Improved models for a single vehicle continuous-time inventory routing problem

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Abstract

We consider an inventory routing problem in which a single vehicle is responsible for the transport of a commodity from a set of supply locations to a set of demand locations. At each location the inventory must be kept within predefined bounds, and the location specific supply and demand rates are constant throughout the time horizon. Each location can be visited several times during the time horizon, and the vehicle can visit the locations in any order as long as the capacity of the vehicle is not exceeded. Two models are presented, each defined on a different extended network. In a *location-event model*, the nodes are indexed by the location and the number of visits made so far to that location, while in a *vehicle-event model* the nodes are indexed by the location and the number of visits so far on the vehicle route. Both models are based on continuous time formulations. They are tightened with valid inequalities, and a new branching algorithm is designed to speed up the solution time of the models. Computational tests based on a set of maritime transportation instances are reported to compare both models and the corresponding tightened variants.

Keywords: Inventory; routing; strong formulations; valid inequalities.

1 Introduction

In this paper we consider an inventory routing problem (IRP) with constant supply and demand rates at supply and demand locations, respectively. A single vehicle is responsible for transporting a single commodity from the supply locations to the demand locations. The vehicle route and the corresponding pickup and delivery operations must be coordinated in order to keep the inventory levels at each location within predefined upper and lower bounds. The vehicle, which has limited capacity, starts from a given initial position, visits the locations in any order along its route, and ends its route at any location. Each location can be visited once or several times during the planning horizon depending on the size of the storage, the supply or demand rate, and the quantity picked up or delivered at each visit. The quantity picked up or delivered at each visit is also variable. Time is regarded as continuous, and the planning horizon has a defined length. The single vehicle continuous-time inventory routing problem with pickups and deliveries (s-CT-IRP-PD) consists of designing routes and schedules for the vehicle in order to minimize the travel and operational costs, and to determine the number of visits at each location including the quantities handled without exceeding the storage limits.

Although the study of this problem is motivated by maritime transportation problems, such problems may also occur in land-based transportation when long travel times and/or long operating times at

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locations are considered. The main distinction compared to many other inventory routing models is that time is treated as continuous and the vehicle may operate continuously, meaning that the distribution plan is not split into discrete time periods, such as hours or days.

As for related problems, such as the travelling salesman problem or the inventory routing problem with pickups and deliveries [12], the single vehicle case can have practical applications on its own. The reasons may be that only one vehicle is available or because the geographical dispersion of the locations to be visited leads to a natural partition and thus an assignment of a subset of the locations to each vehicle. Such assignments can also be used to derive heuristic schemes for the multi-vehicle case, in which a first step is to select the set of locations to be visited by each vehicle, see [7]. The single vehicle-case arises very naturally as a subproblem when column generation approaches are used to solve multi-vehicle inventory routing problems, see for instance [2, 34].

The purpose of this paper is to investigate and present improved mixed integer formulations for the inventory routing problem with constant supply and demand rates that can be used to solve instances with long time horizons.

Inventory routing problems have been studied for almost four decades and have been the subject of several reviews, such as [9, 20].

For the majority of IRPs considered in the literature, the planning horizon is partitioned into periods and it is assumed that arrivals occur at the start of the period, demands take place at the end of the period and that both occur instantaneously; see [15, 20]. However, the quantity picked up or delivered at a particular time depends on the storage capacity and the inventory level at that point in time, and an IRP with discretized time periods may be less accurate than an IRP with continuous time. That is the reason why continuous-time models have been widely used in the past in maritime transportation [8, 17, 19, 30] where several ships transport a commodity between multiple supply ports and multiple demand ports. In such problems, the travel times are usually long and event models are employed in which an event corresponds to a ship visit to a port. Such models are similar to the location-event model presented in Section 3. When the supply and/or demand rate is varying during the planning horizon, a discrete-time model is applied, see [3, 31]. In [4] continuous time is combined with discrete time to model a multi-item maritime inventory routing problem in which continuous time is used to model the visits to ports while discrete time is used to model time-windows in the ports.

Comparisons between continuous-time and discrete-time model are also explored in the literature; see [5] for maritime IRPs. While the discrete-time models tested in [5] proved to provide smaller integrality gaps than those obtained with the continuous-time models, it was also observed that time discretizations may lead to very large sized models when the time horizon is long or if the use of a fine time discretization is required. This scalability issue with the discrete-time models motivates the need to conduct a deeper study of the continuous-time models.

Continuous-time models have also been used in land transportation, particularly for companies in the liquid gas industry. Song and Savelsbergh [32] introduced the IRP with continuous moves in the liquid gas industry. Here, the product was picked up at different facilities and delivered to customers spread over a large geographic area, and the transportation teams were on the road for several days. Avella et al. [13] consider an IRP in which one warehouse supplies a set of fuel pumps using a fleet of trucks. Recently, Fokkema et al. [26] propose a continuous-time model for a practical biogas IRP where containers act as both storage and transportation units. Multiple suppliers and a single facility are assumed. Furthermore, Lagos et al. [29] study a problem typically found in the liquid gas industry, and they propose a dynamic discretization algorithm for IRPs where a time-expanded network formulation is introduced to obtain solutions that are converted to continuous-time solutions. A similar approach can be found in [16] for a continuous-time service network design problem.

A relevant related problem is the cyclic inventory routing problem that aims to find routing schedules of a certain length that are repeated infinitely [1, 24, 26]. In this variant of the IRP the conditions at the end of the planning horizon coincide with those at the beginning (inventory levels and vehicle position). Moreover, in the single-vehicle cyclic inventory routing case, only one supplier is usually considered. Each customer is visited at most once in the cycle and the cycle time of the trip made by the vehicle is to be determined [1]. Hence, the solution techniques proposed in our paper can be used to obtain solutions for the cyclic inventory routing problem if additional constraints forcing the final conditions to match the initial conditions are included (same inventory levels at all locations and the start position of the vehicle is equal to its end position). Also, the requirement that each location is visited at most once can easily be included in the model.

Another important characteristic of an IRP is the network structure, where the basic IRP considers a depot with an unlimited supply of the commodity and many distributed customer nodes demanding the commodity, see for instance [11, 21]. However, the opposite structure with a demand depot and suppliers distributed geographically is also investigated in the literature and called supply-driven IRPs [23, 26]. Inventory routing problems with both pickup and delivery nodes have been extensively studied in the maritime context. We refer to [18, 19] for an introduction and an overview of maritime IRPs with pickup and delivery structure. There are also land-based IRPs with pickup and delivery nodes studied in the literature. One important class of such problems is the closed-loop IRP, which takes into account the return processes as well as the forward flows in order to recover the value from the customers or end users. This means that the locations are simultaneously pickup and delivery locations. Closed-loop inventory routing problems for returnable items with simultaneous pickup and delivery are studied for instance in [28, 33]. The location characteristics deviate from ours because our locations are classified as either a pickup or delivery location, and we do not allow simultaneous pickup and delivery at a customer. Another combined inventory management and pickup and delivery routing problem is studied in [10], where the authors study a real problem of replenishing automated teller machines (ATM). Also for this problem an ATM can act as both a pickup and delivery location and all the vehicle start from a common depot.

The majority of the IRPs studied in the literature, including the c-CT-IRP-PD, consider the transportation of a single product. However, there is also work considering multiple products as in [22] for land based transportation and in [27] for maritime transportation.

One relevant issue in the s-CT-IRP-PD is that each location can be visited several times and that the number of visits is not known in advance. The visits must be coordinated with the inventory levels at the different locations. This means that the vehicle may visit a particular location several times during the time horizon picking up or delivering small quantities of the commodity at each visit or alternatively visit the location just once and pick up or deliver large quantities. This inventory policy is often called a maximum-level policy meaning that the replenishment is flexible, but bounded by the inventory limits, see [20]. An alternative is the order-up-to policy in which the aim during a visit is to fill the storage facility to capacity at a demand location and to empty the facility at a supply location, see [14].

Deriving good formulations for model with a variable number of visits to the locations and variable quantity is challenging. Here, we consider two different models that take the occurrence of multiple visits into account. The first is a *location-event model*, similar to those used in [8, 17, 30], based on an expanded network in which there is a different node for each possible visit to a location. The second, *vehicle-event model*, is based on a layered graph. Here, all locations are present in each layer and the k^{th} layer represents the k^{th} visit of the vehicle. Such models have also been used for related problems, see for instance [25] where a single vehicle is responsible for collecting information that is generated at constant rates in several locations and delivered to a single depot. As these models have large integrality gaps, they are tightened with valid inequalities. Based on a set of instances for maritime inventory routing, a computational study is here conducted to test and compare both models with and without valid inequalities.

In addition to the introduction of the layered vehicle-event model, we describe valid inequalities for the location-event model, which are also valid for the multi-vehicle case, as well as valid inequalities for the vehicle-event model. In addition, we present simple approaches to provide upper bounds on the number of events considered in each model, that is, the number of visits to each of the locations in the location-event model, and the total number of visits made by the vehicle in the vehicle-event model. Providing tight upper bounds allows us to limit the size of the corresponding models. These approaches suggest a new branching algorithm based on a restriction on the number of visits.

The contribution of this work can be summarized as follows:

1. The s-CT-IRP-PD is introduced. The single-vehicle version of the problem is not studied in the literature previously.
2. Two general mathematical formulations of the problem, a location-event model and a vehicle-event model, are presented.
3. New valid inequalities are proposed for both models.
4. New valid inequalities are introduced under the assumption that the vehicle cannot return twice to a demand/supply location without visiting a supply/demand location in between. When this assumption does not hold, these inequalities are used to partition the set of feasible solutions and an exact algorithm is proposed.
5. A branch-and-cut algorithm is described including a new branching algorithm.
6. Benchmark instances for the s-CT-IRP-PD are generated.
7. A computational study gives information about the effectiveness of the models and the valid inequalities and insights into the problem. All the tested instances up to 180 periods are solved to optimality with the best approach that combines the main contributions.

The rest of the paper is organized as follows: In Section 2, we present and discuss the inventory routing problem. In Section 3 we present the location-event model and discuss valid inequalities. In Section 4, the layered vehicle-event model is introduced and tightened. The estimation of the bounds on the number of visits is discussed in Section 5. Computational results are presented in Section 6 and Section 7 contains some concluding remarks.

2 Problem description

In this section, we describe the inventory routing problem in more detail. A single vehicle is transporting a single commodity over a time horizon of length T. Let $G^N = (N, A^N)$ denote a graph in which N is the set of locations to be visited, and A^N is the set of arcs between the locations. For each location i, an initial stock S_i^0, and a constant supply/demand rate R_i are given. The vehicle of limited capacity

C is responsible for picking up the commodity from the supply locations and delivering it to the demand locations to ensure that the stock levels are kept within specified minimum $\underline{S}_i$ and maximum levels $\overline{S}_i$ throughout the time horizon. Initially, the vehicle carries Q^0 units of the commodity and, at the end of the time horizon, it can carry any amount between 0 and its capacity C. For a visit to location i, minimum $\underline{Q}_i$ and maximum $\overline{Q}_i$ pickup or delivery quantities are specified.

The travel time between locations i and j including also any set-up time required to operate at location j is T_{ij}, and the travel time required to travel from the origin to location i is T_i^0. In addition, T_i^Q is the time required to pickup/deliver one unit of the commodity at location i. The vehicle is also allowed to wait before operating at a location.

To resume, we consider the case in which a single routing and distribution plan must be determined for the entire time horizon, see [32]. The vehicle starts from a given initial position, that can be any location (e.g. in maritime transportation the initial position can be a point at sea), can visit any sequence of locations along the route and ends its route at a dummy destination. However, we do not allow that the vehicle makes two consecutive visits to the same node. Each location can be visited multiple times, and the number of visits to each location is a decision resulting from the plan and not an input parameter. The need to visit a particular location several times during the time horizon may be due to the vehicle capacity or the maximum/minimum inventory limits and the amount of the commodity available. After its last visit, the vehicle leaves for an unspecified destination, but the stock levels at all the locations must be feasible up until time T. In Figure 1 we provide an example of a feasible vehicle route that visits locations 2 and 3 twice and location 1 once. Notice that the last location visited by the vehicle before moving to the dummy destination is location 2.

