Inhomogeneous P- and S-wavefields radiated into isotropic elastic solids

Erlend Magnus Viggen[1], Håvard Kjellmo Arnestad[2]

[1] Centre for Innovative Ultrasound Solutions, Department of Circulation and Medical Imaging, Norwegian University of Science and Technology
[2] Department of Informatics, University of Oslo
Contact email: erlend.viggen@ntnu.no

Abstract

A vibrating surface in contact with a solid material will generate P- and S-waves in the solid. When the surface vibration is spatially *attenuated*, we must take into account that the generated waves are always *inhomogeneous*. In an isotropic elastic solid, such inhomogeneous waves are attenuated perpendicularly to their direction of propagation. When the surface vibration's phase speed is lower than the P- and/or S-waves' speed of sound, the inhomogeneity affects the radiation of P- and S-waves in a major but relatively poorly understood way. For a better understanding, finding the total radiated intensity of the two inhomogeneous waves is key. Our work takes a step towards such an understanding by deriving analytical expressions for the velocity, strain, stress, and intensity fields of arbitrarily inhomogeneous P- and S-waves. Furthermore, we investigate whether the total radiated intensity can be found as the sum of the intensities of the individual P- and S-waves. We find that this is only possible when the surface vibration is *unattenuated*; for *attenuated* vibrations, the total radiated intensity should be calculated numerically.

1 Introduction

The most basic type of wave in acoustics is a *homogeneous* plane wave. For a pressure wave in a fluid, it can be expressed as

$$p(\boldsymbol{r}, t) = p_0 \, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \, . \tag{1}$$

Here, $\boldsymbol{k}$ is the real-valued wavenumber vector, $\boldsymbol{r} = (x, y, z)$ is the spatial coordinate, ω is angular frequency, t is time, and the subscripted zero indicates amplitude — the value at $\boldsymbol{r} = \boldsymbol{0}$, $t = 0$. These waves propagate in a direction given by the wavenumber direction $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$. Their phase $(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, and therefore also their pressure value $p(\boldsymbol{r}, t)$, is constant over any plane perpendicular to $\hat{\boldsymbol{k}}$.

Inhomogeneous plane waves is a lesser-known [1] type of plane wave. In such a plane wave, the wavenumber vector $\boldsymbol{k}$ is no longer real-valued, but *complex*:

$$\boldsymbol{k} = \boldsymbol{k}_\mathrm{r} + \mathrm{i}\boldsymbol{k}_\mathrm{i} \in \mathbb{C}^3, \qquad \text{where} \qquad \boldsymbol{k}_\mathrm{r} \in \mathbb{R}^3, \quad \boldsymbol{k}_\mathrm{i} \in \mathbb{R}^3. \tag{2}$$

Here, the subscripted r and i denote the real part and the imaginary part, respectively. Inserted into (1), this leads to

$$p(\boldsymbol{r}, t) = p_0 \, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_\mathrm{r}\cdot\boldsymbol{r}-\omega t)} \, \mathrm{e}^{-\boldsymbol{k}_\mathrm{i}\cdot\boldsymbol{r}} \, . \tag{3}$$

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Hence, the real wavenumber component $\boldsymbol{k}_\mathrm{r}$ represents a *propagation vector*, while the imaginary component $\boldsymbol{k}_\mathrm{i}$ represents an *attenuation vector*: While the wave moves in the $\hat{\boldsymbol{k}}_\mathrm{r}$ direction and still has planes of constant phase perpendicular to $\hat{\boldsymbol{k}}_\mathrm{r}$, the pressure value has an exponential decay in the $\hat{\boldsymbol{k}}_\mathrm{i}$ direction.

The attenuation vector $\boldsymbol{k}_\mathrm{i}$ is often split into two parts [1]: A damping vector parallel with $\hat{\boldsymbol{k}}_\mathrm{r}$, representing the effect of losses in the medium, and an inhomogeneity vector perpendicular to $\hat{\boldsymbol{k}}_\mathrm{r}$, representing an exponential decay along the wavefronts. For a lossless medium there is no damping vector, and $\boldsymbol{k}_\mathrm{r} \perp \boldsymbol{k}_\mathrm{i}$. This can also be proven by inserting (3) into a lossless pressure wave equation, which at the same time can tell us that inhomogeneous waves propagate at a speed

$$c = \frac{\omega}{|\boldsymbol{k}_\mathrm{r}|} = \frac{c_0}{\sqrt{1 + (|\boldsymbol{k}_\mathrm{i}|/k)^2}}. \tag{4}$$

Here, $k = |\boldsymbol{k}| = \omega / c_0$ is the wavenumber magnitude, and c_0 is the medium's speed of sound, which is the speed at which *homogeneous* waves propagate.

The main motivation for working with inhomogeneous waves is their prevalence when dealing with lossy materials such as thermoviscous liquids [2] and viscoelastic solids [1, 3]. Our work, however, is motivated by understanding how leaky Lamb and Rayleigh waves moving at a speed c_v can radiate energy into an adjacent medium in the *subsonic* domain, where $c_v < c_0$. While several works demonstrate this phenomenon [4–7], the literature has widely regarded it as impossible; many references state that this can only occur in the *supersonic* domain, where $c_v > c_0$ [8–13].

However, previous and current work on subsonic radiation [5, 14, 15] has found that properly considering the *inhomogeneity* of radiated waves is the key to understanding how subsonic radiation is possible. This inhomogeneity occurs even if all materials are lossless — it is a consequence of the attenuation of the leaky Lamb or Rayleigh wave, which occurs due to the loss of power into the radiated wave.

The problem can be simplified by considering a surface in the x–z plane on which an attenuated surface vibration propagates with a complex wavenumber

$$k_x = k_{x\mathrm{r}} + \mathrm{i}k_{x\mathrm{i}} \in \mathbb{C}, \qquad \text{where} \qquad k_{x\mathrm{r}} \in \mathbb{R}_{\geq 0}, \quad k_{x\mathrm{i}} \in \mathbb{R}_{\geq 0}, \tag{5}$$

giving a vibrational surface velocity

$$v(x, y = 0, t) = v_0\, \mathrm{e}^{\mathrm{i}(k_x x - \omega t)} = v_0\, \mathrm{e}^{\mathrm{i}(k_{x\mathrm{r}} x - \omega t)}\, \mathrm{e}^{-k_{x\mathrm{i}} x}. \tag{6}$$

Whenever $k_{x\mathrm{i}} > 0$, an inhomogeneous ($\boldsymbol{k}_\mathrm{i} \neq \boldsymbol{0}$) but non-evanescent ($k_{y\mathrm{r}} > 0$) wave is radiated into the fluid [14, 15].

The power associated with a surface vibration can be quantified by its power flow amplitude P_{x0}. For a Lamb wave in a plate, for example, the power flow is the integral over the plate's cross-section of the intensity component parallel to the plate. The power radiated into the fluid can be equated with a loss in power flow per unit length to get an implicit equation for the vibrational attenuation [14, 15]:

$$k_{x\mathrm{i}} = \frac{I_{y0}(k_{x\mathrm{r}}, k_{x\mathrm{i}})}{2P_{x0}(k_{x\mathrm{r}}, k_{x\mathrm{i}})}. \tag{7}$$

Here, I_{y0} is the intensity radiated into the fluid per unit length along the surface. Both the radiated intensity and the power flow can be functions of both $k_{x\mathrm{r}}$ and $k_{x\mathrm{i}}$.

