A Novel Semiparametric Structural Model for Electricity Forward Curves

Marina Dietze, Iago Chávarry, Ana Carolina Freire, Davi Valladão, Alexandre Street, *Senior Member, IEEE,* Stein-Erik Fleten

***Abstract*—We propose a novel semi-parametric structural model to estimate the electricity forward curves based on elementary forward prices. The proposed model (i) explores the non-arbitrage relations between contracts with overlapping delivery periods, (ii) considers a parametric structure for price seasonality and exogenous variables, and (iii) uses non-parametric techniques to extract the remaining inter-temporal and cross-maturity information from data. Thus, our model allows users to estimate and complete the historical prices of any swap contract. We address the multi-objective estimation problem by hierarchical optimization. First, arbitrage levels are minimized. Then, the parametric part of the model is estimated. Finally, smoothness in the maturity and trading date dimensions are jointly considered in the estimation of the non-parametric part of the model. Based on a controlled study with real data from the Nordic power market, we show that our model outperforms benchmarks in terms of estimation error for missing data. We also isolate the effect of accounting for overlaps and smoothing in the trading dates dimension. Results show that these two key features of our model are crucial for improving the model accuracy. Finally, we apply our method to estimate the Brazilian forward curve and reconstruct the historical data.**

***Index Terms*—Data completion, electricity forward curves, smoothing, swap and forward prices, structural model.**

NOMENCLATURE

Sets

$\mathcal{T}$	Set of trading dates $\mathcal{T} = \{t_0, t_1, \ldots, T\}$
$\mathcal{J}$	Set of elementary maturity (number of days) $\mathcal{J} = \{1, 2, \ldots, J\}$
$\mathcal{N}_t$	Set of swaps negotiated on trading date $t \in \mathcal{T}$

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Constants

$F_{t,i}$	Price of the swap i on trading date t
τ_i	Initial delivery time for the swap i
T_i	Terminal delivery time for the swap i
r	Daily discount rate

Decision Variables

$f_{t,j}$	Price of the daily elementary forward contract of maturity j on trading date t
$\varepsilon_{t,j}$	Residual of the elementary forward contract of maturity j on trading date t
$\Delta_{t,i}$	Net present value of the arbitrage level of swap i on trading date t
$\eta_{t,i}$	Adjustment term of swap i on trading date t
$\zeta_{t,i}$	Arbitrage level in prices of the swap i on trading date t
$\boldsymbol{\beta}$	Vector of coefficients of $\boldsymbol{x_{t,j}}$

Vectors

$\boldsymbol{x}_{t,j}$	Vector that defines the structure imposed on the elementary forward contract of maturity j on trading date t
$\boldsymbol{\zeta_t}$	Vector composed by the arbitrage levels on the trading date t

I. INTRODUCTION

FORWARD curves play a central role in power markets worldwide. The whole settlement process, mark-to-market calculations to define margin requirements, and all best practices on risk management must be based on a reliable forward curve estimation. Generators, consumers, and traders use such derivatives to hedge against spot price volatility. These contracts are key for power system agents and become especially more relevant in the context of increasing participation of intermittent resources (e.g., solar and wind power), mostly inelastic demand, and network and other physical constraints. On the regulatory side, contracts reduce the agents' willingness to exercise market power to manipulate spot prices. We refer to [1] and references therein as an updated publication covering and reviewing many relevant aspects of electricity forward market applications. Hence, accurate calculations of forward curves provide relevant gains for almost all segments of the electricity sector. Notwithstanding, to achieve such accuracy, we must consider the effects of power system physical characteristics and the specific financial structure of contracts in the forward curve model.

For a typical commodity, spot seasonality is significantly mitigated in forward prices due to the possibility of physical storage and future delivery. In contrast, the limited storage capacity of power systems can explain the seasonal aspect of electricity forward curves. The large-scale use of batteries is expensive, and the limited capacity of hydro reservoirs still makes electricity highly influenced by seasonal climatic patterns. Consequently, a proper pricing scheme capable of capturing the seasonal profiles of electricity forward contracts is of utmost importance for electricity markets.

Furthermore, electricity is a flow commodity whose forward contract comprises delivery periods instead of fixed delivery dates. Usually referred to in the literature as swaps, forward contracts have differing and overlapping delivery periods (e.g., month, quarter, or year) as illustrated in Fig 1, for monthly, quarterly, semiannual, and yearly contracts.

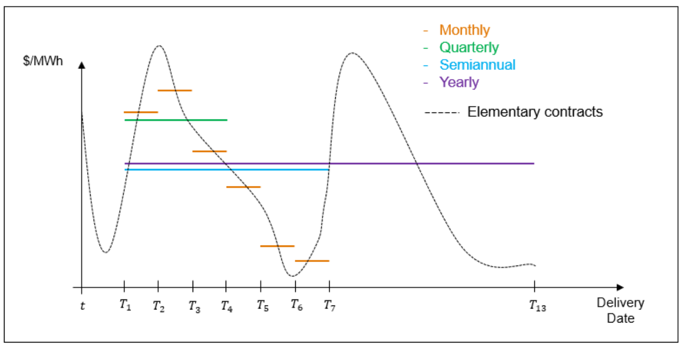

Fig. 1. Example of overlapped structure in electricity forward curves.

These peculiarities suggest that usual forward pricing methods are not directly suitable for the case of electricity. The two most notorious approaches that attempt to adapt such methods to electricity are the spot [2], [3], [4] and forward modeling [5], [6]. The spot modeling approach defines an analytical expression for the dynamics of spot prices, and then a closed-form solution for the forward prices is obtained by no-arbitrage conditions. They allow sophisticated features in the spot equations but with little flexibility to model factors that affect only forward prices. In contrast, forward modeling works directly delineate the forward curve's stochastic representation. In general, forward modeling relies on a portrayal of the electricity forward curve composed of what we refer to as elementary prices, illustrated in Fig 1 by the dashed line. The elementary prices represent the smallest unit of an electricity forward instrument (e.g., an hour, a day) that can reconstruct the price and delivery period of traded contracts. This curve approximates the representation of a fixed-delivery-date commodity (without overlaps), which facilitates the use of traditional derivative pricing models. [1]

Most studies in this field are based on the maximum smoothness criterion, which originated in [7]. In [8], forward market prices are integrated with forecasts from a bottom-up spot-price model. The latter is used to represent the seasonality observed in power markets. A bi-objective quadratic optimization problem is defined, where a combination of the squared errors between the elementary prices and the forecasts of the bottom-up model and the total curvature of the curve is minimized. The optimized prices are constrained by bid and ask observed in the market. In [6], the authors described elementary prices as a sum of a seasonality function and a residual term. Different from [8], the maximum smoothness criterion was imposed on the residuals instead of directly on elementary prices to retain seasonal patterns better. Polynomial splines of order four parameterized the residuals, and a trigonometric function and a bottom-up model were tested as seasonal functions. The authors formulate two different optimization models differing in the constraint that relates the optimized and observed prices: the first is analogous to the one in [8], where prices were constrained between the bid and ask levels; the second; matched the optimized swaps with closing prices.