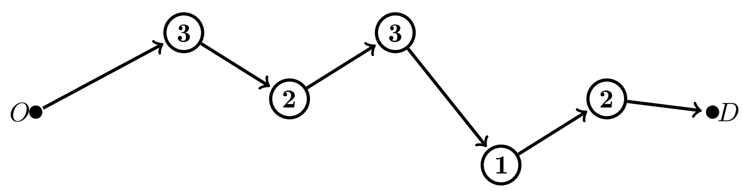


Figure 1: Example of a route with $N = \{1, 2, 3\}$, in which the vehicle departs from its origin (node O) visits location 3 for the first time, then makes a first visit to location 2, returns to location 3 for a second visit, then travels to location 1 followed by a second visit to location 2. Then the vehicle leaves for the destination (node D).

Two types of costs are considered: (i) travel costs C_{ij}^T for a trip from location i to location j and C_i^{T0} for a trip from the initial location of the vehicle to location i and (ii) a fixed set-up/operating cost C_i^S incurred every time the vehicle operates at location i. The objective is to minimize the transportation and operating costs.

Now we present an example showing that the solution to the inventory routing problem can be highly sensitive to the parameters because the inventory bounds are hard constraints.

Example 2.1 *Consider an instance with a time horizon of 40 days, 4 locations, in which location 1 is a supplier and locations 2, 3, 4 are demand locations. Assume the vector of supply/demand rates is given by (6, 2, 2.5, 1.5) and consider two alternatives for the vector of initial stock levels (115, 39, 38, 16) and (115, 39, 38, 15), in which only S_4^0 varies by one unit. The vehicle is located at location 1 at the beginning of the time horizon and the initial load is zero. The pickup/delivery rate T_i^Q is 80 units per time unit for all locations. The travel distances and the travel costs are given by* $T_{12} = 7$, $T_{13} = 8$, $T_{14} = 5$, $T_{23} = 3$, $T_{24} = 5$, $T_{34} = 8$, *and* $C_{12}^T = 70$, $C_{13}^T = 80$, $C_{14}^T = 50$, $C_{23}^T = 30$, $C_{24}^T = 50$, $C_{34}^T = 30$, *respectively. The travel distances T_i^0 and travel costs C_i^{T0} are based on the vehicle's initial position. The* *set-up and operating costs are (7,5,5,5). Figure 2 depicts an optimal solution when* S_4^0 *is* 16 *(in the upper network) and when* $S_4^0 = 15$ *(in the lower network). The corresponding optimal values are 130.7 and 261.6, respectively. In the case when* $S_4^0 = 16$ *the initial inventory level allows location* 4 *to be served at time period 10.3. This allows the vehicle to pick up enough to serve the net demand at the 3 consumer locations. If* S_4^0 *is reduced to* 15*, location 4 needs to be served within time 10 which forces the vehicle to leave the supply location without sufficient quantity to serve all the demand locations. Hence, forcing a second visit to locations 1 and 4 leads to a large increase in cost.*

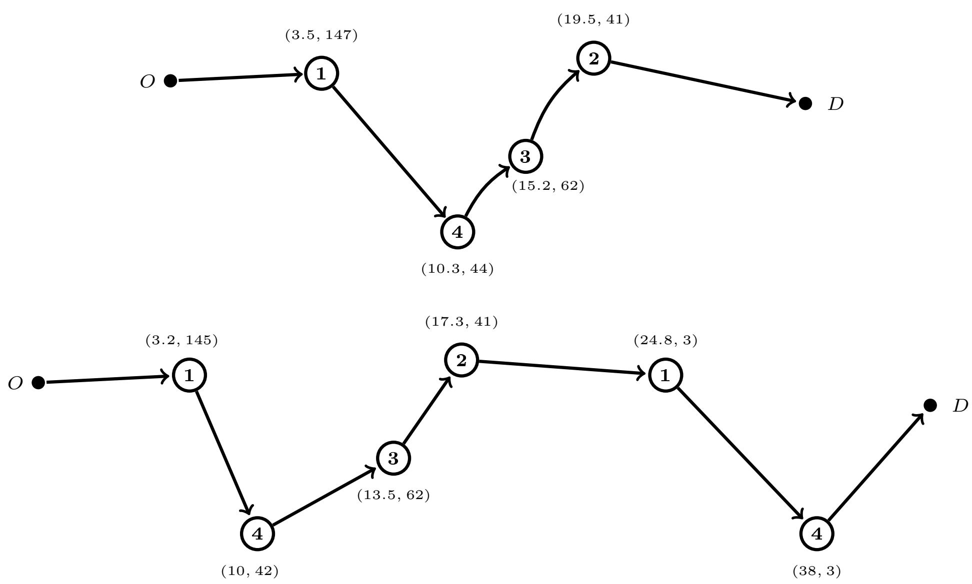


Figure 2: Optimal solutions with $S_4^0 = 16$ (above) and $S_4^0 = 15$ (below). The label next to each node represents (start time of visit, quantity picked up/delivered).

Example 2.1 indicates at least partially the difficulty in coordinating the inventory management with the distribution in inventory routing problems.

Another issue is the end-of-time horizon effect that is often observed in inventory problems. In optimal solutions, the stock level at the end of the time horizon is typically high if the location is a supplier and low if it is a demand location. In order to avoid the end-of-time horizon effect, we introduce a parameter F, that is a fractional value between 0 and 1, to control the inventory level at the end of the time horizon. For a supply location, the stock level at time T should not exceed $(1-F)\overline{S}_i + FS_i^0$, and for a demand location the stock level at time T should be at least $(1-F)\underline{S}_i + FS_i^0$. Hence, setting $F = 0$ we are not imposing any additional restrictions on the inventory levels $\underline{S}_i, \overline{S}_i$, while in the extreme case, $F = 1$, we are imposing that the inventory levels should be at most (least) the initial inventory level at all the supply (demand) locations. In this case, as the inventory levels at the end of the time horizon match the initial inventory levels, the solution can be cyclically repeated (if in addition we force the destination node to coincide with the origin node), solving the corresponding cycle inventory routing variant for cycle time T.

3 The location-event model

In the location-event model, an extended graph is considered in which each node corresponds to a visit to a location. For each location, we consider an ordering of the visits according to the time of the visit. The vehicle path is defined on an extended graph $G^V = (V, A^V)$ in which each node in the set V is represented by a pair (i, m), in which $i \in N$ indicates the location and m indicates the m^{th} visit to location i. Arcs in the graph G^V correspond to direct vehicle movements from node (i, m) to node (j, n).

Thus $((i,m),(j,n)) \in A^V$ if $(i,j) \in A^N$. For ease of notation, arcs $((i,m),(j,n)) \in A^V$ are represented by (i,m,j,n) whenever the meaning is clear from the context. Figure 3 shows how the route shown in Figure 1 is represented in this extended graph.

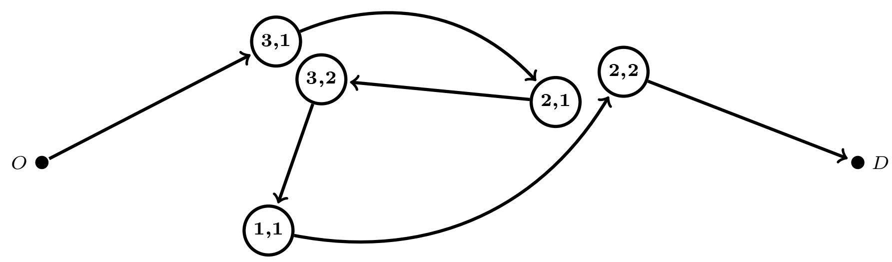


Figure 3: Route given in Figure 1 represented in G^V.

3.1 Formulation

For easy reference, the parameters and the variables for the location-event model are given below. They are followed by the mathematical model.

Parameters

J_i	if location i is a supplier then $J_i = 1$, otherwise $J_i = -1$
$\overline{\mu}_i$	upper bound on the number of visits to location i

Variables

x_{imjn}	1 if the vehicle travels from node (i,m) directly to node (j,n), and 0 otherwise
x^0_{i1}	1 if the vehicle travels directly from its initial position to node $(i,1)$, and 0 otherwise
y_{im}	1 if the vehicle makes the m^{th} visit to location i, and 0 otherwise
z_{im}	1 if the vehicle ends its route at node (i,m), and 0 otherwise
q_{im}	quantity picked up or delivered at node (i,m)
f_{imjn}	quantity transported from node (i,m) to node (j,n)
f^0_{im}	quantity transported from the initial position of the vehicle to node (i,m)
f^D_{im}	quantity transported from node (i,m) to the destination
t_{im}	start time of operation on the m^{th} visit to location i
s_{im}	stock level at the start of operation on the m^{th} visit to location i

Variables x^0_{im} and f^0_{im} are set to zero for all $m > 1$. They are included in the model for ease of notation. The constraints are separated into four groups: routing constraints, pickup and delivery constraints, time constraints and inventory constraints.

Routing constraints

$$\sum_{(i,m)\in V} x^0_{im} = 1, \tag{1}$$

$$y_{im} - \sum_{(j,n)\in V} x_{jnim} - x^0_{im} = 0, \qquad \forall (i,m) \in V, \tag{2}$$

$$y_{im} - \sum_{(j,n)\in V} x_{imjn} - z_{im} = 0, \qquad \forall (i,m) \in V, \tag{3}$$

$$y_{i,m-1} - y_{im} \geq 0, \qquad \forall (i,m) \in V : 2 \leq m \leq \overline{\mu}_i, \tag{4}$$

$$x^0_{im}, y_{im}, z_{im} \in \{0,1\}, \qquad \forall (i,m) \in V, \tag{5}$$

$$x_{imjn} \in \{0,1\}, \qquad \forall (i,m,j,n) \in A^V. \tag{6}$$

Equation (1) ensures that the vehicle leaves the origin. Equations (2) and (3) are the flow conservation constraints, ensuring that if the vehicle arrives at a node, it also leaves that node or ends its route. Constraints (4) state that if location i is visited m times, then it must also have been visited $m-1$ times.

Pickup and delivery constraints

$$f^0_{im} + \sum_{(j,n)\in V} f_{jnim} + J_i q_{im} = \sum_{(j,n)\in V} f_{imjn} + f^D_{im}, \qquad \forall (i,m) \in V, \tag{7}$$

$$f^0_{im} = Q^0 x^0_{im}, \qquad \forall (i,m) \in V, \tag{8}$$

$$f_{imjn} \leq C x_{imjn}, \qquad \forall (i,m,j,n) \in A^V, \tag{9}$$

$$f^D_{im} \leq C z_{im}, \qquad \forall (i,m) \in V, \tag{10}$$

$$\underline{Q}_i y_{im} \leq q_{im} \leq min\{C, \overline{Q}_i\} y_{im}, \qquad \forall (i,m) \in V, \tag{11}$$

$$f_{imjn} \geq 0, \qquad \forall (i,m,j,n) \in A^V. \tag{12}$$

Equations (7) are the flow conservation constraints for the quantity transported by the vehicle. Equations (8) determine the quantity transported from the initial position to node (i,m). Constraints (9) and (10) ensure that the vehicle capacity is not exceeded, while constraints (11) impose lower and upper limits on the pickup/delivery quantities.

Time constraints

$$t_{im} + T^Q_i q_{im} - t_{jn} + (T + T_{ij}) x_{imjn} \leq T, \qquad \forall (i,m,j,n) \in A^V, \tag{13}$$

$$T^0_i x^0_{im} \leq t_{im}, \qquad \forall (i,m) \in V, \tag{14}$$

$$0 \leq t_{im} \leq T, \qquad \forall (i,m) \in V. \tag{15}$$

Constraints (13) link the start time associated with node (i,m) to the start time associated with (j,n) when the vehicle travels directly from (i,m) to (j,n). Constraints (14) ensure that if the vehicle travels from its initial position to (i,m), then the start time at (i,m) is at least the traveling time between the origin and location i. Lower and upper bounds on the start time at each visit are given by (15).

