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Abstract

The notion of cumulants plays a significant role in the combinatorial study of non-commutative probability theory. In this thesis, we study several problems associated with the notions of cumulants for free, Boolean and monotone independence via a unified Hopf algebraic framework for non-commutative probability. In this framework, developed by Ebrahimi-Fard and Patras [EFP15, EFP16, EFP18], the relations between moments and the different brands of cumulants are described by three different exponentials that relate the group of characters and the Lie algebra of infinitesimal characters on a particular word Hopf algebra H.

The first question that we address is to show how the algebraic central limit theorems for free, Boolean and monotone independence enter into the shuffle-algebraic picture for non-commutative probability. We analyze how the different notions of additive convolutions are described in the shuffle framework and prove combinatorial formulas for the powers of the half-shuffle products.

Next, we focus on extending the shuffle framework in order to consider the extension of non-commutative probability called infinitesimal non-commutative probability. We show how infinitesimal cumulants also correspond to infinitesimal characters on H and describe the shuffle-algebraic equations for the infinitesimal moment-cumulant relations. We also prove that the combinatorial relations between infinitesimal cumulants follow via the extended shuffle-algebraic framework.

Afterwards, we concentrate on the problem of writing the multivariate monotone cumulants of random variables in terms of their moments. The starting point to obtain this formula is to compute a logarithm of a certain character on H. Then by investigating a connection of H with a Hopf algebra of decorated rooted trees, we compute the logarithm instead in the Hopf algebra of trees, which yields a new combinatorial formula from moments to monotone cumulants in terms of Schröder trees.

Thereafter, we turn to the problem, left open in [AHLV15], of expressing multivariate monotone cumulants in terms of the free and Boolean cumulants. Our approach relies on a pre-Lie algebraic relation between the three infinitesimal characters associated to the free, Boolean and monotone cumulants. This relation is known as the pre-Lie Magnus expansion and is defined in terms of iterations of a pre-Lie product. Our problem is then transformed into finding a combinatorial formula in terms of non-crossing partitions of the iterated pre-Lie products and identifying and describing the coefficients that govern the transition from free and Boolean to monotone cumulants.

The coefficients obtained in the solution of the previous problem hint at a deeper relation between non-commutative probability, combinatorics of rooted trees and free pre-Lie algebras. Our last question is to have a more systematic understanding of the previous relation. In the process, we develop a concrete and effective method of computations on a specific class of pre-Lie algebras, where iterations of the pre-Lie product can be computed in terms of forest formulas for iterated coproducts. We illustrate our method by retrieving the formulas between monotone and free (Boolean) cumulants, as well as the computation of the pre-Lie Magnus expansion on the generator of the free pre-Lie algebra of rooted trees.

Preface

This thesis is submitted in partial fulfilment of the requirements for the degree of Philosophiae Doctor (PhD) in Mathematical Sciences at the Norwegian University of Science and Technology (NTNU). The research presented here was conducted at the Department of Mathematical Sciences at NTNU, under the supervision of Professor Kurusch Ebrahimi-Fard as the main supervisor and Professor Frédéric Patras as the co-supervisor.

The thesis is a monograph consisting of nine chapters. The first four chapters provide the background and the framework of Hopf and pre-Lie algebras, as well as of non-commutative probability theory for the results of the thesis. The last five chapters explain in detail the results of the joint works [CEFP21, AC21, CEFPP21, CP22] on which the author has worked during his PhD studies.

Acknowledgements

Along these lines, I would like to thank all those who in some way contributed to the realization of this project. This manuscript represents the culmination of a long process that began when I discovered the beauty of mathematics and decided to prepare myself to practice it professionally.

First of all, I would like to express my gratitude to my supervisor, Kurusch Ebrahimi-Fard, for his advice throughout the years of my doctoral studies. I want to thank him for all the insightful discussions and for showing me how to conduct mathematical research. I want to extend my gratitude to my co-supervisor, Frédéric Patras, for his advice and for making my stay in Nice very productive.

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Para mis padres

Carlos y Verónica,

y mi pequeña familia

Gabriela y Rebecca

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Chapter 1

Introduction

1.1 Framework of the thesis

Free Probability Theory [NS06, Voi95, VDN92] was introduced by Dan Voiculescu in the 1980s, aiming to solve the problem of isomorphisms between von Neumann algebras generated by free groups. Later in the 1990s, Roland Speicher [Spe94] proved from a combinatorial point of view that the transition from classical probability to free probability consists in replacing the lattice of all set partitions with the lattice of non-crossing set partitions. For this purpose, Speicher introduced the notion of free cumulant functionals and found fundamental relations between free probability and the combinatorics of non-crossing partitions.

In addition to the notion of free independence, some other notions of independence can be considered: the Boolean independence defined in [SW97] and the monotone independence introduced in [Mur00]. Analogously, the notion of cumulants can be defined for each Boolean [SW97], and monotone independence [HS11b]. Each of these notions of non-commutative cumulants has proven to be a major tool in the combinatorial study of non-commutative probability, where we can find applications, for instance, in the study of non-commutative convolutions of distributions.

The present thesis treats the notion of cumulants in non-commutative probability via a recent Hopf-algebraic point of view. The main contributions are the understanding of several notions in non-commutative probability in the Hopf-algebraic framework as well as the obtainment of new combinatorial formulas that relate different brands of cumulants.

In a series of papers, Ebrahimi-Fard and Patras [EFP15, EFP16, EFP18, EFP19, EFP20] have developed a group-theoretical framework where the relations between moments and the free, monotone, and Boolean cumulants can be explained analogously to the link between a Lie algebra and its corresponding Lie group.

The approach of Ebrahimi-Fard and Patras is based on the identification of a graded, connected, non-commutative, non-cocommutative Hopf algebra, (H, m, Δ), that extends a given non-commutative probability space $(\mathcal{A}, \varphi)$, i.e. $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and

φ is a linear functional on $\mathcal{A}$ such that $\varphi(1_{\mathcal{A}}) = 1$. More precisely, the Hopf algebra H is given by the *double tensor algebra on* $\mathcal{A}$:

$$H := T(T_+(\mathcal{A})) := \bigoplus_{n\geq 0} T_+(\mathcal{A})^{\otimes n}, \quad \text{where} \quad T_+(\mathcal{A}) = \bigoplus_{n\geq 1} \mathcal{A}^{\otimes n}.$$

The main feature of the Hopf algebra H is that its coproduct splits into two half-unshuffle coproducts, $\Delta_{\prec}$ and $\Delta_{\succ}$, which provide H with the structure of *unshuffle bialgebra* [Foi07]. Moreover, these two half-unshuffle coproducts induce a splitting of the convolution product $*$ in the dual space H^* into a sum of two products, denoted by $\prec$ and $\succ$. It turns out that $(H^*, \prec, \succ)$ is a *non-commutative shuffle algebra*, also known as *dendriform algebras*, characterized through the so-called *shuffle identities*:

$$\begin{aligned}
(f \prec g) \prec h &= f \prec (g * h)\\
(f \succ g) \prec h &= f \succ (g \prec h)\\
f \succ (g \succ h) &= (f * g) \succ h,
\end{aligned}$$

for any $f, g, h \in H^*$, where the convolution product writes $f * g = f \prec g + f \succ g$.

Besides the convolution exponential $\exp^*$, in any shuffle algebra, we can consider the half-shuffle exponentials $\mathcal{E}_{\prec}$ and $\mathcal{E}_{\succ}$ with respect to the half-shuffle products $\prec$ and $\succ$, respectively. It turns out that each of the three exponentials provides a bijection between the Lie algebra of infinitesimal characters on H, $\mathfrak{g}$, and the group of characters on H, G. More precisely, if $\Phi \in G$ is a character, then there exists a unique triple of infinitesimal characters $(\kappa, \beta, \rho) \in \mathfrak{g}^3$ such that

$$\Phi = \mathcal{E}_{\prec}(\kappa) = \mathcal{E}_{\succ}(\beta) = \exp^*(\rho). \tag{1.1.1}$$

The link with non-commutative probability, established in [EFP15, EFP18], can be described as follows: if $(\mathcal{A}, \varphi)$ is a non-commutative probability space, we consider the double tensor algebra $H = T(T_+(\mathcal{A}))$ as well as the character $\Phi : H \to \mathbb{C}$ given by

$$\Phi(w) := \varphi(a_1 \cdot_{\mathcal{A}} \dots \cdot_{\mathcal{A}} a_n),$$

for any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. Notice that on the right-hand side of the above equation, the argument of φ is the product on $\mathcal{A}$ of the elements $a_1, \dots, a_n \in \mathcal{A}$. Then, it turns out that the infinitesimal characters given by the three logarithms κ, β and ρ of Φ can be identified with the cumulants on $(\mathcal{A}, \varphi)$. More precisely, if $\{k_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ are the families of free, Boolean and monotone cumulants, respectively, then we have that

$$\kappa(w) = k_n(a_1, \dots, a_n), \quad \beta(w) = b_n(a_1, \dots, a_n), \quad \rho(w) = h_n(a_1, \dots, a_n),$$

for any $n \geq 1$ and $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. In particular, the evaluation of Equation (1.1.1) on w originates the so-called *moment-cumulant relations*:

$$\begin{aligned}
\varphi(a_1 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n) &= \Phi(w) \\
&= \mathcal{E}_{\prec}(\kappa)(w) = \sum_{\pi \in \mathrm{NC}(n)} k_\pi(a_1, \ldots, a_n) \\
&= \mathcal{E}_{\succ}(\beta)(w) = \sum_{\pi \in \mathrm{Int}(n)} b_\pi(a_1, \ldots, a_n) \\
&= \exp^*(\rho)(w) = \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} h_\pi(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}(n)} \frac{1}{t(\pi)!} h_\pi(a_1, \ldots, a_n),
\end{aligned}$$

for any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. Here, $\mathrm{NC}(n)$, $\mathrm{Int}(n)$ and $\mathcal{M}(n)$ stand for the posets of non-crossing partitions, interval partitions and monotone partitions on the set $[n] := \{1, \ldots, n\}$, respectively (see Section 3.4 for the precise definition of the above sets of partitions).

The aim of the thesis is to understand and further develop non-commutative probability theory from the shuffle-algebraic point of view initiated in Ebrahimi-Fard and Patras' work. More precisely, we focus on the following problems through Hopf- and pre-Lie-algebraic techniques:

i) Understanding non-commutative central limit theorems in the shuffle-algebraic framework for non-commutative probability. To this end, we will prove some combinatorial formulas for the iterations of the half-shuffle products of infinitesimal cumulants. This is explained in Chapter 5.

ii) Extending the shuffle-algebraic framework to consider the notion of *infinitesimal non-commutative probability*. We will obtain shuffle identities that describe the infinitesimal moment-cumulant relations as well as combinatorial formulas that relate the different brands of infinitesimal cumulants. This is studied in Chapter 6.

iii) Obtaining a combinatorial formula that writes monotone cumulants of random variables in terms of their moments. Our main tools will be a Hopf algebra based on Schröder trees and a link between such Hopf algebra and the double tensor algebra. This is done in Chapter 7.

iv) Achieving a combinatorial formula, in terms of non-crossing partitions, that expresses monotone cumulants in terms of free and Boolean cumulants. We will describe a shuffle relation between these brands of cumulants in terms of non-crossing partitions originated by iterations of a pre-Lie product on the Lie algebra of infinitesimal characters. The latter is described in detail in Chapter 8.

v) Developing a method to compute the iterations of pre-Lie and brace products via forest formulas for iterated coproducts of the dual of enveloping algebras of certain pre-Lie algebras. By taking the particular case of a pre-Lie algebra of words, we will retrieve the formulas obtained in Chapter 8. Nevertheless, our general method provides further insights into the connections between non-commutative probability, free pre-Lie algebras, and combinatorics of rooted trees. This is exhibited in Chapter 9.

We would like to mention that Ebrahimi-Fard and Patras' work is not the only one that exploits a Hopf algebra approach or, more generally, a higher-algebraic-structure-based approach to non-commutative probability theory. Indeed, several works in this direction have appeared in recent years [MN10, FM15, DCPT15, MS17a]. A further natural question is investigating the possible connections between the different algebraic approaches to non-commutative probability.

1.2 Statements of the results

A more precise description of the main results of the present thesis is provided as follows.

Shuffle-algebraic Central Limit Theorems

The *central limit theorem* (CLT) in probability theory is a fundamental result that states the convergence in distribution of a scaled sum of independent identically distributed real random variables to the Gaussian distribution. When passing to a non-commutative framework $(\mathcal{A}, \varphi)$, convergence in distribution of a sequence of random variables is characterized by the convergence of their moments. These types of central limit theorems are called *algebraic* since we only assume the existence of the moments of the random variables.

The work of Giri and von Waldenfels [GvW78] is one of the first articles dealing with a proof of an algebraic multivariate central limit theorem for tensor independence. Later, in the work of Speicher and von Waldenfels [SvW94], a general central limit theorem is obtained in the sense that the independence conditions on the random variables are replaced by several general assumptions. In particular, every notion of natural independence satisfies such conditions yielding a different limiting distribution.

Chapter 5 of the thesis treats how the central limit theorem for the free, Boolean and monotone independence can be analyzed in the shuffle-algebraic framework for non-commutative probability. More generally, this problem can be addressed by identifying how the respective additive convolutions enter into the picture. For the free and Boolean cases, the characterization of independence in terms of the vanishing mixed cumulants conditions drives to define the characters

$$\Phi_1 \boxplus \Phi_2 := \mathcal{E}_{\prec}(\kappa_1 + \kappa_2), \qquad \Phi_1 \uplus \Phi_2 := \mathcal{E}_{\succ}(\beta_1 + \beta_2),$$

for any characters $\Phi_i = \mathcal{E}_{\prec}(\kappa_i) = \mathcal{E}_{\succ}(\beta_i)$ for $i = 1, 2$. If in particular Φ_1 and Φ_2 are the liftings of linear functionals φ_1 and φ_2, the result in [EFP19, Thm. 32] (Theorem 5.1.2) implies that $\Phi_1 \boxplus \Phi_2$ and $\Phi_1 \uplus \Phi_2$ are the liftings of the free and Boolean product of φ_1 and φ_2, respectively. Remarkably, the lifting monotone product of $\varphi_1 \blacktriangleright \varphi_2$ is given by the convolution product $\Phi_1 * \Phi_2$. Thus, by considering the corresponding scaling map $\Lambda_t(w) = t^{|w|}w$ for any $w \in T(T_+(\mathcal{A}))$ and $t \in \mathbb{R}$, we have the statements of the corresponding shuffle-algebraic central limit theorems.

Theorem (Theorem 5.2.3, Theorem 5.3.5)**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Let $\{a_i\}_{i\geq 1} \subset \mathcal{A}$ be a sequence of random variables whose distribution is extended to a character Φ on $T(T_+(\mathcal{A}))$. Also, assume that $\Phi(a_i) = 0$, for any $i \geq 1$. If ψ is the infinitesimal character on $T(T_+(\mathcal{A}))$ given by*

$$\psi(w) = \begin{cases} \Phi(a_i a_j) & \text{if } w = a_i a_j \text{ for some } i, j \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned} \lim_{m\to\infty} \Phi^{*m} \circ \Lambda_{m^{-1/2}} &= \exp^*(\psi), \\ \lim_{m\to\infty} \Phi^{\boxplus m} \circ \Lambda_{m^{-1/2}} &= \mathcal{E}_{\prec}(\psi), \\ \lim_{m\to\infty} \Phi^{\uplus m} \circ \Lambda_{m^{-1/2}} &= \mathcal{E}_{\succ}(\psi). \end{aligned}$$

In particular, $\Phi^{*m} \circ \Lambda_{m^{-1/2}}$, $\Phi^{\boxplus m} \circ \Lambda_{m^{-1/2}}$ and $\Phi^{\uplus m} \circ \Lambda_{m^{-1/2}}$ converge, as $m \to \infty$, to the lifting of the distribution of an arcsinal family, resp. semicircular family, resp. Bernoulli family, of covariance $\big(\Phi(a_i a_j)\big)_{i,j\geq 1}$.

The precise definitions of an arcsinal family, semicircular family and Bernoulli family are given in Chapter 5. The strategy to prove the above result depends on the type of cumulants due to the different properties that satisfy the coproducts Δ, $\Delta_{\prec}$ and $\Delta_{\succ}$. Actually, for the monotone case, we exploit the fact that Δ is coassociative. Then, we analyze which terms in the expansion of the iterated coproduct remain in the limit. In contrast, the free and Boolean cases are proved by taking advantage of the fact that $\Phi^{\boxplus m} = \mathcal{E}_{\prec}(m\kappa)$ and $\Phi^{\uplus m} = \mathcal{E}_{\succ}(m\beta)$, and proving that the contribution of the partitions with exactly s blocks in the corresponding moment-cumulant relation can be described by the s-th iteration of the respective half-unshuffle coproducts (Lemma 5.3.3, Lemma 5.3.4)

$$\kappa^{\prec s}(w) = \sum_{\pi \in \mathrm{NC}^s(n)} \kappa_\pi(w), \qquad \beta^{\succ s}(w) = \sum_{\pi \in \mathrm{Int}^s(n)} \beta_\pi(w),$$

for any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$.

Infinitesimal Non-commutative Probability via Shuffle Algebras

Chapter 6, based on the joint work [CEFP21], is dedicated to understanding the infinitesimal version of non-commutative probability in the shuffle-algebraic framework of Ebrahimi-Fard and Patras. The theory of infinitesimal free probability is an extension of free probability introduced in the work [BGN03] under the name of *free probability of type B*. The authors' motivation for the previous article was to create a theory of free probability where the combinatorial role of the non-crossing partitions that arise as the non-crossing partitions associated to Coxeter groups of type A is taken by the non-crossing partitions of type B [Rei97].

In a later work, Belinschi and Shlyakhtenko [BS12] introduced the theory of infinitesimal free probability from an analytic point of view and gave analytical tools to treat the free additive convolution of type B. Infinitesimal free probability also has relevant applications in random matrix theory. More specifically, in the study of outliers in the spectra of random matrix models with finite rank perturbations [Shl18], as well as families of matrices with discrete spectrum in the limit [CHS18].

On the other hand, Février and Nica [FN10] introduced a notion of infinitesimal free cumulants in such a way that the notion of infinitesimal free independence is characterized by the vanishing mixed cumulant condition on the free and infinitesimal free cumulants. The notion of infinitesimal freeness can be further extended to consider *higher order infinitesimal freeness* [Fév12] as well as considering the other notions of natural independence [Has11].

The basic object in infinitesimal free probability is the infinitesimal non-commutative probability space. It consists of a triple $(\mathcal{A}, \varphi, \varphi')$ such that $(\mathcal{A}, \varphi)$ is a non-commutative probability space and φ' is a linear functional on $\mathcal{A}$ such that $\varphi'(1_{\mathcal{A}}) = 0$. The key idea in [FN10] in order to define infinitesimal free cumulants is to jointly consider φ and φ' as a $\mathbb{C}$-linear map $\tilde{\varphi}$ from $\mathcal{A}$ to a certain commutative algebra $\mathbb{G} \cong \mathbb{C} \oplus \hbar\mathbb{C}$, where $\hbar^2 = 0$. Then, taking $\tilde{\varphi}$ in the moment-free cumulant formula yields to a family of $\mathbb{G}$-valued cumulants $\tilde{k}_n = k_n + \hbar k'_n$ from which one can obtain the respective infinitesimal free cumulants k'_n. In general, if we have a combinatorial result in free probability, we may have the infinitesimal analogue by considering the $\mathbb{G}$-valued version of the result and then taking the $\hbar$-component. In Chapter 6, we apply this idea for Boolean and monotone cumulants in the shuffle-algebraic framework.

To be more precise, in Section 6.2, we consider a $\mathbb{G}$-valued character $\tilde{\Phi}$ given by the multiplicative and linear extension of $\tilde{\varphi}$ to a map from $T(T_+(\mathcal{A}))$ to $\mathbb{G}$. By looking at the $\hbar$-coordinate, we obtain conditions for an appropriate lifting Φ' of φ'. Furthermore, using the fact that the three exponential bijections are valid when we consider $\mathbb{G}$-valued characters and infinitesimal characters for any commutative algebra $\mathbb{G}$, in Proposition 6.2.6 we obtain that the $\hbar$-components (κ', β', ρ') of the triple of $\mathbb{G}$-valued infinitesimal characters $(\tilde{\kappa}, \tilde{\beta}, \tilde{\rho})$

such that

$$\tilde{\Phi} = \mathcal{E}_{\prec}(\tilde{\kappa}) = \mathcal{E}_{\succ}(\tilde{\beta}) = \exp^*(\tilde{\rho}),$$

are actually the infinitesimal lifting of the infinitesimal free, infinitesimal Boolean, and infinitesimal monotone cumulants $\{k'_n\}_{n\geq 1}$, $\{b'_n\}_{n\geq 1}$ and $\{h'_n\}_{n\geq 1}$, respectively. The corresponding shuffle relations between Φ', κ' and β' are given by

$$\Phi = \epsilon + \kappa \prec \Phi, \qquad \Phi' = \kappa' \prec \Phi + \kappa \prec \Phi',$$

$$\Phi = \epsilon + \Phi \succ \beta, \qquad \Phi' = \Phi \succ \beta' + \Phi' \succ \beta.$$

More interestingly, the fact that $\hbar^2 = 0$ and the shuffle properties allow us to obtain the following non-trivial shuffle relations that describe the infinitesimal moment-cumulant relations in terms of the convolution product.

Theorem (Theorem 6.2.7). *Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space, and let $\tilde{\Phi} = \Phi + \hbar\Phi'$ be the corresponding $\mathbb{G}$-valued lifting of φ and φ' to a $\mathbb{G}$-valued character on $T(T_+(\mathcal{A}))$. Consider the pairs of infinitesimal characters $(\kappa, \kappa'), (\beta, \beta')$ and (ρ, ρ'). Then we have*

$$\begin{aligned} \Phi' &= \Phi * \theta_\kappa(\kappa') \\ &= \theta_{-\beta}(\beta') * \Phi \\ &= \Phi * W_{-\rho}(\rho'). \end{aligned}$$

As a second application of the shuffle-algebraic framework, in Section 6.3 we derive, via purely shuffle algebra relations, the corresponding combinatorial relations between the infinitesimal cumulants, stated in the following result.

Theorem (Theorem 6.3.1). *Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space. The following relations between the infinitesimal free, infinitesimal Boolean and infinitesimal monotone cumulants $\{k'_n\}_{n\geq 1}$, $\{b'_n\}_{n\geq 1}$ and $\{h'_n\}_{n\geq 1}$, respectively, hold:*

$$\begin{aligned} b'_n(a_1, \dots, a_n) &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \partial k_\pi(a_1, \dots, a_n), \\ k'_n(a_1, \dots, a_n) &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1} \partial b_\pi(a_1, \dots, a_n), \\ b'_n(a_1, \dots, a_n) &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!} \partial h_\pi(a_1, \dots, a_n), \\ k'_n(a_1, \dots, a_n) &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{(-1)^{|\pi|-1}}{t(\pi)!} \partial h_\pi(a_1, \dots, a_n), \end{aligned}$$

for any $n \geq 1$ and $a_1, \dots, a_n \in \mathcal{A}$.

We finish the chapter in Section 6.4 by describing the infinitesimal analogue of the algebraic version of the Boolean Bercovici-Pata bijection ([BPB99]). In a few words, the map $\mathcal{B}$ sending a distribution μ with Boolean cumulants $\{b_n\}_{n\geq 1}$ to a distribution $\mathcal{B}(\mu)$ whose free cumulants are $\{b_n\}_{n\geq 1}$, we obtain a special bijection between the set of (analytic) distributions and the set of distributions that are *freely infinitely divisible* ([NS06, Lec. 13]). In the shuffle-algebraic framework, the map $\mathbb{B}$ can be defined by

$$\mathbb{B}(\Phi) = \mathcal{E}_{\prec}(\mathcal{L}_{\succ}(\Phi)),$$

for any character $\Phi : T(T_+(\mathcal{A})) \to \mathbb{C}$. In order to describe its infinitesimal analogue, we extend the above definition to the $\mathbb{G}$-valued case. The main result of the section is the $\mathbb{G}$-valued analogue of the semigroup property described in [BN08b], and we prove the result via purely shuffle relations.

A Monotone Cumulant-Moment Formula via Schröder Trees

The moment-cumulant relations for free and Boolean cumulants on a non-commutative probability space $(\mathcal{A}, \varphi)$ have the special property that the formulas can be inverted. The inversion is done via general Möbius inversion on the lattices of non-crossing and interval partitions, respectively, to obtain combinatorial formulas that write cumulants of random variables in terms of products of their moments:

$$\begin{aligned} k_n(a_1, \ldots, a_n) &= \sum_{\pi \in \mathrm{NC}(n)} \mu_{\mathrm{NC}(n)}(\pi, 1_n)\varphi_\pi(a_1, \ldots, a_n), \\ b_n(a_1, \ldots, a_n) &= \sum_{\pi \in \mathrm{Int}(n)} (-1)^{|\pi|-1}\varphi_\pi(a_1, \ldots, a_n), \end{aligned}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$, where $\mu_{\mathrm{NC}(n)}$ is the Möbius function on $\mathrm{NC}(n)$. However, inverting the moment-monotone cumulant formula via Möbius inversion does not work as in the other cases. The reason is that we have a coefficient $\frac{1}{t(\pi)!}$ for each $\pi \in \mathrm{NC}(n)$ that does not satisfy the multiplicativity condition on the blocks of π that is required to perform Möbius inversion.

Chapter 7, which is based on the joint work [AC21], aims to attack the problem of inverting the moment-cumulant formula. More precisely, we want to describe the family of coefficients $\{\alpha(\pi)\}_{\pi \in \cup_{n\geq 1} \mathrm{NC}(n)}$ which describe the transition from moments to monotone cumulants:

$$h_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}(n)} \alpha(\pi)\varphi_\pi(a_1, \ldots, a_n). \tag{1.2.1}$$

For this purpose, our main combinatorial object will be an algebraic structure associated to a particular type of planar rooted trees known as *Schröder trees*. To be precise, a Schröder tree is a planar rooted tree such that every internal vertex has at least two children. The aforementioned algebraic structure, described in Section 7.1, is a decorated and non-commutative version of the well-known Connes-Kreimer Hopf algebra of rooted trees, introduced by Foissy in his work [Foi02].

The motivation to consider the Hopf algebra on Schröder trees, denoted by $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$, comes from the work of Josuat-Vergès et al. [JVMNT17], where the authors employed an operad of Schröder trees to describe the functional equation between the power series of moments and free cumulants. Most important for us, the authors of [JVMNT17] also introduced an unshuffle bialgebra structure on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ which turns out to be related to the double tensor Hopf algebra $T(T_+(\mathcal{A}))$ through an unshuffle bialgebra morphism $\iota : T(T_+(\mathcal{A})) \to \mathcal{H}_{\mathcal{S}}(\mathcal{A})$ given by

$$\iota(w) = \sum_{t \in \mathrm{ST}(|w|)} t \otimes w,$$

where $w \in T_+(\mathcal{A})$ and $\mathrm{ST}(n)$ stands for the set of Schröder trees with $n+1$ leaves. By extending the linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ to a character $\hat{\Phi}$ on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ in a particular way, the authors of [JVMNT17] described the infinitesimal character $\hat{\kappa}$ such that $\hat{\Phi} = \mathcal{E}_{\prec}(\hat{\kappa})$. Moreover, the authors showed that $\hat{\kappa}$ describes the free cumulants as a sum of moments indexed over Schröder trees.

The strategy in Chapter 7 consists of considering the other two infinitesimal characters $\hat{\beta}$ and $\hat{\rho}$ such that $\tilde{\Phi} = \mathcal{E}_{\succ}(\hat{\beta}) = \exp^*(\hat{\rho})$. In contrast to the fact that analyzing the shuffle equation $\rho = \log^*(\Phi)$ on $T(T_+(\mathcal{A}))$ does not produce a clear combinatorial description of the $\alpha(\pi)$ coefficients in (1.2.1), the equation $\hat{\rho} = \log^*(\hat{\Phi})$ has a nice expression that yields to the main result of the chapter.

Theorem (Theorem 7.3.13)**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and consider $\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n \geq 1}$ the family of monotone cumulants of $(\mathcal{A}, \varphi)$. Then we have that*

$$h_n(a_1, \ldots, a_n) = \sum_{t \in \mathrm{ST}(n)} \omega(\mathrm{sk}(t)) \varphi_{\pi(t)}(a_1, \ldots, a_n),$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$.

The corresponding coefficients ω are indeed defined on any rooted tree and are known as the *Murua coefficient*. The precise definition of these coefficients is given in Definition 7.3.2. Murua coefficients have appeared in the work of Murua [Mur06]. They are in connection with the Baker-Campbell Hausdorff problem, the combinatorics of rooted trees, and free pre-Lie algebras, as we will see in the subsequent Chapter 8 and Chapter 9.

As we have mentioned, in the process of obtaining the main formula in Theorem 7.3.13, we obtained a description of the monotone cumulants in terms of the infinitesimal character $\hat{\rho}$. This description provides the relation to Schröder trees and is a similar result to the main theorem in [JVMNT17]. Completing the picture by analyzing the Boolean case, we obtain the following result, which allows us to write cumulants as a sum of moments indexed by Schröder trees.

Theorem (Theorem 7.3.17)**.** *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and consider $\hat{\Phi}:\mathcal{H}_{\mathcal{S}}(\mathcal{A})\to\mathbb{C}$ the lifting of φ to a character on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$. Let $(\hat{\kappa},\hat{\beta},\hat{\rho})$ be the triple of infinitesimal characters on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ satisfying the identities*

$$\hat{\Phi}=\mathcal{E}_{\prec}(\hat{\kappa})=\mathcal{E}_{\succ}(\hat{\beta})=\exp^*(\hat{\rho}).$$

Then, for any $t\in\mathrm{ST}(n)$ and $a_1,\ldots,a_n\in\mathcal{A}$, the triple of infinitesimal characters is given by

$$\begin{aligned}
\hat{\kappa}(t\otimes a_1\cdots a_n) &= \begin{cases}(-1)^{i(t)-1}\varphi_{\pi(t)}(a_1,\ldots,a_n) & \text{if } t\in\mathrm{PST}(n),\\ 0 & \text{otherwise,}\end{cases}\\
\hat{\beta}(t\otimes a_1\cdots a_n) &= \begin{cases}(-1)^{i(t)-1}\varphi_{\pi(t)}(a_1,\ldots,a_n) & \text{if } t\in\mathrm{BST}(n),\\ 0 & \text{otherwise,}\end{cases}\\
\hat{\rho}(t\otimes a_1\cdots a_n) &= \omega(\mathrm{sk}(t))\varphi_{\pi(t)}(a_1,\ldots,a_n).
\end{aligned}$$

Moreover, the evaluations of $\hat{\kappa}\circ\iota$, $\hat{\beta}\circ\iota$ and $\hat{\rho}\circ\iota$ on a word $w=a_1\cdots a_n\in\mathcal{A}^{\otimes n}$ coincide with the free, Boolean and monotone cumulants of $a_1,\ldots,a_n$, respectively:

$$(\hat{\kappa}\circ\iota)(w)=k_n(a_1,\ldots,a_n),\quad (\hat{\beta}\circ\iota)(w)=b_n(a_1,\ldots,a_n),\quad (\hat{\rho}\circ\iota)(w)=h_n(a_1,\ldots,a_n).$$

The definitions of the subsets of Schröder trees $\mathrm{PST}(n)$ and $\mathrm{BST}(n)$ are precisely stated in Chapter 7.