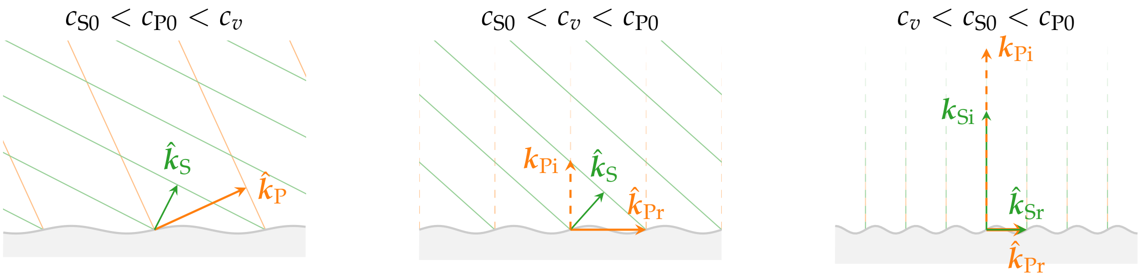

Figure 1: Three domains of P-wave (orange) and S-wave (green) radiation into a solid from an *unattenuated* surface vibration. **Left:** P-supersonic and S-supersonic, where both types of waves radiate into the solid. **Middle:** P-subsonic and S-supersonic, where the P-wave becomes evanescent. **Right:** P-subsonic and S-subsonic, where both waves are evanescent. Unit vectors are indicated by hats and drawn scaled to one wavelength.

For radiation into a fluid, the intensity is relatively straightforward to calculate. We show in an upcoming article [15] (partly summarised in an extended abstract [14]) that (7) predicts subsonic radiation into a fluid in a qualitatively correct way, as well as validate it against leaky Lamb waves.

While this model was derived for radiation into a fluid, it can be straightforwardly repurposed for radiation into a solid by calculating I_{y0} for solids. However, radiation into solids is much less straightforward than radiation into fluids. While fluids only support longitudinal pressure waves, solids support two types of waves: Longitudinal P-waves, with a speed of sound c_{P0}, and transversal S-waves, with a speed of sound $c_{S0} < c_{P0}$. Hence, for a surface vibration radiating into a solid, *two* inhomogeneous wavefields must be determined in order to calculate the total radiated intensity I_{y0}.

Furthermore, in a fluid there is only one coincidence point $c_v = c_0$ that separates two domains of interest: The supersonic domain ($c_v > c_0$) and the subsonic domain ($c_v < c_0$). Solids, with their two types of waves, have *two* coincidence points — one for P-waves ($c_v = c_{P0}$) and one for S-waves ($c_v = c_{S0}$). Hence, solids have *three* domains of interest, also shown in Figure 1:

- P-supersonic and S-supersonic, where $c_{S0} < c_{P0} < c_v$,
- P-subsonic but S-supersonic, where $c_{S0} < c_v < c_{P0}$, and
- P-subsonic and S-subsonic, where $c_v < c_{S0} < c_{P0}$.

In this article, we investigate how to determine the total intensity radiated from a surface vibration into the simplest type of solid, which is isotropic (behaving independently of orientation) and elastic (lossless). In Section 2, we determine the radiated waves' velocity fields and sketch how to determine the strain, stress, and intensity fields. In Section 3, we derive analytical expressions for these fields in a coordinate system aligned with the wave. In Section 4, we investigate in which cases the total radiated intensity I_{y0} can be calculated as the sum $I_{Py0} + I_{Sy0}$ of the P- and S-wave intensities. Section 5 summarises our findings and concludes.

2 Radiated P- and S-wavefields

From the momentum conservation equation of an elastic and isotropic solid, separate wave equations for P- and S-waves can be derived [16–18]. Expressed in terms of the P-and S-waves' particle velocities $\boldsymbol{v}_\mathrm{P}$ and $\boldsymbol{v}_\mathrm{S}$, these wave equations are

$$\frac{1}{c_\mathrm{P0}^2}\frac{\partial^2 \boldsymbol{v}_\mathrm{P}}{\partial t^2} - \nabla^2 \boldsymbol{v}_\mathrm{P} = 0, \qquad \text{with} \qquad c_\mathrm{P0} = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad \text{and} \qquad \nabla \times \boldsymbol{v}_\mathrm{P} = \mathbf{0}, \tag{8a}$$

$$\frac{1}{c_\mathrm{S0}^2}\frac{\partial^2 \boldsymbol{v}_\mathrm{S}}{\partial t^2} - \nabla^2 \boldsymbol{v}_\mathrm{S} = 0 \qquad \text{with} \qquad c_\mathrm{S0} = \sqrt{\frac{\mu}{\rho}} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{v}_\mathrm{S} = 0. \tag{8b}$$

Here, the speeds of sound c_P0 and c_S0 are expressed through the density ρ and Lame's first (λ) and second (μ) parameters, the latter better known as the shear modulus. Each velocity field has a condition applied to it: $\boldsymbol{v}_\mathrm{P}$ is irrotational, and $\boldsymbol{v}_\mathrm{S}$ is divergence-free.

Both wave equations have inhomogeneous plane wave solutions like

$$\boldsymbol{v}_* = \begin{bmatrix} v_{*x0} \\ v_{*y0} \end{bmatrix} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_*\cdot\boldsymbol{r}-\omega t)} = \begin{bmatrix} v_{*x0} \\ v_{*y0} \end{bmatrix} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_{*\mathrm{r}}\cdot\boldsymbol{r}-\omega t)}\, \mathrm{e}^{-\boldsymbol{k}_{*\mathrm{i}}\cdot\boldsymbol{r}}, \tag{9}$$

where the asterisks represent either P or S. (From here on, we treat the problem as two-dimensional, as the wave propagates in the x–y plane and is z invariant.) If the waves are radiated from a surface vibration like (6), the wavenumbers' x components must match the surface vibration's wavenumber, i.e., $k_{*x} = k_x$. The wavenumbers' y components k_{*y} can be calculated from the relation $(\omega/c_{*0})^2 = k_*^2 = k_x^2 + k_{*y}^2$,

$$k_{*y} = k_*\sqrt{1 - \left(\frac{k_x}{k_*}\right)^2} = k_*\sqrt{\left(1 + \left[\frac{k_{x\mathrm{i}}}{k_*}\right]^2 - \left[\frac{k_{x\mathrm{r}}}{k_*}\right]^2\right) - \mathrm{i}\frac{2k_{x\mathrm{r}}k_{x\mathrm{i}}}{k_*^2}} = k_{*y\mathrm{r}} + \mathrm{i}k_{*y\mathrm{i}}. \tag{10}$$

Here, we have chosen the positive sign for the square root, which corresponds to a radiated wave. (The negative sign corresponds to an incoming wave.)