Inventory constraints

$$s_{i1} = S_i^0 + J_i R_i t_{i1}, \qquad \forall i \in N, \tag{16}$$

$$s_{im} = s_{i,m-1} - J_i q_{i,m-1} + J_i R_i (t_{im} - t_{i,m-1}), \qquad \forall (i,m) \in V : m > 1, \tag{17}$$

$$s_{im} + q_{im} - R_i T_i^Q q_{im} \leq \overline{S}_i, \qquad \forall (i,m) \in V | J_i = -1, \tag{18}$$

$$s_{im} - q_{im} + R_i T_i^Q q_{im} \geq \underline{S}_i, \qquad \forall (i,m) \in V | J_i = 1, \tag{19}$$

$$s_{i\overline{\mu}_i} + q_{i\overline{\mu}_i} - R_i(T - t_{i\overline{\mu}_i}) \geq (1-F)\underline{S}_i + F S_i^0, \qquad \forall i \in N | J_i = -1, \tag{20}$$

$$s_{i\overline{\mu}_i} - q_{i\overline{\mu}_i} + R_i(T - t_{i\overline{\mu}_i}) \leq (1-F)\overline{S}_i + F S_i^0, \qquad \forall i \in N | J_i = 1, \tag{21}$$

$$s_{im} \geq \underline{S}_i, \qquad \forall (i,m) \in V | J_i = -1, \tag{22}$$

$$s_{im} \leq \overline{S}_i, \qquad \forall (i,m) \in V | J_i = 1. \tag{23}$$

Equations (16) specify the stock level at the start time of the first visit to a location, and equations (17) relate the stock level at the start time of the m^{th} visit to the stock level at the start time of the previous visit. Constraints (18) and (19) ensure that the stock levels are within their limits at the end of each visit. Constraints (20) impose a lower bound on the inventory level at time T for the demand locations, while constrains (21) impose an upper bound on the inventory level at time T for the supply locations. Notice that if F is a positive number, then the stock level at the end of the time horizon must be greater than the lower bound $\underline{S}_i$ for demand locations and must be lower than the upper bound $\overline{S}_i$ for the supply locations. Finally, constraints (22) and (23) ensure that the stock levels are within their limits at the start of each visit.

Here we consider the value of the variables after the last visit to node i. If κ_i is the number of the last vehicle visit to node i, then the routing constraints (2), (3), and (4) force variables y_{im}, x_{imjn} and x_{jnim} to be zero for all $m > \kappa_i$. Then, using the fact that these variables are zero, constraints (9) and (11), force variables q_{im}, f_{imjn} and f_{jnim} to be zero for all $m > \kappa_i$. The inventory and time variables, s_{im} and t_{im} respectively, for $m > \kappa_i$, are only restricted by their bounds, $\underline{S}_i \leq s_{im} \leq \overline{S}_i$ and $0 \leq t_{im} \leq T$. That means, multiple alternative values can be assigned to these variables. Although such variables have no practical meaning, they are necessary to force the inventory levels at time T to be within the predefined limits. Observe that by adding up constraints (17) for $\kappa_i + 1$ to $\overline{\mu}_i$ and setting q_{im} to zero, we obtain

$$s_{i\overline{\mu}_i} = s_{i\kappa_i} + J_i R_i (t_{i\overline{\mu}_i} - t_{i\kappa_i}).$$

Using this constraint to eliminate variable $s_{i\overline{\mu}_i}$ in constraints (20) and (21) we obtain

$$s_{i\kappa_i} - R_i(T - t_{i\kappa_i}) \geq (1-F)\underline{S}_i + F S_i^0, \qquad \forall i \in N | J_i = -1,$$

$$s_{i\kappa_i} + R_i(T - t_{i\kappa_i}) \leq (1-F)\overline{S}_i + F S_i^0, \qquad \forall i \in N | J_i = 1.$$

This implies that the inventory bounds at the end of the time horizon are also satisfied.

We denote by X the set of feasible solutions satisfying (1) – (23).

Objective function

The objective is to minimize the total travel and operating costs. The objective function is as follows:

$$z = \sum_{(i,m,j,n) \in A^V} C_{ij}^T x_{imjn} + \sum_{(i,m) \in V} C_i^{T0} x_{im}^0 + \sum_{(i,m) \in V} C_i^S y_{im}. \tag{24}$$

3.2 Location-event model tightening

This section presents valid inequalities for the feasible set X. Some inequalities have been used previously, namely, those imposing a minimum number of visits to each node, see [6]. There the problem considered includes several products but does not consider upper bounds on the inventories at the supply locations.

Tighten variable upper bound constraints

Here we consider the tightening of the non-negativity constraints (12):

$$f_{imjn} \geq \underline{Q}_j x_{imjn}, \qquad \forall (i,m,j,n) \in A^V | J_j = -1, \tag{25}$$

$$f_{imjn} \geq \underline{Q}_i x_{imjn}, \qquad \forall (i,m,j,n) \in A^V | J_i = 1, \tag{26}$$

and the tightening of the variable upper bound constraints (9), linking the flow variables with the routing variables for arcs leaving demand locations in (27) and arcs arriving supply locations in (28):

$$f_{imjn} \leq (C - \underline{Q}_i) x_{imjn}, \qquad \forall (i,m,j,n) \in A^V | J_i = -1, \tag{27}$$

$$f_{imjn} \leq (C - \underline{Q}_j) x_{imjn}, \qquad \forall (i,m,j,n) \in A^V | J_j = 1. \tag{28}$$

Lower bounds on the number of visits

A common approach to tighten such a formulation is to include constraints imposing a minimum number of visits to each location. Let $\underline{\mu}_i$ denote a lower bound on the number of visits to location i, $i \in N$.

For each demand location $i \in N$, $J_i = -1$ with $S_i^0 - T \times R_i < \underline{S}_i$, let

$$Q_i^N = max\{T \times R_i - S_i^0 + \underline{S}_i,\ \underline{Q}_i\},$$

denote the net demand over the time horizon. Otherwise $Q_i^N = 0$.

For each supply location $i \in N$, $J_i = 1$ with $S_i^0 + T \times R_i > \overline{S}_i$, let

$$Q_i^N = max\{T \times R_i + S_i^0 - \overline{S}_i,\ \underline{Q}_i\},$$

denote the net supply over the time horizon. Otherwise $Q_i^N = 0$.

The number of visits to location i is at least:

$$\underline{\mu}_i = \left\lceil \frac{Q_i^N}{\min\{\overline{Q}_i, C\}} \right\rceil.$$

If there is only one supply location, then assuming without loss of generality that this location is location 1, we have

$$\underline{\mu}_1 = \left\lceil \frac{\max\{Q_1^N, \sum_{i \in N \setminus \{1\}} Q_i^N - Q^0\}}{\min\{\overline{Q}_1, C\}} \right\rceil.$$

Thus, the following equalities establishing the minimum number of visits can be added:

$$y_{i\underline{\mu}_i} = 1, \qquad \forall i \in N. \tag{29}$$

The following inequalities establish a minimum number of visits that must be made to a subset of locations $S \subseteq N$

$$\sum_{(i,m) \in V | i \in S} y_{im} \geq \left\lceil \frac{\sum_{j \in S} Q_j^N}{C} \right\rceil. \tag{30}$$

Instead of separating over this family of inequalities we include just two inequalities, one for the set of suppliers and one for the set the demand locations, respectively.

$$\sum_{(i,m) \in V | J_i = 1} y_{im} \geq \left\lceil \frac{\sum_{j \in V | J_j = 1} Q_j^N}{C} \right\rceil, \tag{31}$$

$$\sum_{(i,m) \in V | J_i = -1} y_{im} \geq \left\lceil \frac{\sum_{j \in V | J_j = -1} Q_j^N}{C} \right\rceil. \tag{32}$$

End-of-visit inequalities

The following inequalities ensure that if the vehicle makes the m^{th} visit to location i then it cannot have made the last visit to that location in one of the previous visits.

$$\sum_{n<m} z_{in} + y_{im} \leq 1, \qquad i \in N, m > \underline{\mu}_i. \tag{33}$$

Travel time valid inequality

Proposition 3.1 *Let $\underline{I}^{QT}$ denote a lower bound on the time spent picking up and delivering. Then the following travel time inequality is valid for X:*

$$\sum_{(i,m,j,n)\in A^V} T_{ij} x_{imjn} \leq T - \underline{I}^{QT}. \tag{34}$$

A possible value for the lower bound is $\underline{I}^{QT} = \sum_{i\in N} Q_i^N T_i^Q$.

Another set of inequalities results from the assumption that no two consecutive visits can occur at the same node. Let $T_i = \min_{j\in N|(i,j)\in A^N, j\neq i} T_{ij}$. Then the following inequalities are valid.

$$t_{im} \geq t_{i,m-1} + 2T_i, \qquad \forall\ (i,m) \in V \mid 1 < m \leq \underline{\mu}_i, \tag{35}$$

$$t_{im} \geq t_{i,m-1} + 2T_i y_{im}, \qquad \forall\ (i,m) \in V \mid m > \underline{\mu}_i. \tag{36}$$

Example 3.1 *Continuing Example 2.1, for the case $S_4^0 = 16$ we have:*

Location 1, since $S_1^0 + TR_1 = 115 + 40 \times 6 = 355 < \overline{S}_1 = 360$, then $Q_1^N = 0$.

Location 2, $S_2^0 - TR_2 = 39 - 40 \times 2 = -41 < \underline{S}_2 = 0$. Thus $Q_2^N = 41$.

Location 3, $S_3^0 - TR_3 = 38 - 40 \times 2.5 = -62 < \underline{S}_3 = 0$. Thus $Q_3^N = 62$.

Location 4, $S_4^0 - TR_4 = 16 - 40 \times 1.5 = -44 < \underline{S}_4 = 0$. Thus $Q_4^N = 44$.

We have $\sum_{i\in N} Q_i^N = 0 + 41 + 62 + 44 = 147$ and $\underline{I}^{QT} = \sum_{i\in N} Q_i^N T_i^Q = 147/80 = 1.8375$.

Valid inequalities based on hamiltonian dipath

The following, called $(i,m)-(j,n)$ path inequalities result from lifting the following simple inequalities:

$$x_{imjn} + x_{jnim} \leq y_{im}, \qquad \forall (i,m,j,n) \in A^V.$$

Proposition 3.2 *The following inequalities are valid for X.*

$$\sum_{n'\leq n} x_{imjn'} + \sum_{n'\geq n} x_{jn'im} \leq y_{im}, \qquad \forall (i,m,j,n) \in A^V. \tag{37}$$

Proof. If $y_{im} = 0$, then all the variables on the left-hand side are zero.
Let $y_{im} = 1$. For each one of the sums in the left-hand side only one variable can be positive since otherwise, in the first sum there would be multiple arcs leaving node (i,m) and in the second sum there would be multiple arcs entering into node (i,m). If two variables $x_{imjn'}$ with $n' \leq n$ and $x_{j\widehat{n}im}$ with $\widehat{n} \geq n$ are simultaneously one we obtain an incompatibility with one arc preceding the other.

Maximal two-location cliques

Next, we introduce a family of inequalities of the form:

$$\sum_{(i,m,j,n)\in A^V} \pi(i,m,j,n) x_{imjn} \leq 1$$

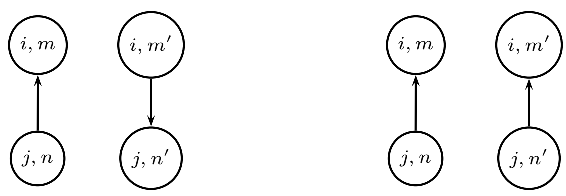
where $\pi \in \{0,1\}^{A^V}$. These inequalities can be regarded as a particular case of clique inequalities on a given conflict graph.

