Conditions for revitalizing the elementary algebra curriculum

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Elementary algebra has been the bedrock of science and technology for centuries. But taught algebra today is much more a set of formal exercises than a modelling tool. The present study focuses on the notion of formula and how its curricular evolution appears to be a witness and a cause of the degradation of taught algebra as a modelling tool. This examination involves careful analyses of curricular facts that seem not to have attracted the full attention of researchers, such as the vanishing of parameters from algebraic equations. Drawing on the anthropological theory of the didactic (ATD), we outline the perspective of an imperative revitalization of the elementary algebra curriculum. Data used in the study include curriculum materials such as textbooks from different countries and various types of publications on school algebra by authors influential in their time.

Keywords: Curriculum, didactic transposition, elementary algebra, formulas, parameters.

Introduction

The passing of time changes curricular contents. This universal process of "curricular aging" can lead any subject matter to lose a large part of its instrumental value and to deteriorate to the point that its study at school becomes but a rite of passage imposed on the younger generations. In this study, we try to set out conditions favourable to making *elementary algebra*, understood as the algebra taught in secondary schools, an effective tool for understanding many "facts" of both the mathematical and the extra-mathematical world. By studying how the possibilities offered by elementary algebra have been greatly reduced by the evolution of the algebra curriculum over the last century, we will highlight the potential of elementary algebra to become a tool for understanding the world around us. Our research question is: *What key conditions should secondary school algebra meet to become an effective modelling tool for our time?*

On the epistemology and methodology of the study

In the framework of the ATD (Chevallard, 2019, 2020), the modelling of didactic phenomena rests on the notions of *person*, *institution*, and *institutional position*.

All human individuals are persons. Any "instituted" reality is an institution, such as a family, a class, a couple, a school, a ministry, the Norwegian society, and the French society. Any institution is organized into a set of institutional positions: In a classroom, there is the teacher position and the position of student; in mathematics education, there are the positions of textbook author, of teacher educator, of "great mathematician", etc. An institutional position is occupied by persons who thus become "subjects" of the institution. Persons are shaped by the set of institutional positions they occupy and have occupied. Persons are thus singular representatives of a position to which they are subjected. At the same time, persons can change the positions they occupy; there is thus a dialectic between persons and institutions in the making of a society.

Consequently, to study any institutional position, one studies the persons who are or have been its subjects; and, conversely, in order to study a person, one studies the positions he or she occupies or has occupied. Hence, in this study, we have studied persons' publications that allow us to enlighten the historical evolution of the teacher and student positions regarding elementary algebra. This has enabled us to identify conditions and constraints that have determined the current didactic transposition of elementary algebra and, most importantly, the vanishing of parameters from it. Our methodology is essentially that of *didactic transposition analysis* (Chevallard, 1991).

The take-off of algebra: From rules to formulas

We shall first highlight the key points of the changes that have affected the algebra curriculum, in order to identify core requirements for revitalizing secondary school algebra. We will focus on an aspect little considered: the role played (or not played), in the algebra curriculum, by the notion of *formula*, seen both as a symptom and as a cause of the impoverishment of school algebra. Here is a typical rule found on the Internet for how to find the area of a trapezoid (How to Find the Area, n.d., Example Question #1 section): "To find the area of a trapezoid, multiply the sum of the bases (the parallel sides) by the height (the perpendicular distance between the bases), and then divide by 2." Once fully algebraized, this rule becomes a formula: $A=\frac{1}{2}(b_1+b_2)h$, where b_1 and b_2 are the lengths of the parallel sides and h the distance between them.

Parameters: From implicit to explicit

To distinguish between "arithmetical rules" and "algebraic formulas," we must use the essential notion of *parameter*. In the formula $A=l\times b$ for the area of a rectangle with length l and breadth b, the letters l and b are parameters specifying the rectangle. While a rule (in words) contains parameters *implicitly* (like "length" and "breadth"), a formula contains *explicit* parameters. Consider the following rule given by Percival Abbott (1869–1954) in his book *Algebra* (Abbott, 1942/1971):

> The area of a rectangle in square metres is equal to the length in metres multiplied by the breadth in metres. This rule is shortened in Algebra by employing letters as symbols, to represent the quantities… [With the letters l, b, and A representing respectively the length, breadth, and area (in metres and square metres)] the above rule can now be written in the form: $A=l\times b$." (pp. 13–14)

A rule thus expressed is therefore called a formula. With this, a question arises. According to Abbott (p. 13), the "classical" doctrine on the arithmetic-algebra divide consists in the fact that, in arithmetic, one employs *definite numbers*, whereas in algebra, "we are, in the main, concerned with *general expression* and *general results*, in which letters or other symbols represent numbers not named or specified." Now this statement is subtly contradictory to the notion of "arithmetical rule": In the rule for the rectangle, we have, not "definite numbers," but "implicit parameters," the length and the breadth, not represented by *letters*.