Monotone Cumulant-Cumulant Formulas from Pre-Lie Magnus Expansion

Due to the combinatorial nature of the moment-cumulant formulas, it is natural to ask about the existence of combinatorial formulas that relate the different brands of non-commutative cumulants. For instance, for the case of two random variables, we have

$$k_2(a_1,a_2)=b_2(a_1,a_2)=h_2(a_1,a_2).$$

However, this is not the case for general n since the posets $\mathrm{NC}(n)$, $\mathrm{Int}(n)$ and $\mathcal{M}(n)$ are substantially different.

The question of finding combinatorial relations between cumulants was initially studied in [Leh02] and further analyzed in [AHLV15], where the authors obtained, via Möbius inversion and other combinatorial techniques, formulas in terms of *irreducible* non-crossing partitions, that write Boolean cumulants in terms of free and monotone cumulants, and free cumulants in terms of Boolean and monotone cumulants. However, the authors of [AHLV15] provided a formula that writes univariate monotone cumulants of a in terms of their free and Boolean cumulants, leaving the multivariate case open.

Chapter 8, based on the joint work [CEFPP21], solves the previous open problem by giving the following description of the coefficients that govern the transition from free and Boolean cumulants to monotone cumulants.

Theorem (Corollary 8.3.7)**.** *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, and consider $\{h_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ and $\{k_n\}_{n\geq 1}$ to be the monotone, Boolean and free cumulants on $(\mathcal{A},\varphi)$, respectively. Then we have*

$$\begin{aligned} h_n(a_1,\dots,a_n) &= \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \omega(t(\pi)) b_\pi(a_1,\dots,a_n),\\ h_n(a_1,\dots,a_n) &= \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1}\omega(t(\pi)) k_\pi(a_1,\dots,a_n), \end{aligned}$$

for any $n\geq 1$ and $a_1,\dots,a_n\in\mathcal{A}$.

The approach of obtaining the above formulas relies on the shuffle-algebraic framework of Ebrahimi-Fard and Patras, or more specifically, on the *pre-Lie algebraic* framework for non-commutative cumulants. More precisely, any shuffle algebra structure yields a pre-Lie algebra structure by

$$a \lhd b = a \prec b - b \succ a.$$

In particular, the dual of the double tensor algebra provides a pre-Lie algebra structure on the Lie algebra of infinitesimal cumulants $\mathfrak{g}$.

Our starting point is the pre-Lie algebraic relation between the three logarithms of a character $\Phi = \mathcal{E}_\prec(\kappa) = \mathcal{E}_\succ(\beta) = \exp^*(\rho)$ via the *pre-Lie Magnus operator* $\Omega:\mathfrak{g}\to\mathfrak{g}$ (Proposition 2.3.28, see [EFP18]):

$$\rho = \Omega(\beta) = -\Omega(-\kappa),$$

where

$$\Omega(\alpha) = \sum_{n\geq 0} \frac{B_n}{n!} r^{(n)}_{\lhd\Omega(\alpha)}(\alpha), \tag{1.2.2}$$

for any $\alpha\in\mathfrak{g}$. In the above expansion, $\{B_n\}_{n\geq 1}$ stands for the sequence of Bernoulli numbers and $r^{(n)}_{\lhd\gamma}(\alpha)$ stands for the *n-th right iterated pre-Lie product*, i.e.

$$r^{(n)}_{\lhd\gamma}(\alpha) = r^{(n-1)}_{\lhd\gamma}(\alpha)\lhd\gamma \quad \text{for } n\geq 1, \text{ with } r^{(0)}_{\lhd\gamma}(\alpha)=\alpha.$$

The strategy to follow is to describe, in terms of non-crossing partitions, the pre-Lie product $\alpha\lhd\gamma$ as well as the iterations $r^{(n)}_{\lhd\gamma}(\alpha)$, for any $n\geq 1$. It will be shown in Proposition 8.2.1 that the n-th iteration of $\triangleleft$ is described by irreducible monotone partitions consisting of exactly $n+1$ blocks:

$$\big((\cdots(\gamma_{n+1} \triangleleft \gamma_n) \triangleleft \cdots) \triangleleft \gamma_1\big)(w) = \sum_{\substack{\pi \in \mathcal{M}^{n+1}_{\mathrm{irr}}(m)\\ \pi=(V_1,\dots,V_{n+1})}} \gamma_{n+1}(w_{V_1}) \cdots \gamma_1(w_{V_{n+1}}), \tag{1.2.3}$$

for any word $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$. We complete our task in Section 8.3, where we identify the coefficients of the corresponding cumulant-cumulant formulas once we replace (1.2.3) in (1.2.2). It turns out that Murua coefficients are the required coefficients that describe the transitions between cumulants. The proof of the main result follows from a known combinatorial identity that describes Murua coefficients of the trees of nestings of irreducible partitions recursively (Proposition 8.3.4).

Pre-Lie Magnus Expansion via Forest Formulas

Forest formulas have appeared in the literature of Quantum Field Theory to compute antipodes of Hopf algebras as part of a process known as *renormalization* ([CP21, Chap. 10]). The main feature of Zimmermann's forest formula is that it is effective in the sense that it considerably reduces the number of terms appearing in the expression of the antipode. Later, Menous and Patras showed in their work [MP18] that Zimmermann's forest formula generalizes in the context of Hopf algebras that are the dual of the enveloping algebras of pre-Lie algebras.

On the other hand, the results obtained in Chapter 8, particularly Proposition 8.2.1, suggest that the iteration of the pre-Lie product $(\cdots(\gamma_{n+1} \triangleleft \gamma_n) \triangleleft \cdots) \triangleleft \gamma_1$ can be considered in general for any pre-Lie algebra of words, where the linear functionals γ_i act on certain decompositions of the word $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$ into $n+1$ subwords.

With the purpose of having a systematic understanding of the iterated pre-Lie products and the Magnus expansion, we are motivated to investigate in the more general framework of *locally finite connected graded pre-Lie algebras*, as well as their associated *symmetric brace algebras* and *right-handed Hopf algebras* (see Section 2.3 for a precise definition of the above notions). Looking into the particular example of the pre-Lie algebra of words, we can observe that the decomposition observed in Proposition 8.2.1 corresponds to the n-th iteration of a certain restriction of the coproduct δ, which is dual to the product on the enveloping algebra of the pre-Lie algebra. It turns out that this iteration can be computed through a forest formula.

In Chapter 9 of this thesis, which is based on the joint work [CP22], we develop an effective method to compute the iteration of the pre-Lie and brace products in terms of forest formulas for iterated coproducts. In particular, our method allows us to compute the pre-Lie Magnus operator and the pre-Lie exponential on L, where L is any locally finite connected graded pre-Lie algebra.

More precisely, the assumptions on $(L, \lhd)$ allow us to identify the dual of its enveloping algebra $(\mathbb{K}[L], *, \Delta)$ with $(\mathbb{K}[L^*], \cdot, \delta)$, where $\cdot$ is the product of polynomials, $*$ is an associative product extending the pre-Lie product $\lhd$, and δ is the coassociative coproduct dual to $*$. Section 9.1 sets the framework to compute iterations of $\lhd$ and the symmetric braces associated to $\lhd$ (Lemma 2.3.17) as stated in Lemma 9.1.7 and Lemma 9.1.8:

$$\langle (\cdots(\alpha_1 \lhd \alpha_2) \lhd \cdots) \lhd \alpha_n | w \rangle = \langle \alpha_1 \otimes \cdots \otimes \alpha_n | \delta_{\mathrm{irr}}^{[n]}(w) \rangle,$$

for any $\alpha_i \in L$ and $w \in L^*$, and

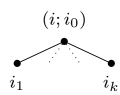
$$\begin{aligned} &\Big\langle (\cdots(\alpha_{1,1}\{\alpha_{1,2}, \ldots \alpha_{m_2,2}\}) \cdots)\{\alpha_{1,n}, \ldots, \alpha_{m_n,n}\} \mid w \Big\rangle \\ &\qquad = \Big\langle \alpha_{1,1} \otimes \alpha_{1,2} \cdots \alpha_{m_2,2} \otimes \cdots \otimes \alpha_{1,n} \cdots \alpha_{m_n,n} \mid \overline{\delta}^{[n]}(w) \Big\rangle \end{aligned}$$

for any $\alpha_{i,j} \in L$ and $w \in L^*$. Above, $\overline{\delta}^{[n]}$ stands for the n-th iteration of the reduced coproduct, and $\delta_{\mathrm{irr}}^{[n]}$ stands for the map obtained by restricting the iterated coproduct $\delta^{[n]}$ to a map from L^* to $(L^*)^{\otimes n}$. Since our ultimate goal is to compute the pre-Lie Magnus operator in the pre-Lie algebra of words, we obtain a concrete expression for the coproduct δ in the corresponding dual Hopf algebra.

Afterwards, we explain the aforementioned forest formulas for iterated coproducts. The basic idea is that in any locally finite connected graded pre-Lie algebra whose graded dual L^* has a countable basis $\mathcal{B} = \{b_i\}_{i \geq 0}$, the irreducible coproduct $\overline{\delta}$ satisfies that

$$\overline{\delta}(b_i) = \sum_{\substack{i_0, I \neq \emptyset \\ I = \{i_1, \ldots, i_k\}}} \lambda_I^{i;i_0} b_{i_0} \otimes b_{i_1} \cdots b_{i_k},$$

where the coefficients $\lambda_I^{i;i_0}$ are indexed by $i_0 \in \mathbb{N}$ and a non-empty multiset $I \subset \mathbb{N}$. Hence, the above formula can be indexed in terms of non-planar decorated rooted trees:

$$\overline{\delta}(b_i) = \sum \lambda\Big($$  $$\Big) b_{i_0} \otimes b_{i_1} \cdots b_{i_k},$$

where the sum above is indexed by the set of decorated corollas such that the decoration of the root of the corolla is given by a pair of non-negative integers $(i; i_0)$, and the leaves are decorated with non-negative integers $i_1, \ldots, i_k$. Theorem 9.2.8 shows that iterations of the reduced coproduct can be written as

$$\overline{\delta}^{[k]}(b_i) = \sum_{T \in \mathcal{T}_i} \sum_{f \in k\text{-lin}(T)} \lambda(T) C(f),$$

where $\mathcal{T}_i$ is the set of decorated trees whose root is decorated by $(i; i_0)$ for some $i_0 \in \mathbb{N}_0$, $\lambda(T)$ is a product of $\lambda_J^{j;j_0}$ coefficients indexed by the decoration of T, and $C(f)$ is a tensor of k components whose components appear according to the decoration and the poset structure of T. The remaining notation is explained in detail in Section 9.2. Such formulas have already appeared in the work [MP18]. However, in Theorem 9.2.8, we give a correct definition of $\lambda(T)$ by introducing a symmetry factor not considered in [MP18].

We finally apply our computation method for the pre-Lie Magnus operator and its inverse in the case of the pre-Lie algebras of words, where we obtain the following result.

Theorem (Theorem 9.3.4, Theorem 9.3.7)**.** *Let L_X be the pre-Lie algebra of words over a finite alphabet X. Then, for $\alpha \in L_X$ and $w_i \in L_X^*$ such that $w_i = a_1 \cdots a_n$, with $a_1, \ldots, a_n \in X$, the actions on w_i of the pre-Lie Magnus operator Ω and its compositional inverse, the pre-Lie exponential*

$$W(\alpha) = \sum_{n\geq 0} \frac{1}{(n+1)!} r^{(n)}_{\triangleleft\alpha}(\alpha),$$

are respectively given by

$$\begin{aligned} \langle \Omega(\alpha) | w_i \rangle &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \omega(t(\pi)) \alpha_\pi(w_i), \\ \langle W(\alpha) | w_i \rangle &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!} \alpha_\pi(w_i). \end{aligned}$$

In the process of converting the forest formula indexed by trees to a sum indexed over irreducible non-crossing partitions, we find in Lemma 9.3.2 a way to construct all the irreducible partitions π associated to a decorated tree T that produces a non-zero contribution on the forest formula for the iterated coproduct.

We close the chapter by applying our method in a different context: the pre-Lie exponential and the Magnus operator on the generator of the free pre-Lie algebra of rooted trees (Proposition 9.4.2, Proposition 9.4.4). The proofs obtained by our method have the feature that they are based on techniques that purely rely on pre-Lie algebra theory. Although these results are known, our approach provides further connections between the combinatorics of rooted trees and free pre-Lie algebras.

1.3 Organization of the thesis

In addition to the preceding introduction, the thesis is organized into eight other chapters. The preliminaries of Hopf algebras and pre-Lie algebras are explained in Chapter 2. On the other hand, Chapter 3 surveys some of the basic theory of non-commutative probability as well as the combinatorics of the non-commutative notions of cumulants. Chapter 4 describes in detail the Ebrahimi-Fard and Patras' Hopf-algebraic framework for non-commutative probability on which this thesis is based. The remaining chapters contain the main contributions of this thesis. Chapter 5 explains the shuffle-algebraic versions of the non-commutative central limit theorems. The extension of the Hopf-algebraic framework that encompasses infinitesimal cumulants is exhibited in Chapter 6. In Chapter 7, we prove a formula that writes monotone cumulants in terms of moments via a decorated Hopf algebra of Schröder trees. Later in Chapter 8, we obtain the combinatorial formula that writes multivariate monotone cumulants in terms of free and Boolean cumulants via pre-Lie algebraic techniques. Finally, in Chapter 9, we develop a general method to compute iterations of pre-Lie and brace products based on a forest formula for iterated coproducts on a particular type of Hopf algebras.

Due to the nature of the first chapters, the reader may skip Chapters 2 and 3 and proceed directly to Chapter 4. Every chapter from Chapter 5 to Chapter 9 can be read independently after Chapter 4. Nevertheless, we suggest reading Chapter 8 before Chapter 9 since several ideas in the latter chapter are inspired by the work done in the former chapter.

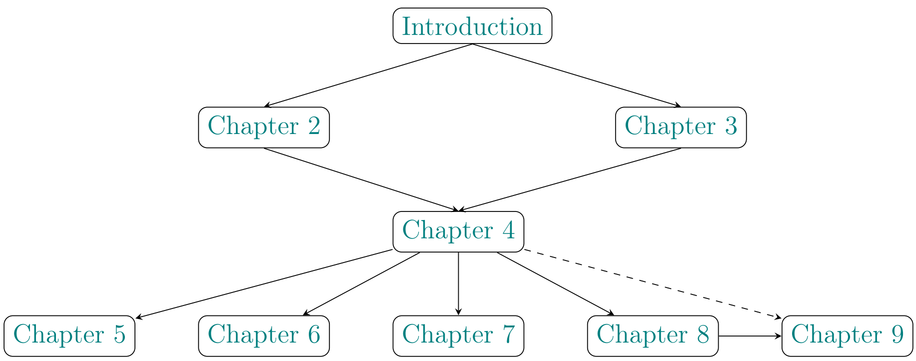


Figure 1.1: Recommended reading of the chapters of the thesis.

Chapter 2

Hopf Algebras and Pre-Lie Algebras

The aim of this chapter is to present the underlying algebraic structures used in this thesis: Hopf and pre-Lie algebras. The forthcoming definitions and examples come from what in the literature is known as *combinatorial Hopf algebras*: algebraic structures based on combinatorial objects such as words, trees and partitions.

Section 2.1 of the present chapter states the classical definition of Hopf algebras as well as some special properties and assumptions that will be relevant to the subsequent chapters, namely the notion of connected graded bialgebra. Section 2.2 describes some classic examples of combinatorial Hopf algebras: word and rooted trees Hopf algebras. Finally, Section 2.3 presents the notions of Lie and pre-Lie algebras. These non-associative algebras are closely related to the notion of Hopf algebra. In particular, we describe the fact that pre-Lie algebras are equivalent to a particular class of Hopf algebras. In the middle, we explain the notion of symmetric brace algebras, which allows a better understanding of iterations of a pre-Lie product. For a reference to the topics in this chapter, the reader can check the classical book of Sweedler [Swe69] as well as the recent books of Grinberg and Reiner [GR20], and Cartier and Patras [CP21].

In what follows, $\mathbb{K}$ denotes a field of characteristic 0 whose unit will be denoted by $1 := 1_{\mathbb{K}}$. In practice, we will take $\mathbb{K} = \mathbb{C}$, the field of complex numbers. Moreover, all the tensor products are taken with respect to $\mathbb{K}$. If V, W are two vector spaces, $\mathrm{Lin}(V, W)$ stands for the vector space of $\mathbb{K}$-linear maps $f : V \to W$.

2.1 Coalgebras, bialgebras and Hopf algebras

The structure of an algebra over a vector space is fundamental in the very definition of a non-commutative probability space. One usually defines an algebra as a vector space together with an operation that satisfies compatibility conditions with the vector space operations. In this chapter, we are interested in having a different understanding of the notion of algebras. The aim of this is to study different useful algebraic structures in a natural and structured way.

Let A be a vector space where a product $(a,b) \mapsto ab$ is defined such that A is an algebra. The main feature of the product on an algebra, besides the compatibility with the sum and the scalar product of A, is the associativity:

$$a(bc) = (ab)c, \qquad \forall\, a,b,c \in A.$$

Representing the associativity via commutative diagrams is a starting point to consider different algebraic structures useful in combinatorics.

Definition 2.1.1 (Algebra)**.** A *unital associative algebra* is a triple (A, m, η) where A is a vector space over $\mathbb{K}$ and $m : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ are linear maps such that the following diagrams commute:

1. Associativity:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

2. Unitary property:

$$\begin{tikzcd} \mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K} \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[d, "\eta \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

The map m is called the *product* on A, and the map η is called the *unit* on A. We will denote $1_A := \eta(1)$, where we recall that $1 = 1_{\mathbb{K}}$.

Notice that, in the second diagram of the above definition, we are using the canonical isomorphisms $\mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K}$. Also, id stands for the identity map on A. When the context is clear, we will denote an algebra just by A instead of a triple (A, m, η), and the product ab instead of $m(a \otimes b)$.

We can formulate the usual properties of algebras in terms of relations coming from commutative diagrams. For instance, we say that an algebra A is *commutative* if $m \circ \tau = m$, where $\tau : A \otimes A \to A \otimes A$ is the flip map $\tau(a \otimes b) = b \otimes a$. In addition, if (A, m_A, η_A) and (B, m_B, η_B) are two algebras, we say that a map $f \in \mathrm{Lin}(A, B)$ is an *algebra morphism* if

$$f \circ m_A = m_B \circ (f \otimes f) \quad \text{ and } \quad \eta_B = f \circ \eta_A.$$

By having defined the notion of algebra in terms of commutative diagrams, it is natural to ask for its dual notion. The notion of coalgebra arises then by reversing the arrows in the definition of algebra.

Definition 2.1.2 (Coalgebra)**.** A *counital coassociative coalgebra* is a triple (C, Δ, ϵ) where C is a vector space over $\mathbb{K}$, and $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ are linear maps such that the following diagrams commute:

1. Coassociativity:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & C \otimes C \otimes C \end{tikzcd}$$

2. Counitary property:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"] \\ C \otimes C \arrow[r, "\epsilon \otimes \mathrm{id}"] & \mathbb{K} \otimes C \cong C \cong C \otimes \mathbb{K} \end{tikzcd}$$

The map Δ is called the *coproduct on* C, and the map ϵ is called the *counit on* C.

One can think a product in an algebra is a rule to combine two elements in order to obtain a single one. Analogously, one can think a coproduct is a recipe to break apart one element into a sum of its parts.

Remark 2.1.3. Let (C, Δ, ϵ) be a coalgebra and $c \in C$. Since in general $\Delta(c)$ is a linear combination of elementary tensors $c_1 \otimes c_2$, one often uses the so-called *Sweedler notation* for Δ

$$\Delta(c) = \sum c_1 \otimes c_2. \tag{2.1.1}$$

With this notation, the coassociativity property can be written as

$$\sum c_1 \otimes (c_2)_1 \otimes (c_2)_2 = \sum (c_1)_1 \otimes (c_1)_2 \otimes c_2.$$

Hence, we can denote the second iterated coproduct simply by

$$(\Delta \otimes \mathrm{id}) \circ \Delta = \sum c_1 \otimes c_2 \otimes c_3.$$

In general, we define the *n-th iterated coproduct* by the recursive formula

$$\Delta^{[n]} := (\Delta \otimes \mathrm{id}^{\otimes(n-2)}) \circ \Delta^{[n-1]}, \quad \text{for } n \geq 3$$

with $\Delta^{[1]} = \mathrm{id}$ and $\Delta^{[2]} := \Delta$. Coassociativity implies that for any $k \leq n-2$

$$\Delta^{[n]} = (\mathrm{id}^{\otimes k} \otimes \Delta \otimes \mathrm{id}^{(n-k-2)}) \circ \Delta^{[n-1]}.$$

We also have dual definitions of concepts associated with algebras. For instance, we say that a coalgebra C is *cocommutative* if $\tau \circ \Delta = \Delta$. We also have the dual notion of algebra morphism.

Definition 2.1.4 (Coalgebra morphism)**.** Let $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ be two coalgebras. We say that $f \in \mathrm{Lin}(C, D)$ is a *coalgebra morphism* if

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f \quad \text{and} \quad \epsilon_D \circ f = \epsilon_C.$$

The aforementioned statement in which it is claimed that coalgebras are dual objects of algebras can be made precise in the following lemma.

Lemma 2.1.5. *Let C be a coalgebra. Then the vector space $C^* = \mathrm{Lin}(C, \mathbb{K})$ is naturally equipped with the structure of an algebra.*

The previous lemma follows from the next more general result.

Proposition 2.1.6. *Let (A, m, η) be an algebra and (C, Δ, ϵ) a coalgebra. The set $\mathrm{Lin}(C, A)$ is equipped with the structure of an algebra with unit given by $\eta \circ \epsilon$ and product*

$$f * g = m \circ (f \otimes g) \circ \Delta, \quad \forall\, f, g \in \mathrm{Lin}(C, A). \tag{2.1.2}$$

The product $f * g$ is called the convolution product of f and g.

Remark 2.1.7. It is not true that the dual of an algebra is automatically a coalgebra. The reason for this is that, in general, the spaces $A^* \otimes A^*$ and $(A \otimes A)^*$ are not isomorphic. However, both spaces are isomorphic when A is finite-dimensional, and one can show that A^* has a natural coalgebra structure in this case.

Remark 2.1.8. Let (A, m_A, η_A) and (B, m_B, η_B) be two algebras. There is a natural algebra structure on $A \otimes B$ given by

$$m_{A\otimes B} := (m_A \otimes m_B) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \quad \text{and} \quad \eta_{A\otimes B} := \eta_A \otimes \eta_B.$$

Analogously, if $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ are two coalgebras, there is a natural coalgebra structure on $C \otimes D$ given by

$$\Delta_{C\otimes D} := (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta_C \otimes \Delta_D) \quad \text{and} \quad \epsilon_{C\otimes D} := \epsilon_C \otimes \epsilon_D.$$

These structures are needed if we want to consider both structures on the same vector space in a compatible way. One of the main relations is nicely described in the following proposition.

Proposition 2.1.9. *Consider (A, m, η) an algebra such that (A, Δ, ϵ) is also a coalgebra. The following statements are equivalent:*

1. *The maps Δ and ϵ are algebra morphisms.*
2. *The maps m and η are coalgebra morphisms.*

The previous proposition allows us to define the notion of bialgebra.

Definition 2.1.10 (Bialgebra)**.** A *bialgebra* is a 5-tuple $(B, m, \eta, \Delta, \epsilon)$ such that

1. (B, m, η) is a unital associative algebra;
2. (B, Δ, ϵ) is a counital coassociative coalgebra;
3. The maps Δ and ϵ are algebra morphisms.

In a similar way, given two bialgebras B and C, we say that $f \in \mathrm{Lin}(B, C)$ is a *bialgebra morphism* if f is an algebra morphism and a coalgebra morphism.

From Proposition 2.1.6, we have that the vector space of linear endomorphisms $\mathrm{End}(B)$ of a bialgebra B is an algebra such that its product is given by the convolution product.

Definition 2.1.11 (Hopf algebra)**.** A bialgebra $(H, m, \eta, \Delta, \epsilon)$ is called a *Hopf algebra* if there exists linear map $S \in \mathrm{End}(H)$ that is the inverse of id with respect the convolution product $*$, i.e.

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{2.1.3}$$

The map S is unique and is called the *antipode of H*.

The following statement collects some properties of the antipode of a Hopf algebra.

Proposition 2.1.12 ([CP21, Prop. 3.1.1])**.** *Let $(H, m, \eta, \Delta, \epsilon, S)$ be a Hopf algebra. Then*

1. $\Delta \circ S = (S \otimes S) \circ \tau \circ \Delta$ *and* $\epsilon \circ S = \epsilon$*;*
2. $S \circ m = m \circ \tau \circ (S \otimes S)$ *and* $S \circ \eta = \eta$*;*
3. *if H is commutative or cocommutative, then $S^2 = \mathrm{id}$.*

Proof. Following [CP21], let us give a proof of part of the proposition. Observe that the second statement states that S is an *algebra anti-homomorphism*, i.e. $S(1_H) = 1_H$ and $S(hl) = S(l)S(h)$ for any $h, l \in H$. By writing (2.1.3) in Sweedler notation, we have that

$$\nu(h) := \eta \circ \epsilon(h) = \sum h_1 S(h_2) = \sum S(h_1) h_2. \tag{2.1.4}$$

Also, since ν is the unit in $\mathrm{End}(H)$, we also have that

$$h = \sum h_1 \nu(h_2), \tag{2.1.5}$$
$$S(h) = \sum S(h_1) \nu(h_2), \tag{2.1.6}$$
$$\nu(hl) = \nu(h)\nu(l). \tag{2.1.7}$$

Recall that the image of ν commutes with any element of H. Hence for any $h, l \in H$ we have

$$\begin{aligned}
S(hl) &= \sum S\Big((hl)_1\Big)\nu(h_2)\nu(l_2) && \text{(by (2.1.5) and (2.1.6))}\\
&= \sum S(h_1l_1)h_2 v(l_2)S(h_3) && \text{(using (2.1.4) on } v(h_2) \text{ and that } v(l_2) \text{ commutes)}\\
&= \sum \Big(S(h_1l_1)h_2l_2\Big)S(l_3)S(h_3) && \text{(using (2.1.4) on } v(l_2))\\
&= \sum \nu\Big((hl)_1\Big)S(l_2)S(h_3) && \text{(using (2.1.4) on } S(h_1l_1)h_2l_2)\\
&= \sum \Big(\nu(l_1)S(l_2)\Big)\Big(\nu(h_1)S(h_2)\Big) && \text{(by (2.1.7) and that the image of } \nu \text{ commutes)}\\
&= S(l)S(h) && \text{(by (2.1.6)).}
\end{aligned}$$

Now, consider the case where H is commutative and $h \in H$. By using the above property, we obtain

$$\begin{aligned}
(S^2 * S)(h) &= \sum S^2(h_1)S(h_2)\\
&= \sum S\Big(h_2 S(h_1)\Big)\\
&= \sum S\Big(S(h_1)h_2\Big)\\
&= S\left(\sum S(h_1)h_2\right)\\
&= S \circ \nu(h)\\
&= \nu(h),
\end{aligned}$$

where in the third equality we used that $h_2S(h_1) = S(h_1)h_2$ since H is commutative. Hence S^2 is a convolution inverse of S. By uniqueness, we conclude that $S^2 = \mathrm{id}$. The cocommutative case is analogous by using the identity $S(h_2S(h_1)) = S(h_1S(h_2))$. □

Not every bialgebra is a Hopf algebra since the existence of the antipode is not automatically granted. However, the existence of the antipode is guaranteed under certain assumptions on a bialgebra.

Definition 2.1.13 (Graded vector spaces and graded linear maps)**.**

1. We say that a vector space V is *graded* if there exists a direct sum decomposition into vector spaces $V = \bigoplus_{n\geq 0} V_n$. The vector space V_n is called the *n-th homogeneous component* of the graded vector space V. If $v \in V_n$, we define $\deg(v) := n$ as the *degree of v*.

2. Let $V = \bigoplus_{n\geq 0} V_n$ and $W = \bigoplus_{n\geq 0} W_n$ be two graded vector spaces. We say that a map $f \in \mathrm{Lin}(V, W)$ is *graded* if $f(V_n) \subset W_n$ for all $n \geq 0$.

Remark 2.1.14. Let V and W two graded vector spaces with decompositions $V = \bigoplus_{n\geq 0} V_n$ and $W = \bigoplus_{n\geq 0} W_n$. The tensor product is endowed with a graded structure where the n-th homogeneous component is given by

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j.$$

The above remark permits stating the definition of the graded versions of algebras and coalgebras.

Definition 2.1.15. We say that an algebra (A, m, η) (or coalgebra (C, Δ, ϵ)) is *graded* if A (or C) is a graded vector space and the maps m and η (Δ and ϵ) are graded.

We also define a *graded bialgebra* B as a bialgebra that is both a graded algebra and graded coalgebra. Observe that the conditions of m and Δ being graded are equivalent to the conditions

$$\begin{aligned} m(B_i \otimes B_j) &\subset B_{i+j}, \quad \forall\, i, j \geq 0, \\ \Delta(B_n) &\subset \bigoplus_{i+j=n} B_i \otimes B_j, \quad \forall\, n \geq 0. \end{aligned}$$

For our final assumption, consider B a graded bialgebra. We say that B is *connected* if $B_0 \cong \mathbb{K}$. A nice property of a connected graded bialgebra is that the coproduct can be expressed in a particular form, allowing us to define an antipode recursively.

Proposition 2.1.16. *Let B be a connected graded bialgebra. Then, if $x \in B_n$, we have that*

$$\Delta(x) = 1_B \otimes x + x \otimes 1_B + \overline{\Delta}(x), \tag{2.1.8}$$

where $\overline{\Delta}(x) \in \sum_{k=1}^{n-1} B_k \otimes B_{n-k}$.