We just consider the digraph restricted to the nodes for locations i and j and the arcs on the vehicle route between two such nodes. Two arcs are said to be incompatible/compatible if they cannot/can both form part of a route. Examples of pairs of incompatible arcs are shown in Figure 4. Thus a) two arcs cannot arrive or leave from the same node, b) two arcs cannot form a 2-cycle, that is, arcs (i,n,j,m) and (j,m,i,n) are incompatible, and c) a pair of arcs (i,n,j,m) and (j,m,i,n') are incompatible if $n > n'$. In Figure 5 we show examples of compatible arc pairs.

$m \leq m'$ and $n \leq n'$
(i)

$m < m', n < n'$ or $m = m', n' \leq n$ or $m' < m, n = n'$
(ii)

Figure 4: Incompatible arc pairs.



$m \leq m', n \leq n'$ or $m \geq m', n \geq n'$
(i)

$m < m', n < n'$
(ii)

Figure 5: Compatible arc pairs with $(m,n) \neq (m',n')$.

Proposition 3.2 *Let A' be a set of arcs linking nodes $(i,m), m = 1,\dots,\overline{\mu}_i$ and $(j,n), n = 1,\dots,\overline{\mu}_j$ such that neither of the configurations in Figure 5 appears. Then the following inequality is valid for X.*

$$\sum_{(i,m,j,n)\in A'} x_{imjn} \leq 1. \tag{38}$$

Proof. Suppose that $(j,n,i,m) \in A'$ and $x_{jnim} = 1$. We show that for all (j,n',i,m'), (i,m',j,n') with $(j,n,i,m) \neq (j,n',i,m')$ either $(j,n',i,m') \notin A'$ or $x_{jn'im'} = 0$ and similarly either $(i,m',j,n') \notin A'$ or $x_{im'jn'} = 0$.

Case 1. Consider (j,n',i,m') with $(j,n',i,m') \neq (j,n,i,m)$.

Case 1a. $m' < m$.

If $n' < n$, $(j,n',i,m') \notin A'$ by (ii) of Figure 5.

If $n' \geq n$, then by (i) of Figure 4, $x_{jn'im'} = 0$.

Case 1b. $m' = m$. Again by (i) of Figure 4, $x_{jn'im'} = 0$.

Case 1c. $m' > m$.

If $n' > n$, $(j,n',i,m') \notin A'$ by (ii) of Figure 5.

Case 2. Consider (i,m',j,n').

Case 2a. $m' < m$.

If $n' \leq n$, $(i,m',j,n') \notin A'$ by (i) of Figure 5.

If $n' > n$, then by (ii) of Figure 4 $x_{im'jn'} = 0$.

Case 2b. $m' = m$.

If $n' > n$, $(i,m',j,n') \notin A'$ by (i) of Figure 5.

If $n' \leq n$, then by (ii) of Figure 4 $x_{im'jn'} = 0$.

Case 2c. $m' > m$.

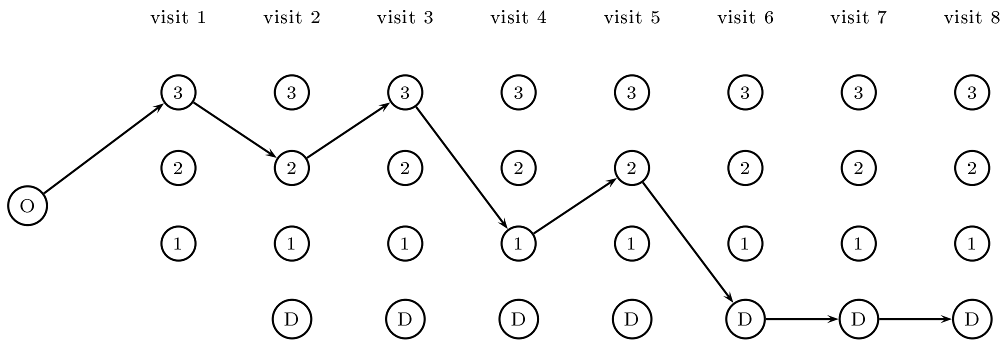


Figure 6: Example of a path in the extended network for a maximum of eight visits and three locations.

If $n' > n$, $(i, m', j, n') \notin A'$ by (i) of Figure 5.

If $n' \leq n$, then by (ii) of Figure 4 $x_{im'jn'} = 0$.

Remark 3.3 *Proposition 3.2 can easily be extended to the case of multiple vehicles by summing up each arc on the left-hand side for all possible vehicles.*

As an efficient separation algorithm for inequalities (38) is not known, we present below two particular polynomial subclasses of the maximal two-location cliques.

Proposition 3.3 *Let $(i, m_1), (i, m_2), (j, n_1), (j, n_2) \in V$ with $m_1 < m_2$ and $n_1 < n_2$. Then the following inequality is valid for X.*

$$\sum_{n \leq n_1} x_{i,m_2,j,n} + \sum_{m \leq m_1} x_{j,n_2,i,m} + x_{i,m_1,j,n_1} + x_{j,n_1,i,m_1} \leq 1. \quad (39)$$

Proposition 3.4 *Let $(i, m_1), (i, m_2), (j, n_1), (j, n_2) \in V$ with $m_1 < m_2$ and $n_1 < n_2$. Then the following inequality is valid for X.*

$$\sum_{m \geq m_2} x_{i,mj,n_1} + \sum_{n \geq n_2} x_{j,ni,m_1} + x_{i,m_1,j,n_2} + x_{j,n_1,i,m_2} \leq 1. \quad (40)$$

4 Layered vehicle-event model

In this section, we propose a different model in which events are linked to the vehicle. The order of the events corresponds to the order of the vehicle visits. In this formulation, the vehicle path is described using a layered graph in which each layer corresponds to the number of visits made by the vehicle. Each layer contains all locations. In Figure 6 we present an example of a path in the extended network for a maximum of eight visits corresponding to the example given in Figure 1. The vehicle leaves the origin to visit location 3 (first visit/layer), then moves to location 2 (second visit/layer), followed by location 3 (third visit/layer), then location 1 (fourth visit/layer) and finally visits location 2 (fifth visit/layer). From the last visit, the vehicle moves to the artificial destination (D).

4.1 Formulation

First, we describe the sets, parameters and variables not defined previously. Then, we describe the mathematical formulation of the vehicle-event model.

Sets and Parameters

$\overline{k}$	maximum number of vehicle visits
K	set of possible visits $\{1, \ldots, \overline{k}\}$

Variables

y_i^k $= 1$ if the k^{th} visit occurs at location i, and 0 otherwise

χ_{ij}^k $= 1$ if the $(k-1)^{th}$ visit is to location i and the k^{th} to location j

z_i^k $= 1$ if the k^{th} visit is to location i and it is the last visit on the route

f_{ij}^k quantity transported by the vehicle from the $(k-1)^{th}$ visit at location i to the k^{th} visit at location j

f_i^0 quantity transported by the vehicle from the origin to the 1^{st} visit at location i

f_i^{Dk} is the amount remaining in the vehicle when the last visit is the k^{th} visit to location i

s_i^k stock level at location i at the start of the k^{th} visit of the vehicle

t^k start time of the k^{th} visit

q_i^k quantity picked up/ delivered at location i during the k^{th} visit of the vehicle

The objective function is again to minimize the travel costs plus the operating costs:

$$Z = \min \sum_{(i,j)\in A^N} \sum_{k\in K} C_{ij}^T \chi_{ij}^k + \sum_{i\in N} C_i^{T0} y_i^1 + \sum_{i\in N} \sum_{k\in K} C_i^S y_i^k. \tag{41}$$

As before, the constraints are presented separately for the main four components: path constraints, pickup and delivery constraints, time constraints and inventory constraints.

Path constraints

$$\sum_{j\in N} y_j^1 = 1, \tag{42}$$

$$y_i^k - \sum_{j\in N|i\neq j} \chi_{ji}^k = 0, \qquad \forall\, i \in N, k \in K \mid k > 1, \tag{43}$$

$$y_i^{k-1} - \sum_{j\in N|j\neq i} \chi_{ij}^k - z_i^{k-1} = 0, \qquad \forall\, i \in N, k \in K \mid k > 1, \tag{44}$$

$$\sum_{i\in N} \sum_{k\in K} z_i^k = 1, \tag{45}$$

$$y_i^k, z_i^k \in \{0,1\}, \qquad \forall\, i \in N, k \in K, \tag{46}$$

$$\chi_{ij}^k \in \{0,1\}, \qquad \forall\, (i,j) \in A^N, k \in K. \tag{47}$$

Equality (42) ensures that the vehicle makes a first visit. Constraints (43) state that if the vehicle travels directly from location j to location i and the visit to location j is the $(k-1)^{th}$, then location i must receive the k^{th} visit. Constraints (44) say that if $y_i^{k-1} = 1$ then the vehicle either travels from location i to another location j or ends its route in i. Constraint (45) ensures that the route terminates at or before the $\bar{k}^{th}$ visit.

Pickup and delivery constraints

$$f_i^0 + J_i q_i^k = \sum_j f_{ij}^{k+1} + f_i^{Dk}, \qquad \forall\, i \in N, k = 1, \tag{48}$$

$$\sum_{i \in N} f_{ij}^k + J_j q_j^k = \sum_\ell f_{j\ell}^{k+1} + f_j^{Dk}, \qquad \forall\, j \in N, k \in K | k > 1, \tag{49}$$

$$f_i^0 = Q^0 y_i^1, \qquad \forall\, i \in N, \tag{50}$$

$$f_{ij}^k \leq C \chi_{ij}^k, \qquad \forall\, (i,j) \in A^N, k \in K, \tag{51}$$

$$f_i^{Dk} \leq C z_i^k, \qquad \forall\, i \in N, k \in K, \tag{52}$$

$$\underline{Q}_i y_i^k \leq q_i^k \leq \min\{C, \overline{Q}_i\} y_i^k, \qquad \forall\, i \in N, k \in K, \tag{53}$$

$$f_{ij}^k \geq 0, \qquad \forall\, (i,j) \in A^N, k \in K, \tag{54}$$

$$f_i^{Dk} \geq 0, \qquad \forall\, i \in N, k \in K. \tag{55}$$

Equations (48)–(49) are the flow balance constraints for the quantity carried by the vehicle. Equations (50) describe the initial load on the vehicle. Inequalities (51) – (52) impose upper bounds on the vehicle load. These variable upper bound constraints also link the binary routing variables to the continuous variables representing the quantities transported. Constraints (53) are the variable lower and upper bound constraints linking the quantity picked up/delivered with the binary variables representing the visits to locations.

Time constraints

$$t^{k-1} + \sum_{i \in N} T_i^Q q_i^{k-1} - t^k + \sum_{(i,j) \in A} T_{ij} \chi_{ij}^k \leq 0, \qquad \forall\, k \in K | k > 1, \tag{56}$$

$$t^1 \geq \sum_{i \in N} T_i^0 y_i^1, \tag{57}$$

$$0 \leq t^k \leq T, \qquad \forall\, k \in K. \tag{58}$$

Constraints (56) guarantee that the start time of the k^{th} visit can only occur after the start time of the $(k-1)^{th}$ visit plus the pickup/delivery time of the $(k-1)^{th}$ visit plus the traveling time between the two locations. Constraint (57) ensures that the first visit cannot be made before the vehicle arrives at the location from the origin. Constraints (58) ensure that the start time at a location is within the time horizon.

