safety

MDPI

Article

Using Tolerance Bounds for Estimation of Characteristic Fatigue Curves for Composites with Confidence

Knut O. Ronold [1] and Andreas T. Echtermeyer [2,*]

[1] DNV, Veritasveien 1, N-1322 Høvik, Norway; ronold@alumni.stanford.edu
[2] Department of Mechanical and Industrial Engineering, Norwegian University of Science and Technology, NTNU, Richard Birkelands vei 2, N-7491 Trondheim, Norway
* Correspondence: Andreas.Echtermeyer@ntnu.no

Abstract: Fatigue *S*–*N* curves provide the number of stress cycles that result in fatigue failure at stress range *S* and need to be measured for new engineering materials where data are not as readily available as they are for well-characterized and widely used metals. A simple statistical method for the estimation of characteristic fatigue curves defined in terms of lower-tail quantiles in probability distributions of dependent variables is presented. The method allows for the estimation of such quantiles with a specified confidence level, taking account of the statistical uncertainty caused by a limited number of experimental test results available for the estimation. The traditional general approach for estimating characteristic *S*–*N* curves by tolerance bounds is complicated and is not much used by engineers. The presented approach allows for calculating the curves with a simple spreadsheet. The only requirement is that the experimental log *S* data for the *S*–*N* curve are fairly uniformly distributed over a finite log*S* interval, where *S* denotes the stress range. Experimental fatigue test programs are often designed such that test data fulfil this assumption. Although developed with fatigue of composite laminates in mind, the presented statistical procedure and the presented associated charts are valid for fatigue curve estimation for any material.

Keywords: design; fatigue curves; tolerance bounds; composite laminates

Citation: Ronold, K.O.; Echtermeyer, A.T. Using Tolerance Bounds for Estimation of Characteristic Fatigue Curves for Composites with Confidence. *Safety* **2022**, *8*, 1. https://doi.org/10.3390/safety8010001

Academic Editor: Raphael Grzebieta

Received: 27 June 2021
Accepted: 18 December 2021
Published: 23 December 2021

1. Introduction

Composites (fibre reinforced plastics) are finding their way into many demanding engineering applications. Demonstrating sufficient mechanical performance in the short and long term is required by design standards, codes and users for such applications. Typically, performance is demonstrated in engineering calculations applying mechanical properties, failure criteria and safety factors. Modern design standards that are based on probabilistic methods use characteristic values of mechanical properties, e.g., [1–3]. For static strength the characteristic value is to be taken as the mean minus *k* standard deviations, where the factor *k* depends on the tolerance level and confidence one wants to obtain and on the number of experimental tests carried out to obtain the strength. This is described in more detail later. A good overview about the meaning of characteristic values can be found in Madsen et al. (1986) [4]. Examples of the characteristic strength values used specifically for composites are the A and B values used in the aerospace industry and defined in the Mil. Standardization Handbook [5] and the characteristic strengths in the standards for offshore applications [2,3]. It is important to note that the exact choice of defining the characteristic value in combination with the safety factors and failure criteria defines the target level of the safety of the design.

A similar approach as for determining the characteristic static strength can, in principle, also be used for fatigue lifetime evaluations. However, this is less established for composites than it is for static strength. Fatigue lifetimes can be calculated for variable loading conditions by combining the *S*–*N* curves with the Miner sum approach and Goodman diagrams, as is well known from standard materials textbooks, e.g., [6,7]. This regular approach has been used successfully for composite laminates for many years, as documented, for example, in references [8–14]. Alternative lifetime calculation methods have also been proposed for composites; good overviews are given in [11,15]. It is beyond the scope of this article to evaluate the alternative methods, but it seems that the regular approach is by far the most widely used in industrial applications. One reason favouring the regular approach is the requirement to use design standards and these standards tend to be based on the regular approach [1–3,16].

The DNV standards for offshore components and thermoplastic composite pipes require the use of characteristic *S*–*N* curves with 97.5% tolerance and 95% confidence. Simple procedures are given to obtain these curves. However, due to the simplicity, the curves are slightly conservative and they can only be extrapolated for a limited range beyond the measured data.

Generally, not much published work is available looking into the probability of failure in fatigue. Methods for structural fatigue design, including probabilistic aspects, are addressed in Lotsberg (2016) [17]. Post [18] developed a reliability-based design methodology, but it is based on a residual strength approach, one of the alternative methods not covered by the official standards so far. Castro, Branner and Dimitrov present an interesting approach that combines the uncertainties of the *S*–*N* curve, Goodman diagram and Miner sum [19]. The uncertainty of the *S*–*N* curve has been addressed by several authors describing probabilistic limits without considering the confidence level, i.e., not taking into account the statistical uncertainty associated with the number of tests being done [20,21]. This is an important limitation, especially if the number of tests in a fatigue test program is low.

Using a tolerance bound approach with confidence is described as being theoretically ideal by Schneider and Maddox [22]. However, the approach is seen to be too complicated. The foundation for the method was first given by Owen [23] and it is summarized for fatigue by Wirsching [24]. A simple solution to obtain tolerance bounds leading to straight lines parallel to the mean *S*–*N* curve on log-log scale was proposed by Wei et al. [25]. This is a somewhat similar, but not identical, approach as used in the current DNV standards [2,3]. The authors of this article have shown previously that a relatively simple and accurate method for tolerance bounds can be found for some special cases [26]. Nijssen [11] shows a similar case in his in-depth review of fatigue prediction methods to characterize *S*–*N* curves. This article takes these developments one step further. It allows for estimating characteristic curves for any combination of tolerance and confidence level. This allows easy and accurate creation of characteristic *S*–*N* curves to be used with the DNV standards and also other standards that use different tolerances and confidence levels.

For many materials, especially commonly used metals, *S*–*N* curves for design can be found in data sheets or standards. There is no need to measure and calculate the characteristic curves for these well characterized materials. The given curves are usually ”characteristic curves”, which means they describe fatigue lifetimes with a specified probability of exceedance, a kind of “guaranteed” lifetime value. A well-known example of a characteristic *S*–*N* curve is on log-log scale the mean curve minus two standard deviations, which is a curve which corresponds to a probability of survival (also referred to as tolerance) of 97.7% when the log$N$ conditional on log$S$ is normally distributed. Note here that N denotes number of stress cycles to failure and S denotes the stress range. Theoretically, the number of cycles in the tests should be infinite, but practically it is well over 100. Many test data are typically only available for widely used well-characterized materials. Methods for structural fatigue design are addressed in Lotsberg (2016) [17].

If the material is not well characterized, its *S*–*N* curve is not readily available and needs to be measured. This is the case for new materials and has been the case for composites (fibre-reinforced plastics) in many projects. Even though composites are not new, they have been widely used since the 1960s–1970s, and no particular composite dominates the market. Different combinations of fibres and resins are used all the time, requiring measurements of new S–N curves. Further, properties change at different environments, requiring even more testing [27]. Establishing the S–N curves for particular projects means that the number of possible tests is limited due to time and cost restrictions. Typically, 10 to 20 tests of one combination of material and environmental condition are carried out. With the limited number of test results, the characteristic curve cannot be obtained by calculating the mean curve minus two standard deviations on a log-log scale, because this approach would not account for the uncertainty created by a limited number of tests. The approach used for limited number of specimens is to estimate the characteristic curve for a given probability of survival at a specified confidence level. Different design standards use different combinations of the survival probability and confidence level to specify characteristic S–N curves for calculation of the fatigue damage to which safety factors are applied. It is important to obtain the data as required by the design standard, otherwise the safety factors specified by the standard would not give the intended level of safety.

Characteristic values of strength are small but measurable quantiles, such as the second, fifth or tenth percentile, or the manufacturer's "guaranteed strength", see Madsen et al. (1986) [4]. The characteristic value of a strength quantity serves as a quality control parameter and is used as a basis for design. The characteristic value for a strength variable Y with a mean value μ and standard deviation σ is often taken as $\mu - 2\sigma$. For a normally distributed strength with a known μ and σ, this implies that 97.7% of all realizations of Y will fall above the characteristic value. This exceedance proportion is often referred to as the survival probability, or just the tolerance. With μ and σ unknown and estimated with statistical uncertainty from a limited number of data, a corresponding characteristic value estimated to be used in design will need to be further away from μ than 2σ to maintain the survival probability of 97.7% with sufficient confidence. The associated factor on σ will thus be higher than 2 and will depend on the number of observations of Y as well as on a required confidence level. The value of the factor on σ can be estimated consistently by means of statistical theory for tolerance bounds and becomes, in practice, a factor on the sample standard deviation s since σ itself is unknown. Examples of the factor on σ for the static strength can be found in DIN55303-5 [28] and DNV-OS-C501 [2]. It is noted in this context that the ASTM standard D2992 [29] describes a widely used method to establish a characteristic S–N curve from test data for use in design. ASTM D2992 does not specify a particular definition of the characteristic S–N curve, but specifies that the curve to be used shall be taken as the lower prediction limit for the next realization of the S–N curve, given the data, and estimated with 95% confidence. It is noted that this approach is not compatible with tolerance bound theory, because no tolerance is associated with the definition of the characteristic S–N curve in ASTM D2992.