How can this seeming discrepancy be explained? A quick answer is: In arithmetic, students are *given* a rule—they only have to *apply* it when the (implicit) parameters in the rule take definite numerical values. *Deriving* a formula (relying on some basic, given rules or formulas) is usually rather easy when done algebraically. In contrast, if you do it "arithmetically," it usually becomes more complex, and beyond the reach of beginners. This can be linked to the notion of *topos* of an institutional position, that is the set of *types of tasks* that persons in that position may have to perform (Chevallard, 2019). Let us repeat: in arithmetic, the teacher provides the rules ready-made, and the students simply apply them to numerical values.

In his *History of Mathematics,* David E. Smith (1925/1958, p. 437) made this comment: "To the student of today, having a good symbolism at his disposal, it seems impossible that the world should ever have been troubled by an equation like $ax+b=0$." Such however was the case. The algebraization of arithmetic was a huge step forward, a game changer for the sciences. But an almost surreptitious drawback happened that greatly reduced the power of the algebra actually taught.

A Pyrrhic victory

The algebraic modelling of arithmetical rules might pertain to the student *topos,* as the exercise set of Abbott's *Algebra* (1942/1971) seems to show. Here are some of Abbott's exercises (pp. 20–21): "Write down expressions for: (1) The number of pence in £x; (2) The number of pounds in n pence… (19) A train travels at v km/h. How far does it go in x hours and how long does it take to go y km?"

The implicit parameters of arithmetical rules are here translated into explicit parameters: n, v, x, and y. However, in the history of elementary algebra, explicit parameters will tend to be replaced by "definite numbers." Here is an example taken from the chapter "Simple Equations" of Abbott's book:

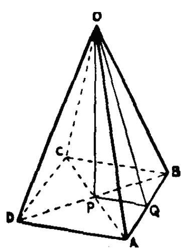
> A motorist travels from town A to town B at an average speed of 64 km/h. On his return journey his average speed is 80 km/h. He takes 9 hours for the double journey (not including stops). How far is it from A to B? (Abbott, 1942/1971, p. 66)

Abbott (1942/1971, p. 76) describes another type of equations "in which the values of the unknown quantities will be found in terms of letters which occur in the equation." He calls them *literal equations* and states that they can be solved by the same methods as simple equations. The examples he gives of literal equations are $5x-a=2x-b$, and $a(x-2)=5x-(a+b)$. The 14 equations in the related exercise set are similar to these: they are quite simple, and nobody knows what they claim to model—they are a mere didactic device, related to the need for tasks for repetitive training. We will come back to this phenomenon in the section "A turning point in the didactic transposition process."

Transformation versus evaluation of formulas: Reaching a demarcation line

The paucity of the material thus presented by Abbott is in striking contrast to the chapter's introduction, which begins with this promising statement (Abbott, 1942/1971):

> One of the most important applications of elementary Algebra is to the use of formulae. In every form of applied science and mathematics… formulae are constantly employed, and their interpretation and manipulation are essential. (p. 69)



The author explains that formulas "involve three operations: (1) Construction; (2) manipulation; (3) evaluation" (p. 69). The construction of a formula does not start from scratch: it relies on formulas previously established, either theoretically or empirically. The first "worked example" given by Abbott is typical: "Find a formula for the total area (A) of the surface of a square pyramid as in Fig. 10 [see figure opposite] when AB$=a$ and OQ$=d$" (p. 70). Here, the use of algebra is genuine but minimalist. By contrast, the second example and all the exercises given by Abbott are just *evaluation* tasks, as this exercise: “The volume of a cone, V, is given by the formula $V=\frac{1}{3}\pi r^2 h$, where r=radius of base, h=height of cone. Find V when $r= 3\cdot 5$, $h=12$, $\pi=\frac{22}{7}$” (Abbott, 1942/1971, p. 70).

