Bachelor's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Solutions to ordinary differential equations at infinity

Bachelor's thesis in Mathematical Sciences
Supervisor: Mats Ehrnström
Co-supervisor: Jun Xue
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Abstract

In this paper we will look at solutions to first order ordinary differential equations as $t \to \infty$. We will discuss three cases of ordinary differential equations and look at what we need to require in order for there to exist a solution. Fixed-point theory, especially Picard-Lindelöf and Banach fixed-point theorem, will be central for this.

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1 Introduction

In this paper, we will discuss if we can solve ordinary differential equations, abbreviated to ODEs, of the form

$$y' + f(t, y(t)) = b(t)$$

as t goes to infinity, where f and b are arbitrary differentiable functions which are not necessarily linear, and y is a dependent variable given by $y = y(t)$, where t is an independent variable.

In order to do so, we will first take a look at what a metric and a metric space are, define what a norm is and we will discuss what a Banach space is. Then, we will talk about what an initial value problem is and take a look at fixed-point theory, more specifically we will state the Peano existence theorem, Banach fixed-point theorem and the Picard-Lindelöf theorem, which helps us guarantee both existence and uniqueness of solutions. In this work, we will also explore Lipschitz continutiy, contractions and Picard iterations. Fixed-point theory will be useful for us to be able to discuss the real matter of this text, namely if we can solve differential equations at infinity. We will look at what kind of requirements that we need in order for there to be a solution. We will go through different cases, advancing in complexity as we go. First, we will look at a special case of Picard-Lindelöf that lay the groundwork for our approach for the next cases. Then we will consider the following ODEs,

$$y'(t) + a(t)y(t) = 0, \quad y'(t) + a(t)y(t) = b(t) \quad \text{and} \quad y'(t) + f(t, y(t) = b(t).$$

on the interval $[t_0, \infty]$. What criterion does a, b, f and y need to satisfy in order for there to be a solution? Is there a solution for the interval $[t_0, \infty]$ or do we need a smaller interval $[t^*, \infty]$ where $t_0 \leq t^*$? These are all things we will analyse in part 5 of this paper. For each case, ensuring that we have a contraction mapping T will be central, which also means we need Lipschitz continuity. Having this mapping T, we can directly apply the Banach fixed-point theorem which tells us that we have a unique solution to the ODE. We will also see that for each case, the solution will converge as $t \to \infty$. We could also study second order ODEs, but we will not be addressing this in this paper.

Before we start, let us state some notation which will be useful later on.

1.1 Notation

In this paper, for $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm on $\mathbb{R}^n$. In other words, for $n \in \mathbb{N}$, we have that

$$|(x_1, x_1, \ldots, x_n)| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

We will also often refer to $y(t)$ as y for simplicity.

2 Preliminaries

2.1 Metrics and metric spaces

In order to talk about fixed-point theory, we first need to define what a metric, a metric space and a banach space are. A metric is a function which is a measure of distance. The formal definition of a metric is stated below.

Definition 2.1. (Metrics and metric spaces). Let X be a nonempty set. A metric on X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$, the following holds:

i) $0 \leq d(x, y) < \infty$ and $d(x, y) = 0 \iff x = y$,

ii) $d(x, y) = d(y, x)$,

iii) $d(x, z) \leq d(x, y) + d(y, z)$.

We then say that (X, d) is a metric space.

We also need the definition of Cauchy sequences and completeness in order to understand what a Banach Space is.

Definition 2.2. (Cauchy sequence). Let (X, d) be a metric space. We call a sequence (X_n) in X Cauchy if for all $\varepsilon > 0$ there exists $N > 0$ such that for all $n, m > N$, we have that

$$d(X_n, X_m) < \varepsilon.$$

Definition 2.3. (Complete metric space). We say that the metric space (X, d) is complete if every Cauchy sequence in (X, d) converges.

A Cauchy sequence does not necessarily need to be convergent. $1/n$ is an example of a sequence where the distance between successive elements of the sequence become arbitrarily small, but it does not converge. A metric space containing this sequence would therefore not be complete.

2.2 Norm

We also need to know what a norm is before defining a Banach space. As for metrics, a norm is a measure of length, but while metrics can be applied on any space, norms are used strictly on vector spaces.

Definition 2.4. (Norm). A norm $\| \cdot \|$ on a vector space V is a function that takes values in V and maps it onto $\mathbb{R}^+$. For all vectors x and y in V and for all scalars a the following conditions must hold:

i) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$,

ii) $\|ax\| = |a|\|x\|$,

iii) $\|x + y\| \leq \|x\| + \|y\|$.

An example is the Euclidean norm on $\mathbb{R}^n$.

Now, we are ready to look at Banach spaces.

2.3 Banach spaces

Definition 2.5. (Banach spaces). A norm space $(X, ||\cdot||)$ is a Banach Space if it is complete, in other words, if every Cauchy sequence in the space converges.

Definition 2.6. (The space BC). BC is the space of all bounded and continuous functions. We can define it over different sets, like for example an open interval $BC(I)$ or over the real numbers $BC(\mathbb{R})$. We can furthermore have even more conditions, like the space of bounded and continuous first order differentiable functions, denoted by $BC^1(\mathbb{R})$.

Now let us prove that $\|\cdot\|_{BC(I)} := \sup_{x\in I} |f(x)|$ is a norm. We need to prove the three conditions in Definition 2.4.

Firstly, if

$$\|f\|_{BC(I)} = 0$$

for some $x \in I$, then

$$\sup_{x\in I} |f(x)| = 0.$$

Since $\|\cdot\|$ is a norm and non-negative, this is congruent to

$$\|f(x)\| = 0 \quad \text{for all} \quad x \in I.$$

Furthermore, since $\|x\| = 0$ is equivalent to $x = 0$, we get that

$$f(x) = 0 \quad \text{for all} \quad x \in I.$$

It then follows that

$$f = 0,$$

so the the first condition holds.