As $\hat{\boldsymbol{k}}_{*\mathrm{r}}$ is the radiation direction, we can use it to calculate the radiation angle. This angle can be expressed in two ways:

$$\theta_* = \arctan\left(\frac{k_{x\mathrm{r}}}{k_{*y\mathrm{r}}}\right), \qquad \alpha_* = \arctan\left(\frac{k_{*y\mathrm{r}}}{k_{x\mathrm{r}}}\right), \tag{11}$$

where θ_* is the radiation angle with respect to the surface normal, and α_* is the radiation angle with respect to the surface.

The amplitudes $\boldsymbol{v}_{*0}$ of the two waves are determined by matching their combined velocities' x and y components with those of the surface vibration from (6) at the surface, as well as relating the x and y components of each wave type in (9) using the conditions in (8). This leads to a set of equations

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -k_{\mathrm{P}y} & k_x & 0 & 0 \\ 0 & 0 & k_x & k_{\mathrm{S}y} \end{bmatrix} \begin{bmatrix} v_{\mathrm{P}x0} \\ v_{\mathrm{P}y0} \\ v_{\mathrm{S}x0} \\ v_{\mathrm{S}y0} \end{bmatrix} = \begin{bmatrix} v_{x0} \\ v_{y0} \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

that can be easily inverted with the help of a computer algebra system, giving

$$\boldsymbol{v}_\mathrm{P0} = \begin{bmatrix} k_x \\ k_{\mathrm{P}y} \end{bmatrix} \frac{k_x v_{x0} + k_{\mathrm{S}y} v_{y0}}{k_x^2 + k_{\mathrm{P}y}k_{\mathrm{S}y}}, \qquad \boldsymbol{v}_\mathrm{S0} = \begin{bmatrix} -k_{\mathrm{S}y} \\ k_x \end{bmatrix} \frac{-k_{\mathrm{P}y} v_{x0} + k_x v_{y0}}{k_x^2 + k_{\mathrm{P}y}k_{\mathrm{S}y}}. \tag{13}$$

Thus, the P and S velocity fields $\boldsymbol{v}_*$ are fully determined. From them, we can calculate the P and S strain tensors as

$$S_{*ij} = \frac{1}{2}\left(\frac{\partial u_{*i}}{\partial x_j} + \frac{\partial u_{*j}}{\partial x_i}\right) = -\frac{1}{2\mathrm{i}\omega}\left(\frac{\partial v_{*i}}{\partial x_j} + \frac{\partial v_{*j}}{\partial x_i}\right) = -\frac{1}{2\omega}\left(k_{*j}v_{*i} + k_{*i}v_{*j}\right). \tag{14}$$

The time-harmonic nature of the wavefields means that the displacement $\boldsymbol{u}_*$ is related to the velocity $\boldsymbol{v}_*$ as $\boldsymbol{v}_* = \partial \boldsymbol{u}_*/\partial t = -\mathrm{i}\omega \boldsymbol{u}_*$. We also use the Einstein summation convention, where we use i, j, and k as generic Cartesian indices, and repeating one of these generic indices in a term implies a summation over all possible values of that index.

From the P and S strain tensors, we can calculate the P and S stress tensors for an isotropic elastic solid as

$$\sigma_{*ij} = \lambda\delta_{ij}S_{*kk} + 2\mu S_{*ij} \overset{(8)}{=} \left(\rho c_{\mathrm{P0}}^2 - 2\rho c_{\mathrm{S0}}^2\right)\delta_{ij}S_{*kk} + 2\rho c_{\mathrm{S0}}^2 S_{*ij}. \tag{15}$$

Here, we express the stresses through the more acoustically relevant speeds of sound c_{P0} and c_{S0} instead of the more traditional Lamé's first and second parameters λ and μ.

Then, we can use the P and S velocity fields and stress tensors to calculate the time-averaged intensities for P- and S-waves as

$$I_{*i} = \mathrm{Re}\left\{-\frac{\sigma_{*ij}\overline{v_{*j}}}{2}\right\}. \tag{16}$$

The bar over the velocity denotes complex conjugation.

The fields of velocity, strain, and stress are all linear quantities. Hence, the corresponding total fields can be found directly as the sum of the P and S components:

$$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{P}} + \boldsymbol{v}_{\mathrm{S}}, \qquad S_{ij} = S_{\mathrm{P}ij} + S_{\mathrm{S}ij}, \qquad \sigma_{ij} = \sigma_{\mathrm{P}ij} + \sigma_{\mathrm{S}ij}. \tag{17}$$

Intensity, however, is a *nonlinear* quantity, as it is calculated through products of stress and velocity. Hence, the total intensity becomes

$$\begin{aligned} I_i &= \mathrm{Re}\left\{-\frac{(\sigma_{\mathrm{P}ij} + \sigma_{\mathrm{S}ij})\overline{(v_{\mathrm{P}j} + v_{\mathrm{S}j})}}{2}\right\} = \mathrm{Re}\left\{-\frac{\sigma_{\mathrm{P}ij}\overline{v_{\mathrm{P}j}} + \sigma_{\mathrm{S}ij}\overline{v_{\mathrm{S}j}} + \sigma_{\mathrm{P}ij}\overline{v_{\mathrm{S}j}} + \sigma_{\mathrm{S}ij}\overline{v_{\mathrm{P}j}}}{2}\right\} \\ &= I_{\mathrm{P}i} + I_{\mathrm{S}i} + \mathrm{Re}\left\{-\frac{\sigma_{\mathrm{P}ij}\overline{v_{\mathrm{S}j}}}{2}\right\} + \mathrm{Re}\left\{-\frac{\sigma_{\mathrm{S}ij}\overline{v_{\mathrm{P}j}}}{2}\right\}. \end{aligned} \tag{18}$$

This would indicate that the total intensity I_i cannot simply be calculated as the sum $I_{\mathrm{P}i} + I_{\mathrm{S}i}$, due to the latter two cross-terms between the P and S wavefields. Even so, Brekhovskikh and Godin proved for fluid-to-solid reflection-transmission problems that a simple summation of the normal component of the P- and S-intensities actually correctly provides the normal component of the total intensity in the solid [10]. In their case, the two cross-terms must necessarily sum to zero. In Section 4, we investigate whether it also holds true for our case.

3 Axis-aligned inhomogeneous wavefields

In Section 2 we fully determined the velocity fields of the inhomogeneous P- and S-waves radiated from a surface vibration, and provided the equations required to calculate the P- and S-fields of strain, stress, and intensity. However, calculating analytical expressions

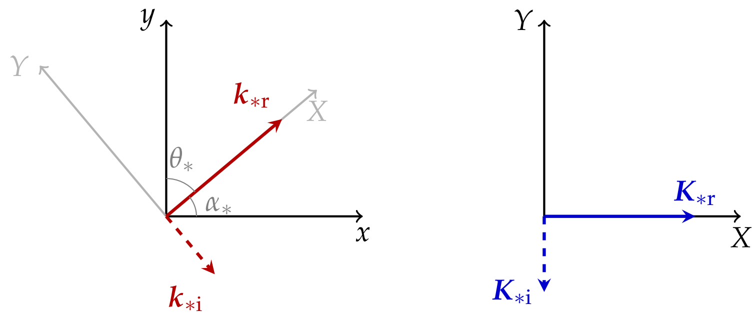

Figure 2: Real and imaginary components of the wavenumber $\boldsymbol{k}_*$ of a wave described in the x–y coordinate system and of the wavenumber $\boldsymbol{K}_*$ described in the wave-aligned X–Y coordinate system.

for the latter fields would be very cumbersome and would likely lead to cluttered expressions, considering that the radiated inhomogeneous waves can have any orientation $0 \leq \alpha_* \leq 2\pi/4$ in the x–y coordinate system.

