which data for this paper come, lies in a well-established, middle-class neighbourhood. The pupils all have Norwegian as their first language. Data consist of video-recordings from the sessions, as well as pupils' written work, collected from all pupils, not only those who were video-recorded.

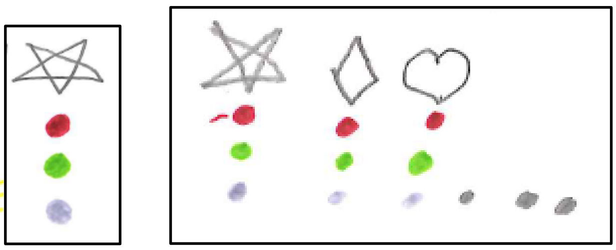
The analysis is based on the thematic development of the dialogue in the pairs, as well as the written work, including work from the pupils not video-recorded, in order to identify statements that show the choice of the representational registers and also serve as evidence for the pupils' possible perception of the situation as multiplicative. Parts of the video-recorded discussions have been transcribed and translated into English. In the analysis I will follow Naomi and Roger in Task 1 and Naomi and Filipa in Task 2. This means that I present data from one pupil (Naomi) in her work with both tasks. Therefore, I will pay most attention to her work in the pairs.

Analysis of the work in pairs

Task 1 (Naomi and Roger)

Naomi and Roger start looking at the task and Roger's first suggestion is that there will be seven different possibilities since there are four shapes and three colours. Then Naomi starts drawing the four shapes in one row and she colours the heart red. She indicates that she can continue to draw new rows with the same shapes and change the colour for each row. She does not complete the drawing in detail but on the video recording it can be seen that she indicates three rows with four shapes in each row (a matrix structure). Then she counts, one-two-three, four-five-six, seven-eight-nine, ten-eleven-twelve, tapping on the drawing column by column as she is counting. I interpret from her gestures and utterances that she has identified a countable unit. She indicates groups of three, but still she counts the shapes one by one. She now considers herself finished with the task and Roger does not object. Since Naomi and Roger solved the first task so quickly, I challenged them to find out what would happen if they had eight shapes and seven colours. They cannot really think of eight different shapes, so Naomi just draws eight circles in a row and fills in with more circles below these. They start to colour each row in one colour (purple, blue, red, …), until they have used all seven colours and hence got seven rows. The result is shown in Figure 4[1]. To the right is shown the calculation the pupils wrote on their worksheet. The drawing shows that they have marked four groups of 14 circles (dots). In the calculation I interpret the first line (14 14) to represent the first two groups, added to get 28 (second line). Then another 14 is added to get 42 and finally another 14 (not written) to get 56.

Figure 4: Naomi and Roger's solution



a) b)

Figure 5: From Naomi's solution in "Our method"

As part of the task, each pupil should fill in a sheet with the heading "Our method for the biscuits task". Naomi wrote (referring to a version of the task with three colours and six shapes):

> We thought that we took a star like this with the three colours beneath [indicated by an arrow pointing to Figure 5 a)]. Then we did the same with all shapes, like this [indicated by an arrow pointing to Figure 5 b)]. Finally, we counted all the dots. We could also take them in small groups like this 3+3+3+3+3+3=6+6+6=18.

In Figure 5 b), Naomi has not drawn all the shapes. Based on her text quoted above, I assume that she has imagined the last three shapes, without a need for drawing them.

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[1] They did not colour the first row of circles, so they put in eight green dots at the bottom. The three encircled shapes at the bottom of the drawing do not belong to this solution.

The representational register used in the formulation of Task 1 consisted of a text and the iconic representation ☆ ◇ ○ ♡. This drawing, endowed with colours, formed the starting point for the work in all the pairs. The pairs worked more or less systematically but the solutions shown in Figures 4 and 5 are representative for many of the pairs, as evidenced by the worksheets. The preferred representations show a matrix structure where each entry consists of one particular shape and one particular colour, mimicking the matrix product in Figure 3. Each entry has the form a_ib_j where a_i represents a shape and b_j represents a colour. An emerging multiplicative structure can be seen, represented graphically as well as numerically. In the solution shown in Figure 4, Naomi and Roger have made groups of two and two columns, and in her description, Naomi writes "[w]e could also make them in small groups like this 3+3+3+3+3+3=6+6+6=18", indicating six groups of three or three groups of six. It is not clear what reasoning lies behind the representation "3+3+3+3+3+3=6+6+6=18", since the only evidence is what Naomi has written. It could be that she groups each 3+3 into 6 and then gets three groups of six. Another possibility is that she sees 3+3+3+3+3+3 by counting on the columns and 6+6+6 by counting on the rows (Figure 5).