Now, let $a \in \mathbb{R}$. Then we have that

$$\|af\|_{BC(I)} = \sup_{x\in I} |af(x)| = |a| \sup_{x\in I} |f(x)| = |a|\|f\|_{BC(I)},$$

so the second condition also holds.

Finally, for $f, g \in BC(I)$, we have the following

$$\|f + g\|_{BC(I)} = \sup_{x\in I} |f(x) + g(x)|,$$

and since $|\cdot|$ is a norm, this is equivalent to

$$\sup_{x\in I} |f(x) + g(x)| \le \sup_{x\in I} (|f(x)| + |g(x)|).$$

Sum of supremum then gives us

$$\sup_{x\in I} (|f(x)| + |g(x)|) \le \sup_{x\in I} |f(x)| + \sup_{x\in I} |g(x)| = \|f\|_{BC(I)} + \|g\|_{BC(I)}.$$

This means the third condition holds and we can conclude that $\| \cdot \|_{BC(I)}$ is a norm.

Therefore, we get that the set

$$X = \{f : I \to \mathbb{R} \quad | \quad \text{continuous and bounded}\}$$

with the norm $\| \cdot \|_{BC(I)}$ is a complete normed space, the Banach space $BC(I$. And this completes the proof.

3 Ordinary differential equations

Ordinary differential equations are differential equations with only one independent variable in contrast to partial differential equations which have multiple. The equations contain one or more functions and its derivatives. In other words, an ordinary differential equation will be on the following form

$$x^{(n)}(t) = g(t, x(t), x^{(1)}(t), \dots, x^{(n-1)}(t)),$$

where g and x are functions and t is the independent variable.

3.1 Inital-value problems

If we have an initial condition alongside with the differential equation, we have an inital-value problem. In other words, let $I \times U$ be an open subset of $\mathbb{R} \times \mathbb{R}^n$ (we will use this definition of $I \times U$ throughout this text). Let

$$f \in C(I \times U, \mathbb{R}^n)$$

be a continuous vector-valued function on this subset and let

$$(t_0, x_0) \in I \times U$$

be a fixed-point. An initial-value problem, abbreviated to IVP, is then the problem of finding a solution $x \in C^1(J, U)$ such that

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases}$$

for some possibly smaller interval $J \subset I$.

3.2 Reformulation of real-valued ODEs as first-order systems

Any ordinary differential equation

$$x^{(n)}(t) = g(t, x(t), x^{(1)}(t), \dots, x^{(n-1)}(t))$$

with initial conditions

$$x(t_0) = x_1, x^{(1)}(t_0) = x_2, \dots, x^{(n-1)}(t_0) = x_n,$$

and g a continuous function in $I \times U$ containing $(t_0, x_1, \dots, x_n)$, can be reformulated to an IVP. We do this in the following way.

Let $y_0 := x, y_1 = x^{(1)}, \dots, y_{n-1} := x^{(n-1)}$. Then

$$\begin{bmatrix} y_0' \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2}' \\ y_{n-1}' \end{bmatrix} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \\ g(t, y_0, \dots, y_{n-1}) \end{bmatrix} \tag{3.1}$$

describes $y = (y_0, \dots, y_{n-1})$ as a function $I \to U \subset \mathbb{R}^n$ for some interval $I \in \mathbb{R}$; the function $f \in C(I \times U, \mathbb{R}^n)$ is the vector-valued function given by the right-hand side of this system. The initial condition is $y(t_0) = (x_1, \dots, x_n)$.

Example 3.1. Let us take a look at the following second-order differential equation

$$\begin{cases} x'' + \sin(x) = 0, \\ x(0) = 1, \\ x'(0) = 2. \end{cases}$$

Using (3.1), we get the equivalent system

$$\begin{bmatrix} y_0' \\ y_1' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} y_1 \\ -\sin(y_0) \end{bmatrix}$$

with

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}_{t=0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

In this case $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ -\sin(y_0) \end{bmatrix}$ is independent of time.

4 General existence theorems

Now that we have stated some basic terminology, we can now move on to some fixed-point theory before we can finally look at ODEs as $x \to \infty$. We will discuss some essential theorems, namely the Peano existence theorem, the Banach fixed-point theorem and the Picard-Lindelöf theorem. This will be useful to guarantee and find a solution to our ODEs.

4.1 The Peano existence theorem

Theorem 4.1. (The Peano existence theorem) Given an IVP, it follows that for any $(t_0, x_0) \in I \times U$ there exists $\varepsilon > 0$, such that the IVP has a local solution x defined for $|t - t_0| < \varepsilon$, which is the solution $x = x(\cdot, t_0, x_0) \in C^1(B_\varepsilon(t_0), U)$.

Note that the Peano existence theorem states the existence of a solution to an IVP, but not how to find it and whether or not it is a unique solution.

Example 4.1. The following initial-value problem

$$\begin{cases} \dot{x} = \frac{3}{2}x^{1/3}, & t \geq 0, \\ \dot{x} = 0, & t < 0, \\ x(0) = 0, \end{cases}$$

has the trivial solution, namely $x \equiv 0$.

But this is not the only possible solution to this initial value problem. Substituting λx in the above equation yields

$$\lambda \dot{x} = \frac{3}{2}(\lambda x)^{1/3}.$$

A direct computation then gives us

$$\lambda^{2/3}\dot{x} = \frac{3}{2}x^{1/3},$$

which means

$$\lambda^{2/3} = 1.$$

This gives us a solution

$$\lambda = \pm 1.$$

Therefore,

$$\begin{cases} x(t) = \pm t^{3/2}, & t \geq 0, \\ x(t) = 0, & t < 0, \\ x(0) = 0, \end{cases}$$

is also a solution to the initial value problem.