Inventory constraints

$$s_i^1 = S_i^0 + J_i R_i t^1, \qquad \forall\, i \in N, \tag{59}$$

$$s_i^k = s_i^{k-1} - J_i q_i^{k-1} + J_i R_i (t^k - t^{k-1}), \qquad \forall\, i \in N, k \in K | k > 1, \tag{60}$$

$$s_i^k + q_i^k - R_i T_i^Q q_i^k \leq \overline{S}_i, \qquad \forall i \in N | J_i = -1, k \in K, \tag{61}$$

$$s_i^k - q_i^k + R_i T_i^Q q_i^k \geq \underline{S}_i, \qquad \forall i \in N | J_i = 1, k \in K, \tag{62}$$

$$s_i^{\overline{k}} + q_i^{\overline{k}} - R_i (T - t^{\overline{k}}) \geq (1 - F) \underline{S}_i + F S_i^0, \qquad \forall i \in N | J_i = -1, \tag{63}$$

$$s_i^{\overline{k}} - q_i^{\overline{k}} + R_i (T - t^{\overline{k}}) \leq (1 - F) \overline{S}_i + F S_i^0, \qquad \forall i \in N | J_i = 1, \tag{64}$$

$$s_i^k \geq \underline{S}_i, \qquad \forall i \in N | J_i = -1, k \in K, \tag{65}$$

$$s_i^k \leq \overline{S}_i, \qquad \forall i \in N | J_i = 1, k \in K. \tag{66}$$

Equations (59) and (60) are the inventory balance constraints at each location. Constraints (61) impose an upper bound on the stock level at each location at the end of each visit for the demand locations, while constraints (62) impose a lower bound on the stock level at each location at the end of each visit for the supply locations. Constraints (63) impose a lower bound on the stock level at the end of the time horizon for the demand locations, and constraints (64) impose an upper bound on the stock level at each supply location at the end of the time horizon. Constraints (65) impose a lower bound on the stock level at each demand location at the beginning of each visit, while constraints (66) impose an upper bound on the stock level at each supply location at the beginning of each visit.

Now, we observe what happens with the value of the variables after the last visit. If κ is the number of the last vehicle visit, then the path constraints force y_i^k and χ_{ij}^k to be zero for all $k > \kappa$, and constraints (53) force the quantity q_i^k to be zero. Constraints (56) then impose $t^k \geq t^{k-1}$. Hence, considering constraints (58), the time of the visits for $k > \kappa$ (visits that are not made) are restricted by $t^{k-1} \leq t^k \leq T$, that means, multiple alternative values can be assigned to these variables. In relation to the inventory variables, for $k > \kappa$, constraints (61), (62), (65), and (66) impose that the inventory levels must be between the inventory bounds, $\underline{S}_i \leq s_i^k \leq \overline{S}_i$, while constraints (60) calculate the inventory levels according to the times assigned to variables t_i^k. These variables t_i^k, s_i^k that have no practical meaning are necessary to force the inventory levels at time T to be within the predefined limits, by constraints (63) and (64).

4.2 Vehicle-event model tightening

Here we describe valid inequalities to tighten the vehicle-event model.

The nonnegativity constraints on the flow variables (54) can be tightened as follows:

$$f_{ij}^k \geq \underline{Q}_j \chi_{ij}^k, \qquad \forall\, (i,j) \in A^N \mid J_j = -1, k \in K, \tag{67}$$

$$f_{ij}^k \geq \underline{Q}_i \chi_{ij}^k, \qquad \forall\, (i,j) \in A^N \mid J_i = 1, k \in K. \tag{68}$$

Also, the variable upper bound constraints (51) can be tightened as follows:

$$f_{ij}^k \leq (C - \underline{Q}_i)\chi_{ij}^k, \qquad \forall\, (i,j) \in A^N \mid J_i = -1, k \in K, \tag{69}$$

$$f_{ij}^k \leq (C - \underline{Q}_j)\chi_{ij}^k, \qquad \forall\, (i,j) \in A^N \mid J_j = 1, k \in K. \tag{70}$$

The end-of-visits inequalities (33) for the location-event model can now be written as follows:

$$\sum_{i \in N} \sum_{k'=1}^{k-1} z_i^{k'} + \sum_{i \in N} y_i^k \leq 1, \qquad \forall\, k \in K. \tag{71}$$

Next, we present a set of inequalities that establishes a minimum number of visits , $\underline{\mu}_i$, that the vehicle must make to each location.

$$\sum_{k \in K} y_i^k \geq \underline{\mu}_i, \qquad \forall\, i \in N. \tag{72}$$

Similarly to inequalities (31) and (32), we define the following inequalities establishing a minimum number of visits that the vehicle must make to the suppliers and to the demand locations, respectively.

$$\sum_{k \in K} \sum_{j \in N \mid J_j = 1} y_j^k \geq \left\lceil \frac{\sum_{j \in N \mid J_j = 1} Q_j^N}{C} \right\rceil, \tag{73}$$

$$\sum_{k \in K} \sum_{j \in N \mid J_j = -1} y_j^k \geq \left\lceil \frac{\sum_{j \in N \mid J_j = -1} Q_j^N}{C} \right\rceil. \tag{74}$$

The following inequalities impose conditions on the first p visits to each location. Let ν_i^p denote the time at which a disruption occurs (i.e. the time when the inventory level reaches the stock limit) at location i if that location received $p-1$ visits picking up/delivering the maximum possible amount $\bar{C}_i = \min\{\overline{S}_i, C\}$. For a supply location we have

$$\nu_i^p = \frac{\overline{S}_i - S_i^0}{R_i} + \frac{(p-1)\bar{C}_i}{R_i},$$

and for a demand location the disruption occurs at time

$$\nu_i^p = \frac{S_i^0 - \underline{S}_i}{R_i} + \frac{(p-1)\bar{C}_i}{R_i}.$$

If the p^{th} visit to node i is the k^{th} vehicle visit, then one must have $t^k \le \nu_i^p$.

Let $T^{min} = \min_{(i,j)\in A^N}\{T_{ij}\}$ be the minimum travel time between two locations, $T_i^{min} = \min_{j\neq i}\{T_{ij}\}$ the minimum travel time between node i and any other node, and let $T^{0min} = \min_{i\in N}\{T_i^0\}$ be the travel time between the origin and the closest location. Hence

$$t^k \ge T^{0min} + (k-p)T^{min} + (p-1)T_i^{min} \qquad \forall\, k \in K \mid k > 1.$$

Therefore, $T^{0min} + (k-p)T^{min} + (p-1)T_i^{min} \le \nu_i^p$.

Hence, for each $i \in N$ and each p, we set

$$k_i^p = p + \left\lfloor \frac{\nu_i^p - T^{0min} - (p-1)T_i^{min}}{T^{min}} \right\rfloor.$$

Proposition 4.1 *For $i \in N$ and each $p \in 1, \ldots, \underline{\mu}_i$, the following inequality is valid:*

$$\sum_{k=1}^{k_i^p} y_i^k \ge p \qquad \forall i \in N. \tag{75}$$

The travel time inequalities (34) can also be adapted for the vehicle-event model as follows:

$$\sum_{k\in K}\sum_{(i,j)\in A^N} T_{ij}\chi_{ij}^k \le T - \underline{I}^{QT}. \tag{76}$$

In most practical situations the following assumption can be assumed.

Assumption 1: The vehicle cannot return twice to a demand location without visiting a supply location in between, nor return twice to a supply location without visiting a demand location in between.

When Assumption 1 is valid, a new set of inequalities can be derived. Let $s = \mid \{i \in N | J_i = 1\} \mid$, and $r = \mid \{i \in N | J_i = -1\} \mid$ denote the number of supply and demand locations, respectively.

First, consider the case $Q^0 = 0$. Then, at least $\left\lceil \frac{\sum_{i\in N|J_i=1} Q_i^N}{C} \right\rceil$ visits must be made to supply locations for pick-up operations and at least $\left\lceil \frac{\sum_{i\in N|J_i=-1} Q_i^N}{C} \right\rceil$ loads must be delivered to the demand locations, which must be picked up at the supply locations. Hence, at least

$$\underline{m} = \left\lceil \frac{\max\{\sum_{i\in N|J_i=1} Q_i^N, \sum_{i\in N|J_i=-1} Q_i^N\}}{C} \right\rceil$$

visits must be made to supply locations. Observe that a first pickup operation must be made before the delivery operations. Hence, the first visit must be to a supply location. Using Assumption 1, the vehicle can make at most $1+r$ visits before making the second visit to a supply location. That is, amongst the first $2+r$ visits, two of them must be made to a supply location. In general, for $p \le \underline{m}$, the vehicle can make at most $p-1+(p-1)r$ visits before returning for the p^{th} time to a supply location. Hence, amongst the first $\kappa^p = p + (p-1)r$ visits, at least p of them must be to a supply location.

If $Q^0 > 0$, then the minimum number of vehicle loads to satisfy the total net demand is $\left\lceil \frac{\sum_{i\in N|J_i=-1} Q_i^N - Q^0}{C} \right\rceil$. In this case $\underline{m} = \left\lceil \frac{\max\{\sum_{i\in N|J_i=1} Q_i^N, \sum_{i\in N|J_i=-1} Q_i^N - Q^0\}}{C} \right\rceil$. Now, we cannot assume that the first visit is made to a supply location. In this case, the vehicle can not perform more than $\kappa^p = rp + (p-1)$ visits before returning for the p^{th} time to a supply location, for each $p \in \{1, \ldots, \underline{m}\}$.

Proposition 4.2 *When Assumption 1 holds, the following inequalities are valid:*

$$\sum_{i \in N | J_i = 1} \sum_{k=1}^{\kappa^p} y_i^k \geq p, \quad \forall p = 1, \ldots, \underline{m}. \tag{77}$$

When Assumption 1 does not hold, these inequalities suggest another interesting way to partition the set of feasible solutions.

Proposition 4.3 *Any feasible solution either satisfies (77) or else there is a unique $q \in \{1, \ldots, \underline{m}\}$ such that*

$$\begin{aligned} &\sum_{i \in N | J_i = 1} \sum_{k=1}^{\kappa^p} y_i^k \geq p, \forall p = 1, \ldots, q-1, \\ &\sum_{i \in N | J_i = 1} \sum_{k=1}^{\kappa^q} y_i^k \leq q. \end{aligned} \tag{78}$$

5 Estimating the number of visits

One of the main challenges when using event based models is to estimate the number of visits. As the size of the corresponding models depends on the number of events, large upper bounds lead to large sized models. On the other hand, restricting the number of events too much may exclude feasible and optimal solutions. Here we propose a scheme to bound the number of events.

5.1 Establishing upper bounds

Next we describe for each model, how to derive upper bounds on the number of events.

Location-event model

An upper bound $\overline{k}$ on the total number of visits can be obtained from the model (1) – (23), tightened with the inequalities introduced in Section 3.2, with the new objective function

$$w = max \sum_{i \in N} \sum_{m=1}^{\overline{\mu}_i} y_{im}. \tag{79}$$

Instead of solving this model to optimality, one can take $\overline{k} = \lfloor \overline{w} \rfloor$ where $\overline{w}$ is an upper bound for w. $\overline{w}$ can be the value of the corresponding linear relaxation or the best upper bound obtained from the branch-and-cut after a given time limit.

An upper bound w_i on the number of visits to node i can be obtained similarly by adding the constraint

$$\sum_{i \in N} \sum_{m=1}^{\overline{\mu}_i} y_{im} \leq \overline{k},$$

and taking as objective function

$$w_i = max \sum_{m=1}^{\overline{\mu}_i} y_{im}. \tag{80}$$

Again, w_i can be replaced by $\lfloor \overline{w}_i \rfloor$, where $\overline{w}_i$ is an upper bound of w_i, and set $\overline{\mu}_i = \lfloor \overline{w}_i \rfloor$.

For models w and w_i, when no initial upper bounds $\overline{\mu}_i, i \in N$ are known, we take $\overline{\mu}_i = \underline{\mu}_i + M$ where M is a large number.

Vehicle-event model

A simple upper bound on the total number of visits can be obtained as follows:

$$\overline{k} \leq \left\lfloor \frac{T - \underline{I}^{QT} - T^{0min}}{T^{min}} \right\rfloor + 1 \tag{81}$$

where T^{0min}, T^{min} are given previously.

Another possible approach is to consider an upper bound for the model (41) - (66) tightened with the inequalities introduced in Section 4.2 with the new objective function:

$$w = \max \sum_{i \in N} \sum_{k \in K} y_i^k \tag{82}$$

and set $\overline{k} = \lfloor \overline{w} \rfloor$, where $\overline{w}$ is an upper bound on w.