Task 2 (Naomi and Filipa)

The two girls start by drawing five sweaters and three pairs of trousers and then they colour each piece, using different colours for each piece in the same category. On the video can be seen that Naomi draws a line from the red sweater to the grey pair of trousers and writes "man" (Monday) above this line. Filipa connects the red sweater with the brown pair of trousers and writes "tir" (Tuesday). The girls continue in the same way, ascertaining that for each new combination they find, it is not already taken. After having written Tuesday for the second time, a break can be observed on the video, and it seems that they struggle to find new combinations. Gradually, they find new combinations and when they have found 14, they think they are done.

1	Naomi:	All the trousers on this [points to the brown sweater] because this has three lines. [Looks at the sheet] I think we have made it.
	Filipa gets a new sheet. Naomi looks further at the drawing.	
2	Naomi:	Oh, we can have one more. [draws a line between the green sweater and the blue pair of trousers]
	Naomi writes 2 weeks and 1 day.	
3	Naomi:	I will ask if it is correct.
	One of the researchers comes to the table and asks if the pupils have found a solution.	
4	Naomi:	We think we have figured it out. We *think* it is two weeks and one day

Naomi puts emphasis on 'think', which I take to mean that they are not sure, and they ask the researcher for confirmation. When the researcher is reluctant to give an answer, the girls call upon one of the other researchers, and the following conversation takes place.

5	Researcher:	Is that what you found? How did you find that out?
6	Naomi:	We took everything together. So there are three lines for each outfit.
7	Researcher:	Three lines for each outfit? On each sweater and each pair of trousers?
8	Naomi:	No, for each sweater and each pair of trousers … For the trousers, it will be … five lines.
9	Researcher:	Are you sure you have drawn lines between all? Have you found all the outfits?
10	Naomi:	I think so. Is it correct?
11	Researcher:	You have to try to convince me. How are you thinking to make sure you have found absolutely all? It can be easy to forget to draw a line, right?

12	Filipa:	Yes.
13	Researcher:	Have you found a strategy to be sure that you really have taken all the sweaters with all the trousers?
14	Naomi:	It is not so easy to see if we have taken all.

The result of Naomi and Filipa’s work can be seen in Figure 6. The lines are marked with abbreviations of the weekdays and to the right is written “2 weeks and 1 day” (2 uker og 1 dag).

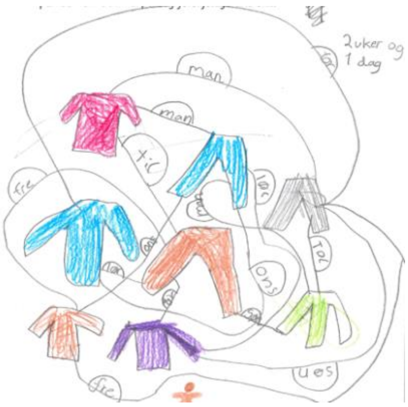

Figure 6: Naomi and Filipa's solution

Figure 7. One of Frances’ five groups

What appears from the pupils’ discussion of Task 2 is a less systematic approach, uncertainty about whether they have identified all possible outfits, and a solution based on counting one by one. There is some evidence of grouping when Naomi says “three lines for each outfit” (#6) and later adjusts to three lines for each sweater and five lines for the trousers, after being questioned by the researcher. Still, there is no evidence of seeing the problem as a situation of five groups of three or three groups of five. The representation chosen by Naomi and Filipa in Task 2 (Figure 6) is much further away from a matrix structure than Naomi’s representation in the solution of Task 1 (Figure 5). As an example of a grouping emerging also in Task 2, I show a solution produced by Frances (Figure 7). She made five groups, each group containing one coloured circle representing a sweater and three coloured circles representing three pairs of trousers. One of these groups is showed in Figure 7. Inside each group she had written “tre dager” (three days). She also wrote “take three times 3 times 3 times 3 times 3, which is 15”. Hence, she got the correct answer but wrote “times” instead of “plus”. Other indications of an emerging multiplicative structure are shown by Roger and Nora, when they say that they have “five lines for each pair of trousers” and “three lines from each sweater”, and then they say that they take “all the sweaters with all the trousers and all the trousers with all the sweaters”.

Discussion

Prediger and Wessel’s (2013, p. 437) model shows a relation between a concrete representational register, a graphical representational register, different verbal registers and a symbolic-algebraic or symbolic-numeric register. Both tasks start with a verbal representation, in Task 1 also a graphical representation. In both tasks, all the pupils made use of a graphical register in the solution process, (evidenced by the worksheets), but also other registers could be identified.