Therefore, we take a similar approach to Poirée [2], defining a X–Y coordinate system that is aligned with the P- or S-wave in question. As Figure 2 shows, the wave propagates along the X axis and decays along the $\pm Y$ axis. In this coordinate system, the wavenumber $\boldsymbol{K}_*$ is

$$K_{*X} = K_{*X\mathrm{r}} + \mathrm{i}0, \quad K_{*Y} = 0 \pm \mathrm{i}|K_{*Y\mathrm{i}}|, \quad \text{where} \quad K_{*X\mathrm{r}} \in \mathbb{R}_{\geq 0}, \quad K_{*Y\mathrm{i}} \in \mathbb{R}. \tag{19}$$

Here, $\pm$ is the sign of the angle between $\boldsymbol{k}_{*\mathrm{r}}$ and $\boldsymbol{k}_{*\mathrm{i}}$, and specifies whether the decay occurs in the $+Y$ or $-Y$ direction. We can express the inhomogeneity magnitude $|K_{*Y\mathrm{i}}|$ more simply as $|K_{*Y}|$.

Expressing the wave in the X–Y coordinate system is useful in at least two circumstances. First, it lets us take a wave, typically expressed in terms of its velocity as in (9), from the x–y coordinate system and analytically determine its other fields, such as strain, stress, or intensity, in a more convenient coordinate system. Second, it lets us us construct an arbitrarily inhomogeneous wave in a convenient axis-aligned coordinate system, and then place it into the x–y coordinate system with the desired orientation. In the former case, we can determine the X–Y wavenumber vector $\boldsymbol{K}_*$ from the original x–y wavenumber vector $\boldsymbol{k}_*$ as

$$\boldsymbol{K}_* = \boldsymbol{R}^{-1}(\alpha_*)\boldsymbol{k}_*, \tag{20}$$

where $\boldsymbol{R}(\alpha_*)$ is the rotation matrix used to transform from the x–y system to the X–Y system, and $\boldsymbol{R}^{-1}(\alpha_*)$ the rotation matrix that performs the opposite transformation:

$$\boldsymbol{R}(\alpha_*) = \begin{bmatrix} \cos\alpha_* & -\sin\alpha_* \\ \sin\alpha_* & \cos\alpha_* \end{bmatrix}, \qquad \boldsymbol{R}^{-1}(\alpha_*) = \boldsymbol{R}^{\mathrm{T}}(\alpha_*) = \begin{bmatrix} \cos\alpha_* & \sin\alpha_* \\ -\sin\alpha_* & \cos\alpha_* \end{bmatrix}. \tag{21}$$

The inhomogeneous wave's speed c_* is always less or equal to the matching speed of sound c_{*0}:

$$c_* = \frac{\omega}{K_{*X}} = \frac{\omega}{\sqrt{K_*^2 + |K_{*Y}|^2}}, \qquad c_{*0} = \frac{\omega}{K_*}. \tag{22}$$

From these two equations, we can derive a useful identity that relates the inhomogeneity magnitude $|K_{*Y}|$ and the resulting change in wave speed c_*:

$$\frac{|K_{*Y}|^2}{\omega^2} = \frac{1}{c_*^2} - \frac{1}{c_{*0}^2} = \frac{1}{c_*^2}\left(1 - \frac{c_*^2}{c_{*0}^2}\right). \tag{23}$$

3.1 Aligned velocity fields

From (9), we already have an expression for the velocity field that can be easily expressed in the X–Y coordinate system as

$$\boldsymbol{v}_* = \begin{bmatrix} v_{*X0} \\ v_{*Y0} \end{bmatrix} \mathrm{e}^{\mathrm{i}(K_{*X}X - \omega t)}\, \mathrm{e}^{\mp|K_{*Y}|Y} . \tag{24}$$

If $\boldsymbol{v}'_{*0}$ is the velocity amplitude vector of an original wave in the x–y coordinate system, we can straightforwardly determine the velocity amplitude vector in (24) as $\boldsymbol{v}_{*0} = \boldsymbol{R}^{-1}(\alpha_*)\boldsymbol{v}'_{*0}$.

However, if we are constructing this wave from the ground up, it is useful to consider the relationship between the X and Y components of $\boldsymbol{v}_{*0}$. The component relationships for P- and S-waves can be found from the P- and S-wave conditions in (8), which give

$$\nabla \times \boldsymbol{v}_\mathrm{P} = \mathrm{i}\boldsymbol{K}_\mathrm{P} \times \boldsymbol{v}_\mathrm{P} = \boldsymbol{0} \quad \Rightarrow \quad K_{\mathrm{P}X}v_{\mathrm{P}Y} - K_{\mathrm{P}Y}v_{\mathrm{P}X} = 0 \quad \Rightarrow \quad v_{\mathrm{P}Y} = \frac{\pm\mathrm{i}|K_{\mathrm{P}Y}|}{K_{\mathrm{P}X}} v_{\mathrm{P}X}, \tag{25a}$$

$$\nabla \cdot \boldsymbol{v}_\mathrm{S} = \mathrm{i}\boldsymbol{K}_\mathrm{S} \cdot \boldsymbol{v}_\mathrm{S} = 0 \quad \Rightarrow \quad K_{\mathrm{S}X}v_{\mathrm{S}X} + K_{\mathrm{S}Y}v_{\mathrm{S}Y} = 0 \quad \Rightarrow \quad v_{\mathrm{S}X} = \frac{\mp\mathrm{i}|K_{\mathrm{S}Y}|}{K_{\mathrm{S}X}} v_{\mathrm{S}Y}. \tag{25b}$$

This shows the well-known fact that homogenous P-waves only move the medium in the X direction, while homogeneous S-waves only move it in the Y direction. As the inhomogeneity grows, the medium's motion becomes elliptical, with an inverse aspect ratio $|K_{*Y}|/K_{*X}$. Hence, it is natural to treat $v_{\mathrm{P}X}$ and $v_{\mathrm{S}Y}$ as scalar wave amplitudes for our P- and S-waves, respectively, and use (25) to calculate the other velocity components.

3.2 Aligned strain fields

Knowing the velocity field, we can use (14) to calculate analytical expressions for the strain field components S_{*XX}, $S_{*XY} = S_{*YX}$, and S_{*YY}, as well as $S_{*KK} = S_{*XX} + S_{*YY}$.