Proof. We outline the ideas of the proof for the convenience of the reader, following the proof in [GR20]. It is clear that $\mathbb{K}$ is graded with 0-th component $\mathbb{K}_0$ equal to $\mathbb{K}$ and the others equal to the null space. Since $\eta : \mathbb{K} \to B$ is graded we have that $\mathbb{K}1_B = \eta(\mathbb{K}_0) \subset B_0$. Since B is connected, then it is possible to show that η is an isomorphism with inverse given by the restriction of ϵ to B_0. In particular $B_0 = \mathbb{K}1_B$.

Now, since ϵ is also graded, then $\epsilon(B_n) \subset \mathbb{K}_n$ for any $n \geq 1$ and thus $\bigoplus_{n>0} B_n \subset \ker \epsilon$. By recalling that ϵ restricted to B_0 is injective, we can conclude the converse containment and hence $\ker \epsilon = \bigoplus_{n>0} B_n$.

For the next step, take $x \in B$. Since $B = \mathbb{K}1_B \oplus \ker \epsilon$, we have

$$\Delta(x) \subset B \otimes \mathbb{K}1_B + B \otimes \ker \epsilon.$$

Then there exist $y \in B \otimes \mathbb{K}1_B$ and $z \in B \otimes \ker \epsilon$ such that $\Delta(x) = y + z$, and $y = y' \otimes 1_B$ for some $y \in B$. By using the counital property $x = (\mathrm{id} \otimes \epsilon)(\Delta(x))$, it is possible to show that $x = y'$ and hence $\Delta(x) = x \otimes 1_B + z$. Similarly, it is possible to show that

$$\Delta(x) = x \otimes 1_B + 1_B \otimes x + z',$$

where $z' \in \ker \epsilon \otimes \ker \epsilon$. We finish by noticing that thanks to the fact that Δ is graded, we obtain that z' lies in $\sum_{k=1}^{n-1} B_k \otimes B_{n-k}$, which concludes the proof. □

Remark 2.1.17. From the proof of the above theorem, one obtains that the counit ϵ in a connected graded bialgebra $B = \bigoplus_{n\geq 0} B_n$ is always given by $\epsilon(1_B) = 1$ and $\epsilon(b) = 0$ for any element $b \in \bigoplus_{n>0} B_n$.

Proposition 2.1.18. *A connected graded bialgebra B has a unique antipode S, which is a graded map $S : B \to B$, endowing B with a Hopf algebra structure. Hence, any connected graded bialgebra is a Hopf algebra.*

Proof. We will define the linear map S on each homogeneous component B_n by induction. For the base case, taking into account item 2 of Proposition 2.1.12, we define $S(1_B) = 1_B$ and so S is the identity on $B_0 = \mathbb{K}$.

Now, for the inductive step we use Proposition 2.1.16: for $x \in B_n$ with $n \geq 1$, we have that $\Delta(x) = x \otimes 1_B + \sum x' \otimes x''$, where each x' and x'' is an element which lies in $\bigoplus_{k=0}^{n-1} B_k$. Since we require that $S * \mathrm{id} = \eta \circ \epsilon$, we must have that $S(x) + \sum S(x')x'' = \eta \circ \epsilon(x) = 0$ since $x \in B_n \subset \ker \epsilon$. Hence $S(x) = -\sum S(x')x''$, where each $S(x')$ have been already defined by the induction hypothesis. Thus S is the left inverse of id with respect to the convolution product. Analogously, we have that S is also the right inverse of id. We conclude then that S is the inverse of id and hence S is the antipode of B. The fact that S is graded is also proved by induction. □

Given a bialgebra B, the map $\overline{\Delta}$ written in the statement of Proposition 2.1.16 receives a special name.

Definition 2.1.19 (Reduced coproduct)**.** Let $(B, m, \eta, \Delta, \epsilon)$ be a bialgebra. The *reduced coproduct* is the linear map $\overline{\Delta} : \ker \epsilon \to \ker \epsilon \otimes \ker \epsilon$ given by

$$\overline{\Delta}(x) := \Delta(x) - x \otimes 1_B - 1_B \otimes x, \qquad \text{for all } x \in \ker \epsilon.$$

Thanks to the coassociativity of Δ, one can check that the reduced coproduct $\overline{\Delta}$ is also coassociative. This motivates to consider the *iterated reduced coproduct* $\overline{\Delta}^{[n]} : \ker \epsilon \to (\ker \epsilon)^{\otimes n}$ defined inductively by $\overline{\Delta}^{[1]} = \mathrm{id}$, $\overline{\Delta}^{[2]} := \overline{\Delta}$ and

$$\overline{\Delta}^{[n]} := (\overline{\Delta} \otimes \mathrm{id}^{\otimes(n-2)}) \circ \overline{\Delta}^{[n-1]}, \qquad \text{for } n \geq 3.$$

If B is in addition connected graded, then $\ker \epsilon = \bigoplus_{n>0} B_n$ and $\overline{\Delta}$ is *conilpotent*, i.e. for any $x \in \bigoplus_{n>0} B_n$, there exists $m \geq 1$ such that $\overline{\Delta}^{[n]}(x) = 0$ for $n \geq m$.

Remark 2.1.20. The notion of reduced coproduct can be defined more generally for the case of a *coaugmented coalgebra*, i.e. a coalgebra (C, Δ, ϵ) together with a morphism of coalgebras $\eta : \mathbb{K} \to C$, and defining $1_C := \eta(1)$.

Let us describe an interesting relation between Hopf algebras and groups that will be useful in the following chapters. Recall that if H is a Hopf algebra, then it is also a coalgebra. Thus, the dual space $H^* = \mathrm{Lin}(H, \mathbb{K})$ is an algebra for the convolution product with unit given by ϵ. The algebra structure allows us to consider the following special subset of H^*.

Definition 2.1.21 (Character)**.** Let H be a Hopf algebra. We say that a map $f \in H^*$ is a *character on H* if it is a unital algebra morphism.

Lemma 2.1.22. *Let H be a Hopf algebra with antipode S. The subset $G(H)$ of characters on H is a group with respect to the convolution product. The inverse (with respect to the convolution product) of an element $f \in G(H)$ is given by*

$$f^{*-1} = f \circ S.$$

Remark 2.1.23. Definition 2.1.21 can be extended for unital algebra morphisms $f \in \mathrm{Lin}(H, A)$, where H is a Hopf algebra and A is a commutative algebra. In this case, we denote $G_A(H)$ the set of characters of $\mathrm{Lin}(H, A)$. The conclusion of Lemma 2.1.22 is still valid, and hence $G_A(H)$ is a group with respect to the convolution product. The commutativity of A is needed when we want to prove that $f * g \in G_A(H)$ for any $f, g \in G_A(H)$.

The second assertion in the previous lemma follows from the following fact about inverses with respect to the convolution product (Proposition 2.1.6) in more general spaces of linear maps.

Proposition 2.1.24. *Let H be a Hopf algebra with antipode S.*

1. *For any algebra A and $f \in \mathrm{Lin}(H, A)$ an algebra morphism, the inverse of f with respect to the convolution product is given by $f^{*-1} = f \circ S$.*

2. *For any coalgebra C and $f \in \mathrm{Lin}(C, H)$ a coalgebra morphism, the inverse of f with respect to the convolution product is given by $f^{*-1} = S \circ f$.*

Proof. We will prove the second statement. Since f is a coalgebra morphism, we have

$$\begin{aligned}(S\circ f)*f &= m_H\circ(S\circ f\otimes f)\circ\Delta_C\\ &= m_H\circ(S\otimes \mathrm{id})\circ(f\otimes f)\circ\Delta_C\\ &= m_H\circ(S\otimes \mathrm{id})\circ\Delta_H\circ f\\ &= (S*\mathrm{id})\circ f\\ &= \eta_H\circ(\epsilon_H\circ f)\\ &= \eta_H\circ\epsilon_C.\end{aligned}$$

Analogously, $f*(S\circ f)=\eta_H\circ\epsilon_C$. By uniqueness we conclude that $f^{*-1}=S\circ f$.

The proof of the first statement is similar. $\square$

Another class of linear maps $f\in H^*$ which are closely related to the group of characters on H is given in the next definition.

Definition 2.1.25 (Infinitesimal character)**.** Let H be a Hopf algebra. We say that a map $f\in H^*$ is an *infinitesimal character on* H if $f(1_H)=0$, and for any $x,y\in H$ we have that

$$f(xy)=f(x)\epsilon(y)+\epsilon(x)f(y).$$

We denote $\mathfrak{g}(H)$ the set of infinitesimal characters on H.

Remark 2.1.26. One can check that $\mathfrak{g}(H)$ is a vector space, but it is not a group. Instead, $\mathfrak{g}(H)$ is closed under the bracket $[f,g]:=f*g-g*f$, for any $f,g\in\mathfrak{g}(H)$.

Lemma 2.1.27. *Let H be a Hopf algebra. The subset $\mathfrak{g}(H)$ of infinitesimal characters on H is a vector space such that $[f,g]\in\mathfrak{g}(H)$ for any $f,g\in\mathfrak{g}(H)$.*

The relation between $G(H)$ and $\mathfrak{g}(H)$ is given through the exponential map with respect to the convolution product. Under the connected graded assumption in the below proposition, we have that the coproduct is conilpotent. Hence, if $f\in H^*$ is such that $f(1_H)=0$ and $x\in H$, then there exists $m\geq 0$ such that

$$f^{*n}(x)=m^{[m]}\circ f^{\otimes m}\circ\overline{\Delta}^{[n]}(x)=0$$

for any $n\geq m$. This implies that

$$\exp^*(f)=\sum_{n\geq 0}\frac{f^{*n}}{n!} \tag{2.1.9}$$

is well-defined. In the same way, if $f=\epsilon+g$ with $g\in H^*$ such that $g(1_H)=0$, then the map

$$\log^*(f)=\sum_{n>0}(-1)^{n-1}\frac{g^{*n}}{n}$$

is also well-defined.

It turns out that the exponential and logarithm maps are set isomorphism in the connected graded case.

Proposition 2.1.28 ([CP21, Cor. 3.4.1]). *If H is a connected graded Hopf algebra, then $G(H)$ and $\mathfrak{g}(H)$ are in bijection:*

$$\log^* : G(H) \leftrightarrows \mathfrak{g}(H) : \exp^* .$$

Remark 2.1.29. In connection with Remark 2.1.23, we can extend the definition of the set of infinitesimal characters $\mathfrak{g}_A(H) \subset \mathrm{Lin}(H, A)$ when A is a commutative algebra. The conclusion of Proposition 2.1.28 also holds in this general framework: the sets $G_A(H)$ and $\mathfrak{g}_A(H)$ are in bijection via the exponential and logarithm maps.

The class of Hopf algebras that will be of interest in this work are precisely the connected graded Hopf algebras. In most of our cases, we will require that each homogeneous component is finite-dimensional. The main advantage of this condition is that we can associate a dual Hopf algebra to a connected graded Hopf algebra.

Definition 2.1.30. Let $V = \bigoplus_{n\geq 0} V_n$ be a graded vector space. We say that V is *locally finite* if each homogeneous component V_n is a finite-dimensional vector space, for any $n \geq 0$. We say that a graded algebra (resp. coalgebra, bialgebra or Hopf algebra) is *locally finite* if its underlying graded vector space is locally finite.

Proposition 2.1.31. *Let $H = \bigoplus_{n\geq 0} H_n$ be a locally finite connected graded Hopf algebra. Define the graded vector space*

$$H^* = \bigoplus_{n\geq 0} (H_n)^*,$$

where $H_n^* := \mathrm{Lin}(H_n, \mathbb{K})$. Then H^* is a Hopf algebra with

1. *product given by the convolution product $*$;*
2. *unit given by the counit of H;*
3. *coproduct given by $\Delta_{H^*}(f) := \rho^{-1}(f \circ m_H)$, where ρ is the canonical isomorphism $H^* \otimes H^* \to (H \otimes H)^*$;*
4. *counit $\epsilon_{H^*} : H^* \to \mathbb{K}$ given by $\epsilon_{H^*}(f) := f(1_H)$*
5. *antipode $S_{H^*} : H^* \to H^*$ given by $S_{H^*}(f) := f \circ S_H$, where S_H is the antipode of H.*

The Hopf algebra H^* is called the graded Hopf algebra dual *of H.*

The above proposition tells us the existence of a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : H \otimes H^* \to \mathbb{K}$ such that the elements of a basis $\{h_i\}_{i\in I}$ of H_n are paired to the elements of the dual basis $\{f_i\}_{i\in I}$ of H_n^* by $\langle h_i, f_j\rangle = \delta_{i,j}$, where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. The duality between H and H^* can be described through the existence of a family of constants $\{c_{j,k}^k\} \subset \mathbb{K}$ such that

$$m_H(h_j \otimes h_k) = \sum_{i\in I} c_{j,k}^i h_i \quad \Leftrightarrow \quad \Delta_{H^*}(f_i) = \sum_{j,k\in I} c_{j,k}^i f_j \otimes f_k.$$

2.2 Examples of Hopf algebras

After introducing the definitions of some of the main algebraic structures used in this manuscript, let us exhibit some of the classic examples which are also relevant for our purposes.

2.2.1 Hopf algebras on tensors

Example 2.2.1 (Tensor Hopf algebra)**.** Let V be a vector space. Define the *tensor algebra over V* as the vector space defined by

$$T(V) := \bigoplus_{n\geq 0} V^{\otimes n}.$$

We consider the elements of $T(V)$ as words on V, i.e. $v_1 \cdots v_n \in V^{\otimes n}$ if $v_1, \dots, v_n \in V$. This allows to define the *concatenation product* on $T(V)$:

$$(v_1 \cdots v_n) \cdot (v_{n+1} \cdots v_{n+m}) = v_1 \cdots v_{n+m} \in V^{\otimes(n+m)}$$

for any $v_1 \cdots v_n \in V^{\otimes n}$ and $v_{n+1} \cdots v_{n+m} \in V^{\otimes m}$. It is easy to check that the concatenation product is associative, making $T(V)$ a unital associative algebra with unit given by the empty word denoted by $\mathbf{1}$.

The tensor algebra $T(V)$ is actually the *free algebra over V*, i.e. for any algebra A and linear map $f : V \to A$, there exists a unique algebra morphism $\tilde{f} : T(V) \to A$ such that $f = \tilde{f} \circ \iota$, where $\iota : V \to T(V)$ is the canonical inclusion.

For the coproduct, define the linear map

$$\Delta(v_1 \cdots v_n) = \sum_{I\subset[n]} v_I \otimes v_{[n]\setminus I}, \quad \forall\, v_1 \cdots v_n \in V^{\otimes n}, \tag{2.2.1}$$

where we introduce the notation $[n] := \{1, \dots, n\}$, $v_\emptyset := \mathbf{1}$, and if $I = \{i_1, \dots, i_s\}$ such that $i_1 < \cdots < i_s$ then $v_I := v_{i_1} \cdots v_{i_s}$. This coproduct is usually called the *unshuffle coproduct*, and we can check that it is cocommutative. Moreover, we can define the linear map $\epsilon : T(V) \to \mathbb{K}$ by

$$\epsilon(\mathbf{1}) = 1 \quad \text{and} \quad \epsilon(v_1 \cdots v_n) = 0 \quad \text{for } n \geq 1,\ v_1, \dots, v_n \in V.$$

These maps make $T(V)$ a coalgebra. We can show that Δ and ϵ are algebra morphisms, which implies that $T(V)$ is a bialgebra.

Finally, one can show that $T(V)$ is a connected graded cocommutative bialgebra, with homogeneous components given by $T(V)_n := V^{\otimes n}$ for any $n \geq 0$. By Proposition 2.1.18, we have that $T(V)$ is a Hopf algebra, that we will call the *tensor Hopf algebra over* V.

In order to compute the antipode, observe that $\Delta(v) = v \otimes \mathbf{1} + \mathbf{1} \otimes v$ and $\epsilon(v) = 1$, by the definition of S we have that $S(v) + v = 0$ and so $S(v) = -v$. In general, we can prove that

$$S(v_1 \cdots v_n) = (-1)^n v_n \cdots v_1.$$

Example 2.2.2 (Shuffle tensor Hopf algebra)**.** The tensor algebra $T(V)$ over a vector space can be endowed with another product and coproduct that provide a new Hopf algebra structure. For the product, we consider the *shuffle product* $\sqcup\!\sqcup$ on $T(V)$ defined by the recursive formula

$$v_1 \cdots v_n \sqcup\!\sqcup\, w_1 \cdots w_m := v_1(v_2 \cdots v_n \sqcup\!\sqcup\, w_1 \cdots w_m) + w_1(v_1 \cdots v_n \sqcup\!\sqcup\, w_2 \cdots w_m), \qquad (2.2.2)$$

with $\mathbf{1} \sqcup\!\sqcup\, \mathbf{1} := \mathbf{1}$ and $\mathbf{1} \sqcup\!\sqcup\, v = v \sqcup\!\sqcup\, \mathbf{1} := v$. The shuffle product is associative and also commutative with the unit given by $\mathbf{1}$.

On the other hand, define the *deconcatenation coproduct* as the linear map

$$\Delta_d(v_1 \cdots v_n) = \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n, \quad \forall\, v_1 \cdots v_n \in V^{\otimes n}, \qquad (2.2.3)$$

and the counit ϵ given in the above example. In this case, $(T(V), \sqcup\!\sqcup, \Delta_d)$ is a connected graded bialgebra, and thus it is a Hopf algebra. One can also show that the formula for the antipode is given by $S(v_1 \cdots v_n) = (-1)^{n-1} v_n \cdots v_1$.

Remark 2.2.3. Let V be a finite-dimensional vector space. Under this assumption, $T(V)$ is locally finite-dimensional and $(T(V)_n)^*$ can be identified with $T(V^*)_n$, for any $n \geq 0$. Hence, the shuffle tensor Hopf algebra is the dual of the tensor Hopf algebra, in the sense of Proposition 2.1.31.

Example 2.2.4 (Polynomial Hopf algebra)**.** Let V be a vector space. We can consider the commutative version of Example 2.2.1. In other words, denote $\mathbb{K}[V]$ the algebra of commutative polynomials over V, i.e. the free commutative unital associative algebra generated by the elements of V. Define the linear map

$$\Delta(v_1 \cdots v_n) = \sum_{I \subset [n]} v_I \otimes v_{[n]\setminus I}, \quad \forall\ v_1, \ldots, v_n \in V \qquad (2.2.4)$$

as in (2.2.1), but observe that now $v_1 \cdots v_n$ is a commutative monomial instead of a non-commutative word. This linear map can also be defined as the algebra morphism such that

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \quad \forall\, v \in V.$$

The counit is given by the linear map $\epsilon : \mathbb{K}[V] \to \mathbb{K}$ given as in Example 2.2.1. If we denote by $\cdot$ the product of polynomials, then the triple $(\mathbb{K}[V], \cdot, \Delta)$ is a connected graded Hopf algebra, where each homogeneous component $\mathbb{K}[V]_n$ consists of the linear span of monomials of degree n, for $n \geq 0$. Due to its importance in this manuscript, we will write the next definition.

Definition 2.2.5. A *polynomial Hopf algebra* over a vector space V is the triple $(\mathbb{K}[V], \cdot, \Delta)$, where $\mathbb{K}[V]$ is the polynomial algebra over V and Δ is the coproduct given in (2.2.4).

2.2.2 Hopf algebras on non-planar rooted trees

The next examples of Hopf algebras are based on central objects in combinatorics: rooted trees and forests. Let us now give a precise definition of such objects.

Definition 2.2.6 (Trees)**.** A (*non-planar*) *rooted tree* is a finite directed graph with a distinguished vertex r such that for any vertex v of the graph, there is exactly one path from r to v. The distinguished vertex of a rooted tree is called the *root* of the tree. A *rooted forest* is a disjoint union of a finite number of rooted trees.

In a rooted tree, the edges are oriented away from the root. Then, any vertex distinct from the root has exactly one incoming edge and any number of outgoing edges. We will represent rooted trees in the following way, where the edges are downward oriented and the root is the topmost placed vertex.

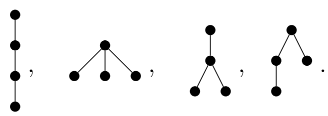


Figure 2.1: Rooted trees with four vertices.

If t is a rooted tree, we will denote by $V(t)$ the set of vertices of t. We will also need to make precise the following concepts associated to trees.

Definition 2.2.7. Let t be a rooted tree and $v, w \in V(t)$.

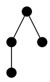
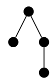
1. We say that v is the *parent* of w if there is an edge from v to w in t. In the same way, we say that w is a child of v if v is the parent of w.

2. More generally, we say that v is an *ascendant* of w if the unique path from the root to w passes through v. Similarly, we say that w is an *descendant* of v if v is an ascendant of w.

3. The vertex v is a *leaf* if v has no children. The set of leaves of t is denoted by $\mathrm{Leaf}(t)$.

4. The vertex v is an *internal vertex* if it is not a leaf. The set of internal vertices of t is denoted by $\mathrm{Intl}(v)$.

Observe that, as non-planar rooted trees, the trees

 and 

are considered the same. If we want to make a distinction between both trees, we need to consider the following notion.

Definition 2.2.8. A *planar rooted tree* is a rooted tree in which each vertex has a specified ordering of its children.

Remark 2.2.9. Associated to a rooted tree t, there is a poset structure on $V(t)$ given by the edges of t. More precisely, if $v, w \in V(t)$, we say that $v \leq w$ if and only if v is an ascendant of w. In this partial order, the root of t is the minimal element, while the leaves of t are the maximal elements.

One advantage of working with rooted trees is that many properties of them can be studied recursively. The latter happens since the children of the root define a rooted forest.

Definition 2.2.10. For a rooted forest f consisting of the non-planar rooted trees $t_1, \dots, t_m$, we define the *grafting of* f, denoted $B^+(f)$, as the non-planar rooted tree obtained by grafting the trees $t_1, \dots, t_m$ onto a common root. Conversely, given a rooted tree t, we denote the *ungrafting of* t $B^-(t)$ as the rooted forest f such that $B^+(f) = t$.

Remark 2.2.11. The above definition extends to the case of planar trees. In this context, we write the grafting of the trees $t_1, \dots, t_m$ as $B^+(t_1, \dots, t_m)$ to emphasize the order in which the trees are grafted.

Observe that in the definition of rooted forest, the grafting of the empty forest is the single-vertex tree. In other words, if $\mathbf{1}$ stands for the empty forest, i.e. the forest containing no trees, then $B^+(\mathbf{1}) = \bullet$.

For what follows, let $\mathcal{T}$ be the set of non-planar rooted trees and take $\mathcal{H}_{\mathrm{CK}} := \mathbb{K}[\mathcal{T}]$ the polynomial algebra over $\mathcal{T}$. It will be useful to identify a monomial $t_1 \cdots t_n$ with the forest consisting of $t_1, \dots, t_n$. Under this identification, $\mathcal{H}_{\mathrm{CK}}$ is the linear span of the set of rooted forests. Also, the product of two forests is given by their disjoint union, with the unit given by the empty forest $\mathbf{1}$. The algebra $\mathcal{H}_{\mathrm{CK}}$ is graded, where the degree of

$t \in \mathcal{T}$ is given by the number of vertices of t, i.e. $\deg(t) = |t| := |V(t)|$. In general, for a forest $f = t_1 \cdots t_m$, we define

$$\deg(f) = \deg(t_1) + \cdots + \deg(t_m).$$

To describe the coproduct, we need the following notion,

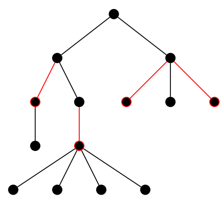
Definition 2.2.12. Let t be a rooted tree. An *admissible cut of* t is a subset (possibly empty) c of edges of t such that for any path from the root to any leaf, there is at most one edge of the path contained in c.

Given an admissible cut, we can construct a forest and another rooted tree.

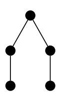
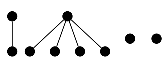
Remark 2.2.13. The notions of trunk and pruning are defined for both non-planar and planar rooted trees. In the latter case, the trunk and the pruning carry the ordering of the vertices in the original planar rooted tree.

Definition 2.2.14. Let t be a rooted tree and c be an admissible cut of t. Also, consider the rooted forest obtained from t by deleting all the edges in c. The *trunk of* t associated to c is the rooted subtree whose root is the same as that of t. The remaining rooted forest is called the *pruning of* t associated to c.

Example 2.2.15. Consider the following rooted tree t together with an admissible cut c whose elements are the edges of t depicted in red colour.

$t =$  .

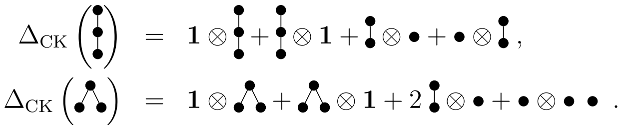
The trunk and the pruning of t associated to c are the following:

$R_c(t) =$  and $P_c(t) =$ 

In what follows, $\mathrm{Adm}(t)$ stands for the set of admissible cuts of a rooted tree t. The coproduct on $\mathcal{H}_{\mathrm{CK}}$ is defined in terms of trunks and prunings. More precisely, define the algebra morphism $\Delta_{\mathrm{CK}} : \mathcal{H}_{\mathrm{CK}} \to \mathcal{H}_{\mathrm{CK}} \otimes \mathcal{H}_{\mathrm{CK}}$ given by $\Delta_{\mathrm{CK}}(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, and for any rooted tree t:

$$\Delta_{\mathrm{CK}}(t) = \mathbf{1} \otimes t + \sum_{c \in \mathrm{Adm}(t)} R_c(t) \otimes P_c(t). \tag{2.2.5}$$

Example 2.2.16. We have the following simple computations of the coproduct Δ_{CK}:



The last ingredient is the counit $\epsilon : \mathcal{H}_{\mathrm{CK}} \to \mathbb{K}$ which is defined to be the algebra morphism such that $\epsilon(\mathbf{1}) = 1$ and $\epsilon(f) = 0$ for any non-empty forest f. With this, we have a Hopf algebra structure on $\mathcal{H}_{\mathrm{CK}}$.

Theorem 2.2.17. *The triple* $(\mathcal{H}_{\mathrm{CK}}, \cdot, \Delta_{\mathrm{CK}})$ *is a connected graded commutative non-cocommutative Hopf algebra.*

The Hopf algebra from the previous theorem is called the *Connes-Kreimer Hopf algebra.*

The coproduct Δ_{CK} is not the unique coproduct that provides $\mathbb{K}[\mathcal{T}]$ with the structure of Hopf algebra. Since $\mathbb{K}[\mathcal{T}]$ is a polynomial algebra, we consider the coproduct in Example 2.2.4: it is the algebra morphism given by

$$\Delta_{\mathrm{GL}}(t_1 \cdots t_n) := \sum_{I \subset [n]} t_I \otimes t_{[n]\setminus I}, \tag{2.2.6}$$

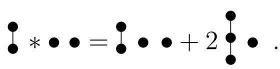
where t_I is the monomial consisting of the trees whose indexes are given by I. In particular, if $I = \emptyset$ then $t_\emptyset = \mathbf{1}$. We have than that $(\mathbb{K}[\mathcal{T}], \cdot, \Delta_{\mathrm{GL}})$ is a polynomial Hopf algebra. However, there exists another associative product $*$ making $(\mathbb{K}[\mathcal{T}], *, \Delta_{\mathrm{GL}})$ a Hopf algebra.

Let us describe the product $*$. For this, consider two commutative monomials $t_1 \cdots t_n$ and $s_1 \cdots s_m$ in $\mathcal{H}_{\mathrm{GL}} := \mathbb{K}[\mathcal{T}]$. We know that we can identify the monomials $t_1 \cdots t_n$ and $s_1 \cdots s_m$ as rooted forests. If $t_1 \cdots t_n$ is the empty forest, define

$$\mathbf{1} * s_1 \cdots s_m := s_1 \cdots s_m.$$

Now assume that $t_1 \cdots t_n$ is not empty and define the rooted tree $s := B^+(s_1 \cdots s_m)$. Then we set $(t_1 \cdots t_n) * (s_1 \cdots s_m)$ as the sum of the ungraftings of the rooted trees obtained by connecting the roots of $t_1, \dots, t_n$ to the vertices of s in all the possible $n^{|s|}$ different ways. One can observe that the identity for this product is given by the empty forest $\mathbf{1}$. It is easy to see that this product is not commutative.

Example 2.2.18. For $t = B^+(\bullet)$ and $s_1 s_2 = \bullet\,\bullet$, their $*$ product is given by



By considering the grading given by the number of vertices of the trees, we can show that the product $*$ and the coproduct Δ_{GL} are also graded. We can then formulate the next result.

Theorem 2.2.19. *The triple $(\mathcal{H}_{\mathrm{GL}}, *, \Delta_{\mathrm{GL}})$ is a connected graded non-commutative co-commutative Hopf algebra.*

The Hopf algebra from the above theorem is called the *Grossman-Larson Hopf algebra*.

It turns out that the Connes-Kreimer Hopf algebra and the Grossman-Larson Hopf algebra are graded dual Hopf algebras to each other, in the sense of Proposition 2.1.31. With the purpose of describing the duality, let us describe the following statistic on rooted trees.

Definition 2.2.20. Let t be a rooted tree. The *internal symmetry factor of* t, denoted $\sigma(t)$, is defined to be the cardinality of the automorphism group of t, i.e $\sigma(t) := |\operatorname{Aut}(t)|$. Alternatively, it can be defined inductively by $\sigma(\bullet) := 1$ and

$$\sigma(t) := \prod_{i=1}^{p} m_i! \sigma(t_i)^{m_i}, \tag{2.2.7}$$

if $t = B^+(t_1^{m_1} \cdots t_p^{m_p})$, where the trees $t_1, \ldots, t_p$ are all distinct and the notation $t_i^{m_i}$ means that the tree t_i has multiplicity m_i, for $1 \leq i \leq p$.

The duality is then described as follows:

Theorem 2.2.21 ([Hof03, Prop. 4.4])**.** *The duality between the Hopf algebras $\mathcal{H}_{\mathrm{CK}}$ and $\mathcal{H}_{\mathrm{GL}}$ is given by the pairing*

$$\langle s_1 \cdots s_m | t_1 \cdots t_n \rangle = \begin{cases} \sigma(B^+(t_1 \cdots t_n)) & \text{if } s_1 \cdots s_n = t_1 \cdots t_n, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2.8}$$

for any monomials $s_1 \cdots s_m \in \mathcal{H}_{\mathrm{GL}}$ and $t_1 \cdots t_n \in \mathcal{H}_{\mathrm{CK}}$.

2.3 Lie algebras and pre-Lie algebras

In the final section of the present chapter, we will study two basic examples of algebraic structures with a product that does not need to be associative, though they also satisfy interesting identities instead.

The exposition of the current section is based on Chapter 6 of the book of Cartier and Patras [CP21], where the interested reader can consult the proofs of the results presented in this section.