Ronold and Echtermeyer (1996) [26] presented the theoretical basis for a means to establish characteristic values from tolerance bounds for independent, as well as dependent variables. They presented an introduction to linear regression for fatigue S–N curves, also known as Wöhler curves. The basis for their work was S–N curves of the form:

$$\log N = \log N_0 - \beta \cdot \log S \quad (1)$$

where N is the number of cycles to failure, N_0 is a constant, β describes the slope of the curve, and S is the applied stress amplitude. In some cases S can also be the stress range or the maximum stress. The definition of S does not influence the statistical procedures described here.

This double logarithmic form of the S–N curve is used for most metals. It has also been successfully used for composite laminates, especially when high numbers of cycles to failure were measured, see Mandell et al. (2003) [30].

The statistical methods described in [26] for the estimation of characteristic S–N curves on the basis of tolerance bounds do also work for the fatigue curves described in the lin–log form

$$\log N = \log N_0 - \beta \cdot S \quad (2)$$

The general statistical procedure described in [26] for the estimation of characteristic *S*–*N* curves based on tolerance bounds is cumbersome in that it depends on how the log*S* values of the test data are distributed over the log*S* range covered by these data. Unlike the corresponding procedure for independent variables, the procedure is not suitable for tabulation or closed-form expressions of the distribution quantiles involved for dependent variables, because there is a new distribution of the log*S* values of the test data each time, i.e., for every new data set. The article [26] showed a simplification of the procedure based on the assumption that the log*S* values of the test data are uniformly distributed over the log*S* range covered by the data. The article presented the results of this simplification graphically for the particular survival probability of 95% and the particular confidence level of 95% and, thereby, made them easily accessible for practical use by designers and engineers. The establishment of such characteristic curves was needed at that time to establish design methods for fatigue calculations giving equivalent safety to similar structures made of steel. (The paper actually showed strain vs. number of cycles *ε*–*N* curve instead of the typically used *S*–*N* curve. The data shown were from a fibre-reinforced composite laminate and for these materials, the *ε*–*N* curve representation has the advantage of being independent of the fibre fraction of the composite and gives similar results for laminates with different reinforcements, see Echtermeyer et al. (1996) [31] and Echtermeyer et al. (1996) [32]. However, for the present article, the principle of the statistical analysis is important. The results can be applied to common *S*–*N* curves and *ε*–*N* curves as well as any other relationship where one variable is linearly dependent on another.)

In the following, the work by Ronold and Echtermeyer (1996) [26] is taken one step further. As stated above, the simplification of the procedure for the estimation of characteristic *S*–*N* curves assumes uniformly distributed log*S* values over the log*S* range covered by the set of fatigue test data available to establish the *S*–*N* curve. The simplification allows the use of easy methods to estimate the characteristic curves for various survival probabilities and confidence levels. First, a procedure is worked out for the simulation of the factor on the sample standard deviation in the downward offset from the sample mean of log*N* to provide an estimate of the characteristic value of log*N* as an estimate of a specified quantile in the distribution of log*N*, where *N* is the number of cycles to failure. Results are presented graphically for various combinations of the survival probability and confidence level. Second, a simple mathematical expression is introduced for the same factor on the sample standard deviation in the downward offset from the sample mean to provide the same estimate of the characteristic value of log*N* as a function of the specified survival probability and confidence level. The simple expression is a function of the length of the log*S* interval covered by the (*S*,*N*) data, and of the deviation from the centre of this interval of the particular log*S* of interest for the estimation. Closed-form expressions for the coefficients in the simple expression for the factor are derived and presented. These expressions are all functions of the specified survival probability, the specified confidence level for the estimation and the number of observations of log*N* available from the fatigue tests.

The introduction of the simple expression for the factor on the sample standard deviation with closed-form expressions for the involved coefficients thus represents a generalization of the simplified method—presented previously [26] for just one set of survival probability and confidence level—to a wide range of survival probabilities and confidence levels. In addition, with its analytical formulas, it also allows for the fast and easy calculation of characteristic fatigue curve estimates, e.g., in a spreadsheet, for use in design.

For completeness in the following, and to place the presented analytical derivations in context, the basics of tolerance bound theory are recapitulated from Ronold and Echtermeyer (1996) [26] and Ronold and Lotsberg [33] before the presentation of the new material for characteristic fatigue curve estimation itself is given.

Throughout this article, upper-case letters are used to denote stochastic variables. For a given stochastic variable, lower-case letters are used to denote realizations, such as specific quantiles. This convention for notation is adopted from Madsen et al. (1986) [4]. The main symbols used are shown in Table 1.

Table 1. Symbols.

Symbol	
α	probability complement of confidence level
$1-\alpha$	statistical confidence level
Γ	gamma function
ε	strain
Φ	standard Gaussian cumulative distribution function
σ	standard deviation
Δx	increment of x
Lx	length of finite interval of x = log$S$
h_n	function of n
m	material property of fiber strength
n	number of fatigue tests
W	chi-square distributed variable with specific number of degrees of freedom
C	non-central t-distributed variable
S	stress range
s	estimate of standard deviation σ
S	stochastic standard deviation
N	number of stress cycles to failure at stress range S
$\overline{\sigma}$	mean fibre strength
σ_f	fibre bundle strength

2. Theoretical Methods

2.1. Theory of Tolerance Bounds for Random Variables

A tolerance interval is a range for a variable Y designed to capture a proportion γ or more of all outcomes of Y with probability $1-\alpha$. The probability $1-\alpha$ is commonly referred to as the confidence level. When a one-sided tolerance interval is considered and γ refers to the proportion of all outcomes of Y larger than the tolerance bound, then the tolerance bound is an estimate of the $1-\gamma$ quantile in the distribution of Y with confidence $1-\alpha$. This is a biased estimate and $1-\alpha$ can be interpreted as the probability that the true but unknown value of the $1-\gamma$ quantile for Y is greater than the value of the tolerance bound. This is useful for the estimation of characteristic values of Y which are defined as quantiles in the probability distribution of Y, when it is required, e.g., by a design standard, that the estimation shall be carried out with confidence.

A normal population Y is considered. The population Y is a stochastic variable and not a fixed number. The distribution of Y is a normal distribution with mean value μ and standard deviation σ. A one-sided random interval for Y with a lower bound Y_k is established through the probability equation

$$P[P[Y_k < Y]] > \gamma = 1-\alpha \tag{3}$$

which implies that, with confidence $1-\alpha$, at least a proportion γ of the population Y will exhibit values in excess of Y_k. The proportion γ is also referred to as the tolerance and, sometimes, as the coverage, and if Y is a strength variable it is also referred to as the survival probability. Y_k will be a random bound. When Y_k takes on a realization y_k, based on n observations of Y, the random interval is turned into a tolerance interval for Y. The realization y_k is referred to as a tolerance bound. The realization y_k can be interpreted as an estimate of the $1-\gamma$ quantile in the distribution of Y with confidence at least $1-\alpha$, i.e., there is a probability of at least $1-\alpha$ that the true, but unknown value of the $1-\gamma$ quantile in the distribution of Y is greater than y_k.

It is of interest to express the random interval bound Y_k in terms of variables with known probability distributions. The random interval bound can be expected to take the form of

$$Y_k = \overline{Y} - c_{1-\alpha} \cdot S \tag{4}$$

where $\overline{Y}$ is the statistically uncertain estimate of the unknown mean value of Y and S is the statistically uncertain estimate of the standard deviation of Y based on n observations of Y.

Hence, it is of interest to find an expression for the factor $c_{1-\alpha}$. Provided n observations of Y are available, the probability equation defining the random estimation interval can be rewritten as follows:

$$\begin{gathered} P\left[P\left[\overline{Y} - c_{1-\alpha} \cdot S < Y\right] > \gamma\right] = 1-\alpha \\ P\left[P\left[\frac{\overline{Y}-c_{1-\alpha}\cdot S-\mu}{\sigma} < \frac{Y-\mu}{\sigma}\right] > 1-\Phi(u_{1-\gamma})\right] = 1-\alpha \\ P\left[1-\Phi\left(\frac{\overline{Y}-c_{1-\alpha}\cdot S-\mu}{\sigma}\right) > 1-\Phi(u_{1-\gamma})\right] = 1-\alpha \\ P\left[\frac{\overline{Y}-c_{1-\alpha}\cdot S-\mu}{\sigma} < u_{1-\gamma}\right] = 1-\alpha \\ P\left[c_{1-\alpha} > \frac{\overline{Y}-\mu}{S} - \frac{\sigma}{S}u_{1-\gamma}\right] = 1-\alpha \end{gathered} \tag{5}$$

This rewriting of the probability equation utilizes that $\Phi()$ is the standard normal cumulative distribution function, $u_{1-\gamma}$ is the $1-\gamma$ quantile in the standard normal distribution function, and $(Y-\mu)/\sigma$ is standard normally distributed.

