Doctoral theses at NTNU, 2022:377

Renny José Arismendi Torres

Piecewise Deterministic Markov Processes for Condition-based Maintenance Modelling

Applications to Critical Infrastructures

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of
Philosophiae Doctor
Faculty of Engineering
Department of Mechanical and Industrial
Engineering

NTNU
Norwegian University of
Science and Technology

Renny José Arismendi Torres

Piecewise Deterministic Markov Processes for Condition-based Maintenance Modelling

Applications to Critical Infrastructures

Thesis for the Degree of Philosophiae Doctor

Trondheim, December 2022

Norwegian University of Science and Technology
Faculty of Engineering
Department of Mechanical and Industrial Engineering

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of Science and Technology

Thesis for the Degree of Philosophiae Doctor

Faculty of Engineering
Department of Mechanical and Industrial Engineering

ISBN 978-82-326-6614-0 (printed ver.)
ISBN 978-82-326-6721-5 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

Doctoral theses at NTNU, 2022:377

Printed by NTNU Grafisk senter

Preface

This thesis is submitted to the Norwegian University of Science and Technology (NTNU), Norway, and the University of Technology of Troyes (UTT), France, for partial fulfillment of the requirements for the degree of Philosophiae Doctor.

This Ph.D. position was funded by NTNU and carried out under a co-tutelle agreement with UTT. The initial appointment was for four years, which entailed a 25% of teaching duties at NTNU and 75% devoted to the PhD research work, making it equivalent to a three years PhD work. The teaching duties at NTNU were later on increased, resulting in an extended appointment period to four years and five months, while the work devoted to the PhD remained equivalent to three years work.

I was originally supposed to stay one full year at UTT, split into two semesters: spring of 2019 and spring of 2020. Unfortunately, the world was hit by the Covid-19 pandemic and the stay of 2020 had to be "postponed" until further notice. During this period, strict travel restrictions were placed all over the world, teaching and conferences were moved to digital solutions, home office became the common rule, and many other changes that have an impact not only on any project but also on every day life. Just like that, about half of my Ph.D. studies were carried out during pandemic restrictions and the second stay in France has not taken place yet. Part of me wants to continue to work on research, have a stay at UTT and participate on international conferences with physical attendance.

Professor Anne Barros (current affiliation: Department of Industrial Engineering at CentraleSupélec, France) and Professor Jørn Vatn have been the main supervisor and co- supervisor, respectively, from the Department of Mechanical and Industrial Engineering, NTNU. Professor Antoine Grall has been the supervisor from the unit of Computer Science and Digital Society, UTT.

The target audience of this thesis includes researchers and practitioners interested in the areas of reliability and maintenance engineering.

Trondheim, Spring 2022

Renny Arimendi Torres

Acknowledgment

I would like to express my gratitude to many people who have been of great support during my pursuit of a PhD degree.

To my main supervisor, professor Anne Barros, I want to express not only my gratitude but my sincere admiration. Before starting my PhD journey I had an idea of how important the supervisor/candidate relationship was for a successful and enjoyable experience. Back then, finding an interesting supervisor to work with was as important to me as finding an interesting topic. Now, at the end of my studies, I can confidently say I made a really good choice when I applied. It is very common to hear the words "ups-and-downs" when people talk about a PhD and research, and yes, there were many of those during my PhD journey, but the quality of Anne's supervision never had any downs, it just kept going up and up. I admire her passion, hard work, ability to encourage, motivate and lead by example. A role model and great friend.

Thanks to my co-supervisors professors Jørn Vatn and Antoine Grall. Every single input from them I have carefully analysed and found great value in. I have learned a lot from our discussions on science and academia. Special thanks to Antoine for all the support on everything related to UTT and my stay in Troyes. I am very grateful for that.

Thanks to the contacts in the industry with whom I have directly and indirectly collaborated during my PhD. Their sincere interest in the topic has been a real motivator. I am grateful for the inputs and discussions around maintenance and reliability. Some of these inputs have found their way into this thesis, some into my lectures but most importantly all of them into my professional formation. Special thanks to Rune Pedersen, Tore Askeland and Inger Lise Johansen from the Norwegian Public Roads Administration and Erling Lunde from Equinor.

To all my colleagues and fellow researchers: thanks for every moment we have shared in and out of the university. Scientific discussions, social activities, lunches, coffees, silly moments and much more. Thanks for bringing in the fun and laughs into this journey: Shenae, Marta, Himanshu, Aibo, Lin, Federico, Nanda, Bahareh, Michael, Tianqi, Ewa, Ariful, Tom Ivar, Jon Martin, Xingheng, Liu, Jie, Juntao, Yun.

To my family, for your unconditional love and support in every decision I have ever made. You are the biggest motivation for me. Thank you for believing in me, reminding me to take breaks when I needed them and reminding me that this effort would be worth it. This simply would not have been possible without you.

R.A.

Summary

This thesis focuses on Piecewise deterministic Markov process (PDMP), a general class of non-diffusion stochastic models, as a framework for modelling condition-based maintenance (CBM) decision problems of critical infrastructures. This model allows to simulate different maintenance strategies for a stochastic deteriorating system and to assess the associated effects, the maintenance costs and the operational performance, in order to determine the best maintenance strategy to implement. From a stochastic models perspective, PDMP represents a canonical model that includes a wide variety of applications as special cases, virtually covering all non-diffusion applications, under a process that combines random jumps and deterministic motion. For CBM modelling, it presents a framework capable of handling a very large number of problems, with different modelling assumptions for both the deterioration and the intervention process, such as non-constant transition rates between discrete states, maintenance delays, different frequency regimes and quality of monitoring and system dependencies.

The application of PDMP as a framework is studied and presented for models of single-items and for multi-component systems, describing the formalism of the process and its evolution while developing a numerical approach for the calculation of quantities of interest such as the probability for the maintained system, to be in a critical or unacceptable state at any time or the maintenance strategy mean cost over a period of time. A simulation approach is also developed for comparison and validation of results of the numerical scheme. The scientific basis of the framework proposed in this thesis is supported by relevant solid theory published in recognized peer-reviewed journals.

The proposed framework is applied to relevant case studies of critical infrastructures to illustrate the modelling and quantification approach. The presented modelling assumptions are based on both literature review and discussions with experts from the critical infrastructures sectors. One case is related to the transport sector with road bridges modelled as a single-unit system, and another case is related to the energy sector with gas compressors, exploring the capabilities for modelling of multi-component systems. Through the case studies, guidelines on how to account for different assumptions such as inspection frequency and quality, system dependencies, as well as maintenance policies are discussed.

The thesis could serve as a basis for further research or engineering applications. A combination of physics-based and data-driven approaches for deterioration modelling and prognostics can be studied with PDMP as framework. Designing and presenting efficient algorithms for the computation of PDMP could allow the development of more advanced simulators than those available today for maintenance planning. Another interesting direction of research could be studying reinforced learning approaches with PDMP as base model, for estimation of model parameters when dealing with limited data characterized by a mixture of qualitative and quantitative information, important problems of censoring, incompleteness, and pollution by maintenance actions.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Background

Modern societies depend on the availability of certain services, facing risks with serious economic consequences or loss of life when such services or products are disrupted or unavailable. Such services are provided by Critical Infrastructures [1].

The Council of the European Union defines a critical infrastructure as "an asset, system or part thereof located in Member States which is essential for the maintenance of vital societal functions, health, safety, security, economic or social well-being of people, and the disruption or destruction of which would have a significant impact in a Member State as a result of the failure to maintain those functions" and establishes a procedure for the identification and designation of European critical infrastructures in [2]. The energy and transport sectors are used for the implementation of the Directive. The sub-sectors are shown in table 1.1.

Table 1.1: List of European Critical Infrastructure sectors [2]

Sector	Subsector	
I Energy	1. Electricity	Infrastructures and facilities for generation and transmission of electricity in respect of supply electricity
	2. Oil	Oil production, refining, treatment, storage and transmission by pipelines
	3. Gas	Gas production, refining, treatment, storage and transmission by pipelines LNG terminals
II Transport	4. Road transport	
	5. Rail transport	
	6. Air transport	
	7. Inland waterways transport	
	8. Ocean and short-sea shipping and ports	

Risks in critical infrastructures may come from different sources, from simple wear of units that can eventually lead to failures to sabotages and terrorist acts and war [1]. This PhD project is mainly concerned with the natural wear of such systems rather than on deliberately induced faults. The risk associated to the deterioration has traditionally pushed the decision makers to take large safety margins and to preventively over inspect or maintain the systems.

In recent decades, the technology and techniques for condition monitoring have experienced a rapid development. However, there is still a need for reducing unnecessary inspections and/or preventive maintenance actions and their associated cost, through optimal design of condition-based maintenance (CBM) strategies. If we consider critical infrastructures in civil engineering or in oil & gas industry, CBM strategies are often carried without any modelling and assessment of their efficiency in the mid or long term. Generally speaking, the problem is about sub-optimal decisions in terms of maintenance cost, availability of production and even reputation [3, 4].

In the bridge management system applied in Denmark, Finland, France, South Africa, United Kingdom, China, South Korea, United States of America, Norway and other countries, inspections are carried out by following pre-defined procedures and a condition rating is assigned to the structure in a discrete scale [5]. The decision about when to perform maintenance is based on the condition rating assigned at these inspections. In Norway, handbooks for management and inspections of bridges [6, 7], establish the types of inspections for the bridges and the period in which they must be performed. For example, the main inspection of a bridge, with an overview of all the elements of the bridge, must (in general) be performed every five years. The handbooks also establish how the inspections must be logged in a database, how the found damages must be reported and when to schedule the repairs for found damages: given the reported damages, the condition of the whole bridge is ranked among a very limited number of global deterioration states and the delay before repair is chosen accordingly. However, the period of inspection and the delay before repair are not optimized according to a time-dependent or a long term safety criteria or maintenance cost.

In the oil & gas industry, the natural gas transportation infrastructure is dependent on high capacity compressors to supply the required flow of gas at any time all over the year. This is especially important during the winter season where a full capacity is needed and one hundred per cent of the equipment is used. Most of the compression systems involve high voltage electrical motors which are subject to deterioration which is assessed by the number of partial discharges in the insulation layers. These electrical motors are periodically inspected and their global deterioration state is ranked among a limited number of discrete states (6 levels according to ABB and Karsten Moholts scale for example [8]). A preventive maintenance can be recommended according to this rank but the production requirements can lead to postpone the execution of the maintenance tasks. For example, a preventive maintenance will not be triggered during the consumption peak in the winter season. Consequently, knowing the electrical motor condition at the inspection date is not enough to make an optimal decision for the maintenance time and task. It is required to model the overall CBM strategy including the maintenance delay due to seasonal constraints and the costs related to maintenance and failures.

In this thesis, we intend to propose a framework for CBM modelling dedicated to critical infrastructures. The purpose is to have a model to assess the performance of CBM strategies. This could allow to challenge those pre-established CBM strategies that might be not optimal and to provide time dependent or long term decision criteria to optimize them. The decision criteria are basically the probability, for the maintained system, to be in a critical or unacceptable state at any time or the maintenance strategy mean cost over a period of time. The main assumptions we keep from the application field of critical infrastructures are: we consider that the deterioration process is stochastic, in the sense that the evolution cannot be appropriately described by means of physical deterministic laws, and the system condition is characterized at a high level of abstraction with discrete states (often given by guidelines in the application field).

1.2 Objectives

The main objective of this thesis is *"to study, develop and demonstrate quantitative models for prognosis and condition-based maintenance assessment of critical infrastructures"*. For this purpose the following sub-objectives are defined:

1. Conduct a literature review on condition-based maintenance, degradation modelling, prognostic approaches and stochastic processes for deterioration modelling.

2. Explore on the capabilities of Piecewise deterministic Markov processes (PDMP) for the modelling and assessment of CBM policies of single-unit systems under certain non-common assumptions.

3. Explore on the capabilities of PDMP for the modelling and assessment of CBM policies of multi-component system subjected to inter-dependencies.

4. Study and develop analytical approaches for the solution of system models.

5. Identify, select and describe relevant case studies for the application of PDMP as a modelling framework and assessment of CBM policies.

6. Apply developed models, validating results of analytical and numerical approaches with Monte Carlo simulations.

1.3 Scientific approach

Research and experimental development (R&D) involves creative and systematic work performed in order to increase the amount of knowledge (including knowledge of humankind, culture and society) and to devise new applications of available knowledge. It can involve three types of activity: basic research, applied research and experimental development [9].

Basic research is experimental or theoretical work performed primarily in order to gain new knowledge of underlying foundations of phenomena and observable facts, without any particular application or use in view. Applied research refers to original investigation performed to acquire new knowledge but directed primarily towards a specific, practical aim or objective. Experimental development is systematic work, taking on knowledge acquired from research and practical experience, which produces additional knowledge directed to produce new products or processes or to improve the existing products and processes.

From the above described types of R&D, the work included in this thesis falls into the category of applied research, i.e. an original activity undertaken to gain new knowledge and insights directed to practical applications in the industry. It involves considering the available knowledge and its extension in order to solve actual problems. In this thesis, existing mathematical frameworks and simulation methods are studied in order to solve actual condition-based maintenance problems, in an original way, contributing to the expansion of the capabilities for modelling of the commonly applied frameworks.

The general basis for this thesis and the topics it contains have been established through literature studies and meetings and discussions with professional experts on the critical infrastructures sector in Norway. The developed models throughout this thesis, based on the framework of PDMP, could in principle be verified empirically of by the collection of field data. However, as with a great deal of applied research works in the field of reliability, availability, maintenance and safety (RAMS), such verification is highly challenging. The reason is that these models deal with undesired events such failures or deterioration phenomena which represents serious risks, especially when the application is within the critical infrastructure sectors. Preventive maintenance tasks are carried out on regular basis in order to avoid those rare failures and to stop or slow down the progression of deterioration phenomena. Hence, the data needed to verify the models in this way is not readily available and it is not practical to carry out experiments in order to collect data. Model evaluation and verification must be done by other methods than empirical or experimental.

Model evaluation and verification

The model evaluation protocol, issued by the European Union in 1994 [10], consist of three main elements:

- Scientific assessment
- Verification

- Validation

The scientific assessment should involve a comprehensive description of the model, assessment of the scientific content, definition of the scope, and advantages and limitations of the model. Verification is the process of showing that a model has scientific basis, that the assumptions are reasonable, the equations are correctly solved. From a general perspective, that the proposed model does what it claims to do. Finally, validation is the process of assessing a model so that its accuracy and usefulness can be determined, often by comparison with other models [11, 12].

In this thesis, the developed framework is based on rational understanding of Markov processes and condition-based maintenance models assumptions and limitations. For verification and validation, the scientific basis of the main framework proposed in this thesis is supported by relevant solid theory published in recognized peer-reviewed journals, as we are dealing with applied research. The proposed assumptions are based on both literature review and discussions with experts from the critical infrastructures sectors. Both analytical and simulation approaches are studied for comparison and validation of results. Case studies have been developed in close collaborations with experts in the energy and transport sectors of critical infrastructures, showing the usefulness of the modelling framework in practical applications.

Scientific quality

The research council of Norway [13] states that quality in science is related to three aspects: originality, solidness and relevance. There can be a trade-off between these aspects, for example strong solidness with thorough theoretical support of the statements and conclusions could be counter productive to the originality and innovation. Similarly, a work with limited originality can still be very practical, useful and hence relevant.

This thesis tries to balance the three aspects. The work aims to explore in solid developed theory in the field of mathematics but with limited applications, with the aim of modelling CBM problems. This is considered an original and novel work, as there are very limited applications in this field today. Relevance of the framework is shown through case studies related to critical infrastructures. Moreover, the scientific quality of the work is also controlled by dissemination in scientific peer-reviewed journals and international conferences. Reviewers are experts in the subject and their comments and feedback have had a very valuable contribution to the quality of the research and work presented here.

1.4 Academic publications

The list of research works that have been submitted, published in international journals or presented in international conferences as part of the work during this PhD project, are:

- Conference 1:
  Renny Arismendi, Anne Barros, Jørn Vatn, Antoine Grall
  Piecewise Deterministic Markov Process for Condition-based Maintenance with Delay (Abstract-only with presentation in special session)
  11th International Conference on Mathematical Methods in Reliability (MMR 2019), Hong Kong, 3-7 June 2019

- Conference 2:
  Renny Arismendi, Anne Barros, Jørn Vatn, Antoine Grall
  Prognostics and Maintenance Optimization in Bridge Management
  29th European Safety and Reliability Conference (ESREL 2019), Hannover, Germany, 22-26 September 2019

- Conference 3:
  Renny Arismendi, Anne Barros, Antoine Grall
  Preventive Maintenance of a Compressor Station: a Modelling Framework for the Assessment of Performance
  30th European Safety and Reliability Conference (ESREL 2020) - 15th Probabilistic Safety Assessment and Management Conference (PSAM15), Venice, Italy, 1-5 November 2020

- Journal 1:
  Renny Arismendi, Anne Barros, Antoine Grall
  Piecewise deterministic Markov process for condition-based maintenance models - Application to critical infrastructures with discrete-state deterioration
  Reliability Engineering & System Safety, Volume 212, August 2021, 107540

- Journal 2:
  Renny Arismendi, Anne Barros, Antoine Grall
  A modelling framework for Condition-based Maintenance of systems with multi-state components - Application to a gas compression system
  Under review by: *Reliability Engineering & System Safety*

1.5 Limitations

This thesis is written from an engineering perspective. As pointed in the previous section, the work included here falls into the category of applied science and research rather than on basic or pure science. Parts of this thesis deal with mathematical models and statistical methods for estimation of model parameters. These are part of the work of considering available knowledge and its extension, with the intention of solving actual problems. In our case, related to condition-based maintenance of critical infrastructures. Hence, the main contributions of this thesis are not necessarily in statistics and mathematics but in applied engineering for maintenance modelling.

In this thesis the focus is placed on Piecewise deterministic Markov processes as a general class of non-diffusion stochastic models, for studying CBM problems with discrete-state deterioration. In this way, the intention is to include different modelling formalisms available in the literature under the same framework. There exists a large collection of special models and methodologies available for studying non-diffusion stochastic models within reliability and maintenance applications. It is not an easy task to compare and organize models in terms of their correctness, as different models may be used to analyze the same systems [14]. The focus of model developing is often place on usefulness. Although the class of PDMP virtually covers all stochastic non-diffusion applications [15], it is not intended to replace the existing models in the literature, many of which present efficient techniques for calculations by making use of the special structure of specific models given the application and its model assumptions.

A model is an abstraction or simplification of the reality it is designed to represent [16, 17]. In systems modelling, there is a popular saying: a model is only as good as its assumptions. It is therefore duty of the model developer to carefully formulate the model assumptions. In this thesis, we take assumptions for deterioration processes and maintenance management inspired from critical infrastructures applications. These assumptions are clearly stated throughout the thesis where they apply.

This thesis includes relevant case studies on the application of the proposed modelling framework for assessment of CBM policies. These case studies have been developed in collaboration with the sector of critical infrastructures. They involved constraints related to limited access to data: current data bases available for critical infrastructures are characterized by a mixture of qualitative and quantitative information, important problems of censoring, incompleteness, and pollution by maintenance actions. Therefore, the model parameters have been estimated from a combination of limited data and expert judgement, and supported from previous internal works of the corresponding sector. The results of these case studies are therefore intended to show the capabilities of the modelling framework in capturing an overview of the systems and some of their interesting features, rather than presenting a definite solution to the specific CBM problem.

1.6 Structure of the thesis

The thesis is structured in chapters. Chapter 2 introduces the theoretical background that serves as basis for the thesis, including the relevant terms, concepts and basic theory for the formulations developed through the thesis. Chapter 3 explores on the application of a Piecewise deterministic Markov process (PDMP) to encompass different modelling assumptions as non-negligible maintenance delays and inspection-based condition monitoring of single-unit systems. A numerical scheme for quantification, as an approximation of the Chapman-Kolmogorov equation, is described. A case study dealing with CBM of road bridges by the NPRA (Norwegian Public Roads Administration) is presented, guiding through the modelling and quantification approach. Chapter 4 explores on the application of PDMP and multi-state systems theory, for modelling CBM of multi-component system, allowing for the performance assessment of maintenance policies at the system level, while taking into account some structural and resources dependencies among the components with non-negligible actions duration and constraints related to the production profile. A case study dealing with CBM of a gas compression system is presented, with modelling and quantification guides to assess the performance of a maintenance policy at the system level Finally, chapter 5 summarizes and concludes the thesis, highlighting features of PDMP as a framework for CBM models and recommending future directions of research and applications.

Chapter 2

Theoretical background

This chapter introduces the theoretical background that serves as basis for the thesis, including the relevant terms, concepts and basic theory for the formulations developed and presented in subsequent chapters.

The chapter starts by introducing the maintenance concept and engineering challenge and types of maintenance in section 2.1. Next, section 2.2 presents the purpose and elements of condition-based maintenance models, and an overview of the common approaches available. This is followed by an introduction to Markov processes, a class of stochastic processes that are the focus of the thesis in section 2.3, while section 2.4 introduces the main framework of the thesis: Piecewise deterministic Markov processes, a more general class of non-diffusion stochastic models. Section 2.5 presents a classification of the system models as single-item and multi-component. Finally, section 2.6 concludes the chapter by highlighting the key concepts, theory and features of this chapter that are key for the developments presented in subsequent chapters.

2.1 Maintenance

The European committee for standardization has prepared the standard: EN 13306:2017 Maintenance - Maintenance terminology [18], with the purpose of defining the generic terms used for all types of maintenance and maintenance management, irrespective of the type of item considered. Such terms may be of particular importance in formulations of maintenance contracts. The standard, defines maintenance as: "combination of all technical, administrative and managerial actions during the life cycle of an item intended to retain it in, or restore it to, a state in which it can perform the required function".

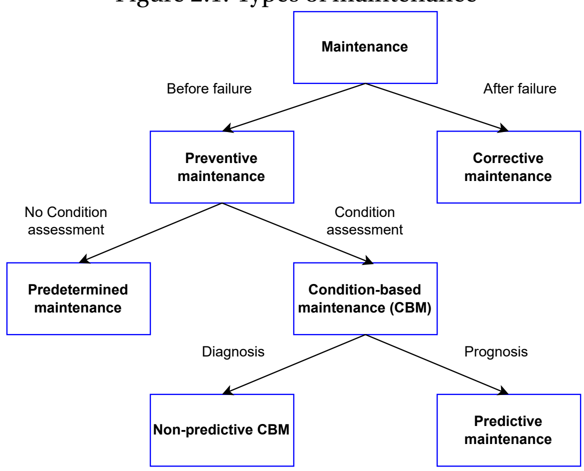
Maintenance management is a problem of decision making under uncertainty. The maintenance decisions may need to account for several criteria that can sometimes be contradictory. Hence, choosing the "best" maintenance strategy is a complex task that is not only dependent on the state of the system, but also in uncertain future factors, such as the consequences of this choice in the long term life of the unit. The objective of such optimization is to determine a maintenance policy that optimizes system performance according to certain criteria (e.g. cost, availability) [19, 20].

Traditionally, maintenance management has been a reverse engineering activity, where the decision process is dependent on the technical and mechanical education of the maintenance staff and their hands on expertise. However, even though expertise is crucial, it should not be the only basis for decisions related to maintenance [19]. By using stochastic models, it may be possible to simulate different maintenance strategies and to assess the associated effects, the maintenance costs and the operational performance. Therefore, these simulations can be used to determine the best maintenance strategy to implement.

2.1.1 Types of maintenance

Maintenance tasks can be classified in different ways. The most common designations are shown in figure 2.1 and described hereafter.

Figure 2.1: Types of maintenance



As shown in figure 2.1, at the highest level maintenance is classified with respect to when the tasks are carried out in corrective and preventive maintenance.

Corrective Maintenance (CM)

Corrective maintenance refers to all actions that are carried out after an item failure or fault has been detected, with the goal of restoring the item to a functioning state.

Preventive Maintenance (PM)

The International Electrotechnical Commission [21] defines Preventive maintenance as "maintenance carried out to mitigate degradation and reduce the probability of failure". PM tasks can include activities like inspections, lubrication, adjustments, replacement of parts, repairs of parts wearing out, among others [22].

Per definition of critical infrastructures, it is crucial to avoid failures, therefore preventive maintenance is imperative for these assets or systems. Preventive maintenance tasks can be further split into Predetermined maintenance and Condition-based maintenance.

Predetermined PM By predetermined PM tasks we refer to scheduled tasks that do not rely on measurements of condition variables of the item. Typical implementations include clock-based and age-based strategies. In clock-based the tasks are carried out at specified calendar times and in age-based the tasks are carried out at a specified age of the item. The age of the item is measured in ways that depend on the application and examples are the time in operation, charge cycles, kilometers driven. Traditionally, predetermined PM has been the most common type of PM applied in the industry, particularly with clock-based tasks due to being easier to manage.

For critical infrastructures, it is not only crucial to avoid failures but also interruptions of their service, including those due to maintenance actions. Hence, there is a need to for reducing unnecessary inspections and/or preventive maintenance actions. The challenge is then to select the most cost efficient maintenance strategy while meeting risk and availability criteria. The development of the technology and techniques for condition monitoring has enabled massive data collection and with this critical infrastructures adopt condition-based maintenance as a way to optimize their maintenance strategies. CBM is the topic of the next section.

2.1.2 Condition-based Maintenance

Condition-based maintenance refers to those tasks that are carried out based on measurements of one or more condition variables of the item. The condition of an item typically exhibits degradation over its lifetime, which eventually leads to failure.

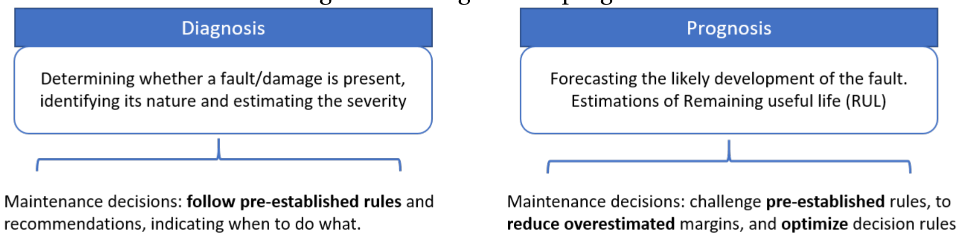
In CBM, a task is started when the degradation or accumulated damage reaches a threshold value [22]. To implement CBM, it is then required to monitor the condition of the item and its degradation. In the recent decades research on condition based maintenance (CBM) has been growing rapidly, assisted by the rapid development of computer based monitoring technologies. Research studies have proved that when CBM is correctly planned and implemented, it can be effective to improve systems reliability/availability with reduced costs [20].

CBM tasks can be classified into predictive and non-predictive as seen in figure 2.2. The principle of this classification is the base for the maintenance decisions, i.e. whether they are based on fault diagnosis or prognosis, as presented on figure 2.2.

CBM that is carried out following prognosis, i.e. forecast analyses taking into account the degradation of the item, conditions about the future as environmental, operational as well as the performance of the maintenance actions, is also known as predictive maintenance.

Currently, the trend in many fields and in critical infrastructures is to move the CBM policy from diagnosis to prognosis. Through fault diagnosis, it is possible to implement mainte-

Figure 2.2: Diagnosis vs prognosis



nance decisions by following pre-established rules and recommendation saying when to do what, i.e. regulations or recommended practices that link a condition to a suggested maintenance action. In this way, the maintenance teams performs fault diagnosis on an item and then follow the recommendation based of the findings.

Failure prognosis allows to take the maintenance to a next step in order to question such pre-established rules, to reduce overestimated margins and to optimize decision rules. Using mathematical models, it can be possible to simulate different maintenance strategies and to assess the associated effects, the maintenance costs and the operational performance. These simulations can then be used to determine the best maintenance strategy to implement [19]. The next section is dedicated to the elements and approaches for CBM modelling and prognostics.

2.2 CBM modelling and prognostics

The purpose of a CBM model is to determine a maintenance strategy that optimises the performance according to some criteria such as cost, availability or others. In general, a model designed to optimise a CBM policy should consist of two elements: (i) a deterioration model and (ii) an intervention model [23] [24]. The deterioration model is used to forecast the actual process of degradation of the health condition of a unit, while the intervention model captures the effect of maintenance and/or inspections in its health condition. Therefore the global model can be used to find the optimal performance under a given intervention strategy (parametric optimisation) or to investigate for an optimal strategy, as in figure 2.3. In this sense, the CBM model involves prognostics in pursuance to support the maintenance decision making.

There is a large amount of literature devoted to CBM optimization with a wide range of modelling frameworks and application areas. A key indicator in failure prognostics is the Remaining Useful Life (RUL). It is commonly understood as the useful life of an asset left at a particular time of operation, given a particular degradation state (defined by a health indicator) and given operational conditions in the future (if available). Its estimation is central for condition based maintenance optimization.

During the last two decades a number of degradation models have been developed to

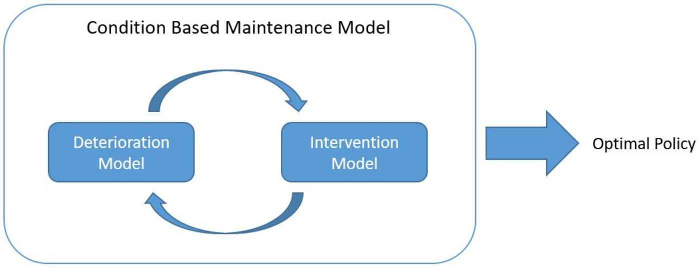
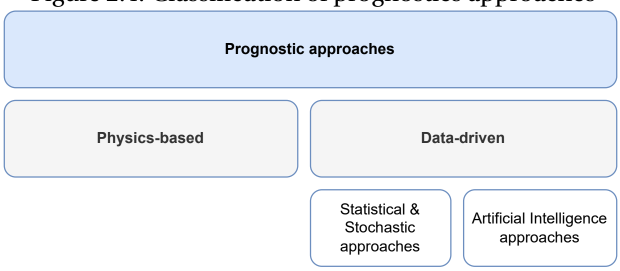


Figure 2.3: Condition-based maintenance model

capture the degradation and assist the decision making [25, 26]. The first step in CBM modelling is the identification and definition of the degradation model. Based on this, the approaches in prognostics can be categorized on physics-based and data-driven [27], as in figure 2.4.

Figure 2.4: Classification of prognostics approaches



Physics-based approaches Physics-based approaches refer to those degradation models that rely on physical laws that describe the behaviour of the damage. The behaviour of the physical model depends model parameters that are usually obtained from laboratory tests or estimated from measured historical operational data up to the current time [27]. The RUL is predicted by progressing the degradation until it reaches a threshold.

The Paris' law is one of the first crack propagation concepts [28] and an example of a physics-based model. The equation 2.1 describes a region of a crack propagation curve in terms of material dependent quantities.

$$\frac{da}{dN} = C(\Delta K)^m \tag{2.1}$$

Where a is the crack length and $\frac{da}{dN}$ is the fatigue crack growth for a load cycle N in terms of material coefficients C and m. These material coefficients are obtained empirically and depend on factors such as: environment, frequency, temperature and stress ratio.

Data-driven approaches In contrast with the physics-based, data-driven approaches are those which do not rely on physical laws or properties but instead use information from observed data to identify the characteristics of the damage progress and to predict its future evolution. Generally, these data-driven approaches are divided in two categories: one involving statistical and stochastic models and one for artificial intelligence approaches.

For the statistical and stochastic models, the aim is to use a data set to fit a degradation model to the items condition from working state until failure. This category can involve trend models, time series and stochastic processes. The law of the RUL is often estimated with these models.

In the case of the artificial intelligence approaches, the aim is to use the data with an automated model building. They rely on machine learning and based on pattern recognition and the theory of computers learning from data, without being programmed for specific tasks. Neural networks, fuzzy logic are some typical approaches of this category used for predictions of RUL. A key difference to the statistical and stochastic models is that the law of the RUL is not estimated but the uncertainty of the prediction is assessed through an error estimated on a validation data set [29].

The focus of this thesis is placed on stochastic processes and these are the approaches which are further discussed.

Stochastic processes

The degradation phenomena is many cases is considered to be random or the factors influencing it are too complex so that the degradation process cannot be described in a deterministic way. In these cases stochastic processes can be used to model the random behaviour.

A stochastic process is any process that describes the evolution of a random phenomenon. From the mathematical perspective, the theory of stochastic processes was established during the 1950s [30]. Stochastic processes have now applications in a wide range of applications in disciplines like physics, control theory, biology, image processing, signal processing, computer science and degradation modelling among others.

Mathematically, a stochastic process is a collection of random variables $\{X(t), t \in T\}$ defined in a common probability space, taking values in a state space χ and indexed by a set T, often thought of as time. For each index t in T, $X(t)$ is called the state of the process at time t. In other words, there is a system for which there are observations at certain times and the observed value at each time is a random variable [31].

In our context, the degradation is considered to follow a continuous-time stochastic process. The state of the process is random and its state space χ can be discrete or continuous. Based on this, the stochastic processes commonly used in degradation and CBM modelling can be classified in discrete-state and continuous-state deterioration.

The choice of discrete-state or continuous-state stochastic processes for CBM modelling, depends mainly on the application field. In many applications, it is reasonable to characterize the condition of the item by a finite set of deterioration states or levels and there is no need to work in every continuous value of the degradation (from an engineering practice perspective). For these applications, discrete-state stochastic process are used for CBM modelling. For other systems, it is practicable to measure and monitor the degradation in a continuous scale, without the need to classify the multiple states. Therefore, continuous-state stochastic processes can be better suited and provide more precision in the results.

A review in CBM modelling for stochastic degrading systems [20] has found that the most common discrete-state deterioration models are Markov chains and extensions of these such as semi-Markov chains and hidden Markov process to account for different model assumptions, examples are found in [32, 33, 34].

On the other hand, the most common continuous-state stochastic processes available in the literature for CBM and degradation modelling are Wiener process, Gamma process and Inverse Gaussian process. A Wiener process [35, 36, 37] can describe degradation that might exhibit increments and decrements. Gamma process [38, 39, 40] are more suitable to monotonic degradation phenomena and Inverse Gaussian process [41, 42, 43] can be more flexible for incorporating random effects and/or co-variates to account for non-homogeneous degradation.

All of these stochastic processes commonly used in degradation and CBM models have something in common: they possess the Markov property and hence, are considered Markov processes (regardless of their designated noun).

There is a good reason why Markov processes are the most commonly applied stochastic processes. In a stochastic process the state of the process $X(t)$ at a certain time, is random and hence has an associated probability to its outcome or observation. In general, this probability depends on what has been obtained in previous observations at previous times. However, this general situation is tedious and very difficult to treat with a tractable formalism [31]. Because of this, simplified processes that are still very relevant are the most commonly used, such as Markov processes which are further described in the following section.

2.3 Markov Processes

A major interest in applied probability is the time-dependent evaluation of the state of random phenomena. The theory of Markov processes have played a key part in these investigations [44] and in particular for the CBM modelling and assessment approach presented in this thesis. The theory of Markov processes take their name from the Russian mathematician Andrey A. Markov who led research on the mathematical description of stochastic processes.

2.3.1 Markov Chains

Markov processes are most regularly associated with Markov Chains and these are a good way to introduce the theory of Markov processes. A Markov chain is a stochastic process $\{X(t), t \geq 0\}$ where t denotes time, that has the Markov property (described next). The state space $\mathcal{X}$ is either finite or countable infinite. When the time domain is continuous, we have a continuous-time Markov chain.

When $X(s) = i$, the chain is said to be in state i at time s. The conditional probability for the chain to be in state j at time $t+s$, given that at time s the state was i, is:

$$Pr(X(t+s)) = j \mid X(s) = i, X(u) = x(u),\ 0 \le u < s) \tag{2.2}$$

Markov property The Markov property states that for a Markov chain $\{X(t), t \ge 0\}$, the following condition is true:

$$\begin{aligned} &Pr(X(t+s)) = j \mid X(s) = i, X(u) = x(u),\ 0 \le u < s) \\ = &Pr(X(t+s)) = j \mid X(s) = i) \text{ for all possible } x(u),\ 0 \le u < s. \end{aligned} \tag{2.3}$$

☞ When the present state of the process is known, the future development of the process is independent of anything that has happened in the past [22]. The stochastic process is said to have no memory, therefore the Markov property is also known as the *memoryless* property.

A Markov chain is said to be homogeneous or stationary if the probability of moving from one state to another in a time interval, depends only on the length of time interval and not on where the interval is on the time axis, i.e. for all states of the process i, j and for any time s, k:

$$Pr(X(t+s) = j \mid X(s) = i) = Pr(X(t+k) = j \mid X(k) = i) \tag{2.4}$$

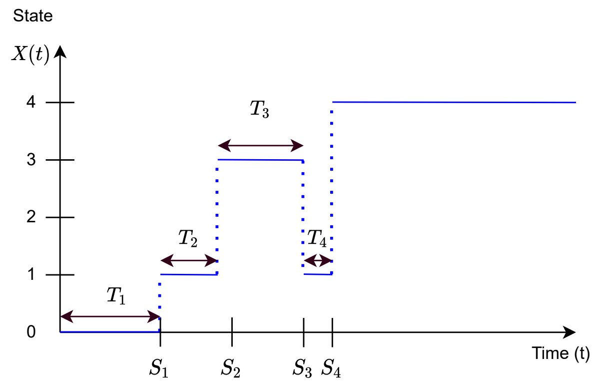
Consider the Markov process $\{X(t), t \ge 0\}$ with state space $\mathcal{X} = \{0, 1, 2, ..., r\}$ and that process begins at state 0 at time 0, i.e. $X(0) = 0$. Let $0 = S_0 \le S_1 \le S_2 \le ...$ be the times at which jumps between states occur and $T_j = S_{j+1} - S_j$ be the time between two consecutive jumps. A possible trajectory of this random process is shown in figure 2.5.

The time spent during a visit to state i is random, e.g. in figure 2.5 it can be observed that state 1 is visited twice during the illustrated trajectory with duration T_2 and T_4 respectively. Let $\tilde{T}_i$ be the random variable denoting the duration of a visit to state i, called *sojourn time*. Let us assume that the process enters state i and s units of time later it remains in state i. We want to find the probability that it will stay in state i for t units of time more, i.e. $Pr(\tilde{T}_i > t+s \mid \tilde{T}_i > s)$. Because of the Markov property, we know that the probability for the process to stay t more units of time in state i depends only on its current state i. We have:

$$Pr(\tilde{T}_i > t+s \mid \tilde{T}_i > s) = Pr(\tilde{T}_i > t) \text{ for } s, t \ge 0 \tag{2.5}$$

The random variable $\tilde{T}_i$ is memoryless. Assuming that the process is homogeneous and a continuous time domain, $\tilde{T}_i$ must be exponentially distributed, i.e. $\tilde{T}_i \sim Exp(a)$. The

Figure 2.5: Trajectory of a Markov chain



amount of time that the process spends in a state i and the next state to visit are independent random variables. Let α_i denote the rate at which the process leaves state i and P_{ij} denote the probability that it goes into state j. Then, the transition rate from state i to state j is:

$$a_{ij} = \alpha_i P_{ij} \text{ for all } i \neq j$$

Let T_{ij} be the time the process spends in state i before a transition to state $j (\neq i)$. For the homogeneous Markov chain we have that this time is random and exponentially distributed with rate a_{ij}. Considering a small time interval Δt, we have:

$$\begin{aligned} P_{ii}(\Delta t) &= Pr(\tilde{T}_i > \Delta t) = e^{-\alpha_i \Delta t} \approx 1 - \alpha_i \Delta t \\ P_{ij}(\Delta t) &= Pr(T_{ij} > \Delta t) = 1 - e^{-a_{ij} \Delta t} \approx a_{ij} \Delta t \end{aligned}$$

Therefore, we have by taking the limit when Δt is small:

$$\lim_{\Delta t \to 0} \frac{1 - P_{ii}(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{Pr(\tilde{T}_i < \Delta t)}{\Delta t} = \alpha_i \quad (2.6)$$

$$\lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{Pr(T_{ij} < \Delta t)}{\Delta t} = a_{ij} \text{ for } i \neq j \quad (2.7)$$

Chapman-Kolmogorov equations

By using the Markov property and the law of total probability we can find the probability that the process is in a state j at a given time. Let us split a time interval $(0, t + \Delta t)$ into two parts. First, we consider that a transition from a state i to a state k occurs in a small interval of time $(0, \Delta t)$, and then a transition from state k to state j occurs in the rest of the interval. We have:

$$P_{ij}(t+\Delta t) = \sum_{k=0}^{r} P_{ik}(\Delta t) P_{kj}(t) \tag{2.8}$$

Equation 2.8 is known as the Chapman-Kolmogorov equation and it was named after the British mathematician Sydney Chapman and the Russian mathematician Andrey Kolmogorov, who independently derived the equation.

To find $P_{ij}(t)$ it is possible to write a set of differential equations. We begin by considering:

$$P_{ij}(t+\Delta t) - P_{ij}(t) = \sum_{\substack{k=0\\k\neq i}}^{r} P_{ik}(\Delta t) P_{kj}(t) - [1 - P_{ii}(\Delta t)] P_{ij}(t)$$

Dividing by Δt and taking the limit when $\Delta t \to 0$:

$$\lim_{\Delta t\to 0} \frac{P_{ij}(t+\Delta t) - P_{ij}(t)}{\Delta t} = \lim_{\Delta t\to 0} \sum_{\substack{k=0\\k\neq i}}^{r} \frac{P_{ik}(\Delta t)}{\Delta t} P_{kj}(t) - \alpha_i P_{ij}(t)$$

Then, replacing by equation 2.7, we get:

$$\frac{d}{dt} P_{ij}(t) = \sum_{\substack{k=0\\k\neq i}}^{r} a_{ik} P_{kj}(t) - \alpha_i P_{ij}(t) = \sum_{k=0}^{r} a_{ik} P_{kj}(t) \tag{2.9}$$

Equations 2.9 are known as the Kolmogorov backwards equations [22] and they describe how the probability that the process is in a certain state changes over time. This characterizes the evolution of the Markov process.