Both [8] and [6] are commonly applied or used as a comparison in several studies. In [9], and [10], a model based on the HJM approach is derived and incorporates features of the previous works. [11] builds an hourly forward curve for electricity by introducing an alternative calibration procedure for the seasonality shape and directly compares their results to those that would generate citeFleten2003 and [6]. In [12], the authors extend the use of [6] for hourly forward curves in a framework that allows the simulation and forecast of electricity spot prices considering the information in forward prices.

As an alternative to the maximum smoothness criterion, [13] develop a parsimonious factor model that automatically describes the seasonal pattern through the estimated factors instead of assuming, in advance, some premise about its form. The multivariate model approximates the delivery dates as the swaps midpoints and acknowledges the distinct impact of contracts of smaller or greater lengths in the forward curve. In [14], an hourly elementary forward price was described as a sum of a periodical pattern and two adjustment terms, fitted with baseload and peak load prices, respectively. The periodic behavior was derived by filtering a macroeconomic trend and periodical components from day-ahead prices. The spot series is submitted to a Hodrick-Prescott filter to remove possible outliers. In addition, future prices were segmented over non-overlapping delivery periods. The development of the proposed model depends on the delivery periods covering the whole time horizon.

To the best of our knowledge, no previous works on the estimation of elementary prices have systematically handled the overlapping seasonal structure of the electricity forward curve. By observing Fig 1, it is possible to realize a clear relationship between the swaps with overlapped delivery periods. For instance, in a competitive market, it should be equivalent to trade the portfolio composed of the three monthly swaps or the quarterly one; otherwise, an arbitrage opportunity takes place. In this scenario, agents would profit without risks and, consequently, prices would eventually reach a new equilibrium without the arbitrage opportunity. This connection

[1] It is important to remark that the literature usually refers to the elementary prices as forward prices, but we adopt this new terminology to differentiate them from the traded contracts.

is an essential source of information that is not considered in any previous studies since all of them disregard overlaps to guarantee the viability of their models.

Although most studies deal with seasonality, they do not have a generic framework that captures relationships between electricity forward prices and other variables, such as spot prices or fuel costs, for instance. Moreover, previous works usually estimate these patterns from historical spot prices and not from a joint representation of all overlapping contracts, which requires a long history for a reliable estimate. Indeed, since they lose important information by disregarding overlaps, the seasonality estimation in forward prices would be inaccurate.

Furthermore, none of the previously reported works addresses the low liquidity of developing markets translated as missing data. In power markets that are not yet very consolidated, such as the Brazilian one, it is common to observe many days without negotiation of specific maturities, especially the longer-term ones. In addition, even in more competitive markets, the liquidity of such contracts may not be high. Therefore, accounting for the absence of data is crucial since the estimation results are affected by this relevant aspect.

Thus, this work proposes a novel framework and estimation procedure that assesses a high-resolution forward curve that acknowledges the relations between swaps with overlapping delivery periods, the seasonality of forward prices and the inter-temporal and cross-maturity dependence.

In this context, the main contributions of this work are:

1) **A novel semiparametric structural model for the forward curve.** The model considers (i) non-arbitrage relations between contracts with overlapping delivery periods, (ii) a parametric structure for price seasonality and exogenous variables, and (iii) non-parametric techniques to extract the remaining inter-temporal and cross-maturity information from data.
2) **Develop a hierarchical optimization procedure addressing the multi-objective estimation of the proposed model.** The estimation is based on the following steps: (i) computation of arbitrage-free prices by minimizing the arbitrage levels through an iterative process, (ii) establishment of a reduced form model that allows the assessment of the parametric coefficients of our model by OLS, (iii) estimation of the non-parametric term by the novel generalized maximum smoothness criterion in both maturity and time series dimensions.
3) **Provide a dataset of elementary prices for the Brazilian and Nordic electricity markets.** From the proposed model and estimation procedure, we can obtain a dataset decomposed in elementary forward contracts with a delivery period of one day. With the daily elementary prices, agents can calculate the prices of contracts of any delivery period and maturity. This is especially important for low liquidity markets, such as the Brazilian.

Among many possible applications, our methodology allows agents to 1) estimate missing prices of low liquidity products, 2) estimate the forward curve and its complete historical data in a transparent and scientifically-based manner, which is crucial for predictive and optimal trading applications, and 3) estimate the reference prices for new products based on the elementary forwards. So, these three possible applications illustrate the relevance of the proposed methodology for market participants, operators, and regulators.

The remainder of this paper is organized as follows. Section II presents our novel semiparametric structural model. Section III develops the hierarchical estimation procedure for elementary forward contracts. Section IV evaluates the impact of overlaps and the smoothing technique in the time series dimension as a crucial source of information. Finally, a case study with the Brazilian electricity market data is presented in Section V and conclusions are drawn in Section VI.

II. Proposed Semiparametric Structural Model

In this work, the proposed framework is based on two mathematical representations governing pricing dynamics. Therefore, this section starts with the algebraic characterization of arbitrage-free swap prices, followed by the structural model for swap prices.

A. Arbitrage-free Swap Prices

In a competitive power market with no arbitrage opportunities, a swap can be recast as a portfolio of contracts with shorter delivery periods.In this context, elementary prices are associated with non-overlapping forward contracts with sufficiently short (the same granularity of the time dimension, in our study, a day) delivery periods that can recover all traded swaps.

In imperfect markets, however, the level of arbitrage can be measured by the net present value (NPV) of a trading strategy where we buy a swap $F_{t,i}$ and sell a portfolio of elementary contracts covering the same delivery period. Mathematically speaking, we define the arbitrage-level NPV as

$$\Delta_{t,i} = \sum_{j=\tau_i - t}^{T_i - t} \frac{F_{t,i} - f_{t,j}}{(1+r)^j}, \qquad \forall t \in \mathcal{T}, i \in \mathcal{N}_t, \qquad (1)$$

where $f_{t,j}$ is the price of the daily elementary forward contract on date t, to be delivered j days ahead. It is important to emphasize that the elementary forwards are hypothetical contracts, not observed in the market, used as building blocks in our methodology. $F_{t,i}$ is the observed price of the swap i on the same date t and the interval $[\tau_i, T_i]$ set its delivery period.