2.1.1. Independent Variables

For an independent variable Y, Equation (5) can be further rewritten as

$$P\left[c_{1-\alpha} > \left(\frac{\overline{Y}-\mu}{\sigma\frac{1}{\sqrt{n}}} - u_{1-\gamma}\sqrt{n}\right) \cdot \sqrt{\frac{n-1}{n}} \cdot \frac{1}{\frac{S}{\sigma}\sqrt{n-1}}\right] = 1-\alpha \tag{6}$$

see Madsen et al. (1986) [4]. From this version of the probability equation, it appears that the factor $c_{1-\alpha}$ can be interpreted as the $1-\alpha$ quantile in the distribution of a variable

$$C = \left(\frac{\overline{X}-\mu}{\sigma\frac{1}{\sqrt{n}}} - u_{1-\gamma}\sqrt{n}\right) \cdot \sqrt{\frac{n-1}{n}} \cdot \frac{1}{\frac{S}{\sigma}\sqrt{n-1}} = \left(U - u_{1-\gamma}\sqrt{n}\right) \cdot \sqrt{\frac{n-1}{n}} \cdot \frac{1}{\sqrt{W}} \tag{7}$$

in which it is recognized that $U = \left(\overline{X}-\mu\right)\sqrt{n}/\sigma$ is the standard that is normally distributed and $W = (n-1)S^2/\sigma^2$ is chi-square distributed with $n-1$ degrees of freedom. It is noted that $C\sqrt{n}$ is a non-central *t*-distributed variable.

The quantile $c_{1-\alpha}$ can be read off from tables; see Resnikoff and Lieberman (1957) [34], Owen (1958) [23] and Pearson and Hartley (1976) [35]. Alternatively, it can be interpreted as the $1-\alpha$ quantile in the simulated distribution of the variable C as obtained from Monte Carlo simulation of its two parent random variables U and W. Yet another alternative is to calculate the quantile $c_{1-\alpha}$ by means of Hald's approximation, see Hald (1952) [36] and Madsen et al. (1986) [4],

$$c_{1-\alpha} \approx \frac{\Phi^{-1}(\gamma) + \Phi^{-1}(1-\alpha)\cdot\sqrt{\frac{1}{n}\cdot\left(1-\frac{\left(\Phi^{-1}(1-\alpha)\right)^2}{2(n-1)}\right) + \left(\Phi^{-1}(\gamma)\right)^2}}{1-\frac{\left(\Phi^{-1}(1-\alpha)\right)^2}{2(n-1)}} \tag{8}$$

in which $\Phi^{-1}()$ denotes the inverse standard normal distribution function.

When the statistically uncertain estimate $\overline{Y}$ of the mean value μ takes on the realization $\overline{y}$ and the statistically uncertain estimate S of the standard deviation σ takes on the realization s, based on n observations of Y, the random interval bound y_k takes on the numerical realization

$$y_k = \overline{y} - c_{1-\alpha} \cdot s \tag{9}$$

which turns the random interval bound into a fixed tolerance bound for the stochastic variable Y. For further details of tolerance bound theory for independent variables, see Guttman (1970) [37], Zacks (1970) [38] and Little (1981) [39].

Note that, according to Ronold and Echtermeyer (1996) [26], this determination of the quantile $c_{1-\alpha}$ leads to the construction of lower tolerance bounds that are identical to those defined according to the method of DIN55303 [28]. The so-called 'A' basis and 'B' basis design allowables, according to MIL-HDBK-5B [5], are also based on this method.

The method can be well applied to estimate the characteristic static strength of a composite laminate. The assumption is then that the static strength of the composite laminate adheres to a normal distribution. The assumption of a normal distribution for the static strength of composite laminates can be based on the following rationale: the static strength of the laminate in the direction of the fibres is dominated by the strength σ_f of the fibre bundles. The strength σ_f of a fibre bundle is proportional to the mean fibre strength $\overline{\sigma}$ of the individual fibres,

$$\sigma_f = \overline{\sigma} \cdot \frac{\exp(1/m)}{m^{1/m} \cdot \Gamma\left(1 + \frac{1}{m}\right)} \tag{10}$$

in which m is a material constant, see Beaumont and Schultz (1990) [40], and Γ denotes the gamma function. The individual fibre strengths very often follow a Weibull distribution; however, regardless of this distribution type, their average will, under the central limit theorem, asymptotically follow a normal distribution, see Ang and Tang (1975) [41]. Hence the mean $\overline{\sigma}$ and then also—through Equation (10)—the fibre bundle strength σ_f and the static laminate strength can be deduced to asymptotically be normally distributed. Although it is very difficult to find data confirming the one or the other distribution type for the laminate strength, there is, thus, theoretical evidence to support the normal distribution and thereby allow for the use of Equation (9) for estimation of the characteristic laminate strength with confidence.

2.1.2. Dependent Variables

Consider now a dependent variable Y, which is normally distributed conditional on an independent variable X, and whose mean μ has a linear variation with X. This would be the case for the fatigue curves on a log-log scale. By a derivation in analogy with that for independent variables above and by capitalization on the results from a linear regression of the linear relationship between μ and X, the probability equation in Equation (5) can be rewritten as

$$P\left[c_{1-\alpha} > \left(\frac{\overline{Y}(x_0) - \mu(x_0)}{\sigma \cdot h_n} - \frac{u_{1-\gamma}}{h_n}\right) \cdot h_n \cdot \sqrt{n-2} \cdot \frac{1}{\frac{S}{\sigma}\sqrt{n-2}}\right] = 1 - \alpha \tag{11}$$

in which

$$h_n = \sqrt{\frac{1}{n} + \frac{(\overline{x} - x_0)^2}{\sum_{i=1}^{n} (x_i - \overline{x})^2}} \tag{12}$$

where x_0 is the particular value of the independent variable X for which the tolerance bound for Y is to be calculated, x_1 is the value of X for which the ith value of Y is observed, and $\overline{x}$ is the mean value of the n x_i values. Further, $\mu(x_0)$ is the mean value of Y for $X = x_0$, σ is the standard deviation of the residuals of Y about the fitted linear relationship between μ and X, and $\overline{Y}(x_0)$ and S are the corresponding uncertain estimators. Note that Y is assumed to be homoscedastic, i.e., the standard deviation σ is assumed to be a constant, independent of the value of X.

From this version of the probability equation it appears that the factor $c_{1-\alpha}$ can be interpreted as the $1-\alpha$ quantile in the distribution of a variable

$$C = \left(\frac{\overline{Y}(x_0)-\mu(x_0)}{\sigma\cdot h_n} - \frac{u_{1-\gamma}}{h_n}\sqrt{n}\right)\cdot h_n\cdot\sqrt{n-2}\cdot\frac{1}{\frac{S}{\sigma}\sqrt{n-2}} = \left(U-\frac{u_{1-\gamma}}{h_n}\right)\cdot h_n\cdot\sqrt{n-2}\cdot\frac{1}{\sqrt{W}} \quad (13)$$

in which it is recognized that $U = \left(\overline{Y}(x_0)-\mu(x_0)/(\sigma h_n)\right)$ is standard normally distributed and $W = (n-2)S^2/\sigma^2$ is chi-square distributed with $n-2$ degrees of freedom.

The quantile $c_{1-\alpha}$ can be interpreted as the $1-\alpha$ quantile in the simulated distribution of the variable C as obtained from Monte Carlo simulation of its two parent random variables U and W.

When the statistically uncertain estimate $\overline{Y}(x_0)$ of the mean value $\mu(x_0)$ takes on the realization $\overline{y}(x_0)$ and the statistically uncertain estimate S of the standard deviation σ takes on the realization s, based on the n observations of Y, the random interval bound Y_k for $X = x_0$ takes on the numerical realization

$$y_k(x_0) = \overline{y}(x_0) - c_{1-\alpha}\cdot s \quad (14)$$

which turns the random interval bound into a fixed tolerance bound for the stochastic variable Y evaluated at $X = x_0$.

It appears that the quantile $c_{1-\alpha}$, which is obtained as the solution of the probability equation, is a function, not only of the proportion γ, the confidence level $1-\alpha$, and the number of observations n of Y, but also of the particular values of X for which Y is observed, and of the value x_0 of X for which the lower tolerance bound for Y is sought.

Thus it appears that the tolerance bound is much dependent on whether the observations of Y are clumped together within a narrow range for X or if they are distributed over a wide range, and whether the current choice for x_0 is some central value within this range or some value outside the range. As a consequence, the tolerance bound does not form a straight line in an X-Y diagram even though the mean value of Y vs. X does. The tolerance bound is curved, and it is more curved the fewer data that are available for its calculation, see Figure 1.

The derivation of the tolerance bound in Equation (14) is based on results from linear regression of data for Y conditioned on X. Other approaches to such tolerance bound estimation for dependent variables exist; for example, Pascual (1997) [42] presents approximate tolerance bounds based on the maximum likelihood results for a set of fatigue test data.

2.2. Tolerance Bounds for Dependent Variables

2.2.1. Theory

Fatigue curves, commonly referred to as S–N curves, are curves which express a dependent variable, log$N$, as a linear function of an independent variable, log$S$, where N denotes number of cycles to failure at stress range S. Interpretation of tolerance bounds for log$N$ form an attractive approach to estimation of lower-tail quantiles for log$N$, used for definition of characteristic fatigue curves, with confidence.

Test programs for fatigue testing are often designed in such a manner that the $\log_{10}S$ values for the various tests are fairly evenly distributed over the interval for log$S$ which is covered by the tests.