As mentioned earlier, for a homogeneous Markov chain the sojourn times ($\tilde{T}_i$) are exponentially distributed. This is a major limitation for their applicability in CBM modelling. As pointed in section 2.2 and figure 2.3, a CBM model must capture not only the stochastic degradation but also the maintenance interventions like inspections, repairs, replacements. For the global model to be a Markov chain, restrictive assumptions must be made, which might not be realistic, e.g. the time for a repair must be random and exponentially distributed, the inspection or condition monitoring must be continuous, among others.

2.3.2 Jump and diffusion Markov processes

Any stochastic process with the Markov property is a Markov process. In the field of RAMS, failures are random events that are assumed to be able to occur at any time and degradation processes are considered to be continuous in time. Therefore, we deal mostly with continuous-time Markov processes. The state space however, can be discrete, continuous, or hybrid and the evolution of the process can also be discrete, continuous or hybrid.

Jumps processes

A jump process is a stochastic process with discrete movements, called jumps, with random arrival times, rather than movement with continuous paths. A jump process with a finite number of jumps in a finite interval is called a pure jump process. A Markov chain is an example of a *pure jump* process that jumps between discrete states.

Gamma and Inverse Gaussian processes are examples of pure jump processes that evolve in a continuous state space, commonly used in degradation and CBM modelling, as mentioned in section 2.2. These processes belong to the family of Lévy processes.

Lévy processes

Lévy processes are named after the french mathematician Paul Lévy [45]. A Lévy process $\{X(t), t \geq 0\}$ is a stochastic process with the following properties:

1. $X(0) = 0$, almost surely
2. It has independent and stationary increments.
3. It has stochastic continuity: for any $\epsilon > 0$ and $t \geq 0$, it holds that: $\lim_{h\to 0} Pr(|X(t+h) - X(h)| \geq \epsilon) = 0$

The distribution of a Lévy process has the property of infinite divisibility, i.e. for any integer n, the law of a Lévy process at time t can be expressed as the law of n independent random variables, that correspond to the increments of the Lévy process over time intervals of lenght t/n, which by condition 2 are independent and identically distributed.

Condition 3, does not imply that the sample paths are continuous. Jump processes such as the Poisson process and the Gamma process satisfy this condition. However, a process with continuous paths can also satisfy these three conditions to be a Lévy process. The Brownian motion, also called Wiener process, is a Lévy process with continuous paths.

Brownian motion and diffusion processes

In some instances, systems are modelled by a process that moves continuously between all possible states that lie in an interval of the real line. An example of such process is the Brownian motion.

Brownian motion Brownian motion was first described in 1828 by the botanist Robert Brown [46] while studying the movement of pollen particles suspended in a fluid. Brown found out that the movement was "chaotic", exhibiting an irregular random behaviour. In 1923 Norbert Wiener [47] established mathematically the foundation of a stochastic process describing the Brownian motion. Consequently, the terms Brownian motion and Wiener process are used interchangeably.

The Brownian motion $\{W(t), t \geq 0\}$ is a stochastic process modelling random continuous motion with the following properties:

1. $W(0) = 0$
2. $W(t)$ is continuous at $t \geq 0$, i.e. it has continuous sample paths with no jumps.
3. It has both stationary and independent increments
4. For $0 \leq s < t$, the random variable $W = W(t) - W(s)$ has a normal distribution with mean 0 and variance $\sigma_w^2 = \sigma^2(t-s)$, i.e. $W \sim \mathcal{N}(0, \sigma^2(t-s))$.

As a process with independent increments, a Brownian motion is also a Markov process. Because of the conditions 1 and 4, we have $W(t) = W(t) - W(0) \sim \mathcal{N}(0, \sigma^2(t-0)) = \mathcal{N}(0, \sigma^2 t)$. $W(t)$ is called the Wiener process or classical Brownian motion. Since a degradation process do not generally have a zero mean, it is common to include a drift measure as:

$$X(t) = \mu t + W(t) \text{ for } t \geq 0 \tag{2.10}$$

Then $\{X(t), t \geq 0\}$ is the Brownian motion with drift rate $\mu > 0$ and infinitesimal variance σ^2, where $W(t)$ is the classical Brownian motion. It follows that $X(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$. This is the Brownian motion or Wiener process commonly used in degradation and CBM modelling mentioned in 2.2.

Diffusion process A diffusion process is a continuous-time, continuous-state process with paths that are continuous everywhere. It can be considered a generalized version of Brownian motion. While Brownian motion originates from the random motion of molecules with random direction of motion, diffusion is the movement of particles from areas of high concentration to areas of low concentration. Thus, diffusion can be seen as occurring when a system is not in equilibrium and random motion tends to bring the system to uniformity.

In probability theory, a diffusion process is defined as a solution to a stochastic differential equation. Consider a continuous-time continuous-state Markov process $\{X(t), t \geq 0\}$ with a probability distribution given by:

$$F(y, t|x, s) = Pr[X(t) \leq y | X(s) = x] \text{ for } s < t \tag{2.11}$$

If the derivative $f(y, t|x, s) = \frac{\partial}{\partial y} F(y, t|x, s)$ exists, it is called the transition density function of the diffusion process, and it satisfies the Chapman-Kolmogorov equation:

$$f(y, t|x, s) = \int_{-\infty}^{\infty} f(y, t|z, u) f(z, u|x, s) dz \tag{2.12}$$

A diffusion process is a Markov process that satisfies the following three conditions [48]:

1. $Pr[|X(t+\Delta t) - X(t)| > \epsilon | X(t)] = o(\Delta t)$, for $\epsilon > 0$, meaning that the sample path is continuous. Alternatively, the process is continuous in probability.

2. $E[X(t+\Delta t)-X(t)|X(t)=x] = a(x,t)\Delta t + o(\Delta t)$, so that:

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{E[X(t+\Delta t)-X(t)|X(t)=x]}{\Delta t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(y-x)f(y,t+\Delta t)|x,t)dy \\ &= a(x,t)\end{aligned}$$

3. $E[\{X(t+\Delta t)-X(t)\}^2|X(t)=x] = b(x,t)\Delta t + o(\Delta t)$ is finite, so that:

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\{E[X(t+\Delta t)-X(t)\}^2|X(t)=x]}{\Delta t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(y-x)^2 f(y,t+\Delta t)|x,t)dy \\ &= b(x,t)\end{aligned}$$

The function $a(x,t)$ is called the infinitesimal drift of $X(t)$ and the function $b(x,t)$ is called the infinitesimal variance of $X(t)$. Let the increment of $X(t)$ over a small interval dt be denoted $dX(t)$. It can be shown that if $W(t)$ is a classical Brownian motion, the above properties can be incorporated into the following stochastic differential equation:

$$dX(t) = a(x,t)dt + b(x,t)dW(t) \text{ for } t \geq 0 \tag{2.13}$$

where $dW(t)$ is the increment of $W(t)$ over the small interval $(t, t+\Delta t)$. Equation 2.13 is know as a Itô stochastic differential equation, whose solution is a diffusion process $\{X(t), t \geq 0\}$. Although there exist different diffusion processes, they differ only in the way the drift and diffusion coefficients are defined [48]. If we make $a(x,t) = 0$ and $b(x,t) = 1$ and solve equation 2.13, we obtain the classical Brownian motion, and by making $a(x,t) = \mu$ and $b(x,t) = 1$, we obtain the Brownian motion with drift from equation 2.10. Hence, Brownian motion is a particular diffusion process.

In general, the techniques used in the analysis of diffusion processes are highly developed and have a unified theory based on Itô calculus and stochastic differential equations. On the contrary, the theory for studying random jumps processes consists of a large collection of special models and methods. This is one of the main motivations to introduce Piecewise deterministic Markov processes as a general class of non-diffusion stochastic models [15], which are the main topic of this thesis and are introduced in the next subsection.

2.4 Piecewise deterministic Markov Processes

Piecewise deterministic Markov Processes (PDMP) were introduced by M.H. Davis in 1984 [15] as a general class of non-diffusion stochastic models. Davis points out that almost all continuous-time stochastic process models of applied probability consists of some combination of:

(a) Diffusion.

(b) Deterministic motion.

(c) Random jumps.

A review on the literature, shows that the techniques used for diffusion processes contrasts radically from the ones usually employed in connection with the other two classes. Moreover, the techniques for diffusion processes form a unified and well developed theory based on the theory of Itô calculus ans stochastic differential equations (as mentioned in section 2.3.2), while the non-diffusion theory is formed by a large and heterogeneous collection of special models and methods applicable for specific problems [15].

For Davis, the ambition when introducing PDMP is to place non-diffusion models (those involving classes (b) and (c) from the list above but not (a)), under a similar foundation to diffusion theory in the availability of both:

1. A generic model capable to encompass a wide selection of special cases.
2. General methods based on stochastic calculus for analysis of the generic model.

2.4.1 Stochastic hybrid systems

Before introducing the formalism of PDMP, it is worth to present the concept of stochastic hybrid systems. A hybrid system is a dynamical system that cannot precisely be represented and analyzed either by methods of continuous systems theory or by methods of the discrete system theory [49].

Discrete systems theory assumes that the system has abrupt changes in its state but the continuous movements of the system cannot be described precisely. Continuous systems theory on the other hand, assumes that the system under consideration is described by some differential equation, which in order to have a unique solution must satisfy a smoothness condition, called *Lipschitz continuity*, a strong form of uniform continuity.

Stochastic hybrid systems arise from the combination of hybrid systems with the theory of probability to deal with randomness. A stochastic hybrid system is a dynamical system with both continuous and discrete behaviour in which some variables cannot be described in a deterministic way, but they involve randomness.

Randomness and hence stochastic hybrid systems are of particular interest in CBM modelling, where we consider stochastic degradation processes, to account for uncertainties of the influencing factors such as operating conditions, loads and strength of components, environmental conditions and so on. Such randomness can be in the form of diffusion or random jumps as described in section 2.3.2. A PDMP is then a hybrid stochastic process which exhibits a combination of continuous state changes and random jumps.

2.4.2 PDMP formalism

A PMDP is a Markov process consisting of a mixture of deterministic motion of random jumps. It is a stochastic hybrid process $\{I(t), X(t)\}\ t \geq 0$ with values in a discrete-continuous space $E \times \mathbb{R}^d$.

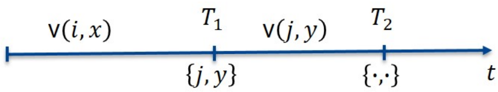
The first component $I(t)$ is discrete and takes values in a finite state space E. The second component $X(t)$ is continuous and takes values in a d-dimensional real coordinate space $\mathbb{R}^d$.

Evolution of the process

The probability law of the PDMP is determined by three local characteristics: the jump rate (λ), the deterministic motion (or flow) (v) and the transition measures (Q) [50].

Consider the process starts in a state $\{I(0), X(0)\} = (i, x)$. In a simple overview the process evolves as follows:

Figure 2.6: Evolution of a PDMP



1. The process follows the deterministic motion $v(i, x)$ until a jump occurs at T_1. Such jump can occur either:
    - Randomly, with rate $\lambda(i, x)$.
    - When the flow hits a boundary in the continuous-state space.
2. The post jump location is selected from the transition measure $Q[(i, x), (j, y)]$.
3. The motion restarts from this state.

Between two consecutive jumps, the process follows the deterministic motion, which in general corresponds to the solution of a set of differential equations for a fixed discrete state, i.e. given that $I = i$ between two jumps, $X(t)$ is solution of:

$$\frac{dx}{dt} = v(i, x)$$

A jump from state (i, x) towards discrete state j occurs with a rate $\lambda(i, x)Q(i, x, j)$, where $\lambda(i, x)$ is the rate at which the process leaves i and $Q(i, x, j)$ is the probability distribution of the jump from i to j. In this sense, the rate at which the process jumps from the discrete state i to a discrete state j is dependent on both the discrete and continuous components before the jump (i, x) and the discrete component after the jump (j). Hence, the rate does not need to be constant as it can be dependent on the continuous component. Likewise, there is no extra generality in allowing a PDMP to be non-stationary, since time can be included as one component of $X(t)$.

2.4.3 Hybrid modelling

The class of PDMP provides a framework for models consisting of a stochastic hybrid system that combines deterministic motion with random jumps. From a modelling perspective this opens up a large range of possibilities. Davis [15] claims that the class of PDMP provides a general family of stochastic models that virtually covers all non-diffusion applications. Moreover, it provides a framework for modelling hybrid phenomena involving state transitions that cannot be represented or analyzed appropriately by the methods developed in continuous or discrete systems theory.

Traditionally, the methods used in reliability analyses assume a constant operated context and are usually supported by the practice of thinking of the worst scenario [51]. The possibility of including bi-directional interactions between $I(t)$ and $X(t)$ is key in the hybrid modelling used in the field of dynamic reliability. Dynamic reliability is an extension of the traditional reliability models and methods, in order to be able of capturing the dynamics of the operational and environmental conditions in which systems evolve.

Hybrid modelling is an approach to solve dynamic reliability problems based on a "separation of concern" approach [52]. The idea is to model two mutually dependent processes: a deterministic and a stochastic, and then couple them by means of shared variables. Under this formulation, the deterministic process can describe the dynamic system in terms of physical laws that determine its physical behaviour, like thermofluid, chemical, rotational, by a set of differential equations. The stochastic process accounts for random events such as component failures that can modify the physical behaviour of the system described by the differential equations, and reversely, the rates of the random jumps can depend on the evolution of the continuous variables described in the deterministic model. Commonly this stochastic process is a random jump process and not a diffusion one, hence, the resulting process from hybrid modelling is a PDMP.

Figure 2.7: Hybrid modelling (adapted from [52])

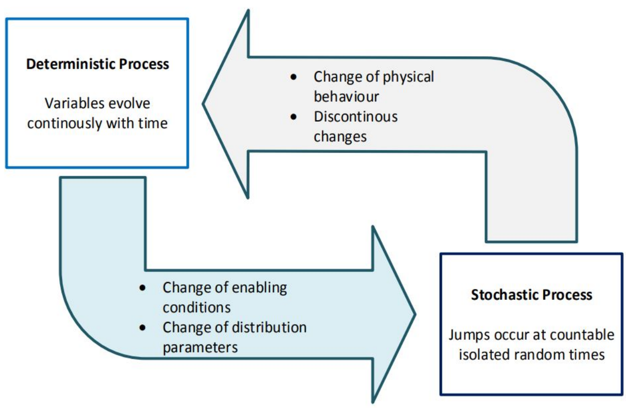


Figure 2.7 shows the interaction between the deterministic and stochastic process. Examples of such interactions, in a PDMP $\{I(t), X(t)\},\ t \geq 0$, for a dynamic reliability problem

are [53]:

- X(t) acts on I(t). A physical variable reaching a boundary or threshold in the continuous-state space can cause a discrete abrupt change such: a tank explosion from over pressure, change in a fluid state due to temperature, activation of an on/off controller due to reaching a set point, among others. A physical variable can also influence the rate of the discrete events, for example, the failure or deterioration rate may depend on variables such as temperature and pressure.

- I(t) acts on X(t). The differential equations governing the deterministic motion may change when discrete events such failures in the system, an on/off control function on the system is activated, and others.

For CBM modelling, this thesis proposes a specific case of PDMP in which the deterministic motion is linear and intended to keep track of time to maintenance interventions (boundary jumps). In this way, the differential equation is reduced to a trivial one. This can be seen as a state augmentation, i.e. a process involving enlargement of the state space [54] that allows to formulate the problem in a Markovian manner. This class of models have also been called Piecewise-linear Markov processes and have been first introduced in 1966 by Gnedenko and Kovalenko [55] and more developed later by Vermes [56] in 1980 with a fairly complete theory of optimal control [15]. Boundaries or threshold are placed in the continuous-state space to mark the time of a maintenance intervention with a corresponding jump in the process. For the sake of generalization, these processes are still called PDMP in this thesis, as the class of PDMP introduced by Davis [15] is a generalized version of the Piecewise-linear processes. It is important to clarify what is meant by single-item model and multi-component model throughout this thesis.

2.5 CBM models for single-item and multi-component systems

The word *"system"* is commonly used in all fields and aspects of life. Examples of its use are: solar system, decimal system, transport system, equation system, digestive system, political system and infinite more. By the use of the word system, people usually refer to either an ordering or to something consisting of interacting parts. Aslaken [57] proposed the following formal definition:

A system consists of three related sets:

- A set of elements.

- A set of internal interactions between elements.

- A set of external interactions between one or more elements and the external world.

In the field of reliability and maintenance, the elements of the system are usually classified as subsystems, subsubsystems, and so on, until the component level [58]. The system can be broken down in levels of hierarchy, called indenture levels, representing a break down structure. An example with three levels of indenture is shown in figure 2.8.

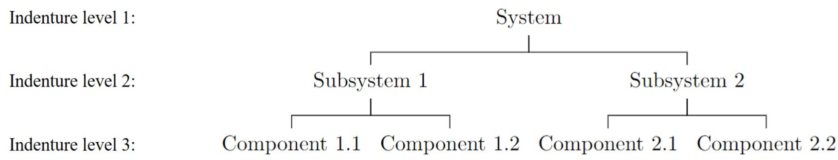


Figure 2.8: System breakdown structure example

System modelling refers to the craft of developing an abstract representation of a real system with the purpose of analyzing its behaviour in terms of performance and dependability, without resorting to measurements on the real system or prototypes [59]. Within CBM modelling, we can consider that the external interactions refer to the interventions and maintenance actions on the system, while the internal interactions between the elements of a system are commonly referred to as dependencies. CBM modelling of systems with discrete state deterioration is related to multi-state systems. In this thesis we distinguish between two ways to approach the CBM modelling for a system:

i. Begin the modelling work at the system level perspective. The deterioration states are directly defined at the system level, while limiting the study to the states that are relevant for the performance assessment.

ii. Begin the modelling work at the components perspective. The deterioration states are defined at components level and then the states at the system level can be computed, in order to assess the performance of a maintenance strategy.

The result from i. is what in this thesis is called *Single-item model*. The word "*item*" is used to refer to any system, subsystem or component that can be considered as an entity [22]. The system is treated as one unit with independent states. The maintenance policies are directly proposed and assessed at the system level. Figure 2.9 shows an overview of a CBM model for a *single-item*.

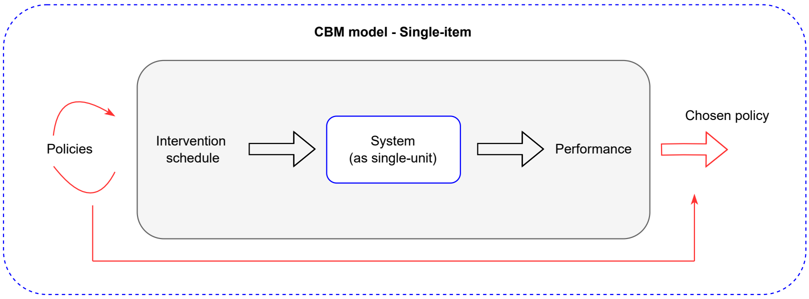


Figure 2.9: Single-item CBM model

On the other hand, with ii. we intend to begin by defining states for each component of the system, then study how they interact to formulate the evolution at the system level and finally assess the performance of the policy at the system level. Here, a model following this approach is called a *multi-component model*.

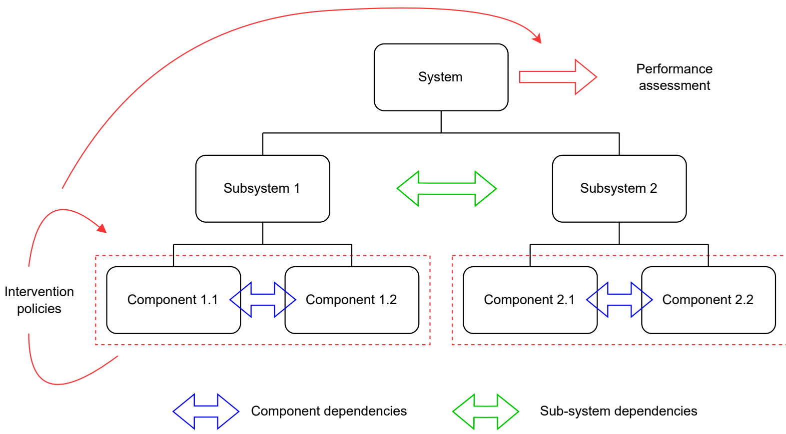


Figure 2.10: Multi-component CBM model

Figure 2.10 gives an overview of a *multi-component model* for CBM. Some of the common dependencies addressed within CBM modelling of systems include structural, stochastic, economic and resource dependencies.

Structural dependence Structural dependence is related to the structural, static relationships among components, from a technical or a performance perspective. From the technical point of view, it relates to systems which are configured in a way that maintaining one component requires or prohibits other to be maintained or at least dismantled. From the performance perspective it refers to systems in which their performance is impacted by the configuration and by the performance of the components, e.g. series, parallel and k-out-of-n relations [60].

Stochastic dependence Stochastic dependence refers to the interaction between failures or deterioration processes of components of the system, as well as the influence of external factors on these. For example the deterioration of one component may influence the rate of deterioration of another component, and if we consider that these deterioration process are stochastic, then the parameters (and the associated stochastic process) are impacted.

Economic dependence Economic dependence indicates that the cost of maintenance is influenced by joining the maintenance tasks of different components.

Resource dependence Resource dependence concerns systems in which maintenance actions can only be executed if the required resources (e.g. spare parts, tools, personnel) are available. It concerns systems in which several components share limited resources, requiring maintenance to be optimized at the system level [61].

Most of the existing CBM strategies for multi-component systems have traditionally been done at the component level, meaning that the optimal CBM policy for a single component is employed per component in the system without taking into account components interdependencies [20].

2.6 Chapter conclusion

The focus of this thesis is placed in studying CBM problems with prognosis, which is also known as predictive maintenance. Within prognostic approaches the central point of this thesis are Piecewise deterministic Markov processes, a stochastic process presented as a general class of non-diffusion stochastic models, in which the future development of the process depends only on a current state of the process and not on what has happened in the past.

The key element of a prognosis approach is the deterioration of the system or unit. Since the focus of this thesis is placed on PDMP, this means that the applications considered through this thesis assume that the deterioration of the system or unit evolves in a discrete-state space with random jumps and continuous-time. This is the case of many engineering applications, where due to practical reasons it is more reasonable to characterize the condition or health of the system by a finite set of deterioration states instead of in a continuous-state space.

The class of PDMP that is the focal point in this thesis is a particular category, also known as Piecewise linear Markov process. To keep things in a general class, we keep the PDMP denomination. In this class, the deterministic evolution of the continuous component is linear. With this background we proceed to propose PDMP for modelling and studying CBM problems of a single-item under certain assumptions and later as a framework for multi-state multi-components systems subjected to some dependencies, together with an approach to assess the performance of maintenance policies at the system level. To solve the models we propose an analytical approach with a numerical scheme and present Monte Carlo simulation for validation.

Chapter 3

Piecewise deterministic Markov process for condition-based maintenance models of single-items

This chapter explores on the application of a Piecewise deterministic Markov process (PDMP) to encompass different modelling assumptions as non-negligible maintenance delays and inspection-based condition monitoring of a single-item. These assumptions are relevant for many critical infrastructures in civil engineering or in oil & gas industry whose deterioration states are classified at a very high level of abstraction among a finite and small set of possible states. A formalism to model this type of problems is proposed in which the deterministic motion of the PDMP is reduced to a trivial differential equation to track the time elapsed between events. A numerical scheme for quantification, as an approximation of the Chapman-Kolmogorov equation, is presented. Later, a case study dealing with CBM of road bridges by the NPRA (Norwegian Public Roads Administration) is presented, guiding through the modelling and quantification approach.

The final objective is to challenge pre-established CBM strategies that might be not optimal and to provide time dependent or long term decision criteria to optimize them. The decision criteria are basically the probability, for the maintained system, to be in a deteriorated or critical state at any time (e.g closure of the bridge), or the maintenance strategy mean cost over a period of time. The main assumptions we keep from the application field of critical infrastructures are: we consider that the system condition is characterized at a high level of abstraction with few discrete states (often given by guidelines in the application field), the complete condition of the system is only known at inspection dates, and the maintenance tasks require a delay before execution.

The main content of this chapter is based on the published articles included in appendices A and B, which have been prepared as part of this PhD project.

3.1 Background and state-of-the-art

There is a large amount of literature devoted to CBM optimization with a wide range of modelling frameworks and application areas. From a generic perspective, we can distinguish two modelling methodologies: i) the approaches based on the description of scenarios, with an exhaustive listing of all the possible sequences of events related to the deterioration evolution and the maintenance effects on a given horizon, ii) the approaches based on the description of the states of the maintained system and the possible transitions between them.

Usually, the first ones are preferred when the number of scenarios is low enough to be described in a tractable way. Analytical solutions for the calculation of associated costs are commonly based on renewal theory and the identification of a renewal process. The second ones are preferred when the number of scenarios is too large. These are the focus of this thesis, in which the deterioration of a system is considered the state of a stochastic process as described in section 2.2. It can be easier to list system states and to model transitions from one state to the possible next ones instead of looking at the complete sequences of possible events on a given horizon. Certainly, such approaches are tractable when the number of states is reasonable or when it can be reduced enough for the modelling and optimization purpose. In this case, analytical solutions for the calculation of associated costs are usually based on the identification of a Markov process or an extension of such a process. Both approaches can be a good support to build Monte Carlo simulation algorithms and to empirically optimize a CBM strategy.

Under these circumstances, the inspections and the delay in the maintenance schedule could increase quite a lot the number of possible scenarios to list, whereas the reduced number of discrete states is a benefit for using a modelling framework based on states and transitions. This is what is proposed here.

In some applications it is practicable to monitor, measure and describe the condition or health of the system, in a continuous-state space. In these cases, the deterioration process can then be modelled by a continuous-space-time stochastic process. CBM models considering this, usually model the deterioration with a diffusion process and commonly used are the gamma process, inverse Gaussian process and Wiener processes. Some recent examples of CBM models that consider continuous-space-time stochastic process are found in [62, 63, 64, 65, 66].

However, for many applications it is more reasonable to characterize the condition or health of the system by a finite set of deterioration states. This thesis is focused on CBM modelling for this type of applications. In these cases, the deterioration is modelled by a jump process between the discrete deterioration states. The time of the jumps cannot be predicted without any uncertainty and are therefore considered to be random. The sequence of deterioration states that the system experiences is then described by a continuous-time discrete-state stochastic process. In addition to these random jumps, the system experiences changes of state according to the maintenance intervention schedule. These types of problems are commonly related to Markov processes [20]. To support the decision-making associated to maintenance, Markov (or semi-Markov) decision processes (MDP) are usually proposed. MDP are controlled stochastic processes in which the outcome has an associated uncertainty.

MDP have been proposed for CBM of different critical infrastructures. Tao et al. [67] consider the problem of optimizing maintenance strategies for highway bridges subject to progressive deterioration and sudden earthquakes. Compare et al. [68] develop a decision-support framework for the management of gas transmission networks subject to degradation modelled as a Markov process. González-Domínguez et al. [69] use Markov-chains to model the deterioration of the roofs of healthcare centers. The maintenance optimization problem on these cases is formulated as a MDP. In some cases, the condition monitoring process does not reveal the true state of the system with certainty. The system dynamics are determined by a MDP but the decision maker may not directly observe the underlying state with certainty. To handle this, observation probabilities over the set of possible system states are introduced in the model and the resulting framework is named a Partially Observable Markov Decision Process. Recent examples can be found in [68], [70], [71].

In most works related to MDP, the action from a decision related to maintenance (modelled as a transition to other discrete states) takes place at the inspection time instantaneously. Some have consider a duration for the maintenance action with the system being stopped immediately and restarted after the intervention. We intend in this chapter to propose a framework capable of addressing cases in which the delay from the time of the decision to the time of the maintenance action may result in further deterioration of the system, which in return will require a different maintenance action than originally planned. In order to characterize the transitions, support the modelling work, and to provide a formalism that can be relevant for a large set of application cases, we propose to use a specific class of Piecewise Deterministic Markov Process (PDMP). This PDMP can serve as a basis to implement a numerical scheme and provide numerical solutions for the maintenance strategy cost evaluation. It can be also a basis to build a Monte Carlo simulation algorithm and provide empirical solutions for the maintenance costs.

A PDMP can be considered as an extension of a Markov chain that incorporates continuous variables to allow a combination of deterministic motion and random jumps. In the framework of dynamic reliability, the continuous variables are used to describe physical phenomena that influences the jump process between discrete states and that can be defined by rather complex differential equations. Some works that have proposed PDMP as a modelling framework for problems that combine deterministic behaviour (described using physics-based knowledge and equations) and stochastic jumps are found in [72], [73], [74]. Other applications of PDMP for CBM focus on problems that require a combination of random jumps and jumps that occur at deterministic times (meeting a maintenance schedule set in advance). Examples of these can be found in [75], [76], [77].

3.2 PDMP

In this section we present in details the modelling framework by defining the modelling assumptions and then the formalism of the corresponding PDMP with linear deterministic motion. At last, a method for numerical calculations of the state probabilities of the PDMP is developed. It is based on a classical Euler numerical scheme. We also provide a formal proof of the convergence of this scheme towards the Chapman-Kolmogorov equations of the PDMP.

3.2.1 Modelling assumptions

As described in section 2.2, a model designed to assess the performance of a CBM policy, consists of two elements: a deterioration model and an intervention model. We propose PDMP as a framework for studying CBM problems with the assumptions explained here under and summarized in figure 3.1.

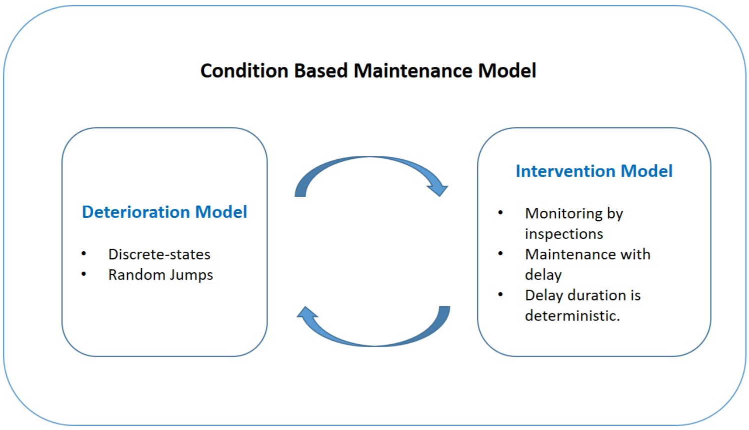


Figure 3.1: Model assumptions for the proposed framework

On the deterioration model side, we assume that the system condition is described by a set of discrete states and the deterioration follows a random jump process between these states. There is no assumption about homogeneity and the jump rate could depend on time as will be presented later.

On the intervention model, there are three main types of monitoring schedules considered in CBM models [20]: continuous monitoring, periodic and non-periodic inspection. When monitoring is continuous the jumps associated to the deterioration process are commonly detected quickly, so information about the health condition of the system could be available on real time or near real time.

When monitoring is performed via inspections (periodic or non-periodic), the condition of the system is unknown to the operator until an inspection is executed. In these cases, the choice of the inspection times obviously influences the performance of the maintenance policy. Our proposed model is aimed at applications of this type, without additional assumptions about the periodicity (or non-periodicity) of the inspection scheme, or the quality of the inspections. Discussions about how these can be handled are presented during the case-study.

In addition to the inspection-based monitoring, we consider that there is a significant time elapsed between the date on the detection and planning of a maintenance task until the date of the execution of the maintenance task. We refer to this elapsed time as maintenance delay. We consider that such delay is mainly due to logistic reasons and as such, its duration is assumed deterministic. It is worth to point out that during this delay the stochastic process is not stopped, which means that further deterioration of the system can occur, requiring a different maintenance task than the originally planned one.

To handle problems with these assumptions, we resort to a specific class of PDMP: the deterministic motion is linear and is intended to keep track of the time to an intervention jump. This will allow compensation of a lack of Markov property and will facilitate the description of the transitions between states.

3.2.2 Formalism

We consider a stochastic hybrid process $\{I(t), X(t)\} t \geq 0$ with values in a discrete-continuous space $E \times R$.

Variables

The first component, $\{I(t)\}$ is discrete and used to represent the deterioration states of the unit. We consider that the deterioration states can be categorized in a finite number N of levels. E is the finite set made of N points.

The second component, $\{X(t)\}$ is continuous, introduced in our case as a way to keep track for the intervention jumps that occur at specified times. We consider $\mathbf{x} = (x_1, x_2, t)$, $\mathbf{x}$ is a vector in which x_1 corresponds to the date of the next inspection, x_2 corresponds to the date of the next maintenance operation, and t stands for time. Hence, the continuous component $\{X(t)\}$ evolves in R, a three-dimensional orthotope of $\mathbb{R}^3$.

The process $\{I(t), X(t)\} t \geq 0$ experiences jumps at random times and jumps at intervention times. Between the times of two consecutive jumps (random or deterministic) the continuous component $X(t)$ evolves with deterministic motion.

Deterministic motion

In general, the deterministic motion of a PDMP corresponds to the solution of a set of differential equations for a fixed discrete state, i.e. given that $I = i$ between two jumps, X is solution of:

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{v}(i, \mathbf{x}) \tag{3.1}$$

In our case, the deterministic evolution of the continuous component between two consecutive jumps is very simple: only the continuous variable t evolves with a speed of one, i.e. $\mathbf{v}(i, \mathbf{x}) = (0, 0, 1)$; $\forall\, i$. This kind of process is a particular case of PDMP, also named piecewise-linear process [56].

Random jumps

Jumps at random times correspond to the stochastic deterioration of the unit. A jump from state $(i, \mathbf{x})$ towards discrete state (j) occurs with a rate $\lambda(i, \mathbf{x})Q(i, \mathbf{x}, j)$, where $\lambda(i, \mathbf{x})$ is the rate at which the process leaves i and $Q(i, \mathbf{x}, j)$ is the probability of the jump from i to j. In this sense, the rate at which the process jumps from the discrete point i to a discrete point j is dependent on both the discrete and continuous component before the jump $(i, \mathbf{x})$ and the discrete component after the jump (j). Hence, the rate does not need to be constant and in the model here proposed it could be time dependent.

Intervention jumps

Jumps at intervention times are associated to the inspections and maintenance operations. To model these jumps a frontier is defined, such as when the continuous component reaches the frontier due to the deterministic motion, a jump occurs. Let $\Gamma = \{x_1 = t\} \cup \{x_2 = t\}$ be the set of points we refer to as the frontier for our case. Such frontier is reached when time (t) reaches the date of an inspection (x_1) or the date of a maintenance operation (x_2). When the frontier Γ is reached at $(i, \mathbf{x})$, a jump occurs in the discrete component towards a point j of E and in the continuous component to a point $\mathbf{y}$ of R equal to $m_\Gamma(i, \mathbf{x}, j)$, with a probability distribution $q(i, \mathbf{x}, j)$. The function $\mathbf{x} \to m_\Gamma(i, \mathbf{x}, j)$ is a function from Γ to R. This means that both the discrete and continuous components jump at the intervention time.

The post jump location $(j, \mathbf{y})$ is dependent on the discrete component before and after the jump and the continuous component before the jump (the frontier).

For example, if the reached frontier corresponds to an inspection of the unit (i.e. when $t = x_1$), a maintenance action and the next inspection of the unit can be scheduled, depending on the deterioration state of the unit, thus a jump in the dates of next inspection (x_1) and next maintenance (x_2) occurs. Similarly, if the reached frontier corresponds to a maintenance action (i.e. when $t = x_2$) a jump occurs in the deterioration state of the unit (i), (usually to a less deteriorated state) and if imperfect maintenance is considered, then a probability distribution can be associated to the post jump location.

3.2.3 Numerical calculations

We calculate now an approximation of the law of this process. This approach is based on the proposed finite volume scheme by Cocozza-Thivent et al. in [78, 79] and some of its applications as [77, 80].

Every term of the continuous state space R is discretized in a finite number of values. In our case, we have $R \subset \mathbb{R}^3$ with time as the quantity for every dimension. Let δ be the continuous state space step. Then, the approximation of the k-ith term has values in $F_k = \{0, \delta, 2\delta, \ldots, n\delta\}$, with $k = \{1,2,3\}$ and n an integer. In this way, R is replaced by the discrete state space $F = F_1 \times F_2 \times F_3$.

The deterministic motion of the process between jumps can be described with a function from F to F by solving equation 3.1 with the Euler method. In our case, between iterations $n\delta$ and $(n+1)\delta$ the continuous component follows the function $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{v}\delta$. When the frontier Γ is reached at $\mathbf{x}$, the continuous component jumps instantaneously from Γ to F as $\mathbf{x} \rightarrow m_\Gamma(i, \mathbf{x}, j)$.

Let $P((i,\mathbf{x}),(j,\mathbf{y}))$ denote the conditional transition probability from state $(i,\mathbf{x})$ to state $(j,\mathbf{y})$ with values in the finite state space $E \times F$, and $\tilde{F}$ denote the set of points in F which are not on the frontier Γ. The arrivals into a state $(j,\mathbf{y})$ may proceed from different paths as represented in figure 3.2.

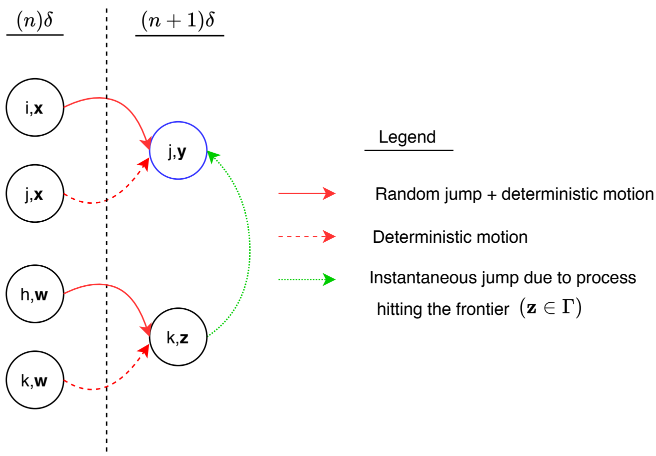


Figure 3.2: Transitions into state $(j,\mathbf{y})$ in $(n\delta, (n+1)\delta]$

Between $t = n\delta$ and $t = (n+1)\delta$, the non-null values of such conditional probabilities due to random jumps and deterministic motion are:

- for any $\mathbf{x}, \mathbf{w}$ in $\tilde{F}$, for any j different from i, for any k different from h:

$$P((i,\mathbf{x}),(j,\mathbf{y})) \approx \lambda(i,\mathbf{x})Q(i,\mathbf{x},j)\delta \text{ with } \mathbf{y} = \mathbf{x} + \mathbf{v}\delta \tag{3.2}$$
$$P((h,\mathbf{w}),(k,\mathbf{z})) \approx \lambda(h,\mathbf{w})Q(h,\mathbf{w},k)\delta \text{ with } \mathbf{z} = \mathbf{w} + \mathbf{v}\delta$$

- for any $\mathbf{x}, \mathbf{w}$ in $\tilde{F}$:

$$P((j,\mathbf{x}),(j,\mathbf{y}) \approx 1-\lambda(j,\mathbf{x})\delta \text{ with } \mathbf{y}=\mathbf{x}+\mathbf{v}\delta \tag{3.3}$$
$$P((k,\mathbf{w}),(k,\mathbf{z}) \approx 1-\lambda(k,\mathbf{w})\delta \text{ with} \mathbf{z}=\mathbf{w}+\mathbf{v}\delta$$

When $\mathbf{z} \in \Gamma$, an instantaneous jump occurs with the conditional probability:

- for any $\mathbf{z}$ in Γ, for any k:

$$P((k,\mathbf{z}),(j,\mathbf{y})) = q(k,\mathbf{z},j) \tag{3.4}$$

Let $\pi_{n\delta}$ denote the law of this stochastic process at the n-ith iteration. By using the law of total probability and the Markov property, we can write the law of the process for state $(j,\mathbf{y})$ at the $(n+1)$-ith iteration based on the transitions showed on figure 3.2. We have for any i,h in E and $\mathbf{x},\mathbf{w}$ in F:

$$\begin{aligned}
\pi_{(n+1)\delta}(j,\mathbf{y}) \quad = \quad & \sum_{\substack{i\neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}}^{N-1} \pi_{n\delta}(i,\mathbf{x})[P((i,\mathbf{x}),(j,\mathbf{y}))] \\
+ \quad & \mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}}\pi_{n\delta}(j,\mathbf{x})[P((j,\mathbf{x}),(j,\mathbf{y}))] \\
+ \quad & \sum_{\substack{k=1 \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \sum_{\substack{h=1 \\ h\neq k \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma}}^{N-1} \pi_{n\delta}(h,\mathbf{w})[P((h,\mathbf{w}),(k,\mathbf{z}))][P((k,\mathbf{z}),(j,\mathbf{y}))] \\
+ \quad & \sum_{\substack{k=1 \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \pi_{n\delta}(k,\mathbf{w})[P((k,\mathbf{w}),(k,\mathbf{z}))][P((k,\mathbf{z}),(j,\mathbf{y}))]
\end{aligned} \tag{3.5}$$

In equation 3.5, the first term accounts for the transitions related to a random jump and deterministic motion; the second term for the transitions related to only deterministic motion; the third term for the transitions with a random jump, deterministic motion and an instantaneous jump from the frontier; and the last term accounts for the transitions with deterministic motion and an instantaneous jump from the frontier; as shown in figure 3.2.