The formulation of Task 1 used a verbal register and an iconic graphical register (Duval, 2006, p. 110), the picture of the shapes. This picture turned out to be instrumental in the pupils’ solutions. All pupils started by copying the picture of the shapes and then they started to colour the shapes. Almost all pupils ended up with a matrix structure similar to what is shown in Figures 4 and 5. Most pupils got several examples to work on, with different number of shapes and colours, and the representations developed into being more systematic and refined for each new example. I described Task 1 as involving a mapping $M_1 \times M_2 \rightarrow M_1^*$, with M_1 containing shapes, M_2 containing colours, and M_1^* containing coloured shapes. The elements of M_1^* are, in a concrete sense, a product (combination) of the elements of M_1 and M_2: "shapes times colours gives coloured shapes". For combinatorial problems, an issue is that the target measure space may not be present from the beginning, and that it is not clear when to stop counting (English, 1991; Shin & Steffe, 2009). The strong relation between the measure spaces in Task 1 may have reduced this challenge.

Task 2 involves a mapping $M_1 \times M_2 \rightarrow M_3 \rightarrow M_4$, where M_3 contains outfits and M_4 contains days of the week. Here, the connection between the measure spaces is weaker than in Task 1. Task 2 was formulated purely in a verbal register, but the dominating register used in solving the task was an iconic graphical register. The solutions were heavily based on drawings of the clothing items. The lack of a stopping strategy was evident in Task 2, as exemplified by Naomi and Filipa: "We *think* it is two weeks and one day" (#4) and "It is not so easy to see if we have taken all" (#14). They stopped because they were not able to find more possibilities, not because they were convinced that the solution was correct. There are some occurrences of statements "three times five" in the data material, but with no clear reasoning about *why* three times five is a representation of the given situation.

Although a graphical register was used in both tasks, the representation chosen for Task 1 was much closer to a symbolic-algebraic representation (matrix) than was the case with Task 2. In Task 1, Naomi also introduced a symbolic-numerical representation by writing 3+3+3+3+3+3=6+6+6=18. A similar representation as in Figure 6 could be found in many of the pupils' worksheets, and some indications of grouping could be found in the graphical representations (Figure 7) as well as in the discussions. However, there is a significant difference in the chosen representations and in the extent to which the situations are perceived as multiplicative. Despite utterances like "three lines for each sweater" and "five lines for each pair of trousers" in Task 2, there are very few indications of grouping and counting of composite units. The counting is done one by one, sometimes using tally marks when a new connection between a sweater and a pair of trousers was discovered. I interpret this difference to be due to two aspects: The difference between the representations in the formulation of the tasks, and the nature of the target measure space. It turned out that in Task 1, many of the pupils could generalise to other numbers, whereas no such generalisation was observed in the work with Task 2. Mathematical symbolism, seen by O'Halloran (2005) as the final stage in a solution process, is generally used to a very limited extent.

In Task 1 the target measure space M_1^* is a variation of the measure space M_1. This makes the situation close to a rectangular array situation (Fosnot & Dolk, 2001), with shapes (biscuits) laid out in a row-column pattern. Therefore, Task 1 is not a genuine *Product of Measures* situation, but more like an *Isomorphy of Measures* situation, and hence less challenging (Vergnaud, 1983).

In this paper, I have shown how the choice of representations and the difference between the measure spaces can influence pupils' solution strategies. In Task 1, the pupils found a systematic solution strategy, which they also could generalise to larger numbers. The nature of the measure spaces in Task 1 made this situation closer to a rectangular array problem than was the case with Task 2.

References

Brousseau, G. (1997). *The theory of didactical situations in mathematics: Didactique des mathématiques, 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. & Trans.). Kluwer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

English, L. D. (1991). Young children's combinatoric strategies. *Educational Studies in Mathematics, 22*(5), 451–474. https://doi.org/10.1007/BF00367908

Fosnot, C. T., & Dolk, M. (2001). *Young mathematicians at work. Constructing multiplication and division.* Heinemann.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). Simon & Schuster Macmillan.

O'Halloran, K. (2005). *Mathematical discourse: Language, symbolism and visual images.* Continuum.

Prediger, S., & Wessel, L. (2013). Fostering German-language learners' construction of meanings for fractions – design and effects of a language- and mathematics-integrated intervention. *Mathematics Education Research Journal, 25*(3), 435–456. https://doi.org/10.1007/s13394-013-0079-2

Rønning, F. (2012). Symmetrisation of an asymmetric multiplication task. In G. H. Gunnarsdóttir, F. Hreinsdóttir, G. Pálsdóttir, M. Hannula, M. Hannula-Sormunen, E. Jablonka, U. T. Jankvist, A. Ryve, P. Valero, & K. Wæge (Eds.), *Proceedings of NORMA 11*, *The Sixth Nordic Conference on Mathematics Education* (pp. 553–563). University of Iceland Press.

Shin, J., & Steffe, L. P. (2009). Seventh graders' use of additive and multiplicative reasoning for enumerative combinatorial problems. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 170–177). Georgia State University.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3–39). State University of New York Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). Academic Press.