Expressing the P-wave strain components in terms of $v_{\mathrm{P}X}$, which we chose as a scalar P-wave amplitude, we find

$$S_{\mathrm{P}XX} = -\frac{K_{\mathrm{P}X}}{\omega} v_{\mathrm{P}X} = -\frac{v_{\mathrm{P}X}}{c_\mathrm{P}}, \tag{26a}$$

$$S_{\mathrm{P}XY} = -\frac{1}{2}\left(\frac{\pm\mathrm{i}|K_{\mathrm{P}Y}|}{\omega} v_{\mathrm{P}X} + \frac{K_{\mathrm{P}X}}{\omega} v_{\mathrm{P}Y}\right) \overset{(25a)}{=} \mp\frac{\mathrm{i}|K_{\mathrm{P}Y}|}{\omega} v_{\mathrm{P}X}, \tag{26b}$$

$$S_{\mathrm{P}YY} = \mp\frac{\mathrm{i}|K_{\mathrm{P}Y}|}{\omega} v_{\mathrm{P}Y} \overset{(25a)}{=} \frac{|K_{\mathrm{P}Y}|^2}{\omega K_{\mathrm{P}X}} v_{\mathrm{P}X} \overset{(23)}{=} \left(1 - \frac{c_\mathrm{P}^2}{c_{\mathrm{P}0}^2}\right)\frac{v_{\mathrm{P}X}}{c_\mathrm{P}}, \tag{26c}$$

$$S_{\mathrm{P}KK} = S_{\mathrm{P}XX} + S_{\mathrm{P}YY} = -\frac{c_\mathrm{P}}{c_{\mathrm{P}0}}\frac{v_{\mathrm{P}X}}{c_{\mathrm{P}0}}. \tag{26d}$$

In a homogeneous P-wave, where $|K_{\mathrm{P}Y}| = 0$ and $c_\mathrm{P} = c_{\mathrm{P}0}$, only $S_{\mathrm{P}XX}$ (and hence $S_{\mathrm{P}KK}$) is nonzero. In an inhomogeneous P-wave, $S_{\mathrm{P}XX}$ is expressed similarly, albeit through the inhomogeneous P-wave speed c_P instead of the homogeneous P-wave speed $c_{\mathrm{P}0}$. The inhomogeneity also leads $S_{\mathrm{P}XY}$ and $S_{\mathrm{P}YY}$ to become nonzero.

We similarly find the strain components of the S-wave, expressed in terms of $v_{\mathrm{S}Y}$, to be

$$S_{SXX} = -\frac{K_{SX}}{\omega} v_{SX} = \pm\frac{\mathrm{i}|K_{SY}|}{\omega} v_{SY}, \tag{27a}$$

$$S_{SXY} = -\frac{1}{2}\left(\frac{\pm\mathrm{i}|K_{SY}|}{\omega} v_{SX} + \frac{K_{SX}}{\omega} v_{SY}\right) \overset{(25b)}{=} -\frac{1}{2}\left(\frac{|K_{SY}|^2}{\omega K_{SX}} + \frac{1}{c_S}\right) v_{SY} \overset{(23)}{=} -\left(2 - \frac{c_S^2}{c_{S0}^2}\right)\frac{v_{SY}}{2c_S}, \tag{27b}$$

$$S_{SYY} = \mp\frac{\mathrm{i}|K_{SY}|}{\omega} v_{SY}, \tag{27c}$$

$$S_{SKK} = S_{SXX} + S_{SYY} = 0. \tag{27d}$$

For a homogeneous S-wave, only the shear strain S_{SXY} is nonzero. In an inhomogeneous wave, the normal strains S_{SXX} and S_{SYY} become nonzero as well. Still, these normal strains do not lead to any change in volume, as the volumetric strain S_{SKK} remains zero.

3.3 Aligned stress fields

The strain field components let us calculate the stress field components σ_{*XX}, $\sigma_{*XY} = \sigma_{*YX}$, and σ_{*YY} directly from (15).

For the P-wave, we get

$$\sigma_{PXX} = \left(-\rho c_P + \frac{2\rho c_P c_{S0}^2}{c_{P0}^2}\right) v_{PX} - \frac{2\rho c_{S0}^2}{c_P} v_{PX} = -\left(1 + \frac{2c_{S0}^2}{c_P^2}\left[1 - \frac{c_P^2}{c_{P0}^2}\right]\right)\rho c_P v_{PX}, \tag{28a}$$

$$\sigma_{PXY} = \mp\frac{2\mathrm{i}|K_{PY}|}{\omega}\rho c_{S0}^2 v_{PX}, \tag{28b}$$

$$\sigma_{PYY} = \left(-\rho c_P + \frac{2\rho c_P c_{S0}^2}{c_{P0}^2}\right) v_{PX} + 2\rho c_{S0}^2\left(\frac{1}{c_P} - \frac{c_P}{c_{P0}^2}\right) v_{PX} = -\left(1 - 2\frac{c_{S0}^2}{c_P^2}\right)\rho c_P v_{PX}. \tag{28c}$$

The inhomogeneity causes changes to the normal stresses σ_{PXX} and σ_{PYY} that can be expressed in terms of the changes in P-wave speed c_P. It also causes the shear stress σ_{PXY} to become nonzero.

For the S-wave, calculating stress from (15) is more straightforward as $S_{SKK} = 0$:

$$\sigma_{SXX} = \pm\frac{2\mathrm{i}|K_{SY}|}{\omega}\rho c_{S0}^2 v_{SY}, \tag{29a}$$

$$\sigma_{SXY} = -\left(2\frac{c_{S0}^2}{c_S^2} - 1\right)\rho c_S v_{SY}, \tag{29b}$$

$$\sigma_{SYY} = \mp\frac{2\mathrm{i}|K_{SY}|}{\omega}\rho c_{S0}^2 v_{SY}. \tag{29c}$$

The inhomogeneity causes a minor change to the shear stress σ_{SXY}, and cause the normal stresses σ_{SXX} and σ_{SYY} to be nonzero. Even so, the wave does not cause a bulk stress σ_{SKK}, as the two normal stresses cancel each other.

3.4 Aligned intensity fields

Now that we know both the velocity fields $\boldsymbol{v}_*$ and the stress fields $\boldsymbol{\sigma}_*$, we can calculate the time-averaged intensity of individual P- or S- waves from (16). Writing it out, we find that we need to calculate two terms for the Ith component of each wave:

$$I_{*I} = -\frac{1}{2}\,\mathrm{Re}\left\{\sigma_{*IX}\overline{v_{*X}} + \sigma_{*IY}\overline{v_{*Y}}\right\}. \tag{30}$$

(Bear in mind that the stress tensors are symmetric, so that $\sigma_{*YX} = \sigma_{*XY}$.) As $\sigma_{\mathrm{P}IJ}$ is expressed through v_{PX} and $\sigma_{\mathrm{S}IJ}$ is expressed through v_{SY}, we will end up with expressions on the form

$$v_{\mathrm{PX}}\overline{v_{\mathrm{PX}}} = |v_{\mathrm{PX}}|^2 \overset{(24)}{=} |v_{\mathrm{PX0}}|^2\,\mathrm{e}^{\mp 2|K_{\mathrm{PY}}|Y}, \qquad v_{\mathrm{SY}}\overline{v_{\mathrm{SY}}} = |v_{\mathrm{SY}}|^2 \overset{(24)}{=} |v_{\mathrm{SY0}}|^2\,\mathrm{e}^{\mp 2|K_{\mathrm{SY}}|Y}. \tag{31}$$