Substituting the conditional probabilities by their approximation or value from equations 3.2, 3.3 and 3.4, gives:

$$
\begin{aligned}
\pi_{(n+1)\delta}(j,\mathbf{y}) \approx & \sum_{\substack{i\neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}}^{N-1} \pi_{n\delta}(i,\mathbf{x})[\lambda(i,\mathbf{x})Q(i,\mathbf{x},j)\delta] \\
+ & \mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}}\pi_{n\delta}(j,\mathbf{x})[1-\lambda(j,\mathbf{x})\delta] \\
+ & \sum_{\substack{k=1 \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \sum_{\substack{h=1 \\ h\neq k \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma}}^{N-1} \pi_{n\delta}(h,\mathbf{w})[\lambda(h,\mathbf{w})Q(h,\mathbf{w},k)\delta][q(k,\mathbf{z},j)] \\
+ & \sum_{\substack{k=1 \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \pi_{n\delta}(k,\mathbf{w})[1-\lambda(k,\mathbf{w})\delta][q(k,\mathbf{z},j)]
\end{aligned}
\tag{3.6}
$$

Computing this equation fully describes the evolution of the PDMP. For Markov processes, it is known that the Chapman-Kolmogorov equation describes the time-evolution of the states probabilities. Equation 3.6 corresponds to an approximation of the known Chapman-Kolmogorov equation. Since this is not very obvious, we proceed to demonstrate it by deriving the Chapman-Kolmogorov equation starting from equation 3.6.

Chapman-Kolmogorov

If $f(i,x)$ is a function from $E\times F$ to $\mathbb{R}$, we can write:

$$
\begin{aligned}
& \sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{(n+1)\delta}(j,\mathbf{y}) = \sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{n\delta}(j,\mathbf{y}) - \sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{n\delta}(j,\mathbf{y}) \\
& + \sum_{j,\mathbf{y}} f(j,\mathbf{y}) \sum_{\substack{i\neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}} \pi_{n\delta}(i,\mathbf{x})\lambda(i,\mathbf{x})Q(i,\mathbf{x},j)\delta \\
& + \sum_{j,\mathbf{y}} f(j,\mathbf{y})\mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}}\pi_{n\delta}(j,\mathbf{x})[1-\lambda(j,\mathbf{x})\delta] \\
& + \sum_{\substack{j,\mathbf{y} \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}} f(j,\mathbf{y}) \sum_{\substack{h\neq k \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta;\, \mathbf{z}\in\Gamma}} \pi_{n\delta}(h,\mathbf{w})\lambda(h,\mathbf{w})Q(h,\mathbf{w},k)q(k,\mathbf{z},j)\delta \\
& + \sum_{j,\mathbf{y}} f(j,\mathbf{y}) \sum_{\substack{k \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta;\, \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}} \pi_{n\delta}(k,\mathbf{w})[1-\lambda(k,\mathbf{w})]q(k,\mathbf{z},j)\delta
\end{aligned}
\tag{3.7}
$$

By changing some indices, we can write:

$$\begin{aligned}
\sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{(n+1)\delta}(j,\mathbf{y}) &= \sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{n\delta}(j,\mathbf{y}) - \sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{n\delta}(j,\mathbf{y}) \\
&+ \sum_{\substack{i,j;\, i\neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}} f(i,\mathbf{y})\, \pi_{n\delta}(j,\mathbf{x})\lambda(j,\mathbf{x})Q(j,\mathbf{x},i)\delta \\
&+ \sum_{\substack{j,\mathbf{x} \\ \mathbf{x}=\mathbf{y}+\mathbf{v}\delta}} f(j,\mathbf{x})\, \pi_{n\delta}(j,\mathbf{y})[1-\lambda(j,\mathbf{y})\delta] \\
&+ \sum_{\substack{i,k,j,\mathbf{y} \\ k\neq j \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta;\, \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},i)}} f(i,\mathbf{y})\, \pi_{n\delta}(j,\mathbf{w})\lambda(j,\mathbf{w})Q(j,\mathbf{w},k)q(k,\mathbf{z},i)\delta \\
&+ \sum_{\substack{i,j \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta;\, \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(j,\mathbf{z},i)}} f(i,\mathbf{y})\, \pi_{n\delta}(j,\mathbf{w})[1-\lambda(j,\mathbf{w})\delta]q(j,\mathbf{z},i)\delta
\end{aligned} \tag{3.8}$$

After grouping, it can be written as:

$$\begin{aligned}
&\sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{(n+1)\delta}(j,\mathbf{y}) = \sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{n\delta}(j,\mathbf{y}) \\
&+ \sum_{\substack{j,\mathbf{y} \\ (\mathbf{y}+\mathbf{v}\delta)\notin\Gamma}} \pi_{n\delta}(j,\mathbf{y})\lambda(j,\mathbf{x})\delta \left(\sum_{\substack{i\neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}} f(i,\mathbf{y})Q(j,\mathbf{x},i) - f(j,\mathbf{y}) \right) \\
&+ \sum_{\substack{j,\mathbf{y} \\ (\mathbf{y}+\mathbf{v}\delta)\notin\Gamma}} \pi_{n\delta}(j,\mathbf{y})[1-\lambda(j,\mathbf{y})][f(j,\mathbf{y}+\mathbf{v}\delta) - f(j,\mathbf{y})] \\
&+ \sum_{\substack{j,\mathbf{w} \\ (\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} \pi_{n\delta}(j,\mathbf{w})\lambda(j,\mathbf{w})\delta \left(\sum_{\substack{i,k;\, k\neq j \\ \mathbf{z}=(\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} f(i,m_\Gamma(k,\mathbf{z},i))Q(j,\mathbf{w},k)q(k,\mathbf{z},i) - f(j,\mathbf{w}) \right) \\
&+ \sum_{\substack{j,\mathbf{w} \\ (\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} \pi_{n\delta}(j,\mathbf{w})(1-\lambda(j,\mathbf{w})\delta) \left(\sum_{\substack{i,\mathbf{z} \\ \mathbf{z}=(\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} f(i,m_\Gamma(j,\mathbf{z},i))q(j,\mathbf{z},i) - f(j,\mathbf{w}) \right)
\end{aligned} \tag{3.9}$$

If $n\delta = t$ by summation of successive differences, we can write:

$$\begin{aligned}
&\sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_{(n+1)\delta}(j,\mathbf{y}) = \sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_0(j,\mathbf{y}) \\
&+ \delta \sum_{m=0}^{n} \sum_{\substack{j,\mathbf{y} \\ (\mathbf{y}+\mathbf{v}\delta)\notin\Gamma}} \pi_{m\delta}(j,\mathbf{y})\lambda(j,\mathbf{y}) \left(\sum_{i\neq j} f(i,\mathbf{y})Q(j,\mathbf{y}-\mathbf{v}\delta,i) - f(j,\mathbf{y}) \right) \\
&+ \delta \sum_{m=0}^{n} \sum_{\substack{j,\mathbf{y} \\ (\mathbf{y}+\mathbf{v}\delta)\notin\Gamma}} \pi_{m\delta}(j,\mathbf{y})(1-\lambda(j,\mathbf{y})\delta) \left(\frac{f(j,\mathbf{y}+\mathbf{v}\delta) - f(j,\mathbf{y})}{\delta} \right) \\
&+ \delta \sum_{m=0}^{n} \sum_{\substack{j,\mathbf{w} \\ (\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} \pi_{m\delta}(j,\mathbf{w})\lambda(j,\mathbf{w}) \left(\sum_{i,k;\, k\neq j} f(i,m_\Gamma(k,\mathbf{w}+\mathbf{v}\delta,i))Q(j,\mathbf{w},k)q(k,\mathbf{w}+\mathbf{v}\delta,i) - f(j,\mathbf{w}) \right) \\
&+ \sum_{m=0}^{n} \sum_{\substack{j,\mathbf{w} \\ (\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} \pi_{m\delta}(j,\mathbf{w})(1-\lambda(j,\mathbf{w}\delta)) \left(\sum_{i} f(i,m_\Gamma(j,\mathbf{w}+\mathbf{v}\delta,i))q(j,\mathbf{w}+\mathbf{v}\delta,i) - f(j,\mathbf{w}) \right)
\end{aligned} \tag{3.10}$$

Finally, by making δ tend towards 0, we get the Chapman-Kolgomorov equation for a following regular function f from $E \times R$ to $\mathbb{R}$, where $\pi_t(j,\mathbf{y})$ denotes the law of the process $\{I(t), X(t)\}$ at time t:

$$
\begin{aligned}
\sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_t(j,\mathbf{y}) &= \sum_{j,\mathbf{y}} f(j,\mathbf{y})\pi_0(j,\mathbf{y}) \\
&+ \int_0^t du \int_{\tilde{F}} \sum_j \pi_u(j,dy)\lambda(j,\mathbf{y})\left(\sum_{i\neq j} f(i,j)Q(j,\mathbf{y},i) - f(j,\mathbf{y})\right) \\
&+ \int_0^t du \int_{\tilde{F}} \pi_u(j,dy)\sum_l \frac{df}{dy_l}(j,\mathbf{y})v_l \\
&+ \int_{(0,t]\times\Gamma} \sum_j \left(\sum_i f(i,m_\Gamma(j,\mathbf{w},i))q(j,\mathbf{w},i) - f(j,\mathbf{w})\right)\sigma(j,d\mathbf{w},ds)
\end{aligned}
\tag{3.11}
$$

The measures $\sigma(j,d\mathbf{w},ds)$ on the time-space $\mathbb{R}_+ \times \Gamma$ describe the way the process reaches the frontier. If t_1 and t_2 are two points in time ($t_1 < t_2$) and Γ_l is a part of the frontier Γ, then $\int_{\Gamma_l \times (t_1,t_2]} \sigma(j,d\mathbf{w},ds)$ is the mean number of times the process reaches the frontier between t_1 and t_2 on the part Γ_l of the frontier with the discrete part being j. In our case, the mean number that the process reaches a part of the frontier corresponds to the mean number of maintenance actions or inspections.

3.3 Case study

This section illustrates the proposed modelling approach through a case study and explores on how different assumptions could be taken into account. The case is related to bridge management, i.e. the planning of inspections and maintenance activities of road bridges, in Norway.

3.3.1 Background

The use of automobiles experienced a rapid growth during the 20th century and with this growth came the development of a massive transportation infrastructures. In [2], the Council of the European Union includes the transport sector in the list of Critical Infrastructures, considering that modern societies depend on the availability of this service and that its disruption or unavailability poses risks with serious consequences to the health, safety, economic or social well-being of people and vital societal functions. A systematic approach to maintenance and rehabilitation strategies for the transportation system was not identified until the late 1960s. The Highway Safety Act of 1968 was a development that resulted from the collapse of the Silver bridge across the Ohio River, USA in 1967, and the concerns related to the bridge management problem [81]. This Act required state road officials to inspect and rate the condition of the bridges.

Bridge management can be understood as the optimal planning of inspections and maintenance activities of road bridges, with the goal of preserving the asset value of the infrastructure by optimizing the costs over its lifetime, while ensuring the safety of users and offering a sufficient quality of service [82]. More than 50 years after the collapse of the Silver bridge, despite the advances in technology, rehabilitation techniques and safety assessments, bridge collapses continue to occur. On August 2018, Ponte Moranti on the A10 motorway in Genoa, Italy (figure 3.3), collapsed resulting in the loss of 43 human lives [83].

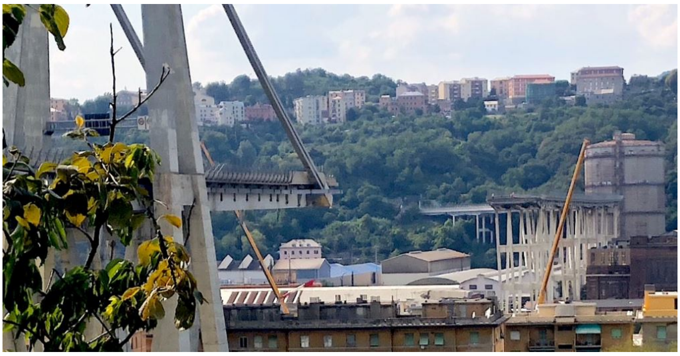


Figure 3.3: Ponte Moranti. Photo credit: Michelle Ferraris, Wikimedia Commons

Bridge collapses have been historically caused by a variety of factors or a combination of these, including poor engineering judgement, use of substandard materials, extreme loading, inadequate maintenance, among others. From all the causes of bridge failure, improper maintenance is the most preventable [84]. Moreover, the construction of new bridges has been slowing down in most countries, which now face a stock of aging bridges, requiring an effective and efficient bridge management.

3.3.2 Problem statement

Road bridges are a vital part of the Norwegian transportation infrastructure. In Norway, there are more than 18,000 road bridges across the country, so an efficient bridge management system is vital for avoiding high costs from over expending and for ensuring safety of the public and availability of the transportation system.

Many factors can make bridge management a challenging task, such as: the varying weight and intensity of the traffic, the evolution of the building codes over the years, the weather influence on the structures, large number of structures spread over a large area, and others [24]. All these factors create uncertainty, which makes the bridge management a problem of decision making under uncertainty.

In the bridge management system applied by the Norwegian Public Roads Administration (NPRA), the agency responsible for planning, building, operating, and maintaining national and country bridges, the inspections are mainly carried out periodically based on predefined rules and the decision about when to perform maintenance is based on the findings of these inspections. The handbooks for management and inspections of bridges [85, 7], establish types of inspections for the bridges and the period in which they must be performed, e.g. a main inspection of a bridge, with an overview of all the elements of the bridge, must (in general) be performed every five years. They also establish how the inspections must be logged in a database, how the findings must be reported and provides guidelines on when to perform the repairs for found damages.

There exists an extensive list of damage mechanisms that can affect a structure. The inspection handbook of the NPRA [7] provides an overview of these mechanisms with guidelines on how to assess their severity. The assessment of the severity consists in a combination of quantitative and qualitative methods. The resulting condition is presented in a scale of one to four, as: (1) Small damage, (2) Medium damage, (3) Large damage and (4) Critical damage.

The regulations dictate a CBM strategy that establishes when the damage must be repaired based on the condition at the inspection. According to the severity of the damage, a maintenance action (or no action) must be scheduled. For small damage (1), no maintenance action is required; for medium damage (2), a maintenance action must take place between four and ten years; for large damage (3), a maintenance action must take place between one and three years and for critical damage (4), a maintenance action must take place in less than six months. Figure 3.4 shows an overview of the described bridge management process.

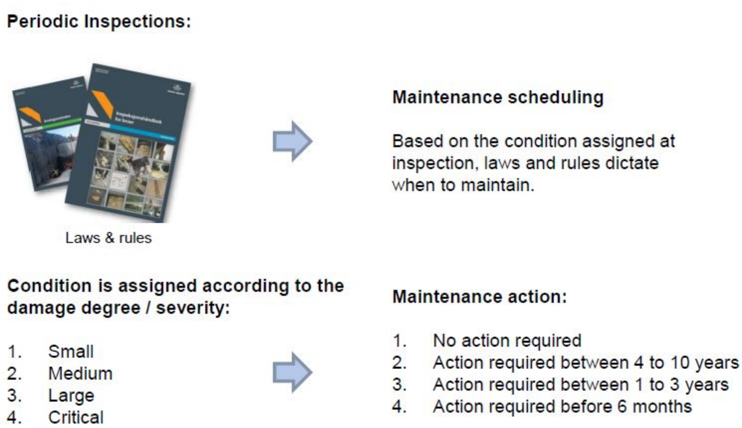


Figure 3.4: Bridge management process

We proceed to build a CBM model illustrating the PDMP formalism described in section 3.2.2. We recall that a state of the PDMP $\{I(t), X(t)\} t \geq 0$ to consider is made of $\{\mathbf{i}, \mathbf{x}\}$ with $\mathbf{x} = (x_1, x_2, t)$, where x_1 corresponds to the date of the next inspection, x_2 corresponds to the date of the next maintenance operation, t stands for time and i corresponds to the deterioration level of the unit.

3.3.3 Deterioration model

For modelling the deterioration process of a structure, we need to define the deterioration states and to describe the jumps between these states.

In the bridge management of the NPRA a condition is assigned to the structure as a level that ranges from one to four. In order to distinguish between a condition not known to the NPRA and a condition which is known based on information from the inspection, we split the deterioration state of the unit in two parts: a real state and a virtual state. In this sense, $\mathbf{i} = (i_1, i_2)$ where $i_1 = \{1,2,3,4\}$ denotes the real state of the structure, and $i_2 = \{1,2,3,4\}$ denotes the virtual state of the structure (known by the operator based on the inspections). A condition (real or virtual) of the structure is assigned as:

- 1 : Small or no damage
- 2 : Medium damage
- 3 : Large damage
- 4 : Critical damage

The deterioration is modelled with random jumps between these states. Since the unit is not continuously monitored, when a deterioration jump occurs, it is not detected until an inspection is performed, so only the real state (i_1) changes and the virtual state (i_2) remains unchanged. In this case, we consider that the structure deteriorates gradually as shown in figure 3.5.

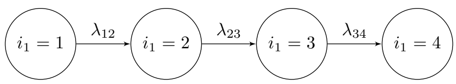


Figure 3.5: Deterioration process.

As described in section 3.2.2, a jump from state $(\mathbf{i}, \mathbf{x})$ towards discrete state $(\mathbf{j})$ occurs with a rate $\lambda(\mathbf{i}, \mathbf{x})Q(\mathbf{i}, \mathbf{x}, \mathbf{j})$. Considering constant transition rates, i.e. dependent only on the discrete components before and after the jump and not on the continuous component, we can write the transition rates out of a discrete component $\mathbf{i}$, as:

- From $i_1 = 1$, $\forall\ i_2$, $\mathbf{x}$:

  $\lambda((1, i_2), \mathbf{x}) = \lambda_{12}$ and $Q((1, i_2), \mathbf{x}, (2, i_2)) = 1$

- From $i_1 = 2$, $\forall\ i_2$, $\mathbf{x}$:

  $\lambda((2, i_2), \mathbf{x}) = \lambda_{23}$ and $Q((2, i_2), \mathbf{x}, (3, i_2)) = 1$

- From $i_1 = 3$, $\forall\ i_2$, $\mathbf{x}$:

$\lambda((3,i_2),\mathbf{x}) = \lambda_{34}$ and $Q((3,i_2),\mathbf{x},(4,i_2)) = 1$

- From $i_1 = 4$, $\forall\ i_2$, $\mathbf{x}$:

$\lambda((4,i_2),\mathbf{x}) = 0$

3.3.4 Intervention model

The inspections and maintenance operations are described by jumps at intervention times as described in section 3.2.2.

Inspections

When a structure is inspected its condition is revealed and a maintenance task and the next inspection are scheduled accordingly. A jump in the PDMP related to an inspection occurs when the part of the frontier Γ made of the points $\{x_1 = t\}$ is reached. To describe a jump at this time we must define the post jump location of the discrete and continuous components and the associated probability distribution of such location.

There are two characteristics about the inspections that can be addressed in the modelling framework: frequency and quality. The frequency of the inspections can be periodic or non-periodic. Currently, due to regulations, the bridge inspections of the NPRA are carried out periodically, i.e. inspections are performed at equal time intervals. Let T denote the constant interval for inspections and M_{i_2} denote the delay for maintenance based on the virtual deterioration condition i_2. As described in section 3.3.2, maintenance is scheduled according to the known deterioration state to the operator (i_2) as: $M_1 = \infty$, $M_2 \in [4,10]$ years, $M_3 \in [1,3]$ years, $M_4 \in [0,0.5]$ years. The post jump location of the continuous component is: $m_\Gamma(\mathbf{i},\mathbf{x},\mathbf{j}) = (t+T, \min(x_B, M_{i_2}), t)$. Non-periodic inspections could for example follow a pre-determined condition-based inspection scheme, in which the time of a next inspection is decided based on the deterioration state of the unit at the current one. To model such inspection scheme, the inspection interval can be set according to the virtual state of the unit (i_2), (similar to the maintenance delay) as T_{i_2} instead a constant interval.

The quality of the inspections can be taken into account by assigning a probability $q(\mathbf{i},\mathbf{x},\mathbf{j})$ distribution to the post-jump location, as described in section 3.2.2. If the inspections are considered perfect, i.e. the real state of the unit is revealed at the inspection without uncertainty, then the virtual state becomes equal to the real state of the unit at the inspection time, with probability of one. In some cases, the inspections may not perfectly reveal the real condition of the unit, due to for example hidden damages or errors in measurements. If the inspections are considered non-perfect or subject to errors, then we can write a conditional probability of the post-jump location of the virtual state (j_2), given the real state of the unit before the jump (i_1), as $P(j_2|i_1) = q(\mathbf{i},\mathbf{x},\mathbf{j})$ and the post jump location of the continuous component would be $m_\Gamma(\mathbf{i},\mathbf{x},\mathbf{j}) = \mathbf{x} + (T_{j_2}, M_{j_2}, 0)$

Maintenance

A maintenance task is scheduled according to the condition of the structure at inspection. The maintenance is arranged to take place after a delay with deterministic duration. When a maintenance action is performed, a jump occurs to a less deteriorated state. The degree of the maintenance is modelled by assigning the post-jump location. For example, if perfect maintenance or replacement is considered, the unit is considered as-good-as-new, thus the discrete component (i_1, i_2) jumps to $(1,1)$. In addition, the date to the next inspection (x_1) does not change, and the date of the next maintenance action (x_2) is set to infinite (no maintenance scheduled).

To consider possible errors in the maintenance operation, a probability distribution $q(\mathbf{i},\mathbf{x},\mathbf{j})$ can be assigned the post-jump location. For example, let θ denote the probability of maintenance error, i.e. the probability that a maintenance action results in a state other than as-good-as-new e.g. a state with medium damage, then the discrete component jumps from $\mathbf{i}$ to $\mathbf{j} = (1,1)$ with probability $q(\mathbf{i},\mathbf{x},\mathbf{j}) = (1-\theta)$ or from $\mathbf{i}$ to $\mathbf{k} = (2,1)$ with probability $q(\mathbf{i},\mathbf{x},\mathbf{k}) = \theta$.

3.3.5 Quantification

The following assumptions are considered in the illustration case for quantification purposes, in addition to those listed in section 3.2.1:

(i) The unit is inspected periodically, i.e. at equal time intervals T

(ii) Inspections are perfect and reveal the true state of the unit.

(iii) Maintenance interventions occur at the scheduled date instantaneously, i.e. the duration of the intervention is neglected

(iv) After a maintenance action, the unit is considered as-good-as-new without error.

From assumptions (i) and (ii), it can be written that if $\mathbf{x} \in \Gamma$ with $x_1 = t$, a jump occurs from state $(\mathbf{i},\mathbf{x})$ to state $(\mathbf{j}, m_\Gamma(\mathbf{i},\mathbf{x},\mathbf{j}))$ with probability $q(\mathbf{i},\mathbf{x},\mathbf{j}) = 1$. The discrete component jumps to $\mathbf{j}$ with $j_1 = j_2 = i_1$, i.e. the virtual deterioration state becomes equal to the real state before the jump, while the continuous component jumps to $m_\Gamma(\mathbf{i},\mathbf{x},\mathbf{j}) = (t+T, \min(x_2, M_{i_2}), t)$.

From assumption (iv), if $\mathbf{z} \in \Gamma$ with $z_2 = t$ then a jump occurs from state $(\mathbf{k},\mathbf{z})$ to state $(\mathbf{j}, m_\Gamma(\mathbf{k},\mathbf{z},\mathbf{j}))$ with probability $q(\mathbf{k},\mathbf{z},\mathbf{j}) = 1$ (no error). The discrete component jumps to $\mathbf{j} = (1,1)$, i.e. as-good-as-new, while the continuous component jumps to $m_\Gamma(\mathbf{k},\mathbf{z},\mathbf{j}) = (z_1, \infty, t)$.

Numerical approach

The states probabilities can be found by iterating on the recursive equation 3.6. At every iteration step, the real deterioration states probabilities $\pi_{n\delta}((i_1,\cdot),\mathbf{x})$ can be found with the summation:

$$\pi_{n\delta}((i_1 = k, \cdot), (\cdot, \cdot, n\delta)) \quad = \quad \sum_{x_2}\sum_{x_1}\sum_{i_2} \pi_{n\delta}((k, i_2), (x_1, x_2, n\delta)) \tag{3.12}$$

Monte Carlo simulation

An alternative quantification method to the numerical approach introduced in 3.2.3 is to perform Monte Carlo simulation of the process to estimate the quantities of interest. The modelling framework described in section 3.2.2 is convenient for setting the structure to simulate problems of CBM with the aforementioned assumptions.

The simulation procedure of the PDMP is shown in figure 3.6. It includes five main steps to simulate a realization of the PDMP until a horizon time t_{hor}.

(i) Set initial system time and initial system state
In our case, initial time is set to zero, the unit is set to be in new condition with no maintenance action scheduled and the date of the first inspection is set to the period. (i.e. $t = 0$, $i_1 = 1$, $i_2 = 1$, $x_1 = T$ and $x_2 = \infty$).

(ii) Sample date of next random jump, if enabled
The date of the next random jump t_{jump} is sampled from the corresponding probability density function and the corresponding parameter(s).

(iii) Identify next event
The date of the next random jump t_{jump} is compared with date of next inspection x_1, the date of next maintenance action x_2 and the horizon time t_{hor}.
The system time is updated as: $t = \min(t_{jump}, x_1, x_2, t_{hor})$. If the simulation time has reached the horizon time, $t \geq t_{hor}$, the simulation continues to step (v), otherwise it continues to step (iv).

(iv) Update system state
The system state is updated according to the jump that takes place at time t: deterioration, inspection or maintenance.

(a) Deterioration: $(t = t_{jump})$
Only i_1 is updated in this jump

(b) Inspection: $(t = x_1)$
The values of i_2, x_1, x_2 are updated. The post-jump values are:
$i_2^+ = i_1^-$;
$x_1^+ = t + T$;
$x_2^+ = t + M_{i_2}$;

(c) Maintenance: $(t = x_2)$
The values of i_1, i_2, x_2 are updated. The post-jump values are:

$i_1^+ = i_2^+ = 1;$
$x_2^+ = \infty.$

(v) Set final system state and time
The final time is t_{hor} and the final system state is the state resulting from the last jump to take place no later than t_{hor}.

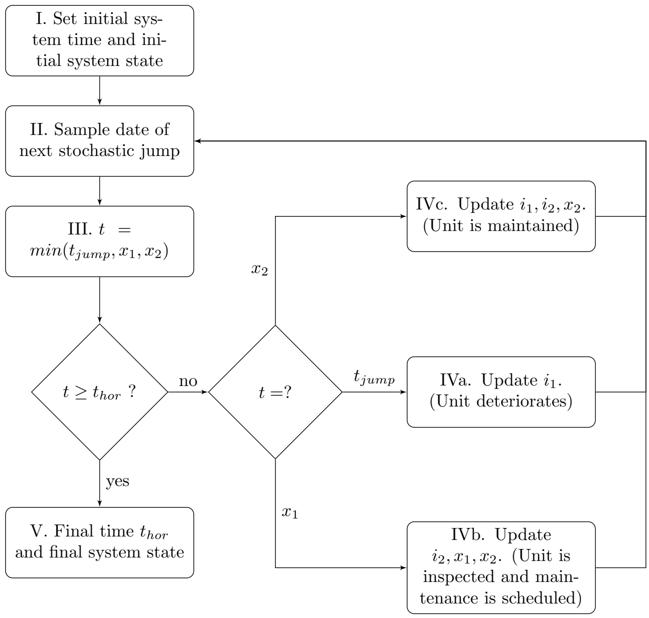


Figure 3.6: Simulation procedure.

This simulation procedure is replicated a high number of times, to approximate quantities of interest, such as deterioration state probabilities and mean numbers of interventions of a given type.

3.3.6 Experiments and results

State probabilities

The model parameters used for quantification are shown in table 3.1. The deterioration rates have been estimated from previous works carried by the NPRA based on the information available on their database for bridge inspections and maintenance actions.

The time dependent real deterioration states probabilities $P(i_1)$ are found using both the numerical approach and Monte Carlo simulation. The results are shown in figure 3.7.

To compare the results of the quantification from both approaches, the residuals or difference between the state probabilities is shown in figure 3.8. It can be observed that the

Deterioration rates (h^{-1})	Maintenance delays (y)	Inspection interval (y)
$\lambda_{12} = 1.5\text{e}{-5}$	$M_1 = \infty$	$T = 5$
$\lambda_{23} = 6\text{e}{-6}$	$M_2 = 8$	
$\lambda_{34} = 1.4\text{e}{-6}$	$M_3 = 3$	
	$M_4 = 0.5$	

Table 3.1: Model parameters

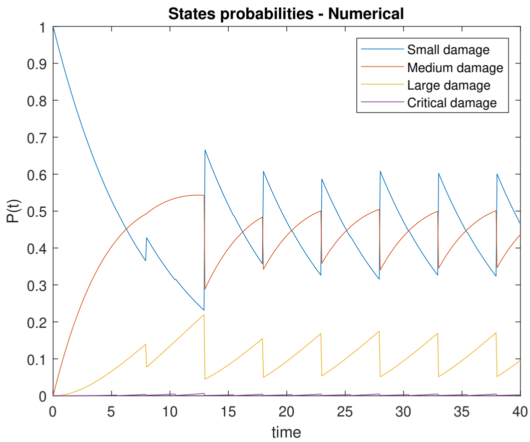


Figure 3.7: Deterioration state probabilities

difference in results is small with an order of magnitude of 10^{-3}. In addition, the difference is reduced by performing a higher number of replications of the Monte Carlo simulation, showing same convergence.

The Monte Carlo simulation method is widely used in practice, conceptually easy to apply and without particular restrictions on the dimension of the PDMP. On the other hand, the numerical scheme has high accuracy with short computation times [73]. In our case, the Monte Carlo simulation with 100,000 replications took approximately one hour to obtain time-dependent probabilities, while with the numerical scheme the results are obtained in approximately ten seconds.

Maintenance optimization

The PDMP allows to test different inspection and maintenance strategies and assess their effect on the structure condition. For example, different periods of inspection can be considered, evaluating the effect on the condition of the structure. Figure 3.9 shows how the critical condition of the unit ($i_1 = 4$), varies with time for different inspection periods. This allows to support the decision process related to inspections by evaluating the associated risk on the structure.

Moreover, to assist the decision process in bridge management, the expected cost per

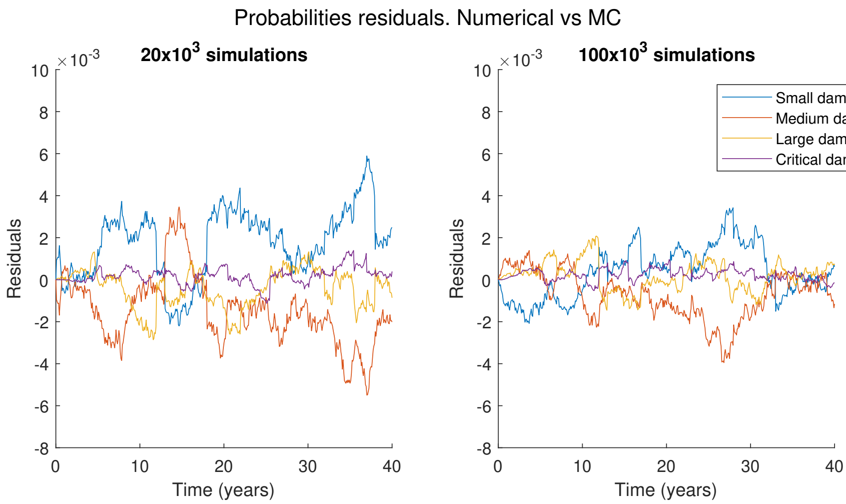


Figure 3.8: Numerical approach and simulation results

unit of time of a given strategy can be assessed in addition to the effect on the condition of a structure. Then a favorable inspection/maintenance strategy which minimizes the cost per unit of time with an acceptable risk for the structure can be chosen. The function for the expected cost can be set as:

$$E[C] = E[N_{in}]C_{in} + E[N_{mr}]C_{mr} + E[N_{lr}]C_{lr} + E[N_{cr}]C_{cr} \tag{3.13}$$

Where C_{in}: cost of inspection, C_{mr}: cost of medium repair (unit with medium damage), C_{lr}: cost of large repair (unit with large damage), C_{cr}: cost of critical repair (unit with critical damage), N_{in}: number of inspections per unit of time, N_{mr}: number of medium repairs per unit of time, N_{lr}: number of large repairs per unit of time, N_{lr}: number of large repairs per unit of time.

The mean number of inspections and repairs can be estimated from Monte Carlo simulations or expressed in terms of the marginal distributions of the PDMP and approximated with the numerical scheme. We look at a long time horizon for the expected cost to be considered asymptotic. For example, the mean number of medium repairs until t, corresponds to the mean number of times the process reaches the part of the frontier related to maintenance ($t = x_2$) with the discrete component $\mathbf{i} = (2,2)$ until time t, which can be approximated by equation 3.14.

$$N_{mr}(t) \approx \sum_{\substack{u=0 \\ \mathbf{x}\in\Gamma_2}}^{t} \pi_u\{(2,2),(\mathbf{x})\} \tag{3.14}$$

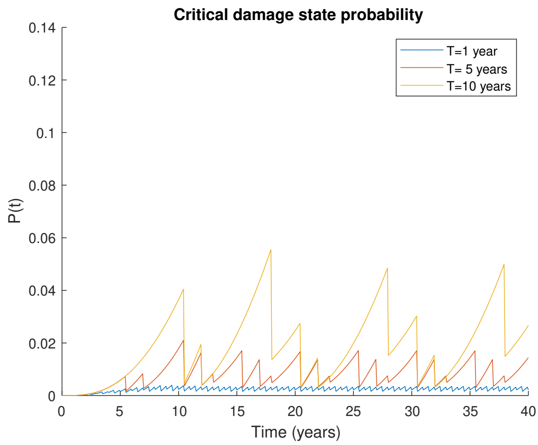


Figure 3.9: Critical damage probability for different inspection intervals

Where Γ_2 refers to the part of the frontier made by the points $\{x_2 = t\}$. To illustrate, we look at the expected cost per unit of time for different inspection intervals (keeping the parameters from table 3.1 fixed, with the exemption of the inspection interval which is varied). We set symbolic values of $C_{in} = 50$, $C_{mr} = 100$, $C_{lr} = 250$, $C_{cr} = 5000$.

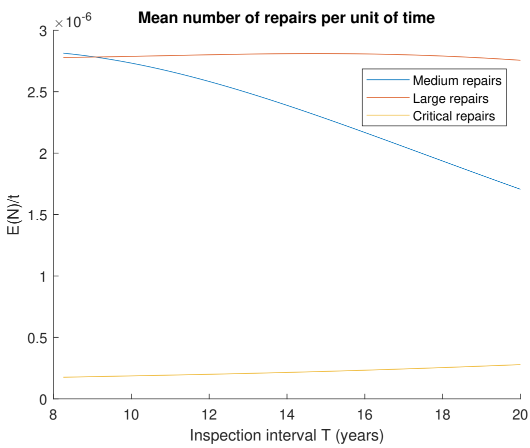


Figure 3.10: Mean number of repairs per unit of time

Figure 3.10 shows the mean number of medium, large and critical repairs per unit of time for different inspection intervals and the resulting cost is shown in figure 3.11. In this case, the expected cost is lowest for an inspection interval $T = 13.75$ years or 13 years and 9 months.

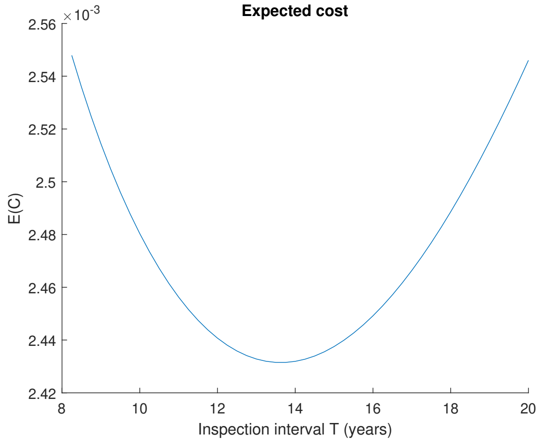


Figure 3.11: Expected cost per unit of time

3.4 Chapter conclusion

A framework for CBM models with discrete-state stochastic deterioration has been proposed based on the theory of PDMP. The proposed model allows to study problems in which the condition monitoring is not continuous but inspection-based and there is an inherent delay for performing maintenance actions. Therefore, the transition law cannot be found by a simple Markov process neither a semi-Markov one. Although this family of problems has been the motivation to propose PDMP, it is worth to mention that PDMP have been introduced as general class of non-diffusion stochastic models and as such can cover a wide variety of applications that involve some combination of random jumps and deterministic motion.

The proposed framework allows the assessment of the probability for the infrastructure to be in a deterioration or a critical state given an inspection period and given a maintenance schedule. In this way, it is possible to evaluate if a given CBM policy is adequate regarding some safety requirements by making variations of the inspection period, the delay before intervention, or the state to which the system is restored after maintenance (imperfect maintenance). It is also possible to assess the cost of a CBM policy and find the optimal parameters of the policy.

A numerical approach for quantification of time dependent probabilities has been developed. This approach is an approximation to the solution of the Chapman-Kolmogorov equation. In comparison, Monte Carlo simulation is in general conceptually easier to apply while the numerical approach could provide better accuracy in the results with faster computation times. However, the system complexity and the number of discrete states can be limitations for this numerical approach while Monte Carlo simulation could offer more flexibility in this aspect. Given that the deterioration of the system can be characterized by a reasonable number of discrete states and that the deterministic motion is reduced to a trivial equation, it is relatively simple to make use of the numerical approach, making it a convenient alternative for problems which require studying different strategies and repeating the quantification procedure several times in order to support the decision-making.

A case study has been presented to illustrate the modelling and quantification approach. Through the case, guidelines on how to account for different assumptions about the inspection frequency and quality as well as maintenance strategies are given.

The proposed modelling framework presented, as well as much of the existing research on CBM focuses on single-item models. Moreover, multistate systems reliability theory usually deals with systems made of independent multistate components. An interesting direction of further works could be to study the application of PDMP under the framework of multistate systems, exploring on the modelling of dependencies among the components such as stochastic, structural and/or economical. In this way, a decision-making process for maintenance at the system level can be considered. This is discussed in the next chapter.

Chapter 4

Piecewise deterministic Markov process for condition-based maintenance models of multi-component systems

This chapter presents a modelling framework for studying the CBM optimization problem of a multi-component system, with application to a gas compressor. The proposed framework is based on a Piecewise deterministic Markov Process (PDMP) and multi-state systems theory, allowing for the performance assessment of maintenance policies at the system level, while taking into account some structural and resources dependencies among the components with non-negligible actions duration and constraints related to the production profile. A case study dealing with CBM of a gas compression system is presented, introducing the assumptions, variables, evolution of the PDMP, numerical calculations of the process and the approach to assess the performance of a maintenance policy at the system level.

The main content of this chapter is based on the published article in appendix C and the article under review in appendix D which have been prepared as part of this PhD project.

4.1 Background and state-of-the-art

The literature on CBM models for multi-component systems is more limited than on single-item. As Alaswad and Xiang [20] have pointed, most of the existing CBM strategies for multi-component systems have traditionally been done at the component level, meaning that the optimal CBM policy for a single component is employed per component in the multi-component system [86, 87, 88] without taking into account components inter-dependencies. However, in practice, multi-component systems are subject to dependencies among the components that should be considered in the decision making for choosing a maintenance policy while assessing the performance at the system level. Neglecting dependencies among components may not guarantee the best maintenance performance. Therefore the study of component dependencies has been gaining attention on CBM models for multi-component systems and they can be classified as hereafter.

Structural dependence is related to the structural, static relationships among components, from a technical or a performance perspective. From the technical point of view, it relates to systems which are configured in a way that maintaining one component requires or prohibits other to be maintained, and from the performance perspective it refers to systems in which their performance is impacted by the configuration and by the performance of the components, e.g. series, parallel and k-out-of-n relations [60]. Resources dependence concerns systems in which maintenance actions can only be executed if the required resources (e.g. spare parts, tools, personnel) are available. It concerns systems in which several components share limited resources, requiring maintenance to be optimized at the system level [61]. Stochastic dependence refers to the interaction between failures or deterioration processes of components of the system, as well as the influence of external factors on these and economic dependence indicates that the cost of maintenance is impacted by joining the maintenance tasks of different components [89].

Some authors have studied maintenance problems of systems while addressing these dependencies. Most of them are based on the identification of a Markov process. Zhang et al. have considered a condition-based maintenance policies for systems whose components are subjected to failure (stochastic) dependency, formulating the problem as a Markov renewal process [62], [90]. Xu et al. [91] and Andersen et al. [92] make use of a Markov Decision process (MDP) to find the optimal maintenance decisions in systems where the dependence among components is characterized by a copula function. MDP have also been developed for joint optimization problems of systems exposed to restrictions such as inventory of spare parts or production schedules [93, 94, 95].

However, the aforementioned works deal with cases in which the duration of the actions (inspections, repairs) is considered negligible. In practice, in systems difficult to maintain, the action duration can be significant and influence the optimal maintenance decision. Some systems are required to operate continuously, with a dynamic demand that alternates between periods that require the system to operate at full capacity while other periods provide better margin for maintenance actions due to the ability of sharing the load. When a maintenance action takes place in part of the system, the rest of the components must ensure the continuous operation and meet the requirements of the system. It is then important to consider the evolution and deterioration of the active components in order to assess the risk in the system's performance while a maintenance action is taking place. For this, we propose a specific class of Piecewise Deterministic Markov Process (PDMP) in order to support the modelling work and to provide a formalism that can be suitable to a larger set of application cases. Such PDMP can serve as a basis to implement a numerical scheme and provide numerical solutions for the maintenance strategy cost evaluation.