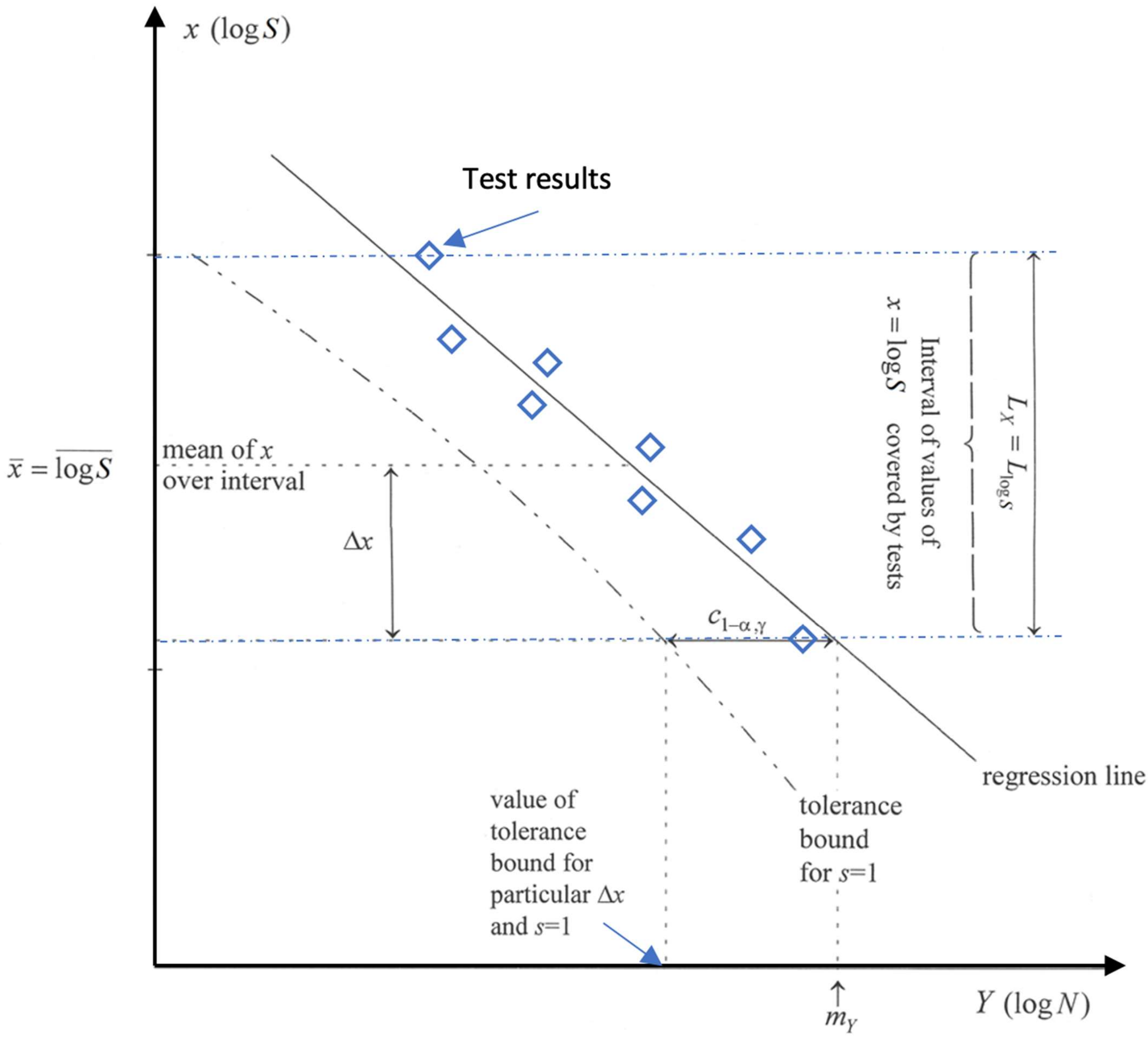

Figure 1. Explanation of measures and symbols. modified from [26].

Let X be used to denote $\log_{10}S$ in the following. With the assumption of uniformly distributed x_i values, $i = 1, \ldots n$, over an interval of length L_x along the x axis, the factor h_n used in the expression for the variable C in Equation (13)

$$C = (U - \frac{u_{1-\gamma}}{h_n}) \cdot h_n \cdot \sqrt{\frac{n-2}{W}} \tag{15}$$

simplifies to

$$h_n = \sqrt{\frac{1}{n} + \frac{12n}{n^2-1} \cdot (\frac{\Delta x}{L_x})^2} \tag{16}$$

in which Δx denotes the positive or negative deviation $x_0 - \overline{x}$, where x_0 is the $\log S$ value of interest for prediction of the $1 - \gamma$ quantile of $\log N$ with confidence, and where the mean value $\overline{x}$ of the n x_i values, under the assumption of uniformly distributed x_{i} values, is also the midpoint of the interval of length L_x that covers the n x_i values. For further explanation, see Figure 1.

Note that under the assumption of uniformly distributed log$S$ values over the log$S$ range covered by the tests, the length L_x of the log$S$ interval can be calculated as

$$L_x = \frac{n}{n-1} \cdot (\log S_{max} - \log S_{min}) \tag{17}$$

where logS_{max} is the maximum and logS_{min} is the minimum of log$S$ in the test set, and n is the number of tests, each leading to a data pair (S, N).

2.2.2. Graphical Representation of the Quantile $c_{1-\alpha}$

For the given tolerance γ and given relative deviation $\Delta x/L_x$ from the midpoint of the considered X = log$S$ interval over which available test data are uniformly distributed, simulations of the variable C in Equation (15) are carried out by simulating the parent variables U and W. A Monte Carlo simulation procedure, as described in Tvedt (2006) [43], is used for this purpose. For each combination of tolerance γ and number of fatigue tests n, 200,000 simulations of C are carried out. The probability distribution of C results from these simulations and the quantile $c_{1-\alpha}$ can be extracted from this distribution for given cumulative probability $1-\alpha$, i.e., for specified confidence level $1-\alpha$.

The results of such simulations are presented graphically in Figure 2 for nine commonly encountered combinations of tolerance (or probability of survival) γ and confidence level $1-\alpha$, viz. γ equal to 0.90, 0.95 and 0.97725 in combination with $1-\alpha$ equal to 0.75, 0.90 and 0.95. The quantile $c_{1-\alpha}$ is plotted against the relative deviation of test data $\Delta x/L_x$ from the midpoint. An example of applying $c_{1-\alpha}$ for a specific $\Delta x/L_x$ is graphically indicated in Figure 1. Figure 2 shows how $c_{1-\alpha}$ increases with $\Delta x/L_x$. The tolerance bound curve in Figure 1 is given by the regression curve of the data shifted horizontally by $c_{1-\alpha}$ for the relevant $\Delta x/L_x$ value. Therefore, the shape of the tolerance bound curve in Figure 1 is directly related to the shape of the $c_{1-\alpha}$ vs. $\Delta x/L_x$ from Figure 2.

2.2.3. Mathematical Representation of the Quantile $c_{1-\alpha}$

Under the prevailing assumption that the n X values pertaining to the data set of n observations of $Y = \log_{10}N$ are uniformly distributed over the interval of length L_x, it can be deduced that the quantile $c_{1-\alpha}$ can be represented well by a hyperbolic expression in $\Delta x/L_x$,

$$c_{1-\alpha} = c_1 + \sqrt{c_2^2 + c_3^2 \cdot (\frac{\Delta x}{L_x})^2} \tag{18}$$

By comparison of the limits of the expressions in Equations (15) and (18) for large values of $\Delta x/L_x$, it can be deduced that

$$c_3 = t_{n-2}(1-\alpha) \cdot \sqrt{\frac{12n}{n^2-1}} \tag{19}$$

in which $t_{n-2}(1-\alpha)$ is the $1-\alpha$ quantile in the Student's t distribution with $n-2$ degrees of freedom. This quantile can be read off from statistical tables, see for example Snedecor and Cochran (1989) [44] and DNV-RP-C207 [45]. The most commonly needed quantiles are summarized in Table 2.