To fix the lack of uniqueness in Peano's theorem, we need to look at the concept of Lipschitz continuity.

4.2 Lipschitz continuity

Let $f \in C(I \times U, \mathbb{R}^n)$ be a continuous function. If for any $(t_0, x_0) \in I \times U$ there exists $\varepsilon,\ L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

holds for all $(t,x),(t,y) \in B_\varepsilon(t_0,x_0)$, then f is said to be locally Lipschitz continious with respect to its second variable $x \in U$.

The set of all Lipschitz functions on $I \times U$ form a vector space, $Lip(I \times U, \mathbb{R}^n)$. A Lipschitz condition is said to be uniform if the Lipschitz constant L does not depend on the point (t_0, x_0). We then say that f is uniformly Lipschitz contionious. A locally Lipschitz function is uniformly Lipschitz contiuous on any closed and bounded set, in other words, a compact set.

Note that any continuously differentiable function is also locally Lipschitz continuous, and thus uniformly Lipschitz on any compact set.

We will later see that if the Lipschitz contant $L < 1$, we call the Lipschitz function a contraction.

Proof. If f is continuously differentiable, then

$$\begin{aligned} f(t,x) - f(t,y) &= \int_0^1 \frac{d}{ds} f(t, sx + (1-s)y)\, ds \\ &= \Big(\int_0^1 D_x f(t, sx + (1-s)y)\, ds \Big)(x-y), \end{aligned}$$

where $D_x f$ is the Jacobian matrix of $f(t,\cdot)$, and $(x-y) \in \mathbb{R}^n$ is a vector. For each fixed triple (t,x,y), the matrix

$$T(t,x,y) := \int_0^1 D_x f(t, sx + (1-s)y)\, ds$$

defines the bounded linear transformation on $\mathbb{R}^n$, so that

$$|T(t,x,y)z| \leq \|T(t,x,y)\| |z|, \quad z \in \mathbb{R}^n.$$

Since $D_x f$ is continuous, so is T (in all its variables); and since norms are continuous, the composition

$$(t,x,y) \mapsto \|T(t,x,y)\|$$

is continuous, which then implies

$$C_{t_0,x_0,y_0} := \max_{\substack{|t-t_0| \leq \varepsilon \\ |x-x_0| \leq \varepsilon \\ |y-y_0| \leq \varepsilon}} \|T(t,x,y)\|_\infty.$$

We then have that

$$|f(t,x) - f(t,y)| = |T(t,x,y)(x-y)| \leq \|T(t,x,y)\| |x-y| \leq C_{t_0,x_0,y_0} |x-y|$$

for all

$$(t,x),(t,y) \in B_\varepsilon(t_0,x_0).$$

(The ball $B_\varepsilon(t_0,x_0)$ is actually a bit smaller than the square describes by $|x - x_0| \le \varepsilon, |t-t_0| \le \varepsilon$, where we have proved the statement, but we do not need more.) $\square$

Example 4.2. Consider $f : \mathbb{R} \to \mathbb{R}$ (one spatial variable, no time).

• Let f be the continuously differentiable function with the mapping $x \mapsto \frac{x}{2}$. Then

$$\left|\frac{x}{2} - \frac{y}{2}\right| \le \frac{1}{2}|x-y|.$$

So we have found a Lipschitz constant less than 1, and this constant is not dependent of x. This implies that $\frac{x}{2}$ is not just Lipschitz continuous, but uniformly Lipschitz continuous.

• $x \mapsto x^2$ is another continuously differentiable function. We have that

$$|x^2 - y^2| = |x+y||x-y|.$$

So, since L is dependent of x, this is only locally Lipshitz continuous.

• $x \mapsto |x|$ is continuous, but not continiuously differentiable at 0. But since

$$||x| - |y|| \le |x-y|,$$

we have a Lipschitz constant of 1, which means it's uniformly Lipschitz, but just barely.

• Now, let's look at an example that is continous, but not locally Lipschitz:

$$x \mapsto \sqrt{|x|}.$$

This is because we cannot find a Lipchitz constant at

$$x_0 =: \frac{|\sqrt{|x|} - \sqrt{|x_0|}|}{|x-x_0|} = \frac{\sqrt{|x|}}{|x|} \to \infty \quad \text{as} \quad x \to 0.$$

This shows that $C^1(\mathbb{R}) \subset Lip(\mathbb{R}) \subset C^0(\mathbb{R})$.

4.3 The Banach fixed-point theorem and its applications

Now, in order to solve it, we are going to reformulate the IVP into

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\,ds, \quad x \in BC(I).$$

Let the right-hand side define a mapping T, which is not necessarily linear,

$$T : BC(J,U) \to BC(J,U), x \mapsto x_0 + \int_{t_0}^{t} f(s,x(s))\,ds$$

for some smaller interval $J = [t_0 - \varepsilon, t_0 + \varepsilon] \subset I$. This is because, if x and f are continuous, so is $s \to f(s, x(s))$, so the integral

$$\int_{t_0}^{t} f(s, x(s))\, ds$$

is continuous, even C^1, and bounded on compact intervals. The idea then is that, if f is also Lipschitz, then T **contracts** points for small $|t - t_0| \leq \varepsilon$:

$$\begin{aligned} |Tx(t) - Ty(t)| &= \Big| \int_{t_0}^{t} [f(s, x(s)) - f(s, y(s))]\, ds \Big| \\ &\leq \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))|\, ds \\ &\leq \int_{t_0}^{t} L|f(s, x(s)) - f(s, y(s))|\, ds. \end{aligned}$$

Thus, if $\varepsilon L < 1$, taking the maximum over $t \in J$ yields

$$\|Tx - Ty\|_{BC(J,U)} \leq \lambda \|x - y\|_{BC(J,U)},$$

for $\lambda = \varepsilon L < 1$, so that Tx and Ty are closer to each other than x and y. As we shall now see, that gives us a local and unique solution of our problem.