PDMP have been proposed in works dealing with maintenance problems. Zhang et al. [75] proposed PDMP for an offshore oil production system and Lair et al. [77] for a train air conditioning system. However, the policy on those cases is not CBM since the health of the components is considered binary, with components being either working or failed. Lin et al. [76] proposed PDMP as a framework for considering dependencies among degradation processes affecting a single item. We propose PDMP for studying CBM problems of a multi-state system made of multi-state components, while addressing some dependencies. At last, in order to have more concise and elegant formulation of the system-level performance for a given maintenance policy, we propose to use a formalism taken from multi-state system theory. This formalism allows for the description of the system performance (level of production) as a function of few system states, by avoiding a complete combinatorial development.

4.2 PDMP

In this section we present the modelling assumptions and then the formalism of the corresponding PDMP with linear deterministic motion. The process evolution is made of linear motion between jumps that can be random or due to the continuous part hitting a boundary, just like in section 3.2.2.

4.2.1 Modelling assumptions

We consider a system that has a breakdown structure with three indentation levels as shown in figure 4.1 and propose PDMP as a framework for studying CBM problems with the assumptions explained here under and summarized in figure 3.1.

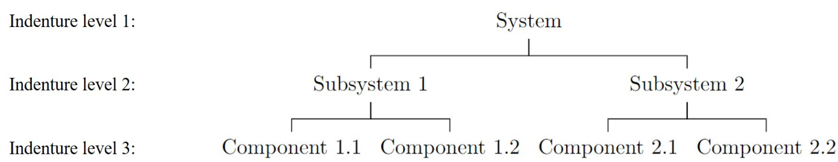


Figure 4.1: Generic breakdown structure

On the deterioration model side, we assume that the system condition is a function of the conditions of its components. The condition of each component is described by a set of discrete states and the deterioration follows a random jump process between these states.

For the interventions, we consider that the system alternates between periods of high demand and periods of low demand. The optimal selection of maintenance activities for systems with these characteristics is known as selective maintenance. Selective maintenance aims to determine which subsystems or components should be maintained and how they should be maintained during a mission break or opportunity, with the available resources, in order to meet the performance requirements of the system during future missions or high demand periods [96].

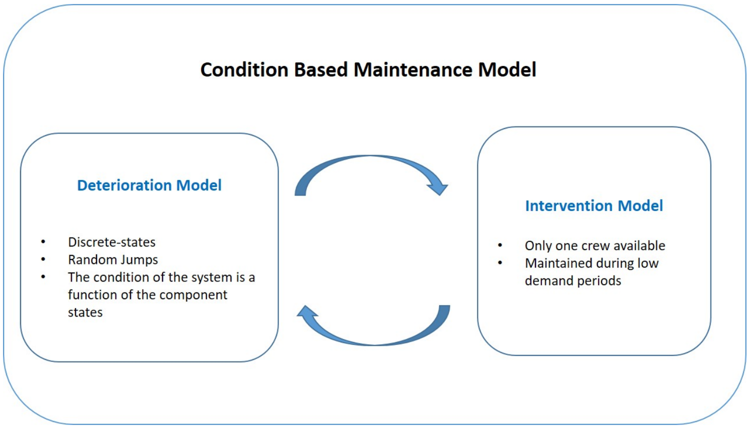


Figure 4.2: Model assumptions for the proposed framework

In addition, we consider that there is only one maintenance crew available, meaning that only one of the subsystems can be maintained at a time. Likewise, we consider that the subsystems are configured in a way that maintaining one of its components requires to take the other component of the same subsystem out of operation.

Once again, we make use of a specific class of PDMP in which the deterministic motion is linear and intended to keep track of the time to an intervention jump. This allows to formulate the problem in a Markovian way.

4.2.2 Formalism

We consider a stochastic hybrid process $\{I(t), X(t)\}t \geq 0$ with values in a discrete-continuous space $E \times R$.

The first component, $\{I(t)\}$ is discrete and used to represent the deterioration states of the components of the system. The system is made of m components and the condition of each component of the system can be categorized in a finite number of states n_i. We consider $\mathbf{i} = (i_1, i_2, ..., i_m, i_{m+1})$ a vector in which $(i_1 ... i_m)$ denote the state of each component of the system. The last term i_{m+1} and i_{m+1} denotes an operational mode of the system and is used to keep track of the availability of the maintenance crew. The discrete state space E is the finite set made of $n_1 \times n_2 \times ... \times n_{m+1}$ points.

The second component, $\{X(t)\}$ is continuous, introduced in our case as a way to keep track for the intervention jumps that occur at specified times. We consider $\mathbf{x} = (x, t)$, $\mathbf{x}$ is a vector in which x is used to mark the time of an intervention or boundary jump and t stands for time. Hence, the continuous component $\{X(t)\}$ evolves in $\mathbb{R}^2$.

The process $\{I(t), X(t)\}t \geq 0$ evolves as what is described in section 3.2.2 (for a more detailed description). It experiences jumps at random times with from a state $(\mathbf{i}, \mathbf{x})$ towards discrete state $(\mathbf{j})$ with rate $\lambda(\mathbf{i}, \mathbf{x})Q(\mathbf{i}, \mathbf{x}, \mathbf{j})$ and jumps at intervention times when t reaches the boundary x. Between the times of two consecutive jumps (random or deterministic) only the continuous variable t evolves with speed of one, i.e. $\mathbf{v}(\mathbf{i},\mathbf{x}) = (0,1)$.

Dependencies

We highlight here how some dependencies are handled from a generic point of view and illustrate them later through the case study in section 4.3.

Structural According to the model assumptions, there are structural dependencies at the subsystem level. From the technical perspective, maintaining one component requires taking the other component of the same sub-system out of operation. This also gives an opportunity for grouping maintenance activities of components of the same subsystem. From the performance perspective, we consider that the performance of the system is dependent on the states and availability of their subsystems.

To handle these dependencies, as later seen in the case study, we first write the states at the subsystem level by developing the full combination states of the components, as in table 4.1, and write a structure functions that outputs the performance at the system level as a function of the states of the maintained components, as section 4.3.4.

Stochastic and resource dependencies There can be stochastic dependence between components of the system in the sense that the rates of deterioration of a component are dependent on the states of the other. Additionally, there is only one maintenance crew available for the subsystems, meaning that only one of the subsystems can be maintained at a time, while also opening the opportunity to work on both components of the subsystem.

It is possible to take these assumptions into account by doing the modelling work of defining the jump rate and transition measures accordingly. The system jumps from a state $(\mathbf{i},\mathbf{x})$ towards discrete state (j) occurs with a rate $\lambda(\mathbf{i},\mathbf{x})Q(\mathbf{i},\mathbf{x},\mathbf{j})$. Is is part of the modelling work to define the random jump rates that are dependent on the states of the components of the system ($\mathbf{i} = (i_1,...,i_{m+1})$) and to map the jumps between states with the transition measures $Q(\mathbf{i},\mathbf{x},\mathbf{j})$. The extra term in the discrete-component allows to prevent a maintenance action starting in a component if the maintenance crew is unavailable. By assigning a value to this term, it is possible to write and map the random jumps as well as the deterministic motion of the process. This is shown through the case study in the description of the PDMP evolution in section 4.3.3.

4.3 Case study

This section illustrates the proposed modelling approach through a case study and explores on how different assumptions could be taken into account. The case is related to a gas compressor system.

4.3.1 Background

Norway is the European Union (EU) second largest gas supplier, covering 27% of the total gas imports of the EU during the third quarter of 2019 [97]. The large majority of these gas exports from Norway to the EU is transported through pipelines. Pipeline networks are the preferred and most efficient method of transporting natural gas. An integral part of a natural gas pipeline network are the compressor stations, which are strategically placed within the network with the function of maintaining the pressure and flow of gas, from the production sites to the end users [98]. Ensuring high availability of compressor stations is of key importance to avoid the large production losses associated to the network downtime.

Usually, high reliability is implemented by robust design at the system design phase. During the operation, rigorous maintenance policies are the means to ensure high availability of the compressor station. Traditionally, these maintenance policies consist on calendar-based preventive maintenance interventions, that may result in high costs from over expending. This situation could be improved by moving towards condition-based maintenance (CBM) policies and a decision process based on prognostics. As pointed out by Kermanshachi et al. [99], the studies on optimal natural gas pipeline maintenance strategies using reliability analysis are limited, leaving a knowledge gap and lack of predictive models to estimate major incidents in natural gas pipeline systems.

4.3.2 System description and model assumptions

We consider a compressor train consisting of a variable speed drive (VSD), an electrical motor (M), a gearbox (GB) and the gas compressor (C), as shown in figure 4.3.

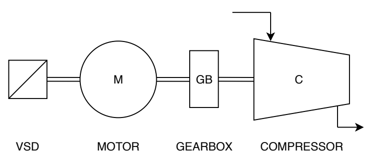


Figure 4.3: Compressor train diagram (adopted from [4])

From these components, the variable speed drive and the gearbox are considered to be much more reliable than the other two, and their repairs are considered to be easy and with short duration, in accordance to [4]. Therefore, their contribution to the unavailability of the train can be considered negligible and here we focus on modelling the deterioration and CBM for the other two components, i.e. the electrical motor and the gas compressor.

In order to propose a framework illustrating the inclusion of dependencies, we consider a system composed of two redundant compressor trains. The structure of the system considered is represented in figure 4.4. In this structure, indenture level one corresponds to the compression system here referred to as the system, indenture level two corresponds to the compressor trains and indenture level three corresponds to the motor and gas compressor, here called components, which are treated as black boxes in our study.

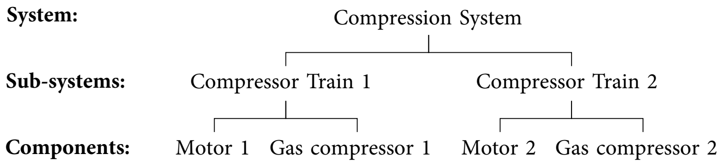


Figure 4.4: Compression system breakdown structure

Condition monitoring is performed on the components: the gas compressors are continuously monitored with online sensors, while the electrical motors can be subject to inspections that require to take them out of operation.

The condition of the gas compressor is described with four discrete states: perfect, low degraded, high degraded and failed. The condition is monitored indirectly but continuously. The production capacity of the compressor drops to 80% when its condition reaches the high degraded state.

The condition of the motor is described with three discrete states: perfect, degraded and failed. The motor is not continuously monitored. In order to detect its degraded state, an inspection must be performed.

For both the motor and the gas compressor, the deterioration process is considered to follow random jumps with constant transition rates, with the deteriorated state gradually increasing from perfect to failed, in accordance to previous works [4, 100]. The state of the motor can jump from perfect to deteriorated with rate λ_{md}, and from deteriorated to failed with rate λ_{mf}. The state of the gas compressor can jump from perfect to low degraded with rate λ_{cd1}, from low degraded to high degraded with rate λ_{cd2}, and from high degraded to failed with rate λ_{cf}.

In addition to the random jumps, the system experiences changes of state according to the maintenance intervention schedule. The production requirements and variations in the demand profile, can lead to delaying the execution of maintenance tasks. For example, the compression system is required at full capacity during peak season, so any preventive maintenance task should not be triggered at that time. On the other hand, periods of low demand present an opportunity to preventively maintain the system in order to ensure an acceptable level of the system reliability during the following high demand periods.

Both structural and resources dependencies are present in our case. At the components level, there is an obvious dependency, e.g. a gas compressor cannot function without a failed drive motor and contrariwise, a failure in the gas compressor would activate a shut down of the motor. At the sub-systems level, although we could consider that the operation and performance of one compressor train does not influence the other, both compressor trains contribute to the performance of the system and when assessing a maintenance policy this should be take into consideration. In addition, we consider that there is only one maintenance crew available and it can only work on one compression train (motor + gas compressor) at a time. Then a maintenance task in one compressor train prohibits a maintenance task on the other train. To manage the structural dependencies at the component level, we propose to model states and transitions at the compressor train (sub-system) level. Later, to deal with the resource dependency to maintain the syb-systems, the maintenance epoch and duration, we propose a PDMP at the system level, mapping the transitions and the process evolution.

We consider that when one of the unit fails, the train immediately stops and no further deterioration on the train occurs. Based on the condition of its components, we define the states of the compression train as in table 4.1.

Train state	Motor state	Gas compressor state	Capacity
11	Perfect	Perfect	100%
10	Degraded	Perfect	100%
9	Perfect	Low degraded	100%
8	Degraded	Low degraded	100%
7	Perfect	High degraded	80%
6	Degraded	High degraded	80%
5	Failed	Perfect	0%
4	Failed	Low degraded	0%
3	Failed	High degraded	0%
2	Perfect	Failed	0%
1	Degraded	Failed	0%

Table 4.1: Compression train states

In a simplification of the production profile, one year can be divided into two seasons, based on the gas demand: winter (high demand) and summer (low demand). Hence, it is preferred to take care of maintenance actions during a summer season since the production losses during winter season are considered to be way too high. We consider there is only one maintenance crew, so only one compressor train can be maintained at a time.

Based on the PDMP framework, we describe the stochastic process at the system level, considering random jumps related to components deterioration, jumps at specific times related to maintenance and the constraint of the resources which require a synchronization of the maintenance tasks.

4.3.3 PDMP Formalism

Consider the process $\{I(t), X(t)\} = \{\mathbf{i}, \mathbf{x}\}$ with values in a discrete-continuous space $E \times R$. Both the discrete and continuous components are vectors and their elements are described next.

Variables

The discrete component for our case is a vector of three elements, $\mathbf{i} = (i_1, i_2, i_3)$, where:

- (i_1, i_2) is a pair denoting the state of both compression trains, as listed in table 4.1.
- $i_3 = \{3, 2, 1\}$ is used to indicate an operation mode, where: 3 indicates that both trains are in operation, and both 2 and 1 indicate that a maintenance task is in progress in one of the trains. This variable allows us to define the appropriate boundary jumps related to interventions, as described later in section 4.3.3.

The discrete state space of the process is then made of $E : \mathscr{S}_{i_1} \times \mathscr{S}_{i_2} \times \mathscr{S}_{i_3}$, where $\mathscr{S}_{i_1}$ and $\mathscr{S}_{i_2}$ correspond to the possible states of the compressor trains and $\mathscr{S}_{i_3}$ corresponds to the possible states of operation, i.e. $\mathscr{S}_{i_1}, \mathscr{S}_{i_2} = \{11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$ (as defined in table 4.1) and $\mathscr{S}_{i_3} = \{3, 2, 1\}$.

The continuous component is a vector of two elements $\mathbf{x} = (x, t)$, where:

- x denotes the date of a next maintenance related transition. This can be a date in which a maintenance starts on a compressor train or finishes.
- t denotes time.

We use x as a variable to keep track of time of an intervention jump. This allows to describe the process in a Markovian way, analogous to the state augmentation mentioned in [54]. Between two consecutive jumps, the deterministic motion is linear with only t evolving with speed of one. In our case, x correspond to a frontier or boundary placed on the time domain. This boundary also experiences jumps in its value that are related to the maintenance scheduling or maintenance actions duration, as later described in the section related to intervention jumps.

Random Jumps ($t \neq x$)

Random jumps correspond to the stochastic deterioration of the system. In general, a jump from state $(\mathbf{i}, \mathbf{x})$ towards discrete state $\mathbf{j}$ occurs with a rate $\lambda(\mathbf{i}, \mathbf{x})Q(\mathbf{i}, \mathbf{x}, \mathbf{j})$, where $\lambda(\mathbf{i}, \mathbf{x})$ is the rate at which the process leaves $\mathbf{i}$ and $Q(\mathbf{i}, \mathbf{x}, \mathbf{j})$ is the probability distribution of the jump from $\mathbf{i}$ to $\mathbf{j}$. In this sense, the rate at which the process jumps from the discrete component $\mathbf{i}$ to a discrete component $\mathbf{j}$ is dependent on both the discrete and continuous component before the jump $(\mathbf{i}, \mathbf{x})$ and the discrete component after the jump $(\mathbf{j})$.

In our case, we consider that the jump rate and the transition measures are homogeneous, meaning that they do not depend on time or the continuous component. Then the transition kernels (given by the pair (λ, Q)) are said to be homogeneous. For the sake of simplicity, we write the transition kernel from discrete component $\mathbf{i}$ to $\mathbf{j}$ as $g(\mathbf{i}, \mathbf{j})$, with: $g(\mathbf{i}, \mathbf{j}) = \lambda(\mathbf{i})Q(\mathbf{i}, \mathbf{j})$ and $g(\mathbf{i}, \mathbf{i}) = 1 - \lambda(\mathbf{i})$.

We distinguish here two cases: the first one when $i_3 \neq 3$, which means that one train is under maintenance and only the train with subscript 1 is in operation, and a second case when $i_3 = 3$, which means that both compression trains are in actual operation.

One compressor train in operation. $i_3 \neq 3$ We consider that the compressor train denoted with subscript 1 is in operation while the compressor train denoted with subscript 2 is under maintenance. Hence, only the state of the unit i_1 might experience random jumps while the state of the unit under maintenance i_2 remains constant, until a boundary jump.

The process might jump from discrete component $\mathbf{i} = (i_1, i_2, i_3)$ to a discrete component $\mathbf{j} = (j_1, j_2, i_3)$ with $i_2 = j_2$. The transition kernels of such jump is given by equation 4.1, where $a_{i,j}$ is the transition rate from i to j of a single compressor train and α_i is the total departure rate from i of a single compressor train. The list of such rates is shown in table 4.2.

$$
\begin{aligned}
&g((i_1, i_2, i_3), (j_1, j_2, j_3)) = a_{i_1, j_1} \text{ for } i_1 \neq j_1,\ i_2 = j_2, \text{ and } i_3 = j_3 \neq 3 \\
&g((i_1, i_2, i_3), (i_1, i_2, i_3)) = 1 - \alpha_{i_1}
\end{aligned}
\tag{4.1}
$$

Both compressor trains in operation. $i_3 = 3$ In this case both compressor trains might experience random jumps related to their deterioration. The transition kernels from the discrete component $\mathbf{i}$ to the discrete component $\mathbf{j}$ are given by the transitions and departure rates for a single compressor train, as shown in equation 4.2.

$$
\begin{aligned}
&g((i_1, i_2, 3), (j_1, j_2, 3)) = \begin{cases} a_{i_1, j_1} a_{i_2, j_2} & \text{for } i_1 \neq j_1 \text{ and } i_2 \neq j_2 \\ (1 - \alpha_{i_1}) a_{i_2, j_2} & \text{for } i_1 = j_1 \text{ and } i_2 \neq j_2 \\ a_{i_1, j_1} (1 - \alpha_{i_2}) & \text{for } i_1 \neq j_1 \text{ and } i_2 = j_2 \end{cases} \\
&g((i_1, i_2, 3), (i_1, i_2, 3)) = 1 - (\alpha_{i_1} + \alpha_{i_2})
\end{aligned}
\tag{4.2}
$$

The departure rate $\alpha(i)$ of a single compressor train from discrete state i and the non-zeros transition rates $(a_{i,j})$ from state i into state j are shown in table 4.2. As mentioned in section 4.3.2, it is assumed that the states $5, 4, 3, 2$ and 1 are absorbing states and hence their total departure rate is zero.

Intervention jumps $(t = x)$

We distinguish here three cases based on the value of i_3 when the boundary is reached. When the boundary is hit at $t = x$, both the discrete and the continuous component of the PDMP experience a jump. We denote $b(\mathbf{i}, \mathbf{j})$ the transition kernel from discrete component $\mathbf{i}$ to $\mathbf{j}$ corresponding to a boundary jump and $m_\Gamma(\mathbf{i}, \mathbf{j})$ the value of the continuous component after the jump.

When the operation mode of the post-jump is different than three $(j_3 \neq 3)$, the date of a next maintenance intervention (x) after the boundary jump corresponds to the time to maintain the compressor train j_2, denoted (M_{j_2}). When $j_3 = 3$, the date of a next maintenance intervention corresponds to a period to maintain the system, denoted τ. The variable time t does not change with the boundary jump, so the value of the continuous component

State	Total departure rate	Non-null transition rates
11	$\alpha_{11} = \lambda_{md} + \lambda_{cd1}$	$a_{11,10} = \lambda_{md}$
		$a_{11,9} = \lambda_{cd1}$
10	$\alpha_{10} = \lambda_{md} + \lambda_{cd1}$	$a_{10,8} = \lambda_{cd1}$
		$a_{10,5} = \lambda_{mf}$
9	$\alpha_{9} = \lambda_{md} + \lambda_{cd2}$	$a_{9,8} = \lambda_{md}$
		$a_{9,7} = \lambda_{cd2}$
8	$\alpha_{8} = \lambda_{mf} + \lambda_{cd2}$	$a_{8,6} = \lambda_{cd2}$
		$a_{8,4} = \lambda_{mf}$
7	$\alpha_{7} = \lambda_{md} + \lambda_{cf}$	$a_{7,6} = \lambda_{md}$
		$a_{7,2} = \lambda_{cf}$
6	$\alpha_{6} = \lambda_{mf} + \lambda_{cf}$	$a_{6,1} = \lambda_{cf}$
		$a_{6,3} = \lambda_{mf}$

Table 4.2: Random jump rates

post-jump are shown in equation 4.3.

$$
\begin{aligned}
m_\Gamma((i_1, i_2, 3), (j_1, j_2, 1)) &= (M_{j_2}, t) \\
m_\Gamma((i_1, i_2, 1), (j_1, j_2, 2)) &= (M_{j_2}, t) \\
m_\Gamma((i_1, i_2, 2), (j_1, j_2, 3)) &= (\tau, t)
\end{aligned}
\tag{4.3}
$$

Start of first maintenance task ($i_3 = 3$ and $t = x$) The maintenance policy must decide which of the two compressor trains should be maintained first, given their states. In our case, the state of the gas compressor unit is known due to continuous monitoring, while the motor might have a hidden degradation. The operator can use the information about the state of the gas compressor as a way to prioritize what to maintain first. However, it is not possible to distinguish between train states with the same gas compressor state and non failed motor. For example, it is not possible to distinguish between train states 11 and 10 before inspecting the motor due to the hidden degradation. This is the case for the (unordered) pairs (10,11), (8,9), (6,7) and (1,2). The transition kernel for the discrete component for these pairs are shown in equation 4.4.

$$
\begin{aligned}
b((i_1, i_2, 3), (j_1, j_2, 1)) &= 0.5; \text{ with } j_1 = i_1 \text{ and } j_2 = i_2 \\
b((i_1, i_2, 3), (j_1, j_2, 1)) &= 0.5; \text{ with } j_1 = i_2 \text{ and } j_2 = i_1
\end{aligned}
\tag{4.4}
$$

For all other compressor train states, the maintenance crew gives priority to the train with the more degraded gas compressor, which is expressed in equation 4.5, meaning that the compressor train with the most degraded gas compressor is maintained while the other is kept in operation.

$$b((i_1,i_2,3),(j_1,j_2),1)) = 1;\ \text{with } j_1 = max(i_1,i_2) \text{ and } j_2 = min(i_1,i_2) \tag{4.5}$$

In both cases, the variable indicating an operation mode (i_3) changes its value from 3 to 1 to express the change from operating both compressor trains to operating only one and placing the first train under maintenance.

Finish first and start second maintenance task ($i_3 = 1$ and $t = x$) The compressor train which has been maintained is placed into operation (with its new state $S(i_2)$) and the other is taken into maintenance. There is a switch between i_1 and i_2 and the operation mode (i_3) changes from 1 to 2, as expressed in equation 4.6.

$$b((i_1,i_2,1),(j_1,j_2,2)) = 1;\ \text{with } j_1 = S(i_2) \text{ and } j_2 = i_1 \tag{4.6}$$

The new state of the maintain compressor train is expressed $S(i_2)$ as a function of the state of the train before the maintenance action. It is a decision or intervention parameter, which corresponds to a jump in the discrete state of the compressor train, mapped $S : i_2 \to j$.

Finish second maintenance task ($i_3 = 2$ and $t = x$) When the second maintenance task is finished, both compressor trains are placed into operation, hence the variable indicating the operation mode (i_3) changes from 2 to 3, as shown in equation 4.7.

$$b((i_1,i_2,2),(j_1,j_2,3)) = 1;\ \text{with } j_1 = i_1 \text{ and } j_2 = S(i_2) \tag{4.7}$$

Numerical approach

We proceed to quantify an approximation of the states probabilities of the underlying Markov process. This approach is based on a finite-volume scheme proposed in previous works [78, 79]. Examples of its application can be found in [77, 80, 101, 73].

Every term of the continuous state space R is discretized in a finite number of values. In our case, we have $R \subset \mathbb{R}^2$ with time as the quantity for both dimensions. Let δ be the continuous state space step. Then, the approximation of the k-ith term has values in $F_k = \{0, \delta, 2\delta, \ldots, n\delta\}$, with $k = \{1,2\}$ and n an integer. In this way, R is replaced by the discrete-state space $F = F_1 \times F_2$.

In our case, we have a simple linear (deterministic) motion. Between $t = n\delta$ and $t = (n+1)\delta$ the continuous component follows the function $\mathbf{x} \to \mathbf{x} + \mathbf{v}\delta$, with $\mathbf{v} = (0,1)$, since between two consecutive jumps, only the variable t (time) evolves with a speed of one. When the frontier Γ is reached at $\mathbf{x}$, the continuous component jumps instantaneously from Γ to F as $\mathbf{x} \to m_\Gamma(\mathbf{i},\mathbf{j})$.

Let $P((\mathbf{i},\mathbf{x}),(\mathbf{j},\mathbf{y}))$ denote the conditional transition probability from state $(\mathbf{i},\mathbf{x})$ to state $(\mathbf{j},\mathbf{y})$ with values in the finite state space $E \times F$, and $\tilde{F}$ denote the set of points in F which are not on the frontier Γ. The arrivals into a state $(\mathbf{j},\mathbf{y})$ may proceed from different paths.

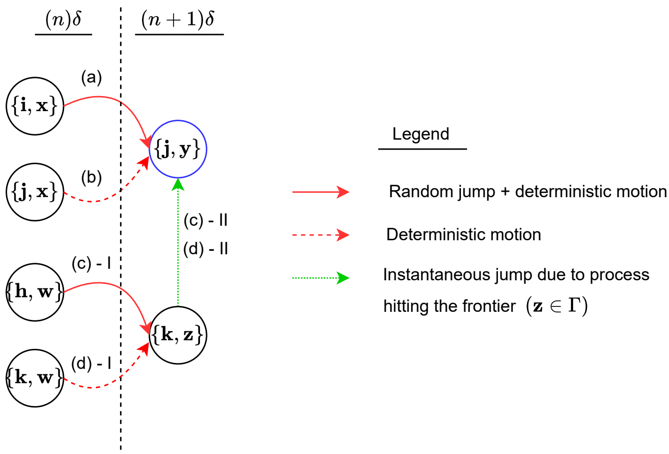


Figure 4.5: Transitions into state $(\mathbf{j},\mathbf{y})$ in $(n\delta,(n+1)\delta]$

Let $\pi_{n\delta}$ denote the marginal probability of this stochastic process at the n-ith iteration. By using the law of total probability and the Markov property, we can write the law of the process for state $(\mathbf{j},\mathbf{y})$ at the $(n+1)$-ith iteration based on the transitions showed on figure 4.5. We have for any $\mathbf{i},\mathbf{h}$ in E and $\mathbf{x},\mathbf{w}$ in F:

$$\begin{aligned}
\pi_{(n+1)\delta}(\mathbf{j},\mathbf{y}) &= \sum_{\substack{\mathbf{i}\neq\mathbf{j}\\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}}^{N-1} \pi_{n\delta}(\mathbf{i},\mathbf{x})[P((\mathbf{i},\mathbf{x}),(\mathbf{j},\mathbf{y}))] \\
&+ \mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}}\pi_{n\delta}(\mathbf{j},\mathbf{x})[P((\mathbf{j},\mathbf{x}),(\mathbf{j},\mathbf{y}))] \\
&+ \sum_{\substack{\mathbf{k}\\ \mathbf{y}=m_\Gamma(\mathbf{k},\mathbf{j})}}^{N} \sum_{\substack{\mathbf{h}\neq\mathbf{k}\\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta\\ \mathbf{z}\in\Gamma}}^{N-1} \pi_{n\delta}(\mathbf{h},\mathbf{w})[P((\mathbf{h},\mathbf{w}),(\mathbf{k},\mathbf{z}))][P((\mathbf{k},\mathbf{z}),(\mathbf{j},\mathbf{y}))] \\
&+ \sum_{\substack{\mathbf{k}\\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta\\ \mathbf{z}\in\Gamma\\ \mathbf{y}=m_\Gamma(\mathbf{k},\mathbf{j})}}^{N} \pi_{n\delta}(\mathbf{k},\mathbf{w})[P((\mathbf{k},\mathbf{w}),(\mathbf{k},\mathbf{z}))][P((\mathbf{k},\mathbf{z}),(\mathbf{j},\mathbf{y}))]
\end{aligned} \tag{4.8}$$

In equation 4.8, the first term accounts for the transitions related to a random jump and deterministic motion (shown as (a) in figure 4.5); the second term for the transitions related to only deterministic motion (shown as (b) in figure 4.5); the third term for the transitions with a random jump, deterministic motion plus an instantaneous jump from the frontier (shown as (c)-I + (c)-II in figure 4.5); and the last term accounts for the transitions with deterministic motion plus an instantaneous jump from the frontier (shown as (d)-I + (d)-II in figure 4.5).

Substituting the conditional probabilities by their approximation or value from equations 4.1-4.7, gives:

$$
\begin{aligned}
\pi_{(n+1)\delta}(\mathbf{j},\mathbf{y}) \approx & \sum_{\substack{\mathbf{i}\neq\mathbf{j} \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}}^{N-1} \pi_{n\delta}(\mathbf{i},\mathbf{x})[g(\mathbf{i},\mathbf{j})\delta] \\
& + \mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}} \pi_{n\delta}(\mathbf{j},\mathbf{x})[g(\mathbf{j},\mathbf{j})\delta] \\
& + \sum_{\substack{\mathbf{k} \\ \mathbf{y}=m_\Gamma(\mathbf{k},\mathbf{j})}}^{N} \sum_{\substack{\mathbf{h}\neq\mathbf{k} \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma}}^{N-1} \pi_{n\delta}(\mathbf{h},\mathbf{w})[g(\mathbf{h},\mathbf{k})\delta][b(\mathbf{k},\mathbf{j})] \\
& + \sum_{\substack{\mathbf{k} \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(\mathbf{k},\mathbf{j})}}^{N} \pi_{n\delta}(\mathbf{k},\mathbf{w})[g(\mathbf{j},\mathbf{j}))\delta][b(\mathbf{k},\mathbf{j})]
\end{aligned} \tag{4.9}
$$

Equation 4.9 corresponds to an approximation of the Chapman-Kolmogorov equation which describes the time evolution of the states probabilities for Markov processes. Demonstration by mathematical proof and validation and comparison with Monte Carlo simulation has been developed in chapter 3.

4.3.4 System performance

The compression system made of the two compressor trains can be categorized according to its capacity as a system. In this case, we have a multi-state system with multi-state components. The total capacity of the system is given by the combination of the capacities of both trains and its operation mode (up or in-maintenance). As shown in table 4.1, the capacity of a single compression train can be of 100%, 80% or 0%, depending on its state. At the system level, we consider not only the state of each train, but also the operation mode of the system, e.g. whether both trains are active or one is unavailable due to maintenance. The total capacity of the system can be classified in six levels: 100%, 90%, 80%, 50%, 40% and 0%. We denote $\mathscr{S}$ the state space of the compression system, with $\mathscr{S} = \{100, 90, 80, 50, 40, 0\}$.

Multi-state systems reliability theory is as a natural extension of the binary theory commonly used in system reliability analysis. In binary theory, Reliability block diagrams (RBD) model a system function with a success oriented graph with a single source and a single terminal. The nodes are called blocks and each one represents a component function. Each block is either up or down and intermediate states are not allowed. The blocks are connected by edges and the system function is up if there exists a path from the source to the terminal through blocks that are up [58]. Each block has an associated binary state variable and the state of the system can be described by a binary function of such variables based on the system structure (structure function).

In a multi-state system case, we can use a similar approach with binary variables and functions, providing an algebraic method to compute the system reliability. Such algebraic method has been studied and proposed with an efficient algorithm for the computation which is applicable to complex systems with independent non-identical components [102].

We can introduce binary variables $\chi_{i_k}^j$ such as:

$$\chi_{i_k}^j = \begin{cases} 1 \text{ if component } i_k \text{ is in state } j \text{ or above} \\ 0 \text{ if component } i_k \text{ is in a state below } j \end{cases} \tag{4.10}$$

Where $k = \{1,2,3\}$ and $j \in \mathscr{S}_{i_k}$. We can then write binary structure functions ϕ^s for each state $s \in \mathscr{S}$, such as:

$$\phi^s = \begin{cases} 1 \text{ if the system is performing at state } s \text{ or above} \\ 0 \text{ if the system is performing at a state lower than } s \end{cases} \tag{4.11}$$

Let us consider a structure for the system to function at state s with associated minimal paths sets $P_1, P_2, ..., P_p$. Every minimal path set P_n has a structure function of a series of components of the type $\chi_{i_k}^j$. We know that the system functions at state s if and only if one of its minimal path series structure is up. Then the system state s has a binary structure function as equation 4.12 that can be interpreted as a parallel structure of the minimal path series structures.

$$\phi^s(\vec{\chi}) = \coprod_{n=1}^{p} \prod_{\chi_{i_k}^j \in P_n} \chi_{i_k}^j \tag{4.12}$$

The system structure function for state s is then a polynomial formula in terms of the states of the components used to obtain the s-reliability of the system. Pascual-Ortigosa et al. [102] proposed associating an algebraic object called a monomial ideal to the coherent system, and by studying the algebraic properties of this ideal obtain information about the system and its reliability. In particular, the numerator of the Hilbert series of an ideal, corresponds to the structure function presented here. In our case we have the system states' structure functions shown in equation 4.13.

$$\begin{aligned}
\phi^{100}(\vec{\chi}) &= \chi_1^8\chi_2^8\chi_3^3 \\
\phi^{90}(\vec{\chi}) &= \chi_1^6\chi_2^8\chi_3^3 + \chi_1^8\chi_2^6\chi_3^3 - \chi_1^8\chi_2^8\chi_3^3 \\
\phi^{80}(\vec{\chi}) &= \chi_1^6\chi_2^6\chi_3^3 \\
\phi^{50}(\vec{\chi}) &= \chi_1^8 + \chi_2^8\chi_3^3 + \chi_1^6\chi_2^6\chi_3^3 - \chi_1^8\chi_2^6\chi_3^3 - \chi_1^6\chi_2^8\chi_3^3 \\
\phi^{40}(\vec{\chi}) &= \chi_1^6 + \chi_2^6\chi_3^3 - \chi_1^6\chi_2^6\chi_3^3
\end{aligned} \tag{4.13}$$

Let R_s denote the probability that the system is performing at level greater than or equal to s. For time dependent probability, we have $R_s(t) = [Pr(\phi^s)(t) = 1]$ which is obtained by assigning probabilities to equation 4.13, which are obtained from the recursive equation 4.9.

4.3.5 Experiments and results

State probabilities

We find the time dependent solution by computing the recursive equation 4.9. That results in a finite number of state vectors at every time point t with an associated probability mass, as: $\pi_t(\vec{j}, \vec{y})$. Then we focus only on the discrete component of the system state, so at every time step, we can use the procedure described in section 4.3.4 and by assigning probabilities to equation 4.13, we obtain the s-reliability of the system, i.e. the probability that the system performance is at least at s capacity, with $s \in \mathscr{S}$ and from there we can directly obtain the system states probabilities as shown in figure 4.6.

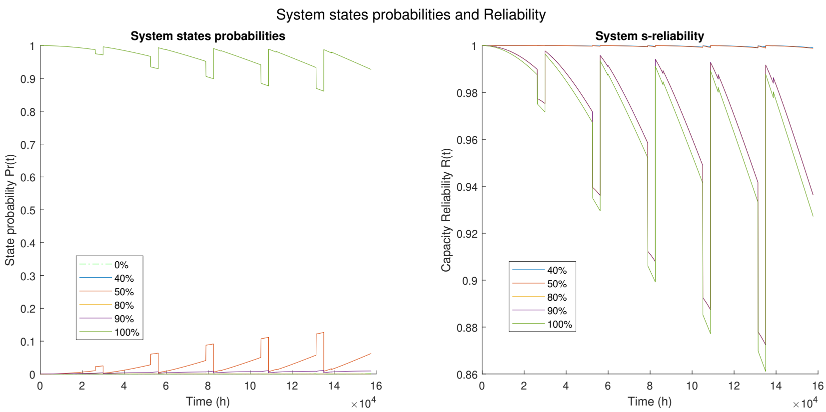


Figure 4.6: System states and reliability

The deterioration parameters used for quantification are shown in table 4.3. In addition, for the interventions we considered a maintenance policy for a compressor train every three years, in which: preventive maintenance is only performed in the gas compressor when in the high degraded state. There is no preventive maintenance actions in the motor (replacements nor inspections) or in the gas compressor when in low degraded state. Corrective maintenance actions are performed in any of both components and when carried out in the motor they are joined with preventive maintenance in the gas compressor if it is in high degraded state. These assumptions are reflected in table 4.4.

Motor	Gas compressor
	$\lambda_{cd1} = 1\text{e}{-6}\ h^{-1}$
$\lambda_{md} = 1\text{e}{-6}\ h^{-1}$	$\lambda_{cd2} = 1\text{e}{-5}\ h^{-1}$
$\lambda_{mf} = 1\text{e}{-5}\ h^{-1}$	$\lambda_{cf} = 1\text{e}{-4}\ h^{-1}$

Table 4.3: Deterioration parameters

State before (i)	State After (S_i)	Duration (M_i)
11	11	N/A
10	10	N/A
9	9	N/A
8	8	N/A
7	11	3650 h
6	10	3650 h
5	11	3650 h
4	9	3650 h
3	11	3650 h
2	11	3650 h
1	10	3650 h

Table 4.4: Scope of Maintenance

In figure 4.6 we can observe how in the long run the performance of the system between maintenance windows tends to decrease since the proposed maintenance policy is not a full renewal of the system, i.e. not all system states are maintained to a good-as-new condition.

Maintenance analysis

The proposed model allows to test different maintenance policies and assess their effect on the system capacity. As previously stated, it is considered that only one maintenance crew is available, so only one compressor train can be intervened at a time. Maintenance tasks on the components of a compressor train are joint. For illustration purposes, we consider three preventive maintenance policies in addition to the one considered in section 4.3.5 and table 4.4 (from now on denoted policy 0), keeping the maintenance windows every three years. We make variations on the scope of the policies, from the less conservative in which no preventive maintenance actions are contemplated, to the most conservative in which preventive maintenance is performed for any condition of the components.

- Policy 1: Only corrective maintenance actions are considered on both components without any preventive maintenance.
- Policy 2: Preventive maintenance performed on the gas compressor from any state but not on the motor.
- Policy 3: Preventive maintenance on the compressor train is performed for both the motor and gas compressor for any state. This means that the motor is inspected and if degradation is found then it is replaced to the perfect state.

In this sense, policy 3 could be considered the most conservative which would result in the highest average maintenance cost, policies 0 and 2 are have a lower degree of preventive maintenance scope and policy 1 is only corrective maintenance for comparison. The parameters of these maintenance policies are shown in table 4.5.

	Policy 1		Policy 2		Policy 3	
i	S_i	M_i	S_i	M_i	S_i	M_i
11	11	N/A	11	N/A	11	120 h
10	10	N/A	10	N/A	11	3770 h
9	9	N/A	11	1820 h	11	1820 h
8	8	N/A	10	1820 h	11	3770 h
7	7	N/A	11	3650 h	11	3650 h
6	6	N/A	10	3650 h	11	3770 h
5	11	3650 h	11	3650 h	11	3650 h
4	9	3650 h	11	3650 h	11	3650 h
3	7	3650 h	11	3650 h	11	3650 h
2	11	3650 h	11	3650 h	11	3650 h
1	10	3650 h	10	3650 h	10	3770 h
i - State before, S_i - State after, M_i - Duration						

Table 4.5: Scope of Maintenance policies

From the system state probabilities we can get the average or expected available system state capacity, for the simulated maintenance policy with equation 4.14. Figure 4.7 on the left subplot shows the average available system capacity of the four maintenance policies considered and on the right subplot shows a zoom into a maintenance window.

$$\mathbb{E}(\mathscr{S}) = \sum sPr(\mathscr{S} = s) \tag{4.14}$$

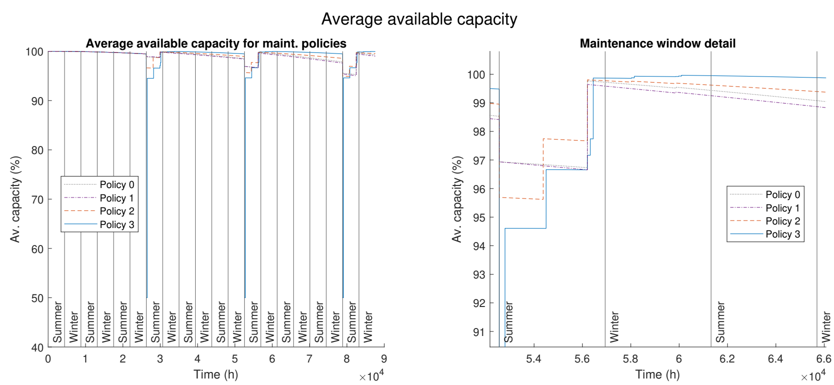


Figure 4.7: Average available system capacity for different maintenance policies

In figure 4.7 it can be observed how while maintenance policy 1 results in higher average available system capacity on intervals between maintenance windows, it also results in lower average available capacity during the summer. For example, the average available system capacity may be unnecessarily reduced in some cases due to inspections on the motor even when it is in perfect state. Since the production demand does not stop, a maintenance policy like this one could involve potential losses during the duration of the action that could be important to quantify, adding up to the costs of performing the maintenance.