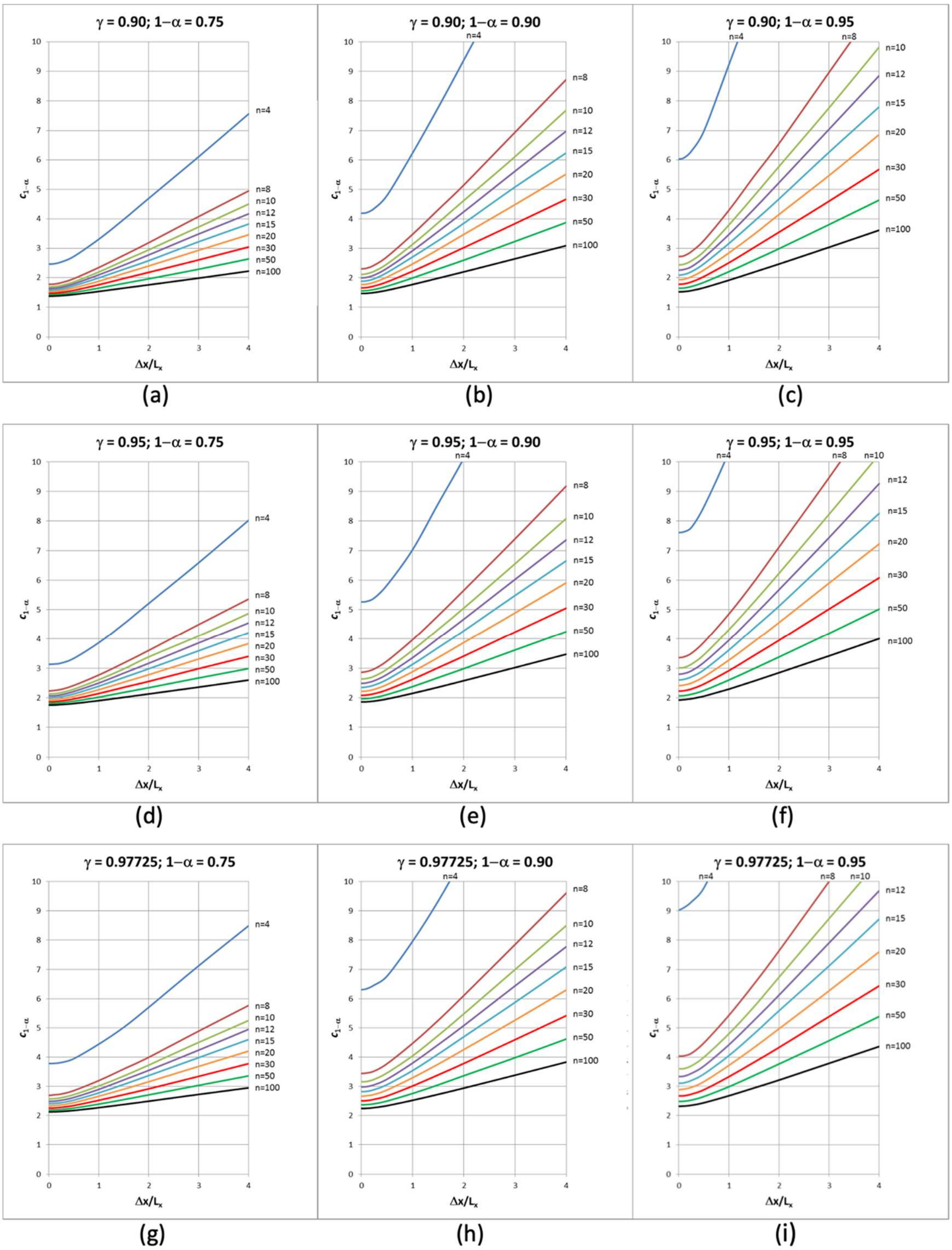

Figure 2. Factor $c_{1-\alpha}$ for nine different combinations of tolerance γ and confidence level $1-\alpha$: (**a**) $\gamma = 0.9$ and $1-\alpha = 0.75$; (**b**) $\gamma = 0.9$ and $1-\alpha = 0.90$; (**c**) $\gamma = 0.9$ and $1-\alpha = 0.95$; (**d**) $\gamma = 0.95$ and $1-\alpha = 0.75$; (**e**) $\gamma = 0.95$ and $1-\alpha = 0.90$; (**f**) $\gamma = 0.95$ and $1-\alpha = 0.95$; (**g**) $\gamma = 0.97725$ and $1-\alpha = 0.75$; (**h**) $\gamma = 0.97725$ and $1-\alpha = 0.90$; (**i**) $\gamma = 0.97725$ and $1-\alpha = 0.95$. The meaning of the axis is explained in Figure 1.

Table 2. Quantiles in the Student's t distribution.

# Test Data	$t_{n-2}(1-\alpha)$		
n	$1-\alpha = 0.75$	$1-\alpha = 0.90$	$1-\alpha = 0.95$
4	0.816	1.886	2.920
8	0.718	1.440	1.943
10	0.706	1.397	1.860
12	0.700	1.372	1.812
15	0.694	1.350	1.771
20	0.688	1.333	1.734
30	0.683	1.313	1.701
50	0.679	1.300	1.676
100	0.677	1.290	1.660
∞	0.674	1.282	1.645

By inspection of the expression for C in Equation (15), the following expression for the sum of the coefficients c_1 and c_2 can be derived

$$c_1 + c_2 = c_{1-\alpha,k}(n-1) \cdot \sqrt{\frac{n-1}{n}} \tag{20}$$

where $c_{1-a,k}(n-1)\sqrt{n-1}$ is the $1-\alpha$ quantile in a non-central t distribution with $n-2$ degrees of freedom and non-centrality parameter $-u_{1-k}\sqrt{n-1}$. u_{1-k} is the $1-k$ quantile of the standard normal distribution function. $c_{1-a,k}(n-1)$ is a function of n, $1-\alpha$ and k and is recognized as the downward offset from the sample mean when estimating the $1-k$ quantile of an independent standard normally distributed variable with confidence $1-\alpha$. This is tabulated in Resnikoff and Lieberman (1957) [34] and in Biometrika [35]. The parameter k is an auxiliary proportion which is a function of the true proportion γ and is calculated as

$$k = 1 - \Phi\left(\sqrt{\frac{n}{n-1}} \cdot \Phi^{-1}(1-\gamma)\right) \tag{21}$$

where γ is the specified survival probability or tolerance, and Φ is the standard Gaussian cumulative distribution function. Table 3 tabulates the inverse function $\Phi^{-1}(\gamma)$ as a function of γ. Table 4 tabulates $c_{1-a,k}(n-1)$ as a function of n, $1-\alpha$ and k. Interpolation may be necessary, if $c_{1-a,k}(n-1)$ is needed for other values of n, $1-\alpha$ and k than those tabulated.

Table 3. The inverse standard Gaussian cumulative distribution function.

Survival Probability (Tolerance) γ	γ Quantile of Standard Normal Variate $\Phi^{-1}(\gamma)$
0.50	0.000
0.75	0.674
0.90	1.282
0.95	1.645
0.97725	2.000
0.99	2.326

There are no separate closed form solutions for c_1 and c_2. However, based on Hald's approximation in Equation (8), generalized to dependent variables, the following expression can be derived and used as an approximation to the coefficient c_1

$$c_1 \approx \frac{\Phi^{-1}(\gamma)}{1 - \frac{\left(\Phi^{-1}(1-\alpha)\right)^2}{2(n-2)}} \tag{22}$$

in which Φ^{-1} denotes the inverse standard Gaussian distribution function. When used together with the expressions for $c_1 + c_2$ and c_3, this approximation will lead to rather accurate results for the sought-after quantile $c_{1-\alpha}$ by Equation (18). The inaccuracy in the prediction of $c_{1-\alpha}$ when using this approximation for c_1 will decrease for increasing sample size n and will eventually vanish as n approaches infinity.

Table 4. Quantiles $c_{1-\alpha,k}(n-1)$ for various auxiliary proportions k and confidence levels $1-\alpha$.

No. of Observations, n	Auxiliary Proportion k = 0.90			Auxiliary Proportion k = 0.95			Auxiliary Proportion k = 0.99		
	$1-\alpha$ = 0.75	$1-\alpha$ = 0.90	$1-\alpha$ = 0.95	$1-\alpha$ = 0.75	$1-\alpha$ = 0.90	$1-\alpha$ = 0.95	$1-\alpha$ = 0.75	$1-\alpha$ = 0.90	$1-\alpha$ = 0.95
4	2.501	4.258	6.158	3.152	5.312	7.657	4.396	7.340	10.552
8	1.791	2.333	2.755	2.251	2.904	3.404	3.126	3.972	4.641
10	1.702	2.133	2.454	2.147	2.660	3.038	2.977	3.641	4.143
12	1.646	2.012	2.275	2.078	2.511	2.825	2.885	3.444	3.852
15	1.591	1.895	2.108	2.012	2.366	2.621	2.796	3.257	3.585
20	1.536	1.781	1.949	1.947	2.237	2.429	2.710	3.078	3.331
30	1.479	1.664	1.788	1.877	2.094	2.233	2.619	2.895	3.079
50	1.428	1.563	1.651	1.817	1.976	2.075	2.540	2.740	2.870
100	1.380	1.471	1.528	1.758	1.862	1.929	2.470	2.601	2.683
∞	1.282	1.282	1.282	1.645	1.645	1.645	2.326	2.326	2.326

3. Results

3.1. Verification

A verification of the hyperbolic model for $c_{1-\alpha}$ in Equation (18) with the expressions for the coefficient sum $c_1 + c_2$ and the coefficient c_3 in Equations (19) to (21) has been carried out for an example case based on $n = 8$ observations of log$N$. The verification was conducted by comparing with the true $c_{1-\alpha}$ values obtained from the distribution of the variable C, established according to Equations (15) and (16) by simulating the parent variables U and W. The verification was carried out for a survival probability $\gamma = 0.95$ and a confidence level $1-\alpha = 0.95$. The results are presented in Figure 3. Excellent agreement with simulation results (red full curve) was found when the coefficient c_1 was back-calculated from the asymptotic value of $c_{1-\alpha}$ for a large value of $\Delta x/L_x$ and used in conjunction with the expressions for $c_1 + c_2$ and c_3 (blue dashed curve), assuming a straight line for the asymptote to the intercept with the $c_{1-\alpha}$ axis. The asymptotic value of $c_{1-\alpha}$ for a large value $\Delta x/L_x$ was taken as the simulated value of $c_{1-\alpha}$, assuming that the true $c_{1-\alpha}$ curve for all practical purposes is "identical" to the asymptote for this large value of $\Delta x/L_x$. Here, $\Delta x/L_x = 4$ was used. Just as excellent an agreement was found when the coefficient c_1 was approximated by the expression in Equation (22) (green dotted curve). Similar verifications have been carried out for other combinations of the sample size n, survival probability γ and confidence level $1-\alpha$ and show the same level of accuracy.