To calculate the P-wave intensity, we require the terms

$$\sigma_{\mathrm{PXX}}\overline{v_{\mathrm{PX}}} = -\left(1 + \frac{2c_{\mathrm{S0}}^2}{c_{\mathrm{P}}^2}\left[1 - \frac{c_{\mathrm{P}}^2}{c_{\mathrm{P0}}^2}\right]\right)\rho c_{\mathrm{P}}|v_{\mathrm{PX}}|^2, \tag{32a}$$

$$\sigma_{\mathrm{PXY}}\overline{v_{\mathrm{PY}}} \overset{(25a)}{=} \left(\mp\frac{2\mathrm{i}|K_{\mathrm{PY}}|}{\omega}\rho c_{\mathrm{S0}}^2 v_{\mathrm{PX}}\right)\left(\frac{\mp\mathrm{i}|K_{\mathrm{PY}}|}{K_{\mathrm{PX}}}\overline{v_{\mathrm{PX}}}\right) \overset{(23)}{=} -\frac{2c_{\mathrm{S0}}^2}{c_{\mathrm{P}}^2}\left(1 - \frac{c_{\mathrm{P}}^2}{c_{\mathrm{P0}}^2}\right)\rho c_{\mathrm{P}}|v_{\mathrm{PX}}|^2, \tag{32b}$$

$$\sigma_{\mathrm{PXY}}\overline{v_{\mathrm{PX}}} = \mp\frac{2\mathrm{i}|K_{\mathrm{PY}}|}{\omega}\rho c_{\mathrm{S0}}^2|v_{\mathrm{PX}}|^2, \tag{32c}$$

$$\sigma_{\mathrm{PYY}}\overline{v_{\mathrm{PY}}} \overset{(25a)}{=} \left(-\left[1 - 2\frac{c_{\mathrm{S0}}^2}{c_{\mathrm{P}}^2}\right]\rho c_{\mathrm{P}}\right)\left(\frac{\mp\mathrm{i}|K_{\mathrm{PY}}|}{K_{\mathrm{PX}}}\right)|v_{\mathrm{PX}}|^2. \tag{32d}$$

Together, they lead to a P-wave intensity

$$I_{\mathrm{PX}} = \left(1 + \frac{4c_{\mathrm{S0}}^2}{c_{\mathrm{P}}^2}\left[1 - \frac{c_{\mathrm{P}}^2}{c_{\mathrm{P0}}^2}\right]\right)\frac{\rho c_{\mathrm{P}}}{2}|v_{\mathrm{PX}}|^2, \qquad I_{\mathrm{PY}} = 0. \tag{33}$$

The last equality follows because $\sigma_{\mathrm{PXY}}\overline{v_{\mathrm{PX}}}$ and $\sigma_{\mathrm{PYY}}\overline{v_{\mathrm{PY}}}$ are purely imaginary, with no real component. Therefore, energy only flows along the direction of wave propagation for inhomogeneous P-waves, in the same way as for homogeneous ones.

For S-waves, we require the terms

$$\sigma_{\mathrm{SXX}}\overline{v_{\mathrm{SX}}} \overset{(25b)}{=} \left(\pm\frac{2\mathrm{i}|K_{\mathrm{SY}}|}{\omega}\rho c_{\mathrm{S0}}^2 v_{\mathrm{SY}}\right)\left(\frac{\pm\mathrm{i}|K_{\mathrm{SY}}|}{K_{\mathrm{SX}}}\overline{v_{\mathrm{SY}}}\right) \overset{(23)}{=} -2\left(\frac{c_{\mathrm{S0}}^2}{c_{\mathrm{S}}^2} - 1\right)\rho c_{\mathrm{S}}|v_{\mathrm{SY}}|^2, \tag{34a}$$

$$\sigma_{\mathrm{SXY}}\overline{v_{\mathrm{SY}}} = -\left(2\frac{c_{\mathrm{S0}}^2}{c_{\mathrm{S}}^2} - 1\right)\rho c_{\mathrm{S}}|v_{\mathrm{SY}}|^2, \tag{34b}$$

$$\sigma_{\mathrm{SXY}}\overline{v_{\mathrm{SX}}} \overset{(25b)}{=} \left(-\left[2\frac{c_{\mathrm{S0}}^2}{c_{\mathrm{S}}^2} - 1\right]\rho c_{\mathrm{S}}\right)\left(\frac{\pm\mathrm{i}|K_{\mathrm{SY}}|}{K_{\mathrm{SX}}}\right)|v_{\mathrm{SY}}|^2, \tag{34c}$$

$$\sigma_{\mathrm{SYY}}\overline{v_{\mathrm{SY}}} = \mp\frac{2\mathrm{i}|K_{\mathrm{SY}}|}{\omega}\rho c_{\mathrm{S0}}^2|v_{\mathrm{SY}}|^2, \tag{34d}$$

which lead to an S-wave intensity

$$I_{\mathrm{SX}} = \left(4\frac{c_{\mathrm{S0}}^2}{c_{\mathrm{S}}^2} - 3\right)\frac{\rho c_{\mathrm{S}}}{2}|v_{\mathrm{SY}}|^2, \qquad I_{\mathrm{SY}} = 0. \tag{35}$$

As with the P-waves, the Y component of the S-wave intensity is zero because the real parts of $\sigma_{\mathrm{SXY}}\overline{v_{\mathrm{SX}}}$ and $\sigma_{\mathrm{SYY}}\overline{v_{\mathrm{SY}}}$ are zero, so that energy only flows in the X direction.

Closer inspection of the expressions for P- and S-wave intensities reveals that they are related. By the substitutions $|v_{\mathrm{PX}}|^2 \to |v_{\mathrm{SY}}|^2$, $c_{\mathrm{P}} \to c_{\mathrm{S}}$, and $c_{\mathrm{P0}} \to c_{\mathrm{S0}}$, (33) can be transformed into (35).