The non-arbitrage condition is the most important premise adopted in financial models, and it establishes that agents shouldn't be able to make a risk-free profit with a self-financed strategy. In a market without arbitrage, the value of $\Delta_{t,i}$ would be zero, but that can be unrealistic due to the low liquidity, especially in the long term of the curve. A simple manner of handling arbitrage-free swap prices consists of modeling only contracts without overlap. However, this approach neglects critical information regarding the intersection of delivery periods that would be exploited to improve model accuracy. Instead of disregarding overlaps, our approach uses (1) to assemble swaps as a function of the non-observed elementary prices, i.e.,

$$F_{t,i} = \sum_{j=\tau_i - t}^{T_i - t} f_{t,j}\, w_{i,j} + \zeta_{t,i}, \qquad \forall t \in \mathcal{T}, \forall i \in \mathcal{N}_t, \tag{2}$$

where

$$w_{i,j} = \frac{(1+r)^{-j}}{\sum_{k=\tau_i - t}^{T_i - t} (1+r)^{-k}}$$

and

$$\zeta_{t,i} = \frac{\Delta_{t,i}}{\sum_{k=\tau_i - t}^{T_i - t} (1+r)^{-k}}$$

is the arbitrage level in prices. Therefore, a swap is then characterized by the associate arbitrage-free price plus the arbitrage level in price basis. The arbitrage-free swaps are indeed a convex combination of elementary price, i.e., $\sum_{j=\tau_i - t}^{T_i - t} w_{i,j} = 1$.

B. Structural Model for Swap Prices

Structural models are a time series framework that acknowledges the existence of non-observed variables (e.g., elementary prices) but imposes a predetermined structure to govern their dynamics. These unobserved quantities, so-called state variables, act as time-varying coefficients and are estimated from a relationship to other observed time series. In our context, we impose a semi-parametric structure to the non-observed elementary forward prices (hereinafter referred to as elementary prices), and use the arbitrage relationship (2) to connect them with traded swaps. Hence, we propose the following state-space model for the swaps and elementary prices:

$$F_{t,i} = \sum_{j=\tau_i - t}^{T_i - t} f_{t,j}\, w_{i,j} + \zeta_{t,i}, \qquad \forall t \in \mathcal{T}, \forall i \in \mathcal{N}_t \tag{3}$$

$$f_{t,j} = \boldsymbol{x}_{t,j}^T \boldsymbol{\beta} + \varepsilon(t,j), \qquad \forall t \in \mathcal{T}, \forall j \in \mathcal{J}. \tag{4}$$

Expression (3) is the measurement equation and translates the relation between the observation (traded swap prices, $F_{t,i}$) and state variables (elementary prices, $f_{t,j}$). Equation (4) is the transition/state equation, where the time evolution of the state variable is defined. The error term in the measurement equation $\zeta_{t,i}$ is the arbitrage level (in price units) while the residual $\varepsilon(t,j)$ is a function of time and maturity. The latter is not the standard uncorrelated errors usually considered in state space models since it may carry some lingering structure. In this study, we do not assume a parametric form for $\varepsilon(t,j)$. Rather, we assume that function ε belongs to a specific set of smooth functions.

Besides the residuals $\varepsilon(t,j)$, a variety of possible structures could be inserted in the vector $\boldsymbol{x}_{t,j}$, e.g., dummies to address calendar effects such as the impact of weekdays and weekends on prices, dummies or sine and cosine functions to address seasonality, trends, and exogenous variables in general. The vector $\boldsymbol{\beta}$ defines the associated coefficients to be estimated.

III. Estimation Procedure

The estimation procedure defined in this chapter aims to calculate the elementary forward prices induced by swap contracts traded on electricity markets. The proposed semi-parametric structural model is based on the relationship between the swap prices $F_{t,i}$ and the elementary contracts, $f_{t,j}$. We propose a hierarchical approach based on the following steps to address the challenge of estimating a semiparametric structural model. First, we estimate arbitrage-free prices $Y_{t,i}$ by selecting the elementary prices that minimize the worst-case arbitrage level for each trading date. Then, we derive a reduced form to estimate the parametric part of the model, namely the coefficient $\boldsymbol{\beta}$. With fixed $\boldsymbol{\beta}$, we estimate the non-parametric residuals, $\varepsilon(t,j)$, via maximum smoothness in time and maturity dimensions. The following subsections detail the steps mentioned above.

A. Computing Arbitrage-Free Prices

Let arbitrage-free prices be defined as

$$Y_{t,i} = F_{t,i} - \zeta_{t,i}, \qquad \forall t \in \mathcal{T}, \forall i \in \mathcal{N}_t. \tag{5}$$

To compute $Y_{t,i}$, we must calculate the values of $\zeta_{t,i}$. To do that, we use the following optimization problem for each trading date:

$$\min_{\boldsymbol{\zeta}, \boldsymbol{f}} \theta \tag{6}$$

$$\text{s.t. } F_{t,i} = \sum_{j=\tau_i - t}^{T_i - t} f_{t,j}\, w_{i,j} + \zeta_{t,i}, \qquad \forall i \in \mathcal{N}_t \tag{7}$$

$$\theta \geq |\zeta_{t,i}|, \qquad \forall i \in \hat{\mathcal{N}}_t, \ : \gamma_i, \tag{8}$$

which aims to minimize the sup-norm of $\boldsymbol{\zeta_t}$. To avoid degeneracy (multiple solutions for $\boldsymbol{\zeta}$) in the sup-norm minimization, we propose the iterative process described in **Algorithm 1**.

Algorithm 1 Calculate arbitrage-free prices

for all t $\in \mathcal{T}$ **do**
  Initialize $\hat{\mathcal{N}}_t \leftarrow \mathcal{N}_t$
  while $|\hat{\mathcal{N}}_t| > 0$ **do**
    Solve: (6)–(8)
    Find: $i^* \in \arg\max_{i \in \hat{\mathcal{N}}_t} \gamma_i$
    Add: Constraint $\zeta_{t,i} = \zeta_{t,i^*}$ for $i = i^*$
    Update: $\hat{\mathcal{N}}_t \leftarrow \hat{\mathcal{N}}_t \setminus \{i^*\}$
  end while

The sup-norm is justified by the need to assess a minimum level of arbitrage opportunities before estimating the model. We also argue that this procedure can be naturally used as an outlier detector. It is reasonable to say that a high absolute value of $\zeta_{t,i}$ may be associated with an abnormal trade eligible to be excluded from the analysis. This might occur when there are negotiations that have not been completed and, therefore, do not represent the equilibrium prices. More objectively, one can pre-specify a maximum arbitrage level for the trade to be included in the estimation.