2.3.1 Lie algebras

The first example of non-associative algebras is the well-known notion of Lie algebras.

Definition 2.3.1 (Lie algebra)**.** A *Lie algebra* is a vector space L together with a bilinear map $[\,,] : L \otimes L \to L$, $x \otimes y \mapsto [x, y]$, satisfying the following conditions:

1. Anti-symmetry: $[x, x] = 0$ for any $x \in L$;

2. Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for any $x, y, z \in L$.

Example 2.3.2. Let A be an associative algebra. We define the bracket $[a, b] = ab - ba$, for any $a, b \in A$. It is easy to see that associativity implies the Jacobi identity. Hence, $(A, [\,,])$ is a Lie algebra.

Example 2.3.3. Let B be a bialgebra. We say that $b \in B$ is *primitive* if

$$\Delta(b) = b \otimes 1_B + 1_B \otimes b. \tag{2.3.1}$$

Let $\mathrm{Prim}(B)$ be the set of primitives of B. One can show that $\mathrm{Prim}(B)$ is a vector subspace of B. Moreover, by considering the Lie bracket by the associative product of B, $[a, b] = ab-ba$, we have that $\mathrm{Prim}(B)$ is actually a Lie algebra. Indeed, if $a, b \in \mathrm{Prim}(B)$:

$$\begin{aligned} \Delta([a, b]) &= \Delta(a)\Delta(b) - \Delta(b)\Delta(a) \\ &= (a \otimes 1_B + 1_B \otimes a)(b \otimes 1_B + 1_B \otimes b) - (b \otimes 1_B + 1_B \otimes b)(a \otimes 1_B + 1_B \otimes a) \\ &= [a, b] \otimes 1_B + 1_B \otimes [a, b], \end{aligned}$$

which implies that $[a, b] \in \mathrm{Prim}(B)$.

Example 2.3.4. Let H be a Hopf algebra and consider $\mathfrak{g}$ the set of infinitesimal characters on H (Definition 2.1.25). Remark 2.1.26 implies that $\mathfrak{g}$ is a Lie subalgebra of H^*.

The bracket construction described in Example 2.3.2 is a functor from associative algebras to Lie algebras. Its adjoint displays a nice example of Hopf algebras.

Definition 2.3.5 (Enveloping algebra)**.** Let L be a Lie algebra. We say that the pair $(U(L), \iota)$ is an *enveloping algebra of* L if $U(L)$ is an associative algebra and $\iota : L \to U(L)$ is such that

1. $U(L)$ is the associative algebra generated by the image of L by ι;

2. $\iota([a, b]) = \iota(a)\iota(b) - \iota(b)\iota(a)$, for any $a, b \in L$;

3. for any associative algebra A and a Lie algebra morphism $f : L \to A$, i.e. f is linear such that $f([a, b]) = f(a)f(b) - f(b)f(a)$, then there exists a unique algebra morphism $\tilde{f} : U(L) \to A$ such that $f = \tilde{f} \circ \iota$.

By a usual argument of universal properties, one shows that enveloping algebras are unique up to isomorphism. On the other hand, given a Lie algebra L, a usual explicit construction of $U(L)$ is given as a quotient of the tensor algebra over L. More precisely, we can show that $T(L)/I$ is the enveloping algebra of L, where I is the ideal generated by the elements $[a, b] - ab + ba$, for any $a, b \in L$.

Under the above setting, we define the linear map $\delta : L \to U(L) \otimes U(L)$ by $\delta(x) := x \otimes \mathbf{1} + \mathbf{1} \otimes x$, for $x \in L$. One can show that δ is a Lie algebra morphism, and by the universal property, it can be extended to an algebra morphism $\Delta : U(L) \to U(L) \otimes U(L)$. Finally, by setting the algebra morphisms $\epsilon : U(L) \to \mathbb{K}$ which vanishes on L, and $S : U(L) \to U(L)$ by $S(x) = -x$, we obtain:

Proposition 2.3.6. *Let L be a Lie algebra. The enveloping algebra $U(L)$ is a Hopf algebra.*

Let H be a Hopf algebra. Example 2.3.3 shows a particular Lie algebra associated to H. Furthermore, Proposition 2.3.6 describes another Hopf algebra in which the elements of the Lie algebra are primitive elements. The following important result provides conditions in which the converse holds, i.e. when the Hopf algebra is actually the enveloping algebra of their primitive elements.

Theorem 2.3.7 (Cartier-Milnor-Moore [CP21, Thm. 4.3.1])**.** *Let H be a locally finite connected graded cocommutative Hopf algebra and consider* $\mathrm{Prim}(H)$ *the Lie algebra of primitive elements of H. Then the inclusion of* $\mathrm{Prim}(H)$ *into H extends to an isomorphism of Hopf algebras $\Phi : U(\mathrm{Prim}(H)) \to H$.*

2.3.2 Pre-Lie algebras

The most important example of non-associative algebra for this thesis is the notion of pre-Lie algebra. They are also called Vinberg algebras and appear in different contexts in mathematics, such as differential geometry, operads, and, most important for us, combinatorics.

Definition 2.3.8 (Left pre-Lie algebra)**.** A *left pre-Lie algebra* is a vector space L together with a bilinear map $\rhd : L \otimes L \to L$, $x \otimes y \mapsto x \rhd y$, satisfying the identity

$$x \rhd (y \rhd z) - (x \rhd y) \rhd z = y \rhd (x \rhd z) - (y \rhd x) \rhd z, \quad \forall\, x, y, z \in L. \tag{2.3.2}$$

Definition 2.3.9 (Right pre-Lie algebra)**.** A *right pre-Lie algebra* is a vector space L together with a bilinear map $\lhd : L \otimes L \to L$, $x \otimes y \mapsto x \lhd y$, satisfying the identity

$$x \lhd (y \lhd z) - (x \lhd y) \lhd z = x \lhd (z \lhd y) - (x \lhd z) \lhd y, \quad \forall\, x, y, z \in L. \tag{2.3.3}$$

It is clear that both previous notions are equivalent to each other. Indeed, for any right pre-Lie product $\lhd$, we can define a left pre-Lie product by $x \rhd y := y \lhd x$. Analogously, we can construct a right pre-Lie product from a left pre-Lie product. For this reason, we will only treat right pre-Lie algebras in this section.

One of the first properties that one can obtain directly from the definition of pre-Lie algebras is a nice connection with more classical Lie algebras.

Lemma 2.3.10. *Let L be a pre-Lie algebra. Then the bracket*

$$[x,y] := x \lhd y - y \lhd x, \quad \forall\, x,y \in L$$

is a Lie bracket, providing L with a Lie algebra structure.

Now we introduce an algebraic structure that will be useful for understanding several aspects of pre-Lie algebras. For the corresponding definition, recall, from Example 2.2.4, that the polynomial algebra $\mathbb{K}[V]$ over a vector space is a cocommutative Hopf algebra whose coproduct is such that $\Delta(v) = v \otimes 1 + 1 \otimes v$, for any $v \in V$. In this section, we will omit the sum symbol in Sweedler notation for the iterated coproduct:

$$\Delta^{[k]}(P) := P^{(1)} \otimes \cdots \otimes P^{(k+1)}, \quad \forall\, P \in \mathbb{K}[V].$$

Definition 2.3.11 (Symmetric brace algebra)**.** A *symmetric brace algebra* is a vector space V together with a linear map $V \otimes \mathbb{K}[V] \to V$, $v \otimes P \mapsto v\{P\}$ satisfying the following identities for any $n \geq 1$ and $v, w_1, \ldots, w_n \in V$, $P \in \mathbb{K}[V]$:

$$v\{1\} = v, \tag{2.3.4}$$
$$\big(v\{w_1,\ldots,w_n\}\big)\{P\} = v\big\{w_1\{P^{(1)}\}, w_2\{P^{(2)}\}, \ldots, w_n\{P^{(n)}\}, P^{(n+1)}\big\}, \tag{2.3.5}$$

where $\Delta^{[n+1]}(P) = P^{(1)} \otimes \cdots \otimes P^{(n+1)}$.

Remark 2.3.12. For elements $v, w_1, \ldots, w_n$ in a symmetric brace algebra V, notice that we have used the notation $v\{w_1,\ldots,w_n\}$ for $v \otimes w_1 \cdots w_n \in V \otimes \mathbb{K}[V]$ with the purpose of pointing out that each w_i is an element of V instead of a monomial of length greater than 1 in $\mathbb{K}[V]$

The fact that symmetric brace algebras are actually equivalent to pre-Lie algebras is due to the following two results by Oudom and Guin [OG08].

Lemma 2.3.13 ([CP21, Lemma 6.2.2])**.** *Let V be a symmetric brace algebra. Then the linear map defined by*

$$v \lhd w = v\{w\} \tag{2.3.6}$$

provides a pre-Lie algebra structure on V.

Proof. We verify the right pre-Lie identity: if $u, v, w \in V$, then we have

$$u \lhd (v \lhd w) - (u \lhd v) \lhd w = u\{v\{w\}\} - (u\{v\})\{w\}.$$

By (2.3.5) and the fact that $\Delta(w) = w \otimes 1 + 1 \otimes w$, we get

$$(u\{v\})\{w\} = u\{v\{w\},1\} + u\{v\{1\},w\} = u\{v\{w\}\} + u\{v,w\},$$

and thus

$$u \lhd (v \lhd w) - (u \lhd v) \lhd w = -u\{v, w\}. \tag{2.3.7}$$

Since this expression is symmetrical in v and w, the pre-Lie identity (2.3.3) holds. □

By an induction argument and the repeatedly use of the pre-Lie identity, we can show that a pre-Lie product can be extended to a symmetric brace product.

Lemma 2.3.14 (Oudom-Guin [OG08] [CP21, Lemma 6.2.3]). *Let $(L, \lhd)$ be a pre-Lie algebra. Then L is equipped with the structure of a symmetric brace algebra by the formulas*

$$\begin{aligned} v\{w\} &= v \lhd w, \\ v\{w_1, \dots, w_n\} &= \big(v\{w_1, \dots, w_{n-1}\}\big)\{w_n\} - \sum_{i=1}^{n-1} v\{w_1, \dots, w_i\{w_n\}, \dots, w_{n-1}\}, \end{aligned} \tag{2.3.8}$$

for all $v, w_1, \dots, w_n \in L$.

Corollary 2.3.15 (Oudom-Guin). *The categories of pre-Lie algebras and symmetric brace algebras are isomorphic.*

One of the main characteristics of pre-Lie algebras is that their associated Lie algebras have enveloping algebras with nice features. Roughly speaking, they are polynomial Hopf algebras with a second associative product equivalent to the pre-Lie product. This motivates the following definition.

Definition 2.3.16 (Right-handed polynomial Hopf algebra). A *right-handed polynomial Hopf algebra* is a polynomial Hopf algebra $(\mathbb{K}[V], \cdot, \Delta)$ that is equipped with a second associative product $*$ with unit $1 \in \mathbb{K}[V]_0 = \mathbb{K}$ making $(\mathbb{K}[V], *, \Delta)$ a cocommutative Hopf algebra such that $\bigoplus_{i \geq 2} \mathbb{K}[V]_i$ is a right ideal of $\mathbb{K}[V]$ with respect to the product $*$.

The relation between right-handed polynomial Hopf algebras and symmetric brace algebras is that they are indeed equivalent notions.

Lemma 2.3.17 ([CP21, Lemma 6.2.1]). *Let V be a vector space. Then V is a symmetric brace algebra if and only if $\mathbb{K}[V]$ is a right-handed polynomial Hopf algebra.*

From the proof of Lemma 6.2.1 in [CP21], given a right-handed polynomial Hopf algebra $\mathbb{K}[V]$ with second associative product $*$, the symmetric braces are constructed by restricting the image of $V \otimes \mathbb{K}[V]$ on V. In other words, the braces are the composition of $*|_{V \otimes \mathbb{K}[V]}$ and the projection map on $\mathbb{K}[V]_1 = V$. Conversely, given a symmetric brace operation on V, the second associative product $*$ on $\mathbb{K}[V]$ can be written as

$$(a_1 \cdots a_l) * (b_1 \cdots b_m) = \sum_f B_0 (a_1\{B_1\}) \cdots (a_l\{B_l\}), \tag{2.3.9}$$

for any $a_1, \dots, a_l, b_1, \dots, b_m \in V$, where the sum is over all the maps $f : \{1, \dots, m\} \to \{0, \dots, l\}$, and for any $0 \leq i \leq l$, we define $B_i = \prod_{j \in f^{-1}(i)} b_j$.

Example 2.3.18. For some small values of l and m in (2.3.9), we have the following expansions:

$$\begin{aligned} a * b &= ab + a\{b\}, \\ a_1a_2 * b &= a_1a_2b + (a_1\{b\})a_2 + a_1(a_2\{b\}), \\ a * b_1b_2 &= ab_1b_2 + b_1(a\{b_2\}) + b_2(a\{b_1\}) + a\{b_1b_2\} \\ &= ab_1b_2 + b_1(a\{b_2\}) + b_2(a\{b_1\}) + (a\{b_1\})\{b_2\} - a\{b_1\{b_2\}\}. \end{aligned}$$

Let $(L, \lhd)$ be a pre-Lie algebra. By Lemma 2.3.14, we can construct a symmetric brace product out of $\lhd$. Moreover, by Lemma 2.3.17, we can construct an associative product $*$ out of the braces following the recipe in (2.3.9). Using the first computation in the last example, we obtain

$$v\{w_1 * w_2\} = v\{w_1, w_2\} + v\{w_1\{w_2\}\}.$$

From (2.3.7), the first term on the right-hand side of the above equation is equal to

$$v\{w_1, w_2\} = (v\{w_1\})\{w_2\} - v\{w_1\{w_2\}\}.$$

Hence we obtain that $v\{w_1 * w_2\} = (v\{w_1\})\{w_2\}$. This relation generalizes in the following result.

Proposition 2.3.19. *Let $(L, \lhd)$ be a pre-Lie algebra and consider the symmetric braces associated to $\lhd$. For any $v, w_1, \ldots, w_n \in L$ we have that*

$$v\{w_1 * w_2 * \cdots * w_n\} = (\cdots((v\{w_1\})\{w_2\})\cdots)\{w_n\}. \tag{2.3.10}$$

Proof. By induction on n. The case $n = 1$ is trivial, and the case $n = 2$ has already been proved. Assume that the result holds for $n - 1$. We claim that

$$v\{a_1 \cdots a_l * b\} = (v\{a_1 \cdots a_l\})\{b\},$$

for any $a_1, \ldots, a_l, b \in L$. Indeed, by (2.3.9), we can write

$$a_1 \cdots a_l * b = a_1 \cdots a_l b + \sum_{i=1}^{l} a_1 \cdots a_{i-1}(a_i\{b\})a_{i+1} \cdots a_l.$$

Then, by (2.3.8), we obtain the proof of our claim

$$v\{a_1 \cdots a_l * b\} = v\{a_1, \ldots, a_l, b\} + \sum_{i=1}^{l} v\{a_1, \ldots, a_i\{b\}, \ldots, a_l\} = (v\{a_1, \ldots, a_l\})\{b\}.$$

The claim extends by linearity to the case that the monomial $a_1 \cdots a_l$ is replaced by a general polynomial $P \in \mathbb{K}[L]$. Therefore, by taking $P = w_1 * \cdots * w_{n-1}$ and $b = w_n$ and applying the induction hypothesis, we deduce that

$$v\{(w_1 * \cdots * w_{n-1}) * w_n\} = (v\{w_1 * \cdots * w_{n-1}\})\{w_n\} = ((\cdots(v\{w_1\})\cdots)\{w_{n-1}\})\{w_n\}.$$

□

Using the first computation in the last example, one notices that the Lie bracket associated to $*$ is given by

$$a * b - b * a = ab + a\{b\} - ab - b\{a\} = a\{b\} - b\{a\} = a \triangleleft b - b \triangleleft a, \qquad \forall\, a, b \in L.$$

It follows that the Lie bracket associated to $*$ coincides with the Lie bracket associated to $\triangleleft$ defined in Lemma 2.3.10. The Cartier-Milnor-Moore Theorem implies the following result.

Theorem 2.3.20 ([CP21, Theorem 6.2.2])**.** *Let $(L, \triangleleft)$ be a pre-Lie algebra and consider L as a Lie algebra where the bracket is given by Lemma 2.3.10. Then the cocommutative Hopf algebra $(\mathbb{K}[L], *, \Delta)$ where $*$ is the associative product extending $\triangleleft$ given in (2.3.9) is the enveloping algebra of L.*

2.3.3 The pre-Lie algebra of rooted trees

Our next objective is to describe a central example of pre-Lie algebras. As in the previous section, let $\mathcal{T}$ be the set of non-planar rooted trees and let $L^{\mathcal{T}}$ be the linear span of $\mathcal{T}$. Next, given two trees $t, t' \in \mathcal{T}$ and $v \in V(t)$, we denote by $t \leftarrow_v t'$ the element in $\mathcal{T}$ obtained by grafting the root of t' via a new edge to v in t.

The previous grafting operation allows to define a bilinear map $\triangleleft$ on $L^{\mathcal{T}}$ given by the sum of the trees obtained by grafting the root of t' to the vertices of t:

$$t \triangleleft t' := \sum_{v \in V(t)} t \leftarrow_v t', \quad \forall\, t, t' \in \mathcal{T}. \tag{2.3.11}$$

It is not difficult to see that $\triangleleft$ satisfies (2.3.3) and hence $L^{\mathcal{T}}$ is a pre-Lie algebra. It turns out that $L^{\mathcal{T}}$ is the free pre-Lie algebra generated by the single-vertex tree $\bullet$. Here, the word "free" means the universal property for pre-Lie algebras: for any set X and $i : X \to L$ an injective map from X into a pre-Lie algebra L, we say that L is the *free pre-Lie algebra* generated by X if for any pre-Lie algebra A and any map $f : X \to A$, there is a unique pre-Lie algebra morphism $\tilde{f} : L \to A$ such that $f = \tilde{f} \circ i$.

Proposition 2.3.21 ([CL01])**.** *The pair $(L^{\mathcal{T}}, \triangleleft)$ is the free pre-Lie algebra over the generator $\bullet$ (the single-vertex tree).*

Associated with the free pre-Lie algebra $L^{\mathcal{T}}$, we can consider the right-handed polynomial Hopf algebra $\mathbb{K}[L^{\mathcal{T}}]$ given by Lemma 2.3.17. It turns out that the product $*$ in (2.3.9) is precisely the associative product of the Hopf algebra of Theorem 2.2.19. We have then the following result.

Proposition 2.3.22. *The right-handed polynomial Hopf algebra $\mathbb{K}[L^{\mathcal{T}}]$ associated to the free pre-Lie algebra $L^{\mathcal{T}}$ is isomorphic to the Grossman-Larson Hopf algebra.*

2.3.4 The pre-Lie exponential and the Magnus operator

The last part of this chapter is devoted to studying some properties of exponentials and logarithms associated with pre-Lie products. In this section, we assume that L is a graded pre-Lie algebra, i.e. $L = \bigoplus_{n\geq 0} L_n$ is graded as a vector space, and the pre-Lie product is graded. We also need that the pre-Lie algebra is *locally finite*, i.e. every graded component L_n is a finite-dimensional vector space. Finally, we assume that L is *connected*, which means $V_0 = 0$.

By Theorem 2.3.20, the enveloping algebra of L is the connected graded cocommutative Hopf algebra $(\mathbb{K}[L], *, \Delta)$. Moreover, we also have the Hopf algebra structure $(\mathbb{K}[L], \cdot, \Delta)$ given by the product of polynomials $\cdot$. Both associative products provide a group structure on the following particular set.

Definition 2.3.23. We define the *completion of* $\mathbb{K}[L]$ with respect to the graduation as the vector space

$$\widehat{\mathbb{K}[L]} := \prod_{n\geq 0} \mathbb{K}[L]_n.$$

We can extend the products $\cdot, *$ and the coproduct Δ to the completion $\widehat{\mathbb{K}[L]}$. We then consider the set of *group-like elements in* $\widehat{\mathbb{K}[L]}$ as the set

$$G(L) := \{v \in \widehat{\mathbb{K}[L]} \,:\, \Delta(v) = v \otimes v\}.$$

One can show that the products $\cdot$ and $*$ endow a group structure to $G(V)$. Also, by the initial assumptions on L (see Section 4.6 in [CP21] for details), we have that the exponentials and logarithms defined in terms of $*$ and $\cdot$ are bijections. More precisely, if

$$\exp^*(v) = \sum_{n\geq 0} \frac{1}{n!} v^{*n}, \qquad \text{and} \qquad \log^*(w) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} w^{*n},$$

then we have the bijective correspondence

$$\exp^* : L \leftrightarrows G(L) : \log^*,$$

and analogously

$$\exp^{\cdot} : L \leftrightarrows G(L) : \log^{\cdot},$$

are bijections, with $\exp^{\cdot}$ and $\log^{\cdot}$ being the exponential and logarithm, respectively, with respect to the product $\cdot$. The first isomorphism allows to define a group structure on L as follows.

Definition 2.3.24. For any $v, w \in L$, define the *Baker-Campbell-Hausdorff group law* as the element

$$\mathrm{BCH}(v, w) := \log^*(\exp^*(v) * \exp^*(w)). \tag{2.3.12}$$

The above bijections can be used to define set automorphism on the pre-Lie algebra L, for instance:

Definition 2.3.25. The *Agrachev-Gamkrelidze operator* ([AG81]) is the map $W : L \to L$ given by

$$W(v) = \log^{\cdot} \circ \exp^*(v), \quad \forall\, v \in L. \tag{2.3.13}$$

The Agrachev-Gamkrelidze operator is also called the *pre-Lie exponential* since one can show that W has an expansion given in the following proposition.

Proposition 2.3.26 ([CP21, Prop. 6.6.1])**.** *The Agrachev-Gamkrelidze operator $W : L \to L$ can be written in terms of the pre-Lie product as*

$$\begin{aligned} W(v) &= v + \frac{1}{2} v \triangleleft v + \frac{1}{6}(v \triangleleft v) \triangleleft v + \cdots \\ &= \sum_{n \geq 0} \frac{1}{(n+1)!} r^{(n)}_{\triangleleft v}(v), \qquad \forall\, v \in L, \end{aligned}$$

where $r^{(0)}_{\triangleleft w}(v) = v$, and $r^{(n)}_{\triangleleft w}(v) = r^{(n-1)}_{\triangleleft w}(v) \triangleleft w$ for $n \geq 1$.

It is easy to see that W is invertible with respect to the composition of maps. The inverse of W is, of course, another set automorphism and receives the following definition.

Definition 2.3.27 (Pre-Lie Magnus operator)**.** The *pre-Lie Magnus operator* is the bijective map $\Omega : L \to L$ given by

$$\Omega(v) = \log^* \circ \exp^{\cdot}(v), \quad \forall\, v \in L. \tag{2.3.14}$$

As well as the Agrachev-Gamkrelidze operator, the Magnus operator writes in terms of iterated pre-Lie products.

Proposition 2.3.28 (Pre-Lie Magnus expansion [CP21, Prop. 6.5.1])**.** *The Magnus operator satisfies the fixed point equation*

$$\Omega(v) = \sum_{n \geq 0} \frac{B_n}{n!} r^{(n)}_{\triangleleft \Omega(v)}(v), \qquad \forall\, v \in L, \tag{2.3.15}$$

where $\{B_n\}_{n\geq 0}$ is the sequence of Bernoulli numbers defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n\geq 0} \frac{B_n}{n!} x^n. \tag{2.3.16}$$

The Magnus operator allows us to describe another group law on L obtained by transporting the group law on $G(V)$ using the exponential $\exp^{\cdot}$.

Definition 2.3.29. The *Agrachev-Gamkrelidze group law on L* is defined by

$$v\#w := \log^{\cdot}\left(\exp^{\cdot}(v) * \exp^{\cdot}(w)\right), \qquad \forall\, v, w \in L. \tag{2.3.17}$$

By definition, it readily follows that

$$W(\mathrm{BCH}(\Omega(v), \Omega(w))) = v\#w. \tag{2.3.18}$$

Furthermore, we have

Proposition 2.3.30 ([CP21, Prop. 6.6.2])**.** *For any $v, w \in L$, we have that*

$$v\#w = w + v\{\exp^*(\Omega(w))\} = w + v\{\exp^{\cdot}(w)\}. \tag{2.3.19}$$

Proof. The second equality follows from the definition of Ω since

$$\exp^*(\Omega(w)) = \exp^* \circ(\log^* \circ \exp^{\cdot}(w)) = \exp^{\cdot}(w).$$

On the other hand, the first equality in the statement of the proposition is equivalent to the equality

$$W(\mathrm{BCH}(v, w)) = W(w) + W(v)\{\exp^*(w)\} \tag{2.3.20}$$

since (2.3.18) holds and W is a bijection with inverse Ω, and the braces extend the pre-Lie product $\triangleleft$. We now introduce a notation for the W map. Notice that pre-Lie algebra can be augmented with a unit by setting $1\{1\} = 1 \triangleleft 1 := 1$ and $1\{v\} = 1 \triangleleft v = v$. Extending Proposition 2.3.19, we also set

$$1\{v_1 * \cdots * v_n\} := (\cdots((1\{v_1\})\{v_2\})\cdots)\{v_n\}.$$

The previous notation, together with $r_{\triangleleft v}(w) := r^{(1)}_{\triangleleft v}(w) = w \triangleleft v$, permits to write

$$W(v) = \frac{e^{r_{\triangleleft v}} - \mathrm{id}}{r_{\triangleleft v}}(v) = e^{r_{\triangleleft v}}(1) - 1 = 1\{\exp^*(v)\} - 1. \tag{2.3.21}$$

For this reason, the Agrachev-Gamkrelidze operator is also known as the *pre-Lie expo-*

nential. In particular, by the definition of the BCH group law, we have that

$$\begin{aligned} W(\mathrm{BCH}(v,w)) &= 1\{\mathrm{BCH}(v,w)\} - 1 \\ &= 1\{\exp^*(v) * \exp^*(w)\} - 1 \\ &= (1\{\exp^*(v)\})\{\exp^*(w)\} - 1 \\ &= (1\{\exp^*(v) - 1\})\{\exp^*(w)\} + 1\{\exp^*(w)\} - 1 \\ &= W(v)\{\exp^*(w)\} + W(w), \end{aligned}$$

which is precisely (2.3.20). $\square$

Using the iterated pre-Lie product, we can find another expression for the # group law on L.

Corollary 2.3.31. *For any $v, w \in L$, we have that*

$$v\#w = w + e^{r_{\triangleleft\Omega(w)}}(v). \tag{2.3.22}$$

Proof. From the proof of Proposition 2.3.30, we know that

$$W(v) = e^{r_{\triangleleft v}}(1) - 1 = 1\{\exp^*(v)\} - 1.$$

We obtain the desired expression for $v\#w$ by applying Proposition 2.3.19 in the following way:

$$\begin{aligned} v\#w &= w + v\{\exp^*(\Omega(w))\} \\ &= w + \sum_{n\geq 0} \frac{1}{n!} v\{\Omega(w)^{*n}\} \\ &= w + \sum_{n\geq 0} \frac{1}{n!} r_{\triangleleft\Omega(w)}(v) \\ &= w + e^{r_{\triangleleft\Omega(w)}}(v). \end{aligned}$$

$\square$

We finish this section by giving a description of the formula (2.3.14) that defines the pre-Lie Magnus operator that will be of fundamental importance in the computations in Chapter 9. First, let us introduce some notation. First, we say that an *ordered partition of* $[n]$ *of length* k is a k-tuple $(I_1, \ldots, I_k)$ such that $I_1, \ldots, I_k$ are pairwise disjoint non-empty subsets of $[n] = \{1, \ldots, n\}$ with $I_1 \cup \cdots \cup I_k = [n]$. The set of ordered partitions of $[n]$ of length k is denoted by $\mathcal{OP}^k(n)$.

Definition 2.3.32. For any $n \geq 1$ and $v_1, \ldots, v_n \in L$, consider the monomial $v_1 \cdots v_n \in$

$\mathbb{K}[L]$ (i.e. using the commutative polynomial product on $\mathbb{K}[L]$). We then define

$$\mathrm{sol}_1(v_1 \cdots v_n) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \sum_{\substack{\pi \in \mathcal{OP}^k(n) \\ \pi=(I_1,\ldots,I_k)}} v_{I_1} * \cdots * v_{I_k},$$

where, for each $1 \leq j \leq k$, v_{I_j} is the monomial $v_{I_j} = v_{i_1} \cdots v_{i_r}$ if $I_j = \{i_1, \ldots, i_r\}$. Then we linearly extend sol_1 to be defined on L and on the completion $\hat{L}$.

The reader may notice that sol_1 resembles the expansion of a logarithm with respect to the associative product $*$. The following theorem shows that is indeed the case.

Theorem 2.3.33 ([CP13, Thm. 4.2])**.** *For $v \in L$, we have, in $\hat{L}$:*

$$\Omega(v) = \mathrm{sol}_1(\exp^{\cdot}(v)), \tag{2.3.23}$$

where $\exp^{\cdot}(v) = \sum\limits_{n \geq 0} \frac{v^{\cdot n}}{n!}$*, with $v^{\cdot n}$ the n-th commutative polynomial power of v in $\mathbb{K}[L]$.*

Chapter 3

Non-commutative Probability

The present chapter aims to explain the central notions and results in non-commutative probability theory, emphasizing the combinatorial point of view via cumulants and set partitions. We first recall the basics of classical probability theory in Section 3.1 and see how they are translated in a non-commutative setting in Section 3.2. In particular, the non-commutative analogue of one of the most relevant notions in classical probability, the concept of independence, is studied in Section 3.3, where we present the five natural notions of non-commutative independence. Next, in Section 3.4, we present the different posets of set partitions that allow defining the non-commutative cumulants for each notion of independence. Afterwards, the cumulant functionals are applied in Section 3.5 to study the several notions of additive convolutions for each notion of independence. We finish in Section 3.6 by explaining the definition of an extension of non-commutative probability, known as conditionally free independence, which can be understood by the methods and techniques of this thesis.

For the convenience of the reader unfamiliar with non-commutative probability theory, we have included some proofs that exemplify the combinatorial tools in the theory. The interested reader is suggested to consult the book of Nica and Speicher [NS06], which describes the combinatorial point of view of free probability, as well as the book of Mingo and Speicher [MS17b], which treats the connection between free probability and random matrix theory.

3.1 Classical probability

We begin the chapter by reviewing basic concepts of classical probability theory, in order to make the analogy with non-commutative probability more transparent.