Besides considering maintenance policies which keep the interval every three years, it is also possible to make variations on the interval between maintenance windows and assess the performance to choose a favorable interval for the given policy.

To assess the cost-benefit of the maintenance policies it is necessary to quantify the expected number of actions of a certain type (e.g. motor inspections, small preventive compressor repairs, large preventive compressor repairs, corrective maintenance) in a given time horizon and its cost. In addition, it is possible to quantify the expected losses in production by comparing the average time-dependent capacity to a predicted time-dependent demand of the system in the given time horizon. With this information maintenance policies can be properly assessed and compared. The lower the negative impact of a maintenance policy the better.

4.4 Chapter conclusion

A framework for CBM models with multi-state components based on the theory of PDMP has been presented as an approach to simulate maintenance policies assessing the performance at the system level. State augmentation allows to reformulate problems that initially do not posses the Markov property into one that can be described in a Markovian form, and since a PDMP is a stochastic hybrid model, it widely opens the possibilities for what stochastic processes can be formulated in this form, as the case presented in this chapter. Modelling a problem as a Markov process facilitates the description of its evolution, which can serve as a basis to implement a numerical scheme or to build a Monte Carlo simulation algorithm for its solution.

Through the application on the compression system, resource dependency is contemplated as a constraint on the availability of a maintenance crew and structural dependency is considered since the performance (available capacity) of the system is determined by the condition of the components. For this purpose, we have proposed a PDMP in which the discrete component is a vector containing information on the state of both compressor trains plus a variable denoting an operational mode. This last variable allows us simulate a synchronisation on the maintenance tasks of the system by enabling and restricting certain transitions of the stochastic process depending on its value.

In the case presented here, the implementation of the numerical approach has been found to be an efficient method for computing the time dependent probabilities, concurring with previous studies. Nevertheless, the discretization of the continuous state space results in high computer memory consumption. Increasing the dimension of the process or the dynamic of the deterministic flow can make this numerical approach a challenge to implement and compute.

The states of the proposed PDMP are developed based on the condition or health of the components in order to allow the simulation of different CBM policies. However, to assess the performance of the policies, the available capacity of the system can be compared to a production or demand plan, which by nature can be dynamic and time-dependent. This demand plan is considered fixed in the case study presented in this chapter, but in reality it can be subject to unexpected fluctuations and uncertainties of the market. The available capacity of the system depends on the condition of the trains and on its operation or maintenance status, with structures that are not homogeneous across the system states, e.g. one system state has a series structure of component states while other has a k-out-of-n structure. To assess the performance at the system level, an algebraic approach from multi-state systems reliability theory has been described. This is an efficient approach to describe and calculate the combinatorial problem for our case.

The proposed framework allows to assess the performance of maintenance policies for systems which are required to operate continuously, while taking into account the duration of the maintenance actions according to the component states and its impact on the performance of the system. A direction of further works could be to study the scalability of the framework for larger systems, with special attention on the capabilities of the implementation of the numerical approach compared to Monte Carlo simulation.

Chapter 5

Highlights, conclusion and further works

Throughout the thesis we have resorted to a simplified class of PDMP that can be called Piecewise-linear Markov Processes for modelling of the studied CBM problems. As formulated by Vermes [56], a Piecewise-linear process is a hybrid Markov process $\{I(t), X(t)\}$ where the primary component $I(t)$ takes integer values and the secondary component $X(t)$ takes values in an interval of the real line. When the process starts, the primary component keeps a fixed value while the secondary component moves to the right with unit speed, until a jump occurs. A jump in the process can occur either randomly or when the secondary component hits a boundary in its interval. Then the motion restarts from the new state. Thus, the law of the process is determined by specifying the boundaries, the jump rate and the transition measures.

By making use of the relatively simple structure and evolution of this model, we are able to greatly increase the flexibility of the model assumptions that can be handled for CBM problems, when compared to a Markov chain. In this chapter the modelling capabilities for CBM problems of this specific class are highlighted, with the intention of serving as a guide for CBM model developers. As previously described, we separate the CBM model into two dependent models, a deterioration model and an intervention model.

5.1 Highlights of PDMP as a framework for CBM modelling

5.1.1 Deterioration model

The main requirement on the deterioration model is that the deterioration is characterized as a jump process with random transition times between discrete states.

Time-dependent jump rates

An interesting feature of the proposed PDMP, is that the jump rate $\lambda(i, x)$ is dependent on both the discrete and continuous component of the process. In the case of a Piecewise-linear process, this allows to model time-dependent transition rates between the states. For example, when sojourn times follow a Weibull distribution with shape parameter α and scale parameter θ, the jump rate of the process, for a discrete state i, is given by:

$$\lambda(i,x) = \alpha\theta^{-\alpha}t^{\alpha-1} \tag{5.1}$$

where t, denoting time, is included in the continuous component of the process, as in the proposed formalism in this thesis ($\mathbf{x} = (x_1,...,t)$). It is possible to map the random jumps such that a change in the discrete component also entails a restart of time, or a jump to zero in t, such as that time is not global but keeping track of the time spent in the discrete state i, say state-age.

For the numerical approach, as an example, one could write the non-null conditional probabilities keeping the notation from section 3.2.3, between $t = n\delta$ and $t = (n+1)\delta$, as:

$$\begin{aligned} P((i,\mathbf{x}),(j,\mathbf{y})) &\approx \lambda(i,\mathbf{x})Q(i,\mathbf{x},j)\delta \text{ with } \mathbf{y} = (x_1,...,0) \\ P((j,\mathbf{x}),(j,\mathbf{y}) &\approx 1-\lambda(j,\mathbf{x})\delta \text{ with } \mathbf{y} = \mathbf{x}+\mathbf{v}\delta \end{aligned} \tag{5.2}$$

It is worth to point out that for the numerical approach, having time-dependent (or continuous component-dependent) jump rates, requires recomputing the jump rate at each time-step and increases greatly the amount of vector-states with non-null probability mass at each time-step. This may require large computational memory, and could result in problems for the numerical approach depending on the computer memory and software capacity, depending on the complexity or dimensionality of the process. In these cases, Monte Carlo simulations are a useful method for analyzing the process with the trade-off of computational time. Other solutions could be also studied, such as Phase-type distributions [103]. It could be possible to account for non-exponential sojourn times by introducing additional virtual discrete state (with no physical meaning), with constant jump rates. The trade-off between adding discrete states vs time-dependent jump rates, must unfortunately be assessed for the specific cases, and it is difficult to generalize the solution.

Beyond the Piecewise-linear process, on PDMP the jump rate for the deterioration process can be dependent in physical variables included in the model in the continuous component, such as temperature, pressure, flow and others. Such physical variables evolve in deterministic fashion as solution of differential equations for a fixed discrete state. Again, for implementation of the numerical approach, the jump rate must be recomputed at each time-step, which in some cases and depending on the complexity of the model, could result in computational challenges related to memory and simulations of the process could be a good alternative.

Stochastic dependencies

For a multi-component system, the interesting feature in deterioration modelling is the capability of including stochastic dependencies among components. In section 4.2.2, the discrete component of the process is proposed as a vector $\mathbf{i} = (i_1, ..., i_{m+1})$ that contains the state of every component. Stochastic dependencies can be accounted in the model by writing the corresponding jump rates $\lambda(\mathbf{i}, \mathbf{x})$ for every component states combination. This is part of the modelling work and depending on the dimension of the system, the number of components and their number of states, it can be a challenging task. No shortcut has been envisioned for this and therefore the dimension of the system can be pose a limitation for the use of this modelling approach. Object-oriented programming methods are worth exploring for complex systems modelling.

5.1.2 Intervention model

One of the aspects of CBM models is capturing the condition monitoring of the system. Two relevant properties of the monitoring are important for the model, frequency and quality. Other features explored in this thesis in the intervention model are maintenance delays and resources dependencies.

Monitoring frequency

The frequency of the monitoring can be continuous or inspection-based (non-continuous). In continuous monitoring, the system is constantly monitored, by sensors and a setup that assess the condition of the system and triggers an alarm or warning when a threshold is met. Since continuous monitoring is not always technically possible or financially beneficial, some system are inspected at specified times in order to assess their condition. CBM models are useful to study the optimal frequency of inspections. These are the applications considered in this thesis in chapter 3.

The continuous component proposed in section 3.2.2, is a vector $\mathbf{x} = (x_1, x_2, t)$ in which x_1 corresponds to the date of the next inspection, x_2 corresponds to the date of the next maintenance operation, and t stands for time. In this way, x_1 is used in the model as a boundary to keep track to the time in which the system is inspected, allowing to specify in the model the appropriate transition as a jump. When this boundary is reached by the process, then the time of the next inspection is specified in $m_\Gamma(i, \mathbf{x}, j)$, the function for the location of the continuous component of the process after the jump. This allows to have a predetermined inspection scheme in which the period can be condition-based, that is, depending on the condition of the system (in the discrete component i, j) at the time of an inspection, the date of the next inspection is set.

Inspection quality

The CBM model should also capture the quality of the inspections. The PDMP framework allows to account for the degree of the quality of the inspection in form of a probability to reveal the true condition of the system. In a perfect inspection the exact state of the system is revealed without error. However, in some applications it is more realistic to assume that the inspections are imperfect and might not reveal the true state of the system.

The quality of the inspections can be taken into account by assigning a probability $q(\mathbf{i},\mathbf{x},\mathbf{j})$ distribution to the post-jump location, as described in section 3.2.2. If the inspections are considered perfect, then the virtual state becomes equal to the real state of the unit at the inspection time, with probability of one. If the inspections are considered imperfect or subject to errors, then we can write a conditional probability of the post-jump location of the virtual state (j_2), given the real state of the unit before the jump (i_1), as:

$$P(j_2|i_1) = q(\mathbf{i},\mathbf{x},\mathbf{j}) \tag{5.3}$$

The post jump location of the continuous component is given by:

$$m_\Gamma(\mathbf{i},\mathbf{x},\mathbf{j}) = \mathbf{x} + (T_{j_2}, M_{j_2}, 0) \tag{5.4}$$

These terms are included in the iterative equation of the numerical approach shown in equation 3.6.

Maintenance delay

Maintenance delay in this thesis is understood as the time elapsed from when a maintenance task is ordered until it is carried out. Maintenance lead time is another used term to refer to this. The framework of PDMP with linear motion gives the opportunity to model maintenance delay in the process, while keeping the process Markovian.

A maintenance task can be ordered when certain condition of the system is detected by continuous monitoring or by a scheduled inspection. When a task is ordered, its date is placed as a boundary for the time variable. From the process perspective, this is similar to scheduling the next inspection as previously described. The time of the maintenance action is specified in $m_\Gamma(i,\mathbf{x},j)$, which is a deterministic function, meaning that the maintenance delay is predetermined by the modeller in order to assess its performance.

The evolution of the process continues as usual during this delay, random jumps might occur or the boundary could be reached in the inspection dimension. Some scenarios that can take place during the delay are:

1. There is no further deterioration of the system (no random jump) and no inspection. In this case, the maintenance task takes place as planned on the corresponding date.

2. There is an inspection scheduled during the delay but no further deterioration has occurred. Then, the ordered maintenance is kept as planned and there is an extra cost of the inspection.

3. There is further deterioration of the system (random jumps) and an inspection. This gives the opportunity to re-schedule the maintenance task accordingly with the associated logistic, resources and date.

4. There is further deterioration of the system (random jumps) but no inspection. In this case, when the process hits the maintenance date boundary, additional resources might be required as the maintenance tasks is different than originally planned.

These scenarios impact the performance of the maintenance policy. Scenario 4 can result in a penalty cost due to wrong planning of the required maintenance task. In scenario 3, there is the added cost of the extra inspection, the correct maintenance task is planned, but no penalty is added. In scenario 2, there is the added cost of the extra inspection and in scenario 1 there is no added cost to the maintenance task. These quantities would impact the expected cost of the overall maintenance policy.

The number of wrongly planned maintenance tasks of scenario 4, could be quantified similarly to equation 3.14. For example, lets consider that when the maintenance boundary (Γ_2) is reached, the real deterioration state of the system is state $i_1 = 4$, but the maintenance that was ordered was for the system in state 2, i.e. the virtual state is $i_2 = 2$. Then, the number of these actions $N(t)$ can be quantified with the numerical scheme as:

$$N(t) \approx \sum_{\substack{u=0 \\ \mathbf{x}\in\Gamma_2}}^{t} \pi_u\{(4,2),(\mathbf{x})\} \tag{5.5}$$

Resource and structural dependencies

By resource dependency we have considered that sub-systems share a maintenance crew, so that one sub-system cannot be maintained while a maintenance action is taking place in other subsystem. To achieve this, we propose that the discrete component of the PDMP is a vector $i = (i1,...,im+1)$ that contains the state of every component and the last term i_{m+1} denotes an operational mode of the system and it is used to keep track of the availability of the maintenance crew. In this way, this last term can be used as a mean to count when a maintenance action is taking place in a sub-system or to indicate that the maintenance crew is available to initiate an action. Again, as with the stochastic dependencies, it is part of the modelling work to write the corresponding non-null transitions for every vector of states of components, including the maintenance crew state. Then manually, one can describe the process such as that a maintenance action can only initiate on given states of the maintenance crew operation mode.

To assess the overall performance of the system we consider the impact of the combination of the states of the components. This is considered structural dependence. For this, we write the multi-state structure functions for each performance level of the system, as can be seen in equation 4.13. This is an elegant and efficient way to communicate and aggregate a large amount of combination of component states into a few system states of interest.

These are the highlights from the capabilities of PDMP with linear motion as a framework for modelling CBM problems, based on the contents of this thesis. All of these features can be quantified with the numerical approach presented throughout this thesis. However, the dimension and complexity of the model might present a challenge for the numerical computation and implementation, as has been discussed. On the other hand, the framework of PDMP can provide a systematic approach to model the problem in terms of variables, random jumps, linear motion, while setting a foundation for simulations. It is worth to point out that although the class of PDMP virtually covers all stochastic non-diffusion applications, it is not intended to replace existing models in the literature, many of which present efficient techniques for calculations by making use of the special structure of specific models given the application and its model assumptions.

5.2 Conclusion

The focus of this thesis has been placed in studying CBM problems with prognosis, also called predictive maintenance. More specifically, Piecewise deterministic Markov processes are the central focus of the thesis for modelling and assessing CBM policies. PDMP is a stochastic process presented as a general class of non-diffusion stochastic models, in which the future development of the process depends only on a current state of the process and not on what has happened in the past. PDMP can cover a wide variety of applications that involve some combination of random jumps and deterministic motion. State augmentation allows to reformulate problems that initially do not posses the Markov property into one that can be described in a Markovian form. The structure of PDMP as a stochastic hybrid model, widely opens the possibilities for what stochastic processes can be formulated in this form.

A key element of a prognosis model is the deterioration of the system or unit. For PDMP, this means that the applications considered through this thesis assume that the deterioration of the system or unit evolves in a discrete-state space with random jumps and continuous-time. This is the case of many engineering applications, that due to practical reasons it is more reasonable to characterize the condition or health of the system by a finite set of deterioration states instead of in a continuous-state space. The PDMP mainly studied in this thesis can be seen as particular category of the class, in which the deterministic evolution of the continuous component is linear. This class has also been called Piecewise linear Markov process.

First, a framework for the modelling and assessment of CBM policies of single-items has been proposed and studied based on the theory of PDMP. The proposed model allows to study problems in which the condition monitoring is not continuous but inspection-based and there is an inherent delay for performing maintenance actions. Therefore, the transition law cannot be found by a simple Markov chain. The proposed framework allows the assessment of the probability for the infrastructure to be in certain deterioration state given an inspection period and given a maintenance schedule. In this way, it is possible to evaluate if a given CBM policy is adequate regarding some safety requirements by making variations of the inspection period, the delay before intervention, or the state to which the system is restored after maintenance. It is also possible to assess the cost of a CBM policy to find the optimal parameters of the policy.

Then, the capabilities of PDMP for the modelling and assessment of CBM policies of multi-component system subjected to inter-dependencies has been explored. Resource dependency has been contemplated as constraints on the availability of maintenance crew and structural dependency has been considered in the sense that the overall performance of the system is determined by the condition of the multi-state components. For this, we have proposed a PDMP in which the discrete component is a vector containing information on the state of each component plus a variable denoting an operational mode. This last variable allows to simulate a synchronisation on the maintenance tasks of the system by enabling and restricting certain transitions of the stochastic process depending on its value. The available capacity of the system depends on the condition of the trains and on its operation or maintenance status, with structures that are not homogeneous across the system states, e.g. one system state has a series structure of component states while other has a k-out-of-n structure. To assess the performance at the system level, an algebraic approach from multi-state systems reliability theory has been described. This is an efficient approach to describe and calculate the combinatorial problem of component states. The proposed framework allows to assess the performance of maintenance policies for systems which are required to operate continuously, while taking into account the duration of the maintenance actions according to the component states and its impact on the performance of the system.

Modelling a problem as a Markov process facilitates the description of its evolution, which can serve as a basis to implement a numerical scheme or to build a Monte Carlo simulation algorithm for its solution. A numerical approach for quantification of time dependent probabilities has been developed in this thesis. This approach is an approximation to the solution of the Chapman-Kolmogorov equation. The implementation of the numerical approach has been found to be an efficient method for computing the time dependent probabilities, concurring with previous studies. In comparison, Monte Carlo simulation is in general conceptually easier to apply while the numerical approach could provide better accuracy in the results with faster computation times. , However, the system complexity and the number of discrete states can be limitations for this numerical approach while Monte Carlo simulation could offer more flexibility in this aspect. Given that the deterioration of the system can be characterized by a reasonable number of discrete states and that the deterministic motion is reduced to a trivial equation, it is relatively simple to make use of the numerical approach, making it a convenient alternative for problems which require studying different strategies and repeating the quantification procedure several times in order to support the decision-making. Nevertheless, the discretization of the continuous state space results in high computer memory consumption. Increasing the dimension of the process or the dynamic of the deterministic flow can make this numerical approach a challenge to implement and compute.

The proposed framework has been applied to relevant case studies of critical infrastructures to illustrate the modelling and quantification approach. One case was related to the transport sector with road bridges modelled as a single-item, and another case was related to the energy sector with gas compressors, exploring the capabilities for modelling of multi-component systems. Through the case studies, guidelines on how to account for different assumptions such as inspection frequency and quality, system dependencies, as well as maintenance policies are discussed.

5.3 Further works

This thesis could serve as a basis for further research or engineering applications. The hybrid modelling capabilities of PDMP provides a framework for studying CBM problems in which the deterioration process can be categorized by a mixture of random jumps and deterministic motion. In this sense, a combination of physics-based and data-driven approaches for deterioration can be studied with PDMP as framework.

Although potential challenges have been mentioned in the application of the numerical approach for quantification of PDMP models of complex systems, it is worth to find out more clearly the applications in which these problems arise, to find a sort of boundary of when it is more efficient to move to simulations. In addition, the numerical approach has been presented in form of a forward equation. Computing this equation requires to have an overview of the possible paths of the process and designing an algorithm that stores the non-null state-vector probabilities at each time step. It could be useful to work in designing and presenting efficient algorithms for this computation.

The application cases presented have been intended more as an illustration of the modelling approach and capabilities which are the main focus of the thesis, rather than as real problem to solve and optimize the CBM. Hence, dealing with parameters estimation and uncertainty has been left outside of the scope of the thesis. It is however recognized that this is an important part of modelling and in the assessment of the performance of CBM policies. A direction of research could be placed on advanced statistical approaches for estimation of parameters of the process when dealing with limited data that is characterized by a mixture of qualitative and quantitative information, important problems of censoring, incompleteness, and pollution by maintenance actions. This could include a combination of data and elicitation approaches from expert judgement. An interesting direction could be studying reinforced learning approaches with PDMP as base model.

Bibliography

[1] J. Lopez, R. Setola, and S. Wolthusen, *Critical Infrastructure Protection: Advances in Critical Infrastructure Protection: Information Infrastructure Models, Analysis, and Defense*, vol. 7130. Springer, 2012.

[2] E. Comission, "Council directive 2008/114/ec of 8 december 2008 on the identification and designation of european critical infrastructures and the assessment of the need to improve their protection (text with eea relevance)," *Official Journal of the European Union*, 2008.

[3] R. Arismendi, A. Barros, J. Vatn, and A. Grall, "Prognostics and maintenance optimization in bridge management," *Proceedings of the 29th European Safety and Reliability Conference (ESREL), September 22-26, 2019, Hannover, Germany*, 2019.

[4] Y. Zhang, A. Barros, A. Rauzy, and E. Lunde, "A modelling methodology for the assessment of preventive maintenance on a compressor drive system," *Safety and Reliability–Safe Societies in a Changing World. Proceedings of ESREL 2018, June 17-21, 2018, Trondheim, Norway*, 2018.

[5] I. Srikanth and M. Arockiasamy, "Deterioration models for prediction of remaining useful life of timber and concrete bridges–a review," *Journal of traffic and transportation engineering (English edition)*, 2020.

[6] Vegdirektoratet, *Bruforvaltning. Handbook R411.* Statens Vegvesen, Norway, 2014.

[7] Vegdirektoratet, *Inspeksjonshåndbok for bruer. Handbook V441.* Statens Vegvesen, Norway, 2014.

[8] M. Heimdal, "Remaining useful lifetime modeling of a compressor system," Master's thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2017.

[9] OECD, *Frascati Manual: Guidelines for Collecting and Reporting Data on Research and Experimental Development. The measurement of Scientific, Technological and Innovation Activities.* OECD Publishing, Paris, 2015.

[10] K. E. Petersen, "The eu model evaluation group," *Journal of Hazardous Materials*, vol. 65, no. 1, pp. 37–41, 1999.

[11] H. Srivastav, *Optimizing condition monitoring for dynamic health and risk management - Quantification of added value of condition information, optimization of inspection and monitoring strategies.* PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2021.

[12] T. Welte, *Deterioration and Maintenance Models for Components in Hydropower Plants.* PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2008.

[13] The Research Council of Norway, "Kvalitet i norsk forskning (quality in norwegian research)," 2000.

[14] D. P. Thunnissen, *Propagating and mitigating uncertainty in the design of complex multidisciplinary systems.* PhD thesis, California Institute of Technology, Pasadena, California, USA, 2005.

[15] M. H. Davis, "Piecewise-deterministic markov processes: A general class of non-diffusion stochastic models," *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 46, no. 3, pp. 353–376, 1984.

[16] G. W. Parry, "The characterization of uncertainty in probabilistic risk assessments of complex systems," *Reliability Engineering & System Safety*, vol. 54, no. 2-3, pp. 119–126, 1996.

[17] M. Abrahamsson, *Uncertainty in quantitative risk analysis: characterisation and methods of treatment*, vol. 1024. Citeseer, 2002.

[18] NS-EN13306:2017, "Maintenance - maintenance terminology," standard, European committee for Standardization, Brussels, BE, Mar. 2017.

[19] M. Rausand and A. Høyland, *System Reliability Theory. Models, Statistical Methods, and Applications.* John Wiley & Sons, 2004.

[20] A. Suzan and X. Yisha, "A review on condition-based maintenance optimization models for stochastically deteriorating system," *Reliability Engineering & System Safety*, vol. 157, pp. 54 – 63, 2017.

[21] I. E. Commission, "Preventive maintenance - electropedia - iev online," 2015. Online; accessed 8-February-2022.

[22] M. Rausand, A. Barros, and A. Høyland, *System Reliability Theory. Models, Statistical Methods, and Applications.* John Wiley & Sons, 2021.

[23] D. M. Frangopol, M.-J. Kallen, and J. M. v. Noortwijk, "Probabilistic models for life-cycle performance of deteriorating structures: review and future directions," *Progress in structural engineering and Materials*, vol. 6, no. 4, pp. 197–212, 2004.

[24] M. J. Kallen, *Markov processes for maintenance optimization of civil infrastructure in the Netherlands.* PhD thesis, Delft University of Technology, Delft, Netherlands, 2007.

[25] X.-S. Si, W. Wang, C.-H. Hu, and D.-H. Zhou, "Remaining useful life estimation–a review on the statistical data driven approaches," *European journal of operational research*, vol. 213, no. 1, pp. 1–14, 2011.

[26] Z.-S. Ye and M. Xie, "Stochastic modelling and analysis of degradation for highly reliable products," *Applied Stochastic Models in Business and Industry*, vol. 31, no. 1, pp. 16–32, 2015.

[27] D. An, N. H. Kim, and J.-H. Choi, "Practical options for selecting data-driven or physics-based prognostics algorithms with reviews," *Reliability Engineering & System Safety*, vol. 133, pp. 223–236, 2015.

[28] H. Richard and M. Sander, *Fatigue Crack Growth Under Cyclic Loading with Constant Amplitude*, pp. 113–151. Springer, Cham, 2016.

[29] A. Barros, "Data driven prognostic and predictive maintenance." Course Compendium, 2018.

[30] F. Baudoin, "Stochastic processes," in *International Encyclopedia of Education (Third Edition)* (P. Peterson, E. Baker, and B. McGaw, eds.), pp. 451–452, Oxford: Elsevier, third edition ed., 2010.

[31] C. Blomberg, "§ 18 - stochastic processes," in *Physics of Life* (C. Blomberg, ed.), pp. 178–182, Amsterdam: Elsevier Science B.V., 2007.

[32] M. Kurt and J. P. Kharoufeh, "Monotone optimal replacement policies for a markovian deteriorating system in a controllable environment," *Operations Research Letters*, vol. 38, no. 4, pp. 273–279, 2010.

[33] E. Byon and Y. Ding, "Season-dependent condition-based maintenance for a wind turbine using a partially observed markov decision process," *IEEE Transactions on Power Systems*, vol. 25, no. 4, pp. 1823–1834, 2010.

[34] M. L. Neves, L. P. Santiago, and C. A. Maia, "A condition-based maintenance policy and input parameters estimation for deteriorating systems under periodic inspection," *Computers & Industrial Engineering*, vol. 61, no. 3, pp. 503–511, 2011.

[35] C. Guo, W. Wang, B. Guo, and X. Si, "A maintenance optimization model for mission-oriented systems based on wiener degradation," *Reliability Engineering & System Safety*, vol. 111, pp. 183–194, 2013.

[36] M. Zhang, Z. Ye, and M. Xie, "A condition-based maintenance strategy for heterogeneous populations," *Computers & Industrial Engineering*, vol. 77, pp. 103–114, 2014.

[37] A. H. Elwany, N. Z. Gebraeel, and L. M. Maillart, "Structured replacement policies for components with complex degradation processes and dedicated sensors," *Operations research*, vol. 59, no. 3, pp. 684–695, 2011.

[38] N. Caballé, I. Castro, C. Pérez, and J. M. Lanza-Gutiérrez, "A condition-based maintenance of a dependent degradation-threshold-shock model in a system with multiple degradation processes," *Reliability Engineering & System Safety*, vol. 134, pp. 98–109, 2015.

[39] P. Do, A. Voisin, E. Levrat, and B. Iung, "A proactive condition-based maintenance strategy with both perfect and imperfect maintenance actions," *Reliability Engineering & System Safety*, vol. 133, pp. 22–32, 2015.

[40] E. Deloux, B. Castanier, and C. Bérenguer, "Predictive maintenance policy for a gradually deteriorating system subject to stress," *Reliability Engineering & System Safety*, vol. 94, no. 2, pp. 418–431, 2009.

[41] Z.-S. Ye and N. Chen, "The inverse gaussian process as a degradation model," *Technometrics*, vol. 56, no. 3, pp. 302–311, 2014.

[42] X. Wang and D. Xu, "An inverse gaussian process model for degradation data," *Technometrics*, vol. 52, no. 2, pp. 188–197, 2010.

[43] N. Chen, Z.-S. Ye, Y. Xiang, and L. Zhang, "Condition-based maintenance using the inverse gaussian degradation model," *European Journal of Operational Research*, vol. 243, no. 1, pp. 190–199, 2015.

[44] J. Pukite and P. Pukite, *Foundations of Probability Theory*, pp. 21–27. John Wiley & Sons, 1998.

[45] E. Jondeau, S.-H. Poon, and M. Rockinger, *Financial modeling under non-Gaussian distributions*. Springer Science & Business Media, 2007.

[46] O. C. Ibe, "3 - introduction to markov processes," in *Markov Processes for Stochastic Modeling (Second Edition)* (O. C. Ibe, ed.), pp. 49–57, Oxford: Elsevier, second edition ed., 2013.

[47] N. Wiener, "Differential-space," *Journal of Mathematics and Physics*, vol. 2, no. 1-4, pp. 131–174, 1923.

[48] O. C. Ibe, "10 - diffusion processes," in *Markov Processes for Stochastic Modeling (Second Edition)* (O. C. Ibe, ed.), pp. 295–327, Oxford: Elsevier, second edition ed., 2013.

[49] J. Lunze, "What is a hybrid system?," in *Modelling, Analysis, and Design of Hybrid Systems* (S. Engell, G. Frehse, and E. Schnieder, eds.), (Berlin, Heidelberg), pp. 3–14, Springer Berlin Heidelberg, 2002.

[50] R. Azaïs, J.-B. Bardet, A. Génadot, N. Krell, and P.-A. Zitt, "Piecewise deterministic markov process—recent results," in *ESAIM: Proceedings*, vol. 44, pp. 276–290, EDP Sciences, 2014.

[51] F. Chiacchio, D. D'Urso, G. Manno, and L. Compagno, "Stochastic hybrid automaton model of a multi-state system with aging: reliability assessment and design consequences," *Reliability Engineering & System Safety*, vol. 149, pp. 1–13, 2016.

[52] G. Manno, A. Zymaris, F. Chiacchio, L. Compagno, and D. D'urso, "Hybrid-pair modelling in dynamic reliability: Concepts, tool implementation and applications," in *Proc. 25th Eur. Saf. Rel. Conf. Saf. Rel. Complex Engineered Syst.*, pp. 713–723, 2015.

[53] M. Bouissou, "Comparison of two monte carlo schemes for simulating piecewise deterministic markov processes," *Proceedings of the Mathematical Methods in Reliability, Glasgow, UK*, pp. 1–4, 2007.

[54] K. G. Papakonstantinou and M. Shinozuka, "Planning structural inspection and maintenance policies via dynamic programming and markov processes. part i: Theory," *Reliability Engineering & System Safety*, vol. 130, pp. 202–213, 2014.

[55] B. Gnedenko and I. Kovalenko, "Introduction to the theory of mass service," *Nau (trad, English: 1968, Jerusalem)*, 1966.

[56] D. Vermes, "Optimal dynamic control of a useful class of randomly jumping processes," professional paper, IIASA, Laxenburg, Austria, December 1980.

[57] E. W. Aslaksen, *The system concept and its application to engineering*. Springer Science & Business Media, 2012.

[58] M. Rausand, A. Barros, and A. Hoyland, *System reliability theory: models, statistical methods, and applications*. John Wiley & Sons, third ed., 2021.

[59] A. Puliafito and K. S. Trivedi, "Systems modelling: Methodologies and tools," in *Systems Modeling: Methodologies and Tools*, pp. 1–7, Springer, 2019.

[60] Y. Zhao and Y. Liu, "Condition-based maintenance for systems with dependencies: Related concepts, challenges and opportunities," *Proceedings of the 31st European Safety and Reliability Conference, Angers, France*, 2021.

[61] M. C. O. Keizer, S. D. P. Flapper, and R. H. Teunter, "Condition-based maintenance policies for systems with multiple dependent components: A review," *European Journal of Operational Research*, vol. 261, no. 2, pp. 405–420, 2017.

[62] N. Zhang, M. Fouladirad, A. Barros, and J. Zhang, "Condition-based maintenance for a k-out-of-n deteriorating system under periodic inspection with failure dependence," *European Journal of Operational Research*, 2020.

[63] S. Hao, J. Yang, and C. Berenguer, "Condition-based maintenance with imperfect inspections for continuous degradation processes," *Applied Mathematical Modelling*, 2020.

[64] C. Duan, Z. Li, and F. Liu, "Condition-based maintenance for ship pumps subject to competing risks under stochastic maintenance quality," *Ocean Engineering*, vol. 218, p. 108180, 2020.

[65] W. Fauriat and E. Zio, "Optimization of an aperiodic sequential inspection and condition-based maintenance policy driven by value of information," *Reliability Engineering & System Safety*, vol. 204, p. 107133, 2020.

[66] Y. Wang, Y. Liu, J. Chen, and X. Li, "Reliability and condition-based maintenance modeling for systems operating under performance-based contracting," *Computers & Industrial Engineering*, vol. 142, p. 106344, 2020.

[67] W. Tao, P. Lin, and N. Wang, "Optimum life-cycle maintenance strategies of deteriorating highway bridges subject to seismic hazard by a hybrid markov decision process model," *Structural Safety*, vol. 89, p. 102042, 2021.

[68] M. Compare, P. Baraldi, P. Marelli, and E. Zio, "Partially observable markov decision processes for optimal operations of gas transmission networks," *Reliability Engineering & System Safety*, p. 106893, 2020.

[69] J. González-Domínguez, G. Sánchez-Barroso, and J. García-Sanz-Calcedo, "Preventive maintenance optimisation of accessible flat roofs in healthcare centres using the markov chain," *Journal of Building Engineering*, vol. 32, p. 101775, 2020.

[70] H. E. van Staden and R. N. Boute, "The effect of multi-sensor data on condition-based maintenance policies," *European Journal of Operational Research*, 2020.

[71] K. T. Nguyen, P. Do, K. T. Huynh, C. Bérenguer, and A. Grall, "Joint optimization of monitoring quality and replacement decisions in condition-based maintenance," *Reliability Engineering & System Safety*, vol. 189, pp. 177–195, 2019.

[72] S. Jiang and Y. F. Li, "The simulation and dynamic reliability estimation of multiple-crack system," in *2020 Asia-Pacific International Symposium on Advanced Reliability and Maintenance Modeling (APARM)*, pp. 1–6, 2020.

[73] Y.-H. Lin, Y.-F. Li, and E. Zio, "A comparison between monte carlo simulation and finite-volume scheme for reliability assessment of multi-state physics systems," *Reliability Engineering & System Safety*, vol. 174, pp. 1–11, 2018.

[74] J. Demgne, S. Mercier, W. Lair, and J. Lonchampt, "Modelling and numerical assessment of a maintenance strategy with stock through piecewise deterministic markov

processes and quasi monte carlo methods," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 231, no. 4, pp. 429–445, 2017.

[75] H. Zhang, F. Innal, F. Dufour, and Y. Dutuit, "Piecewise deterministic markov processes based approach applied to an offshore oil production system," *Reliability Engineering & System Safety*, vol. 126, pp. 126–134, 2014.

[76] Y. Lin, Y. Li, and E. Zio, "A framework for modeling and optimizing maintenance in systems considering epistemic uncertainty and degradation dependence based on pdmps," *IEEE Transactions on Industrial Informatics*, vol. 14, no. 1, pp. 210–220, 2018.

[77] W. Lair, S. Mercier, M. Roussignol, and R. Ziani, "Piecewise deterministic markov processes and maintenance modeling: application to maintenance of a train air-conditioning system," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 225, no. 2, pp. 199–209, 2011.

[78] C. Cocozza-Thivent, R. Eymard, and S. Mercier, "A finite-volume scheme for dynamic reliability models," *IMA journal of numerical analysis*, vol. 26, no. 3, pp. 446–471, 2006.

[79] C. Cocozza-Thivent, R. Eymard, S. Mercier, M. Roussignol, *et al.*, "Characterization of the marginal distributions of markov processes used in dynamic reliability," *International Journal of Stochastic Analysis*, vol. 2006, pp. 1–18, 2006.

[80] W. Lair, R. Ziani, S. Mercier, and M. Roussignol, "Maintenance effect modelling and optimization of a two-components system," in *Advances in Safety, Reliability and Risk Management*, pp. 902–909, CRC Press, 2011.

[81] W. T. Scherer and D. M. Glagola, "Markovian models for bridge maintenance management," *Journal of Transportation Engineering*, vol. 120, no. 1, pp. 37–51, 1994.

[82] R. Woodward, P. Vassie, and M. Godart, "Bridge management in europe (brime): overview of project and review of bridge management systems," in *Bridge Management 4-Inspection, Maintenance, Assessment and Repair*, 2000.

[83] J. B. Chen, "The failure of ponte morandi - private mismanagement of public infrastructure," in *Political engineering*, 2019.

[84] A. Schultz and A. Gastineau, "Chapter 31 - bridge collapse," in *Innovative Bridge Design Handbook* (A. Pipinato, ed.), pp. 795–815, Boston: Butterworth-Heinemann, 2016.

[85] Statens Vegvesen, Norway, *Bruforvaltning*, 2014.

[86] Y. Zhu, E. Elsayed, H. Liao, and L. Chan, "Availability optimization of systems subject to competing risk," *European Journal of Operational Research*, vol. 202, no. 3, pp. 781–788, 2010.

[87] Z. Tian and H. Liao, "Condition based maintenance optimization for multi-component systems using proportional hazards model," *Reliability Engineering & System Safety*, vol. 96, no. 5, pp. 581–589, 2011.

[88] Z. Tian, T. Jin, B. Wu, and F. Ding, "Condition based maintenance optimization for wind power generation systems under continuous monitoring," *Renewable Energy*, vol. 36, no. 5, pp. 1502–1509, 2011.

[89] D. Fan, A. Zhang, Q. Feng, B. Cai, Y. Liu, and Y. Ren, "Group maintenance optimization of subsea xmas trees with stochastic dependency," *Reliability Engineering & System Safety*, vol. 209, p. 107450, 2021.

[90] N. Zhang, K. Cai, J. Zhang, and T. Wang, "A condition-based maintenance policy considering failure dependence and imperfect inspection for a two-component system," *Reliability Engineering & System Safety*, vol. 217, p. 108069, 2022.

[91] J. Xu, Z. Liang, Y.-F. Li, and K. Wang, "Generalized condition-based maintenance optimization for multi-component systems considering stochastic dependency and imperfect maintenance," *Reliability Engineering & System Safety*, vol. 211, p. 107592, 2021.

[92] J. F. Andersen, A. R. Andersen, M. Kulahci, and B. F. Nielsen, "A numerical study of markov decision process algorithms for multi-component replacement problems," *European Journal of Operational Research*, 2021.

[93] M. Zheng, H. Ye, D. Wang, and E. Pan, "Joint optimization of condition-based maintenance and spare parts orders for multi-unit systems with dual sourcing," *Reliability engineering & system safety*, vol. 210, p. 107512, 2021.

[94] J. Wang and X. Zhu, "Joint optimization of condition-based maintenance and inventory control for a k-out-of-n: F system of multi-state degrading components," *European Journal of Operational Research*, vol. 290, no. 2, pp. 514–529, 2021.

[95] H. Yang, W. Li, and B. Wang, "Joint optimization of preventive maintenance and production scheduling for multi-state production systems based on reinforcement learning," *Reliability Engineering & System Safety*, vol. 214, p. 107713, 2021.

[96] C. D. Dao and M. J. Zuo, "Selective maintenance of multi-state systems with structural dependence," *Reliability engineering & system safety*, vol. 159, pp. 184–195, 2017.

[97] E. Commission, *Quarterly Report on European Gas Markets*, vol. 13 (issue 3, third quarter of 2019). Market Observatory for Energy, 2019.

[98] D. Messersmith, D. Brockett, and C. Loveland, "Understanding natural gas compressor stations," *Penn State Extension*, 2015.

[99] S. Kermanshachi, B. Rouhanizadeh, M. M. Cobanoglu, and I. Damnjanovic, "Optimal pipeline maintenance strategies in the united states: Stochastic reliability analysis of gas pipeline network failures," *Journal of Pipeline Systems Engineering and Practice*, vol. 11, no. 1, p. 04019041, 2020.

[100] R. Arismendi, A. Barros, and A. Grall, "Preventive maintenance of a compressor station: A modeling framework for the assessment of performance," *Proceedings of the 30th European Safety and Reliability Conference and the 15th Probabilistic Safety Assessment and Management Conference*, 2020.

[101] R. Arismendi, A. Barros, J. Vatn, and A. Grall, "Prognostics and maintenance optimization in bridge management," in *Proceedings of the 29th European Safety and Reliability Conference (ESREL). 22–26 September 2019 Hannover, Germany*, Research Publishing Services, 2019.

[102] P. Pascual-Ortigosa, E. Sáenz-de Cabezón, and H. P. Wynn, "Algebraic reliability of multi-state k-out-of-n systems," *Probability in the Engineering and Informational Sciences*, pp. 1–24, 2020.

[103] M. Bladt, "A review on phase-type distributions and their use in risk theory," *ASTIN Bulletin: The Journal of the IAA*, vol. 35, no. 1, pp. 145–161, 2005.

[104] R. Arismendi, A. Barros, and A. Grall, "Piecewise deterministic markov process for condition-based maintenance models—application to critical infrastructures with discrete-state deterioration," *Reliability Engineering & System Safety*, vol. 212, p. 107540, 2021.

[105] R. Arismendi, A. Barros, and A. Grall, "A modelling framework for condition-based maintenance of systems with multi-state components - application to a gas compression system," *"Submitted" Reliability Engineering & System Safety*.