4.3.1 Contractions

Let (X, d) be a Metric space. A mapping $T : X \to X$ is called a contraction if there exists $\lambda < 1$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. In particular, contractions are continuous.

Note that the uniformity of the constant $\lambda < 1$ is important. It is not enough that $d(T(x), T(y)) < d(x, y)$ for each pair $(x, y) \in X \times X$.

4.3.2 The Banach fixed-point theorem

Theorem 4.2. (The Banach fixed-point). Let T be a contraction on a complete metric space (X, d) with $X \neq \emptyset$. Then there exists a unique $x \in X$ such that $T(x) = x$. We call this x a fixed-point for T, since T sends x to itself.

Proof. We need to prove three things, namely the existence and uniqueness of x and that x is a fixed-point for T.

Existence of x. Let $x_0 \in X$, $x_1 := T(x_0)$, $x_{n+1} := T(x_n) = T^{n+1}(x_0)$, $n \in \mathbb{N}$. For $n > m \geq n_0$, we have that

$$d(x_n, x_m) \leq \sum_{k=m+1}^{n} d(x_k, x_{k-1}),$$

by using the triangle inequality. Furthermore, by the defintion of x_n, we get the following

$$\sum_{k=m+1}^{n} d(x_k, x_{k-1}) = \sum_{k=m+1}^{n} d(T^k(x_0), T^{k-1}(x_0)).$$

Since T is a contraction, this again implies that

$$\begin{aligned}\sum_{k=m+1}^{n} d(T^k(x_0), T^{k-1}(x_0)) &\leq \sum_{k=m+1}^{n} \lambda^{k-1} d(x_1, x_0) \\ &= d(x_1, x_0)\lambda^m \sum_{k=m+1}^{n} \lambda^k.\end{aligned}$$

The formula for a geometric series then gives us

$$\begin{aligned}d(x_1, x_0)\lambda^m \sum_{k=m+1}^{n} \lambda^k &= d(x_1, x_0)\lambda^m \frac{1-\lambda^{n-m}}{1-\lambda} \\ &\leq \frac{\lambda^{n_0}}{(1-\lambda)} d(x_1, x_0) \to 0 \quad \text{as} \quad n_0 \to \infty.\end{aligned}$$

Thus the sequence $\{x_n\}$ is Cauchy. By assumption, (X, d) is complete, so there exists $x := \lim_{n\to\infty} x_n \in X$.

x **is a fixed-point for** T**.** Then we need the distance between x and $T(x)$ to go to zero. A simple calculation gives us the following

$$\begin{aligned}0 &\leq d(x, T(x)) \\ &\leq d(x, x_n) + d(x_n, T(x_n)) + d(T(x_n), T(x)) \\ &\leq d(x, x_n) + d(x_n, x_n + 1) + \lambda d(x_n, x) \\ &\leq d(x, x_n)\lambda^n d(x_0, x_1) + \lambda d(x_n, x)\end{aligned}$$

Since $d(x, x_n) \to 0$, $\lambda^n \to 0$ and $d(x_n, x) \to 0$ as $n \to \infty$, we get that

$$d(x, x_n)\lambda^n d(x_0, x_1) + \lambda d(x_n, x) \to 0 \quad as \quad n \to \infty,$$

which is what we wanted.

Uniqueness of x**.** Assume that $y = T(y)$. Then

$$0 \leq d(x, y) = d(T(x), T(y)) \leq \lambda d(x, y),$$

and since $\lambda < 1$, this implies that

$$d(x, y) = 0$$

which again gives us

$$y = x.$$

□

4.4 The Picard-Lindelöf theorem

The follwoing theorem will be very useful later in this text.

Theorem 4.3. Let $f : I \times U \to \mathbb{R}^n$ be locally Lipchitz continious with respect to the second variable x and let (t_0, x_0) be the initial condition. Then, for $\eta > 0$, there exists $\varepsilon > 0$ such that the IVP has a unique solution

$$x \in C^1(\overline{\mathrm{B}_\varepsilon(\mathrm{t}_0)}, \overline{\mathrm{B}_\eta(\mathrm{x}_0)}).$$

Note that this theorem guarantees both uniqueness and existence of a fixed-point as opposed to Peano's which only guarantees existence of a solution.

Proof. :

Equivalence of formulations. If $x \in C^1(I, U)$ solves the IVP, then

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \tag{4.1}$$

by integration. Conversely, if $x \in C(I, U)$ satisfies (4.1), then it follows from the Fundamental Theorem of Calculus that x is a $C^1(I, U)$-solution. Thus, the initial-value problem for $x \in C^1(I, U)$ is equivalent to (4.1) for $x \in C^0(I, U)$.

Some constants. Let $\delta > 0$ be such that $[t_0 - \delta, t_0 + \delta] \subset I$. Fix an arbitrary constant $\eta > 0$. Let

$$R := [t_0 - \delta, t_0 + \delta] \times \overline{B_\eta(x_0)}, \quad M := \max_{(t,x)\in R} |f(t, x)|,$$

$$\varepsilon := \{\delta, \frac{\eta}{M}, \frac{1}{2L}\} \quad J := [t_0 - \varepsilon, t_0 + \varepsilon],$$

where L denotes the Lipschitz constant for T in R (since R is compact, f is uniformly Lipschitz continuous on R).