3.2. Numerical Example

As mentioned earlier, *S*–*N* curves for composite materials are often not available and need to be tested for individual projects. A set of fatigue test data from tests on interlaminar shear specimens of composites under dry conditions is available and used here as basis for a numerical example. The materials used for the interlaminar shear specimens were typical marine/offshore laminates. The glass reinforcement was WR 9622 R24-810 (woven roving) from Ahlström AB and the polyester resin was Norpol 200-M800 from Reichold AS.

Cyclic fatigue testing was carried out on special shear lap specimens in a servo-hydraulic test machine from about $2{\cdot}10^3$ to $2{\cdot}10^6$ cycles in the air. During cyclic fatigue testing the specimens were loaded in both tension and compression, at a load ratio of $R = -1$, i.e., the maximum stress in compression is equal to the (negative) maximum stress in tension, and the mean stress is zero. For further details about the material and test set-up, see Echtermeyer et al. (2004) [46].

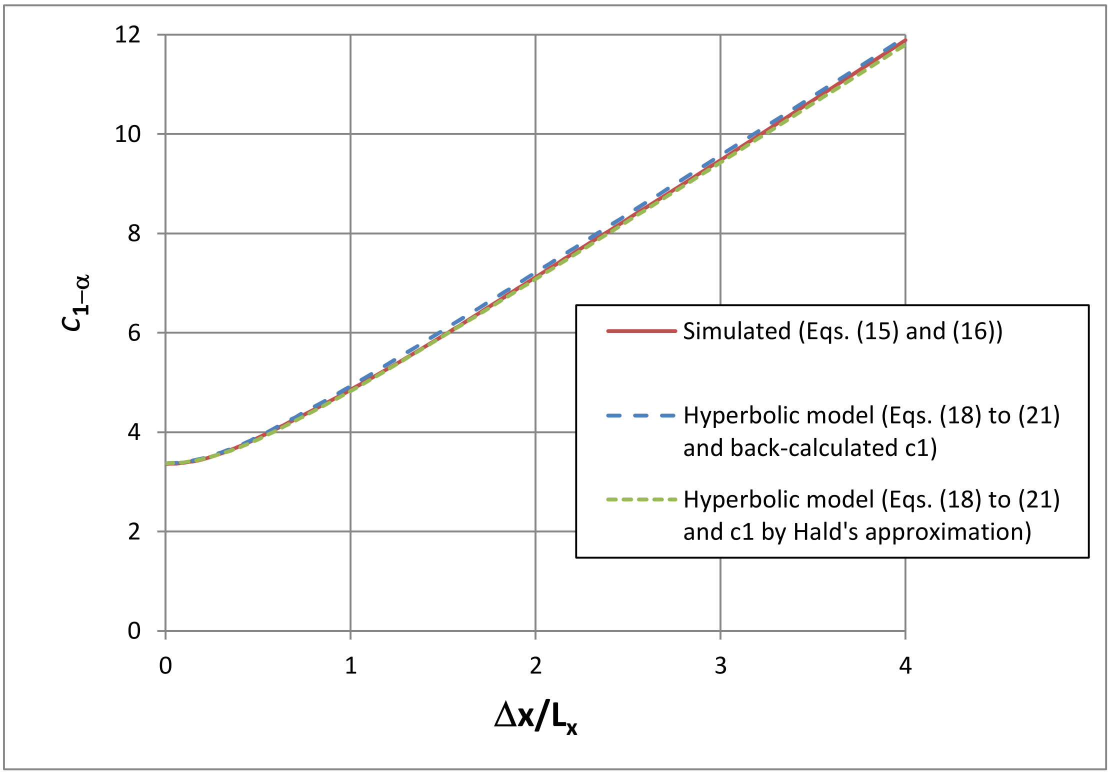

Figure 3. Comparison between the true (simulated) $c_{1-\alpha}$ values and the $c_{1-\alpha}$ values obtained by the hyperbolic model for n = 8, γ = 0.95 and $1 - \alpha$ = 0.95.

The dataset consists of 11 pairs of shear stress amplitude S and number of cycles to failure N, see Table 5.

Table 5. Example test data.

Stress Amplitude S (MPa)	Number of Cycles to Failure N	log$S$	log$N$
2.60	1,591,872	0.415	6.202
3.20	1,140,319	0.505	6.057
3.20	2,680,000	0.505	6.428
3.85	19,550	0.585	4.291
3.85	802,398	0.585	5.904
3.85	204,100	0.585	5.310
5.80	15,639	0.763	4.194
6.45	4595	0.810	3.662
6.45	2137	0.810	3.330
6.45	2330	0.810	3.367
7.10	2034	0.851	3.308

A linear regression of log$N$ on log$S$ in accordance with the procedure given in ASTM E739 [47] leads to the following estimated mean S–N curve

$$E[\hat{\log} N] = 9.755 - 7.648 \cdot \log S \tag{23}$$

and an estimated standard deviation in the residuals of log$N$ equal to s = 0.473. The experimental data is shown in Figure 3. Based on a simple visual evaluation, the data are linear and should be suitable for an evaluation based on the methods of this paper. The scatter is also typical for the fatigue curves of composite materials.

Assume now, as an example, that the characteristic *S*–*N* curve is defined as the curve with a survival probability of 97.725%, i.e., the characteristic value of log*N* is the mean value minus two standard deviations under a normal distribution assumption for log*N* conditional on log*S*. Assume also that this characteristic *S*–*N* curve is to be estimated with 95% confidence.

The procedure in Equation (12) through (14) is now used to estimate the characteristic *S*–*N* curve with 95% confidence. This would be the theoretical or true characteristic *S*–*N* curve estimate, given the data. Under the assumption that the log*S* values of the available test data are uniformly distributed over the log*S* interval covered by the tests, Figure 2 is now used to establish an alternative, approximate characteristic *S*–*N* curve estimate with 95% confidence. It is noted that this assumption is not quite fulfilled by this data set. The results of these exercises are presented in Table 6 and Figure 4. It appears that, despite the assumption, uniformly distributed log*S* values are not quite fulfilled for this data set, and the two *S*–*N* curve estimates come out as practically identical over the range of log*S* values covered by the data, and a little beyond. This serves to demonstrate that even if the assumption of uniformly distributed log*S* values over the log*S* interval covered by the tests is not quite fulfilled, the simplifying approach implied by using Figure 2 rather than the more cumbersome procedure in Equation (12) through (14) may suffice for practical purposes.

Table 6. Example tests; factor $c_{1-\alpha}$ for $1-\alpha = 0.95$.

Stress Amplitude *S* (MPa)	log*S* Covered by Tests	By Theory (Simulation by Equation (13)) $c_{1-\alpha}$	By Figure 2 $\frac{\Delta x}{L_x}$	By Figure 2 $c_{1-\alpha}$
2.60	0.415	3.79	0.455	3.75
3.20	0.505	3.59	0.267	3.57
3.85	0.585	3.48	0.099	3.46
5.80	0.763	3.52	0.272	3.57
6.45	0.810	3.59	0.368	3.66
7.10	0.851	3.67	0.455	3.77