B. Reduced Form Model

Embedding (5) and (4) in (3), we obtain the following reduced form model:

$$Y_{t,i} = \boldsymbol{X}_{t,i}\boldsymbol{\beta} + \eta_{t,i}, \qquad \forall t \in \mathcal{T}, \forall i \in \mathcal{N}_t, \tag{9}$$

where,

$$\boldsymbol{X}_{t,i} = \sum_{j=\tau_i - t}^{T_i - t} \boldsymbol{x}_{t,j}^T\, w_{i,j} \quad \text{and} \quad \eta_{t,i} = \sum_{j=\tau_i - t}^{T_i - t} \varepsilon(t,j)\, w_{i,j}.$$

Equation (9) depicts how the reduced-form model resembles a simple linear regression, where the regressors $\boldsymbol{X}_{t,i}$ of the arbitrage-free price $\boldsymbol{Y}_{t,i}$ are defined as a vector with weighted averages of explanatory variables of the elementary prices. They are deterministic values for each trading date t and contract i. The error term $\eta_{t,i}$ refers to the parcel of the swap price not explained by the parametric structure imposed on elementary forwards. They are a weighted average of the residuals $\varepsilon(t,j)$ during the swap delivery period.

The computation of coefficients $\boldsymbol{\beta}$ is directly related to the error term, $\eta_{t,i}$, as observed in (9). Therefore, the values of $\boldsymbol{\beta}$ are estimated by the ordinary least square (OLS) problem

$$\boldsymbol{\beta}^* \in \arg\min_{\boldsymbol{\beta}} \quad \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{N}_t} \left(Y_{t,i} - \boldsymbol{X}_{t,i}\,\boldsymbol{\beta}\right)^2. \tag{10}$$

C. Generalized Maximum Smoothness for Scarce Data Sets

Previous studies regarding estimating electricity forward curves through maximum smoothness criteria only address the cross-maturity relations, i.e., the smoothness is guaranteed only in the maturity dimension. To understand the impact of this approach, following [6], we apply the maximum smoothness criteria in the residuals of (4). Here, there is no premise about the parametric representation for $\varepsilon(t,j)$. Instead, we explicitly refer to the elements of its image, hereinafter $\varepsilon_{t,j}$.

Assume that, on a trading date of August 2020, monthly swaps with delivery for September 2020 and November 2020 were traded. As the objective function of the usual maximum smoothness criteria is to minimize the curvature of the forward curve along the maturity dimension, j, it is imperative to calculate the residuals $\varepsilon_{t,j}$ that will compose the swap for October 2020, which has no observed price even though it is the link between both negotiated contracts. In cases like this, the maximum smoothness criterion connects the maturities that constitute the available and missing swaps. This produces an interpolation effect, a salient feature of the proposed smoothing framework.

Now, consider a more critical situation. Suppose that only the monthly contract for September 2020 was traded on a given date t, and we want to infer the prices that would have been settled for October and November 2020. As the objective function minimizes the slope variation on the maturity dimension, the residuals $\varepsilon_{t,j}$ for $j > T_i$, where i is the swap for September 2020, will be estimated to maintain the final slope of the maturities composing the traded contract. This means that if $\varepsilon_{t,T_{t,i}}$ increases, decreases, or persists, the same trend will be kept indefinitely for all higher maturities. And this is a potentially undesirable shape for the boundary of the forward curve.

In similar cases, it is reasonable to assume that the information of the forward curves for adjacent trading periods (the days before and after) could be used to interpolate the elementary prices in the presence of missing data. This is especially important when dealing with data from low liquidity markets like Brazil. Therefore, we reformulate the maximum smoothness approach as a bi-objective optimization problem that minimizes a linear combination of quadratic penalties for the concavity over time and maturity. The resulting optimization model can be written as follows:

$$\min_{\boldsymbol{\varepsilon}} \quad \lambda_1 \left[\sum_{t\in\mathcal{T}} \sum_{j\in\hat{\mathcal{J}}} (\varepsilon_{t,j+1} - 2\varepsilon_{t,j} + \varepsilon_{t,j-1})^2\right] +$$

$$\lambda_2 \left[\sum_{t\in\hat{\mathcal{T}}} \sum_{j\in\mathcal{J}} (\varepsilon_{t+1,j} - 2\varepsilon_{t,j} + \varepsilon_{t-1,j})^2\right] \tag{11}$$

$$\text{s.t.} \quad Y_{t,i} = \boldsymbol{X}_{t,i}\,\boldsymbol{\beta}^* + \sum_{j=\tau_i - t}^{T_i - t} \varepsilon_{t,j}\, w_{i,j}, \; \forall t \in \mathcal{T}, i \in \mathcal{N}_t, \tag{12}$$

where $\hat{\mathcal{T}} = \mathcal{T} \setminus \{1, T\}$ and $\hat{\mathcal{J}} = \mathcal{J} \setminus \{1, J\}$.

Parameters λ_1 and λ_2 must be fixed a priori. To ensure the same order of magnitude in both objectives, the parameters λ_1 and λ_2 were normalized by their respective objective function values, calculated previously by two separated optimization problems – one with only the maximum smoothness in the maturity and the other in the time dimension. Mathematically, let us denote by $g(\lambda_1, \lambda_2)$ the optimal value of the problem (11)–(12) as a function of the weights λ_1 and λ_2. Then, we can choose a single scaled weight $\lambda \in [0,1]$ whereby we define $\lambda_1 = (1-\lambda)/g(1,0)$ and $\lambda_2 = \lambda/g(0,1)$ to be used problem (11)–(12). The calibration of λ can be made through out-of-sample cross-validation tests as reported in our case study.

IV. Computational Experiments

In this section, we study the performance of our model to complete missing data using a real data set from the Nordic market. We also evaluate the trade-off between the weights λ_1 and λ_2 in the bi-objective function. To run the case studies, we used a personal computer Intel(R) Core(TM) i7-8565U CPU @ 1.80GHz with four cores and 8 GB of RAM. The framework was programmed in Julia language, and the optimization was solved with Gurobi. We provide the implementation of our model as an open source code at https://github.com/LAMPSPUC/ForwardCurveSmoother.