Definition of T. For $v \in BC(J, \mathbb{R}^n)$, define

$$T(v)(t) := x_0 + \int_{t_0}^{t} f(s, v(s))\, ds, \quad t \in J,$$

and consider

$$X := \{v \in BC(J, \mathbb{R}^n) : v(t_0) = x_0, \sup_{t\in J} |x_0 - v(t)| \leq \eta\},$$

which is a closed subset of $BC(J, \mathbb{R}^n)$. Note that $BC(J, \mathbb{R}^n)$ is a complete metric space with respect to the metric

$$d(v_1, v_2) := \max_{t\in J} |v_1(t) - v_2(t)|,$$

so that (X, d) is a complete metric space because it's a closed metric subspace of a complete metric space.

T maps X into X. If $v \in X$, then $T(v)(t_0) = x_0$ and by the definitions of R, M, ε and J, we have

$$|x_0 - T(v)(t)| = \Big| \int_{t_0}^{t} f(s, v(s))\, ds \Big| \leq |t_0 - t| \max_{t \in J} |f(t, v(t))| \leq \varepsilon M \leq \eta,$$

where the second inequality follows from the fact that $v(t) \in \overline{B_\eta(x_0)}$.

T **is a contraction on** X. Let $v_1, v_2 \in X$. Then

$$\begin{aligned} |T(v_1)(t) - T(v_2)(t)| &= \Big| \int_{t_0}^{t} [f(s, v_1(s)) - f(s, v_2(s))]\, ds \Big| \\ &\leq \varepsilon \max_{|s-t_0| \leq |t-t_0|} |f(s, v_1(s)) - f(s, v_2(s))| \\ &\leq \varepsilon L \max_{|s-t_0| \leq |t-t_0|} |v_1(s) - v_2(s)| \\ &\leq \frac{1}{2} \max_{s \in J} |v_1(s) - v_2(s)|, \end{aligned}$$

where we have used $\varepsilon \leq \frac{1}{2}L$ in the last step. Now, taking the maximum over all $t \in J$ yields

$$d(T(v_1), T(v_2)) \leq \frac{1}{2} d(v_1, v_2).$$

Thus, according to Banach fixed-point theorem, there exists a unique solution $x \in BC(J, \overline{B_\eta(x_0)})$.

□

4.5 Picard iteration

Proposition 4.4. Under the assumptions of the Picard-Lindelöf theorem, the sequence given by

$$x_0 = x(t_0), \quad x_n = Tx_{n-1} \quad n \in \mathbb{N}$$
$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

converges uniformly and exponentially fast to the unique solution x on $J = [t_0 - \varepsilon, t_0 + \varepsilon]$,

$$\|x_n - x\|_{BC(J, \mathbb{R}^n)} \leq \frac{\lambda^n}{1 - \lambda} \|x_1 - x_0\|_{BC(J, \mathbb{R}^n)},$$

where $\lambda = \varepsilon L$ is the contraction constant used in the proof of the Picard-Lindelöf theorem.

Proof. According to the proof of the Banach fixed-point theorem, if $m \geq n$, one has

$$d(x_n, x_m) \leq \frac{\lambda^n}{1 - \lambda} d(x_1, x_0),$$

where $\lambda \in (0,1)$ is the contraction constant. We apply this to the operator T, the metric d, and the constants ε and L defined in the proof of the Picard-Lindelöf theorem. Since $\lim_{m\to\infty} x_m = x$ and $d(x_n, \cdot) = \|x_n - \cdot\|_{BC(J,\mathbb{R}^n)}$ is continuous, the proposition follows.

□

Example 4.3. If we have the initial-value problem

$$\dot{x} = \sqrt{x} + x^3, \quad x(1) = 2,$$

then the first Picard iteration is given by

$$x_1(t) = 2 + \int_1^t [\sqrt{2} + 2^3]\, ds = 2 + (\sqrt{2} + 8)(t - 1).$$

The second iteration is

$$x_2(t) = 2 + \int_1^t [\sqrt{x_1(s)} + (x_1(s))^3]\, ds.$$

(This indicates that the Picard iteration, in spite of its simplicity and fast convergence, is better suited as a theoretical and computer-aided tool, than as a way to solve ODE's by hand.)

5 Solution of ODEs

Now we are ready to discuss solutions of ODEs, which is the main purpose of this text. Let us first look at a special case of Picard-Lindelöf which will become useful for our later proofs.

5.1 A special case of Picard-Lindelöf

Let a be a continous function idenpendent from y. Then, let's see if we can find a solution for $y(t)$ to the following ODE

$$y'(t) + a(t)y(t) = 0. \tag{5.1}$$

By rewriting this, we get

$$y'(t) = -a(t)y(t),$$

and since contiuous functions on compact intervals are integrable, this is equivalent to

$$y(t) = y_0 - \int_{t_0}^{t} a(s)y(s)\,ds, \tag{5.2}$$

where $y_0 = y(t_0)$. This follows from the second part of the fundamental theorem of calculus.

Let T be the the function defined by the right side of (5.2). Then, if there exists a solution of (5.1), this will be a fixed-point for T. Let X be a Banach space. If

$$T_B(X, X)$$

and

$$\|T(y_1) - T(y_2)\|_X \leq \lambda \|y_1 - y_2\|_X,$$

then according to Banach's fixed-point theorem there exists a unique solution to $y = T(y)$, and we can find this by iteration using

$$y_{n+1} = T(y_n).$$

Let u be a candidate for a solution. Then

$$T : u \to y_0 - \int_{t_0}^{t} a(s)y(s)\,ds. \tag{5.3}$$

A natural choice of u would be $u \in X = BC(I_\varepsilon)$, where I_ε is the closed subset $[t_0, t_0 + \varepsilon]$.