5.2 Two branching schemes

The size of the location-event and vehicle-event models depends on the upper bound on the number of visits. By limiting this number, we obtain restricted subproblems which become smaller and, therefore, can be solved faster, as shown in the computational section. Here we propose a two-level branching scheme to solve the inventory routing problem to optimality. In the first level we split the problem into several subproblems by restricting the domain of the total number of visits, and in the second level we solve each subproblem. A good choice of the number of subproblems may depend on the length of the time horizon and the expected total number of visits. Here we present the case for two subproblems, since this choice performed well on the instances tested. First, we determine an upper bound on the number of visits, $\overline{k}$, using the procedure described in Section 5.1. Then we split the problem into two subproblems, one with the constraint $\sum_{i \in N} \sum_{k \in K} y_i^k \leq \lceil \overline{k}/2 \rceil$, and the second with $\sum_{i \in N} \sum_{k \in K} y_i^k \geq \lceil \overline{k}/2 \rceil + 1$. The first subproblem is solved by branch-and-cut. The value of the best feasible solution found is added as a cut-off value for the second subproblem. The full algorithm is detailed in Algorithm 1.

Algorithm 1 A two-level branching approach for the inventory routing problem.

1: Determine an upper bound for the number of visits $\overline{k}$

2: Add constraint $\sum_{i \in N} \sum_{k \in K} y_i^k \leq \lceil \overline{k}/2 \rceil$

3: Solve the resulting model with a time limit of β seconds

4: Set $\bar{z}^1$ to the value of the best feasible solution found and $+\infty$ if no solution is found

5: Replace constraint given in 2 by constraint $\sum_{i \in N} \sum_{k \in K} y_i^k \geq \lceil \overline{k}/2 \rceil + 1$

6: Add the cutoff value $\bar{z}^1$ to the model

7: Solve the resulting model with a time limit of β seconds with optimal value $\bar{z}^2$

8: Let $\underline{z}^i$ for $i = 1, 2$ be the value of the best lower bound obtained for each subproblem. Then the best lower bound is $\min(\underline{z}^1, \underline{z}^2)$ and the best upper bound $\min(\bar{z}^1, \bar{z}^2)$.

The second algorithm is based on Proposition 4.3.

Algorithm 2 An $\underline{m} + 1$ branch approach for the inventory routing problem.

1: Determine the values $\underline{m}$ and κ^p as in Proposition 4.2.

2: Add constraint (77) and solve the resulting model with a time limit of β_1 seconds

3: Set $\bar{z}^0$ to the value of the best feasible solution found and $+\infty$ if no solution is found, and $\underline{z}^0$ to be the best lower bound found

4: For $q = 1, \ldots, \underline{m}$, replace (77) by constraints (78)

5: Add the cutoff value $\bar{z} = \bar{z}^0$ to the model

6: Solve the resulting model with a time limit of β_2 seconds with best upper and lower bounds $\bar{z}^q$ and $\underline{z}^q$, respectively. Update the cutoff value $\bar{z} \leftarrow \min[\bar{z}, \bar{z}^q]$

7: On termination, $\bar{z}$ is the value of the best feasible solution found and the best lower bound is $\min(\underline{z}^0, \underline{z}^1, \ldots, \underline{z}^{\underline{m}})$

6 Computational results

This section presents computational experiments carried out to compare the two models and the tightening strategies. The formulations are written in Mosel and implemented in Xpress-IVE Version 1.25.02, with 64 bits. All the tests were run on a computer with a CPU Intel(R) Core i7-10510U, with 16GB RAM and using the Xpress Optimizer Version 34.01.03 with the default options.

Part of the motivation for the current research is related to the results presented in [3] for a maritime inventory routing problem in which only instances with a short time horizon were solved to optimality. Here, a set of fourteen instances based on the original seven given in [3] are considered. In contrast to most of the original instances that involved multiple vehicles (ships), these fourteen instances are developed assuming a single vehicle is available. Both the constant supply and demand rates and the initial stock levels have been changed. For each of these 14 instances, two values for the end-of-time horizon inventory levels are considered, $F = 0$ and $F = 0.2$. First, we consider two different lengths of the time horizon: 60 and 120 days. Travel and operating costs are time invariant. Some of these instances are infeasible. By considering such instances we also aim to test whether the models can prove infeasibility quickly. Later, in Section 6.2, we conduct further tests with a horizon of 180 and 240 days with $F = 0$ and with additional adjustments to the rates.

6.1 Medium size instances

In Table 1 we present some basic information regarding the set of instances, namely, the number of locations $|N|$, the corresponding optimal objective function value (columns "Obj"), the total number of visits in the optimal solution (columns "opt"), the upper bound on the number of visits obtained with formula (81) (columns "U"), the upper bounds on the number of visits obtained from the linear relaxation of (82) (column $\bar{w}$), and columns w report the optimal value of (82). Note that computing w requires the solution of a problem similar to the s-CT-IRP-PD, which is very hard. The gains from computing w instead of the linear relaxation $\overline{w}$ are minor (on average a little more than one visit for $T = 60$ and $T = 120$) which clearly indicates that the computational effort to obtain w is not compensated by a significant reduction in the size of the model. The "INF" symbol means that the instance is infeasible.

The bounds on the number of visits using the location-event model are worse than those with the vehicle-event model and therefore are omitted.

In Table 2 we present the computational results for the location-event model. Two formulations are tested. A location-event formulation (1) – (23) with no additional valid inequalities, called the weak formulation, and the same formulation tightened with all the inequalities (25) – (29), (31)– (37), (39), and (40) (where m_1, m_2, n_1, n_2 are bounded by $\underline{\mu}_i$+3), called the strong formulation. The number of visits is determined as described in Section 5.1 by taking the best upper bounds obtained with branch-and-cut when solving the problems with objective functions w and w_i for 3 seconds for both models $T = 60$ and for the weak formulation with $T = 120$. For the remaining case (corresponding to the strong formulation and $T = 120$) we run for 5 seconds to ensure a bound is obtained. Columns "BCw" and "BCs" report the results obtained with the solver using the branch-and-cut with the default options and a time limit of one hour on the weak and strong formulations, respectively. Columns "ALG1" report the results with Algorithm 1 using the strong formulation and $\beta = 3600$ on each branch. The running times presented in columns "Time" include both the running time of the corresponding approach and the time spent to obtain the bounds on the number of the visits. For $T = 60$, all instances are solved to optimality. Hence, only the running times are reported for each approach. For $T = 120$, we report the running time in

Table 1: Data and information on the optimal solution and number of visits for instances with a time horizon consisting of 60 and 120 days.

			T = 60					T = 120				
Instance	F	|N|	Obj	opt	U	$\bar{w}$	w	Obj	opt	U	$\bar{w}$	w
A1	0	3	331.3	8	18	16	15	898.2	19	36	30	27
	0.2		462.3	10	18	16	14	933.2	20	35	30	27
A2	0	3	331.3	8	19	17	16	877.5	19	37	32	30
	0.2		331.3	8	19	17	15	898.2	19	37	32	30
B1	0	4	310.7	7	19	14	14	INF	INF	37	26	INF
	0.2		310.7	7	19	14	14	INF	INF	36	24	INF
B2	0	4	310.7	7	19	14	14	779	15	36	24	22
	0.2		310.7	7	19	14	14	779	15	36	24	22
C1	0	4	295.8	6	20	17	16	859.1	13	37	28	27
	0.2		397.1	6	19	15	14	859.8	13	36	25	25
C2	0	4	381.3	7	19	16	16	954.1	14	37	27	25
	0.2		402.1	7	19	16	15	975.4	16	36	25	25
D1	0	5	320.3	8	15	13	12	714.5	16	27	25	23
	0.2		329.5	8	14	13	12	764.9	18	27	24	22
D2	0	5	320.3	8	15	13	12	INF	INF	27	24	INF
	0.2		366.7	9	14	13	12	INF	INF	27	24	INF
E1	0	5	273.3	7	29	24	20	572.4	14	56	44	37
	0.2		273.3	7	29	23	19	599.4	15	55	43	36
E2	0	5	298.9	7	29	24	18	662.5	16	56	45	34
	0.2		332.3	9	29	23	18	683.3	17	56	44	34
F1	0	4	313.9	6	14	12	12	858.1	15	25	20	16
	0.2		313.9	6	14	12	11	INF	INF	25	17	INF
F2	0	4	318.7	7	14	12	11	INF	INF	24	INF	INF
	0.2		383	8	13	11	9	INF	INF	23	INF	INF
G1	0	6	208.5	5	14	13	13	INF	INF	26	18	INF
	0.2		375.3	6	14	11	10	INF	INF	26	17	INF
G2	0	6	158.6	12	15	14	14	804.5	12	29	24	21
	0.2		234	12	15	14	14	804.5	12	29	23	23
Average				7.6	18.4	15.4	14.1		15.7	34.8	27.7	26.6

seconds, the final lower and upper bounds (in columns "LB" and "UB", respectively). For example, for instance B1, $F = 0$ and $T = 120$, the BCs approach proves that the instance is infeasible after 2870.9 seconds, while the BCw approach reaches the time limit without proving infeasibility. For this instance, since no feasible solution is found, the upper bound is set to $+\infty$ and the best proven lower bound 772.2 is reported.

Table 2: Computational results for the location-event model.

		T=60			T=120								
		BCw	BCs	ALG1	BCw			BCs			ALG1		
Instance	F	Time	Time	Time	Time	LB	UB	Time	LB	UB	Time	LB	UB
A1	0	31.6	21.7	20.4	3615	656.4	$+\infty$	3625.0	842.9	$+\infty$	3922.5	845.4	$+\infty$
	0.2	546	26.2	27.2	3614	719.1	$+\infty$	3625.6	831.6	$+\infty$	3691.4	847.7	$+\infty$
A2	0	35.2	21.7	17.3	3614	596.7	$+\infty$	3624.5	724.4	$+\infty$	5113.1	767.3	947.8
	0.2	33.5	23.8	19.6	3615	614.3	$+\infty$	3625.6	735.3	$+\infty$	5067.7	769.4	932.5
B1	0	11.5	7.7	8.0	3612	772.2	$+\infty$	2870.9	INF	INF	3619.4	938.9	$+\infty$
	0.2	12	3.8	3.3	3612	810.0	$+\infty$	443.0	INF	INF	451.5	INF	INF
B2	0	13.3	9.2	9.0	195	779.0	779.0	19.5	779.0	779.0	21.0	779.0	779.0
	0.2	13.4	5.2	5.7	114	779.0	779.0	19.1	779.0	779.0	21.0	779.0	779.0
C1	0	20.5	19.9	29.1	204	859.1	859.1	104.2	859.1	859.1	115.6	859.1	859.1
	0.2	29.4	21.3	24.9	78	859.9	859.9	39.9	859.9	859.9	32.6	859.9	859.9
C2	0	20.4	14.0	16.5	2907	954.1	954.1	242.2	954.1	954.1	320.9	954.1	954.1
	0.2	32.6	19.3	17.8	1983	975.3	975.4	77.1	975.4	975.4	214.1	975.4	975.4
D1	0	89.8	32.5	24.3	3618	530.6	$+\infty$	3629.8	633.0	780.1	3736.1	645.7	$+\infty$
	0.2	135.6	37.1	34.4	3618	557.9	$+\infty$	3630.3	650.8	$+\infty$	3672.4	659.6	$+\infty$
D2	0	195.1	37.2	37.0	3618	567.7	$+\infty$	3629.3	645.4	$+\infty$	3698.9	664.9	$+\infty$
	0.2	283.9	37.3	36.1	3617	574.6	$+\infty$	3629.4	664.5	$+\infty$	3659.1	674.2	$+\infty$
E1	0	56.7	30.2	59.4	3617	437.6	$+\infty$	2795.0	572.4	572.4	5566.7	572.4	572.4
	0.2	85.7	33.7	65.1	3618	449.3	$+\infty$	3630.2	532.7	599.4	5892.5	599.4	599.4
E2	0	166.3	43.0	72.2	3617	446.1	$+\infty$	3629.8	549.5	677.5	7229.6	547.6	697.9
	0.2	235.3	47.7	98.3	3617	455.7	$+\infty$	3630.1	561.6	$+\infty$	7229.2	567.2	$+\infty$
F1	0	31.4	19.0	23.8	3614	785.5	$+\infty$	180.2	858.1	858.1	210.7	858.1	858.1
	0.2	55.1	15.9	20.5	3614	894.8	$+\infty$	31.6	INF	INF	31.5	INF	INF
F2	0	27.8	16.0	17.6	3615	999.3	$+\infty$	3.9	INF	INF	3.4	INF	INF
	0.2	23.5	17.0	16.9	3615	1060.0	$+\infty$	4.1	INF	INF	3.1	INF	INF
G1	0	20.5	11.5	13.0	3621	882.0	$+\infty$	617.8	INF	INF	198.2	INF	INF
	0.2	21.7	10.3	10.6	3621	920.9	$+\infty$	47.7	INF	INF	45.9	INF	INF
G2	0	21	13.7	13.2	3621	621.1	$+\infty$	3634.1	673.8	$+\infty$	6451.4	798.5	804.5
	0.2	24.4	23.3	35.9	3621	654.8	$+\infty$	3634.8	744.8	804.5	5247.2	777.5	804.5
Average		81.2	22.1	27.8	3037.2			1952.7			2695.2		

The results from solving the location-event model show that all instances are solved for $T = 60$. For these smaller instances, using Algorithm 1 on the model with all inequalities is on average much faster than using the BCw approach. For $T = 120$, only six instances are solved with the BCw approach whereas the search is completed for 15 instances when using the two other approaches. Although these last two approaches are not directly comparable, as the overall running time limits are different, there is no clear indication that Algorithm 1 is better than BCs when working with the location-event model.