Appendix A

Prognostics and maintenance optimization in bridge management

R. Arismendi, A. Barros, J. Vatn, and A. Grall, “Prognostics and maintenance optimization in bridge management,” *Proceedings of the 29th European Safety and Reliability Conference (ESREL), September 22-26, 2019, Hannover, Germany*, 2019

Prognostics and Maintenance Optimization in Bridge Management

Renny Arismendi, Anne Barros, Jørn Vatn

Department of Mechanical and Industrial Engineering, NTNU, Norway. E-mail: renny.j.arismendi@ntnu.no, anne.barros@ntnu.no, jorn.vatn@ntnu.no

Antoine Grall

Institute Charles Delaunay, LM2S, University of Technology of Troyes, France. E-mail: antoine.grall@utt.fr

This paper has been written in collaboration with the Norwegian Public Roads Administration (NPRA). In Norway, bridges are a vital part of the transportation infrastructure. With more than 18,000 road bridges across the country, an efficient bridge management system is of critical importance to avoid high costs from over expending, to ensure safety of the public and availability of the transportation system. In the bridge management system applied by NPRA, the inspections are mainly carried out periodically based on pre-defined rules and the decision about when to perform the maintenance is based on the findings of these inspections. The objective of this paper is to propose a modelling framework that makes it possible to challenge these pre-defined rules by doing degradation prognostic and maintenance optimization. We propose to use a Piecewise Deterministic Markov Process to encompass different modelling assumptions as non-negligible maintenance delays and time dependent inspections. State probabilities and performance indicators are assessed through Monte Carlo simulations and a numerical scheme. The experimental values provided at the end show that optimal maintenance and optimization strategies should be investigated and further developed.

Keywords: Bridge management, stochastic modelling, piecewise deterministic Markov process, prognostics, numerical assessment, Monte Carlo simulation, road bridges.

1. Introduction

The use of automobiles experienced a rapid growth during the 20th century and with this growth came the development of a massive transportation infrastructures. In Comission (2008), the Council of the European Union includes the transport sector in the list of Critical Infrastructures, considering that modern societies depend on the availability of this service and that its disruption or unavailability poses risks with serious consequences to the health, safety, economic or social well-being of people and vital societal functions. A systematic approach to maintenance and rehabilitation strategies for the transportation system was not identified until the late 1960s. The Highway Safety Act of 1968 was a development that resulted from the collapse of the Silver bridge across the Ohio River, USA in 1967, and the concerns related to the bridge management problem. This Act required state road officials to inspect and rate the condition of the bridges as mentioned by Scherer and Glagola (1994).

Bridge management can be understood as the optimal planning of inspections and maintenance activities of road bridges, with the goal of preserving the asset value of the infrastructure by optimizing the costs over its lifetime, while ensuring the safety of users and offering a sufficient quality of service, as Woodward et al. (2000). More than 50 years after the collapse of the Silver bridge, despite the advances in technology, rehabilitation techniques and safety assessments, bridge collapses continue to occur. Moreover, the construction of new bridges has been slowing down in most countries, which now face a stock of aging bridges, requiring an effective and efficient bridge management.

1.1. *Bridge management in Norway*

In Norway, bridges are a vital part of the transportation infrastructure. With more than 18,000 road bridges across the country, an efficient bridge management system is of critical importance to avoid high costs from over expending and to ensure safety of the public and availability of the transportation system.

As pointed by Kallen (2007), there are many factors that make bridge management a challenging task, such as: the varying weight and intensity of the traffic, the evolution of the building codes over the years, the weather influence on the structures, large number of structures spread over a large area, and others. All these factors create uncertainty, which makes the bridge management a problem of decision making under uncertainty.

In the bridge management system applied

Proceedings of the 29th European Safety and Reliability Conference.
Edited by Michael Beer and Enrico Zio

Published by Research Publishing, Singapore.
ISBN: 978-981-11-2724-3; doi:10.3850/978-981-11-2724-3_0081-cd

by the Norwegian Public Roads Administration (NPRA), the agency responsible for planning, building, operating, and maintaining national and country bridges in Norway, the inspections are mainly carried out periodically based on pre-defined rules and the decision about when to perform the maintenance is based on the findings of these inspections. The handbooks for management and inspections of bridges, Statens Vegvesen (2014a,b), establish types of inspections for the bridges and the period in which they must be performed, e.g. a main inspection of a bridge, with an overview of all the elements of the bridge, must (in general) be performed every five years. Statens Vegvesen (2014b) also establishes how the inspections must be logged in a database, how the findings must be reported and provides guidelines on when to perform the repairs for found damages.

When an inspection is performed on a bridge, the severity of the found damages is assessed in a scale of one to four, as:

- 1 - Small damage
- 2 - Medium damage
- 3 - Large damage
- 4 - Critical damage

Based on the severity of the damage, a maintenance action is scheduled:

- Severity: 1 - No maintenance action is required
- Severity: 2 - A maintenance action must take place between four and ten years
- Severity: 3 - A maintenance action must take place between one and three years
- Severity: 4 - A maintenance action must take place in less than six months

This bridge management system can be characterized as a condition-based maintenance program, in which the maintenance decisions are based on recommendations from the information gathered through condition monitoring. However, following this program is a challenging task for the NPRA. With such a large stock of bridges throughout the country, it is difficult to keep up to date the inspection program due to budget and resources constraints.

A problem raised for some years by the NPRA is to question if this bridge management system can be optimized by moving from diagnostics to prognostics.

1.1.1. *Diagnostics to Prognostics*

The current trend in many fields and with critical infrastructures is to move the decision making in condition-based maintenance from diagnostics to prognostics.

Diagnostics involve the techniques and practice of determining whether a fault is present, identifying its nature and estimating its severity. Prognostics on the other hand, is the practice of forecasting the likely development of such fault.

Through fault diagnosis, it is possible to implement maintenance decisions by following pre-established rules and recommendations saying when to perform what. This process tends to be dependent on the technical and mechanical education of the maintenance staff and their hands on expertise, and as pointed out by Rausand and Høyland (2004), although the expertise is key in maintenance management and performance, it should not be the only basis for making the decisions.

Prognostics allow to take the analysis one step further in order to question such pre-established rules, to reduce overestimated margins and to optimize decision rules. With the use of mathematical models, it may be possible to simulate different maintenance strategies and to assess the associated effects, the maintenance costs and the operational performance in the long run. Therefore, these simulations can be very helpful for deciding the most appropriate maintenance strategy to implement.

In this sense, the maintenance decision-making in the bridge management of the NPRA may be improved by using information available in a national data base (BRUTUS), the NPRAs tool for management and supervision of bridge-related work tasks, and a model capable of describing the deterioration of the bridge and the effect of decision criteria, such as: inspection interval, condition thresholds for performing preventive repairs, and type of repair (complete renewal or partial repairs).

The objective of this paper is to demonstrate the implementation of a Piecewise-Deterministic Markov Process (PDMP) as a framework to model the deterioration process of a structure and maintenance strategies applicable by the NPRA, in order to assess the effects of such strategies and assist the decision-making process. The remainder of this paper is organized as follows: section 2 states the assumptions, the problem statement and model formulation. Section 3 describes the implementation and quantification of the model in terms of next-event simulation and Monte Carlo simulation. Section 4 presents discussions around the framework and results.

2. Modelling Framework and Assumptions

In the field of civil engineering and bridge management, it is widely common to assess the severity and condition of the structures in a discrete scale similar to the one used by the NPRA. To quantify for the uncertainties involved in the deterioration process of a structure, described in a discrete scale, finite-state Markov processes have been applied often for modelling the deterioration of bridges, as Kallen (2007), Cesare et al. (1992), and Morcous (2006). More recently, semi-Markov processes have been studied in order to account for the aging of the structures as Mašović et al. (2015), Thomas and Sobanjo (2016) and Zambon et al. (2019).

2.1. *Assumptions*

For modelling the deterioration process of a structure and inspections and maintenance strategy consistent with the bridge management of the NPRA, the following assumptions are made:

(i) The observed condition of the unit is represented by a discrete variable ranging from small or no damage to critical damage
(ii) The deterioration process of the unit can be modelled with a homogeneous Markov chain with constant transition rates
(iii) The unit is periodically inspected and not continuously monitored
(iv) Inspections are perfect and reveal the true state of the unit
(v) When an inspection reveals a damage with severity medium or higher, a maintenance action is scheduled
(vi) There is a significant delay before a maintenance is performed
(vii) The duration of the delay is deterministic
(viii) Maintenance interventions occur at the scheduled date instantaneously, i.e. the duration of the intervention is null
(ix) After a maintenance action, the unit is as good as new

A Markov process is not suitable to model the inspection and maintenance strategy of the NPRA due to assumptions iii and vii. Here, we propose a PDMP, as a framework to model the deterioration of the structure and the effect of inspection and maintenance strategies.

2.2. *Modelling framework*

A PDMP is an extension of a Markov chain that incorporates continuous states with evolution that follow discrete state-dependent deterministic differential equations. The resulting stochastic process is a Markov process with a mixture of random jumps and deterministic motion. They were introduced by Davis (1984), as a general class of non-diffusion stochastic models that provides a framework for studying optimization problems.

A PDMP is a hybrid process $\{I_t, X_t\}_{t>0}$ with values in a discrete-continuous space $E \times R$, as described by Lair et al. (2011, 2012). The first component I_t is discrete, with values in a finite state space E and corresponds to the unit states. The second component X_t takes values in a Borel subset $R \subset \mathbb{R}^k$ and it stands for the environmental conditions, which in our case will refer to the time until next inspection and next maintenance action.

2.2.1. *Discrete component I_t*

The discrete component I_t of the PDMP in our case, is used to model the deterioration process of the structure and to indicate a type of maintenance that has been scheduled.

First, a variable indicating the condition of the structure can be denoted $i_A(t)$.

$i_A(t) = \{1, 2, 3, 4\}$, where:

- $i_A = 1$: Small or no damage
- $i_A = 2$: Medium damage
- $i_A = 3$: Large damage
- $i_A = 4$: Critical damage

Only when the unit is inspected, the degree of deterioration of the unit is detected, and a maintenance is scheduled accordingly. The type of scheduled maintenance can be denoted $i_B(t)$.

$i_B(t) = \{1, 2, 3, 4\}$, where:

- $i_B = 1$: No maintenance is scheduled
- $i_B = 2$: Slow maintenance is scheduled, (i.e. a maintenance intervention takes place between four and ten years)
- $i_B = 3$: Medium maintenance is scheduled, (i.e. a maintenance intervention takes place between one and three years)
- $i_B = 4$: Fast maintenance is scheduled, (i.e. a maintenance intervention before six months)

The discrete component of the PDMP is then $I_t = i$, with $i = (i_A, i_B)$, given that all the combinations are not possible and should be taken only in the finite state space E of the PDMP, E = {(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3),(4,4)}. To simplify, we denote hereafter $i = (i_A, i_B)$, without reminding that the possible couples of values (i_A, i_B) are limited to E.

2.2.2. *Continuous component X_t*

The continuous component here is not related to any physical phenomena, but it is used as an artefact to model a process that requires a combination of stochastic random jumps and continuous variables to count time. The environmental condition in this case, stands for the date of the next inspection, the date of the next maintenance action and time.

Let $X_t = x$, with $x = (x_A, x_B, t)$, where:

- x_A : date of next inspection
- x_B : date of next maintenance action
- t : time

2.2.3. *PDMP*

The complete process to consider $\{I_t, X_t\}$ is made of $\{(i_A, i_B), (x_A, x_B, t)\}$. The process may experience jumps at random or at deterministic times.

Jumps at random times are used in our case to simulate the deterioration of the unit. The unit makes a transition to a more degraded state. This degradation is not detected immediately, so the scheduled type of maintenance does not change. The discrete component jumps from $(i_A, i_B) = (j, k)$ to $(i_A, i_B) = (m, k)$, while the continuous component does not change. The deterioration process of the unit with random jumps is shown in figure 1.

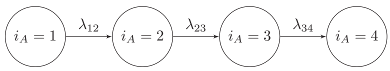


Fig. 1. Deterioration process.

Jumps at deterministic times are used to model the inspection and maintenance actions.

When an inspection is performed, the date to the next inspection (x_A) is updated, a maintenance action is scheduled (x_B) and the type of scheduled maintenance (i_B) is updated according the condition of deterioration of the unit.

When a maintenance action is performed, the discrete component (i_A, i_B) jumps to $(1, 1)$ (as good as new), the date to the next inspection (x_A) does not change, and the date of the next maintenance action (x_B) is set to infinite (no maintenance scheduled).

Between two consecutive jumps, only the continuous variable t evolves, with speed of one.

3. Quantification

Solving the PDMP analytically is generally impossible due to complex system behaviour. For reliability assessments, Monte Carlo simulation and numerical scheme based on finite-volume methods are two commonly used approaches to solve PDMP. In our case, both approaches are used for validating the results and compare the advantages or disadvantages from each.

3.1. *Monte Carlo simulation*

The simulation procedure of the PDMP is shown in figure 2. It includes five main steps to simulate a realization of the PDMP until the horizon time t_{hor}.

(i) Set initial system time and initial system state
In our case, initial time is set to zero, the unit is set to be in new condition with no maintenance action scheduled and the date of the first inspection is set to the period. (i.e. $t = 0$, $i_A = 1$, $i_B = 1$, $x_A = T$ and $x_B = \infty$), where T is the inspection period.

(ii) Sample date of next stochastic jump, if enabled
The date of the next stochastic jump t_{jump} is sampled from the corresponding probability density function and the corresponding parameter(s). In our case, exponential distribution is considered with rates as shown in figure 1.

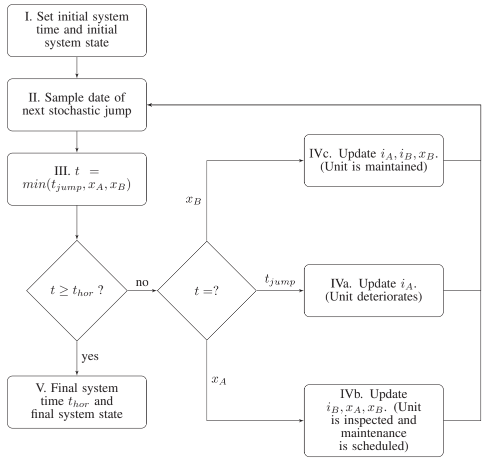


Fig. 2. Simulation procedure.

(iii) Identify next event
The date of the next stochastic jump t_{jump} is compared with date of next inspection x_A, the date of next maintenance action x_B and the horizon time t_{hor}.
The system time is updated as: $t = \min(t_{jump}, x_A, x_B, t_{hor})$. If the simulation time has reached the horizon time, $t = t_{hor}$, the simulation continues to step v, otherwise it continues to step iv.

(iv) Update system state
The system state is updated according to the jump that takes place at time t, deterioration, inspection or maintenance.

(a) Deterioration: ($t = t_{jump}$)
Only i_A is updated in this jump

(b) Inspection: ($t = x_A$)
Variables i_B, x_A, x_B are updated. The after jump values ($^+$) are:
$i_B^+ = i_A$;
$x_A^+ = t + T$;
$x_B^+ = t + M_{i_B}$; where M_{i_B} is the delay for maintenance action of the type i_B.

(c) Maintenance: ($t = x_B$)
Variables i_A, i_B, x_B are updated. The after jump values ($^+$) are:
$i_A^+ = i_B^+ = 1$;
$x_B^+ = \infty$.

(v) Set final system time and final system state
The final system time is t_{hor} and the final system state is the state resulting from the last jump to take place no later than t_{hor}.

This simulation procedure is replicated N

times, to approximate quantities of interest, such as deterioration state probabilities.

3.2. *Numerical scheme*

The probability of the state of the system of a PDMP can be completely described by the Chapman-Kolmogorov equations, as demonstrated by Cocozza-Thivent et al. (2006). A numerical scheme based in finite-volume methods to approximate these probability measures is proposed by Cocozza-Thivent et al. (2006), with proof of the convergence to the unique solution.

The principle of the scheme is the discretization of the continuous component X_t into cells. The time evolution of the probability masses in each cell of the environmental space is followed, and at each step, a balance between the in-coming and out-going probability masses is written, allowing us to solve a linear system, as Lair et al. (2012).

Let $\mathcal{M}$ denote the mesh of the discretization of the environmental state space R and δ_t denote the environmental state space step (we use the same step for x_A, x_B and t in our case, since x_A, x_B and t have units of time). A cell w of $\mathcal{M}$ has cubic shape $w = [n_1\delta_t; (n_1+1)\delta_t) \times [n_2\delta_t; (n_2+1)\delta_t) \times [n_3\delta_t; (n_3+1)\delta_t)$, with $(n_1, n_2, n_3) \in N^3$.

The evolution of the process, between t and $t + \delta_t$ can be written as:

$$p_{t+\delta_t}\{i, x\} = \sum_{\substack{u \in E \\ w \in \mathcal{M}}} p_t\{u, w\} G^{\{i,x\}}_{\{u,w\}} \qquad (1)$$

Where $G^{\{i,x\}}_{\{u,w\}}$ is the probability that the system moves from state $\{u, w\}$ to state $\{i, x\}$ in the time interval $[t; t + \delta_t)$. The conditional probabilities for this model are included in the appendix.

The probability for the unit to be in the state of deterioration j, $Pr(i_A = j)$, at time t is:

$$Pr(i_A = j)_t = \sum_{k,r,s} p_t((j, k), (r, s, t)) \qquad (2)$$

4. Results and Discussions

Both quantification approaches are used to approximate the deterioration states probabilities shown in figure 3. The parameters used are shown in table 1. The deterioration rates have been estimated from previous works carried by the NPRA based on the information available on their database for inspections and maintenance actions, BRUTUS.

4.1. *Monte Carlo simulation vs numerical scheme*

To compare the results of the quantification from both approaches, the residuals or difference between the state probabilities is shown in figure 4.

Table 1. Model parameters.

Deterioration rates (h^{-1})	Maintenance delays (y)	Inspection interval (y)
$\lambda_{12} = 1.5\text{e}{-5}$	$M1 = \infty$	$T = 5$
$\lambda_{23} = 6\text{e}{-6}$	$M2 = 8$	
$\lambda_{34} = 1.4\text{e}{-6}$	$M3 = 3$	
	$M4 = 0.5$	

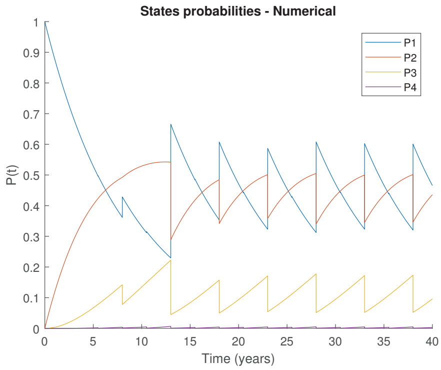


Fig. 3. Deterioration states probabilities

It can be observed that the difference in results is small, with an order of magnitude of 10^{-3}. In addition, the difference is reduced by performing a higher number of replications of the Monte Carlo simulation, showing same convergence.

The Monte Carlo simulation method is widely used in practice, conceptually easy to apply and without particular restrictions on the dimension of the PDMP. On the other hand, the numerical scheme has high accuracy with short computation times, as pointed by Lin et al. (2018). In our case, the Monte Carlo simulation with 100,000 replications took approximately one hour to obtain time-dependent probabilities, while with the numerical scheme the results are obtained in one second.

4.2. *Strategy assessment*

The PDMP allows to test different inspection and maintenance strategies and assess their effect on the structure condition. In a first attempt, we can challenge the inspection period, evaluating the effect on the condition of the structure. Figure 5 shows how the critical condition of the unit ($i_A = 4$), varies with time for different inspection periods. This allows to support the decision process related to inspections by evaluating the associated risk on the structure.

The PDMP framework supports the modelling of a strategy in which the inspection is not performed periodically, but that can instead be dependent on the condition of the structure. The

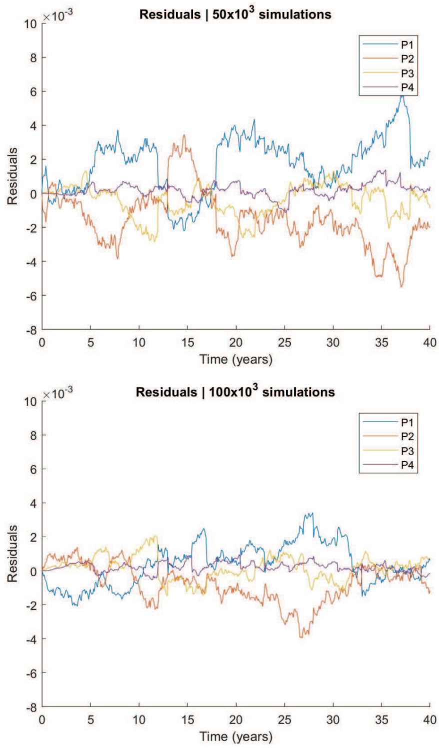


Fig. 4. Residuals between quantification approaches

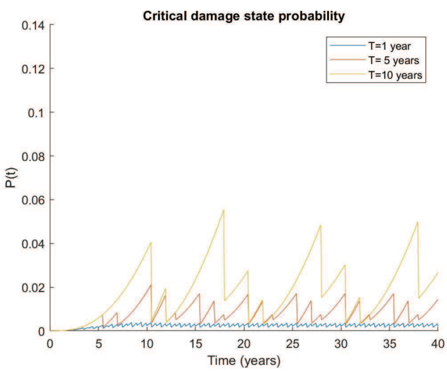


Fig. 5. Critical damage probability for different inspection intervals

model proposed here can be modified to allow for this strategy, in a similar way to how different maintenance delays have been set dependent on the condition of the unit.

Moreover, to assist the decision process in bridge management, the cost of a strategy can be evaluated in addition to the effect on the condition of a structure. In this way the strategy can be optimized, by finding an inspection/maintenance strategy that minimizes the mean cost over a time period, with acceptable risk for the structure. The cost function can be set as:

$$C(t) = N_{insp}(t) \cdot C_{insp} + N_{mr}(t) \cdot C_{mr} + N_{lr}(t) \cdot C_{lr}(t) + N_{cr}(t) \cdot C_{cr}(t) \quad (3)$$

Where:

- C_{insp}: Cost of inspection
- C_{mr}: Cost of medium repair (unit with medium damage)
- C_{lr}: Cost of large repair (unit with large damage)
- C_{cr}: Cost of critical repair (unit with critical damage)
- $N_{insp}(t)$: Mean number of inspections until t
- $N_{mr}(t)$: Mean number of medium repairs until t
- $N_{lr}(t)$: Mean number of large repairs until t
- $N_{cr}(t)$: Mean number of critical repairs until t

The number of inspections and repairs can be counted from Monte Carlo simulations or expressed in terms of the marginal distributions of the PDMP and approximated with the numerical scheme. For example, the mean number of medium repairs until t, can be approximated as the probability that the system jumps from state: $\{u, w\}$ to state $\{i, x\}$ with $u = (2, 2)$ and $i = (1, 1)$ before time t, when δt is small so that the probability of two or more medium repairs in $(t, t + \delta t]$ is negligible, as:

$$N_{mr}(t) \approx \sum_{z=0}^{t} p_t\{(2,2), w\} G_{\{(2,2),w\}}^{\{(1,1),x\}} \quad (4)$$

5. Conclusions and Further Works

Diagnostics allow the application of condition-based maintenance by following pre-established rules and guidelines that state when to perform inspection and maintenance activities. Prognostics empower the decision makers by enabling them to evaluate the effect and cost of a given strategy, therefore allowing to allocate resources in a more efficient manner and optimize the bridge management.

In this paper, we propose a PDMP as a statistical data driven approach to model the deterioration of a structure as a stochastic process, relying on available past observed data, and to make prognosis for a unit that is not monitored continuously but periodically and with significant delay for a maintenance action to be performed.

Two approaches for solving the PDMP are presented. In general, the Monte Carlo simulation approach is conceptually easier to apply while the numerical scheme can provide better accuracy in the results with faster computation times. In the PDMP presented here, the evolution of the continuous component is reduced to a trivial equation. This makes it relatively simple to apply the numerical scheme, presenting a convenient alternative for optimization problems which require testing different strategies, thus repeating the quantification procedure several times.

With support from the NPRA, the work presented here can be developed further. More advanced estimation of parameters for the PDMP can be explored, with sensitivity analysis. Other strategies can be evaluated, such as a condition-based inspection policy rather than inspections performed at equal time intervals, and other maintenance alternatives than as-good-as-new replacements. A PDMP is a framework suitable to model such strategies. In addition, the proposed cost function needs to be addressed together with the definition of constraints on the risk, to optimize the bridge management.

A PDMP presents a framework for hybrid models prognostics, a combination between data-driven and physics-based models, that could be explored for bridge management. It is also of interest to study the application of the PDMP for maintenance models for multi-units systems, accounting for their dependencies, and evaluating the advantages and disadvantages of the numerical scheme and Monte Carlo simulation in these applications.

Acknowledgement

The authors would like to thank the Norwegian Public Roads Administration and the E39 Coastal Highway Route project for their contribution to the problem statement, granting access to their database and their involvement and discussions with experts.

Appendix. Conditional probabilities for the numerical scheme

Consider the environmental state space cells $w = w_1 \times w_2 \times w_3$ and $x = x_1 \times x_2 \times x_3$. Where w_j, x_j are intervals, e.g. $w_j = [n_j\delta_t, (n_j+1)\delta_t)$ with $n_j \in N$, $j = \{1,2,3\}$. For simplicity, we denote: $w_j = [\underline{w_j}, \overline{w_j})$, where $\overline{w_j} = \underline{w_j} + \delta_t$

Due to the deterioration of the unit, modelled with random jumps, the probability masses move from w to x, which are neighboring cells of the mesh $\mathcal{M}$, i.e. $x_1 = w_1$, $x_2 = w_2$ and $x_3 = w_3 + [\delta_t, \delta_t)$, since only the environmental variable t evolves with speed of one between two consecutive jumps.

The non-null transition probabilities due to a random jump, can be written as:

- $G^{\{(1,1),(x)\}}_{\{(1,1),(w)\}} = 1 - (\lambda_{12}\delta_t)$
- $G^{\{(2,1),(x)\}}_{\{(1,1),(w)\}} = \lambda_{12}\delta_t$
- $G^{\{(2,1),(x)\}}_{\{(2,1),(w)\}} = 1 - (\lambda_{23}\delta_t)$
- $G^{\{(3,1),(x)\}}_{\{(2,1),(w)\}} = \lambda_{23}\delta_t$
- $G^{\{(2,2),(x)\}}_{\{(2,2),(w)\}} = 1 - (\lambda_{23}\delta_t)$
- $G^{\{(3,2),(x)\}}_{\{(2,2),(w)\}} = \lambda_{23}\delta_t$
- $G^{\{(3,1),(x)\}}_{\{(3,1),(w)\}} = 1 - (\lambda_{34}\delta_t)$
- $G^{\{(4,1),(x)\}}_{\{(3,1),(w)\}} = \lambda_{34}\delta_t$
- $G^{\{(3,2),(x)\}}_{\{(3,2),(w)\}} = 1 - (\lambda_{34}\delta_t)$
- $G^{\{(4,2),(x)\}}_{\{(3,2),(w)\}} = \lambda_{34}\delta_t$
- $G^{\{(3,3),(x)\}}_{\{(3,3),(w)\}} = 1 - (\lambda_{34}\delta_t)$
- $G^{\{(4,3),(x)\}}_{\{(3,3),(w)\}} = \lambda_{34}\delta_t$
- $G^{\{(4,1),(x)\}}_{\{(4,1),(w)\}} = 1$
- $G^{\{(4,2),(x)\}}_{\{(4,2),(w)\}} = 1$
- $G^{\{(4,3),(x)\}}_{\{(4,3),(w)\}} = 1$
- $G^{\{(4,4),(x)\}}_{\{(4,4),(w)\}} = 1$

In our case, jumps between non-neighboring cells of the environmental space occur only at inspection and maintenance dates.

At inspection dates ($w_1 = w_3$), the probability masses may move from cell w to cell x, when a maintenance is scheduled or re-scheduled, or may move from cell w to cell y when no maintenance action needs to be scheduled or re-scheduled, with: $x_1 = w_1 + [T, T)$, $\underline{x_2} = min(\underline{w_2}, \underline{x_3} + M_{iB})$, $x_3 = w_3$, $y_1 = w_1 + [T, T)$, $y_2 = w_2$ and $y_3 = w_3$. The non-null transition probabilities of this type are:

- $G^{\{(1,1),(y)\}}_{\{(1,1),(w)\}} = 1$
- $G^{\{(2,2),(x)\}}_{\{(2,1),(w)\}} = 1$
- $G^{\{(2,2),(y)\}}_{\{(2,2),(w)\}} = 1$
- $G^{\{(3,3),(x)\}}_{\{(3,1),(w)\}} = 1$
- $G^{\{(3,3),(x)\}}_{\{(3,2),(w)\}} = 1$
- $G^{\{(3,3),(y)\}}_{\{(3,3),(w)\}} = 1$
- $G^{\{(4,4),(x)\}}_{\{(4,1),(w)\}} = 1$

- $G_{\{(4,2),(w)\}}^{\{(4,4),(x)\}} = 1$
- $G_{\{(4,3),(w)\}}^{\{(4,4),(x)\}} = 1$
- $G_{\{(4,4),(w)\}}^{\{(4,4),(y)\}} = 1$

At maintenance dates, ($w_2 = w_3$), the probability masses move from cell w to cell x, with $x_1 = w_1$, $x_2 = \infty$ and $x_3 = w_3$. The non-null transition probabilities of this type are:

- $G_{\{(2,2),(w)\}}^{\{(1,1),(x)\}} = 1$
- $G_{\{(3,2),(w)\}}^{\{(1,1),(x)\}} = 1$
- $G_{\{(3,3),(w)\}}^{\{(1,1),(x)\}} = 1$
- $G_{\{(4,2),(w)\}}^{\{(1,1),(x)\}} = 1$
- $G_{\{(4,3),(w)\}}^{\{(1,1),(x)\}} = 1$
- $G_{\{(4,4),(w)\}}^{\{(1,1),(x)\}} = 1$

References

Cesare, M. A., C. Santamarina, C. Turkstra, and E. H. Vanmarcke (1992). Modeling bridge deterioration with markov chains. *Journal of Transportation Engineering 118*(6), 820–833.

Cocozza-Thivent, C., R. Eymard, and S. Mercier (2006). A finite-volume scheme for dynamic reliability models. *IMA journal of numerical analysis 26*(3), 446–471.

Cocozza-Thivent, C., R. Eymard, S. Mercier, M. Roussignol, et al. (2006). Characterization of the marginal distributions of markov processes used in dynamic reliability. *International Journal of Stochastic Analysis 2006*, 1–18.

Comission, E. (2008). Council directive 2008/114/ec of 8 december 2008 on the identification and designation of european critical infrastructures and the assessment of the need to improve their protection (text with eea relevance). *Official Journal of the European Union*.

Davis, M. H. (1984). Piecewise-deterministic markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society. Series B (Methodological)*, 353–388.

Kallen, M. J. (2007). *Markov processes for maintenance optimization of civil infrastructure in the Netherlands*. Ph. D. thesis, Delft University of Technology, Delft, Netherlands.

Lair, W., S. Mercier, M. Roussignol, and R. Ziani (2011). Piecewise deterministic markov processes and maintenance modeling: application to maintenance of a train air-conditioning system. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability 225*(2), 199–209.

Lair, W., R. Ziani, S. Mercier, and M. Roussignol (2012). Maintenance effect modelling and optimization of a two-components system. *Advances in Safety, Reliability and Risk Manag*.

Lin, Y.-H., Y.-F. Li, and E. Zio (2018). A comparison between monte carlo simulation and finite-volume scheme for reliability assessment of multi-state physics systems. *Reliability Engineering & System Safety 174*, 1–11.

Mašović, S., S. Stošić, and R. Hajdin (2015). Application of semi-markov decision process in bridge management. In *IABSE Symposium Report*, Volume 105, pp. 1–8. International Association for Bridge and Structural Engineering.

Morcous, G. (2006). Performance prediction of bridge deck systems using markov chains. *Journal of performance of Constructed Facilities 20*(2), 146–155.

Rausand, M. and A. Høyland (2004). *System Reliability Theory. Models, Statistical Methods, and Applications.* John Wiley & Sons.

Scherer, W. T. and D. M. Glagola (1994). Markovian models for bridge maintenance management. *Journal of Transportation Engineering 120*(1), 37–51.

Statens Vegvesen (2014a). *Bruforvaltning. Handbook R411*. Norway: Statens Vegvesen.

Statens Vegvesen (2014b). *Inspeksjonshndbok for bruer. Handbook V441*. Norway: Statens Vegvesen.

Thomas, O. and J. Sobanjo (2016). Semi-markov models for the deterioration of bridge elements. *Journal of Infrastructure Systems 22*(3), 04016010.

Woodward, R., P. Vassie, and M. Godart (2000). Bridge management in europe (brime): overview of project and review of bridge management systems. In *Bridge Management 4-Inspection, Maintenance, Assessment and Repair*.

Zambon, I., A. Vidović, A. Strauss, and J. Matos (2019). Condition prediction of existing concrete bridges as a combination of visual inspection and analytical models of deterioration. *Applied Sciences 9*(1), 148.

Appendix B

Piecewise deterministic Markov process for condition-based maintenance models - Application to critical infrastructures with discrete-state deterioration

R. Arismendi, A. Barros, and A. Grall, “Piecewise deterministic markov process for condition-based maintenance models—application to critical infrastructures with discrete-state deterioration,” *Reliability Engineering & System Safety*, vol. 212, p. 107540, 2021

Reliability Engineering and System Safety 212 (2021) 107540

Contents lists available at ScienceDirect

Reliability Engineering and System Safety

journal homepage: www.elsevier.com/locate/ress

Piecewise deterministic Markov process for condition-based maintenance models — Application to critical infrastructures with discrete-state deterioration

Renny Arismendi [a,*], Anne Barros [b], Antoine Grall [c]

[a] MTP, Norwegian University of Science and Technology, Trondheim, Norway
[b] LGI, CentraleSupélec, France
[c] LM2S, University of Technology of Troyes, France

ARTICLE INFO

Keywords:
Condition-based maintenance
Piecewise deterministic Markov process
Stochastic modelling
Numerical analysis
Monte Carlo simulation
Road bridges maintenance

ABSTRACT

In recent decades, the technology and techniques for condition monitoring have experienced a rapid development. However, there is still a need for reducing unnecessary inspections and/or preventive maintenance actions and their associated cost, through optimal design of condition-based maintenance (CBM) strategies. Accordingly, mathematical modelling and optimization of CBM has become of interest for industry managers and researchers. This work explores on the application of a piecewise deterministic Markov process (PDMP) to encompass different modelling assumptions as non-negligible maintenance delays and inspection-based condition monitoring. These assumptions are relevant for many critical infrastructures in civil engineering or in oil & gas industry whose deterioration states are classified at a very high level of abstraction among a finite and small set of possible states. A formalism to model this type of problems is proposed in which the deterministic motion of the PDMP is reduced to a trivial differential equation to track the time elapsed between events. A numerical scheme for quantification, as an approximation of the Chapman–Kolmogorov equation, is presented. Later, an illustration case dealing with CBM of road bridges by the NPRA (Norwegian Public Roads Administration) is presented, guiding through the modelling and quantification approach.

1. Introduction

In recent decades, the technology and techniques for condition monitoring have experienced a rapid development. However, there is still a need for reducing unnecessary inspections and/or preventive maintenance actions and their associated cost, through optimal design of condition-based maintenance (CBM) strategies. If we consider critical infrastructures in civil engineering or in oil & gas industry, CBM strategies are often carried without any modelling and assessment of their efficiency in the mid or long term. This fact usually pushes the decision makers to take large safety margins and to over-inspect, or to postpone maintenance tasks taking the risk of facing up critical situations. Generally speaking, the problem is about sub-optimal decisions in terms of maintenance cost, availability of production and even reputation [1,2].

In the bridge management system applied in Denmark, Finland, France, South Africa, United Kingdom, China, South Korea, United States of America, Norway and other countries, inspections are carried out by following pre-defined procedures and a condition rating is assigned to the structure in a discrete scale [3]. The decision about when to perform maintenance is based on the condition rating assigned at these inspections. In Norway, handbooks for management and inspections of bridges [4,5], establish the types of inspections for the bridges and the period in which they must be performed. For example, the main inspection of a bridge, with an overview of all the elements of the bridge, must (in general) be performed every five years. The handbooks also establish how the inspections must be logged in a database, how the found damages must be reported and when to schedule the repairs for found damages: given the reported damages, the condition of the whole bridge is ranked among a very limited number of global deterioration states and the delay before repair is chosen accordingly. However, the period of inspection and the delay before repair are not optimized according to a time-dependent or a long term safety criteria or maintenance cost.

In the oil & gas industry, the natural gas transportation infrastructure is dependent on high capacity compressors to supply the required flow of gas at any time all over the year. This is especially important during the winter season where a full capacity is needed and one

* Corresponding author.
E-mail address: renny.j.arismendi@ntnu.no (R. Arismendi).

https://doi.org/10.1016/j.ress.2021.107540
Received 24 June 2020; Received in revised form 28 January 2021; Accepted 6 February 2021
Available online 8 March 2021

hundred per cent of the equipment is used. Most of the compression systems involve high voltage electrical motors which are subject to deterioration which is assessed by the number of partial discharges in the insulation layers. These electrical motors are periodically inspected and their global deterioration state is ranked among a limited number of discrete states (6 levels according to ABB and Karsten Moholts scale for example [6]). A preventive maintenance can be recommended according to this rank but the production requirements can lead to postpone the execution of the maintenance tasks. For example, a preventive maintenance will not be triggered during the consumption peak in the winter season. Consequently, knowing the electrical motor condition at the inspection date is not enough to make an optimal decision for the maintenance time and task. It is required to model the overall CBM strategy including the maintenance delay due to seasonal constraints and the costs related to maintenance and failures.

In this paper, we intend to propose a framework for CBM modelling dedicated to critical infrastructures. The final objective is to challenge pre-established CBM strategies that might be not optimal and to provide time dependent or long term decision criteria to optimize them. The decision criteria are basically the probability, for the maintained system, to be in a deteriorated or critical state at any time (e.g closure of the bridge), or the maintenance strategy mean cost over a period of time. The main assumptions we keep from the application field of critical infrastructures are: we consider that the system condition is characterized at a high level of abstraction with few discrete states (often given by guidelines in the application field), the complete condition of the system is only known at inspection dates, and the maintenance tasks require a delay before execution.

The paper is organized as follows: Section 2 gives a presentation of the proposed modelling framework with the background and the state of the art. Section 3 describes the model in detail and in light of the modelling assumptions. A numerical approach is proposed for the quantification and its relation to the Chapman–Kolmogorov equation is shown. Section 4 presents an illustration case based on road bridges maintenance in Norway. It is intended to clarify the proposed model and quantification method. Guidance on how to account for different aspects of inspection and maintenance is included. Finally, concluding remarks and future directions of research are provided in Section 5.

2. Modelling framework: background and state of the art

2.1. Background

The purpose of a CBM model is to determine an intervention strategy that optimizes the performance according to some criteria such as cost, availability or others. In general, a model designed to optimize a CBM policy should consist of two elements: (i) a deterioration model and (ii) an intervention model [7,8]. The deterioration model is used to forecast the actual process of degradation of the health condition of a unit, while the intervention model captures the effect of maintenance and/or inspections in its health condition. Therefore, the global model can be used to find the optimal performance under a given intervention strategy (parametric optimization) or to investigate for an optimal strategy, as in Fig. 1.

There is a large amount of literature devoted to CBM optimization with a wide range of modelling frameworks and application areas. From a very generic point of view, we can distinguish two main categories: (i) the approaches based on the description of scenarios, with an exhaustive listing of all the possible sequences of events related to the deterioration evolution and the maintenance effects on a given horizon, (ii) the approaches based on the description of the states of the maintained system and the possible transitions between them.

Usually, the first ones are preferred when the number of scenarios is low enough to be described in a tractable way. Analytical solutions for the calculation of associated costs are commonly based on renewal theory and the identification of a renewal process. The second ones are preferred when the number of scenarios is too large. It can be easier to list system states and to model transitions from one state to the possible next ones instead of looking at the complete sequences of possible events on a given horizon. Certainly, such approaches are tractable when the number of states is reasonable or when it can be reduced enough for the modelling and optimization purpose. In this case, analytical solutions for the calculation of associated costs are usually based on the identification of a Markov process or an extension of such a process. Both approaches can be a good support to build Monte Carlo simulation algorithms and to empirically optimize a CBM strategy.

Under these circumstances, the inspections and the delay in the maintenance schedule could increase quite a lot the number of possible scenarios to list, whereas the reduced number of discrete states is a benefit for using a modelling framework based on states and transitions. This is what we propose in this paper.

2.2. State of the art

In some applications it is practicable to monitor, measure and describe the condition or health of the system, in a continuous-state space. In these cases, the deterioration process can then be modelled by a continuous-space–time stochastic process. CBM models considering this, usually model the deterioration with a diffusion process and commonly used are the gamma process, inverse Gaussian process and Wiener processes. Some recent examples of CBM models that consider continuous-space–time stochastic process are found in [9–13].

However, for many applications it is more reasonable to characterize the condition or health of the system by a finite set of deterioration states. This paper is focused on CBM modelling for this type of applications. In these cases, the deterioration is modelled by a jump process between the discrete deterioration states. The time of the jumps cannot be predicted without any uncertainty and are therefore considered to be random. The sequence of deterioration states that the system experiences is then described by a continuous-time discrete-state stochastic process. In addition to these random jumps, the system experiences changes of state according to the maintenance intervention schedule. These types of problems are commonly related to Markov processes [14]. To support the decision-making associated to maintenance, Markov (or semi-Markov) decision processes (MDP) are usually proposed. MDP are controlled stochastic processes in which the outcome has an associated uncertainty.