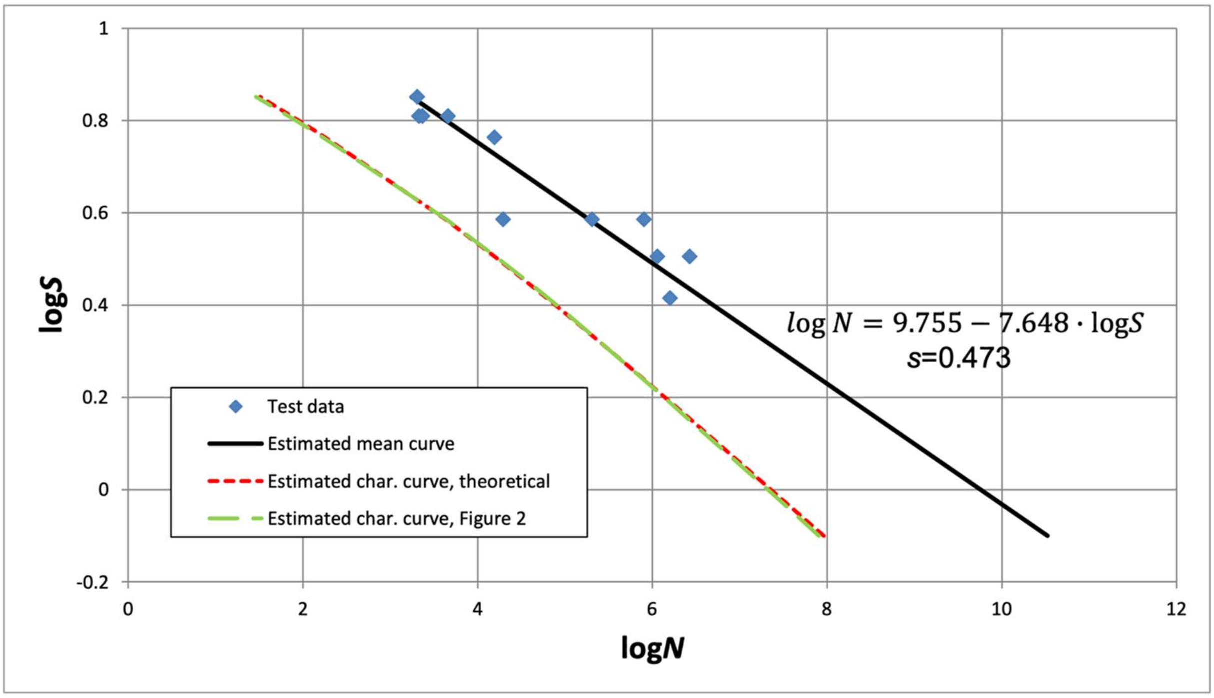

Figure 4. Estimated mean *S*–*N* curve on log-log scale and estimated characteristic *S*–*N* curve (1) by theoretically correct approach and (2) by Figure 2.

4. Discussion

For practical application of the results, the characteristic fatigue curves to be used in design are conservatively given in standards as two straight lines on a log-log scale, parallel to the fitted mean curve based on the test data; see DNV-OS-C501 [2] and DNVGL-RP-F119 [3]. One line is used within the range of log$S$ covered by the measured data; the other line is used for extrapolation beyond this range. The extrapolation is limited to half the width of the range of log$S$ covered by the data. The linear characteristic fatigue curves specified in DNV-OS-C501 and DNVGL-RP-F119 have been calibrated by means of the procedure outlined above such that some specified minimum confidence level for the calculated characteristic log$N$ value is met or exceeded for all log$S$ values within the specified range for log$S$. It is noted that these linear curves come out as similar to the parallel straight line tolerance limits resulting from the approach presented by Little and Jebe (1975) [48], based on work by Bowden (1968) [49].

The approach given here, using the curved tolerance bounds based on Equation (14) and Figure 2 rather than the straight lines given in the standards, allows the determination of the characteristic fatigue curve in a more accurate and still fairly simple way, and there is no restriction on how far out on the log$S$ axis one can extrapolate. Within the log$S$ range of applicability of the straight lines specified in the standards, the approach given here will allow for higher utilization than these lines. Significant extrapolation beyond the range of log$S$ covered by the test data will produce characteristic fatigue life estimates which will be relatively short owing to the curvature of the tolerance bounds, thus reflecting the uncertainty associated with such extrapolation.

In addition, the approach using the curved tolerance bounds based on Equation (14) and Figure 2, rather than the true curved tolerance bounds based on Equation (12) through (14), will in most cases suffice for practical purposes, i.e., the designer can obtain the parameters from Figure 2 rather than having to carry out cumbersome and elaborate calculations according to Equations (12) and (13).

The limit of extrapolation of fatigue data in the standards is given, because the simplified straight characteristic curve would become non-conservative beyond the limit. But sometimes longer lifetimes need to be estimated and absolutely no indication is given in the standards on how these lifetimes should be obtained. Calculating the lifetimes from basic theory as described here in Equations (12)–(14) is cumbersome and not widely known. Having now a simple procedure to predict long time fatigue performance beyond the extrapolations allowed by the current design standards is very beneficial.

The new method also allows for using small data sets and shorter test times for predicting performance at long lifetimes. The characteristic curves based on a few data give lower lifetimes than characteristic curves based on more tests and longer test times, but at least a proper estimate can be made. Possibly the estimate is good enough for a particular design. Otherwise it also allows for evaluating whether it is beneficial to perform further and expensive long-term testing for obtaining better estimates of the lifetimes.

It should be noted that the approach presented here for fatigue curves under cyclic loading can also be applied to stress rupture curves under sustained loading simply by replacing the log of number of cycles to failure with the log of time to failure.

The results can be applied to S–N curves and ε–N curves for other materials than composites, as well as to any other relationship where one variable is linearly dependent on another. The reference to composites is merely by coincidence. A very good, completely different example is soil strength as a function of depth within a soil layer in a stratified soil deposit.

5. Conclusions

S–N curves need to be measured for engineering materials that are not as well characterized as widely used metals. A versatile, simple and accurate method has been presented for calculating characteristic curves for fatigue S–N curves or other experimental results with one variable dependent on another, independent variable. The basis is a statistical method for the estimation of characteristic values of uncertain variables, founded on the theory for tolerance bounds of random variables. This statistical method has been extended to produce tolerance bounds for dependent variables. The method has been applied in conjunction with regression analysis results to establish estimates of characteristic values for the fatigue life of general materials, such as composites or adhesive joints for use in design. The method takes into account that the fatigue life (the dependent variable) is a function of the applied stress amplitude (the independent variable).

A general statistical method existed before and is correct, but is not very user-friendly for dependent variables. By introducing the simplification of requiring uniformly distributed test data over a finite log$S$ interval, a mathematically fairly simple solution has been found. Fatigue test programs are often designed such that test data roughly come in this way. Based on this simplification, the method has been used to create a set of nine easy-to-use charts corresponding to nine commonly encountered combinations of survival probability (for definition of characteristic fatigue life) and confidence level (for estimation). The curves in these charts may be used for the simplified estimation of characteristic S–N curves for use in design. A numerical example based on a real dataset has been presented and shows that very accurate results are achieved, even if the assumption of uniformly distributed test data over a finite log$S$ interval is not quite fulfilled. The curves in the charts may be well represented by hyperbolic functions. Analytical expressions for the coefficients in these functions have been derived and are presented.

The method allows the structural engineer to estimate fatigue lifetimes beyond the range given in the existing standards and literature based on limited sets of data. The method is relatively simple. It allows for evaluating the benefits of more and expensive long-term testing vs. required estimated lifetimes.

The methodology can be applied to statistical analysis also of other types of data than fatigue life, for example data for time to rupture (creep rupture) of composite laminates or polymers subjected to sustained loading—time to rupture (or rather its logarithm) is then the dependent variable whereas the sustained load (or its logarithm) is the independent variable. Results of the methodology under the idealized assumptions about the test data were used as the basis for calibrating requirements for linear characteristic fatigue curves on log-log scale in the DNV standard for composite components, DNV-OS-C501 [2].

Author Contributions: K.O.R. was the main responsible for the statistical aspects and calculations in this paper. A.T.E. was the main responsible for composites and evaluation of test results. All authors have read and agreed to the published version of the manuscript.

Funding: Most of this research received no external funding. Andreas T. Echtermeyer would like to express thanks for some financial support from The Research Council of Norway (C-PIMS Project 309238 in the Petromaks 2 program).

Data Availability Statement: All required data is directly reported in the paper.

Acknowledgments: This work is based on the experience gained from several projects and was based on the desire to improve fatigue analysis. The authors would like to thank their employers DNV and NTNU for providing an environment in which this work could be done.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. DNV Standard. *Submarine Pipeline Systems*; DNV-OS-F101; DNV Offshore Standard: Høvik, Norway, 2015
2. DNV Standard. *Composite Components*; DNVGL-OS-C501; DNV Offshore Standard: Høvik, Norway, 2013.
3. DNV Standard. *Thermoplastic Composite Pipes*; DNVGL-ST-F119; DNV Standard: Høvik, Norway, 2018.
4. Madsen, H.O.; Krenk, S.; Lind, N.C. *Methods of Structural Safety*; Prentice-Hall Inc.: Englewood Cliffs, NJ, USA, 1986.
5. Metallic Materials and Elements for Aerospace Vehicle Structures. In *Military Standardization Handbook*; MIL-HDBK-5B; Department of Defense: Washington, DC, USA, 1971.
6. Jones, D.R.; Ashby, M.F. *Engineering Materials 1: An Introduction to Properties, Applications and Design*; Elsevier: Amsterdam, The Netherlands, 2011.