Definition 3.1.1. A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a set, $\mathcal{F}$ is a σ-algebra over Ω, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{F})$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Recall that a *random variable* is a $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{C}$. Associated with X, we can define its *distribution* as the probability measure μ_X on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ given by $\mu_X(A) := \mathbb{P}(X \in A)$, for any A element of the Borel σ-algebra $\mathcal{B}(\mathbb{C})$.

Another fundamental concept associated to a random variable is the expected value.

Definition 3.1.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{C}$ be a random variable. For any $f : \mathbb{C} \to \mathbb{C}$ bounded measurable function, we define the *expected value of* $f(X)$, denoted by $\mathbb{E}(f(X))$, by

$$\mathbb{E}(f(X)) = \int_{\Omega} f(X(\omega))\, d\mathbb{P}(\omega) = \int_{\mathbb{C}} f(x)\, d\mu_X(x) \tag{3.1.1}$$

provided that the integral exists.

The above definition allows us to define a fundamental sequence of real numbers associated to a real random variable $X : \Omega \to \mathbb{R}$, called the *sequence of moments of* X.

Definition 3.1.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ be a real random variable. For $n \geq 0$, we define the *n-th moment of* X by

$$m_n(X) := \mathbb{E}(X^n) = \int_{\mathbb{R}} x^n d\mu_X(x),$$

whenever the integral exists.

Remark 3.1.4. The sequence of moments $\{m_n(X)\}_{n\geq 0}$ of a random variable X is always well-defined when X has bounded support. In the latter case, all the moments of X exist and the Stone-Weierstrass theorem implies that the sequence of moments uniquely determines the distribution of X.

Another fundamental concept in the theory of classical probability is the notion of independence. Let us recall the definition.

Definition 3.1.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{X_i\}_{i\in I}$ be a family of random variables. We say that the family is *independent* if

$$\mathbb{P}\left(\bigcap_{j\in J}\{X_j \in A_j\}\right) = \prod_{j\in J} \mathbb{P}(X_j \in A_j),$$

for all $A_j \in \mathcal{B}(\mathbb{C})$ with $j \in J$ and for all $J \subset I$ finite.

An important property of the previous definition is that if X, Y are independent random variables and $f, g : \mathbb{C} \to \mathbb{C}$ are bounded measurable functions, independence implies that

$$\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y)). \tag{3.1.2}$$

In particular, if we have two independent random variables X and Y with all their moments, then

$$\mathbb{E}(X^m Y^n) = \mathbb{E}(X^m)\mathbb{E}(Y^n), \quad \forall\, m, n \geq 0.$$

Observe that if X and Y have bounded support, the sequences of moments $\mathbb{E}(X^m)$ and $\mathbb{E}(Y^n)$ determine the joint distribution of X and Y. From an algebraic point of view, the independence of two random variables can be seen as a "recipe" for computing moments of X^mY^n from the moments of X^m and Y^n.

Independence of random variables allows us to define the notion of convolution. More precisely, given two independent random variables with bounded support X and Y, we define its *additive convolution* as the probability measure $\mu_X * \mu_Y$ given by the distribution of $X+Y$. By independence, the moments of $X+Y$ are determined by the moments of X and Y. In other words, the probability measure μ_{X+Y} is indeed determined by the probability measures μ_X and μ_Y.

A very effective tool to treat the additive convolution is the *characteristic function*

$$\hat{\mu}_X(t) := \mathbb{E}\left(e^{itX}\right), \qquad \forall\, t \in \mathbb{R}.$$

Notice that if X and Y are independent random variables, then

$$\hat{\mu}_{X+Y} = \mathbb{E}\left(e^{it(X+Y)}\right) = \mathbb{E}\left(e^{itX}\right)\mathbb{E}\left(e^{itY}\right) = \hat{\mu}_X \hat{\mu}_Y.$$

Then, if we define the *C-transform of X* by $C_X(t) := \log \hat{\mu}_X(t)$, we get that

$$C_{X+Y}(t) = C_X(t) + C_Y(t),$$

for any $t \in \mathbb{R}$. In other words, the C-transform has the property of linearizing the additive convolution. Moreover, under the assumption that X and Y have all their moments, the characteristic functions and the C-transform have the following power series expansion:

$$\begin{aligned} \hat{\mu}_X(t) &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} m_n(X), \\ C_X(t) &= \sum_{n=1}^{\infty} \frac{(it)^n}{n!} c_n(X), \end{aligned}$$

for a certain sequence of complex numbers $\{c_n(X)\}_{n\geq 1}$.

Definition 3.1.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a random variable with all their moments. The sequence of complex numbers $\{c_n(X)\}_{n\geq 1}$ such that

$$\log \hat{\mu}_X(t) = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} c_n(X)$$

is called the *sequence of (classical) cumulants of X*.

One can observe that the information provided by the sequence of cumulants of a random variable is equivalent to the information given by the sequence of moments. In particular, the sequence of cumulants uniquely determines the distribution of X if and only if the sequence of moments uniquely determines the distribution of X. However, the cumulants behave nicer than the moments of X when dealing with additive convolution since

$$c_n(X+Y) = c_n(X) + c_n(Y),$$

for any $n \geq 1$ when X and Y are independent random variables.

3.2 Non-commutative probability theory

The natural setup to develop non-commutative probability theory appears when considering a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ from an operator-algebraic point of view. In other words, the information of the probability space is encoded in the algebra of random variables $X : \Omega \to \mathbb{C}$ together with their expected values $\mathbb{E}(X)$. The mathematical object that abstracts the above information is formally given in the next definition.

Definition 3.2.1 (Non-commutative probability space)**.** A *non-commutative probability space* is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. An element $a \in \mathcal{A}$ is called a *random variable*. The linear functional φ is sometimes called the *expectation*. With this analogy, the sequence of complex numbers $\{\varphi(a^n)\}_{n\geq 0}$ is called the *sequence of moments* of $a \in \mathcal{A}$.

Example 3.2.2. 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We denote $L^{\infty-} := L^{\infty-}(\Omega, \mathbb{P})$ the vector space of all the random variables on Ω with all their moments. By Cauchy-Schwarz inequality, it is easy to see that the product of two elements of $L^{\infty-}$ is a random variable with all their moments. Thus $L^{\infty-}$ is an algebra with unital element given by the constant function 1. On the other hand, $\mathbb{E}$ can be regarded as linear functional on $L^{\infty-}$ such that $\mathbb{E}(1) = 1$. Hence $(L^{\infty-}, \mathbb{E})$ is a non-commutative probability space.

2. For $d \in \mathbb{N}$, we consider $M_d(\mathbb{C})$ the algebra of $d \times d$ complex matrices along with the usual multiplication of matrices. In this way, $M_d(\mathbb{C})$ is a unital algebra. In addition, if we define the linear functional $\mathrm{tr} : M_d(\mathbb{C}) \to \mathbb{C}$ by

$$\mathrm{tr}(A) = \frac{1}{d}\sum_{i=1}^{d} a_{ii}, \quad \forall\, A = (a_{ij})_{i,j=1}^{d} \in M_d(\mathbb{C}),$$

then $(M_d(\mathbb{C}), \mathrm{tr})$ is a non-commutative probability space.

3. We can combine the latter two examples in the following one. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider $\mathcal{A} = L^{\infty-}$. For $d \in \mathbb{N}$, we consider the algebra of random matrices $M_d(\mathcal{A})$. Then$(M_d(\mathcal{A}), \mathrm{tr} \circ \mathbb{E})$ is a non-commutative probability space.

Note that we have the algebra isomorphism $M_d(\mathcal{A}) \cong M_d(\mathbb{C}) \otimes \mathcal{A}$. Actually, it is possible to show that if $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ are probability spaces, then $(\mathcal{A} \otimes \mathcal{B}, \varphi \otimes \psi)$ is again a non-commutative probability space.

It is desirable that the analogous notion of distribution in the non-commutative context contains all the information of a random variable. Before giving the corresponding definition, we denote by $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ the unital algebra freely generated by non-commutative indeterminates $X_1, \ldots, X_n$.

Definition 3.2.3. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $a_1, \ldots, a_n \in \mathcal{A}$. The *joint distribution of* $\{a_1, \ldots, a_n\}$ is the linear functional $\mu_{a_1,\ldots,a_n} : \mathbb{C}\langle X_1, \ldots, X_n\rangle \to \mathbb{C}$ defined by

$$\mu_{a_1,\ldots,a_n}(X_{i_1} \cdots X_{i_m}) = \varphi(a_{i_1} \cdots a_{i_m}), \quad \forall\, i_1, \ldots, i_m \in [n],\, m \geq 1.$$

In the above definition, recall that $[n]$ stands for the set $\{1, \ldots, n\}$. The notion of non-commutative probability space can be enhanced to consider positivity. Recall that $\mathcal{A}$ is a $*$-*algebra* if $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$ with a map $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ such that $(ab)^* = b^*a^*$ and $(\lambda a)^* = \overline{\lambda} a^*$, for any $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Definition 3.2.4. A $*$-*probability space* is a non-commutative probability space $(\mathcal{A}, \varphi)$ such that $\mathcal{A}$ is a $*$-algebra and φ is *positive*, i.e. $\varphi(a^*a) \geq 0$ for any $a \in \mathcal{A}$.

Remark 3.2.5. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Several elementary consequences can be obtained from the fact that φ is positive. To mention a few, we have that φ is *selfadjoint* in the sense that $\varphi(a^*) = \overline{\varphi(a)}$ for any $a \in \mathcal{A}$. Also, we have the Cauchy-Schwarz inequality

$$|\varphi(b^*a)| \leq \varphi(a^*a)\varphi(b^*b), \quad \forall\, a, b \in \mathcal{A}.$$

The concept of distribution also extends to the $*$-case. Indeed, if $(\mathcal{A}, \varphi)$ is a $*$-probability space and $a_1, \ldots, a_n \in \mathcal{A}$, we define the *joint* $*$-*distribution of* $\{a_1, \ldots, a_n\}$ as the joint distribution of $\{a_1, a_1^*, \ldots, a_n, a_n^*\}$. In particular, the sequence of moments of $a \in \mathcal{A}$ is given by the collection of complex numbers

$$\{\varphi(a^{\epsilon_1} \cdots a^{\epsilon_m}) \,:\, m \geq 1,\, (\epsilon_1, \ldots, \epsilon_m) \in \{1, *\}^m\}.$$

In the context of $*$-probability spaces, we say that a random variable $a \in \mathcal{A}$ is *normal* if $a^*a = aa^*$. We also say that a is *selfadjoint* if $a^* = a$. These types of random variables become relevant when $\mathcal{A}$ has also a C^*-structure, i.e. $\mathcal{A}$ is endowed with a norm $\|\cdot\|$ making it a complete normed vector space and $\|aa^*\| = \|a\|^2$ for any $a \in \mathcal{A}$. The concept of distribution is translated into a definition with a more analytic flavour.

Definition 3.2.6. Leg $(\mathcal{A}, \varphi)$ be a $*$-probability space and $a \in \mathcal{A}$ be a normal random variable. If there exists a probability measure μ with a compact support on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} z^k \bar{z}^\ell \, d\mu(z) = \varphi\left(a^k (a^*)^\ell\right), \quad \forall\, k, \ell \in \mathbb{N}, \tag{3.2.1}$$

we say that μ is the *distribution (in the analytic sense) of a.*

The analytic distribution of a random variable does not necessarily exist. However, it exists for a good number of important examples. Furthermore, if we add an extra analytic structure to $\mathcal{A}$, for instance, the previously mentioned C^*-algebra structure, we can guarantee the existence of such analytic distribution.

Theorem 3.2.7 ([NS06, Cor. 3.14])**.** *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space such that $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$ be a normal random variable. Then a has a distribution in the analytic sense.*

Remark 3.2.8. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and $a \in \mathcal{A}$ be a selfadjoint random variable. From the previous theorem, we know that a has analytic distribution μ whose support is compact and contained in $\mathbb{R}$. Thus μ is determined by its sequence of moments $\{m_n = \int_{\mathbb{R}} t^n \, d\mu(t)\}_{n \geq 1}$. Hence, in the framework of C^*-spaces, the analytic and algebraic distributions have the same information.

3.3 Types of non-commutative independence

A fundamental concept in probability theory is the notion of independence. From the algebraic point of view of non-commutative probability, a notion of independence can be considered as a concrete rule to compute mixed moments $\varphi(a^{m_1} b^{n_1} \cdots a^{m_s} b^{n_s})$ in terms of the sequences of moments of a and b when they are, in some sense, independent random variables.

The notions of independence that we will consider in this section are the following.

Definition 3.3.1 (Notions of non-commutative independence)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $\{\mathcal{A}_i\}_{i=1}^N$ be a family of subalgebras of $\mathcal{A}$. For $n \geq 1$, consider random variables $a_j \in \mathcal{A}_{i_j}$ for $j = 1, \ldots, n$, with $i_1, \ldots, i_n \in [N]$ and $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$.

1. We say that the family $\{\mathcal{A}_i\}_{i=1}^N$ is *tensor independent* if for any $a_1, \ldots, a_n \in \mathcal{A}$ as above we have
$$\varphi(a_1 \cdots a_n) = \prod_{i=1}^N \varphi\left(\overrightarrow{\prod_{j:\, a_{i_j} \in \mathcal{A}_i}} a_{i_j}\right).$$

2. If in addition the subalgebras $\{\mathcal{A}_i\}_{i=1}^N$ are unital, we say that the family $\{\mathcal{A}_i\}_{i=1}^N$ is *freely independent* if for any $a_1, \ldots, a_n \in \mathcal{A}$ as above we have

$$\varphi(a_1 \cdots a_n) = 0$$

whenever $\varphi(a_j) = 0$ for any $j = 1, \ldots, n$.

3. We say that the family $\{\mathcal{A}_i\}_{i=1}^N$ is *Boolean independent* if for any $a_1, \ldots, a_n \in \mathcal{A}$ as above we have

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n).$$

4. We say that the family $\{\mathcal{A}_i\}_{i=1}^N$ is *monotone independent* if for any $a_1, \ldots, a_n \in \mathcal{A}$ as above we have

$$\varphi(a_1 \cdots a_\ell \cdots a_n) = \varphi(a_\ell)\varphi(a_1 \cdots a_{\ell-1}a_{\ell+1} \cdots a_n)$$

whenever $i_{\ell-1} < i_\ell$ and $i_\ell > i_{\ell+1}$.

5. We say that the family $\{\mathcal{A}_i\}_{i=1}^N$ is *anti-monotone independent* if for any $a_1, \ldots, a_n \in \mathcal{A}$ as above we have

$$\varphi(a_1 \cdots a_\ell \cdots a_n) = \varphi(a_\ell)\varphi(a_1 \cdots a_{\ell-1}a_{\ell+1} \cdots a_n)$$

whenever $i_{\ell-1} > i_\ell$ and $i_\ell < i_{\ell+1}$.

Remark 3.3.2. According to the definition, the tensor, free, and Boolean independence are commutative in the sense that the formulas do not depend on the order of the subalgebras in the list $\{\mathcal{A}_i\}_{i=1}^N$. The opposite case occurs with the monotone and anti-monotone independence: it is not true that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are (anti-)monotone independent subalgebras, then so are $\mathcal{A}_2$ and $\mathcal{A}_1$. For this reason, in the particular case that $N = 2$ we will say that $\mathcal{A}_1$ *is monotone independent of* $\mathcal{A}_2$ if the family $\{\mathcal{A}_1, \mathcal{A}_2\}$ is monotone independent. Furthermore, it is clear that $\mathcal{A}_1, \ldots, \mathcal{A}_N$ are monotone independent subalgebras if and only if $\mathcal{A}_N, \ldots, \mathcal{A}_1$ are anti-monotone independent subalgebras.

Remark 3.3.3. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. For any of the previous notions of independence, we can generalize it by stating that the subsets $\mathcal{X}_1, \ldots, \mathcal{X}_N \subseteq \mathcal{A}$ are independent if and only if $\mathcal{A}_1, \ldots, \mathcal{A}_N$ are independent, with $\mathcal{A}_i$ being the subalgebra of $\mathcal{A}$ generated by $\mathcal{X}_i$, for every $i = 1, \ldots, N$. In the unital case, we consider the unital subalgebras generated by every $\mathcal{X}_i$ instead.

Remark 3.3.4. 1. The notion of tensor independence arises from a natural non-commutative generalization of classical independence since it models the usual formula

$$\mathbb{E}(X^{m_1}Y^{n_1} \cdots X^{m_s}Y^{n_s}) = \mathbb{E}\left(X^{\sum_{i=1}^s m_i}\right)\mathbb{E}\left(Y^{\sum_{i=1}^s n_i}\right),$$

for X and Y independent classical random variables.

2. Free independence has been particularly studied among the previous notions of non-commutative independence since it has fruitful relations and applications with combinatorics and random matrix theory. It was introduced by Dan Voiculescu in [Voi85] in the 1980s, aiming to solve the problem of isomorphisms between von Neumann algebras generated by free groups. More precisely, if $\mathbb{F}_n$ stands for the free group on n generators and $L(\mathbb{F}_n)$ stands for the von Neumann algebra generated by $\mathbb{F}_n$, the isomorphisms problem consists of proving or disproving whether $L(\mathbb{F}_m)$ and $L(\mathbb{F}_n)$ non-isomorphic if $m \neq n$.

3. Speicher and Woroudi introduced Boolean independence in [SW97]. Finally, the notions of monotone and anti-monotone independence were introduced by Muraki in [Mur00] based on previous work on the arcsine Brownian motion in the monotone Fock space.

The fact that the notions of independence in Definition 3.3.1 are indeed rules for computing mixed moments is formally stated in the next proposition.

Proposition 3.3.5. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and, for each notion of independence in Definition 3.3.1, let $\{\mathcal{A}_i\}_{i=1}^N$ be a family of independent subalgebras of $\mathcal{A}$ (unital subalgebras in the free case). If, for $n \geq 1$, $a_j \in \mathcal{A}_{i_j}$ for each $j = 1, \ldots, n$, then $\varphi(a_1 \cdots a_n)$ is uniquely determined by the restrictions $\{\varphi|_{\mathcal{A}_j}\}_{j=1}^N$.*

Proof. Since the result is clear for the tensor, Boolean, monotone and anti-monotone independence, we will only prove the result for the free case. We will prove the result by induction on n. Observe that the base case $n = 1$ is trivial. First, we observe that we can assume that consecutive elements belong to different subalgebras, i.e. $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$: otherwise, we can consider $a_j a_{j+1} \in \mathcal{A}_j$ and apply induction hypothesis.

Recalling that the subalgebras are unital, we can consider the elements $a_j^o := a_j - \varphi(a_j)1_{\mathcal{A}}$, for any $j = 1, \ldots, n$. It is clear that $a_j^o \in \mathcal{A}_{i_j}$ and $\varphi(a_j^o) = 0$, for any j. Hence we can write

$$\begin{aligned}
\varphi(a_1 \cdots a_n) &= \varphi\left((a_1^o + \varphi(a_1)1_{\mathcal{A}}) \cdots (a_n^o + \varphi(a_n)1_{\mathcal{A}})\right) \\
&= \varphi(a_1^o \cdots a_n^o) + \sum_{S \subsetneq [n]} \varphi(a_S^o) \prod_{j \in [n] \setminus S} \varphi(a_j),
\end{aligned}$$

where if $S = \{i_1 < \cdots < i_s\}$, then $a_S^o := a_{i_1}^o \cdots a_{i_s}^o$. Observe that a_S^o is a product of less than n elements of the subalgebras $\{\mathcal{A}_i\}_{i=1}^N$, for every $S \subsetneq [n]$. Then we can apply the induction hypothesis to obtain that each $\varphi(a_S^o)$ is uniquely determined by the restrictions $\{\varphi|_{\mathcal{A}_j}\}_{j=1}^N$. On the other hand, the first term in the right-hand side of the above equation satisfies that $i_j \neq i_{j+1}$ for every $j < n$ and $\varphi(a_j^o) = 0$ for every $j = 1, \ldots, n$. Hence we can apply the definition of free independence in order to conclude that $\varphi(a_1^o \cdots a_n^o) = 0$. By combining both conclusions, we get that $\varphi(a_1 \cdots a_n)$ is determined by $\{\varphi|_{\mathcal{A}_j}\}_{j=1}^N$, as we wanted to show. $\square$

The proof of the above proposition implies that the notion of free independence can be characterized in the following way.

Proposition 3.3.6. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $\{\mathcal{A}_i\}_{i=1}^N$ be a family of unital subalgebras of $\mathcal{A}$. We have that the family $\{\mathcal{A}_i\}_{i=1}^N$ is freely independent if and only if, for any $n \geq 1$ and random variables $a_j \in \mathcal{A}_{i_j}$ for $j = 1, \ldots, n$ with $i_1, \ldots, i_n \in [N]$ and $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$, we have that*

$$\varphi(a_1 \cdots a_n) = \sum_{S \subsetneq [n]} (-1)^{n-|S|+1} \varphi(a_S) \prod_{j \in [n] \setminus S} \varphi(a_j), \tag{3.3.1}$$

where $a_S = a_{i_1} \cdots a_{i_s}$ if $S = \{i_1 < \cdots < i_s\}$.

The importance of the previous notions of independence is that we can find analogous notions to several of the main concepts of classical probability. Indeed, for each independence, we can have the concepts of product space, convolutions, cumulants, central limit theorems and Lévy processes. In the forthcoming sections, we will review some of these ideas.

3.3.1 Non-commutative types of independence as natural products

Continuing with the construction of analogous concepts to classical probability theory, we have that the notion of independence can be described through the product space of probability spaces. Indeed, if $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ are probability spaces, the product probability space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$ is defined by the relation

$$(\mathbb{P}_1 \times \mathbb{P}_2)(A_1 \times A_2) = \mathbb{P}_1(A_1)\mathbb{P}_2(A_2),$$

for all $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$. In particular, if $X_i : \Omega_i \to \mathbb{C}$ is a $\mathcal{F}_i$-random variable for $i = 1, 2$, we can find copies $X_1', X_2' : \Omega_1 \times \Omega_2 \to \mathbb{C}$ of X_1 and X_2 respectively, such that X_1', X_2' are independent random variables in the product space.

When we are interested in considering a non-commutative analogous of independence, one can ask about the existence of products of non-commutative probability spaces satisfying some properties motivated by the classical product of spaces, for instance, the factorization property $\varphi(ab) = \varphi(a)\varphi(b)$ for a, b (in some sense) independent random variables. The problem of classification of types of independence was initiated by Schürmann [Sch95], where he defined three products of non-commutative probability spaces that correspond to the notion of tensor, free, and Boolean independence and conjectured that these three notions are the only possible. Later, Speicher [Spe97] proved the conjecture by introducing the notion of a universal product of non-commutative probability spaces. The same conjecture was proved by Ben Ghorbal and Schürmann [GS02] in a categorical setting for universal products.

Let us explain the categorical framework for universal products introduced in [GS02]. Let $\mathcal{K}$ be the category of pairs $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is an associative algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional. Observe that we do not require that the algebra $\mathcal{A}$ is unital. Also, recall that in the category of associative algebras over $\mathbb{C}$, the (categorical) coproduct $\mathcal{A}_1 \sqcup \mathcal{A}_2$ is given by the free product of algebras

$$\mathcal{A}_1 * \mathcal{A}_2 = \bigoplus_{n=1}^{\infty} \bigoplus_{\substack{i_1,\dots,i_n \in \{1,2\} \\ i_1 \neq i_2, \dots, i_{n-1} \neq i_n}} \mathcal{A}_{i_1} \otimes \cdots \otimes \mathcal{A}_{i_n},$$

with injective algebra homomorphisms $i_1 : \mathcal{A}_1 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ and $i_2 : \mathcal{A}_2 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$. By the universal property of the categorical coproduct, for any pair of algebra homomorphisms $j_1 : \mathcal{B}_1 \to \mathcal{A}_1$ and $j_2 : \mathcal{B}_2 \to \mathcal{A}_2$ and if $(\mathcal{B}_1 \sqcup \mathcal{B}_2, \iota_1, \iota_2)$ is the coproduct of $\mathcal{B}_1$ and $\mathcal{B}_2$, we can find a unique algebra homomorphism $j_1 \sqcup j_2 : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{B}_1 \arrow[r, "j_1"] \arrow[dl, "\iota_1"'] & \mathcal{A}_1 \arrow[dr, "i_1"] & \\
\mathcal{B}_1 \sqcup \mathcal{B}_2 \arrow[rrr, "j_1 \sqcup j_2"] & & & \mathcal{A}_1 \sqcup \mathcal{A}_2 \\
 & \mathcal{B}_2 \arrow[r, "j_2"'] \arrow[ul, "\iota_2"] & \mathcal{A}_2 \arrow[ur, "i_2"'] &
\end{tikzcd}$$

Definition 3.3.7 ([GS02, Mur03])**.** A *universal product* is a map $\mathcal{K} \times \mathcal{K} \to \mathcal{K}$ given by

$$((\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)) \mapsto (\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \odot \varphi_2),$$

such that the following conditions hold:

1. Under the identification $\mathcal{A}_1 \sqcup \mathcal{A}_2 \cong \mathcal{A}_2 \sqcup \mathcal{A}_1$, we have
$$\varphi_1 \odot \varphi_2 = \varphi_2 \odot \varphi_1.$$

2. Under the identification $(\mathcal{A}_1 \sqcup \mathcal{A}_2) \sqcup \mathcal{A}_3 \cong \mathcal{A}_1 \sqcup (\mathcal{A}_2 \sqcup \mathcal{A}_3)$, we have
$$(\varphi_1 \odot \varphi_2) \odot \varphi_3 = \varphi_1 \odot (\varphi_2 \odot \varphi_3).$$

3. For any pair of algebra homomorphisms $j_1 : \mathcal{B}_1 \to \mathcal{A}_1$ and $j_2 : \mathcal{B}_2 \to \mathcal{A}_2$, we have
$$(\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2) = (\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2).$$

4. We have

$$\begin{aligned}(\varphi_1 \odot \varphi_2) \circ i_1 &= \varphi_1,\\ (\varphi_1 \odot \varphi_2) \circ i_2 &= \varphi_2,\end{aligned}$$

and

$$(\varphi_1 \odot \varphi_2)(i_1(a)i_2(b)) = (\varphi_1 \odot \varphi_2)(i_2(b)i_1(a)) = \varphi_1(a)\varphi_2(b), \qquad \text{for } a \in \mathcal{A}_1, b \in \mathcal{A}_2.$$

We can find the interpretation of each of the aforementioned conditions as analogous to the following notions in classical probability:

1. Condition 1 corresponds to the fact that classical independence is a commutative notion: if X and Y are independent random variables, then Y and X are also independent.

2. Condition 2 corresponds to the fact that if $\{X_1, X_2\}$ and X_3 are independent random variables, with X_1 and X_2 being independent random variables, then X_1 and $\{X_2, X_3\}$ are independent random variables, with X_2 and X_3 being independent.

3. Condition 3 corresponds to the fact that if X and Y are independent random variables, then $f(X)$ and $g(Y)$ are independent random variables for any f and g measurable functions.

The following definition provides examples of universal products.

Definition 3.3.8. Let $(\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)$ be elements in $\mathcal{K}$ and consider $a_1, \ldots, a_n \in \mathcal{A}_1 \sqcup \mathcal{A}_2$ for any $n \geq 2$ such that $a_k = i_{j_k}(a_k^{(j_k)})$ with $a_k^{(j_k)} \in \mathcal{A}_{j_k}$, $j_k \in \{1, 2\}$, for $k = 1, \ldots, n$, and $j_1 \neq j_2$, $j_2 \neq j_3, \ldots, j_{n-1} \neq j_n$.

1. We define the *tensor product* $\varphi_1 \otimes \varphi_2$ as the linear functional on $\mathcal{A}_1 \sqcup \mathcal{A}_2$ given by

$$(\varphi_1 \otimes \varphi_2)(a_1 \cdots a_n) = \varphi_1 \left(\overrightarrow{\prod_{k\colon j_k=1}} a_k^{(1)} \right) \varphi_2 \left(\overrightarrow{\prod_{\ell\colon j_\ell=2}} a_\ell^{(2)} \right), \tag{3.3.2}$$

where the notation $\overrightarrow{\prod_k} a_k$ refers to the ordered product of the a_k's, where the order is the same as the elements have in the product $a_1 \cdots a_n$.

2. We define the *Boolean product* $\varphi_1 \diamond \varphi_2$ as the linear functional on $\mathcal{A}_1 \sqcup \mathcal{A}_2$ given by

$$(\varphi_1 \diamond \varphi_2)(a_1 \cdots a_n) = \left(\prod_{k\colon j_k=1} \varphi_1 \left(a_k^{(1)} \right) \right) \left(\prod_{\ell\colon j_\ell=2} \varphi_2 \left(a_\ell^{(2)} \right) \right). \tag{3.3.3}$$

For the next example of universal product, we consider $\mathcal{K}_1$ the subclass of $(\mathcal{A},\varphi)\in\mathcal{K}$ such that $\mathcal{A}$ is unital and $\varphi(1_{\mathcal{A}})=1$, where $1_{\mathcal{A}}$ stands for the unit of $\mathcal{A}$. In this setting, the categorical coproduct in $\mathcal{K}_1$ can be described as follows: if $(\mathcal{A}_1,\varphi_1),(\mathcal{A}_2,\varphi_2)\in\mathcal{K}_1$, we consider the subspace of codimension 1 given by $\mathcal{A}_i^o:=\ker(\varphi_i)$, for $i=1,2$. Hence we can write the free product with identification of units as

$$\mathcal{A}_1*\mathcal{A}_2=\mathbb{C}1_{\mathcal{A}_1*\mathcal{A}_2}\oplus\bigoplus_{n=1}^{\infty}\bigoplus_{\substack{j_1,\dots,j_n\in\{1,2\}\\ j_1\neq j_2,\dots,j_{n-1}\neq j_n}}\mathcal{A}_{j_1}^o\otimes\cdots\otimes\mathcal{A}_{j_n}^o. \tag{3.3.4}$$

In this way, we define the free product in the following manner.

Definition 3.3.9. Let $(\mathcal{A}_1,\varphi_1),(\mathcal{A}_2,\varphi_2)$ be elements in $\mathcal{K}_1$ and consider $a_1,\dots,a_n\in\mathcal{A}_1\sqcup\mathcal{A}_2$ for any $n\geq 2$ such that $a_k=i_{j_k}(a_k^{(j_k)})$ with $a_k^{(j_k)}\in\mathcal{A}_{j_k}$, $j_k\in\{1,2\}$, for $k=1,\dots,n$, and $j_1\neq j_2$, $j_2\neq j_3,\dots,j_{n-1}\neq j_n$. We define the *free product* as the linear functional on $\mathcal{A}_1\sqcup\mathcal{A}_2$ given by

$$(\varphi_1*\varphi_2)(a_1\cdots a_n)=0 \tag{3.3.5}$$

whenever $\varphi_{j_k}(a_k^{(j_k)})=0$ for every $k=1,\dots,n$.