MDP have been proposed for CBM of different critical infrastructures. Tao et al. [15] consider the problem of optimizing maintenance strategies for highway bridges subject to progressive deterioration and sudden earthquakes. Compare et al. [16] develop a decision-support framework for the management of gas transmission networks subject to degradation modelled as a Markov process. González-Domínguez et al. [17] use Markov-chains to model the deterioration of the roofs of healthcare centres. The maintenance optimization problem on these cases is formulated as a MDP. In some cases, the condition monitoring process does not reveal the true state of the system with certainty. The system dynamics are determined by a MDP but the decision maker may not directly observe the underlying state with certainty. To handle this, observation probabilities over the set of possible system states are introduced in the model and the resulting framework is named a Partially Observable Markov Decision Process. Recent examples can be found in [16,18,19].

In most works related to MDP, the action from a decision related to maintenance (modelled as a transition to other discrete states) takes place at the inspection time instantaneously. Some have consider a duration for the maintenance action with the system being stopped immediately and restarted after the intervention. We intend in this paper to propose a framework capable of addressing cases in which the delay from the time of the decision to the time of the maintenance action may result in further deterioration of the system, which in return

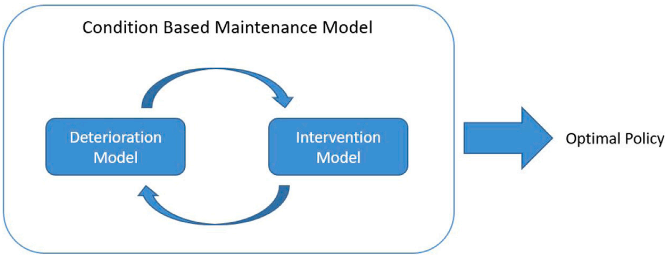


Fig. 1. Condition-based maintenance model.

will require a different maintenance action than originally planned. In order to characterize the transitions, support the modelling work, and to provide a formalism that can be relevant for a large set of application cases, we propose to use a specific class of Piecewise Deterministic Markov Process (PDMP). This PDMP can serve as a basis to implement a numerical scheme and provide numerical solutions for the maintenance strategy cost evaluation. It can be also a basis to build a Monte Carlo simulation algorithm and provide empirical solutions for the maintenance costs.

PDMPs were introduced in 1984 by Davis [20], as a general class of non-diffusion stochastic models that provides a framework for studying optimization problems. A PDMP can be considered as an extension of a Markov chain that incorporates continuous variables to allow a combination of deterministic motion and random jumps. In the framework of dynamic reliability, the continuous variables are used to describe physical phenomena that influences the jump process between discrete states and that can be defined by rather complex differential equations. Some works that have proposed PDMPs as a modelling framework for problems that combine deterministic behaviour (described using physics-based knowledge and equations) and stochastic jumps are found in [21–23]. Other applications of PDMPs for CBM focus on problems that require a combination of random jumps and jumps that occur at deterministic times (meeting a maintenance schedule set in advance). Examples of these can be found in [24–26].

3. Piecewise deterministic Markov processes

In this section we present in details the modelling framework by defining the modelling assumptions and then the formalism of the corresponding PDMP with linear deterministic motion. At last, a method for numerical calculations of the state probabilities of the PDMP is developed. It is based on a classical Euler numerical scheme. We also provide a formal proof of the convergence of this scheme towards the Chapman–Kolmogorov equations of the PDMP.

3.1. Modelling assumptions

We propose PDMP as a framework for studying CBM problems with the following modelling assumptions:

- The deterioration of the unit can be described by a discrete system
- Monitoring of the deterioration of the unit is performed by means of an inspection scheme
- There is a significant time elapsed from the date of detection and planning of a maintenance operation until the date of actually performing the operation. This elapsed time is referred to as maintenance delay
- The maintenance delay is mainly due to logistics reasons and hence its duration considered deterministic

To handle problems with these assumptions, we resort to a specific class of PDMP: the deterministic motion is linear and is intended to keep track of the time to an intervention jump. This will allow compensation of a lack of Markov property and will facilitate the description of the transitions between states.

3.2. Formalism

We consider a hybrid stochastic process $\{I_t, X_t\}$ with values in a discrete-continuous space $E \times R$.

The first component, $\{I_t\}$ is discrete and used to represent the deterioration states of the unit. We consider that the deterioration states can be categorized in a finite number N of levels. E is the finite set made of N points.

The second component, $\{X_t\}$ is continuous, introduced in our case as a way to keep track for the intervention jumps that occur at specified times. We consider $\mathbf{x} = (x_1, x_2, t)$, $\mathbf{x}$ is a vector in which x_1 corresponds to the date of the next inspection, x_2 corresponds to the date of the next maintenance operation, and t stands for time. Hence, the continuous component $\{X_t\}$ evolves in R, a three-dimensional orthotope of $\mathbb{R}^3$.

The process $\{I_t, X_t\}$ experiences jumps at random times and jumps at intervention times. Between the times of two consecutive jumps (random or deterministic) the continuous component X_t evolves with deterministic motion.

3.2.1. Deterministic motion

In general, the deterministic motion of a PDMP corresponds to the solution of a set of differential equations for a fixed discrete state, i.e. given that $I_t = i$ between two jumps, X_t is solution of:

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{v}(i, \mathbf{x}) \tag{1}$$

In our case, the deterministic evolution of the continuous component between two consecutive jumps is very simple: only the continuous variable t evolves with a speed of one, i.e. $\mathbf{v}(i, \mathbf{x}) = (0, 0, 1);\ \forall i$. This kind of process is a particular case of PDMP, also named piecewise-linear process [27].

3.2.2. Jumps at random times

Jumps at random times correspond to the stochastic deterioration of the unit. A jump from state $(i, \mathbf{x})$ towards discrete state (j) occurs with a rate $\lambda(i, \mathbf{x})Q(i, \mathbf{x}, j)$, where $\lambda(i, \mathbf{x})$ is the rate at which the process leaves i and $Q(i, \mathbf{x}, j)$ is the probability distribution of the jump from i to j. In this sense, the rate at which the process jumps from the discrete point i to a discrete point j is dependent on both the discrete and continuous component before the jump $(i, \mathbf{x})$ and the discrete component after the jump (j). Hence, the rate does not need to be constant and in the model here proposed it could be time dependent.

3.2.3. Jumps at intervention times

Jumps at intervention times are associated to the inspections and maintenance operations. To model these jumps a frontier is defined, such as when the continuous component reaches the frontier due to the deterministic motion, a jump occurs. Let $\Gamma = \{x_1 = t\} \cup \{x_2 = t\}$ be the set of points we refer to as the frontier for our case. Such frontier is reached when time (t) reaches the date of an inspection (x_1) or the date of a maintenance operation (x_2). When the frontier Γ is reached at $(i, \mathbf{x})$, a jump occurs in the discrete component towards a point j of E

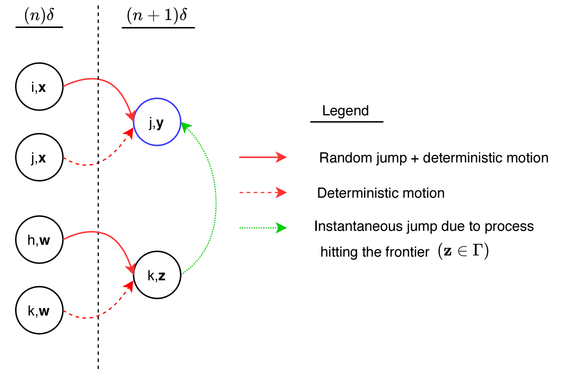


Fig. 2. Transitions into state $(j, \mathbf{y})$ in $(n\delta, (n+1)\delta]$.

and in the continuous component to a point $\mathbf{y}$ of R equal to $m_\Gamma(i, \mathbf{x}, j)$, with a probability distribution $q(i, \mathbf{x}, j)$. The function $\mathbf{x} \to m_\Gamma(i, \mathbf{x}, j)$ is a function from Γ to R. This means that both the discrete and continuous components jump at the intervention time.

The post jump location $(j, \mathbf{y})$ is dependent on the discrete component before and after the jump and the continuous component before the jump (the frontier).

For example, if the reached frontier corresponds to an inspection of the unit (i.e. when $t = x_1$), a maintenance action and the next inspection of the unit can be scheduled, depending on the deterioration state of the unit, thus a jump in the dates of next inspection (x_1) and next maintenance (x_2) occurs. Similarly, if the reached frontier corresponds to a maintenance action (i.e. when $t = x_2$) a jump occurs in the deterioration state of the unit (i), (usually to a less deteriorated state) and if imperfect maintenance is considered, then a probability distribution can be associated to the post jump location.

3.3. Numerical calculations

We calculate now an approximation of the law of this process. This approach is based on the proposed finite volume scheme by Cocozza-Thivent et al. in [28,29] and some of its applications as [26,30].

Every term of the continuous state space R is discretized in a finite number of values. Let δ be the continuous state space step. Then, the approximation of the k-ith term has values in $F_k = \{0, \delta, 2\delta, \ldots, n\delta\}$. In this way, R is replaced by the discrete state space $F = F_1 \times F_2 \times F_3$.

The deterministic motion of the process between jumps can be described with a function from F to F by solving Eq. (1) with the Euler method. In our case, between iterations $n\delta$ and $(n+1)\delta$ the continuous component follows the function $\mathbf{x} \to \mathbf{x} + \mathbf{v}\delta$. When the frontier Γ is reached at $\mathbf{x}$, the continuous component jumps instantaneously from Γ to F as $\mathbf{x} \to m_\Gamma(i, \mathbf{x}, j)$.

Let $P((i, \mathbf{x}), (j, \mathbf{y}))$ denote the conditional transition probability from state $(i, \mathbf{x})$ to state $(j, \mathbf{y})$ with values in the finite state space $E \times F$, and $\tilde{F}$ denote the set of points in F which are not on the frontier Γ. The arrivals into a state $(j, \mathbf{y})$ may proceed from different paths as represented in Fig. 2.

Between $t = n\delta$ and $t = (n+1)\delta$, the non-null values of such conditional probabilities due to random jumps and deterministic motion are:

- for any $\mathbf{x}, \mathbf{w}$ in $\tilde{F}$, for any j different from i, for any k different from h:

$$P((i,\mathbf{x}),(j,\mathbf{y})) \approx \lambda(i,\mathbf{x})Q(i,\mathbf{x},j)\delta \text{ with } \mathbf{y} = \mathbf{x} + \mathbf{v}\delta$$
$$P((h,\mathbf{w}),(k,\mathbf{z})) \approx \lambda(h,\mathbf{w})Q(h,\mathbf{w},k)\delta \text{ with } \mathbf{z} = \mathbf{w} + \mathbf{v}\delta \tag{2}$$

- for any $\mathbf{x}, \mathbf{w}$ in $\tilde{F}$:

$$P((j,\mathbf{x}),(j,\mathbf{y})) \approx 1 - \lambda(j,\mathbf{x})\delta \text{ with } \mathbf{y} = \mathbf{x} + \mathbf{v}\delta \tag{3}$$
$$P((k,\mathbf{w}),(k,\mathbf{z})) \approx 1 - \lambda(k,\mathbf{w})\delta \text{ with } \mathbf{z} = \mathbf{w} + \mathbf{v}\delta$$

When $\mathbf{z} \in \Gamma$, an instantaneous jump occurs with the conditional probability:

- for any $\mathbf{z}$ in Γ, for any k:

$$P((k,\mathbf{z}),(j,\mathbf{y})) = q(k,\mathbf{z},j) \tag{4}$$

Let $\underline{\pi}_{n\delta}$ denote the law of this stochastic process at the n-ith iteration. By using the law of total probability and the Markov property, we can write the law of the process for state $(j, \mathbf{y})$ at the $(n+1)$-ith iteration based on the transitions showed on Fig. 2. We have for any i, h in E and $\mathbf{x}, \mathbf{w}$ in F:

$$\begin{aligned}
\underline{\pi}_{(n+1)\delta}(j,\mathbf{y}) = & \sum_{\substack{i \neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}}^{N-1} \underline{\pi}_{n\delta}(i,\mathbf{x})[P((i,\mathbf{x}),(j,\mathbf{y}))] \\
& + \mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}}\, \underline{\pi}_{n\delta}(j,\mathbf{x})[P((j,\mathbf{x}),(j,\mathbf{y}))] \\
& + \sum_{\substack{k=1 \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \sum_{\substack{h=1 \\ h\neq k \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma}}^{N-1} \underline{\pi}_{n\delta}(h,\mathbf{w})[P((h,\mathbf{w}),(k,\mathbf{z}))][P((k,\mathbf{z}),(j,\mathbf{y}))] \\
& + \sum_{\substack{k=1 \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \underline{\pi}_{n\delta}(k,\mathbf{w})[P((k,\mathbf{w}),(k,\mathbf{z}))][P((k,\mathbf{z}),(j,\mathbf{y}))]
\end{aligned} \tag{5}$$

In Eq. (5), the first term accounts for the transitions related to a random jump and deterministic motion; the second term for the transitions related to only deterministic motion; the third term for the transitions with a random jump, deterministic motion and an instantaneous jump from the frontier; and the last term accounts for the transitions with deterministic motion and an instantaneous jump from the frontier; as shown in Fig. 2.

Substituting the conditional probabilities by their approximation or value from Eqs. (2), (3) and (4), gives:

$$\begin{aligned}
\underline{\pi}_{(n+1)\delta}(j,\mathbf{y}) \approx & \sum_{\substack{i \neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}}^{N-1} \underline{\pi}_{n\delta}(i,\mathbf{x})[\lambda(i,\mathbf{x})Q(i,\mathbf{x},j)\delta] \\
& + \mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}}\, \underline{\pi}_{n\delta}(j,\mathbf{x})[1-\lambda(j,\mathbf{x})\delta] \\
& + \sum_{\substack{k=1 \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \sum_{\substack{h=1 \\ h\neq k \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma}}^{N-1} \underline{\pi}_{n\delta}(h,\mathbf{w})[\lambda(h,\mathbf{w})Q(h,\mathbf{w},k)\delta][q(k,\mathbf{z},j)] \\
& + \sum_{\substack{k=1 \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta \\ \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}}^{N} \underline{\pi}_{n\delta}(k,\mathbf{w})[1-\lambda(k,\mathbf{w})\delta][q(k,\mathbf{z},j)]
\end{aligned} \tag{6}$$

Computing this equation fully describes the evolution of the PDMP. For Markov processes, it is known that the Chapman–Kolmogorov equation describes the time-evolution of the states probabilities. Eq. (6) corresponds to an approximation of the known Chapman–Kolmogorov equation. Since this is not very obvious, we proceed to demonstrate it by deriving the Chapman–Kolmogorov equation starting from Eq. (6).

Chapman–kolmogorov

If $f(i, x)$ is a function from $E \times F$ to $\mathbb{R}$, we can write:

$$\begin{aligned}
\sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{(n+1)\delta}(j,\mathbf{y}) = & \sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{n\delta}(j,\mathbf{y}) - \sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{n\delta}(j,\mathbf{y}) \\
& + \sum_{j,\mathbf{y}} f(j,\mathbf{y}) \sum_{\substack{i\neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}} \underline{\pi}_{n\delta}(i,\mathbf{x})\lambda(i,\mathbf{x})Q(i,\mathbf{x},j)\delta \\
& + \sum_{j,\mathbf{y}} f(j,\mathbf{y})\mathbb{1}_{\{\mathbf{y}=\mathbf{x}+\mathbf{v}\delta\}}\underline{\pi}_{n\delta}(j,\mathbf{x})[1-\lambda(j,\mathbf{x})\delta]
\end{aligned} \tag{7}$$

$$+ \sum_{\substack{j,\mathbf{y} \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}} f(j,\mathbf{y}) \sum_{\substack{h\neq k \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta;\ \mathbf{z}\in\Gamma}} \underline{\pi}_{n\delta}(h,\mathbf{w})\lambda(h,\mathbf{w})Q(h,\mathbf{w},k)q(k,\mathbf{z},j)\delta$$

$$+ \sum_{j,\mathbf{y}} f(j,\mathbf{y}) \sum_{\substack{k \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta;\ \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},j)}} \underline{\pi}_{n\delta}(k,\mathbf{w})[1-\lambda(k,\mathbf{w})]q(k,\mathbf{z},j)\delta$$

By changing some indices, we can write:

$$\sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{(n+1)\delta}(j,\mathbf{y}) = \sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{n\delta}(j,\mathbf{y}) - \sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{n\delta}(j,\mathbf{y}) \tag{8}$$

$$+ \sum_{\substack{i,j;\ i\neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}} f(i,\mathbf{y})\ \underline{\pi}_{n\delta}(j,\mathbf{x})\lambda(j,\mathbf{x})Q(j,\mathbf{x},i)\delta$$

$$+ \sum_{\substack{j,\mathbf{x} \\ \mathbf{x}=\mathbf{y}+\mathbf{v}\delta}} f(j,\mathbf{x})\ \underline{\pi}_{n\delta}(j,\mathbf{y})[1-\lambda(j,\mathbf{y})\delta]$$

$$+ \sum_{\substack{i,k,j,\mathbf{y} \\ k\neq j \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta;\mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(k,\mathbf{z},i)}} f(i,\mathbf{y})\underline{\pi}_{n\delta}(j,\mathbf{w})\lambda(j,\mathbf{w})Q(j,\mathbf{w},k)q(k,\mathbf{z},i)\delta$$

$$+ \sum_{\substack{i,j \\ \mathbf{z}=\mathbf{w}+\mathbf{v}\delta;\ \mathbf{z}\in\Gamma \\ \mathbf{y}=m_\Gamma(j,\mathbf{z},i)}} f(i,\mathbf{y})\ \underline{\pi}_{n\delta}(j,\mathbf{w})[1-\lambda(j,\mathbf{w})\delta]q(j,\mathbf{z},i)\delta$$

After grouping, it can be written as:

$$\sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{(n+1)\delta}(j,\mathbf{y}) = \sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{n\delta}(j,\mathbf{y}) \tag{9}$$

$$+ \sum_{\substack{j,\mathbf{y} \\ (\mathbf{y}+\mathbf{v}\delta)\notin\Gamma}} \underline{\pi}_{n\delta}(j,\mathbf{y})\lambda(j,\mathbf{x})\delta\left(\sum_{\substack{i\neq j \\ \mathbf{y}=\mathbf{x}+\mathbf{v}\delta}} f(i,\mathbf{y})Q(j,\mathbf{x},i) - f(j,\mathbf{y})\right)$$

$$+ \sum_{\substack{j,\mathbf{y} \\ (\mathbf{y}+\mathbf{v}\delta)\notin\Gamma}} \underline{\pi}_{n\delta}(j,\mathbf{y})[1-\lambda(j,\mathbf{y})][f(j,\mathbf{y}+\mathbf{v}\delta) - f(j,\mathbf{y})]$$

$$+ \sum_{\substack{j,\mathbf{w} \\ (\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} \underline{\pi}_{n\delta}(j,\mathbf{w})\lambda(j,\mathbf{w})\delta\left(\sum_{\substack{i,k;\ k\neq j \\ \mathbf{z}=(\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} f(i,m_\Gamma(k,\mathbf{z},i))Q(j,\mathbf{w},k)q(k,\mathbf{z},i)\right.$$

$$\left. -\ f(j,\mathbf{w})\right)$$

$$+ \sum_{\substack{j,\mathbf{w} \\ (\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} \underline{\pi}_{n\delta}(j,\mathbf{w})(1-\lambda(j,\mathbf{w})\delta)\left(\sum_{\substack{i,\mathbf{z} \\ \mathbf{z}=(\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} f(i,m_\Gamma(j,\mathbf{z},i))q(j,\mathbf{z},i)\right.$$

$$\left. -\ f(j,\mathbf{w})\right)$$

If $n\delta = t$ by summation of successive differences, we can write:

$$\sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_{(n+1)\delta}(j,\mathbf{y}) = \sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_0(j,\mathbf{y}) \tag{10}$$

$$+ \delta\sum_{m=0}^{n} \sum_{\substack{j,\mathbf{y} \\ (\mathbf{y}+\mathbf{v}\delta)\notin\Gamma}} \underline{\pi}_{m\delta}(j,\mathbf{y})\lambda(j,\mathbf{y})\left(\sum_{i\neq j} f(i,\mathbf{y})Q(j,\mathbf{y}-\mathbf{v}\delta,i) - f(j,\mathbf{y})\right)$$

$$+ \delta\sum_{m=0}^{n} \sum_{\substack{j,\mathbf{y} \\ (\mathbf{y}+\mathbf{v}\delta)\notin\Gamma}} \underline{\pi}_{m\delta}(j,\mathbf{y})(1-\lambda(j,\mathbf{y})\delta)\left(\frac{f(j,\mathbf{y}+\mathbf{v}\delta) - f(j,\mathbf{y})}{\delta}\right)$$

$$+ \delta\sum_{m=0}^{n} \sum_{\substack{j,\mathbf{w} \\ (\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} \underline{\pi}_{m\delta}(j,\mathbf{w})\lambda(j,\mathbf{w})\left(\sum_{i,k;\ k\neq j} f(i,m_\Gamma(k,\mathbf{w}\right.$$

$$\left. +\ \mathbf{v}\delta,i))Q(j,\mathbf{w},k)q(k,\mathbf{w}+\mathbf{v}\delta,i) - f(j,\mathbf{w})\right)$$

$$+ \sum_{m=0}^{n} \sum_{\substack{j,\mathbf{w} \\ (\mathbf{w}+\mathbf{v}\delta)\in\Gamma}} \underline{\pi}_{m\delta}(j,\mathbf{w})(1-\lambda(j,\mathbf{w}\delta))\left(\sum_{i} f(i,m_\Gamma(j,\mathbf{w}\right.$$

$$\left. +\ \mathbf{v}\delta,i))q(j,\mathbf{w}+\mathbf{v}\delta,i) - f(j,\mathbf{w})\right)$$

Finally, by making δ tend towards 0, we get the Chapman–Kolmogorov equation for a following regular function f from $E \times R$ to $\mathbb{R}$, where $\pi_t(j,\mathbf{y})$ denotes the law of the process (I_t, X_t) at time t :

$$\sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_t(j,\mathbf{y}) = \sum_{j,\mathbf{y}} f(j,\mathbf{y})\underline{\pi}_0(j,\mathbf{y}) \tag{11}$$

$$+ \int_0^t du \int_{\tilde{F}} \sum_j \underline{\pi}_u(j,d\mathbf{y})\lambda(j,\mathbf{y})\left(\sum_{i\neq j} f(i,j)Q(j,\mathbf{y},i) - f(j,\mathbf{y})\right)$$

$$+ \int_0^t du \int_{\tilde{F}} \underline{\pi}_u(j,d\mathbf{y}) \sum_l \frac{df}{dy_l}(j,\mathbf{y})v_l$$

$$+ \int_{(0,t]\times\Gamma} \sum_j \left(\sum_i f(i,m_\Gamma(j,\mathbf{w},i))q(j,\mathbf{w},i) - f(j,\mathbf{w})\right)\sigma(j,d\mathbf{w},ds)$$

The measures $\sigma(j,d\mathbf{w},ds)$ on the time–space $\mathbb{R}_+\times\Gamma$ describe the way the process reaches the frontier. If t_1 and t_2 are two times $(t_1 < t_2)$ and Γ_l is a part of the frontier Γ, $\int_{\Gamma_l\times(t_1,t_2]}\sigma(j,d\mathbf{w},ds)$ is the mean number of times the process reaches the frontier between t_1 and t_2 on the part Γ_l of the frontier with the discrete part equals j. In our case, the mean number that the process reaches a part of the frontier corresponds to the mean number of maintenance actions or inspections.

4. Illustration case. Bridge management in Norway

This section illustrates the proposed modelling approach through a case study and explores on how different assumptions could be taken into account. The case is related to bridge management, i.e. the planning of inspections and maintenance activities of road bridges, in Norway.

4.1. Problem statement

Road bridges are a vital part of the Norwegian transportation infrastructure. In Norway, there are more than 18,000 road bridges across the country, so an efficient bridge management system is vital for avoiding high costs from over expending and for ensuring safety of the public and availability of the transportation system.

Many factors can make bridge management a challenging task, such as: the varying weight and intensity of the traffic, the evolution of the building codes over the years, the weather influence on the structures, large number of structures spread over a large area, and others [8]. All these factors create uncertainty, which makes the bridge management a problem of decision making under uncertainty.

In the bridge management system applied by the Norwegian Public Roads Administration (NPRA), the agency responsible for planning, building, operating, and maintaining national and country bridges, the inspections are mainly carried out periodically based on pre-defined rules and the decision about when to perform maintenance is based on the findings of these inspections. The handbooks for management and inspections of bridges [4,5], establish types of inspections for the bridges and the period in which they must be performed, e.g. a main inspection of a bridge, with an overview of all the elements of the bridge, must (in general) be performed every five years. They also establish how the inspections must be logged in a database, how the findings must be reported and provides guidelines on when to perform the repairs for found damages.

There exists an extensive list of damage mechanisms that can affect a structure. The inspection handbook of the NPRA [5] provides an overview of these mechanisms with guidelines on how to assess their severity. The assessment of the severity consists in a combination

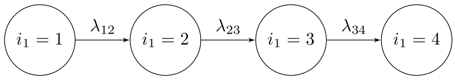


Fig. 3. Deterioration process.

of quantitative and qualitative methods. The resulting condition is presented in a scale of one to four, as: (1) Small damage, (2) Medium damage, (3) Large damage and (4) Critical damage.

The regulations dictate a CBM strategy that establishes when the damage must be repaired based on the condition at the inspection. According to the severity of the damage, a maintenance action (or no action) must be scheduled. For small damage (1), no maintenance action is required; for medium damage (2), a maintenance action must take place between four and ten years; for large damage (3), a maintenance action must take place between one and three years and for critical damage (4), a maintenance action must take place in less than six months.

We proceed to build a CBM model illustrating the PDMP formalism described in Section 3.2. We recall that a state of the PDMP $\{I_t, X_t\}$ to consider is made of $\{\mathbf{i}, \mathbf{x}\}$ with $\mathbf{x} = (x_1, x_2, t)$, where x_1 corresponds to the date of the next inspection, x_2 corresponds to the date of the next maintenance operation, t stands for time and i corresponds to the deterioration level of the unit.

4.2. Deterioration model

For modelling the deterioration process of a structure, we need to define the deterioration states and to describe the jumps between these states.

In the bridge management of the NPRA a condition is assigned to the structure as a level that ranges from one to four. In order to distinguish between a condition not known to the NPRA and a condition which is known based on information from the inspection, we split the deterioration state of the unit in two parts: a real state and a virtual state. In this sense, $\mathbf{i} = (i_1, i_2)$ where $i_1 = \{1, 2, 3, 4\}$ denotes the real state of the structure, and $i_2 = \{1, 2, 3, 4\}$ denotes the virtual state of the structure (known by the operator based on the inspections). A condition (real or virtual) of the structure is assigned as:

- 1 : Small or no damage
- 2 : Medium damage
- 3 : Large damage
- 4 : Critical damage

The deterioration is modelled with random jumps between these states. Since the unit is not continuously monitored, when a deterioration jump occurs, it is not detected until an inspection is performed, so only the real state (i_1) changes and the virtual state (i_2) remains unchanged. In this case, we consider that the structure deteriorates gradually as shown in Fig. 3.

As described in Section 3.2.2, a jump from state $(\mathbf{i}, \mathbf{x})$ towards discrete state $(\mathbf{j})$ occurs with a rate $\lambda(\mathbf{i}, \mathbf{x})Q(\mathbf{i}, \mathbf{x}, \mathbf{j})$. Considering constant transition rates, i.e. dependent only on the discrete components before and after the jump and not on the continuous component, we can write the transition rates out of a discrete component $\mathbf{i}$, as:

- From $i_1 = 1$, $\forall$ i_2, $\mathbf{x}$:
  $\lambda((1, i_2), \mathbf{x}) = \lambda_{12}$ and $Q((1, i_2), \mathbf{x}, (2, i_2)) = 1$
- From $i_1 = 2$, $\forall$ i_2, $\mathbf{x}$:
  $\lambda((2, i_2), \mathbf{x}) = \lambda_{23}$ and $Q((2, i_2), \mathbf{x}, (3, i_2)) = 1$
- From $i_1 = 3$, $\forall$ i_2, $\mathbf{x}$:
  $\lambda((3, i_2), \mathbf{x}) = \lambda_{34}$ and $Q((3, i_2), \mathbf{x}, (4, i_2)) = 1$
- From $i_1 = 4$, $\forall$ i_2, $\mathbf{x}$:
  $\lambda((4, i_2), \mathbf{x}) = 0$

4.3. Intervention model

The inspections and maintenance operations are described by jumps at intervention times as described in Section 3.2.3.

4.3.1. Inspections

When a structure is inspected its condition is revealed and a maintenance task and the next inspection are scheduled accordingly. A jump in the PDMP related to an inspection occurs when the part of the frontier Γ made of the points $\{x_1 = t\}$ is reached. To describe a jump at this time we must define the post jump location of the discrete and continuous components and the associated probability distribution of such location.

There are two characteristics about the inspections that can be addressed in the modelling framework: frequency and quality. The frequency of the inspections can be periodic or non-periodic. Currently, due to regulations, the bridge inspections of the NPRA are carried out periodically, i.e. inspections are performed at equal time intervals. Let T denote the constant interval for inspections and M_{i_2} denote the delay for maintenance based on the virtual deterioration condition i_2. As described in Section 4.1, maintenance is scheduled according to the known deterioration state to the operator (i_2) as: $M_1 = \infty$, $M_2 \in [4, 10]$ years, $M_3 \in [1, 3]$ years, $M_4 \in [0, 0.5]$ years. The post jump location of the continuous component is: $m_\Gamma(\mathbf{i}, \mathbf{x}, \mathbf{j}) = (t + T, \min(x_B, M_{i_2}), t)$. Non-periodic inspections could for example follow a pre-determined condition-based inspection scheme, in which the time of a next inspection is decided based on the deterioration state of the unit at the current one. To model such inspection scheme, the inspection interval can be set according to the virtual state of the unit (i_2), (similar to the maintenance delay) as T_{i_2} instead a constant interval.

The quality of the inspections can be taken into account by assigning a probability $q(\mathbf{i}, \mathbf{x}, \mathbf{j})$ distribution to the post-jump location, as described in Section 3.2.3. If the inspections are considered perfect, i.e. the real state of the unit is revealed at the inspection without uncertainty, then the virtual state becomes equal to the real state of the unit at the inspection time, with probability of one. In some cases, the inspections may not perfectly reveal the real condition of the unit, due to for example hidden damages or errors in measurements. If the inspections are considered non-perfect or subject to errors, then we can write a conditional probability of the post-jump location of the virtual state (j_2), given the real state of the unit before the jump (i_1), as $P(j_2|i_1) = q(\mathbf{i}, \mathbf{x}, \mathbf{j})$ and the post jump location of the continuous component would be $m_\Gamma(\mathbf{i}, \mathbf{x}, \mathbf{j}) = \mathbf{x} + (T_{j_2}, M_{j_2}, 0)$

4.3.2. Maintenance

A maintenance task is scheduled according to the condition of the structure at inspection. The maintenance is arranged to take place after a delay with deterministic duration. When a maintenance action is performed, a jump occurs to a less deteriorated state. The degree of the maintenance is modelled by assigning the post-jump location. For example, if perfect maintenance or replacement is considered, the unit is considered as-good-as-new, thus the discrete component (i_1, i_2) jumps to (1,1). In addition, the date to the next inspection (x_1) does not change, and the date of the next maintenance action (x_2) is set to infinite (no maintenance scheduled).

To consider possible errors in the maintenance operation, a probability distribution $q(\mathbf{i}, \mathbf{x}, \mathbf{j})$ can be assigned the post-jump location. For example, let θ denote the probability of maintenance error, i.e. the probability that a maintenance action results in a state other than as-good-as-new e.g. a state with medium damage, then the discrete component jumps from $\mathbf{i}$ to $\mathbf{j} = (1, 1)$ with probability $q(\mathbf{i}, \mathbf{x}, \mathbf{j}) = (1 - \theta)$ or from $\mathbf{i}$ to $\mathbf{k} = (2, 1)$ with probability $q(\mathbf{i}, \mathbf{x}, \mathbf{k}) = \theta$.

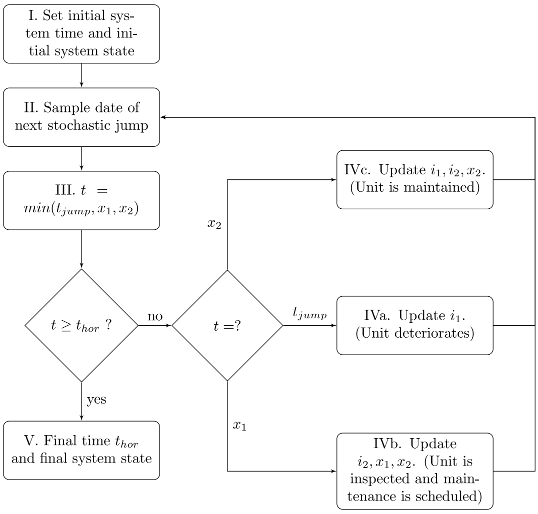


Fig. 4. Simulation procedure.

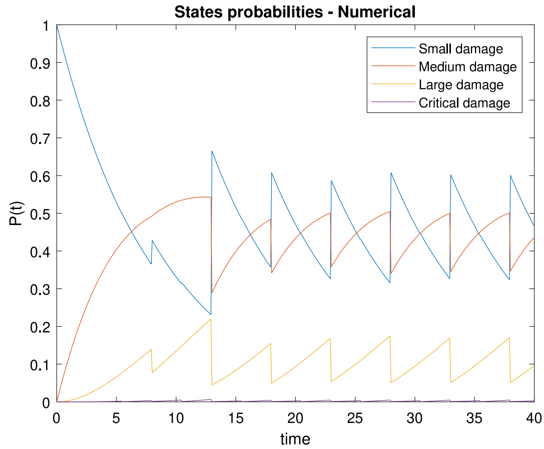


Fig. 5. Deterioration state probabilities.

4.4. Quantification

The following assumptions are considered in the illustration case for quantification purposes, in addition to those listed in Section 3:

(i) The unit is inspected periodically, i.e. at equal time intervals T
(ii) Inspections are perfect and reveal the true state of the unit.
(iii) Maintenance interventions occur at the scheduled date instantaneously, i.e. the duration of the intervention is neglected
(iv) After a maintenance action, the unit is considered as-good-as-new without error.

From assumptions (i) and (ii), it can be written that if $\mathbf{x} \in \Gamma$ with $x_1 = t$, a jump occurs from state $(\mathbf{i}, \mathbf{x})$ to state $(\mathbf{j}, m_\Gamma(\mathbf{i}, \mathbf{x}, \mathbf{j}))$ with probability $q(\mathbf{i}, \mathbf{x}, \mathbf{j}) = 1$. The discrete component jumps to $\mathbf{j}$ with $j_1 = j_2 = i_1$, i.e. the virtual deterioration state becomes equal to the real state before the jump, while the continuous component jumps to $m_\Gamma(\mathbf{i}, \mathbf{x}, \mathbf{j}) = (t + T, \min(x_2, M_{i_2}), t)$.

From assumption (iv), if $\mathbf{z} \in \Gamma$ with $z_2 = t$ then a jump occurs from state $(\mathbf{k}, \mathbf{z})$ to state $(\mathbf{j}, m_\Gamma(\mathbf{k}, \mathbf{z}, \mathbf{j}))$ with probability $q(\mathbf{k}, \mathbf{z}, \mathbf{j}) = 1$ (no error). The discrete component jumps to $\mathbf{j} = (1, 1)$, i.e. as-good-as-new, while the continuous component jumps to $m_\Gamma(\mathbf{k}, \mathbf{z}, \mathbf{j}) = (z_1, \infty, t)$.

4.4.1. Numerical approach

The states probabilities can be found by iterating on the recursive Eq. (6). At every iteration step, the real deterioration states probabilities $\underline{\pi}_{n\delta}((i_1, \cdot), \mathbf{x})$ can be found with the summation:

$$\underline{\pi}_{n\delta}((i_1 = k, \cdot), (\cdot, \cdot, n\delta)) = \sum_{x_2} \sum_{x_1} \sum_{i_2} \underline{\pi}_{n\delta}((k, i_2), (x_1, x_2, n\delta)) \quad (12)$$

4.4.2. Monte Carlo simulation

An alternative quantification method to the numerical approach introduced in 3.3 is to perform Monte Carlo simulation of the process to estimate the quantities of interest. The modelling framework described in Section 3.2 is convenient for setting the structure to simulate problems of CBM with the aforementioned assumptions.

The simulation procedure of the PDMP is shown in Fig. 4. It includes five main steps to simulate a realization of the PDMP until a horizon time t_{hor}.

(i) Set initial system time and initial system state
In our case, initial time is set to zero, the unit is set to be in new condition with no maintenance action scheduled and the date of the first inspection is set to the period. (i.e. $t = 0$, $i_1 = 1$, $i_2 = 1$, $x_1 = T$ and $x_2 = \infty$).
(ii) Sample date of next random jump, if enabled
The date of the next random jump t_{jump} is sampled from the corresponding probability density function and the corresponding parameter(s).
(iii) Identify next event
The date of the next random jump t_{jump} is compared with date of next inspection x_1, the date of next maintenance action x_2 and the horizon time t_{hor}.
The system time is updated as: $t = \min(t_{jump}, x_1, x_2, t_{hor})$. If the simulation time has reached the horizon time, $t \geq t_{hor}$, the simulation continues to step (v), otherwise it continues to step (iv).
(iv) Update system state
The system state is updated according to the jump that takes place at time t: deterioration, inspection or maintenance.

(a) Deterioration: $(t = t_{jump})$
Only i_1 is updated in this jump
(b) Inspection: $(t = x_1)$
The values of i_2, x_1, x_2 are updated. The post-jump values are:
$i_2^+ = i_1^-$;
$x_1^+ = t + T$;
$x_2^+ = t + M_{i_2}$;
(c) Maintenance: $(t = x_2)$
The values of i_1, i_2, x_2 are updated. The post-jump values are:
$i_1^+ = i_2^+ = 1$;
$x_2^+ = \infty$.

(v) Set final system state and time
The final time is t_{hor} and the final system state is the state resulting from the last jump to take place no later than t_{hor}.

This simulation procedure is replicated a high number of times, to approximate quantities of interest, such as deterioration state probabilities and mean numbers of interventions of a given type.

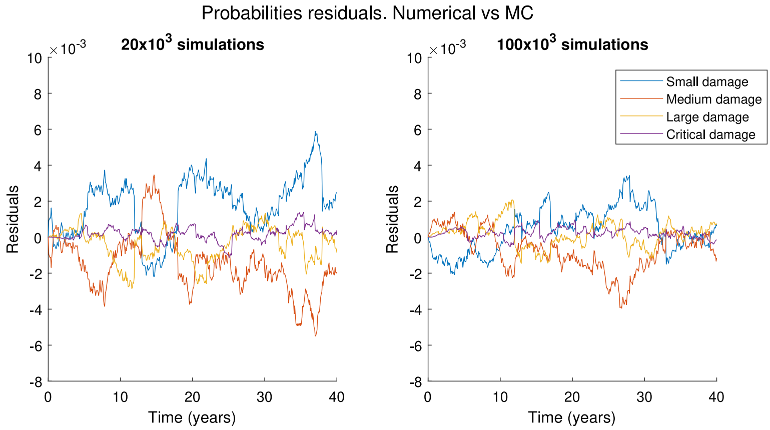


Fig. 6. Deterioration state probabilities.

Table 1
Model parameters.

Deterioration rates (h^{-1})	Maintenance delays (y)	Inspection interval (y)
λ_{12} = 1.5e−5	$M_1 = \infty$	$T = 5$
λ_{23} = 6e−6	$M_2 = 8$	
λ_{34} = 1.4e−6	$M_3 = 3$	
	$M_4 = 0.5$	

4.5. Experiments and results

4.5.1. State probabilities

The model parameters used for quantification are shown in Table 1. The deterioration rates have been estimated from previous works carried by the NPRA based on the information available on their database for bridge inspections and maintenance actions.

The time dependent real deterioration states probabilities $P(i_1)$ are found using both the numerical approach and Monte Carlo simulation. The results are shown in Fig. 5.

To compare the results of the quantification from both approaches, the residuals or difference between the state probabilities is shown in Fig. 6. It can be observed that the difference in results is small with an order of magnitude of 10^{-3}. In addition, the difference is reduced by performing a higher number of replications of the Monte Carlo simulation, showing same convergence.

The Monte Carlo simulation method is widely used in practice, conceptually easy to apply and without particular restrictions on the dimension of the PDMP. On the other hand, the numerical scheme has high accuracy with short computation times [22]. In our case, the Monte Carlo simulation with 100,000 replications took approximately one hour to obtain time-dependent probabilities, while with the numerical scheme the results are obtained in approximately ten seconds.

4.5.2. Maintenance optimization

The PDMP allows to test different inspection and maintenance strategies and assess their effect on the structure condition. For example, different periods of inspection can be considered, evaluating the effect on the condition of the structure. Fig. 7 shows how the critical condition of the unit ($i_1 = 4$), varies with time for different inspection periods. This allows to support the decision process related to inspections by evaluating the associated risk on the structure.