While we are not aware of any literature providing expressions for inhomogeneous strain and stress that we can compare our corresponding expressions with, a 1985 article by Borcherdt and Wennerberg provides a general expression for the intensity of inhomogeneous P- or S-waves in a lossy medium [3]. Their medium is characterised by a complex shear modulus $\mu = \mu_\mathrm{r} + \mathrm{i}\mu_\mathrm{i}$, where the real part is tied to elasticity and the imaginary part is tied to loss. Adapted to our notation, their expression for time-averaged intensity is

$$\begin{aligned} \boldsymbol{I}_* &= |\boldsymbol{G}_{*0}|^2\, \mathrm{e}^{-2\boldsymbol{k}_{*\mathrm{i}}\cdot\boldsymbol{r}}\, \frac{\omega}{2}\left(\rho\omega^2\boldsymbol{k}_{*\mathrm{r}} + 4\left[\boldsymbol{k}_{*\mathrm{r}}\times\boldsymbol{k}_{*\mathrm{i}}\right]\times\left[\mu_\mathrm{i}\boldsymbol{k}_{*\mathrm{r}} - \mu_\mathrm{r}\boldsymbol{k}_{*\mathrm{i}}\right]\right) \\ &\overset{\text{lossless}}{=} \left(1 + 4\frac{c_{\mathrm{S}0}^2}{c_*^2}\left[1 - \frac{c_*^2}{c_{*0}^2}\right]\right)\frac{\rho\omega^3}{2}|\boldsymbol{G}_{*0}|^2\,\mathrm{e}^{-2\boldsymbol{k}_{*\mathrm{i}}\cdot\boldsymbol{r}}\,\boldsymbol{k}_{*\mathrm{r}}. \end{aligned} \tag{36}$$

In the last equality, we assumed the solid to be lossless, so that $\mu_\mathrm{i} = 0$, $\mu = \mu_\mathrm{r} = \rho c_{\mathrm{S}0}^2$, and $\boldsymbol{k}_{*\mathrm{r}} \perp \boldsymbol{k}_{*\mathrm{i}}$. In the reference, $|\boldsymbol{G}_{\mathrm{P}0}|$ represents the scalar displacement potential amplitude, and $\boldsymbol{G}_{\mathrm{S}0} = G_{\mathrm{S}z0}\hat{\boldsymbol{z}}$ represents the vector displacement potential amplitude. We can relate these potential amplitudes to our velocity amplitudes as

$$\boldsymbol{v}_{\mathrm{P}0} = \omega\boldsymbol{k}_\mathrm{P}|\boldsymbol{G}_{\mathrm{P}0}| \qquad \text{and} \qquad \boldsymbol{v}_{\mathrm{S}0} = \omega\boldsymbol{k}_\mathrm{S}\times\boldsymbol{G}_{\mathrm{S}0}. \tag{37}$$

If we choose this wave to be aligned along the X and Y axes ($\boldsymbol{k}_* \to \boldsymbol{K}_*$), we find that Borcherdt and Wennerberg's general expression for intensity matches our (33) and (35) when the solid is lossless.

3.5 Reorienting the wavefields

The expressions for velocity $\boldsymbol{v}$, strain $\boldsymbol{S}$, stress $\boldsymbol{\sigma}$, and intensity $\boldsymbol{I}$ found in Sections 3.1–3.4 were all expressed in the X–Y coordinate system, which is aligned with the wave. We can re-express these quantities in the x–y coordinate system using the rotation matrix from (21), so that

$$\boldsymbol{v}_*' = \boldsymbol{R}\boldsymbol{v}_*, \qquad \boldsymbol{S}_*' = \boldsymbol{R}\boldsymbol{S}_*\boldsymbol{R}^{-1}, \qquad \boldsymbol{\sigma}_*' = \boldsymbol{R}\boldsymbol{\sigma}_*\boldsymbol{R}^{-1}, \qquad \boldsymbol{I}_*' = \boldsymbol{R}\boldsymbol{I}_*. \tag{38}$$

Here, the primed quantities are expressed in the x–y system, while the unprimed quantities are expressed in the X–Y system.

3.6 Summary of axis-aligned inhomogeneous wavefields

In this section, we have derived generic analytical expressions of the basic fields — velocity, strain, stress, and intensity — of individual inhomogeneous P- and S-waves propagating along the X axis and decaying along the $\pm Y$ axis as shown in Figure 2. While these fields are aligned to an artificial coordinate system, they can be transferred to the x–y plane with any orientation, as shown in (38).

Not only are the final expressions for intensity equivalent to those provided in [3], but every expression provided in this section can be (and has been) verified through numerical calculation. With the velocity field specified as in (24), the strain, stress, and intensity amplitudes can be computed numerically using (14), (15), and (16). These values can then be compared against values calculated from the analytical expressions in Sections 3.2–3.4.

4 Total radiated intensity

At this point, we know three equivalent ways to calculate the intensity of the individual P- and S-waves radiated from an attenuated surface vibration. Starting with the P and S velocity fields from (9) and (13), we can:

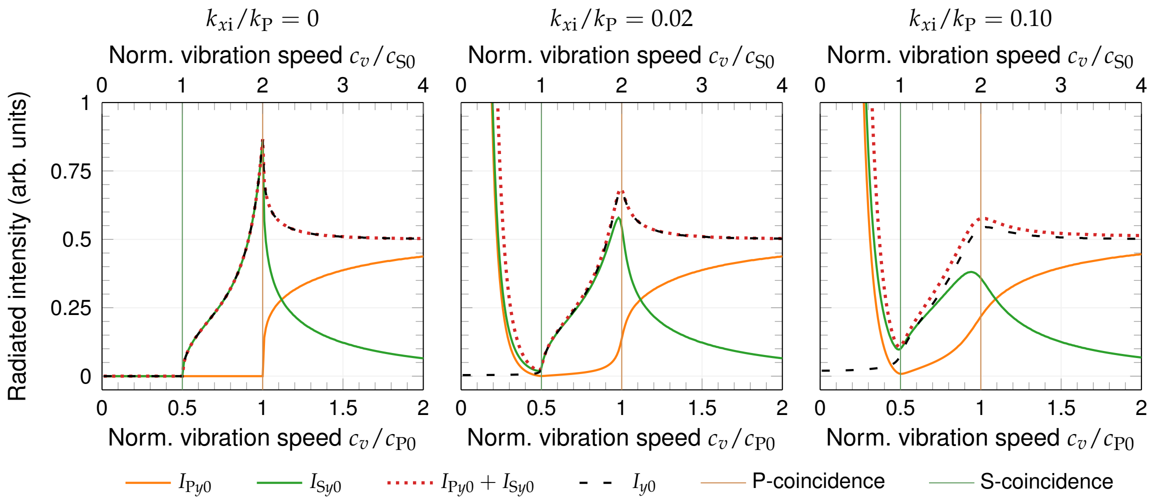

Figure 3: Radiated intensity in arbitrary units against vibration speed for the P-wave, the S-wave, the sum of individual P- and S-waves, and the total field, for three different attenuations of the surface vibration. For either wave type, the waves are subsonic at vibration speeds below coincidence and supersonic above.

1. Calculate the intensities in the X–Y coordinate system by the analytical expression in (33) and (35) and use (38) to express them in the x–y system.

2. Calculate the intensities numerically, using (14)–(16).

3. Calculate the intensities using Borcherdt and Wennerberg's expression [3] in (36).

However, this only gives us the intensities of the individual P- and S-waves. What we need for (7) is the *total* radiated intensity I_{y0}, which must be calculated from the combined P- and S-wavefields as shown in (18). As we explained at the end of Section 2, (18) shows that I_{y0} is not necessarily simply a sum of I_{Py0} and I_{Sy0}, although Brekhovskikh and Godin [10] show such a summation to be valid for the case of wave transmission from a fluid into a solid.

To investigate whether this summation is valid for wave radiation into solids from an attenuated surface vibration, we calculate and compare I_{Py0}, I_{Sy0}, and I_{y0} for different attenuations of the surface vibration. In all cases, we choose a purely normal surface velocity of arbitrary amplitude, so that $v_{x0} = 0$, and $v_{y0} = 1$. Furthermore, we choose arbitrary parameters $\rho = 1$, $c_{P0} = 1$, $c_{S0} = 0.5$, and $\omega = 2\pi$.