Theorem 3.3.10 ([GS02])**.** *The tensor, Boolean, and free products are the only universal products over $\mathcal{K}$.*

From the work of Muraki with respect to the arcsine Brownian motion in the monotone Fock space, the notions of monotone and anti-monotone products can be defined in the following way.

Definition 3.3.11. Let $(\mathcal{A}_1,\varphi_1),(\mathcal{A}_2,\varphi_2)$ be elements in $\mathcal{K}$ and consider $a_1,\dots,a_n\in\mathcal{A}_1\sqcup\mathcal{A}_2$ for any $n\geq 2$ such that $a_k=i_{j_k}(a_k^{(j_k)})$ with $a_k^{(j_k)}\in\mathcal{A}_{j_k}$, $j_k\in\{1,2\}$, for $k=1,\dots,n$, and $j_1\neq j_2$, $j_2\neq j_3,\dots,j_{n-1}\neq j_n$.

1. We define the *monotone product* $\varphi_1\blacktriangleright\varphi_2$ as the linear functional on $\mathcal{A}_1\sqcup\mathcal{A}_2$ given by
$$(\varphi_1\blacktriangleright\varphi_2)(a_1\cdots a_n)=\varphi_1\left(\overrightarrow{\prod_{k:\,j_k=1}}a_k^{(1)}\right)\left(\prod_{\ell:\,j_\ell=2}\varphi_2\left(a_l^{(2)}\right)\right). \tag{3.3.6}$$

2. We define the *anti-monotone product* $\varphi_1\blacktriangleleft\varphi_2$ as the linear functional on $\mathcal{A}_1\sqcup\mathcal{A}_2$ given by
$$(\varphi_1\blacktriangleleft\varphi_2)(a_1\cdots a_n)=\left(\prod_{k:\,j_k=1}\varphi_1\left(a_k^{(1)}\right)\right)\varphi_2\left(\overrightarrow{\prod_{\ell:\,j_\ell=2}}a_\ell^{(2)}\right). \tag{3.3.7}$$

The above products, however, are not universal by the main result of [GS02]. Indeed, it can be shown that the monotone and anti-monotone products satisfy conditions 2, 3 and

4 of Definition 3.3.7 but not the commutativity condition. Muraki in his work [Mur03] introduced the notion of *natural product over* $\mathcal{K}$, namely a map $\mathcal{K} \times \mathcal{K} \to \mathcal{K}$ that satisfies the conditions 2, 3 and 4 of Definition 3.3.7. The main result in [Mur03] is an extension of the result of [GS02] in the sense that Muraki's result establishes the classification of natural products over $\mathcal{K}$.

Theorem 3.3.12 ([Mur03])**.** *The tensor product, the Boolean product, the free product, the monotone product, and the anti-monotone product are the only natural products over* $\mathcal{K}$*.*

The next result states how each notion of natural product identifies with its corresponding notion of independence. As a consequence, the main result in [Mur03] provides a complete description of the types of independence in a non-commutative probability space which satisfy the axioms of a natural product.

Theorem 3.3.13. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_N$ be non-unital associative algebras over $\mathbb{C}$, and for each $i = 1, \ldots, N$, let φ_i be a linear functional on $\mathcal{A}_i$. Consider $\mathcal{A} = \mathbb{C}1_{\mathcal{A}} \oplus *_{i=1}^{N} \mathcal{A}_i$ the unitization of the free product of $\mathcal{A}_1, \ldots, \mathcal{A}_N$, and for a natural product $\odot \in \{\otimes, \diamond, *, \blacktriangleright, \blacktriangleleft\}$, define the linear functional $\varphi_\odot : \mathcal{A} \to \mathbb{C}$ given by $\varphi_\odot(1_{\mathcal{A}}) = 1$ and $\varphi_\odot|_{*_{i=1}^{N}\mathcal{A}_i} = \bigodot_{i=1}^{N} \varphi_i$ (for the free product case, assume that the algebras $\mathcal{A}_i$ are unital and the free product identifies the units $1_{\mathcal{A}_i}$ with $1_{\mathcal{A}}$). Under the identification of $\mathcal{A}_i$ in $\mathcal{A}$, the subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_N$ are $\odot$-independent in the non-commutative probability space $(\mathcal{A}, \varphi_\odot)$.*

3.4 Cumulants in non-commutative probability

The five notions of natural independence discussed in the previous section have the nice property that they are rich enough to define a non-commutative probability theory for each of them. In other words, it is possible to obtain non-commutative versions of definitions and theorems in classical probability that deal with classical independence. In this section, we discuss one of them that has proved to be the major combinatorial tool in non-commutative probability, especially in free probability: the notion of cumulants. We will present the combinatorial definition of cumulants, as well as a few main properties of these objects.

Recall that, in a classical probability space, given the sequence of moments of a random variable, the sequence of cumulants can be obtained as the coefficients of the logarithm of its characteristic function. Also, as we mentioned before, one of the advantages of cumulants over the moments is that cumulants linearize the additive convolution of random variables. Our objective here is to explain a combinatorial approach in which cumulants can be described for each notion of independence.

Let us begin the discussion with the classical case. Consider X a classical random variable, with moments and cumulants, $\{m_n\}_{n\geq 1}$ and $\{c_n\}_{n\geq 1}$, respectively. Consider the moment exponential generating function and the cumulant exponential generating function of X, M and C, respectively, as the formal power series given by

$$\begin{aligned} M(x) &= 1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} x^n, \\ C(x) &= \sum_{n=1}^{\infty} \frac{c_n}{n!} x^n. \end{aligned}$$

Observe that, for the moment, we are not treating convergence issues. Now, by definition of the coefficients of C, we have that $M(x) = \exp(C(x))$, where $\exp(C(x)) = \sum_{k=0}^{\infty} \frac{C(x)^k}{k!}$. By standard theory of exponential generating functions, the coefficient of x^n in $C(x)^k$, denoted by $[x^n](C(x)^k)$, is equal to

$$[x^n]C(x)^k = \sum_{\substack{r_1,\dots,r_k \geq 1 \\ r_1+\cdots+r_k=n}} \binom{n}{r_1,\dots,r_k} c_{r_1} c_{r_2} \cdots c_{r_k}. \tag{3.4.1}$$

Thus

$$m_n = \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{r_1,\dots,r_k \geq 1 \\ r_1+\cdots+r_k=n}} \binom{n}{r_1,\dots,r_k} c_{r_1} c_{r_2} \cdots c_{r_k}, \quad \forall n \geq 1. \tag{3.4.2}$$

We will come back to the previous expression later. First, we will describe the main combinatorial objects to define cumulants in non-commutative probability.

3.4.1 Partitions and non-crossing partitions

Definition 3.4.1. Let $n \in \mathbb{N}$.

1. A *partition of* $[n]$ is a collection of disjoint non-empty subsets $V_1, \dots, V_r$ of $[n]$ such that $\bigcup_{i=1}^{n} V_i = [n]$. The set of all partitions of $[n]$ is denoted by $\mathcal{P}(n)$.

2. Let $\pi \in \mathcal{P}(n)$. The elements of π are called the *blocks of* π. The number of blocks of π is denoted by $|\pi|$.

3. A *non-crossing partition of* $[n]$ is a partition $\pi \in \mathcal{P}(n)$ such that there are no blocks $V, W \in \pi$, $V \neq W$ with elements $a < b < c < d$ in $[n]$ such that $a, c \in V$ and $b, d \in W$. The set of all non-crossing partitions of $[n]$ is denoted by $\mathrm{NC}(n)$. We will also denote $\mathrm{NC} := \bigcup_{n\geq 1} \mathrm{NC}(n)$.

4. An *interval partition of* $[n]$ is a partition $\pi \in \mathcal{P}(n)$ such that every block of π is of the form $\{i, i+1, \dots, i+j\}$ for some integers $1 \leq i \leq i+j \leq n$. The set of all interval partitions of $[n]$ is denoted by $\mathrm{Int}(n)$.

Remark 3.4.2. The previous definition generalizes in a clear way when we substitute $[n]$ by a finite totally ordered set X. In other words, we can consider the sets of partitions, non-crossing partitions, and interval partitions on X, respectively, $\mathcal{P}(X), \mathrm{NC}(X)$ and $\mathrm{Int}(X)$.

Remark 3.4.3. Partitions and non-crossing partitions can be represented by arcs, as it is displayed in the figure below. The non-crossing condition of the blocks of a partition means that there are no intersections of the arcs.

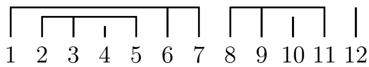
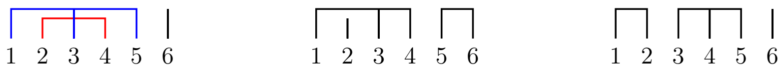


Figure 3.1: Representation of $\pi = \{\{1,6,7\},\{2,3,5\},\{4\},\{8,9,11\},\{10\},\{12\}\} \in \mathrm{NC}(12)$ by arcs.



a) Crossing partition b) Non-crossing partition c) Interval partition

Figure 3.2: Different types of partitions of the set $[6] = \{1, \dots, 6\}$.

Remark 3.4.4. The collection $\mathcal{P}(n)$ is not merely a set of partitions. It is endowed with a poset structure as follows: for two partitions $\pi, \sigma \in \mathcal{P}(n)$, we define $\pi \leq \sigma$ if and only if every block in π is contained in a block of σ. Equivalently, we say that $\pi \leq \sigma$ if and only if every block in σ can be constructed by the union of some blocks in π. The defined order is called the *reverse refinement order on* $\mathcal{P}(n)$.

We can easily observe that $\mathrm{Int}(n) \subset \mathrm{NC}(n) \subset \mathcal{P}(n)$. The reverse refinement order can be considered on collections $\mathrm{Int}(n)$ and $\mathrm{NC}(n)$. Hence we have that $\mathrm{Int}(n), \mathrm{NC}(n)$ and $\mathcal{P}(n)$ are posets. Even more, it can be shown that the three collections of partitions are indeed lattices, with maximal element given by the single block partition $1_n = \{\{1, \dots, n\}\}$, and minimal element given by the partition with n blocks $0_n = \{\{1\}, \dots, \{n\}\}$.

Now, let us restrict our attention to the poset of non-crossing partitions $\mathrm{NC}(n)$. If $\pi \in \mathrm{NC}(n)$ with blocks $\pi = \{V_1, \dots, V_r\}$, we can see π as a poset by declaring that $V_i \leq V_j$ for $V_i, V_j \in \pi$ if and only if V_j is nested in V_i, i.e. $a \in [\min(V_i), \max(V_i)]$ for any $a \in V_j$. It is clear that the maximal elements in π are the blocks that are intervals. The minimal blocks with respect to this partial order are called *outer blocks*. If a block is not outer, we will call it *inner block*.

By considering a linear order in π that is consistent with the previous partial order, we obtain an interesting class of non-crossing partitions.

Definition 3.4.5. A *monotone non-crossing partition* of $[n]$ is a pair (π, λ) such that $\pi \in \mathrm{NC}(n)$ and $\lambda : \pi \to \{1, \dots, |\pi|\}$ is a bijective function such that if $V, W \in \pi$ and $W \leq V$ and $V \neq W$, then $\lambda(W) < \lambda(V)$. The set of all monotone non-crossing partitions of $[n]$ is denoted by $\mathcal{M}(n)$.

For simplicity, we will refer to monotone non-crossing partitions as *monotone partitions*. We will often write $\pi \in \mathcal{M}(n)$ instead of $(\pi, \lambda) \in \mathcal{M}(n)$ when we are considering a general monotone partition. Also, if π is a monotone partition with k blocks, we write $\pi = (V_1, \dots, V_k)$ to denote $\lambda(V_i) = i$, for any $1 \leq i \leq k$.

Remark 3.4.6. A celebrated result establishes that the cardinality of $\mathrm{NC}(n)$ is counted by the Catalan numbers: $|\mathrm{NC}(n)| = C_n$, with

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad \forall\, n \geq 0.$$

It can also be shown ([AHLV15, Prop. 3.4]) that

$$|\mathcal{M}(n)| = \frac{(n+1)!}{2}, \quad \forall\, n \geq 0.$$

Another quantity of interest is the number of monotone labellings of π, i.e. the number of monotone partitions associated to a given $\pi \in \mathrm{NC}(n)$, denoted by $m(\pi)$. To find an expression for $m(\pi)$, we need the following notions associated to a particular class of non-crossing partitions.

Definition 3.4.7. An *irreducible partition of* $[n]$ is a partition $\pi \in \mathrm{NC}(n)$ such that $1, n$ belong to the same block in π. The set of all irreducible partitions of $[n]$ is denoted by $\mathrm{NC}_{\mathrm{irr}}(n)$.

Remark 3.4.8. Let $\pi \in \mathrm{NC}_{\mathrm{irr}}(n)$. It is clear that π has a unique minimal element when we see it as a poset. For a general $\pi \in \mathrm{NC}(n)$ and a minimal block $V \in \pi$, we say that $\pi' := \{W \in \pi : W \leq V\}$ is an *irreducible component of* π. In other words, π' is the non-crossing partition obtained by restricting π to the interval $[\min(V), \max(V)]$. It readily follows that π has a unique irreducible component if and only if $\pi \in \mathrm{NC}_{\mathrm{irr}}(n)$.

Remark 3.4.9. Associated with a $\pi \in \mathrm{NC}_{\mathrm{irr}}(n)$, we can build a planar rooted tree $t(\pi)$ with $|\pi|$ vertices. This is done by decorating the vertices of $t(\pi)$ by the blocks of π and drawing a directed edge from a vertex v to a vertex w if and only if the corresponding block V associated to v *covers* the block W associated to w, i.e. $V < W$ and there is no other block $U \in \pi$ such that $V < U < W$. The tree obtained by the described process is called the *nesting tree of* π, and it will be denoted by $t(\pi)$. For the general case $\pi \in \mathrm{NC}(n)$, we can construct the *nesting forest of* π as the ordered forest whose trees are given by the nesting tree of the irreducible components of π. The order of the irreducible components is given by the total order determined by the minimum element of each component.

Example 3.4.10. Let π and σ be non-crossing partitions represented by arcs below. The nesting forest $t(\pi)$ and $t(\sigma)$ are also displayed. Each vertex of the forest is decorated with the minimal element of their associated block in the partition.

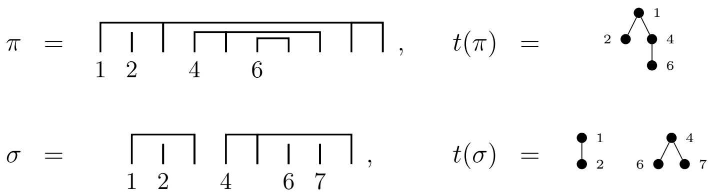


Figure 3.3: Example of the nesting forest.

Definition 3.4.11. Let t be a rooted tree. The *tree factorial of* t is recursively defined for any rooted tree t by $t! = 1$ if t is the rooted tree consisting of a single vertex. Otherwise, if t is a rooted tree that can be obtained by grafting the subtrees $s_1, \dots, s_m$ to the root vertex, we define

$$t! := |t| s_1! \cdots s_m!, \tag{3.4.3}$$

where $|t|$ is the number of vertices of t. More generally, if f is a forest of rooted trees $t_1, \dots, t_n$, we define $f! := t_1! \cdots t_n!$.

A critical feature of the nesting forest is that the number of monotone labellings $m(\pi)$ of a non-crossing partition $\pi \in \mathrm{NC}(n)$ can be described in terms of $t(\pi)!$. More precisely, we have the following result.

Proposition 3.4.12 ([AHLV15, Prop. 3.3])**.** *Let $\pi \in \mathrm{NC}(n)$. The number $m(\pi)$ of monotone labellings of a non-crossing partition π depends only on its nesting forest $t(\pi)$ and is given by*

$$m(\pi) = \frac{|\pi|!}{t(\pi)!}. \tag{3.4.4}$$

3.4.2 Definition of non-commutative cumulants

Let us return to the formal power series of moments and cumulants in the classical setting described at the beginning of the present section. From (3.4.2) and $M(x) = \exp(C(x))$, we have that

$$m_n = \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{r_1,\dots,r_k \geq 1 \\ r_1+\dots+r_k=n}} \binom{n}{r_1, \dots, r_k} c_{r_1} c_{r_2} \cdots c_{r_k}.$$

The main observation is that the above sum can be written in terms of partitions by using that there are $\binom{n}{r_1,\dots,r_k}$ *ordered* partitions $\pi \in \mathcal{P}(n)$ with exactly k blocks $\pi = (V_1, \dots, V_k)$

such that $|V_i| = r_i$ for $i = 1, \ldots, k$. Hence

$$\begin{aligned} m_n &= \sum_{k=1}^{n} \sum_{\substack{\pi \in \mathcal{P}(n) \\ |\pi|=k}} \prod_{V\in\pi} c_{|V|} \\ &= \sum_{\pi\in\mathcal{P}(n)} \prod_{V\in\pi} c_{|V|}. \end{aligned}$$

The appearance of a product indexed by the blocks of a partition motivates the following notation.

Notation 3.4.13. Let $\mathcal{A}$ be a vector space and consider $\{f_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ a family of multilinear functionals. For each $n \geq 1$ and $\pi \in \mathcal{P}(n)$, we denote

$$f_\pi(a_1, \ldots, a_n) = \prod_{V\in\pi} f_{|V|}(a_1, \ldots, a_n|V), \quad \forall\, a_1, \ldots, a_n \in \mathcal{A},$$

where if $V = \{i_1 < \cdots < i_s\}$, then $f_{|V|}(a_1, \ldots, a_n|V) := f_s(a_{i_1}, \ldots, a_{i_s})$.

Example 3.4.14. Let $\mathcal{A}$ be a vector space and consider $\{f_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ a family of multilinear functionals. Consider the partition $\pi = \{\{1, 4, 5\}, \{2, 3\}, \{6, 8, 10\}, \{7\}, \{9\}\} \in \mathcal{P}(10)$. Then

$$f_\pi(a_1. \ldots, a_{10}) = f_3(a_1, a_4, a_5)f_2(a_2, a_3)f_3(a_6, a_8, a_{10})f_1(a_7)f_1(a_9)$$

for any $a_1, \ldots, a_{10} \in \mathcal{A}$.

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. The linear functional φ provides an example of a family of multilinear functionals $\{\varphi_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ defined by

$$\varphi_n(a_1, \ldots, a_n) := \varphi(a_1 \cdots a_n), \quad \forall\, n \geq 1,\ a_1, \ldots, a_n \in \mathcal{A}. \tag{3.4.5}$$

The relation between moments and classical cumulants in terms of partitions motivates the non-commutative version of cumulants for tensor independence.

Definition 3.4.15 (Tensor cumulant functionals ([Spe83]))**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. The *tensor cumulant functionals*, or simply *tensor cumulants*, form the family of multilinear functionals $\{c_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ recursively defined by the following formula:

$$\varphi_n(a_1, \ldots, a_n) = \sum_{\pi\in\mathcal{P}(n)} c_\pi(a_1, \ldots, a_n), \tag{3.4.6}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$. The relation (3.4.6) is called the *moment-tensor cumulant relation*.

Example 3.4.16. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. For $n = 1$, it is clear that $c_1(a_1) = \varphi(a_1)$. For $n = 2$, we have that $\mathcal{P}(2) = \{0_2, 1_2\}$. Hence

$$\varphi_2(a_1, a_2) = \varphi(a_1 a_2) = c_{1_2}(a_1, a_2) + c_{0_2}(a_1, a_2) = c_2(a_1, a_2) + c_1(a_1)c_1(a_2),$$

where we conclude that $c_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$, i.e. $c(a_1, a_2)$ is the *covariance* of a_1 and a_2.

Remark 3.4.17. Observe that (3.4.6) actually define the tensor cumulant functionals c_n, for any $n \geq 1$. Indeed, by considering $\pi = 1_n$, the term associated to 1_n in the sum of the right-hand side of (3.4.6) is precisely $c_n(a_1, \ldots, a_n)$. The other terms associated to the remaining $\pi \neq 1_n$ are products of at least two linear functionals c_k for $1 \leq k < n$. By induction, these c_k are well defined in terms of φ_k for $1 \leq k < n$, and so is c_n.

Roland Speicher [Spe97] introduced the notion of free cumulants from a combinatorial point of view. His remarkable idea states that the transition from tensor to free probability can be described by considering the set of non-crossing partitions instead of the set of partitions. This idea leads us to the next definition.

Definition 3.4.18 (Free cumulant functionals ([Spe94]))**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. The *free cumulant functionals* form the family of multilinear functionals $\{k_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ recursively defined by the following formula:

$$\varphi_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}(n)} k_\pi(a_1, \ldots, a_n) \tag{3.4.7}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$. The relation (3.4.7) is called the *moment-free cumulant relation.*

The Boolean counterpart is defined in [SW97] by considering the set of interval partitions.

Definition 3.4.19 (Boolean cumulant functionals ([SW97]))**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. The *Boolean cumulant functionals* form the family of multilinear functionals $\{b_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ recursively defined by the following formula:

$$\varphi_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{Int}(n)} b_\pi(a_1, \ldots, a_n) \tag{3.4.8}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$. The relation (3.4.8) is called the *moment-Boolean cumulant relation.*

A crucial property of tensor, free, and Boolean cumulants is that they characterize independence through vanishing mixed cumulants. More precisely, we have the following result.

Theorem 3.4.20. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Consider Γ one of the notions of tensor, free and Boolean independence, and $\mathcal{A}_1, \ldots, \mathcal{A}_N \subset \mathcal{A}$ subalgebras of $\mathcal{A}$ (unital subalgebras if we are in the free case). We have the following equivalent statements:*

1. *$\mathcal{A}_1, \ldots, \mathcal{A}_N$ are Γ-independent.*
2. *Let $\{r_n\}_{n\geq 1}$ be the sequence of cumulants associated to Γ. For each $n \geq 2$ and $a_j \in \mathcal{A}_{i_j}$ with $i_j \in [N]$ for $j = 1, \ldots, n$, we have that*

$$r_n(a_1, \ldots, a_n) = 0$$

whenever there exist $1 \leq r < s \leq n$ such that $i_r \neq i_s$.

For the remaining notion of natural independence, Hasebe and Muraki [HS11b, HS11a] introduced the monotone cumulants by using monotone partitions.

Definition 3.4.21 (Monotone cumulant functionals ([HS11b, HS11a]))**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. The *monotone cumulant functionals* form the family of multilinear functionals $\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ recursively defined by the following formula:

$$\varphi_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} h_\pi(a_1, \ldots, a_n) \tag{3.4.9}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$. The relation (3.4.9) is called the *moment-monotone cumulant relation.*

Remark 3.4.22. Observe that each term in the sum on the right-hand side of (3.4.9) does not depend on the monotone labelling of the partition $\pi \in \mathcal{M}(n)$. Hence, we can reorganize the sum and use Proposition 3.4.12 to obtain

$$\varphi_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}(n)} \frac{1}{t(\pi)!} h_\pi(a_1, \ldots, a_n), \tag{3.4.10}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$.

Recall that the sets $\mathcal{P}(n), \mathrm{NC}(n)$ and $\mathrm{Int}(n)$ admit a partial order given by the reverse refinement order given in Remark 3.4.4. In general, given a poset $(P, \leq)$, we can consider the incidence algebra over P. This algebra is defined as the space of functions $f : P^{(2)} \to \mathbb{C}$ where

$$P^{(2)} = \{(\pi, \sigma) : \pi, \sigma \in P, \pi \leq \sigma\}$$

with operation $f * g$ given by

$$(f * g)(\pi, \sigma) = \sum_{\pi \leq \tau \leq \sigma} f(\pi, \tau) g(\tau, \sigma).$$

The incidence algebra is actually a unital associative algebra with unit given by the function $\delta(\pi,\pi) = 1$ and $\delta(\pi,\sigma) = 0$ if $\pi < \sigma$. The *Möbius function on P*, denoted by μ_P, is defined to be the inverse of the map $\zeta(\pi,\sigma) = 1$ for any $\pi \leq \sigma$. In other words, we have that $\zeta * \mu_P = \delta$. For the particular case of our lattices of partitions, it can be shown that

$$\begin{aligned}
\mu_{\mathcal{P}(n)}(\pi, 1_n) &= (-1)^{|\pi|-1}(|\pi|-1)!, \\
\mu_{\mathrm{NC}(n)}(\pi, 1_n) &= (-1)^{|\pi|-1}C_{|\pi|-1}, \\
\mu_{\mathrm{Int}(n)}(\pi, 1_n) &= (-1)^{|\pi|-1}.
\end{aligned}$$

and these are extended multiplicatively to any interval (π,σ) (see [NS06, Lec. 10] for details on the last statement). Möbius function allows us to give an equivalent definition of the notion of tensor, free and Boolean cumulants. More precisely, the respective moment-cumulant relations are equivalent to

$$\begin{aligned}
m(0_n,\pi) &= c * \zeta_{\mathcal{P}(n)}(0_n,\pi), \\
m(0_n,\pi) &= k * \zeta_{\mathrm{NC}(n)}(0_n,\pi), \\
m(0_n,\pi) &= b * \zeta_{\mathrm{Int}(n)}(0_n,\pi),
\end{aligned}$$

where $m(0_n,\pi) := \varphi_\pi$, $c(0_n,\pi) := c_\pi$, $k(0_n,\pi) := k_\pi$ and $b(0_n,\pi) := b_\pi$. Multiplying by the corresponding Möbius function, we obtain the cumulant-moment relations

$$c_n(a_1,\dots,a_n) = \sum_{\pi\in\mathcal{P}(n)} \varphi_\pi(a_1,\dots,a_n)\mu_{\mathcal{P}(n)}(\pi,1_n), \tag{3.4.11}$$

$$k_n(a_1,\dots,a_n) = \sum_{\pi\in\mathrm{NC}(n)} \varphi_\pi(a_1,\dots,a_n)\mu_{\mathrm{NC}(n)}(\pi,1_n), \tag{3.4.12}$$

$$b_n(a_1,\dots,a_n) = \sum_{\pi\in\mathrm{Int}(n)} \varphi_\pi(a_1,\dots,a_n)\mu_{\mathrm{Int}(n)}(\pi,1_n). \tag{3.4.13}$$

It must be remarked that this cannot be performed in the case of monotone cumulants. Regardless of having (3.4.10) as a sum in terms of non-crossing partitions, the corresponding coefficients $\frac{1}{t(\pi)!}$ do not satisfy certain multiplicative conditions required to apply Möbius inversion.

Remark 3.4.23. One should remark that the monotone analogue for Theorem 3.4.20 does not hold. For instance, considering $n = 1, 2$ and 3 in (3.4.10) we obtain

$$\begin{aligned}
\varphi(a) &= h_1(a), \\
\varphi(ab) &= h_2(a,b) + h_1(a)h_1(b), \\
\varphi(abc) &= h_3(a,b,c) + h_2(a,b)h_1(c) + h_2(b,c)h_1(a) + \frac{1}{2}h_2(a,c)h_1(b) + h_1(a)h_1(b)h_1(c).
\end{aligned}$$

Hence we can deduce

$$h_3(a,b,c) = \varphi(abc) - \varphi(a)\varphi(bc) - \varphi(c)\varphi(ab) - \frac{1}{2}\varphi(b)\varphi(ac) + \frac{3}{2}\varphi(a)\varphi(b)\varphi(c).$$

Now, assume that a is monotone independent from b. By definition we get that $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(aba) = \varphi(b)\varphi(a^2)$. In this case, the monotone cumulant of (a,b,a) is given by

$$h_3(a,b,a) = \frac{1}{2}\varphi(b)\left(\varphi(a^2) - \varphi(a)^2\right),$$

and then $h_3(a,b,a)$ is not necessarily equal to 0.

In [HS11a], the authors replaced the vanishing mixed cumulants condition by a weaker condition that holds for monotone cumulants. They called this property *extensivity* and it is defined by using the *dot operation*. More precisely, let us fix a notion of natural independence (tensor, free, Boolean, monotone) in a non-commutative probability space $(\mathcal{A},\varphi)$ and a family of random variables $\{a_i\}_{i\in I} \subset \mathcal{A}$. Then, consider a sequence of random variables $\{a_i^{(j)}\}_{j\geq 1}$ for each $i \in I$ such that any $a_i^{(j)}$ has the same distribution as a_i, for every $j \geq 1, i \in I$. We also assume that the subsets $\{a_i^{(j)}\}_{i\in I}$ with $j \geq 1$ are independent, and $\varphi(a_{i_1}^{(j)} \cdots a_{i_n}^{(j)}) = \varphi(a_{i_1} \cdots a_{i_n})$, for every $j \geq 1$, $n \geq 1$ and $i_1,\dots,i_n \in I$. With this notation, we define the dot operation by

$$N.a := a^{(1)} + \cdots + a^{(N)},$$

for any $N \geq 1$ and $a \in \mathcal{A}$. The authors of [HS11a] described the basic properties that a notion of cumulants should satisfy as follows:

- (MK1) Multilinearity: $r_n : \mathcal{A}^n \to \mathbb{C}$ is multilinear,
- (MK2) Polynomiality: There exists a polynomial P_n such that

$$r_n(a_1,\dots,a_n) = \varphi(a_1\cdots a_n) + P_n\left(\{\varphi(a_{i_1}\cdots a_{i_p})\}_{\substack{1\leq p\leq n-1\\ i_1<\cdots<i_p}}\right),$$

- (MK3) Extensivity: $r_n(N.a_1,\dots,N.a_n) = N r_n(a_1,\dots,a_n)$.

The results of [HS11a] establish:

Theorem 3.4.24 ([HS11a, Thm. 3.1]). *Cumulants associated to tensor, free, Boolean, and monotone independence are the only ones that satisfy (MK1), (MK2) and (MK3) for their respective notions of independence.*

Looking into the different moment-cumulant relations, the question of how the cumulants are related naturally arises. For instance, it is easy to see that the cumulants of order two coincide:

$$c_2(a,b) = k_2(a,b) = b_2(a,b) = h_2(a,b),$$

and are equal to $\varphi(ab) - \varphi(a)\varphi(b)$, namely, the covariance of a and b. We can compare a slightly more complicated example of order 3. For instance, by using (3.4.13) to write the Boolean cumulants in terms of moments and then using (3.4.7) to write the moments in terms of their free cumulants, we obtain that

$$\begin{aligned} b_3(a_1,a_2,a_3) &= \varphi(a_1a_2a_3) - \varphi(a_1a_2)\varphi(a_3) - \varphi(a_1)\varphi(a_2a_3) + \varphi(a_1)\varphi(a_2)\varphi(a_3) \\ &= k_3(a_1,a_2,a_3) + k_1(a_1)k_2(a_2,a_3) + k_1(a_2)k_2(a_1,a_3) + k_1(a_3)k_2(a_1,a_2) \\ &\quad + k_1(a_1)k_1(a_2)k_1(a_3) - \big(k_2(a_1,a_2) + k_1(a_1)k_1(a_2)\big)k_1(a_3) \\ &\quad - k_1(a_1)\big(k_2(a_2,a_3) + k_1(a_2)k_1(a_3)\big) + k_1(a_1)k_1(a_2)k_1(a_3) \\ &= k_3(a_1,a_2,a_3) + k_1(a_2)k_2(a_1,a_3). \end{aligned}$$

Observe that $b_3(a_1,a_2,a_3)$ can be written as a sum, over certain non-crossing partitions in NC(3), of the free cumulants of a_1, a_2 and a_3. One can ask about the existence of combinatorial formulas, analogue to the moment-cumulant formulas, that relate different brands of cumulants. Arizmendi and collaborators studied this problem in detail in the work [AHLV15]. In this work, the authors obtained relations between classical, free, Boolean and monotone cumulants by using Möbius inversion in the several lattices of set partitions and other combinatorial and algebraic techniques. Several of their main results can be encompassed in the following statement.