Moreover, to assist the decision process in bridge management, the expected cost per unit of time of a given strategy can be assessed in addition to the effect on the condition of a structure. Then a favourable inspection/maintenance strategy which minimizes the cost per unit of time with an acceptable risk for the structure can be chosen. The function for the expected cost can be set as:

$$E[C] = E[N_{in}]C_{in} + E[N_{mr}]C_{mr} + E[N_{lr}]C_{lr} + E[N_{cr}]C_{cr} \quad (13)$$

Where C_{in}: cost of inspection, C_{mr}: cost of medium repair (unit with medium damage), C_{lr}: cost of large repair (unit with large damage), C_{cr}: cost of critical repair (unit with critical damage), N_{in}: number of inspections per unit of time, N_{mr}: number of medium repairs per unit of time, N_{lr}: number of large repairs per unit of time, N_{lr}: number of large repairs per unit of time.

The mean number of inspections and repairs can be estimated from Monte Carlo simulations or expressed in terms of the marginal distributions of the PDMP and approximated with the numerical scheme. We look at a long time horizon for the expected cost to be considered asymptotic. For example, the mean number of medium repairs until t, corresponds to the mean number of times the process reaches the part of the frontier related to maintenance ($t = x_2$) with the discrete component $\mathbf{i} = (2, 2)$ until time t, which can be approximated by Eq. (14).

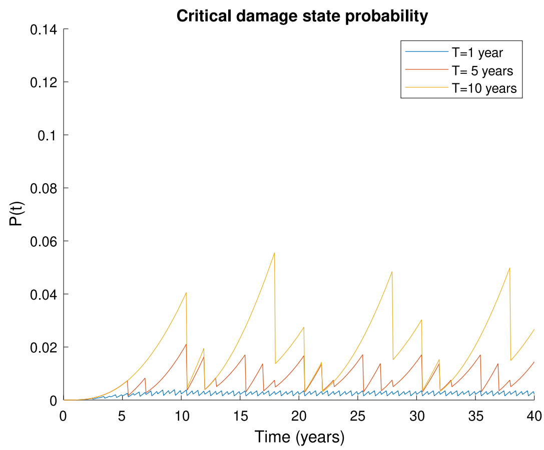


Fig. 7. Critical damage probability for different inspection intervals.

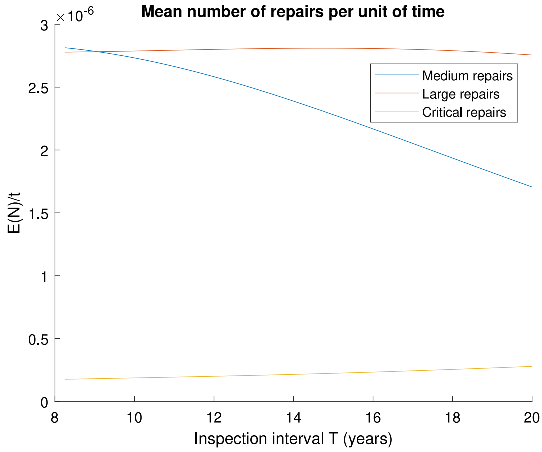


Fig. 8. Mean number of repairs per unit of time.

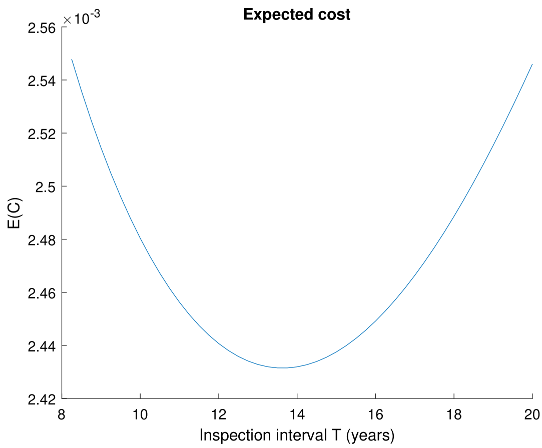


Fig. 9. Expected cost per unit of time.

$$N_{mr}(t) \approx \sum_{\substack{u=0 \\ \mathbf{x}\in\Gamma_2}}^{t} \pi_u\{(2,2),(\mathbf{x})\} \tag{14}$$

Where Γ_2 refers to the part of the frontier made by the points $\{x_2 = t\}$. To illustrate, we look at the expected cost per unit of time for different inspection intervals (keeping the parameters from Table 1 fixed, with the exemption of the inspection interval which is varied). We set symbolic values of $C_{in} = 50$, $C_{mr} = 100$, $C_{lr} = 250$, $C_{cr} = 5000$.

Fig. 8 shows the mean number of medium, large and critical repairs per unit of time for different inspection intervals and the resulting cost is shown in Fig. 9. In this case, the expected cost is lowest for an inspection interval $T = 13.75$ years or 13 years and 9 months.

5. Conclusion

A framework for CBM models with discrete-state stochastic deterioration has been proposed based on the theory of PDMPs. The proposed model allows to study problems in which the condition monitoring is not continuous but inspection-based and there is an inherent delay for performing maintenance actions. Therefore, the transition law cannot be found by a simple Markov process neither a semi-Markov one. Although this family of problems has been the motivation to propose PDMP, it is worth to mention that PDMPs have been introduced as general class of non-diffusion stochastic models and as such can cover a wide variety of applications that involve some combination of random jumps and deterministic motion.

The proposed framework allows the assessment of the probability for the infrastructure to be in a deterioration or a critical state given an inspection period and given a maintenance schedule. In this way, it is possible to evaluate if a given CBM policy is adequate regarding some safety requirements by making variations of the inspection period, the delay before intervention, or the state to which the system is restored after maintenance (imperfect maintenance). It is also possible to assess the cost of a CBM policy and find the optimal parameters of the policy.

A numerical approach for quantification of time dependent probabilities has been developed. This approach is an approximation to the solution of the Chapman–Kolmogorov equation. In comparison, Monte Carlo simulation is in general conceptually easier to apply while the numerical approach could provide better accuracy in the results with faster computation times. However, the system complexity and the number of discrete states can be limitations for this numerical approach while Monte Carlo simulation could offer more flexibility in this aspect. Given that the deterioration of the system can be characterized by a reasonable number of discrete states and that the deterministic motion is reduced to a trivial equation, it is relatively simple to make use of the numerical approach, making it a convenient alternative for problems which require studying different strategies and repeating the quantification procedure several times in order to support the decision-making.

A case study has been presented to illustrate the modelling and quantification approach. Through the case, guidelines on how to account for different assumptions about the inspection frequency and quality as well as maintenance strategies are given.

The proposed modelling framework presented, as well as much of the existing research on CBM focuses on a single-unit system. Moreover, multistate systems reliability theory usually deals with systems made of independent multistate components. An interesting direction of further works could be to study the application of PDMPs under the framework of multistate systems, exploring on the modelling of dependencies among the components such as stochastic, structural and/or economical. In this way, a decision-making process for maintenance at the system level can be considered.

CRediT authorship contribution statement

Renny Arismendi: Conceptualization, Methodology, Investigation, Software, Validation, Formal analysis, Writing - original draft, Visualization. **Anne Barros:** Conceptualization, Methodology, Validation, Formal analysis, Writing - review & editing, Visualization, Supervision. **Antoine Grall:** Validation, Formal analysis, Writing - review & editing, Visualization, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors would like to thank the Norwegian Public Roads Administration and the E39 Coastal Highway Route project for their contribution to the problem statement, granting access to their database and their involvement and discussions with experts. This work has been supported by the Norwegian University of Science and Technology.

References

[1] Arismendi R, Barros A, Vatn J, Grall A. Prognostics and maintenance optimization in bridge management. In: Proceedings of the 29th European safety and reliability conference (ESREL). Research Publishing; 2019.

[2] Zhang Y, Barros A, Rauzy A, Lunde E. A modelling methodology for the assessment of preventive maintenance on a compressor drive system. In: Safety and reliability–safe societies in a changing world. proceedings of ESREL 2018. Taylor & Francis; 2018.

[3] Srikanth I, Arockiasamy M. Deterioration models for prediction of remaining useful life of timber and concrete bridges–a review. J Traff Transp Eng (Engl Ed) 2020.

[4] Vegdirektoratet. Bruforvaltning. Handbook R411. Norway: Statens Vegvesen; 2014.

[5] Vegdirektoratet. Inspeksjonshåndbok for bruer. Handbook V441. Norway: Statens Vegvesen; 2014.

[6] Heimdal M. Remaining useful lifetime modeling of a compressor system [Master's thesis], Trondheim, Norway: Norwegian University of Science and Technology; 2017.

[7] Frangopol DM, Kallen M-J, Noortwijk JMv. Probabilistic models for life-cycle performance of deteriorating structures: review and future directions. Progr Struct Eng Mater 2004;6(4):197–212.

[8] Kallen MJ. Markov processes for maintenance optimization of civil infrastructure in the Netherlands. [Ph.D. thesis], Delft, Netherlands: Delft University of Technology; 2007.

[9] Zhang N, Fouladirad M, Barros A, Zhang J. Condition-based maintenance for a K-out-of-N deteriorating system under periodic inspection with failure dependence. European J Oper Res 2020.

[10] Hao S, Yang J, Berenguer C. Condition-based maintenance with imperfect inspections for continuous degradation processes. Appl Math Model 2020.

[11] Duan C, Li Z, Liu F. Condition-based maintenance for ship pumps subject to competing risks under stochastic maintenance quality. Ocean Eng 2020;218:108180.

[12] Fauriat W, Zio E. Optimization of an aperiodic sequential inspection and condition-based maintenance policy driven by value of information. Reliab Eng Syst Saf 2020;204:107133.

[13] Wang Y, Liu Y, Chen J, Li X. Reliability and condition-based maintenance modeling for systems operating under performance-based contracting. Comput Ind Eng 2020;142:106344.

[14] Alaswad S, Xiang Y. A review on condition-based maintenance optimization models for stochastically deteriorating system. Reliab Eng Syst Saf 2017;157:54–63.

[15] Tao W, Lin P, Wang N. Optimum life-cycle maintenance strategies of deteriorating highway bridges subject to seismic hazard by a hybrid Markov decision process model. Struct Saf 2021;89:102042.

[16] Compare M, Baraldi P, Marelli P, Zio E. Partially observable Markov decision processes for optimal operations of gas transmission networks. Reliab Eng Syst Saf 2020;106893.

[17] González-Domínguez J, Sánchez-Barroso G, García-Sanz-Calcedo J. Preventive maintenance optimisation of accessible flat roofs in healthcare centres using the Markov chain. J Build Eng 2020;32:101775.

[18] van Staden HE, Boute RN. The effect of multi-sensor data on condition-based maintenance policies. European J Oper Res 2020.

[19] Nguyen KT, Do P, Huynh KT, Bérenguer C, Grall A. Joint optimization of monitoring quality and replacement decisions in condition-based maintenance. Reliab Eng Syst Saf 2019;189:177–95.

[20] Davis MH. Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. J R Stat Soc Ser B Stat Methodol 1984;46(3):353–76.

[21] Jiang S, Li Y-F. The simulation and dynamic reliability estimation of multiple-crack system. In: 2020 Asia-Pacific international symposium on advanced reliability and maintenance modeling (APARM). 2020, p. 1–6. http://dx.doi.org/10.1109/APARM49247.2020.9209425.

[22] Lin Y-H, Li Y-F, Zio E. A comparison between Monte Carlo simulation and finite-volume scheme for reliability assessment of multi-state physics systems. Reliab Eng Syst Saf 2018;174:1–11.

[23] Demgne J, Mercier S, Lair W, Lonchampt J. Modelling and numerical assessment of a maintenance strategy with stock through piecewise deterministic Markov processes and quasi Monte Carlo methods. Proc Inst Mech Eng O 2017;231(4):429–45.

[24] Zhang H, Innal F, Dufour F, Dutuit Y. Piecewise deterministic Markov processes based approach applied to an offshore oil production system. Reliab Eng Syst Saf 2014;126:126–34.

[25] Lin Y, Li Y, Zio E. A framework for modeling and optimizing maintenance in systems considering epistemic uncertainty and degradation dependence based on PDMPs. IEEE Trans Ind Inf 2018;14(1):210–20. http://dx.doi.org/10.1109/TII.2017.2743820.

[26] Lair W, Mercier S, Roussignol M, Ziani R. Piecewise deterministic Markov processes and maintenance modeling: application to maintenance of a train air-conditioning system. Proc Inst Mech Eng O 2011;225(2):199–209.

[27] Vermes D. Optimal dynamic control of a useful class of randomly jumping processes. Professional Paper, Laxenburg, Austria: IIASA; 1980.

[28] Cocozza-Thivent C, Eymard R, Mercier S. A finite-volume scheme for dynamic reliability models. IMA J Numer Anal 2006;26(3):446–71.

[29] Cocozza-Thivent C, Eymard R, Mercier S, Roussignol M, et al. Characterization of the marginal distributions of Markov processes used in dynamic reliability. Int J Stoch Anal 2006;2006:1–18.

[30] Lair W, Ziani R, Mercier S, Roussignol M. Maintenance effect modelling and optimization of a two-components system. In: Advances in safety, reliability and risk management. CRC Press; 2011, p. 902–9.

Appendix C

Preventive Maintenance of a Compressor Station: a Modeling Framework for the Assessment of Performance

R. Arismendi, A. Barros, and A. Grall, “Preventive maintenance of a compressor station: A modeling framework for the assessment of performance,” *Proceedings of the 30th European Safety and Reliability Conference and the 15th Probabilistic Safety Assessment and Management Conference*, 2020

Preventive Maintenance of a Compressor Station: a Modeling Framework for the Assessment of Performance

Renny Arismendi

MTP, Norwegian University of Science and Technology, Norway. E-mail: renny.j.arismendi@ntnu.no

Anne Barros

LGI, CentraleSupélec, France. E-mail: anne.barros@centralesupelec.fr

Antoine Grall

LM2S, University of Technology of Troyes, France. E-mail: antoine.grall@utt.fr

Compressor stations are an essential part of natural gas networks, enabling the transportation of natural gas from producing wells to the final users. An efficient scheduling of maintenance tasks is key for ensuring high availability of the system and avoiding production losses and high costs from over expending. The demand of natural gas is highly dependent on weather and seasonal conditions, being higher during winter and lower during summer. Thus, the operation of the compressor stations alternates between periods of full capacity and periods of reduced capacity. The optimal planning of maintenance tasks needs to take into account such constraints from the production plans.
Currently, preventive maintenance tasks are usually carried out periodically, following a calendar-based strategy. This maintenance strategy can be improved by moving to predictive maintenance in which the decisions are based on prognostics that consider both the deterioration of the system and the production profile.
This work explores on the application of Piecewise-deterministic Markov Process (PDMP) as a framework for the optimization problem. The system level model captures the deterioration process of the units, the condition monitoring techniques, maintenance strategies and production profile in order to assess the performance of a maintenance strategy.

Keywords: Preventive maintenance, maintenance modeling, Piecewise-deterministic Markov process, numerical approach, compressor, natural gas.

1. Introduction

Norway is the European Union (EU) second largest gas supplier, covering 27% of the total gas imports of the EU during the third quarter of 2019, according to Commission (2019). The large majority of these gas exports from Norway to the EU is transported through pipelines. Pipeline networks are the preferred and most efficient method of transporting natural gas.

An integral part of a natural gas pipeline network are the compressor stations, which are strategically placed within the network with the function of maintaining the pressure and flow of gas, from the production sites to the end users (Messersmith et al. (2015)). Ensuring high availability of compressor stations is of key importance to avoid the large production losses associated to the network downtime.

Usually, high reliability is implemented by robust design at the system design phase. During the operation, rigorous maintenance policies are the means to ensure high availability of the compressor station. Traditionally, these maintenance policies consist on calendar-based preventive maintenance interventions, that may result in high costs from over expending. This situation could be improved by moving towards condition-based maintenance (CBM) policies and a decision process based on prognostics.

As pointed out by Kermanshachi et al. (2020), the studies on optimal natural gas pipeline maintenance strategies using reliability analysis are limited, leaving a knowledge gap and lack of predictive models to estimate major incidents in natural gas pipeline systems.

The objective of this paper is to propose a modeling framework for a CBM optimization problem of a gas compressor station, based on prognosis that take into account the condition monitoring techniques and the production profile. The case study is inspired by a real compressor station in Norway with modifications due to confidentiality.

1.1. *System description*

The compressor station to consider is composed of six compressor trains. Each compressor train

Proceedings of the 30th European Safety and Reliability Conference and the 15th Probabilistic Safety Assessment and Management Conference
Edited by Piero Baraldi, Francesco Di Maio and Enrico Zio

ISBN: 978-981-14-8593-0; doi:10.3850/978-981-14-8593-0

consist of a variable speed drive (VSD), an electrical motor (M), a gearbox (GB) and the gas compressor (C), as shown in figure 1.

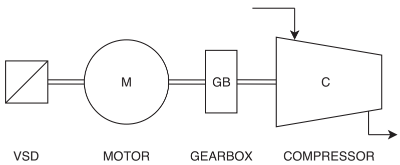


Fig. 1. Compressor train diagram

In a simplification of the production profile, a year can be divided into two seasons, based on the gas demand: winter (high demand) and summer (low demand). During winter, it is assumed that all the six compressor trains are required to operate in order to meet the market demand, while during summer season three out of the six compressor trains are enough to meet the requirements.

Assumptions about the lifetime of the units of the compressor station (e.g. Weibull distributed) and about the maintenance interventions (e.g. inspection based monitoring, deterministic delays for maintenance preparations) prevent the use of a Markov chain as modeling framework. A previous work on this system by Zhang et al. (2018), has explored on a modeling methodology based on object-oriented language and Montecarlo simulation for the assessment of a maintenance strategy. In our case, the units of the system are modelled as piecewise-deterministic Markov processes and the availability of the system is assessed by the structure function with support of a numerical scheme.

The rest of this paper is organized as follows: section 2 describes the framework of PDMP and the numerical scheme used for quantification, section 3 describes the assumptions and models of the compressor station units, section 4 presents numerical experiments and results and section 5 presents the conclusions and further works.

2. Modeling framework

A PDMP is an extension of a Markov chain that incorporates continuous states with evolution that follow discrete state-dependent deterministic differential equations. The resulting stochastic process is a Markov process with a mixture of random jumps and deterministic motion. They were introduced by Davis (1984), as a general class of non-diffusion stochastic models that provides a framework for studying optimization problems.

A PDMP is a hybrid process $\{I_t, X_t\}_{t>0}$ with values in a discrete-continuous space $E \times R$, as described by Lair et al. (2011, 2012). The first component I_t is discrete, with values in a finite state space E and corresponds to the unit condition. The second component X_t is continuous, taking values in a Borel subset $R \subset \mathbb{R}^k$, which in our case will be used to model a process which requires to keep track of time for interventions.

As described by Azaïs et al. (2014), the motion of a PDMP $\{I_t, X_t\}_{t>0}$ is determined by three local characteristics: the jump rate λ, the flow ϕ and the transition measure Q. The process starts from (i, x) and follows the flow $\phi(i, x)$ until a first jump occurs at T_1. A jump can occur either randomly with the rate $\lambda(i, x)$ or when the flow hits a boundary in the continuous state space R. The post-jump location of the jump at time T_1, denoted $Z_1 = \{I_{T_1}, X_{T_1}\}$ is selected from the transition measure $Q[(i, x), (\cdot, \cdot)]$ (conditional probability that the process jumps from (i, x) to a new location). Then the motion restarts from this point. This describes the evolution for $\{I_t, X_t\}$ with jump times T_k, post-jump locations Z_k and evolving according to the flow ϕ between two consecutive jumps. The flow of a PDMP is deterministic and in general described by differential equations.

Once the trajectory of a PDMP is described, the challenge is to solve for quantities of interest. Analytically solving the PDMP is a difficult task due to the complexity in the system behavior. In reliability assessments Monte Carlo simulation and a finite-volume numerical scheme are two common approaches used for solving such problems (Lin et al. (2018)). In this paper, we make use of the numerical scheme based on the finite-volume method.

2.1. *Numerical scheme for quantification*

The probability of the state of the system of a PDMP can be completely described by the Chapman-Kolmogorov equations, as demonstrated by Cocozza-Thivent et al. (2006). A numerical scheme based in finite-volume methods to approximate these probability measures has been proposed by Cocozza-Thivent et al. (2006), with proof of the convergence to the unique solution.

The principle of the scheme is the discretization of the continuous component X_t into cells. The time evolution of the probability masses in each cell of the environmental space is followed, and at each step, a balance between the in-coming and out-going probability masses is written, allowing us to solve a linear system, as Lair et al. (2012).

Let δ_t denote the step for the discretization of the continuous state space R and $\mathcal{M}$ be the resulting discrete state space. The part of $\mathcal{M}$ which forms the boundary is denoted Γ and the part which is not in the boundary is denoted $\tilde{\Gamma}$.

Let $\pi_t\{i, x\}$ denote the probability that the process state is $\{i, x\}$ at time t. By using the law of total probability and since the process is Markovian, the probability that the process is in state $\{j, y\}$ at time $(t + \delta_t)$ can be written as:

$$\begin{aligned}\pi_{t+\delta_t}\{j,y\} &= \sum_{\substack{i\in E\\ x\in\tilde{\Gamma}}} \pi_t\{i,x\}G^{\{j,y\}}_{\{i,x\}} \\ &+ \sum_{\substack{k\in E\\ z\in\Gamma}} \sum_{\substack{h\in E\\ w\in\tilde{\Gamma}}} \pi_t\{h,w\}G^{\{k,z\}}_{\{h,w\}}B^{\{j,y\}}_{\{k,z\}}\end{aligned} \tag{1}$$

Where $G^{\{j,y\}}_{\{i,x\}}$ and $G^{\{k,z\}}_{\{h,w\}}$ are the probabilities that the process moves from state $\{i, x\}$ to states $\{j, y\}$ and $\{k, z\}$ respectively, in the time interval $(t; t + \delta_t]$, and $B^{\{j,y\}}_{\{k,z\}}$ is the probability that the process moves from $\{k, z\}$ to $\{j, y\}$ instantaneously at time $(t + \delta_t)$ due to the process hitting the boundary.

These conditional probabilities are determined by the jump rate λ, the flow ϕ and the transition measure Q as described in the previous section. Since in our case, for every of the unit models, the flow of the process is linear, then $y = x + \phi(i)\delta_t$, where $\phi(i)$ is the speed of the linear motion for a fixed discrete component i. The conditional probabilities for the interval $(t, t + \delta_t]$ can be written according to possible transition scenarios in the interval, as:

(i) Random jump + flow ($y = x + \phi(i)\delta_t$)

$$G^{\{j,y\}}_{\{i,x\}} \approx \lambda(i,x)Q[(i,x),(j,y)]\delta_t$$

(ii) Only flow ($y = x + \phi(i)\delta_t$)

$$G^{\{j,y\}}_{\{i,x\}} \approx 1 - \lambda(i,x)\delta_t$$

(iii) Random jump + flow ($z = w + \phi(h)\delta_t$) + boundary jump ($z \in \Gamma$)

$$G^{\{k,z\}}_{\{h,w\}} \approx \lambda(h,w)Q[(h,w),(k,z)]\delta_t$$
$$B^{\{j,y\}}_{\{k,z\}} = Q[(k,z),(j,y)]$$

(iv) Flow ($z = w + \phi(h)\delta_t$) + boundary jump ($z \in \Gamma$)

$$G^{\{k,z\}}_{\{k,w\}} \approx 1 - \lambda(k,w)\delta_t$$
$$B^{\{j,y\}}_{\{k,z\}} = Q[(k,z),(j,y)]$$

For every unit of the system we propose a PDMP model which is determined by defining: the variables, the random jump rates, the flow and the transition measures.

3. System modeling

3.1. *VSD and gearbox*

3.1.1. *Assumptions*

The lifetime of the VSD and the gearbox are assumed to be Weibull distributed with shape parameter higher than one. A failure of these units is immediately detected and a corrective maintenance action begins. The duration of the maintenance action (ρ_{mn}) is deterministic and after the action the unit is considered to be as-good-as-new (AGAN). Only corrective maintenance is considered for these units.

3.1.2. *Variables, flow, jumps and transition measures*

The discrete component of the PDMP for these units is a binary variable, I, that indicates the condition of the unit: working (1) or failed (0) at time t. In this sense, $I_t = \{1, 0\}$.

The continuous component is a variable X whose value x corresponds to the amount of time spent in a given discrete state i at time t. In this way, when the unit is working, the failure rate λ is a function of x, and when the unit is in failed state, x corresponds to the time spent in maintenance. The complete process to consider is made of $\{I_t, X_t\}_{t>0}$.

Between two consecutive jumps, x evolves with a speed of one, i.e. $\phi(i, x) = 1, \ \forall i, t \geq 0$. The jump rate (failure) for this process is:

$$\lambda(1,x) = \frac{\alpha}{\mu}\left(\frac{x}{\mu}\right)^{(\alpha-1)} \tag{2}$$

Where α and μ are the shape and scale parameter of the Weibull distribution respectively. The measure of the random jump is $Q[(1, x), (0, 0)] = 1$.

In addition to the random failures, the process can jump at intervention times that correspond to a maintenance action finishing (after ρ_{mn}). The flow of the process is bounded with ρ_{mn} for $i = 0$ and not bounded for $i = 1$. The measure of a jump related to maintenance is $Q[(0, \rho_{mn}), (1, 0)] = 1$.

The non-null conditional transition probabilities for the numerical scheme can then be written as:

$$G^{\{(1,x+\delta_t)\}}_{\{(1,x))\}} \approx 1 - \lambda(1,x)\delta_t$$
$$G^{\{(0,0)\}}_{\{(1,x))\}} \approx \lambda(1,x)\delta_t$$
$$G^{\{(0,x+\delta_t)\}}_{\{(0,x))\}} = 1$$
$$B^{\{(1,0)\}}_{\{(0,\rho_{mn}))\}} = 1$$

The availability of the VSD or gearbox at time t is found by using eq. 1 recursively until time t, and:

$$A_{VSD}(t) = \sum_{x \in \mathcal{M}} \pi_t\{1, x\} \qquad (3)$$

3.2. *Motor*

3.2.1. *Assumptions*

The condition of the motor is described with three discrete states: new (2), degraded (1) and failed (0). The deterioration process is considered to be gradually increasing from (2) to (0) with constant transition rates λ_{21} and λ_{10}.

The motor is periodically inspected with an interval of τ. In case that degradation is detected during an inspection, a preventive maintenance is planned and a spare unit is ordered for the replacement. There is a delay for the maintenance action due to the preparation of the work and logistics involved. A failure of the motor is detected immediately and the spare unit for the corrective replacement is ordered (if it has not been ordered earlier as result of an inspection, shortening the delay).

The duration of the inspection, the delay for maintenance and the maintenance action are considered deterministic and last for ρ_{in}, δ_{mn} and ρ_{mn} units of time, respectively. During the delay for maintenance the unit is placed back into operation, while an inspection or maintenance action require to take the motor out of operation. After a maintenance action the motor is assumed to be AGAN.

3.2.2. *Variables, flow, jumps and transition measures*

The discrete component of the PDMP for the motor is composed of two variables I_a, I_b. The first one I_a indicates the condition of the unit: new (2), degraded (1) and failed (0) at time t, so: $I_{a_t} = \{2, 1, 0\}$. The second one I_b represents three distinct modes in which the unit can be: in operation (2), in inspection (1) or in maintenance (0), then: $I_{b_t} = \{2, 1, 0\}$.

The continuous component is composed of X_a which represents the date of a next inspection, X_b which represents the date of a next maintenance action and t stands for time. Then $X = (X_a, X_b, t)$. Viewed in another way, x_a and x_b represent boundaries for t, marking the time for an intervention jump.

Between two consecutive jumps, only the continuous variable t evolves with a speed of one, i.e. $\phi(\mathbf{i}, \mathbf{x}) = (0, 0, 1),\ \forall \mathbf{i}$

The process $\{(I_a, I_b), (X_a, X_b, t)\}$, experiences jumps at random times due to deterioration of the unit. We assume that the unit can only deteriorate while in operation mode, i.e. when $I_{b_t} = 2$. Then, the non-null random jump rates for this process are:

$$\lambda((2,2), \mathbf{x}) = \lambda_{21}$$
$$\lambda((1,2), \mathbf{x}) = \lambda_{10}$$

The transition measures of random jumps are:

$$Q[((2,2),(x_a,x_b,t)),((1,2),(x_a,x_b,t))] = 1$$
$$Q[((1,2),(x_a,x_b,t)),((0,2),(x_a,t+\delta_{mn},t))] = 1$$

The process also jumps at intervention times when $t = x_a$ or $t = x_b$. The transition measures related to an inspection ($t = x_a$) are:

$$Q[((i_a,2),(t,x_b,t)),((i_a,1),(t+\rho_{in},x_b,t))] = 1$$
$$Q[((2,1),(t,x_b,t)),((2,2),(t+\tau,x_b,t))] = 1$$
$$Q[((1,1),(t,x_b,t)),((1,2),(t+\tau,t+\delta_{mn},t))] = 1$$

The transition measures related to a maintenance action ($t = x_b$) are:

$$Q[((i_a,2),(x_a,t,t)),((i_a,0),(t,t+\rho_{mn},t))] = 1$$
$$Q[((i_a,0),(t,x_b,t)),((2,2),(t,\infty,t))] = 1$$

With the given information, the non-null conditional probabilities (G) of transitions in a time interval $(t, t+\delta_t]$ could be written (similarly to the case of the VSD and gearbox in section 3.1.2).

The availability of the motor at time t is found by using eq. 1 recursively until time t, and:

$$A_M(t) = \sum_{i_a=1}^{2} \sum_{\mathbf{x} \in \mathcal{M}} \pi_t\{(i_a, 2), \mathbf{x}\} \qquad (4)$$

3.3. *Compressor*

3.3.1. *Assumptions*

The condition of the compressor is described with four discrete states: new (3), low degradation (2), high degradation (1) and failed (0). The condition is monitored indirectly but continuously. The deterioration process is considered to be gradually increasing from (3) to (0) with constant transition rates λ_{32}, λ_{21} and λ_{10}.

When the condition of the compressor reaches the low degradation level (2), a minor preventive maintenance is planned and when it reaches the high degradation level (1) or failed level (0) a major preventive (or corrective) maintenance is planned. The duration of a minor preventive maintenance action is ρ_{mn} and the duration of a major preventive maintenance or corrective maintenance action is ρ_{mj}. The delay for maintenance due to preparation is δ_{mn} or δ_{mj} respectively. The duration of a maintenance action and the delay are considered deterministic. After a maintenance action the compressor is assumed to be AGAN.

3.3.2. *Variables, flow, jumps and transition measures*

The discrete component of the PDMP for the compressor is composed of two variables I_a, I_b. The first one I_a indicates the condition of the unit: new (3), low degradation (2), high degradation (1) and failed (0) at time t, so: $I_{a_t} = \{3, 2, 1, 0\}$. The second one I_b represents two distinct modes in which the unit can be: in operation (1) or in maintenance (0), then: $I_{b_t} = \{1, 0\}$.

The continuous component is composed of variable X whose value x denotes the date of a next maintenance, and t which stands for time. The process to consider is $\{(I_a, I_b), (X, t)\}$

Between two consecutive jumps, only the continuous variable t evolves with a speed of one, i.e. $\phi(\mathbf{i}, x) = (0, 1),\ \forall \mathbf{i}$

Considering that the compressor only deteriorates when in operation ($i_b = 1$), the process $\{(I_a, I_b), (X, t)\}$ experiences random jumps related to the deterioration process with jump rates:

$$\lambda((3,1), \mathbf{x}) = \lambda_{32}$$
$$\lambda((2,1), \mathbf{x}) = \lambda_{21}$$
$$\lambda((1,1), \mathbf{x}) = \lambda_{10}$$

The transition measures of random jumps are:

$$Q[((3,1),(x,t)),((2,1),(t+\delta_{mn},t))] = 1$$
$$Q[((2,1),(x,t)),((1,1),(t+\delta_{mj},t))] = 1$$
$$Q[((1,1),(x,t)),((0,1),(x,t))] = 1$$

The transition measures related to a maintenance action ($t = x$) are:

$$Q[((2,1),(t,t)),((2,0),(t+\rho_{mn},t))] = 1$$
$$Q[((1,1),(t,t)),((1,0),(t+\rho_{mj},t))] = 1$$
$$Q[((0,1),(t,t)),((0,0),(t+\rho_{mj},t))] = 1$$
$$Q[((i_a,0),(t,t)),((3,1),(\infty,t))] = 1$$

With the given information, the non-null conditional probabilities (G) of transitions in a time interval $(t, t+\delta_t]$ could be written (similarly to the case of the VSD and gearbox in section 3.1.2).

The availability of the compressor at time t is found by using eq. 1 recursively until time t, and:

$$A_C(t) = \sum_{i_a=1}^{3} \sum_{\mathbf{x}\in\mathcal{M}} \pi_t\{(i_a, 1), \mathbf{x}\} \qquad (5)$$

3.4. *System - compressor station*

Assuming that all units in a compressor train are independent, a compressor train is modelled with a series structure and its availability ($A_{CT}(t)$) is found by:

$$A_{CT}(t) = A_{VSD}(t) \cdot A_M(t) \cdot A_{GB}(t) \cdot A_C(t) \quad (6)$$

Where $A_{VSD}(t)$, $A_M(t)$, $A_{GB}(t)$ and $A_C(t)$ are the availability of the VSD, motor, gearbox and compressor given by equations 3, 4 and 5.

During winter season, the compressor system can be modelled with a series structure of six compressor trains, and during summer season, the system can be modelled as a 3-out-of-6 structure. Assuming that all compressor trains are identical and independent, the availability of the whole compression system ($A_{CS}(t)$) can be written as:

$$A_{CS}(t) = \begin{cases} A_{CT}(t)^6 & t \in HD \\ 1 - (1 - A_{CT}(t)^3)^{\binom{6}{3}} & t \in LD \end{cases} \quad (7)$$

Where HD stands for high demand season (winter) and LD stands for low demand season (summer).

4. Numerical experiment and results

The described numerical scheme is applied to find the time-dependent availability for each unit of the compressor train and equation 7 is used to find the time dependent availability for the whole compression station. A set of parameters chosen for illustration purposes only (not necessarily realistic) are shown in table 1.

Table 1. Model parameters

Parameter	VSD / Gearbox	Motor	Compressor
α	3		
$\mu\ (h)$	9×10^8		
$\lambda_{32}\ (h^{-1})$			1×10^{-6}
$\lambda_{21}\ (h^{-1})$		1×10^{-4}	1×10^{-5}
$\lambda_{10}\ (h^{-1})$		1×10^{-3}	1×10^{-4}
$\rho_{mn}\ (h)$	10	180	180
$\delta_{mn}\ (h)$		360	500
$\rho_{mj}\ (h)$			360
$\delta_{mj}\ (h)$			1000
$\tau\ (h)$		730	
$\rho_{in}\ (h)$		10	

It is assumed that the initial state for every unit is new and that the initial season is summer (low demand). Figure 2 shows the time-dependent availability for the whole compressor system during the first two years. From the figure it can be observed that the frequent periodic inspections of the motor are a contributor to the system unavailability, since the inspection requires to take the motor out of operation for its duration (ρ_{in}).

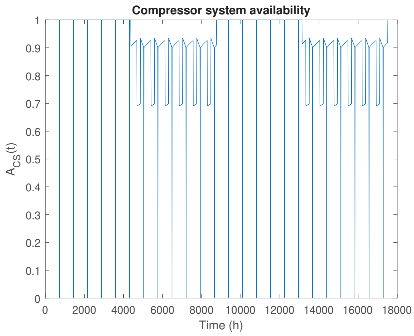


Fig. 2. Availability of the compressor system

To assess the impact of the motor inspection period in the system availability, different values of the inspection period are considered, $(180\,h \leq \tau \leq 2160\,h)$. Figures 3 and 4 show the average availability of the system with respect to the motor inspection period.

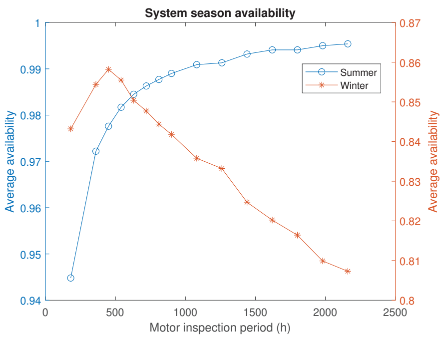


Fig. 3. Seasonal average availability of the compressor system

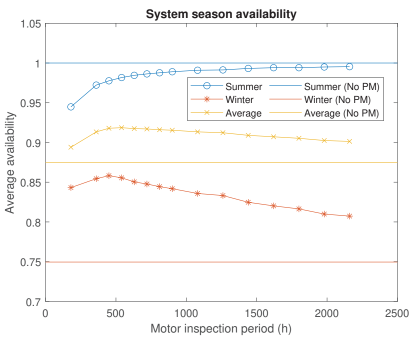


Fig. 4. Average availability of the compressor system

Figure 3 shows the impact of variations in the motor inspection period on the average availability of the system by season. Due to the compressor system structure, the average availability for summer is strictly increasing while the average availability for the winter has a maximum around $\tau = 450\,h$, within the given range of τ.

Figure 4 shows the average availability of the compressor system together with the seasonal availability. The horizontal lines show the seasonal and average availability of the system for a maintenance strategy of the motor consisting on only corrective maintenance (no inspections and no preventive maintenance interventions). Within the given range, the average availability for the system is maximum around $\tau = 550\,h$, indicating that the inspection interval is at its optimum at $\tau \approx 550\,h$.

Inspections and preventive maintenance of the motor are a strategy to increase the average availability of the system during winter season and as result, the overall average availability. However, during summer, the inspections and preventive maintenance of the motor may reduce the availability of the system when compared to an only corrective maintenance strategy, as shown for the given parameters. Cost analyses of maintenance strategies must be carried out in order to find optimal an optimal policy for the system, while considering the cost of downtime and production losses.

5. Conclusions and further works

Predictive maintenance is appealing because it allows to make maintenance decisions based not only on the current information about the system, but on projections of the condition of the system into the future, taking into account estimations of the future use profile.

In this paper, we propose PDMP models for each unit of a compressor station as an approach to model the deterioration of the unit with a stochastic process and maintenance interventions, considering different assumptions, such as an aging unit, maintenance delay and periodic inspections. The numerical scheme presented allows for the assessment of availability quantities in relatively short computing times and it is well suited for the optimization of maintenance policies.

For the compressor system, an illustration of finding an optimal inspection interval for the motor has been shown. However, there are limitations for the proposed model. First, all units are considered independent, both in terms of stochastic deterioration and for maintenance interventions. Second, the models at the unit level make no distinction of the production season profiles. Hence, there are no constrains about the availability of maintenance crews and system maintenance strategies like opportunistic maintenance have not been considered. In practice, any preventive maintenance intervention should be carried out during the summer season when the demand can be satisfied without all compressor trains in operation, while aiming to reduce the risk of production losses for the winter season. The authors intend to further explore on modifications to the proposed models to overcome these limitations. Estimating the remaining useful life of the system at the start of the winter could allow to make maintenance decisions for the summer. The optimization problem should also include a cost analysis that considers not only unit replacements but also minimal repairs.

References

Azaïs, R., J.-B. Bardet, A. Génadot, N. Krell, and P.-A. Zitt (2014). Piecewise deterministic markov process—recent results. In *ESAIM: Proceedings*, Volume 44, pp. 276–290. EDP Sciences.

Cocozza-Thivent, C., R. Eymard, and S. Mercier (2006). A finite-volume scheme for dynamic reliability models. *IMA journal of numerical analysis 26*(3), 446–471.

Cocozza-Thivent, C., R. Eymard, S. Mercier, M. Roussignol, et al. (2006). Characterization of the marginal distributions of markov processes used in dynamic reliability. *International Journal of Stochastic Analysis 2006*, 1–18.

Commission, E. (2019). *Quarterly Report on European Gas Markets*, Volume 13 (issue 3, third quarter of 2019). Market Observatory for Energy.

Davis, M. H. (1984). Piecewise-deterministic markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society. Series B (Methodological)*, 353–388.

Kermanshachi, S., B. Rouhanizadeh, M. M. Cobanoglu, and I. Damnjanovic (2020). Optimal pipeline maintenance strategies in the united states: Stochastic reliability analysis of gas pipeline network failures. *Journal of Pipeline Systems Engineering and Practice 11*(1), 04019041.

Lair, W., S. Mercier, M. Roussignol, and R. Ziani (2011). Piecewise deterministic markov processes and maintenance modeling: application to maintenance of a train air-conditioning system. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability 225*(2), 199–209.

Lair, W., R. Ziani, S. Mercier, and M. Roussignol (2012). Maintenance effect modelling and optimization of a two-components system. *Advances in Safety, Reliability and Risk Manag*.

Lin, Y.-H., Y.-F. Li, and E. Zio (2018). A comparison between monte carlo simulation and finite-volume scheme for reliability assessment of multi-state physics systems. *Reliability Engineering & System Safety 174*, 1–11.

Messersmith, D., D. Brockett, and C. Loveland (2015). Understanding natural gas compressor stations. *Penn State Extension*.

Zhang, Y., A. Barros, A. Rauzy, and E. Lunde (2018). A modelling methodology for the assessment of preventive maintenance on a compressor drive system. *Safety and Reliability–Safe Societies in a Changing World. Proceedings of ESREL 2018, June 17-21, 2018, Trondheim, Norway*.

Appendix D

A modelling framework for Condition-based Maintenance of systems with multi-state components - Application to a gas compression system

R. Arismendi, A. Barros, and A. Grall, “A modelling framework for condition-based maintenance of systems with multi-state components - application to a gas compression system,” *"Submitted" Reliability Engineering & System Safety*

This paper is awaiting publication and is not included in NTNU Open

ISBN 978-82-326-6614-0 (printed ver.)
ISBN 978-82-326-6721-5 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

NTNU
Norwegian University of
Science and Technology