The *manipulation* of formulas is only present in the section entitled “Transformation of Formulae.” About the volume of a cone ($V=\frac{1}{3}\pi r^2 h$), Abbott (1942/1971) writes:

> It may be necessary to express the height of the cone in terms of the volume and the radius of the base. In that case we would write the formula in the form: $h=\frac{3V}{\pi r^2}$, that is, the formula has been transformed. When one quantity is expressed in terms of others, as in $V=\frac{1}{3}\pi r^2 h$, the quantity thus expressed, in this case V, is sometimes called the *subject of the formula*… This process of transformation has been termed by Prof. Sir Percy Nunn *“changing the subject of the formula.”* (pp. 71–72)

About such a change, Abbott (1942/1971) adds this caveat: “The transformation of formulae often requires skill and experience in algebraical manipulation” (p. 72). He then illustrates the “methods” to be followed by five “worked examples.” In one of them (p. 73), he transforms the formula $L=l+\frac{8d^2}{3l}$ to find d in terms of L and l and arrives at $d=\sqrt{\frac{3l(L-l)}{8}} = \sqrt{\frac{3lL-3l^2}{8}}$. Readers are not asked to find the expression of l in terms of L and d—the answers are $l=\frac{L}{2} \pm \sqrt{\frac{L^2}{4}-\frac{8d^2}{3}}$—, which would require solving the *quadratic* equation $l^2 - Ll + \frac{8d^2}{3} = 0$. Here we reach the demarcation line drawn by the traditional didactic transposition of elementary algebra.

This line draws a curricular curiosity. Firstly, the quadratic equations with parameters considered have only one parameter. Secondly, students are not asked to give the expression of their solutions (which, in the general case, would include the parameter), but simply to specify, according to the value of the parameter, when they have 0, 1 or 2 roots. This sudden change of didactic contract (Brousseau, 1997)—an equation is no longer “something to be solved” but to be “studied” or “discussed”—was (and still is) a source of difficulty for students. In spite of this, the question of the “manipulation” and transformation of formulas, alongside their “construction” and “evaluation,” which are much less problematic, is at the heart of what algebra can consist of. In Abbott’s Example 5 (p. 74), readers are asked to find the length l of a simple pendulum in terms of the other quantities when its time of vibration is given by $t=2\pi\sqrt{\frac{l}{g}}$. Exercise 13 (No. 2) is about expressing the radius r of a sphere in terms of its volume V (p. 74). The usefulness of these transformations seems obvious. Now the big problem is that the type of tasks in question— “changing the subject of a formula”—has become marginalized in most secondary curriculums. A study of Norwegian textbooks used in recent decades is a clear testimony to this fact (Strømskag & Chevallard, 2021). In one of the textbooks for Grade 11 (Sandvold et al., 2006, p. 26), the authors consider the formula $v=\frac{d}{t}$ (where v is the speed, d is the distance travelled, and t is the time) and explain how to “solve the formula with respect to the time t" as if the students were complete beginners in elementary algebra. The authors then explain how to solve for c the (fabricated) formula $p=a+\frac{1}{2}bc^2$ to arrive at $c=\pm\sqrt{\frac{2-2a}{b}}$. Then follow tasks that ask to solve for t these formulas: $d=vt$; $d=\frac{1}{2}at^2$; $v=v_0+at$; $d=\frac{(v_0+v)t}{2}$. This is a limited viaticum (e.g., there are no quadratic equations with parameters) for a further journey into elementary algebra. The same phenomenon prevails in French textbooks (Chevallard & Bosch, 2012) and in German textbooks (e.g., Brandt & Reinelt, 2009). The latter is in line with the German Mathematics Standards for the expected level by the end of upper secondary education (Kultusministerkonferenz, 2012), where parameters are present in some of the example tasks but never in equations to be solved.

The pitfalls of didactic transposition

The marginalization of formula transformation

Let us consider the following exercise proposed by Abbott (1942/1971, p. 75): "There is an electrical formula $I=\frac{V}{R}$. Express this (1) as a formula for V and (2) as a formula for R. Find I if $V=2$ and $R=20$." A number of teaching institutions choose to "spare" their subjects the algebraic "work" needed to go from the formula $I=\frac{V}{R}$ to the formulas $V=R\cdot I$ and $R=\frac{V}{I}$. One of the most widespread techniques, it seems, consists in substituting to the algebra needed a graphic "mnemonic trick" which takes the form of a triangle in which the parameters V, I, R are displayed (see example in Figure 1).