Theorem 3.4.25 ([AHLV15])**.** *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, and let $\{k_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, and $\{h_n\}_{n\geq 1}$ be the families of free cumulants, Boolean cumulants, and monotone cumulants, respectively. Then, for any $n \geq 1$ and elements $a_1,\dots,a_n \in \mathcal{A}$ we have*

$$b_n(a_1,\dots,a_n) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} k_\pi(a_1,\dots,a_n), \tag{3.4.14}$$

$$k_n(a_1,\dots,a_n) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1} b_\pi(a_1,\dots,a_n), \tag{3.4.15}$$

$$b_n(a_1,\dots,a_n) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!} h_\pi(a_1,\dots,a_n), \tag{3.4.16}$$

$$k_n(a_1,\dots,a_n) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{(-1)^{|\pi|-1}}{t(\pi)!} h_\pi(a_1,\dots,a_n). \tag{3.4.17}$$

Remark 3.4.26. In the work [AHLV15], the authors also obtained formulas that write the n-th monotone cumulant of a single random variable a in terms of the free and Boolean cumulants. In other words, they provided a description of the coefficients $\{\alpha_1(\pi)\}_{\pi\in\cup_{n\geq 1}\mathrm{NC}_{\mathrm{irr}}(n)}$ and $\{\alpha_2(\pi)\}_{\pi\in\cup_{n\geq 1}\mathrm{NC}_{\mathrm{irr}}(n)}$ such that the relations

$$h_n(a) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \alpha_1(\pi)k_\pi(a), \qquad \text{and} \qquad h_n(a) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \alpha_2(\pi)b_\pi(a)$$

hold. One of the contributions of this thesis is the obtainment of the description of such coefficients for the multivariable case $h_n(a_1, \dots, a_n)$. This will be explained in detail in Chapter 8.

3.5 Additive convolutions

3.5.1 Definition of non-commutative additive convolutions

In this section, we will describe the additive convolutions associated to every notion of natural independence. We will show how cumulants give us a simple description of additive convolutions. Recall that a notion of independence can be considered as a rule of computing mixed moments by using the moments of each random variable. In particular, for any independent random variables a, b, independence will give a recipe to obtain the moments of one of the most simple functions in a, b, their sum $a + b$.

Definition 3.5.1 (Additive convolution)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $a, b \in \mathcal{A}$ be random variables with distributions (Definition 3.2.3) $\mu, \nu : \mathbb{C}[X] \to \mathbb{C}$, respectively.

1. If a and b are tensor independent, the *tensor additive convolution*, denoted by $\mu * \nu$, is the linear functional on $\mathbb{C}[X]$ defined to be the distribution of $a + b$.

2. If a and b are freely independent, the *free additive convolution*, denoted by $\mu \boxplus \nu$, is the linear functional on $\mathbb{C}[X]$ defined to be the distribution of $a + b$.

3. If a and b are Boolean independent, the *Boolean additive convolution*, denoted by $\mu \uplus \nu$, is the linear functional on $\mathbb{C}[X]$ defined to be the distribution of $a + b$.

4. If a is monotone independent of b, the *monotone additive convolution*, denoted by $\mu \blacktriangleright \nu$, is the linear functional on $\mathbb{C}[X]$ defined to be the distribution of $a + b$.

Remark 3.5.2. Additive convolutions can be defined as operations on the set of real probability measures. For instance, if μ, ν are two probability measures, we can find a C^*-probability space $(\mathcal{A}, \varphi)$ and two free selfadjoint random variables a, b such that μ is the distribution of a and ν is the distribution of b. Then, the free additive convolution $\mu \boxplus \nu$ is defined as the analytic distribution of $a + b$.

Remark 3.5.3. Monotone convolution is not commutative, i.e. if $a, b \in \mathcal{A}$ are random variables with distributions μ, ν, respectively, and a is monotone independent of b, then $\nu \blacktriangleright \mu \neq \mu \blacktriangleright \nu$ in general. The reason is that a being monotone independent of b does not imply that b is monotone independent of b. Also, by definition of anti-monotone independent, we can define the anti-monotone convolution by $\mu \blacktriangleleft \nu := \nu \blacktriangleright \mu$ when a is anti-monotone independent of b.

A straightforward consequence of Theorem 3.4.20 is that the tensor, free, and Boolean additive convolutions are linearized by their respective cumulants. To make more precise this statement, recall that given a random variable a in a non-commutative probability space $(\mathcal{A}, \varphi)$ and with moments $m_n(a) = \varphi(a^n)$, we define the *n-th free (respectively tensor, Boolean, monotone) cumulant of a* by the complex number $k_n(a) := k_n(a, \dots, a)$ (respectively $c_n(a)$, $b_n(a)$, $h_n(a)$) for any $n \geq 1$. In other words, $\{k_n(a)\}_{n\geq 1}$ is the sequence of complex numbers recursively defined by the moment-cumulant relation

$$m_n(a) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{V \in \pi} k_{|V|}(a).$$

Now, in the case that $a, b \in \mathcal{A}$ are *free random variables*, i.e. freely independent random variables, the moments of $a + b$ are somewhat complicated expressions in terms of the moments $m_n(a)$ and $m_n(b)$. However, the free cumulant of $a + b$ is precisely the sum of the cumulants of a and b.

Theorem 3.5.4. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $a, b \in \mathcal{A}$ be random variables.*

1. *If a and b are freely independent, then*

$$k_n(a + b) = k_n(a) + k_n(b), \quad \forall\, n \geq 1.$$

2. *If a and b are tensor independent, then*

$$c_n(a + b) = c_n(a) + c_n(b), \quad \forall\, n \geq 1.$$

3. *If a and b are Boolean independent, then*

$$b_n(a + b) = b_n(a) + b_n(b), \quad \forall\, n \geq 1.$$

Proof. We will only give the proof in the free case since by Theorem 3.4.20, the proofs for the other cases follow exactly the same way. For $n = 1$ it readily follows that

$$k_1(a + b) = m_1(a + b) = m_1(a) + m_1(b) = k_1(a) + k_1(b).$$

Now, for any $n \geq 2$, by using the multilinearity of the free cumulants and the vanishing mixed cumulant condition implied by Theorem 3.4.20, we obtain

$$\begin{aligned} k_n(a + b) &= k_n(a + b, \dots, a + b) \\ &= k_n(a, \dots, a) + k_n(b, \dots, b) + \sum_{\substack{\varepsilon(a,b) \in \{a,b\}^n \\ \varepsilon(a,b) \text{ contains at least one } a \text{ and one } b}} k_n(\varepsilon(a, b)) \end{aligned}$$

$$\begin{aligned} &= k_n(a, \dots, a) + k_n(b, \dots, b) \\ &= k_n(a) + k_n(b), \end{aligned}$$

where we used that the mixed cumulant vanishes in the third equality. □

Remark 3.5.5. The previous theorem does not hold for monotone independence since monotone cumulants do not satisfy in general that $h_n(a+b) = h_n(a)+h_n(b)$ for any $n \geq 1$ if a is monotone independent of b. Nevertheless, the additivity of monotone cumulants is satisfied when a and b have the same distribution. Indeed, if a is monotone independent of b and both have the same distribution, then a and b have the same moments, and thus they have the same monotone cumulants. Moreover, from the condition (MK3) in Theorem 3.4.24 we get that

$$h_n(a+b) = h_n(a+b, \dots, a+b) = h_n(2.a, \dots, 2.a) = 2h_n(a, \dots, a) = h_n(a) + h_n(b).$$

3.5.2 Non-commutative Central Limit Theorems

Now we discuss another nice and straightforward application of the properties of cumulants: the non-commutative central limit theorems for each notion of independence. In the classical setting, recall that a sequence of real random variables $\{X_n\}_{n\geq 1}$ *converges in distribution to* X if for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) d\mu_{X_n}(x) = \int_{\mathbb{R}} f(x) d\mu_X(x),$$

where μ_{X_n} and μ_X stand for the distribution of X_n and X, respectively. It is customary to write $X_n \xrightarrow[n\to\infty]{d} X$ when $\{X_n\}_{n\geq 1}$ converges in distribution to X.

Theorem 3.5.6 (Classical central limit theorem)**.** *If $\{X_n\}_{n\geq 1}$ is a sequence of independent identically distributed classical random variables such that $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^2) = 1$, then we have the convergence in distribution*

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow[n\to\infty]{d} Z$$

where Z is a random variable with standard Gaussian distribution given by $d\mu_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$.

One can show, for instance, by using Carleman's condition, that the standard Gaussian distribution is determined by its moments. On the other hand, a well-known result in measure theory establishes that if a distribution of a random variable X is determined by its moments and there is a sequence of random variables $\{X_n\}_{n\geq 1}$ that *converges in moments to* X, i.e. such that $\mathbb{E}(X_n^m) \to \mathbb{E}(X^m)$ as $n \to \infty$, for any $m \geq 1$, then $\{X_n\}_{n\geq 1}$ converges in distribution to X. Therefore, one way to prove the classical central limit theorem is to show that the moments of $\frac{X_1+\cdots+X_n}{\sqrt{n}}$ converge to the moments of the standard Gaussian distribution.

The previous discussion promotes the non-commutative counterpart to the notion of convergence in distribution.

Definition 3.5.7. Let $\{(\mathcal{A}_n, \varphi_n)\}_{n\geq 1}$ and $(\mathcal{A}, \varphi)$ be non-commutative probability spaces. For an index set I, consider the random variables $a_i(n) \in \mathcal{A}_n$ and $a_i \in \mathcal{A}$ for each $i \in I$, $n \geq 1$. We say that the family $\{a_i(n)\}_{i\in I}$ *converges in distribution to* $\{a_i\}_{i\in I}$ if for any $m \geq 1$ and $i_1, \ldots, i_m \in I$, we have that

$$\lim_{n\to\infty} \varphi_n \left(a_{i_1}(n) \cdots a_{i_m}(n)\right) = \varphi\left(a_{i_1} \cdots a_{i_m}\right). \tag{3.5.1}$$

In the scope of the above definition, convergence in distribution is denoted by

$$(a_i(n))_{i\in I} \xrightarrow{d} (a_i)_{i\in I}.$$

Also, notice that, when $I = [m]$ is finite, Definition 3.5.7 is equivalent to the convergence of the sequence of distributions $\{\mu_{a_1(n),\ldots,a_m(n)}\}_{n\geq 1}$ given in Definition 3.2.3 to the distribution $\mu_{a_1,\ldots,a_m}$.

With the purpose of stating the corresponding central limit theorems, we define the special random variables that arise in the limiting distribution.

Definition 3.5.8 (Limiting random variables for central limit theorems)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $x \in \mathcal{A}$ be a random variable.

1. We say that x is a *standard Gaussian random variable* if the moments of x are given by
$$\varphi(x^n) = \begin{cases} \frac{(2k)!}{2^k k!} & \text{if } n = 2k, \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.5.2}$$

2. We say that x is a *standard semicircular random variable* if the moments of x are given by
$$\varphi(x^n) = \begin{cases} C_k & \text{if } n = 2k, \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.5.3}$$

3. We say that x is a *standard Bernoulli random variable* if the moments of x are given by
$$\varphi(x^n) = \begin{cases} 1 & \text{if } n = 2k, \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.5.4}$$

4. We say that x is a *standard arcsine random variable* if the moments of x are given by
$$\varphi(x^n) = \begin{cases} \frac{1}{2^k}\binom{2k}{k} & \text{if } n = 2k, \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.5.5}$$

Remark 3.5.9. The sequence of moments of each of the random variables described in the previous definition is indeed the sequence of moments of a real probability measure:

$$\begin{aligned}
\text{Gaussian} \quad &\leftrightarrow \quad d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, \\
\text{semicircular} \quad &\leftrightarrow \quad d\mu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\chi_{[-2,2]}(x)dx, \\
\text{Bernoulli} \quad &\leftrightarrow \quad \mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1, \\
\text{arcsine} \quad &\leftrightarrow \quad d\mu(x) = \frac{1}{\pi\sqrt{2-x^2}}\chi_{-[\sqrt{2},\sqrt{2}]}(x)dx,
\end{aligned}$$

where χ_A stands for the indicator function on a subset $A \subset \mathbb{R}$ and δ_x stands for the Dirac measure on $x \in \mathbb{R}$. By using Carleman's condition, one can show that every of the four previous probability measures is determined by its moments.

The non-commutative central limit theorems for the notions of natural independence can be stated in the following way.

Theorem 3.5.10 (Non-commutative central limit theorems)**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and $\{a_n\}_{n\geq 1}$ be a sequence of random variables with the same distribution. Assume that for every $n \geq 1$ we have that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$.*

1. *If $\{a_n\}_{n\geq 1}$ is a family of tensor independent random variables, then*

$$\frac{a_1 + \cdots + a_n}{\sqrt{n}} \xrightarrow{d} g,$$

where g is a standard Gaussian random variable.

2. *If $\{a_n\}_{n\geq 1}$ is a family of freely independent random variables, then*

$$\frac{a_1 + \cdots + a_n}{\sqrt{n}} \xrightarrow{d} s,$$

where s is a standard semicircular random variable.

3. *If $\{a_n\}_{n\geq 1}$ is a family of Boolean independent random variables, then*

$$\frac{a_1 + \cdots + a_n}{\sqrt{n}} \xrightarrow{d} b,$$

where b is a standard Bernoulli random variable.

4. *If $\{a_n\}_{n\geq 1}$ is a family of monotone independent random variables, then*

$$\frac{a_1 + \cdots + a_n}{\sqrt{n}} \xrightarrow{d} \mathfrak{a},$$

where $\mathfrak{a}$ is a standard arcsine random variable.

Proof. We will prove the monotone case. Since cumulants are polynomials on the moments, convergence in distribution, i.e. convergence of moments, is equivalent to the convergence of cumulants. Denote $S_n := \frac{a_1+\cdots+a_n}{\sqrt{n}}$, for each $n \geq 1$. By multilinearity (MK1) and extensivity (MK3) (Remark 3.5.5) we have that

$$h_m(S_n) = n^{-m/2} h_m(a_1 + \cdots + a_n) = n^{-m/2+1} h_m(a_1), \quad \forall\, m \geq 1.$$

We have then the following cases:

- $m = 1$: since $h_1(a_1) = \varphi(a_1) = 0$, we get that $h_1(S_n) = 0$ for any $n \geq 1$.
- $m = 2$: since $h_2(a_1) = \varphi(a_1^2) - \varphi(a_1)^2 = 1$ by Remark 3.4.23, we obtain that

$$h_2(S_n) = n^{-1+1} h_2(a_1) = 1.$$

- $m \geq 3$: in this case, observe that $-n/2 + 1 \leq -3/2 + 1 < 0$, then $\lim_{m \to \infty} h_m(S_n) = 0$.

We have proved that the sequence of random variables $\{S_n\}_{n\geq 1}$ converges in distribution to a random variable a whose monotone cumulants are $h_2(a) = 1$ and $h_m(a) = 0$ for any $m \neq 2$. Finally, it remains to find the moments of a. By the moment-monotone cumulant formula (3.4.9), it follows that

$$\varphi(a^n) = \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} h_\pi(a, \ldots, a).$$

Observe that, for m being an odd integer and $\pi \in \mathcal{M}(n)$, the partition π will have at least one block of odd size, and hence one of the factors in $h_\pi(a) = \prod_{V \in \pi} h_{|V|}(a)$ is zero, then the contribution of any π is 0. This allows us to conclude that $\varphi(a^n) = 0$ when n is odd.

Now assume that $n = 2k$ and consider $\pi \in \mathcal{M}(2k)$. Notice that in the case that there exists $W \in \pi$ such that $|W| \neq 2$, then $h_\pi(a) = 0$. Hence, the only partitions that give a non-zero contribution in the moment-cumulant formula are those whose blocks have size 2. Hence

$$\varphi(a^{2k}) = \sum_{\pi \in \mathcal{M}_2(2k)} \frac{1}{k!} \cdot 1 = \frac{|\mathcal{M}_2(2k)|}{k!},$$

where $\mathcal{M}_2(2k)$ stands for the set of *pair monotone partitions* of $[2k]$, i.e. partitions in $\mathcal{M}(2k)$ whose all blocks have size 2. In the last equality we have used that $h_2(a, a) = 1$. We will follow the idea of [AHLV15, Prop. 3.4] to find $|\mathcal{M}_2(2k)|$. First, denote $IB(2k)$ the set of intervals of the set $[2k]$ of size 2, more precisely

$$IB(2k) = \big\{\{i, i+1\} \,:\, 1 \leq i < 2k\big\}.$$

Let $(\pi, \lambda) \in \mathcal{M}_2(2k)$. Consider the block $W \in \pi$ such that $\lambda(W) = k$. Since π is a non-crossing partition and λ is increasing with respect to the partial order of the blocks of π, one can easily see that $W \in IB(2k)$ and $\pi' := \pi\backslash\{W\}$ can be considered as an element of $\mathcal{M}_2(2(k-1))$ (after applying the natural increasing bijection $[2k]\backslash W \to [2(k-1)]$). On the other hand, given a partition $\pi' \in \mathcal{M}_2(2(k-1))$ and an interval $W \in IB(2k)$, we can construct $\pi \in \mathcal{M}_2(2k)$ by inserting the interval W in π' such that the new partition π contains the block W labelled by k. Hence, we have a bijection

$$\mathcal{M}_2(2k) \leftrightarrow \mathcal{M}_2(2(k-1)) \times IB(2k).$$

Since $|\mathcal{M}_2(2)| = 1$ and $|IB(2k)| = 2k-1$ for any $k \geq 1$, we inductively obtain that

$$|\mathcal{M}_2(2k)| = (2k-1)(2k-3)\cdots 5\cdot 3\cdot 1 = \frac{(2k)!}{2^k k!}, \quad \forall\, k \geq 1.$$

Hence

$$\varphi(a^{2k}) = \frac{\mathcal{M}_2(2k)}{k!} = \frac{(2k)!}{2^k k! k!} = \frac{1}{2^k}\binom{2k}{k}.$$

We conclude that a has the same moments as those described in (3.5.5). Thus, $\{S_n\}_{n\geq 1}$ converges in distribution to a standard arcsine random variable, as we wanted to show.

The proofs for the other notions of independence follow in the same way as in the monotone case by using the corresponding moment-cumulant formula. □

3.5.3 Analytic transforms and subordination convolution

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. In the case that the distributions μ and ν of two random variables identify with compactly supported probability measures, several transforms permit addressing convolutions from an analytic perspective. We will briefly describe some of them and mention some important properties related to convolutions.

Let μ be a probability measure on $\mathbb{R}$. The *Cauchy transform of* μ is the analytic function $G_\mu(z)$ defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t), \quad \forall\, z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.5.6}$$

We also consider the *reciprocal Cauchy transform* $F_\mu(z)$ to be the map

$$F_\mu(z) = \frac{1}{G_\mu(z)}, \quad \forall\, z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.5.7}$$

For a study of the properties of the Cauchy transform, the reader is encouraged to check [Akh20, Maa92]. Here, one of the properties of $G_\mu(z)$ that will be useful is that there exists a domain D such that $G_\mu(z)$ can be inverted with respect to composition ([BV93]). In this domain, we can define the $\mathcal{R}$*-transform* of μ, denoted by $\mathcal{R}_\mu(z)$ as the analytic map such that

$$G_\mu\left(\mathcal{R}_\mu(z) + \frac{1}{z}\right) = z. \tag{3.5.8}$$

It can be shown [NS06, Theorem 12.7] that if $\{k_n(\mu)\}_{n\geq 1}$ is the sequence of free cumulants of a (or simply, the free cumulants of μ), then

$$\mathcal{R}_\mu(z) = \sum_{n=0}^{\infty} \kappa_{n+1}(\mu) z^n.$$

In particular

$$\mathcal{R}_{\mu \boxplus \nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z), \tag{3.5.9}$$

for z in a certain domain, for any real compactly supported probability measures μ and ν. In other words, the $\mathcal{R}$-transform linearizes the free additive convolution.

The Cauchy transform provides another description of the free convolution of two real probability measures μ and ν, which is stated in the following result proved first by Voiculescu [Voi93] and later by Biane in [Bia98].

Theorem 3.5.11. *Let μ, ν be Borel probability measures on $\mathbb{R}$. Then there exist two analytic maps $F_1, F_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that*

1. *$G_\mu(F_1(z)) = G_\nu(F_2(z)) = G_{\mu\boxplus\nu}(z)$, for any $z \in \mathbb{C}^+$,*
2. *$F_1(z) + F_2(z) = z + F_{\mu\boxplus\nu}(z)$, for any $z \in \mathbb{C}^+$.*

Remark 3.5.12. The functions F_1 and F_2 given by the above theorem are called the *subordination functions* of the free convolution $\mu \boxplus \nu$. Moreover, it is possible to show that F_1 and F_2 are indeed the reciprocal Cauchy transform of two real probability measures σ and σ', respectively.

On the other hand, the F-transform also provides a nice characterization of Boolean and monotone convolutions of real probability measures μ and ν. Indeed, from [SW97] and [Mur00], we have that the following equations characterize Boolean and monotone convolutions, respectively:

$$F_{\mu \uplus \nu}(z) = F_\mu(z) + F_\nu(z) - z, \quad \text{and} \quad F_{\mu \blacktriangleright \nu}(z) = F_\mu(F_\nu(z)),$$

for any $z \in \mathbb{C}^+$. Comparing with Theorem 3.5.11, we arrive at the following result.

Theorem 3.5.13. *Let μ and ν be compactly supported probability measures on $\mathbb{R}$. Then, there exists a compactly supported measure σ on $\mathbb{R}$ such that*

$$\mu \boxplus \nu = \mu \blacktriangleright \sigma.$$

The distribution σ associated to μ and ν by the previous theorem is called the *subordination distribution of $\mu \boxplus \nu$*, and it is denoted by $\mu \boxminus \nu$.

Remark 3.5.14. The previous results encompass different notions of additive convolutions in the case of probability measures over $\mathbb{R}$. Let us now explain how these results can be extended to the multivariate case from the more general algebraic point of view of joint distributions $\mu : \mathbb{C}\langle X_1, \dots, X_k\rangle \to \mathbb{C}$. The relations between moments and cumulants can also be related via formal power series. More precisely, let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and consider a k-tuple of random variables $a = (a_1, \dots, a_k)$ in $\mathcal{A}$. We then construct the *multivariate moment series* M_a of $a = (a_1, \dots, a_k)$ as the formal power series in non-commuting indeterminates $z_1, \dots, z_k$ with coefficients in $\mathbb{C}$ given by

$$M_a(z_1, \dots, z_k) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{k} \varphi_n(a_{i_1}, \dots, a_{i_n}) z_{i_1} \cdots z_{i_n}. \tag{3.5.10}$$

Equivalently, if μ is the distribution of a, we will define M_μ as the moment series of a. Analogously, we define the multivariate R-transform and B-transform, respectively, as follows:

$$R_a(z_1, \dots, z_k) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{k} k_n(a_{i_1}, \dots, a_{i_n}) z_{i_1} \cdots z_{i_n},$$
$$B_a(z_1, \dots, z_k) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{k} b_n(a_{i_1}, \dots, a_{i_n}) z_{i_1} \cdots z_{i_n}.$$

The moment-cumulant formulas allow us to obtain the following functional equations between the previous generating functions:

$$M_a(z_1, \dots, z_k) = R_a\Big(z_1(1 + M_a(z_1, \dots, z_k)), \dots, z_k(1 + M_a(z_1, \dots, z_k))\Big), \tag{3.5.11}$$

$$M_a(z_1, \dots, z_k) = \big(1 + M_a(z_1, \dots, z_k)\big) B_a(z_1, \dots, z_k). \tag{3.5.12}$$

Following [Nic09], consider the set of distributions of k-tuples of random variables in $\mathcal{A}$:

$$\mathcal{D}_{\mathrm{alg}}(k) = \{\mu : \mathbb{C}\langle X_1, \dots, X_k\rangle \to \mathbb{C} \,:\, \mu \text{ is the distribution of } (a_1, \dots, a_k) \in \mathcal{A}^k\}.$$

Equation (3.5.11) allows us to define the free additive convolution of $\mu, \nu \in \mathcal{D}_{\mathrm{alg}}(k)$ as the distribution $\mu \boxplus \nu \in \mathcal{D}_{\mathrm{alg}}(k)$ whose moment series $M_{\mu\boxplus\nu}$ satisfies (3.5.11) with R-transform given by $R_\mu + R_\nu$. Analogously, we define the Boolean additive convolution $\mu \uplus \nu$ by using the B-transform and (3.5.12).

We are interested in defining the subordination distribution for $\mu, \nu \in \mathcal{D}_{\mathrm{alg}}(k)$. We first look at the case $k = 1$. From the previous discussion, we have that $\mu \boxplus \nu$ can be defined by using the Cauchy transform G of $\mu \boxplus \nu$. Observe that, as formal power series, we have that

$$1 + M_\mu(1/z) = zG_\mu(z).$$

Then, it is possible to show that (3.5.8) is equivalent to (3.5.11) for $k=1$

$$R_\mu\big(z(1+M_\mu(z))\big) = M_\mu(z).$$

Since $\mu \boxright \nu$ is such that $G_{\mu\boxplus\nu} = G_\nu \circ F_{\mu\boxright\nu}$, the authors of [Nic09] obtained the following functional equation for the subordination convolution:

$$R_{\mu\boxright\nu}(z) = \frac{R_\mu\big(z(1+M_\nu(z))\big)}{1+M_\nu(z)}.$$

The previous equation motivates the following definition.

Definition 3.5.15 ([Nic09])**.** Let $\mu,\nu \in \mathcal{D}_{\mathrm{alg}}(k)$. The *subordination distribution* of $\mu \boxplus \nu$ with respect to ν is the distribution $\mu \boxright \nu \in \mathcal{D}_{\mathrm{alg}}(k)$ whose R-transform satisfies that

$$R_{\mu\boxright\nu}(z_1,\dots,z_k) = R_\mu\Big(z_1(1+M_\nu),\dots,z_k(1+M_\nu)\Big)\cdot(1+M_\nu)^{-1}(z_1,\dots,z_k), \quad (3.5.13)$$

where $(1+M_\nu)^{-1}$ is the multiplicative inverse of $1+M_\nu$ in the algebra $\mathbb{C}\langle\langle z_1,\dots,z_k\rangle\rangle$ of power series in non-commutative indeterminates $z_1,\dots,z_k$.

The following formulas appearing in [Nic09] exhibit the combinatorial relations between the moments and free cumulants of $\mu \boxright \nu$ and the free cumulants of μ and ν.

Theorem 3.5.16 ([Nic09])**.** *Let $\mu,\nu \in \mathcal{D}_{\mathrm{alg}}(k)$. If for every $n \geq 1$ and $1 \leq i_1,\dots,i_n \leq k$, $k_{(i_1,\dots,i_n)}$ and $r_{(i_1,\dots,i_n)}$ stand for the coefficients of $z_{i_1}\cdots z_{i_n}$ in R_μ and R_ν, respectively, then*

$$(\mu\boxright\nu)(X_{i_1}\cdots X_{i_n}) = \sum_{\pi\in \mathrm{NC}(n)} \left(\prod_{\substack{V\in\pi \\ \text{outer block}}} k_{(i_1,\dots,i_n)|V}\right)\left(\prod_{\substack{W\in\pi \\ \text{inner block}}} k_{(i_1,\dots,i_n)|W} + r_{(i_1,\dots,i_n)|W}\right).$$

The family of free cumulants $\{k_n^{\mu\boxright\nu}(X_{i_1},\dots,X_{i_n})\}_{n\geq 1}$ of $\mu\boxright\nu$ is given by

$$k_n^{\mu\boxright\nu}(X_{i_1},\dots,X_{i_n}) = \sum_{\substack{\pi\in \mathrm{NC}_{\mathrm{irr}}(n) \\ 1,n\in V\in\pi}} k_{(i_1,\dots,i_n)|V} \prod_{\substack{W\in\pi \\ W\neq V}} r_{(i_1,\dots,i_n)|W}.$$

The subordination convolution also shows a nice description of the Boolean cumulants of the free convolution.

Proposition 3.5.17 ([Nic09])**.** *For $\mu,\nu \in \mathcal{D}_{\mathrm{alg}}(k)$, we have that*

$$B_{\mu\boxplus\nu} = B_{\mu\boxright\nu} + B_{\nu\boxright\mu} \quad (3.5.14)$$

Finally, let us recall an important concept in the study of the connections between free and Boolean probability.

Definition 3.5.18 (Boolean Bercovici-Pata bijection ([BN08a, BN09]))**.** Consider the bijective map $\mathbb{B} : \mathcal{D}_{\mathrm{alg}}(k) \to \mathcal{D}_{\mathrm{alg}}(k)$ defined as follows: for any $\mu \in \mathcal{D}_{\mathrm{alg}}(k)$, the element $\mathbb{B}(\mu) \in \mathcal{D}_{\mathrm{alg}}(k)$ is the distribution ν such that their free cumulants are given by the Boolean cumulants of μ, i.e.

$$R_{\mathbb{B}(\mu)} = B_{\mu}.$$

The map $\mathbb{B}$ is called the *Boolean Bercovici-Pata bijection.*

Its analytic importance is shown in the context of infinite divisibility: $\mathbb{B}$ is also a bijection between the set of analytic distributions that are free infinite divisible and the set of analytic distributions. See [BN08a] for the definition of infinite divisibility and the precise statements and proofs of the previous claims. In particular, the Boolean Bercovici-Pata bijection appears as an element of a family of transformations as follows: for every $t \geq 0$, consider the map $\mathbb{B}_t : \mathcal{D}_{\mathrm{alg}}(k) \to \mathcal{D}_{\mathrm{alg}}(k)$ given by

$$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(1+t)}\right)^{\uplus \frac{1}{1+t}}, \quad \forall\, \mu \in \mathcal{D}_{\mathrm{alg}}(k). \tag{3.5.15}$$

Proposition 3.5.19 ([BN09])**.** *The family of transformations $(\mathbb{B}_t)_{t\geq 0}$ satisfies that*

1. $\mathbb{B}_{s+t} = \mathbb{B}_s \circ \mathbb{B}_t$*, for any* $s, t \geq 0$*,*

2. $\mathbb{B}_1 = \mathbb{B}$ *is the Boolean Bercovici-Pata bijection.*

The relation between the Boolean Bercovici-Pata bijection and subordination convolution appears from (3.5.13) by taking $\mu = \nu$, the definition of the B-transform, and the definition of $\mathbb{B}$.