In order for

$$T : X \to X, \tag{5.4}$$

we need T to be continuous and bounded. Since y_0 is a constant, a is continuous and we choose u to be continuous, Tu is continuous. Now, if we require that I is a compact set, we get that u is bounded, and therefore also Tu is bounded and we get that (5.4) is the case. (We can also show this by taking the supremum on u.

$$\sup_{t\in I_\varepsilon}|u(t)| \leq M \implies |u_0 - \int_{t_0}^{t} a(s)u(s)\,ds| \leq |u_0| + |\int_{t_0}^{t} a(s)u(s)\,ds|$$
$$\leq |u_0| + \varepsilon \max_{t\in I_\varepsilon}|a(t)|M < \infty,$$

where we first use the triangle inequality and then the fact that ε is the maximum length of the integral and $\max_{t\in I_\varepsilon}|a(t)|$ and M are the maximal values of a and u, respectively.)

Now we want T to be a contraction, in other words we want

$$\|Tu - Tv\|_X \leq \lambda\|u - v\|_X$$

where $\lambda < 1$. We start by taking the absolute value, because if this is not bounded, then neither is the sup-norm. We have

$$\begin{aligned}|Tu - Tv| &= \left|u_0 - \int_{t_0}^{t} a(s)u(s)\,ds - u_0 + \int_{t_0}^{t} a(s)v(s)\,ds\right|\\ &= \left|\int_{t_0}^{t} a(s)[v(s) - u(s)]\,ds\right|\\ &= \int_{t_0}^{t} |a(s)||v(s) - u(s)|\,ds\\ &= \int_{t_0}^{t} |a(s)||u(s) - v(s)|\,ds.\end{aligned}$$

This again is less than the supremum of $u - v$ and by taking the integral over the maximum length, which is when $t = t_0 + \varepsilon$, we get that

$$\int_{t_0}^{t} |a(s)||u(s) - v(s)|\,ds| \leq \sup_{t\in I_\varepsilon}|u(t) - v(t)| \int_{t_0}^{t_0+\varepsilon} |a(s)|\,ds.$$

Let

$$\lambda = \int_{t_0}^{t_0+\varepsilon} |a(s)|\,ds$$

and we know that $\sup_{t\in I_\varepsilon}|u(t) - v(t)| = \|u(t) - v(t)\|_X$, which then implies that

$$|Tu - Tv| \leq \lambda\|u(t) - v(t)\|_X.$$

Then, we also have that

$$\sup_{t\in I_\varepsilon}|Tu - Tv| \leq \lambda\|u(t) - v(t)\|_X,$$

which is equivalent to

$$\|Tu - Tv\|_X \leq \lambda\|u(t) - v(t)\|_X.$$

Now, choosing ε such that

$$\lambda = \int_{t_0}^{t_0+\varepsilon} |a(s)|\, ds < 1$$

gives us the desired contraction. Then the Banach fixed-point theorem tells us that there exists a solution $y \in X$ such that $y = Ty$.

5.2 Solution of first order ODEs

Now, what if instead of integrating over the interval $I_\varepsilon = [t_0, t_0 + \varepsilon]$, we now integrate over the interval $I = [t_0, \infty)$?

5.2.1 y'(t)+a(t)y(t)=0

If we take a closer look at

$$y(t) = y_\infty - \int_t^\infty a(s)y(s)\, ds,$$

we can see that in order for there to be a solution to this equation, we need

$$\lim_{t\to\infty} y(t) = k$$

for some constant $k \in \mathbb{R}$. This implies that

$$\lim_{t\to\infty} a(t) = 0. \tag{5.5}$$

Note that $y'(t)$ does not necessarily have to tend to 0, but it is likely that this would be the case here. Now let's state the formal theorem and prove it.

Theorem 5.1. Given a first order ODE

$$y'(t) + a(t)y(t) = 0 \tag{5.6}$$

on the interval $I = [t_0, \infty)$ and $a(t)$ continuous satisfying the following conditions

$$\lim_{t\to\infty} a(t) = 0, \quad \int_t^\infty |a(s)|\, ds < \infty.$$

Let

$$\lim_{t\to\infty} y(t) = k,$$

then there exists a unique solution $m = k$.

Proof. So, given an ODE on the form

$$y'(t) + a(t)y(t) = 0,$$

and $a(t)$ and $y(t)$ satisfying the conditions of the theorem. Let's begin with finding a contraction mapping T. Define T as

$$T : u \to y_\infty - \int_t^\infty a(s)y(s)\, ds,$$

and let Y be the Banach space defined by

$$Y = BC([t_0, \infty), \mathbb{R}).$$

As for (5.6), we have that y_∞ is a constant and a and y are continuous, so then Tu is also continuous. Here we see that requiring (5.5) is necessary for a bounded interval, but not sufficient.

$$\int_t^\infty |a(s)|\, ds < \infty \tag{5.7}$$

is also necessary for boundedness. $\frac{1}{x}$ is an example of a function satifying (5.5) but not (5.8) since

$$\frac{1}{x} \to 0 \quad \text{as} \quad x \to \infty$$

but

$$\lim_{R\to\infty} \int_1^R \frac{1}{x}\, dx = \ln R \to \infty \quad \text{as} \quad R \to \infty.$$

But if we have fulfilled both (5.5) and (5.8), and we know that y_∞ is a constant, we get that Tu is bounded, and $T : Y \to Y$.