A natural question concerns the usefulness of the valid inequalities added in the strong formulation. Though the values z_{LP} of the LP relaxations are not reported in the Table, the average values of the gap $\frac{z_{LP}-z_I}{Z_{LP}} * 100\%$ were calculated, where z_I is the optimal value. For $T = 60$ and $T = 120$ the gaps with the weak formulation were 28 and 25% respectively and with the strong formulation 27 and 24%, so the bounds are hardly improved. However, the results in the Table show that the valid inequalities lead to a significant improvement in the total run times.

Table 3 presents the computational results obtained with the vehicle-event model. Again, we consider two formulations, the weak vehicle-event formulation, given by the inequalities (42) – (66) without valid inequalities, and the strong vehicle-event formulation in which the inequalities (67) – (76) are added. The number of visits is determined as described in Section 5.1 by solving the corresponding maximization problem for 3 seconds. Columns "BCw" and "BCs" report the results obtained with the branch-and-cut algorithm with the default options, using the weak and strong formulations, respectively. A time limit of one hour is imposed. Columns "ALG1" and "ALG2" report the results with Algorithm 1 (with $\beta = 3600$ sec.) and Algorithm 2 (with $\beta_1 = \beta_2 = 3600$ sec.), respectively. Both algorithms are based on the strong vehicle-event formulation. Columns "ALG1+2" present the results obtained by combining Algorithms 1 and 2. Specifically, in Step 2 of Algorithm 2, the strong formulation with the additional inequalities (77) is solved using Algorithm 1. Except for the two instances using the BCw approach (whose times are given in bold), all the instances are solved to optimality or infeasibility is verified, so only the running times are presented.

Table 3: Running times using the vehicle-event model.

		T = 60					T = 120				
Inst.	F	BCw	BCs	ALG1	ALG2	ALG1+2	BCw	BCs	ALG1	ALG2	ALG1+2
A1	0	5.1	4.9	4.5	1.4	3.1	113	60.7	62.5	9.8	17
	0.2	7	6.8	8.1	1.6	4.3	63.5	54.4	59.1	10.9	11.4
A2	0	5.9	5.1	4.3	1.8	2.6	150.6	104	134.5	31.8	24.5
	0.2	6.2	10.3	3.6	1.1	3	128.9	103.3	144.2	32.7	27.5
B1	0	2.9	3.8	2	0.1	0.9	9.6	10.5	8.4	2.5	5.3
	0.2	3.5	3.7	1.8	0.2	0.4	13.4	7.3	10.3	2.2	3.9
B2	0	3.7	3.6	2.5	0	0.6	4.9	5.8	7.3	1.5	2.5
	0.2	3.9	3.3	2.4	0	0.3	4.2	5.1	5.3	0.8	1
C1	0	4.1	6.3	2.5	1.6	2.7	43.9	35.1	28.7	12.7	9.8
	0.2	4.9	5.7	3.3	1.4	3.1	18.1	9	9.1	4	6.5
C2	0	4.4	3.9	1.3	1.3	3.1	40.4	40.6	30.5	22.6	28.4
	0.2	5	5.4	3.7	1	3.2	36.4	28.5	19.1	9.9	27.8
D1	0	7.2	7.2	6.1	4.5	5.3	468.1	212.2	278.6	174	246.7
	0.2	8.5	7.2	7.3	3.3	5.5	517.2	435.7	465.2	134.3	267.8
D2	0	9.6	7.8	7.6	2.8	6.9	494.1	95.6	103.8	262.2	206.5
	0.2	9	7.8	8.3	3.7	6.7	636.9	99.3	171.1	93.7	504.4
E1	0	16.4	12.9	8.5	7.3	5.2	1171.7	742.1	125.9	401.1	109.6
	0.2	13.8	11.1	8.5	5.9	7.1	1559.2	629.5	163	738.7	197.2
E2	0	25.1	19.9	8.1	7.6	7	3510.3	1330.8	326.2	462.6	243.1
	0.2	34.8	23.1	10.2	11.6	7	2097.9	1651.3	442.3	1255.4	225.2
F1	0	4	5.7	3.7	2.1	3.2	13.8	10.6	11.7	7.1	8.7
	0.2	4.8	4.6	3.6	0.8	2.8	5.6	6.6	5.6	1.9	4.1
F2	0	6.1	5.3	4.3	1.4	2.7	2.3	0.9	0.9	0.6	0.6
	0.2	4.7	5.5	6	1.1	3.1	2.6	1	0.9	1	0.8
G1	0	5.9	3.3	2.7	0.3	2	2290.2	56	54	45.7	38.8
	0.2	7.2	8.8	5	3.2	2.7	175.4	18.8	18.5	13.1	18.4
G2	0	6.3	2.9	2.5	0.1	2.7	**3602.6**	528.7	209.8	756.1	141.6
	0.2	6.6	4.9	5.6	1.8	4.5	**3602.7**	697.4	243.4	496	158.9
Average		8.1	7.2	2.5	4.9	3.6	742.1	249.3	178.0	112.1	90.6

Clearly, comparing Tables 2 and 3, we observe that the solution procedures based on the vehicle-event formulations outperform the corresponding procedures based on the location-event formulations. In particular, this is true for the infeasible instances. We can see that the branch-and-cut on the location-event strong formulation fails to prove infeasibility on instances C2 and the running times for proving infeasibility on B1 are quite large when compared with the corresponding times obtained with the same approach using the vehicle-event strong formulation.

Considering Table 3, we see that when $T = 60$ all five approaches work. For $T = 120$ we observe that the running times using the model with inequalities BCs are, on average, around one third of the times for the model without inequalities BCw (note that for the two unsolved instances with the weak formulation we consider the truncated time). Algorithm 1 leads to running times that are between half and one third of the corresponding running times using branch-and-cut with default options. The running times of the $\underline{m}$ subproblems solved in Steps 4-6 of Algorithm 2 are negligible for $T = 60$ and $T = 120$. Hence, by comparing the times in columns ALG2 with those in columns BCs we see the impact of adding inequalities (77) to the strong formulation. For $T = 120$, these inequalities allow us to reduce the average running times of the strong formulation by half. For all the feasible instances the optimal solutions satisfy Assumption 1.

To help in visualizing the results, Figure 7 presents the boxplots of the running times obtained with the five approaches (indicated on the horizontal axis), where each box is limited by the lower and upper quartiles of the corresponding running time data.

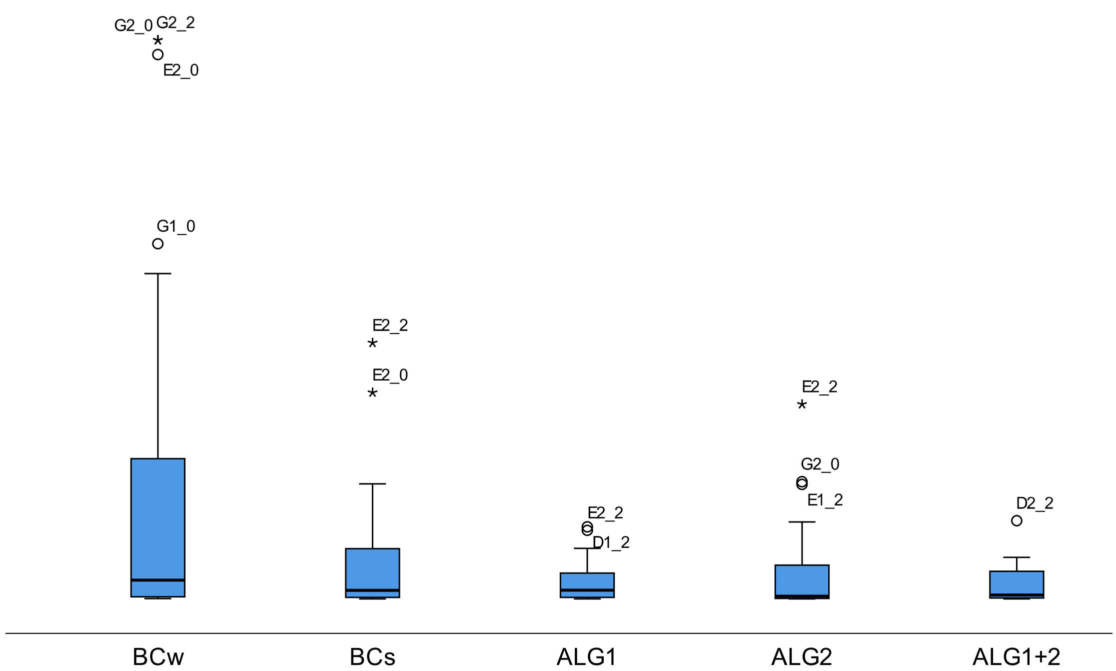


Figure 7: Boxplots for the running times (on the vertical axis) using the vehicle event model and $T = 120$.

The boxplots show that the most significant differences between the "BCs" and the "ALG1" approaches occur in the right tail, that is, for the harder instances, since the median times, indicating the maximum time needed to solve half of the instances, are similar. The approach "ALG1+2" has the lowest average time (from the table), a small median time, and is the one that presents the smallest number of outliers, meaning that it is the most effective approach to control the time of the hardest instances.

Regarding the linear relaxation of the vehicle-event formulation (results not reported in the table), for $T = 60$ the average gap ($\frac{z_{LP} - z_I}{Z_{LP}} * 100\%$) is 48% with the weak and 25% with the strong formulation. As for the location-event formulation, the valid inequalities are not able to provide tight bounds but lead to a significant improvement in the total run times. Comparing the strong location-event and strong vehicle-event formulations, neither formulation leads to an uniformly better linear relaxation bound than the other.

As the size of the vehicle-event model varies linearly as function of the upper bound on the number of visits, we expect the running times to increase with the increase of this parameter. Above, we motivated the choice of using the linear relaxation $\overline{w}$ to determine the value of this parameter. Additional tests were conducted with the vehicle-event model for $T = 120$, in which the upper bound on the number of visits was determined using the formula (81). As expected, the results are worse than those presented in Table 3. Using model BCw, three instances are not solved to optimality within the time limit of 1 hour, and the running times are on average 14% higher than those presented in Table 3. Using model BCs, two instances are not solved to optimality and the running times are on average 135.5% higher than those in Table 3. For algorithms ALG1, ALG2, and ALG1+2, all the instances were solved to optimality and the running times were 126.1%, 31.7% and 24% higher than the corresponding values presented in Table 3. These results show clearly the importance of using tight bounds on the number of visits in defining the vehicle-event model. Further tests on the impact of this parameter on the vehicle-event model are discussed in the next section.