7. Callister, W.D., Jr.; Rethwisch, D.G. *Fundamentals of Materials Science and Engineering: An Integrated Approach*; John Wiley & Sons: Hoboken, NJ, USA, 2012.
8. Tsai, S.W. *Theory of Composites Design*; Think Composites: Dayton, OH, USA, 1992.
9. Carlsson, L.A.; Gillespie, J.W.; Zweben, C.H. *Delaware Composites Design Encyclopedia: Mechanical Behavior and Properties of Composite Materials*; CRC: Boca Raton, FL, USA, 1989.
10. Mayer, R.M. (Ed.) *Design of Composite Structures against Fatigue: Applications to Wind Turbine Blades*; Mechanical Engineering Publications: Bury St Edmunds, UK, 1996.
11. Nijssen, R.P.L. *Fatigue Life Prediction and Strength Degradation of Wind Turbine Rotor Blade Composites*; Sandia Report, SAND2006-7810P; Sandia National Laboratories: Albuquerque, NM, USA, 2007.
12. Bond, I.P. Fatigue life prediction of GRP subject to variable amplitude loading. *Compos. Part A* **1999**, *30*, 961–970. [CrossRef]
13. Westphal, T.; Nijssen, R.P.L. Fatigue life prediction of rotor blade composites: Validation of constant amplitude formulations with variable amplitude experiments. *J. Phys. Conf. Ser.* **2014**, *555*, 012107. [CrossRef]
14. Sanchez, H.; Sankararaman, S.; Escobet, T.; Puig, V.; Frost, S.; Goebel, K. Analysis of two modeling approaches for fatigue estimation and remaining useful life predictions of wind turbine blades. In Proceedings of the Third European PHM, Bilbao, Spain, 5–8 July 2016.
15. Passipoularidis, V.A.; Brønsted, P. *Fatigue Evaluation Algorithms: Review*; Risø-R-Report-1740 (EN); NLSE: Risø, Denmark, 2009.
16. API; Recommended Practice. *Design of Risers for Floating Production Systems (FPS) and Tension-Leg Platforms (TLPs)*; American Petroleum Institute: Washington, DC, USA, 1998.
17. Lotsberg, I. *Fatigue Design of Marine Structures*; Cambridge University Press: New York, NY, USA, 2016.
18. Post, N.L. Reliability Based Design Methodology Incorporating Residual Strength Prediction of Structural Fiber Reinforced Polymer Composites under Stochastic Variable Amplitude Fatigue Loading. Ph.D. Dissertation, Virginia Polytechnic Institute and State University, Blacksburg, VA, USA, 2008.
19. Castro, O.; Branner, K.; Dimitrov, N. Assessment and propagation of mechanical property uncertainties in fatigue life predictions of composite laminates. *J. Compos. Mater.* **2018**, *52*, 3381–3398. [CrossRef]
20. Nishijima, S. Statistical Fatigue Properties of Some Heat-Treated Steeis for Macliine Structural Use. In *Statistical Analysis of Fatigue Data*; ASTM STP 744; Little, R.E., Ekvall, J.C., Eds.; American Society for Testing and Materials: Philadelphia, PA, USA, 1981; pp. 75–88.
21. Sudret, B.; Hornet, P.; Stephan, J.M.; Guede, Z.; Lemaire, M. Probabilistic assessment of fatigue life including statistical uncertainties in the S-N curve. In Proceedings of the 17th International Conference on Paper # M03-4 Structural Mechanics in Reactor Technology (SMiRT 17), Prague, Czech Republic, 17–22 August 2003.
22. Schneider, C.R.A.; Maddox, S.J. *Best Practice Guide on Statistical Analysis of Fatigue Data*; TWI: Cambridge, UK, February 2003.
23. Owen, D.B. *Tables of Factors of One-Sided Tolerance Limits for a Normal Distribution*; SCR-13; Sandia Corporation Monograph: Washington, DC, USA, 1958.
24. Wirsching, P.H. *Statistical Summaries of Fatigue Data for Design Purposes*; NASA Contractor Report 3697; NASA: Washington, DC, USA, 1983.
25. Wei, Z.; Dogan, B.; Luo, L.; Lin, B.; Konson, D. Design Curve Construction Based on Tolerance Limit Concept. In Proceedings of the ASME 2012 Pressure Vessels & Piping Conference (PVP 2012), Toronto, ON, Canada, 15–19 July 2012.
26. Ronold, K.O.; Echtermeyer, A.T. Estimation of fatigue curves for design of composite laminates. *J. Compos. Part A* **1996**, *27A*, 485–491. [CrossRef]
27. Echtermeyer, A.T. Integrating Durability in Marine Composite Certification. In *Durability of Composites in a Marine Environment*; Solid Mechanics and Its Applications 208; Springer Science + Business Media: Dordrecht, The Netherlands, 2014; pp. 179–194. ISBN 978-94-007-7416-2.
28. DIN. *Statistische Auswertung von Daten, Bestimmung Eines Statistischen Anteilsbereichs (Statistical Interpretation of Data*; Determination of a Statistical Tolerance Interval); DIN55303, Part 5; Deutsches Institut für Normung E.V. (DIN): Berlin, Germany, 1987. (In Germany)
29. ASTM. *Standard Practice for Obtaining Hydrostatic or Pressure Design Basis for "Fiberglass (Glass-Fiber-Reinforced Thermosetting-Resin) Pipe and Fittings*; ASTM Standard D2992-12; American Society for Testing Materials: Philadelphia, PA, USA, 2012. [CrossRef]
30. Mandell, J.F.; Samborsky, D.D.; Wahl, N.K.; Sutherland, H.J. *Testing and Analysis of Low Cost Composite Materials Under Spectrum Loading and High Cycle Fatigue Condition*. Conference Paper, ICCM14, Paper # 1811; SME/ASC. 2003. Available online: https://www.semanticscholar.org/paper/Testing-and-Analysis-of-Low-Cost-Composite-Under-Mandell-Samborsky/8cef98d9ccec02d1e54489c597c0eb536de26bdb (accessed on 12 November 2021).
31. Echtermeyer, A.T.; Engh, B.; Buene, L. Influence of matrix and fabric. In *Design of Composite Structures Against Fatigue: Applications to Wind Turbine Blades*; Mayer, R.M., Ed.; Mechanical Engineering Publications Ltd.: Suffolk, UK, 1996; pp. 33–49.
32. Echtermeyer, A.T.; Kensche, C.; Bach, P.; Poppen, M.; Lilholt, L.; Andersen, S.I.; Brønsted, P. Method to Predict Fatigue Lifetimes of GRP Wind Turbine Blades and Comparison with Experiments. In Proceedings of the 1996 European Union Wind Energy Conference and Exhibition, Gothenburg, Sweden, 20–24 May 1996; H.P. Stephens & Associates: Bedford, UK, 1996; pp. 907–913.
33. Ronold, K.O.; Lotsberg, I. On the estimation of characteristic *S*–*N* curves with confidence. *J. Mar. Struct.* **2012**, *27*, 29–44. [CrossRef]
34. Resnikoff, G.J.; Lieberman, G.J. *Tables of the Noncentral t-Distribution*; Stanford University Press: Stanford, CA, USA, 1957.

35. Pearson, E.S.; Hartley, H.O. *Biometrika Tables for Statisticians*, 3rd ed.; Biometrika Trust: London, UK, 1976.
36. Hald, A. *Statistical Theory with Engineering Applications*; John Wiley: Hoboken, NJ, USA, 1952.
37. Guttman, I. *Statistical Tolerance Regions: Classical and Bayesian*; Griffin: London, UK, 1970.
38. Zacks, S. *The Theory of Statistical Inference*; John Wiley and Sons, Inc.: New York, NY, USA, 1970.
39. Little, R.E. Review of statistical analysis of fatigue life data using one-sided lower statistical tolerance limits. In *Statistical Analysis of Fatigue Data*; ASTM STP 744; Little, R.E., Ekvall, J.C., Eds.; American Society for Testing and Materials: Philadelphia, PA, USA, 1981; pp. 3–23.
40. Beaumont, P.W.R.; Schultz, J.M. Statistical Aspects of Fracture. In *Failure Analysis of Composite Materials*; Chapter 4.6; Beaumont, P.W.R., Schultz, J.M., Friedrich, K., Eds.; Technomic Publishing Co., Inc.: Lancaster, PA, USA, 1990; Volume 4.
41. Ang, A.H.-S.; Tang, W.H. *Probability Concepts in Engineering Planning and Design, Vol. I, Basic Principles*; John Wiley & Sons: New York, NY, USA, 1975.
42. Pascual, F.G. Planning Fatigue Experiments and Analyzing Fatigue Data with the Random Fatigue-Limit Model and Modified Sudden Death Tests. Ph.D. Thesis, Iowa State University, Ames, IA, USA, 1997.
43. Tvedt, L. Proban—Probabilistic Analysis. *Struct. Saf.* **2006**, *28*, 150–163. [CrossRef]
44. Snedecor, G.W.; Cochran, W.G. *Statistical Methods*; Iowa State University Press: Ames, IA, USA, 1989.
45. Veritas, D.N. *Statistical Representation of Soil Data*; DNV-RP-C207; Recommended Practice: Høvik, Norway, 2012.
46. Echtermeyer, A.T.; Ekeberg, T.S.; Sund, O.E. *Long-Term Testing of Composite Through-Thickness Properties*; Research Report 131; Health & Safety Executive, HSE: Norwich, UK, 2004.
47. American Society for Testing and Materials. *Standard Practice for Statistical Analysis of Linear or Linearized Stress-Life (S-N) and Strain-Life (ε-N) Fatigue Data*; ASTM E739–91; American Society for Testing and Materials: West Conshohocken, PA, USA, 2004.
48. Little, R.E.; Jebe, E.H. *Statistical Design of Fatigue Experiments*; Applied Science Publishers Ltd.: London, UK, 1975.
49. Bowden, D.C. Query 26: Tolerance Interval in Regression. *Technometrics* **1968**, *10*, 207–209. [CrossRef]