Now we need T to be a contraction. As before, we have that

$$\begin{aligned} |Tu - Tv| &= \left| u_0 - \int_t^\infty a(s)u(s)\, ds - u_0 + \int_t^\infty a(s)v(s)\, ds \right| \\ &= \int_t^\infty |a(s)||u(s) - v(s)|\, ds. \end{aligned}$$

This time, the maximum length of the integral is $[t_0, \infty)$, so we get the following inequality

$$\int_t^\infty |a(s)||u(s) - v(s)|\, ds \le \sup_{t\in I} |u(t) - v(t)| \int_{t_0}^\infty |a(s)|\, ds.$$

This means that our λ is defined as

$$\lambda = \int_{t_0}^\infty |a(s)|\, ds$$

this time. Here we have two possibilities. We can fix t_0, and then have a condition on a in order to obtain $\lambda < 1$, or we can define a and then have a requirement on t_0 in order to get our desired contraction. Since we have (5.8), both of this cases works. In other words, given a function $a(t) \in C^1(I, \mathbb{R})$ and a $t_0 \in \mathbb{R}$ satisfying

$$\int_{t_0}^{\infty} |a(s)|\, ds < \infty,$$

there exists a t^* such that

$$\int_{t^*}^{\infty} |a(s)|\, ds < 1.$$

Likewise, we have for every given t_0 that there exists a function $b(t)$ for which

$$\int_{t_0}^{\infty} |b(s)|\, ds < 1.$$

I want to focus on finding a solution to any given function, so therefore I choose to have condition on t_0, given my function $a(t)$. So, choose t_0 such that

$$\lambda = \int_{t_0}^{\infty} |a(s)|\, ds < 1.$$

Then we get

$$\sup_{t \in I} |u(t) - v(t)| \int_{t_0}^{\infty} |a(s)|\, ds = \lambda \|u(t) - v(t)\|_X .$$

This means that

$$\sup_{t \in I} |Tu - Tv| = \|Tu - Tv\|_X \leq \lambda \|u(t) - v(t)\|_X ,$$

and again by Banach fixed-point theorem there exists a unique solution satisfying $y = Ty$.

Now we want to prove that this solution is $m = k$. Given the definition of Y, we have that for all $y \in Y$ there exists

$$\|y\|_{\infty} < \infty.$$

By

$$\int_{t_0}^{\infty} a(s)\, ds < \infty$$

and

$$\int_{t^*}^{\infty} a(s)\, ds < 1,$$

this again implies that

$$\Big| \int_{t}^{\infty} a(s)y(s)\, ds \Big| \leq \int_{t}^{\infty} |a(s)y(s)|\, ds \leq \|y\|_{\infty} \int_{t}^{\infty} |a(s)|\, ds.$$

Since the last integral converges to 0 as $t \to \infty$, the whole equation goes to 0 as $t \to \infty$. This again implies that our solution $y(t)$ to (5.6) converges to m,

$$y(t) = m - \int_{t}^{\infty} a(s)y(s)\, ds \to m \quad \text{as} \quad t \to \infty.$$

$\square$

Now, what happens if we replace 0 in (5.6) with $b(t)$, giving us the equation

$$y(t) + a(t)y(t) = b(t)?$$

5.2.2 y'(t)+a(t)y(t)=b(t)

Given

$$y(t) + a(t)y(t) = b(t),$$

we can rewrite this into

$$y(t) = y_\infty + \int_t^\infty [b(t) - a(s)y(s)]\, ds, \tag{5.8}$$

which again is equivalent to

$$y(t) = y_\infty + \int_t^\infty b(t)\, ds - \int_t^\infty a(s)y(s)\, ds.$$

As before, a solution to $y(t)$ for the interval I implies that

$$\lim_{t\to\infty} y(t) = k.$$

So naturally, we also need

$$\lim_{t\to\infty} \left[y_\infty + \int_t^\infty b(t)\, ds - \int_t^\infty a(s)y(s)\, ds \right] = k.$$

This requires that both

$$\lim_{t\to\infty} b(t) = 0$$

and

$$\lim_{t\to\infty} a(t) = 0.$$

Theorem 5.2. Given an ODE on the form

$$y'(t) + a(t)y(t) = b(t)$$

where $a(t)$ and $b(t)$ is continuous, $a(t)$ satisfying the following conditions

$$\lim_{t\to\infty} a(t) = 0, \quad \int_t^\infty |a(s)|\, ds < \infty,$$

and $b(t)$

$$\lim_{t\to\infty} b(t) = 0, \quad \int_t^\infty |b(s)|\, ds < \infty.$$

If

$$\lim_{t\to\infty} y(t) = k,$$

then there exists a unique solution as $t \to \infty$, namely $m = k$.

Proof. Let Y be the Banach space defined by $Y = BC([t_0, \infty), \mathbb{R})$. Let's define a contraction map T for the ODE. A natural candidate is the following,

$$T : u \to y_\infty + \int_t^\infty [b(s) - a(s)y(s)]\, ds,$$

If T is a contraction map, then $T : Y \to Y$. In order for this to happen, we need a few conditions to ensure boundedness, which is

$$\lim_{t\to\infty} a(t) = 0, \quad \lim_{t\to\infty} b(t) = 0$$

and

$$\int_t^\infty |a(s)|\, ds < \infty, \quad \int_t^\infty |b(s)|\, ds < \infty.$$

Assuming this is satisfied, we have that Tu is bounded and $T : Y \to Y$. Now, we check if T is a contraction. We have

$$\begin{aligned} |Tu - Tv| &= \left| u_0 + \int_t^\infty [b(s) - a(s)u(s)]\, ds - u_0 - \int_t^\infty [b(s) - a(s)v(s)]\, ds \right| \\ &= \int_t^\infty |a(s)||u(s) - v(s)|\, ds. \end{aligned}$$