A. Data set and experiment setup

We use real data from the Nordic market, available in [15]. The data is composed of observed forward prices from the beginning of 2017 to the end of 2018, as a case of a consolidated power market with high liquidity and no missing prices. This allows us to perform a controlled study where a parcel of the data set is assumed to be missing, and our

TABLE I
MAPE AND OPTIMAL WEIGHT λ FOR EACH CASE AND MODEL TREATMENTS.

CASES	PROBABILITY OF MISSING PRICES	MODEL TREATMENTS			
		No overlaps + No smoothing in time (Benchmark)	No overlaps + Smoothing in time	Overlaps + No smoothing in time	Overlaps + Smoothing in time
CASE 1 Removing observations of the maturity M+1	**p = 0 %**	0.00% (0.0)	0.00% (0.1-0.9)	0.025% (0.0)	0.025% (0.1-0.9)
	p = 50%	2.40% (0.0)	1.00% (0.1)	1.78% (0.0)	0.69% (0.1)
	p = 90%	4.25% (0.0)	2.79% (0.5)	3.12% (0.0)	2.08% (0.4)
CASE 2 Removing observations of the maturity M+5	**p = 0 %**	0.00% (0.0)	0.00% (0.1-0.9)	0.12% (0.0)	0.12% (0.1-0.9)
	p = 50%	2.62% (0.0)	0.74% (0.1)	1.35% (0.0)	0.52% (0.7)
	p = 90%	4.44% (0.0)	1.87% (0.5)	2.11% (0.0)	0.87% (0.7)
CASE 3 Removing observations of the maturity Q+1	**p = 0 %**	0.00% (0.0)	0.014% (0.1)	0.12% (0.0)	0.12% (0.1-0.9)
	p = 50%	2.33% (0.0)	0.83% (0.1)	0.17% (0.0)	0.17% (0.1-0.9)
	p = 90%	3.87% (0.0)	2.20% (0.1)	0.21% (0.0)	0.21% (0.1-0.9)
CASE 4 Removing observations of the maturity Q+4	**p = 0 %**	0.00% (0.0)	0.00% (0.1)	0.15% (0.0)	0.15% (0.1-0.2)
	p = 50%	2.66% (0.0)	0.64% (0.1)	0.64% (0.0)	0.52% (0.2)
	p = 90%	5.14% (0.0)	2.23% (0.1)	1.06% (0.0)	0.97% (0.3)
CASE 5 Removing observations of the maturity Y+1	**p = 0 %**	0.00% (0.0)	0.12% (0.1)	0.15% (0.0)	0.15% (0.1-0.6)
	p = 50%	3.48% (0.0)	0.70% (0.1)	0.22% (0.0)	0.22% (0.1-0.4)
	p = 90%	7.79% (0.0)	2.55% (0.9)	0.28% (0.0)	0.28% (0.1-0.6)
CASE 6 Removing observations of the maturity Y+3	**p = 0 %**	0.00% (0.0)	0.021% (0.1)	0.00% (0.0)	0.021% (0.1)
	p = 50%	6.46% (0.0)	0.47% (0.1)	6.46% (0.0)	0.47% (0.1)
	p = 90%	15.40% (0.0)	2.97% (0.3)	15.40% (0.0)	2.97% (0.3)

model is used to estimate and complete these prices. Then, we can use the real data to assess error metrics for different particularizations of the model. The Nordic price data set consists of daily prices of monthly, quarterly, and yearly contracts.

To run the controlled study for data completion, we randomly removed a group of tradings from specific maturities to reproduce different levels of data scarcity. Thus, for each chosen maturity series, we simulate a random variable from a *Bernoulli(p)* distribution, where p is the probability of the observation being disregarded. So, if the Bernoulli variable sampled for a given period and contract values 1, the observation is considered missing and is not used in the estimation process. Different values of p were tested, namely, $p = 0\%, 50\%,$ and 90%. Furthermore, for each p, we have also tested distinct values of λ to study the effect of the bi-objective smoothing technique; we varied λ between $0\% - 90\%$, with a step size of 10%.

Recalling the mathematical definition of elementary forward prices in (4), we must first specify their structure besides the residuals to perform our estimation procedure. For the following results, seasonality on delivery and trading dates were included through sine and cosine functions, specifically a truncated Fourier series with only one harmonic. Let us define

$$\begin{aligned} f_{t,j} = \mu + \beta^s sin\left[\frac{2\pi(t+j)}{365}\right] + \beta^c cos\left[\frac{2\pi(t+j)}{365}\right] \\ + \phi^s sin\left[\frac{2\pi t}{365}\right] + \phi^c cos\left[\frac{2\pi t}{365}\right] \\ + \varepsilon(t,j), \qquad \forall t \in \mathcal{T}, \forall j \in \mathcal{J}, \quad (13) \end{aligned}$$

Consequently,

$$\boldsymbol{x}_{t,j} = \Bigg[1,\ \sin\left[\frac{2\pi(t+j)}{365}\right], \cos\left[\frac{2\pi(t+j)}{365}\right], \\ \sin\left[\frac{2\pi t}{365}\right], \cos\left[\frac{2\pi t}{365}\right]\Bigg]^T$$

$$\boldsymbol{\beta} = [\mu, \quad \beta^s, \quad \beta^c, \quad \phi^s, \quad \phi^c]^T.$$

Besides the structure depicted above, we also ensure two additional characteristics in the resulting forward curve. First, we impose that the elementary price with maturity zero equals the spot price for the correspondent trading date. This is the same as considering a swap whose delivery period is just the current date. Then, we also add the constraint $\varepsilon_{t,j} - \varepsilon_{t,j-1} = 0$, for j equal to the final maturity, to translate the fact that the long term of the curve is less sensitive [6].

The names of the electricity contract price time series carry information about the swap's length and maturity. We use the letters M, Q, and Y for monthly, quarterly, and yearly contracts, respectively, while an integer represents the maturities. Thus, always using the trading date as the reference, we refer to a monthly contract to be delivered in the next month as M+1, two months ahead as M+2, and so forth. The same reasoning is used for quarterly and yearly contracts, but now considering the correspondent delivery time.