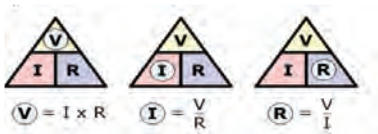


Figure 1: A graphic mnemonic trick (retrieved from Nimar_geek, 2020)

This "triangle technique" is pushed forward by institutions. The institutional enforcement of this technique seems to send the following message: "You don't need to know algebra *at all*."

There is also another, widespread technique, which is implemented more by people—students, in particular—than by institutions. This technique consists in avoiding any literal calculation. Suppose we are given the formula $I=\frac{V}{R}$ and values for I and R, and are asked to calculate V. If $I=1.2$ and $R=20$, the formula gives rise to the equality $1.2=\frac{V}{20}$, which is a linear equation in V that the student can therefore easily solve. This technique consists in first transforming a formula into a "simple" numerical equation.

A turning point in the didactic transposition process

How has this demarcation line been drawn? The answer must involve the conditions and constraints that have historically determined the didactic transposition of elementary algebra. Two influential textbook authors who have taken part in this transpositive work are Abbott and Nunn. About the phrase "Change the subject of a formula," Nunn writes in his book *The Teaching of Algebra* (1914):

> He [the author himself] believes that it was used for the first time in his lectures to teachers of mathematics in 1909. It was subsequently adopted in the Report on the Teaching of Algebra by the Committee of the Mathematical Association. (p. 78)

Why did Nunn introduce this way of saying, which was adopted by Abbott and others, when what is required is simply to "solve the equation $A=\pi r^2$ for r?" Nunn seems to have been quite aware of the change he wanted to popularize. Thus, he launches an attack against the position of strength given to equations, which he calls "conundrums" that "the school tradition has not lifted … to a much higher level of intellectual dignity" (Nunn, 1914, p. 77).

Nunn's degradation of equations leads to the coming apart of two distinct topics: equations and formulas. Formulas such as $V = R{\cdot}I$, which become equations once an unknown has been chosen (we can solve $V = R{\cdot}I$ for I for example), are indispensable in many fields of science and technology. Paradoxically, they were going to be marginalized by their very "promotion."

This detail of the didactic transposition process is linked to two great constraints. The first constraint is that of *simplicity*: the transposed content must be "simple" enough to offer students a *topos* that they can actually occupy. In France, in the early 1960s, a demanding exercise textbook for Grade 10 still proposed the following, highly artificial exercise (what exactly is it modelling?): "Solve the equation $(a+b)^2x^2 - (a-b)(a^2-b^2)\,x - 2ab(a^2+b^2) = 0$" (Combes, 1961, p. 124). But parameters in equations were officially deemed "undesirable" in 1981 (Chevallard & Bosch, 2012, p. 16). The second constraint is that, for didactic reasons of repetitive training, the teacher must be able to produce *at will* tasks of any type he or she has to teach. However, because of their origin in specific domains (geometry, physics, technology, etc.), it seems that the list of formulas to be "solved" is limited. In order to make it easier to create formula transformation tasks, it is accepted to break the link between a formula and what it models. The constraints mentioned therefore contributed to making the algebra taught a separate field, almost foreign to the other fields of mathematics and science.

Systems and models: Algebra for the future

So, what should elementary algebra consist of? To answer this question, we must first introduce two basic notions of the ATD: the notions of *system* and *model*. A system $\mathcal{S}$ is any entity subject to laws of its own. For example, a (geometric) sphere is a system whose "laws" are generally called the *properties* of the sphere, such as the following: "A great circle… of a sphere is the intersection of the sphere and a plane that passes through the centre point of the sphere" ("Great Circle," 2021). Any formula is a system as well. The formulas for the volume and the surface area of a sphere of radius r, that is, $V=\frac{4}{3}\pi r^3$ and $A=4\pi r^2$, are systems in their own right, which themselves have properties (we have $V=A\times\frac{r}{3}$ or $A=\frac{3V}{r}$ or $3V-rA = 0$, etc.). Given a system $\mathcal{S}$, a system $\mathcal{S}'$ is said to be a model of $\mathcal{S}$ if, by studying $\mathcal{S}'$, one can answer certain questions Q about $\mathcal{S}$. In practice, given a question Q relating to $\mathcal{S}$ which one wants to answer, one tries to build up a model $\mathcal{S}'$ of $\mathcal{S}$ (or choose one already existing) whose study with respect to the question Q is easier, safer, quicker than by a "direct" study of $\mathcal{S}$. For example, if the radius r of a sphere increases by 20%, the new surface area A' will be $4\pi r'^2=4\pi(1.2r)^2=1.44A$, so that the surface area will increase by 44%—a tricky result to obtain experimentally.