As before, the maximum length of the integral is $[t_0, \infty)$, so we get the following inequality

$$\int_t^\infty |a(s)||u(s) - v(s)|\, ds \le \sup_{t\in I} |u(t) - v(t)| \int_{t_0}^\infty |a(s)|\, ds.$$

So λ is defined as

$$\lambda = \int_{t_0}^\infty |a(s)|\, ds.$$

Again, we have two possibilities, namely we can fix t_0, and then have a condition on a in order to obtain $\lambda < 1$, or we can define a and then have a requirement on t_0 in order to get our desired contraction. Once again, they both work, so choose t_0 such that

$$\lambda = \int_{t_0}^\infty |a(s)|\, ds < 1.$$

Then we get

$$\sup_{t\in I} |u(t) - v(t)| \int_{t_0}^\infty |a(s)|\, ds = \lambda \|u(t) - v(t)\|_X.$$

This means that

$$\sup_{t\in I} |Tu - Tv| = \|Tu - Tv\|_X \le \lambda \|u(t) - v(t)\|_X,$$

and again by Banach fixed-point theorem there exists a unique solution satisfying $y = Ty$.

Now we want to prove that this solution is $m = k$. Given the definition of Y, we have that for all $y \in Y$ there exists

$$\|y\|_\infty < \infty.$$

Since

$$\int_{t_0}^{\infty} a(s)\, ds < \infty$$

and

$$\int_{t^*}^{\infty} a(s)\, ds < 1,$$

this again implies that

$$\Big|\int_t^{\infty} a(s)y(s)\, ds\Big| \le \int_t^{\infty} |a(s)y(s)|\, ds \le \|y\|_\infty \int_t^{\infty} |a(s)|\, ds.$$

Since the last integral converges to 0 as $t \to \infty$, the whole equation goes to 0 as $t \to \infty$. This again implies that (5.8) converges,

$$y(t) = m - \int_t^{\infty} a(s)y(s)\, ds \to m \quad \text{as} \quad t \to \infty.$$

□

5.2.3 y'(t)+f(t,y(t))=b(t)

Now, let us look at

$$y'(t) + f(t, y(t)) = b(t).$$

The difference between $a(t)y(t)$ and $f(t, y(t)$ is that $f(t, y(t)$ may become non-linear, so we can have a $y(t)^2$ for example, which makes this ODE more complex. Let us begin by stating the theorem.

Theorem 5.3. Given an ODE on the form

$$y'(t) + f(t, y(t)) = b(t), \tag{5.9}$$

where $b(t)$ is continuous and satisfies the following conditions

$$\lim_{t\to\infty} b(t) = 0, \qquad \int_t^{\infty} |b(s)|\, ds < \infty,$$

and $f \in BC(I)$ and f is Lipschitz.

If

$$\lim_{t\to\infty} y(t) = k,$$

then there exists a unique solution as $t \to \infty$, and the solution is $m = k$.

Proof. As before, we want to find a contraction mapping T. Let it be defined as the following,

$$T : u \to y_\infty + \int_t^\infty [b(s) - f(t, y(t))]\, ds.$$

To ensure boundedness, we need the following conditions to be satisfied,

$$\lim_{t\to\infty} f(t, y(t)) = M < \infty$$

and

$$\lim_{t\to\infty} b(t) = 0 \qquad \int_t^\infty |b(s)|\, ds < \infty.$$

Then we have that Tu is bounded and $T : Y \to Y$. Now, we check if T is a contraction. A direct computation shows that

$$\begin{aligned} |Tu - Tv| &= \left| u_0 + \int_t^\infty [b(s) - f(s, u(s))]\, ds - u_0 - \int_t^\infty [b(s) - f(s, v(s))]\, ds \right| \\ &= \int_t^\infty |f(s, u(s)) - f(s, v(s))|\, ds. \end{aligned}$$

Since f is Lipshitz, we have

$$\int_t^\infty |f(s, u(s)) - f(s, v(s))|\, ds \le \int_t^\infty |a(s)||u(s) - v(s)|\, ds.$$

Since

$$|u(s) - v(s)| \le |u - v|_{BC(I)},$$

it follows that

$$\int_t^\infty |a(s)||u(s) - v(s)|\, ds \le |u - v|_{BC(I)} \int_t^\infty |a(s)|\, ds.$$

We have

$$\int_{t_0}^\infty a(s)\, ds < \infty,$$

$a \ge 0$ and a is continuous. This implies that there exists $t^* \ge t_0$ such that

$$\int_{t^*}^\infty a(s)\, ds < 1.$$

Let us call this integral λ. Then we get

$$|Tu - Tv| \le \lambda |u - v|_{BC(I)},$$

which again implies that

$$\|Tu - Tv\|_X \le \lambda \|u - v\|_X,$$

and T is our desired contraction mapping. By the Banach fixed-point theorem, we therefore have a unique solution such that $y = Ty$.

Now, let's prove that this solution is $m = k$. By the definition of Y, we have that for all $y \in Y$ there exists

$$\|y\|_\infty < \infty.$$

Since

$$\int_{t_0}^{\infty} a(s)\, ds < \infty$$

and

$$\int_{t^*}^{\infty} a(s)\, ds < 1,$$

this again implies that

$$\Big| \int_t^\infty f(s, y(s))\, ds \Big| \leq \int_t^\infty |a(s)||y(s)|\, ds \leq \|y\|_\infty \int_t^\infty |a(s)|\, ds.$$

Since the last integral converges to 0 as $t \to \infty$, the whole equation (5.9) goes to 0 as $t \to \infty$. This again implies that,

$$y(t) = m - \int_t^\infty a(s)y(s)\, ds \to m \quad \text{as} \quad t \to \infty.$$

□

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