We selected the following time series to test the capability of our model to retrieve data: M+1, M+5, Q+1, Q+4, Y+1, and Y+3. This allows us to evaluate our model's behavior in contracts of different delivery periods and levels of overlaps. More specifically, we focused on the following characteristics:

- M+1: Monthly contract with the lowest level of overlap;
- M+5: Monthly contract that always composes the delivery period of a quarterly contract;
- Q+1: Quarterly contract with its delivery period always overlapped with monthly contracts;

- Q+4: Quarterly contract that always composes the delivery period of a yearly contract;
- Y+1: Yearly contract with its delivery period always overlapped with quarterly contracts;
- Y+3: Yearly contract without overlaps.

B. General results

The salient new features of the proposed framework in comparison to previously reported works are the following: our model does not disregard overlaps, and the maximum smoothness in the time dimension is used in combination with smoothness in maturity. These two features are crucial to creating a link among the whole data set within a unified non-parametric model for the residuals, $\varepsilon(t, j)$. Hence, the following results aim to study the benefits of these features in different combinations.

Table I shows the mean absolute percentage error (MAPE) values for each analyzed data set and the respective model treatment. We define a treatment as a given combination of features (overlap and smooth in time) in our framework. So, by comparing the results of the different treatments, we can isolate and identify the effects and contributions of each feature alone and in combination with the other. The first column specifies the time series that had part of its data removed with probability p. Then, parameter p is varied in the second column for each time series. The first treatment is analogous to the one in the literature, where both overlaps and time smoothing are disregarded, thereby serving as a benchmark for comparison purposes. To deal with the superposition, in this case, all contracts overlapping the delivery period of the examined time series were removed. Furthermore, each feature was added separately in the second and third treatments and then considered in combination to compose the fourth treatment, which constitutes the proposed framework. Values in the parentheses identify the optimal smoothing weights in time. In cases where multiple values of weights result in the same performance, the interval with these values is presented.

For all cases where $p = 0\%$, the metric values were low, not exceeding 0.15%. This is especially true when we do not consider overlaps and smoothing in time. Since the parametric part of the framework has many degrees of freedom, the benchmark can perfectly fit in-sample observed prices if the delivery periods have no intersection. When only overlaps are considered, the MAPE reflects the arbitrage relationships and exhibits a slight growth. In this case, the remaining low metric indicates Nordic's market liquidity and robustness; otherwise, the increment would be higher.

In the presence of missing data notwithstanding, ignoring overlaps and time dependency implies considerably worse MAPE values. Adding one of the features separately brings significant improvements compared to the benchmark. However, when simultaneously accounting for the two features (the proposed approach) in the presence of missing data, the performance is always better or equal to the alternatives.

In terms of computational burden, our proposed approach is more computationally intensive than the benchmark because it relies on a larger optimization problem that can not be decomposed per period. Notwithstanding, problems involved are essentially convex least-squares regression-like problems, which can be solved in polynomial time by off-the-shelf software. For instance, the computational time of the larger problem solved in our case study, i.e., with no missing data and the complete set of contracts comprising 8500 observations was equal to 697.7 seconds. For the same data set, the benchmark consumed 177.5 seconds.

Next, we discuss the benefits of each feature and how they differ depending on the analyzed maturity.

C. Impact of Smoothing in Time

Following the model treatments order in Table I, we evaluate the impact of adding the maximum smoothness criteria in the time dimension solely (column two of Table I). The maturities that are more affected by adding only this feature instead of only the overlaps (column three) were M+1, M+5, and Y+3. Regarding the M+1 contract, although its delivery period sometimes overlaps with quarterly and/or yearly contracts, this occurs at specific months. Therefore, the arbitrage relationship is not strong enough to help fill the missing prices. A more extreme case happens with the Y+3, as it does not intersect with any other contract, regardless of the trading date. This means that adding overlaps makes no difference in this case, and only the time dependency can help improve the price's recovery and reduce the MAPEs. For the yearly contract, when $p = 90\%$, the metric reduces from 15.40% to 2.97% when the smoothness is added in the time dimension with the weight $\lambda = 0.3$. Fig. 2 shows the relationship between the metric values and different weights λ for the three sparse data sets where Y+3 negotiations were removed. Fixing $\lambda = 0$ means that smoothing in time is not present. Based on this result, the benefit of considering the second objective (smooth in time) is clear, as we find considerably lower MAPE metrics.

Fig. 3 illustrates the Y+3 estimated prices from the benchmark (red curve) and the proposed model (green curve), with $\lambda = 0.3$ and $p = 90\%$ in comparison with the prices observed in the market. The original prices correspond to the observations in the data set during the estimation process, while the missing ones are removed. Examining the benchmark curve makes clear the practical benefit of considering the smoothness in the time dimension. The elementary prices of maturities that recover the missing Y+3 swap prices do not have any observed price to support their calculation. Thus, they are solely obtained based on a sum of the parametric shape of our model and a smooth error in maturity, which, in this case, maintains the slope of the last maturities with an associated observed price. However, for the trading dates with observed Y+3 prices, the model's fit is almost perfect due to the high degree of freedom for non-parametric models. This results in the erratic behavior observed in Fig. 3 for the benchmark. This issue is addressed for the time series where maximum smoothness in time is applied, and the recovered prices are much more consistent with the real-time series.

Regarding the M+5 contract, although its delivery period is always overlapped with a quarterly contract and sometimes with a yearly one, the non-arbitrage relationship between overlapped contracts does not produce the same benefit as

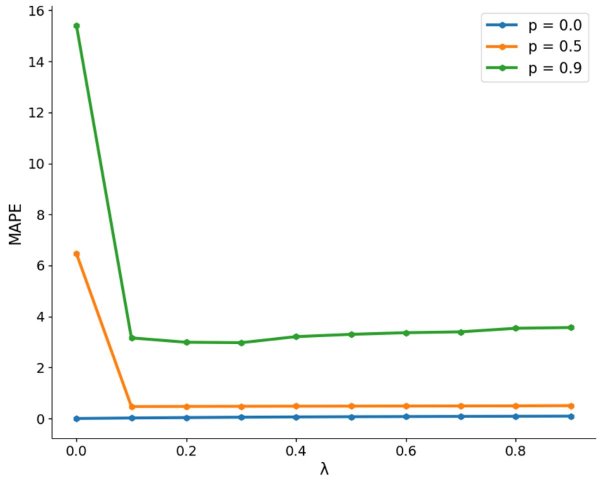

Fig. 2. Curves λ x MAPE for the three different levels of sparsity ($p = 0\%, p = 50\%$ and $p = 90\%$) of the Y+3 contracts.

Fig. 3. Comparison between the estimated Y+3 prices with the benchmark model and the our framework, with $\lambda = 0.3$.

the time link created by smoothing in time. Furthermore, the combination of both effects provides additional gains, albeit the marginal contribution of the time link is higher. The marginal contribution is the loss of performance when one of the features is not considered compared to the performance of the model considering both features.