The great catastrophe which historically disorganized and denatured elementary algebra resulted from the generalized rupture of the link between the systems $\mathcal{S}$ to be modelled algebraically and their algebraic models $\mathcal{S}'$ relating to some question Q about $\mathcal{S}$. In most textbooks, this link has disappeared entirely. The vanishing of parameters from algebraic expressions goes together with the purely formal existence of algebraic expressions, which consequently lose their functional role, that is, the role of elements of a model of a system.

Conclusion: Conditions for a more authentic algebra curriculum

How can we pave the way towards a curricular reconstruction that revitalizes elementary algebra? The answer rests on two components: 1) the notions of system and model explained above and 2) rescuing the notion of formula and reintroducing expressions in several indeterminates in order to be able to model a diversity of mathematical or extra-mathematical systems. In a mathematics class, it is essential to study triples ($\mathcal{S}$, Q, $\mathcal{S}'$) composed of a mathematical or extra-mathematical system $\mathcal{S}$, a question Q raised about $\mathcal{S}$, and a model $\mathcal{S}'$ (related to $\mathcal{S}$ and Q) which contains mathematical elements that are key to constructing an answer to Q.

Mathematics education is therefore potentially concerned with all situations in which mathematics is or can be used to better understand the situation in question. In this respect, let us remind the reader that, from about 1600 to 1800, mathematics was divided into two branches, that of *pure* mathematics and the widely embracing branch of *mixed* mathematics (see e.g., Bacon, 1605/1901, pp. 172–174).

So, what key conditions should school algebra meet to be an effective modelling tool for our time? By way of a conclusion, we shall sum up the core of a more “authentic” study and use of algebra identified in the course of this inquiry:

> 1) The students start from a system $\mathcal{S}$ and a question Q raised about it, whose adequate treatment seems to involve mathematical elements; 2) These students build up a model $\mathcal{S}'$ of $\mathcal{S}$, relative to the question Q, which will be built with elementary algebra (and will include as many parameters as seems useful); 3) They work on $\mathcal{S}'$ to derive an answer A deemed adequate to the question Q; 4) At the same time, prompted by this process of inquiring about $\mathcal{S}$, they discover the resources of algebra, study or restudy them in order to make an efficient use of the tools thus garnered.

A brief example is in order here. The starting point is the theorem which says that when the sum of three numbers a, b, and c is constant, then the expression $ab+bc+ca$ is maximal when $a=b=c$. What can this result be used for? One answer concerns the prices of diamonds, when assumed to be proportional to the *square* of their weight. If the price of a diamond of weight w is equal to kw^2, where $k>0$, and if a diamond is broken into three pieces of weight a, b, and c, respectively, the price of each of these pieces is ka^2, kb^2, and kc^2 while the price of the original diamond of weight w_0 was $kw_0^2=k(a+b+c)^2$. We have: $(a+b+c)^2-(a^2+b^2+c^2)=2(ab+bc+ca)>0$. The price of the original diamond is therefore greater than the sum of the prices of the three diamonds obtained. As a consequence of the equality $3(ab+bc+ca)=(a+b+c)^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=w_0^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$, the loss of value caused by the breaking of the diamond into three parts is maximal when $a=b=c$, that is when the three pieces have the same weight. Here, the system $\mathcal{S}$ is a diamond and its selling price, and a key element of the model $\mathcal{S}'$ is the equality

$3(ab+bc+ca)=(a+b+c)^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$ with three parameters. For a more detailed discussion of this example, see Strømskag and Chevallard (2021). The four points listed above outline a research and innovation programme to which the present study is a contribution in order to help develop, in the decade to come, the full collaboration of researchers, teachers, and teacher educators.

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