6.2 Large size instances

In this section, we test the vehicle-event model on larger instances with $T = 180$ and $T = 240$. For these tests, we consider only the case $F = 0$ given that the ending inventory level is probably less relevant after such a long period. Additionally, the instances are adapted since many of them are infeasible for such long time horizons. This may be due to the fact that several instances are unbalanced since the aggregate supply rate is different from the aggregate demand rate. The instances are modified by multiplying both the supply and demand rates by a parameter ρ $(0 < \rho \leq 1)$.

For these large instances, we make a few changes to Algorithm 1 resulting from the observation that the two branches can result in two significantly unbalanced subproblems. Moreover, the number of visits estimated by the procedure described in Section 5.1 can be quite large.

To motivate these changes we present bounds based on the number of visits and linear relaxations in Table 4. Column "ρ" gives the value of ρ, column "Obj" gives the corresponding optimal objective function value, column V gives the number of visits in the optimal solution, and columns $\bar{w}$ and $\underline{w}$ give the number of visits obtained from the linear relaxation of (82), $\bar{w}$ obtained by rounding down the value with the maximization problem and $\underline{w}$ obtained by rounding up the value with the minimization problem. Columns $\alpha\underline{w}$ for $\alpha \in \{1, 1.25, 1.5, 1.75, 2\}$ give the bound obtained with the linear relaxation of the vehicle-event model tightened with all valid inequalities (strong formulation) and with the additional constraint

$$\sum_{i \in N} \sum_{m=1}^{\overline{\mu}_i} y_{im} \geq \alpha \underline{w} \tag{83}$$

imposing a minimum number of visits. Columns LB1 and LB2 give the linear relaxation of the two subproblems obtained in Algorithm 1 with $\bar{k} = \bar{w}$, respectively. Columns LB1A and LB2A give the corresponding lower bounds obtained from running the branch-and-bound for 30 seconds.

We observe that the bounding constraint (83) has a significant impact on the objective function value of the linear relaxation of the strong formulation with this additional constraint. We also observe that LB1 and LB2 differ significantly for several instances, which may indicate that the size of the branches may be unbalanced. Additionally, LB1A is significantly greater than LB1 (observe that the average of LB1A is calculated only over the feasible instances, which explains that the resulting average is lower than the average of LB1) while LB2A is, in general, slightly greater than LB2, which indicates that it is worth spending 30 seconds to improve the bounds on the first branch, but not on the second.

Table 4: Bounds based on the number of visits and linear relaxations.

Inst.	ρ	Opt	V	$\bar{w}$	$\underline{w}$	$1\underline{w}$	$1.25\underline{w}$	$1.5\underline{w}$	$1.75\underline{w}$	$2\underline{w}$	LB1	LB1A	LB2	LB2A
A1	0.7	829.2	17	25	11	646.5	655.2	735.5	802.0	920.0	646.5	681.6	1041.2	1058.7
A2	0.8	933.2	20	50	10	553.5	570.9	649.6	710.3	813.6	553.5	678.5	1020.5	1028.9
B1	0.6	811.4	20	47	12	604.6	663.3	736.1	813.7	902.9	604.6	669.0	924.8	942.1
B2	1	1126.1	21	35	21	1119.4	1217.3	1315.3	INF	INF	1119.4	INF	1119.4	1126.1
C1	1	1316.9	19	38	19	1257.5	1333.0	1461.9	1633.8	1813.4	1257.5	1316.9	1262.5	1321.9
C2	1	1452.4	22	37	21	1338.1	1426.7	1574.1	1751.7	INF	1338.1	INF	1338.1	1361.2
D1	0.7	705.7	16	39	11	451.9	518.7	628.7	741.8	852.4	451.9	523.7	814.0	819.1
D2	0.6	610.2	13	40	9	386.0	439.2	511.3	585.6	697.2	386.0	447.4	807.8	814.7
E1	0.9	727.7	18	69	10	419.4	444.1	504.8	546.0	607.5	419.4	459.7	938.5	949.1
E2	0.8	753.5	18	71	9	365.9	404.0	444.9	485.9	547.1	365.9	435.3	940.2	951.4
F1	0.6	609.6	10	37	10	575.4	635.9	727.5	793.7	903.4	575.4	609.6	903.4	907.6
F2	0.55	713.3	12	35	12	672.1	763.1	855.2	968.5	1078.1	672.1	713.3	886.1	910.1
G1	0.75	1030.8	13	31	13	940.3	975.8	1074.3	1192.8	1364.2	940.3	999.6	1000.0	1000.0
G2	0.9	947.7	14	36	11	675.4	719.4	834.4	964.5	1085.4	675.4	710.9	940.1	943.9
Average		897.7	16.6	22.6	12.8	714.7	769.1	861.0	922.3	891.2	714.7	687.1	995.5	1009.6

Based on these observations we propose a refinement of Algorithm 1, denoted ALG1r, for the larger instances. ALG1r deviates from Algorithm 1 only in the rule used to split the problem into two subproblems. ALG1r uses the information provided by LB1A and LB2 to obtain more balanced subproblems. First, we split the problem into two subproblems, by multiplying $\bar{k}$ by r as in Steps 2 and 5 of Algorithm 1 (Note that there $r = 0.5$).

Using the bounds LB1A and LB2, the parameter r is chosen as follows: If the first subproblem is infeasible ($LB1A = +\infty$), we take $r = 2/3$. Otherwise, set $r = 0.6$ if $LB2 < 1.1LB1A$; $r = 0.45$ if $1.1LB1A \leq LB2 \leq 1.5LB1A$; $r = 0.4$ if $1.5LB1A < LB2 < 2LB1A$ and $r = 0.3$ if $LB2 \geq 2LB1A$. Notice that for small size instances the time spent in these adjustments may not compensate for the subsequent gains.

Table 5: Computational results for large size instances with $T = 180$ and $T = 240$.

		BCs				ALG1r				ALG1r+2			
		T=180		T=240		T=180		T=240		T=180		T=240	
Inst.	rho	Time	UB	Time	UB	Time	UB	Time	UB	Time	UB	Time	UB
A1	0.7	209	829.2	7203	**1133.9**	127	829.2	3631	**1129.5**	112	829.2	3634	**1129.5**
A2	0.8	1279	933.2	7203	**1385.4**	210	933.2	7232	**1309.5**	63	933.2	3924	**1299.5**
B1	0.6	9	669	702	1099.7	3	669	62	1099.7	4	669	80	1099.7
B2	1	7	1126.1	10	1594.4	4	1126.1	4	1594.4	2	1126.1	5	1594.4
C1	1	66	1316.9	636	1846	15	1316.9	201	1846.0	28	1316.9	256	1846.0
C2	1	119	1452.4	4157	2045.1	73	1452.4	4092	**2045.1**	76	1452.4	2619	2045.1
D1	0.7	1805	705.7	7203	**1103.2**	418	705.7	7236	**1167.0**	311	705.7	7234	**1155.0**
D2	0.6	529	601.8	7203	**901.3**	110	601.8	3636	**900.5**	102	601.8	3634	**888.0**
E1	0.9	7202	**727.7**	7203	**1003.6**	621	727.7	3644	**1061.8**	850	727.7	3637	**1104.2**
E2	0.8	7203	**760.1**	7203	**1208.7**	3637	**761.5**	7244	**1181.1**	2819	753.5	7237	**1179.7**
F1	0.6	14	609.6	22	1052.7	9	609.6	13	1052.7	7	609.6	20	1052.7
F2	0.55	19	713.3	204	1282.5	9	713.3	52	1282.5	6	713.3	41	1282.5
G1	0.75	475	1030.8	7203	**1524.3**	119	1030.8	783	1508.3	119	1030.8	213	1508.3
G2	0.9	7203	**947.7**	7203	**1415.8**	699	947.7	7241	**1384.7**	945	947.7	7240	**1476.0**
Average		1867	887.4	4525	1328.3	432	887.5	3219	1325.9	390	886.9	2841	1332.9

In Table 5 we compare the standard branch-and-cut on the strong formulation (approach BCs) to the adjusted Algorithm 1 (approach ALG1r) and the approach obtained by combining Algorithms ALG1r and 2 (approach ALG1r+2) where in Step 2 of Algorithm 2, the strong formulation with the additional inequalities (77) is solved using algorithm ALG1r. In addition, Steps 4-5 of Algorithm 2 are omitted when in Step 2 the problem is not solved to optimality. Notice that, in general, the $\underline{m}$ subproblems are solved mainly to prove optimality, since in most practical situations we expect that Assumption 1 is satisfied. Columns "UB" give the value of the best feasible solution found. When the number is in bold it means the algorithm stops without proving optimality.

Table 5 shows that by using the BCs approach for $T = 180$, only three instances are not solved to optimality (E1, E2 and G2) and for $T = 240$ eight instances are not solved. Using the ALG1r+2 approach all instances for $T = 180$ and seven instances for $T = 240$ are solved to optimality. We can also observe that the ALG1r+2 approach is much faster than the BCs approach for all the solved instances. Comparing the ALG1r and ALG1r+2 approaches, we observe that ALG1r+2 is on average faster. ALG1r+2 solves to optimality two more instances than ALG1r ($E2$ for $T = 180$ and $C2$ for $T = 240$). On the unsolved instances it provides a better bound than ALG1r on 5 instances and a worse upper bound on two instances (E1 and G2 for $T = 240$).

7 Conclusions and future research

A general single-vehicle inventory routing problem (IRP) with pickups and deliveries is studied. Compared to the majority of land-based IRPs considered in the literature, where the planning horizon is partitioned into periods and it is assumed that the routes are made within a time period, the time is here considered as continuous due to constant supply and demand rates at the supply and demand locations. The quantity picked up or delivered at a location depends on the storage capacity at the location, the inventory level at the visit time at the location, the quantity on the vehicle as well as the capacity of the vehicle. This type of inventory routing problems is particularly complex due to the high degree of freedom concerning the variable number of visits to each location during the time horizon and the variable quantity picked up and delivered at each visit. Deriving strong formulations for the problem is a challenge. In this paper we have presented two improved models. One is defined on an extended graph in which the nodes correspond to visits to locations (location-event model) and the other in which the nodes correspond to vehicle visits (vehicle-event model). The size of both models depends on the number of events considered. We propose a simple method to bound the number of nodes in each extended graph. Additionally, we propose new valid inequalities to tighten the two models. For each model, a new exact algorithm (Algorithm 1) combining all contributions is proposed to solve the inventory routing problem.

Computational tests based on a set of instances from a maritime inventory routing problem are presented showing that the branch-and-cut algorithm based on the vehicle-event model performs better than the location-event model. The results also show that the method to bound the number of events as well as the inequalities is important to reduce the running time. In addition, the vehicle-event model outperforms, in general, the location-event model when it comes to verifying infeasibility. Using the vehicle-event model our most effective algorithm solved to optimality all instances with a horizon of 180 days and half the instances with a 240 day horizon.

We have developed valid inequalities for instances satisfying the assumption that the vehicle cannot return twice to a demand/supply location without visiting a supply/demand location in between. When this assumption does not hold, these inequalities are used to partition the set of feasible solutions (Algorithm 2). Running Algorithm 1 on the initial problem with these inequalities can be seen as a heuristic. As all the solved instances satisfy this assumption, this heuristic potentially generates the optimal solution to all the instances. In addition, Algorithm 2 is valid whatever values are selected for the split factor (vehicle visit number), so for other classes of instances another choice may well be appropriate.

As to future research it would be interesting to investigate further the polyhedral structure of the two proposed models, even in the restricted Hamiltonian case in which the number of visits to each location is fixed. Extending the models to deal with multiple vehicles is perhaps the major challenge. It might also be of interest to examine other problems in which some crucial parameter, in our case the total number of visits, can be used to speed up the solution process and investigate related branching schemes based on partitions of the set of possible values for that parameter.

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