D. Impact of Overlaps

Accounting only for overlaps might be reasonable if the missing data belong to a maturity whose delivery period is always overlapped with other contracts, which is the case of the Q+1, Q+4, and Y+1 time series. Regarding the Q+1 and Y+1 cases, we observe that their prices can be consistently described as a combination of three monthly and four quarterly contracts, respectively. Consequently, in a competitive environment, the price information of monthly and quarterly contracts should be enough to derive the Q+1 and Y+1 prices; otherwise, an agent could profit without facing any risk. Hence, the MAPE value for both cases is lower when only the overlaps are added, and incorporating the maximum smoothness in time does not bring any marginal contribution for both $p = 50\%$ and $p = 90\%$.

Fig. 4 illustrates the Y+1 estimated prices obtained with the benchmark and the proposed model (with $\lambda = 0.1$) for $p = 90\%$. The shape of the benchmark curve has the same interpretation as Fig. 3. The MAPE decreased from 7.79% to 0.28%. It is possible to notice that the prices reconstructed with our framework (or only adding overlaps) are very close to those observed in the market, whereas the benchmark estimated prices follow the same noisy pattern described for the Y+3.

Fig. 4. Comparison between the estimated Y+1 prices with the benchmark model and the our framework, with $\lambda = 0.1$.

Although we understood the impact of adding overlaps and time smoothing separately in different maturities, considering both attributes simultaneously always led to better or equal metrics when only one of the features was contemplated. It is important to emphasize that we performed a controlled study where only one maturity had its observations removed at a time. Therefore, the no-arbitrage relationships were sometimes enough. However, when missing data occurs in multiple time series, the marginal contribution of both features tends to increase.

V. Case Study: Brazilian Electricity Market

This section presents a case study with real data from the Brazilian over-the-counter (OTC) Energy, BBCE (Balcão Brasileiro de Comercialização de Energia) [16]. This case study data set is available at [15]. Table II shows the percentage of missing prices of each available time series between 2018 and 2019.

TABLE II
PERCENTAGE OF MISSING PRICES OF EACH TIME SERIES.

Contracts	Missing Prices (%)
M+1	1.1%
M+2	5.8%
M+3	30%
M+4	59%
M+5	84%
M+6	94%
Q+1	36%
Q+2	54%

We used the previous data set from the Nordic electricity market to choose the weights of our hierarchical procedure's bi-objective function. We've estimated our semi-parametric structural model with only M+1 to M+6 and Q+1 to Q+2 contracts. Additionally, we built a scarce data set where the tradings of each maturity faced the same probability of being removed as those presented in Table II. We tested the same values for λ as in Section IV. Because the optimal weight can differ for each maturity, we calculated an average MAPE between all-time series and selected $\lambda = 0.3$, which resulted in the lowest aggregated metric as depicted in Fig. 5.

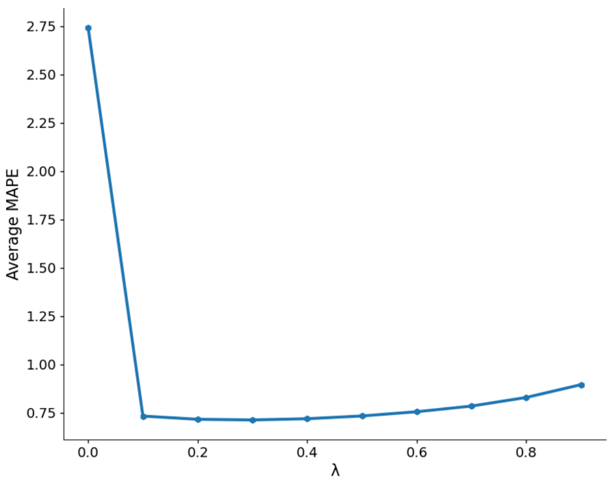

Fig. 5. Average MAPE between all maturities of the Nordic set, when considering the Brazilian percentage of missing prices.

To verify the liquidity of the Brazilian forward market, we calculate the arbitrage levels associated with each contract in each trading day from the first step of our estimation procedure. This allows us to investigate the market's degree of competitiveness since we expect that opportunities to profit without taking any risk do not exist in a competitive environment. The maximum arbitrage observed was equal to 21.6 R$/MWh, which is considerably high and a direct consequence of the market's low liquidity.

One interesting and obvious statement is that arbitrage levels different from zero are only observed on completely overlapped contracts. To illustrate this fact, Fig. 6 shows the composition between the quarterly and three monthly swaps on August 1st of, 2018, the day with the highest levels of arbitrage in the Brazilian data set.

Approximately each third of a quarterly swap's delivery period is overlapped with a monthly contract, so it is correct to expect the former's price to be essentially an average of the monthly assets. Looking at Fig. 6, we can readily conclude that this is unfeasible for the trading day under analysis, as the monthly prices are all lower than the quarterly one. This is reflected in their absolute arbitrage values, which were around 21.6R$/MWh. One could argue why do not just cancel the arbitrage of monthly contracts and keep only the quarterly with a value different from zero and higher than 21.6R$. This impossibility is a consequence of the algorithm presented in Section III, which aims to minimize the maximum

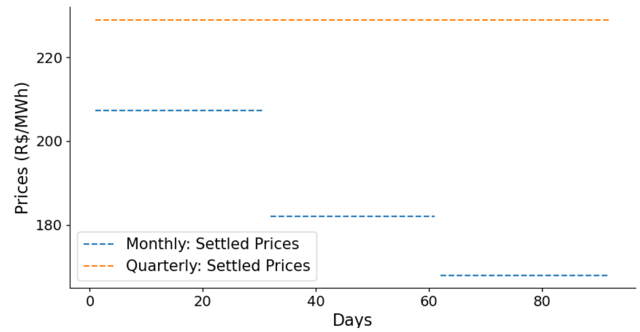

Fig. 6. Settled prices of contracts with the highest arbitrage on the Brazilian's forward market.

arbitrage level on a date t and, as a solution, has a not null arbitrage associated with all the overlapped swaps. The resulting arbitrage-free prices are presented in Fig. 7.