Figure 3 compares the radiated intensities for three different vibrational attenuations k_{xi}. As the attenuation increases, the radiated intensities are smoothed, similarly to when a surface vibration radiates into a fluid [14]. With a nonzero attenuation ($k_{xi} > 0$), the intensities of the individual P- and S-waves diverge when we go into in the S-subsonic domain. (Even when choosing different values of c_{S0}/c_{P0}, this divergence only occurs when $c_v < c_{S0}$.) Despite this divergence of the individual waves' intensity, the total intensity I_{y0} remains finite and small.

So, can we confirm Brekhovskikh and Godin's result [10] that $I_{y0} = I_{Py0} + I_{Sy0}$? For *unattenuated* surface vibrations ($k_{xi} = 0$), this holds exactly true. For *attenuated* surface vibrations ($k_{xi} > 0$), however, $I_{y0} \neq I_{Py0} + I_{Sy0}$. The discrepancy is higher for higher attenuations and becomes extreme as we go into the S-subsonic domain. As for the x-component, $I_{x0} \neq I_{Px0} + I_{Sx0}$ in all cases.

Hence, Brekhovskikh and Godin only found their result because they were looking at a reflection-transmission problem with incoming plane waves from a *lossless* medium. In this case, the incoming wave amplitude is the same across the entire interface. For incoming waves from a *lossy* medium, however, the incoming wave amplitude would be exponentially attenuated along the interface, leading to an inhomogeneous transmitted wave [1]. This would be a situation similar to our attenuated surface vibration, so that $I_{y0} \neq I_{\mathrm{P}y0} + I_{\mathrm{S}y0}$.

Even so, how can $I_{y0} = I_{\mathrm{P}y0} + I_{\mathrm{S}y0}$ even for $k_{x\mathrm{i}} = 0$, when (18) shows that there are cross-terms between the P and S wavefields? A closer numerical investigation shows that all of these cross-terms are nonzero, but that $\sigma_{\mathrm{P}yx0}\overline{v_{\mathrm{S}x0}} = -\sigma_{\mathrm{S}yy0}\overline{v_{\mathrm{P}y0}}$ and $\sigma_{\mathrm{P}yy0}\overline{v_{\mathrm{S}y0}} = -\sigma_{\mathrm{S}yx0}\overline{v_{\mathrm{P}x0}}$. Thus, all the cross-terms in (18) end up cancelling. For $k_{x\mathrm{i}} > 0$, these relations no longer hold. However, a closer investigation of this phenomenon is outside the scope of this article.

5 Conclusion

In this article, we have taken an important step towards understanding subsonic radiation into a solid through the explanatory model underlying (7). We have investigated how to correctly determine the required total intensity radiated into a solid by a vibrating surface as specified in Section 2.

In Section 3, we derived analytical expressions for the velocity, strain, stress, and intensity fields of arbitrarily inhomogeneous individual P- and S-waves in lossless isotropic solids. However, we also found in Section 4 that the individual P- and S-wave intensities are not generally sufficient to find the total radiated intensity, due to the cross-terms between the P- and S-wavefields shown in (18). Whenever the surface vibration is attenuated due to loss of vibrational energy, the total radiated intensity should be calculated numerically instead. Fortunately, given the P and S velocity fields, it is quite straightforward, as well as exact to machine precision, to calculate the total radiated intensity via the velocity, strain, and stress of the P- and S-waves. While the analytical wavefield expressions in Section 3 are not as useful for our end goal as we had hoped, we have still included them in the hope that they may be useful to other researchers.

We aim to follow up this investigation with a future article that explores the behaviour of subsonic radiation into solids as described by (7) and the explanatory model underlying it.

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References

[1] N. Declercq, R. Briers, J. Degrieck, and O. Leroy, "The history and properties of ultrasonic inhomogeneous waves," *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, vol. 52, no. 5, pp. 776–791, May 2005.

[2] B. Poirée, "Complex Harmonic Plane Waves," in *Physical Acoustics: Fundamentals and Applications*, O. Leroy and M. A. Breazeale, Eds. Boston, MA: Springer US, 1991, pp. 99–117.

[3] R. D. Borcherdt and L. Wennerberg, “General P, type-I S, and type-II S waves in anelastic solids; inhomogeneous wave fields in low-loss solids,” *Bulletin of the Seismological Society of America*, vol. 75, no. 6, pp. 1729–1763, 1985.

[4] H. Dabirikhah and C. W. Turner, “The coupling of the A0 and interface Scholte modes in fluid-loaded plates,” *The Journal of the Acoustical Society of America*, vol. 100, no. 5, pp. 3442–3445, Nov. 1996.

[5] V. G. Mozhaev and M. Weihnacht, “Subsonic leaky Rayleigh waves at liquid–solid interfaces,” *Ultrasonics*, vol. 40, no. 1-8, pp. 927–933, May 2002.

[6] D. A. Kiefer, M. Ponschab, S. J. Rupitsch, and M. Mayle, “Calculating the full leaky Lamb wave spectrum with exact fluid interaction,” *The Journal of the Acoustical Society of America*, vol. 145, no. 6, pp. 3341–3350, Jun. 2019.

[7] D. A. Kiefer, M. Ponschab, and S. J. Rupitsch, “From Lamb waves to quasi-guided waves: On the wave field and radiation of elastic and viscoelastic plates,” 2020, preprint on ResearchGate.

[8] P. M. Morse and K. U. Ingard, *Theoretical Acoustics*. McGraw-Hill, 1968.

[9] N. Brower, D. Himberger, and W. Mayer, “Restrictions on the existence of leaky Rayleigh waves,” *IEEE Transactions on Sonics and Ultrasonics*, vol. 26, no. 4, pp. 306–307, Jul. 1979.

[10] L. M. Brekhovskikh and O. A. Godin, *Acoustics of Layered Media I*, 1st ed., ser. Springer Series on Wave Phenomena. Springer-Verlag Berlin Heidelberg, 1990, vol. 5.

[11] G. Williams, *Fourier Acoustics: Sound Radiation and Nearfield Acoustical Holography*. Elsevier Science, 1999.

[12] T. E. Vigran, *Building Acoustics*. CRC Press, 2008.

[13] A. D. Pierce, *Acoustics*, 3rd ed. Springer Nature Switzerland AG, 2019.

[14] E. M. Viggen and H. K. Arnestad, “Understanding sound radiation from surface vibrations moving at subsonic speeds,” in *Proceedings of the 44th Scandinavian Symposium on Physical Acoustics*. Norwegian Physical Society, 2021, p. 4, extended abstract.

[15] ——, 2022, journal article to be submitted.

[16] K. F. Graff, *Wave Motion in Elastic Solids*, 1st ed. Dover Publications, 1975.

[17] J. D. N. Cheeke, *Fundamentals and Applications of Ultrasonic Waves*, 1st ed. CRC Press, 2002.

[18] J. L. Rose, *Ultrasonic Guided Waves in Solid Media*. Cambridge University Press, 2014.