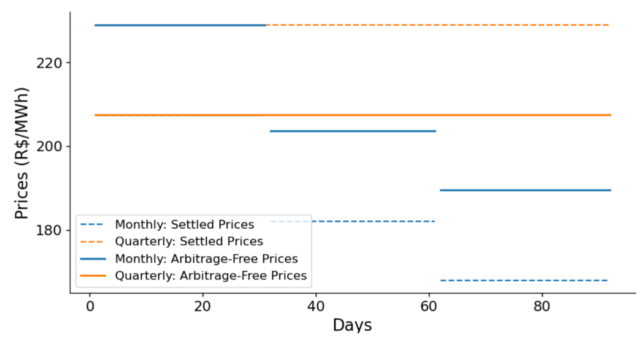

Fig. 7. Contracts with the highest arbitrage on the Brazilian's forward market and their arbitrage-free prices.

Another observed arrangement between overlapped contracts is similar to those presented in Fig. 6 and 7, but with one of the monthly swaps being unavailable. If it occurs, the elementary forward prices that would compose the missing asset could be determined to make the others' arbitrage equal to zero.

To illustrate the estimated prices from our model, Fig. 8 and Fig. 9 show the monthly contracts' outcomes M+6 and M+1, respectively. In the former, we can understand the smoothing impact in the time dimension in the series with the highest data scarcity. It ensures less noisy estimated prices and is more coherent with what we would expect in reality. In contrast, the latter highlights that, when we deal with high liquidity maturities, our model's fit is almost perfect, even with only the maturity dependency. Visually, it is impossible to differentiate between both curves' reconstructed prices.

Finally, we illustrate in Fig. 10 the estimated forward curve of a trading day in the Brazilian market, which is the primary goal of the proposed framework. We provide the estimated high-resolution (daily) forward curve composed of elementary prices without an overlapping structure. We show a specific trading date from the Brazilian power market where all swaps

Fig. 8. Comparison between the estimated M+6 prices with ($\lambda = 0.3$) and without ($\lambda = 0.0$) the maximum smoothness in the time dimension.

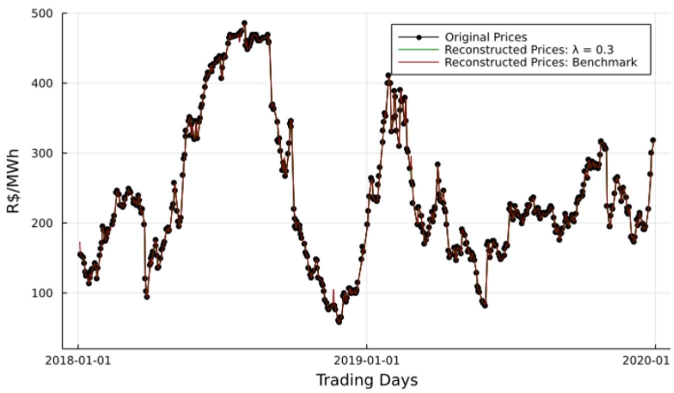

Fig. 9. Comparison between the estimated M+1 prices with ($\lambda = 0.3$) and without ($\lambda = 0.0$) the maximum smoothness in the time dimension.

were traded. The monthly and quarterly arbitrage-free swaps are also highlighted. Here, we ensured that the elementary prices at maturity zero equal the spot price on the analyzed trading date. The data set of estimated elementary forward prices is available at [15]. Note that these data can be used to either estimate the price of new contracts or feed a time series model to produce probabilistic forecasts for the whole forward curve. These two fronts are part of our ongoing research and will be addressed in future publications.

VI. Conclusion

In this paper, we presented a new semiparametric forward curve model that (i) explores the non-arbitrage relations between contracts with overlapping delivery periods, (ii) considers a parametric structure for price seasonality and exogenous variables, and (iii) uses non-parametric techniques to extract the remaining inter-temporal and cross-maturity information from data.

In a controlled experiment with the forward prices of the Nordic power market, we evaluate the importance of the two main features considered in our model, namely, contract overlaps and maximum smoothness in the time dimension. Results

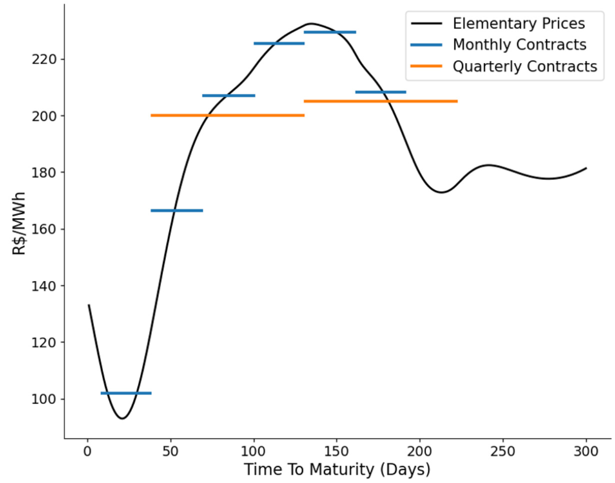

Fig. 10. High-resolution forward curve and arbitrage-free swaps from Brazil.

corroborate the benefits of adding both features. In the case of maturities whose delivery period is a composition of smaller observed contracts, adding the maximum smoothness criterion in the time provided no marginal contribution. In contrast, incorporating the second objective considerably increases the model's accuracy in maturities that are not always intersected with other contracts. In such cases, the marginal contribution to the model's accuracy of the smoothness in time can achieve higher values than that observed for the overlap.

The proposed semiparametric structural model is highly flexible, and its use can be extended to multiple purposes. The most direct one is its application associated with traditional forward pricing models. Another avenue of research is the probabilistic forecasting of the whole forward curve as an integrated (unified) model. Electricity forward curve forecasting is a very unexplored field. The tied dependency between contracts with overlapped delivery periods makes this task even more complex and scarce in the literature. Our model bypasses this complexity. Preliminary results with principal component analysis indicate promising results for forecasting a lower dimensional model.

Furthermore, it is relevant to mention that the proposed modeling framework may be of interest to other products inside or outside electricity market domains. The benefit of the capacity to extract relevant non-parametric information from overlapping contracts and time dependencies exhibited by our model could be valuable to other related products with similar forward structures (different maturities, delivery periods, etc.). Additionally, we highlight that our model could be extended and studied in other contexts, such as outlier detection and market monitoring. So, we highlight these topics as relevant future research avenues.

Besides improving the previous applications, the ability to recover missing data with high accuracy can be a valuable instrument for market players, operators, and exchanges. In less competitive markets, such as the Brazilian one, the benefits of such methodology is even more significant. Making available a consistent forward curve, even in the absence of the prices of some maturities, increases the market's transparency and can foster the development of central clearing counterparts, which results in more liquid and robust markets.

VII. Acknowledgment

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