Proposition 3.5.20 ([Nic09])**.** *For any $\mu \in \mathcal{D}_{\mathrm{alg}}(k)$ we have that*

$$\mathbb{B}(\mu) = \mu \boxminus \mu. \tag{3.5.16}$$

3.6 Conditionally free probability

Variations and extensions of free probability have arisen from both theoretical and applied problems. To finish this chapter, let us recall an extension that emerges from considering an additional linear functional on $\mathcal{A}$.

Definition 3.6.1 (Conditional non-commutative probability space)**.** A *c-non-commutative probability space* is a triple $(\mathcal{A}, \varphi, \chi)$ where $(\mathcal{A}, \varphi)$ is a non-commutative probability space and $\chi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\chi(1_{\mathcal{A}}) = 1$.

The main construction in this framework is the relation between both linear functionals φ and χ through the concept of conditional freeness.

Definition 3.6.2 (Conditional freeness)**.** Let $(\mathcal{A}, \varphi, \chi)$ be a c-non-commutative probability space. Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. We say that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are *conditionally free* (or simple *c-free*) if the following holds: for $n \geq 2$, any sequence of indices $i_1, \ldots, i_n \in [k]$ such that $i_j \neq i_{j+1}$ for $1 \leq j < n$, and elements $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ such that $\chi(a_j) = 0$ for $j = 1, \ldots, n$, the following is satisfied:

1. $\chi(a_1 \cdots a_n) = 0$,
2. $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$.

Conditional freeness has been widely studied in the works [BS91, BLS96, PW11]. In particular, in [BLS96] the authors introduced a notion of cumulants so that c-free independence of subalgebras is characterized by the vanishing mixed conditions for the free cumulants and c-free cumulants.

Definition 3.6.3 (Conditionally free cumulant functionals [BLS96])**.** Let $(\mathcal{A}, \varphi, \chi)$ be a c-non-commutative probability space. The *c-free cumulants* form the family of multilinear functionals $\{k_n^{(c)} : \mathcal{A} \to \mathbb{C}\}_{n \geq 1}$ recursively defined by the following formula:

$$\chi(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \left(\prod_{\substack{V \in \pi \\ \text{inner block}}} k_{|V|}(a_1, \ldots, a_n | V) \right) \left(\prod_{\substack{W \in \pi \\ \text{outer block}}} k_{|W|}^{(c)}(a_1, \ldots, a_n | W) \right), \tag{3.6.1}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$, where $\{k_n\}_{n \geq 1}$ is the sequence of free cumulant functionals of $(\mathcal{A}, \varphi)$.

As mentioned in [FMNS19], it is not so straightforward to find a formula that writes the c-free cumulants in terms of the moments of φ and χ. However, it is possible to find a nice combinatorial formula in terms of the Boolean cumulants with respect to φ and χ.

Proposition 3.6.4 ([FMNS19])**.** *Let $(\mathcal{A}, \varphi, \chi)$ be a c-non-commutative probability space. Then the c-free cumulants $\{k_n^{(c)}\}_{n \geq 1}$ of $(\mathcal{A}, \varphi, \chi)$ can be written in terms of the Boolean cumulants $\{b_{n;\varphi}\}_{n \geq 1}$ and $\{b_{n;\chi}\}_{n \geq 1}$ associated to $(\mathcal{A}, \varphi)$ and $(\mathcal{A}, \chi)$, respectively, as follows:*

$$k_n^{(c)}(a_1, \ldots, a_n) = \sum_{\substack{\pi \in \mathrm{NC}_{\mathrm{irr}}(n) \\ 1, n \in V_1}} (-1)^{|\pi| - 1} b_{|V_1|;\chi}(a_1, \ldots, a_n | V_1) \prod_{\substack{W \in \pi \\ W \neq V_1}} b_{|W|;\varphi}(a_1, \ldots, a_n | W), \tag{3.6.2}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$.

Chapter 4

Shuffle Algebras and Non-commutative Probability

The objective of the present chapter is to study the group-theoretical framework for non-commutative probability introduced by K. Ebrahimi-Fard and F. Patras in the series of articles [EFP15, EFP16, EFP18, EFP19]. In contrast to the combinatorial framework of Speicher, where he defined free cumulants in terms of Möbius inversion in the lattice of non-crossing partitions, Ebrahimi-Fard and Patras considered cumulants as elements of a (pre-)Lie algebra of infinitesimal characters on a specific Hopf algebra. This point of view allows us to recognize the relations between moments and cumulants as the exponential relation between a (Lie) group and its corresponding Lie algebra.

First, in Section 4.1, we state the definitions of shuffle algebras and its dual notion, unshuffle coalgebras. We also state the definition of the half-shuffle exponentials associated to the two non-associative products on a shuffle algebra. The previous notions are specialized in Section 4.2, where we expose the primary example of the chapter: the double tensor algebra of an algebra and its associated unshuffle coalgebra and dual shuffle algebra. In this example, we explain that the classical convolution exponential and the two half-shuffle exponentials are three bijections from the group of characters and the Lie algebra of infinitesimal characters. Afterwards, in Section 4.3 we show how the link with non-commutative probability emerges when considering the double tensor algebra of a non-commutative probability space $(\mathcal{A}, \varphi)$, being Theorem 4.3.2 the fundamental basis for the results of this thesis. The main conclusion of the theorem is that non-commutative cumulants appear as the logarithms of a particular character extending the linear functional φ. Finally, we close the chapter by referring to some recent works in the context of non-commutative probability and shuffle algebras.

4.1 Shuffle algebras and unshuffle coalgebras

Roughly speaking, the abstract notion of shuffle product consists of a splitting of an associative product on an algebra. The splitting emerges naturally in several combinatorial Hopf algebras where the product has a suitable combinatorial definition, also with a combinatorial meaning. The precise definition of shuffle algebra is given as follows.

Definition 4.1.1 (Shuffle algebra)**.** A *shuffle algebra* is a vector space D together with two bilinear maps $\prec: D \otimes D \to D$ and $\succ: D \otimes D \to D$ satisfying the identities

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y * z), && (4.1.1)\\ (x \succ y) \prec z &= x \succ (y \prec z), && (4.1.2)\\ x \succ (y \succ z) &= (x * y) \succ z, && (4.1.3)\end{aligned}$$

for any $x, y, z \in D$, where $x * y := x \prec y + x \succ y$.

Shuffle algebras, also known as *dendriform algebras* in the literature, are examples of non-associative algebras. However, an associative product can be easily obtained from the two bilinear maps $\prec$ and $\succ$.

Remark 4.1.2. Let $(D, \prec, \succ)$ be a shuffle algebra and $x, y, z \in D$. From (4.1.1)-(4.1.3), we get

$$\begin{aligned} (x * y) * z &= (x * y) \prec z + (x * y) \succ z\\ &= (x \prec y) \prec z + (x \succ y) \prec z + x \succ (y \succ z)\\ &= x \prec (y * z) + x \succ (y \prec z) + x \succ (y \succ z)\\ &= x \prec (y * z) + x \succ (y * x)\\ &= x * (y * z).\end{aligned}$$

Thus, the operation $*$ makes D a non-unital associative algebra.

In a shuffle algebra $(D, \prec, \succ)$, the maps $\prec$ and $\succ$ are called the *left half-shuffle product* and the *right half-shuffle product*, respectively. The associative product $*$ is called the *shuffle product*. The reader may recall that the terminology shuffle has already been mentioned in Example 2.2.2. Indeed, the shuffle tensor algebra provides an example of a shuffle algebra.

Example 4.1.3. Let $T(V)$ be the tensor algebra over a vector space V, and consider the shuffle product $\sqcup\!\sqcup$ defined in Equation (2.2.2). Consider the non-unital tensor algebra over V, i.e. $T_+(V) = \bigoplus_{n \geq 1} V^{\otimes n}$. If we define

$$\begin{aligned} v_1 \cdots v_n \prec w_1 \cdots w_m &:= v_1(v_2 \cdots v_n \sqcup\!\sqcup w_1 \cdots w_m),\\ v_1 \cdots v_n \succ w_1 \cdots w_m &:= w_1(v_1 \cdots v_n \sqcup\!\sqcup w_2 \cdots w_m),\end{aligned}$$

for any $v_1, \dots, v_n, w_1, \dots, w_m \in V$, then we have that $(T_+(V), \prec, \succ)$ is a shuffle algebra.

Observe that the commutativity of $\sqcup\!\sqcup$ implies that $a \prec b = b \succ a$, for any $a, b \in T_+(V)$. In general, shuffle algebras that satisfy the previous relation are called *commutative shuffle algebras* or *Zinbiel algebras*.

Proposition 4.1.4. *Let $(D, \prec, \succ)$ be a shuffle algebra. Define the bilinear map $\lhd : D \otimes D \to D$ by the recipe*

$$x \lhd y := x \prec y - y \succ x, \qquad \forall\, x, y \in D. \tag{4.1.4}$$

Then $(D, \lhd)$ is a right pre-Lie algebra.

Proof. The pre-Lie identity follows from the shuffle identities (4.1.1)-(4.1.3). Indeed, the right pre-Lie identity is equivalent to

$$x \lhd (y \lhd z) - x \lhd (z \lhd y) = (x \lhd y) \lhd z - (x \lhd z) \lhd y.$$

First, the left-hand side of the above equation is equal to

$$x \lhd (y \lhd z - z \lhd y) = x \lhd (y \prec z - z \succ y - z \prec y + y \succ z) = x \lhd (y * z - z * y),$$

by definition of $* = \prec + \succ$. On the other hand, the right-hand side can be expanded as

$$\begin{aligned}
(x \lhd y) \lhd z - (x \lhd z) \lhd y &= (x \prec y - y \succ x) \lhd z - (x \prec z - z \succ x) \prec y \\
&= (x \prec y) \prec z - z \succ (x \prec y) - (y \prec x) \succ z + z \succ (y \succ x) \\
&\quad -(x \prec z) \prec y + y \succ (x \prec z) + (z \succ x) \prec y - y \succ (z \succ x) \\
&= x \prec (y * z) + (z * y) \succ x - x \prec (z * y) - (y * z) \succ x \\
&= x \lhd (y * z) - x \lhd (z * y),
\end{aligned}$$

where in the third equality, we used the shuffle identities. □

It is important to mention that a shuffle algebra can be augmented with a unit $\overline{D} = D \oplus \mathbb{K}\mathbf{1}$ together with the following relations:

$$\mathbf{1} * \mathbf{1} = \mathbf{1}, \quad x \prec \mathbf{1} = x = \mathbf{1} \succ x, \quad \mathbf{1} \prec x = 0 = x \succ \mathbf{1}, \quad \forall\, x \in D. \tag{4.1.5}$$

However, it is not possible to define $\mathbf{1} \prec \mathbf{1}$ and $\mathbf{1} \succ \mathbf{1}$ in a consistent way such that the previous relations hold.

In a unital shuffle algebra $\overline{D}$, one can consider the formal exponential and logarithm with respect to the shuffle product $*$ as

$$\exp^*(x) = \sum_{n \geq 0} \frac{x^{*n}}{n!} \quad \text{and} \quad \log^*(\mathbf{1} + x) = \sum_{n \geq 1} (-1)^{n-1} \frac{x^{*n-1}}{n}, \quad \forall\, x \in D,$$

where $x^{*0} := \mathbf{1}$. The adjective "formal" previously stated means that we are not considering convergence issues. In addition, the examples where we will apply this machinery fall in the realm of connected graded bialgebras so that our sums for exponentials and logarithms will be finite.

On the other hand, the non-associative half-shuffle products motivate the definition of suitable notions of half-shuffle exponentials in a unital shuffle algebra.

Definition 4.1.5 (Half-shuffle exponentials)**.** Let $(\overline{D}, \prec, \succ)$ be a unital shuffle algebra. For $x \in D$, we define the *left half-shuffle exponential* as the element

$$\mathcal{E}_{\prec}(x) = \mathbf{1} + \sum_{n>0} x^{\prec n}, \tag{4.1.6}$$

where $x^{\prec 0} := \mathbf{1}$ and $x^{\prec n} := x \prec x^{\prec n-1}$ for $n \geq 1$. Analogously, we define the *right half-shuffle exponential* as the element

$$\mathcal{E}_{\succ}(x) = \mathbf{1} + \sum_{n>0} x^{\succ n}, \tag{4.1.7}$$

where $x^{\succ 0} := \mathbf{1}$ and $x^{\succ n} := x^{\succ n-1} \succ x$.

Observe that the left and right half-shuffle exponentials can be equivalently defined as the solutions of the following fixed-point equations, respectively:

$$X = \mathbf{1} + x \prec X, \qquad Y = \mathbf{1} + Y \succ x. \tag{4.1.8}$$

The axioms of the half-shuffle products produce a nice relation between a half-shuffle exponential and the inverse, with respect to the shuffle product, of the other half-shuffle exponential.

Lemma 4.1.6 ([EFP15, Lem. 2])**.** *Let $(\overline{D}, \prec, \succ)$ be a unital shuffle algebra. Then for any $x \in D$, we have that*

$$\mathcal{E}_{\prec}(x)^{*-1} = \mathcal{E}_{\succ}(-x). \tag{4.1.9}$$

Let $\overline{D}$ be a unital shuffle algebra and $x \in D$. Since $X = \mathcal{E}_{\prec}(x)$ satisfies that $X = \mathbf{1} + x \prec X$, we can write

$$X - \mathbf{1} = x \prec X \Rightarrow (X - \mathbf{1}) \prec X^{*-1} = (x \prec X) \prec X^{*-1} = x \prec (X * X^{*-1}) = x \prec \mathbf{1} = x.$$

Then we can define the *left half-shuffle logarithm* of $X \in \overline{D}$ by

$$\mathcal{L}_{\prec}(X) = (X - \mathbf{1}) \prec X^{*-1}.$$

In the same way, we define the *right half-shuffle logarithm* of $X \in \overline{D}$ by

$$\mathcal{L}_{\succ}(X) = X^{*-1} \succ (X - 1).$$

Before presenting the main example of shuffle algebras in this work, it is necessary to consider the dual notion of a shuffle algebra, introduced by Foissy in his work [Foi07].

Definition 4.1.7 (Unshuffle coalgebra)**.** A *counital unshuffle coalgebra* is a coaugmented coassociative coalgebra $(\overline{C} = C \oplus \mathbb{K}\mathbf{1}, \Delta)$ whose reduced coproduct

$$\overline{\Delta}(x) = \Delta(x) - x \otimes \mathbf{1} - \mathbf{1} \otimes x$$

can be split in the form $\overline{\Delta} = \Delta_{\prec} + \Delta_{\succ}$ and the following identities hold:

$$\begin{aligned} (\Delta_{\prec} \otimes \mathrm{id}) \circ \Delta_{\prec} &= (\mathrm{id} \otimes \overline{\Delta}) \circ \Delta_{\prec}, && (4.1.10)\\ (\Delta_{\succ} \otimes \mathrm{id}) \circ \Delta_{\prec} &= (\mathrm{id} \otimes \Delta_{\prec}) \circ \Delta_{\succ}, && (4.1.11)\\ (\overline{\Delta} \otimes \mathrm{id}) \circ \Delta_{\succ} &= (\mathrm{id} \otimes \Delta_{\succ}) \circ \Delta_{\succ}. && (4.1.12) \end{aligned}$$

The linear maps $\Delta_{\prec}$ and $\Delta_{\succ}$ are called the *left half-unshuffle* and *right half-unshuffle*, respectively.

The previous notion can be extended to consider compatibility with the product if C is also a bialgebra.

Definition 4.1.8 (Unshuffle bialgebra)**.** An *unshuffle bialgebra* is a unital and counital bialgebra $(\overline{B} = B \oplus \mathbb{K}\mathbf{1}, m, \Delta)$ whose reduced coproduct splits $\overline{\Delta} = \Delta_{\prec} + \Delta_{\succ}$ such that $(\overline{B}, \Delta_{\prec}, \Delta_{\succ})$ is a counital unshuffle coalgebra, and in addition, we have the following compatibility with the product of $\overline{B}$, for any $x, y \in B$:

$$\begin{aligned} \Delta^{+}_{\prec}(m(x \otimes y)) &= m_{B\otimes B}\left(\Delta^{+}_{\prec}(x), \Delta(y)\right), && (4.1.13)\\ \Delta^{+}_{\succ}(m(x \otimes y)) &= m_{B\otimes B}\left(\Delta^{+}_{\succ}(x), \Delta(y)\right), && (4.1.14) \end{aligned}$$

where $\Delta^{+}_{\prec}(x) := \Delta_{\prec}(x) + x \otimes \mathbf{1}$ and $\Delta^{+}_{\succ}(x) := \Delta_{\succ}(x) + \mathbf{1} \otimes x$.

4.2 The double tensor algebra

We now introduce one of the crucial examples in the work [EFP15]. Let $\mathcal{A}$ stand for a unital associative algebra in the present section. Consider the tensor algebra and non-unital tensor algebra over $\mathcal{A}$

$$T(\mathcal{A}) = \bigoplus_{n\geq 0} \mathcal{A}^{\otimes n}, \qquad T_{+}(\mathcal{A}) = \bigoplus_{n>0} \mathcal{A}^{\otimes n},$$

respectively. We define the *double tensor algebra over* $\mathcal{A}$ as the space

$$T(T_+(\mathcal{A})) = \bigoplus_{n\geq 0} T_+(\mathcal{A})^{\otimes n}. \tag{4.2.1}$$

Recall that the elements of the tensor algebra are written as words $a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$ instead of tensor elements $a_1 \otimes \cdots \otimes a_n$. On the other hand, the elements in the double tensor algebra will be denoted by the bar notation

$$w_1|w_2|\cdots|w_n \in T_+(\mathcal{A})^{\otimes n}.$$

It is clear that the double tensor algebra is actually a unital associative algebra with respect to the concatenation product

$$(v_1|\cdots|v_m)(w_1|\cdots|w_n) := v_1|\cdots|v_m|w_1|\cdots|w_n \in T_+(\mathcal{A})^{\otimes m+n},$$

for any $v_1|\cdots|v_m \in T_+(\mathcal{A})^{\otimes m}$ and $w_1|\cdots|w_n \in T_+(\mathcal{A})^{\otimes n}$. The unit is given by the empty word, denoted by $\mathbf{1}$.

As an algebra, $T(T_+(\mathcal{A}))$ is graded, where the homogeneous components are given by

$$T(T_+(\mathcal{A}))_n = \bigoplus_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\cdots+n_k=n}} \mathcal{A}^{\otimes n_1} \otimes \cdots \otimes \mathcal{A}^{\otimes n_k}, \tag{4.2.2}$$

for $n \geq 1$ and $T(T_+(\mathcal{A}))_0 = \mathbb{K}\mathbf{1}$. In particular, the *degree* of a word is given precisely by its length:

$$\deg(w) := |w| := n, \quad \text{if } w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}.$$

Now, we describe the coproduct as follows. Take a word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. If $S \subseteq [n]$ is of the form $S = \{i_1 < \cdots < i_s\}$, we denote the subword $a_S := a_{i_1} \cdots a_{i_s}$. In the case that $S = \emptyset$, then $a_S := \mathbf{1}$. In addition, let $J_1, \dots, J_r$ be the maximal subintervals in $[n]\backslash S$, ordered in an increasing way according to their minimal elements. For example, if $n = 4$ and $S = \{3\}$, then $[n]\backslash S = \{1, 2, 4\}$ and thus $J_1 = \{1, 2\}$ and $J_2 = \{4\}$. With this notation, we proceed to define the linear map

$$\Delta : T(T_+(\mathcal{A})) \to T(T_+(\mathcal{A})) \otimes T(T_+(\mathcal{A}))$$

by the rules $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$,

$$\Delta(a_1 \cdots a_n) = \sum_{S\subseteq[n]} a_S \otimes a_{J_1}|\cdots|a_{J_r}, \tag{4.2.3}$$

and $\Delta(w_1|\cdots|w_m) = \Delta(w_1)\cdots\Delta(w_m)$, for any $w_1, \dots w_m \in T_+(\mathcal{A})$.

Example 4.2.1. Let us compute the above coproduct for small values of n.

$$\begin{aligned}
\Delta(a_1) &= a_1 \otimes \mathbf{1} + \mathbf{1} \otimes 1, \\
\Delta(a_1 a_2) &= a_1 a_2 \otimes \mathbf{1} + a_1 \otimes a_2 + a_2 \otimes a_1 + \mathbf{1} \otimes a_1 a_2, \\
\Delta(a_1 a_2 a_3) &= a_1 a_2 a_3 \otimes \mathbf{1} + a_1 \otimes a_2 a_3 + a_2 \otimes a_1 | a_3 + a_3 \otimes a_1 a_2 \\
&\quad + a_1 a_2 \otimes a_3 + a_1 a_3 \otimes a_2 + a_2 a_3 \otimes a_1 + \mathbf{1} \otimes a_1 a_2 a_3.
\end{aligned}$$

Finally, we define the linear map $\epsilon : T(T_+(\mathcal{A})) \to \mathbb{K}$ as the projection on $T(T_+(\mathcal{A}))_0$, i.e. the algebra morphism given $\epsilon(\mathbf{1}) = 1$ and $\epsilon(a_1 \cdots a_n) = 0$ for any non-empty word $a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. As the reader can expect, all the ingredients have been settled to state the following result.

Theorem 4.2.2 ([EFP15, Thm. 4])**.** *The double tensor algebra $T(T_+(\mathcal{A}))$ is a connected graded non-commutative and non-cocommutative bialgebra. Hence it is a Hopf algebra.*

The proof of the above theorem easily follows from the construction and definition of the algebra morphism Δ and ϵ. In particular, the coassociativity of Δ is obtained in the following way: if $a_1 \cdots a_n$ is a word, $S \subseteq [n]$ and $J_1, \ldots, J_r$ are the maximal subintervals of $[n] \backslash S$, then we write

$$a_{J^S_{[n]}} := a_{J_1} | \cdots | a_{J_r}.$$

In general, we can consider the maximal subintervals of $U \subset S$ and define $a_{J^S_U}$ as a product of subwords of a_U in an analogous way. Thus, the two possible iterated coproducts write

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a_1 \cdots a_n) = \sum_{S \subseteq U \subseteq [n]} a_S \otimes a_{J^S_U} \otimes a_{J^U_{[n]}} = (\mathrm{id} \otimes \Delta) \circ \Delta(a_1 \cdots a_n). \tag{4.2.4}$$

The fundamental property of the coproduct Δ for our purposes is that it can be easily split as

$$\Delta^+_{\prec}(a_1 \cdots a_n) = \sum_{1 \in S \subseteq [n]} a_S \otimes a_{J^S_{[n]}}, \tag{4.2.5}$$

$$\Delta^+_{\succ}(a_1 \cdots a_n) = \sum_{1 \notin S \subseteq [n]} a_S \otimes a_{J^S_{[n]}}. \tag{4.2.6}$$

Thus, the reduced coproduct can be written as $\overline{\Delta} = \Delta_{\prec} + \Delta_{\succ}$, with

$$\begin{aligned}
\Delta_{\prec}(w) &= \Delta^+_{\prec}(w) - w \otimes \mathbf{1}, \\
\Delta_{\succ}(w) &= \Delta^+_{\succ}(w) - \mathbf{1} \otimes w.
\end{aligned}$$

Finally, the maps $\Delta^+_{\prec}$ and $\Delta^+_{\succ}$ are extended multiplicatively by the rule

$$\Delta^+_{\prec}(v_1 | v_2 | \cdots | v_n) = \Delta^+_{\prec}(v_1) \Delta(v_2 | \cdots | v_n) \quad \text{and} \quad \Delta^+_{\succ}(v_1 | v_2 | \cdots | v_n) = \Delta^+_{\succ}(v_1) \Delta(v_2 | \cdots | v_n),$$

for any words $v_1, \dots, v_n \in T_+(\mathcal{A})$. As the notation persuades, we have our example of unshuffle bialgebra. The proof consists of verifying the three unshuffle identities by doing a similar analysis as the one done in (4.2.4).

Theorem 4.2.3 ([EFP15, Thm. 5])**.** *Let $T(T_+(\mathcal{A}))$ be the double tensor algebra over $\mathcal{A}$ and consider the splitting of the coproduct Δ defined in* (4.2.5) *and* (4.2.6). *Then $(T(T_+(\mathcal{A})), \Delta_\prec, \Delta_\succ)$ is an unshuffle bialgebra.*

The previous theorem automatically provides an example of unital shuffle algebra. Indeed, recall that the convolution product on $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{K})$ is the dual of the coassociative coproduct Δ on $T(T_+(\mathcal{A}))$. The two half-unshuffles also induce non-associative convolution products by the formulas

$$f \prec g = m_{\mathbb{K}} \circ (f \otimes g) \circ \Delta_\prec, \tag{4.2.7}$$

$$f \succ g = m_{\mathbb{K}} \circ (f \otimes g) \circ \Delta_\succ, \tag{4.2.8}$$

for any $f, g \in \mathrm{Lin}(T_+(T_+(\mathcal{A})), \mathbb{K})$, where $T_+(T_+(\mathcal{A})) = \bigoplus_{n>0} T_+(\mathcal{A})^{\otimes n}$ is the non-unital double tensor algebra over $\mathcal{A}$. The unshuffle identities imply a key shuffle algebra for this work.

Theorem 4.2.4. *Consider the non-associative products $\prec$ and $\succ$ defined in* (4.2.7) *and* (4.2.8) *associated to the unshuffle bialgebra $(T(T_+(\mathcal{A})), \Delta_\prec, \Delta_\succ)$. Then $(\mathrm{Lin}(T_+(T_+(\mathcal{A})), \mathbb{K}), \prec, \succ)$ is a shuffle algebra.*

Remark 4.2.5. The shuffle algebra from the previous theorem can be enhanced with the counit $\epsilon : T(T_+(\mathcal{A})) \to \mathbb{K}$ in order to have a unital shuffle algebra structure on $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{K})$ following the identities (4.1.5). In particular, the definition of the half-shuffles for the unital case will require to replace $\Delta_\prec$ and $\Delta_\succ$ by $\Delta_\prec^+$ and $\Delta_\succ^+$, respectively. However, we will keep the notation without the superscript for simplicity.

Let us come back to the three exponentials that can be formally defined in the unital shuffle algebra $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{K})$. Since this is an algebra of linear functionals, we can apply the results described in Section 2.1. Indeed, let $G := G(T(T_+(\mathcal{A})))$ and $\mathfrak{g} := \mathfrak{g}(T(T_+(\mathcal{A})))$ be the group of characters of $T(T_+(\mathcal{A}))$ and the Lie algebra of infinitesimal characters of $T(T_+(\mathcal{A}))$, respectively. By Proposition 2.1.28, the exponential with respect to the convolution product is a set isomorphism from $\mathfrak{g}$ to G.

Now, take $\alpha \in \mathfrak{g}$. It is easy to see that the two half-shuffle exponentials $\mathcal{E}_\prec(\alpha)$ and $\mathcal{E}_\succ(\alpha)$ are well-defined since $\alpha(\mathbf{1}) = 0$. It turns out that the left and right half-shuffle exponentials are also set isomorphisms from $\mathfrak{g}$ to G.

Theorem 4.2.6 ([EFP15, EFP18])**.** *For $\Phi \in G$ a character, there exist a unique triple of infinitesimal characters $(\kappa, \beta, \rho) \in \mathfrak{g}^3$ such that*

$$\Phi = \exp^*(\rho) = \mathcal{E}_\prec(\kappa) = \mathcal{E}_\succ(\beta). \tag{4.2.9}$$

The infinitesimal characters κ and β are the unique solutions of the fixed point equations

$$\Phi = \epsilon + \kappa \prec \Phi \quad \text{and} \quad \Phi = \epsilon + \Phi \succ \beta, \tag{4.2.10}$$

respectively. Conversely, given $\alpha \in \mathfrak{g}$ then $\exp^*(\alpha), \mathcal{E}_\prec(\alpha), \mathcal{E}_\succ(\alpha) \in G$.

Remark 4.2.7. For the computations in the subsequent sections, the reader should observe that the fact that α is an infinitesimal character on $(T_+(\mathcal{A}))$ is equivalent to the fact that $\alpha(\mathbf{1}) = 0$ and $\alpha(w_1|w_2) = 0$, for any $w_1, w_2 \in T_+(T_+(\mathcal{A}))$.

There exist non-trivial relations between the three logarithms of a character Φ. Indeed, from the fixed point equations (4.2.10), we obtain $\kappa \prec \Phi = \Phi \succ \beta$. Multiplying the right half-shuffle product by Φ^{*-1} from the left and using the shuffle identities, we obtain

$$\Phi^{*-1} \succ \kappa \prec \Phi = \Phi^{*-1} \succ (\Phi \succ \beta) = (\Phi^{*-1} * \Phi) \succ \beta = \epsilon \succ \beta = \beta.$$

Hence we have

Proposition 4.2.8. *Let Φ be a character and let κ and β the solutions of* (4.2.10)*. Hence κ and β satisfy*

$$\beta = \Phi^{*-1} \succ \kappa \prec \Phi \quad \text{and} \quad \kappa = \Phi \succ \beta \prec \Phi^{*-1}. \tag{4.2.11}$$

The relation between the shuffle logarithm $\rho = \log^*(\Phi)$ and the other half-shuffle logarithms is even more intricate. With the purpose of describing it, let us introduce a similar notation to the iterated products (used in the statement of Proposition 2.3.26). Let $(D, \prec, \succ)$ be a shuffle algebra. For $x, y \in D$, we define

$$r_{\prec y}(x) = r^{(1)}_{\prec y}(x) := x \prec y, \qquad r^{(n)}_{\prec y}(x) = r^{(n-1)}_{\prec y}(x) \prec y, \quad \text{for } n \geq 2.$$

In the same way, we define $\ell_{y\succ}(x) = y \succ x$ and the corresponding iterations. On the other hand, Proposition 4.1.4 implies that $x \lhd y = x \prec y - y \succ x = (r_{\prec y} - \ell_{y\succ})(x)$ is a pre-Lie product. We also write $r_{\lhd y}(x) = x \lhd y$.

Returning to our shuffle algebra $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{K})$, the set of infinitesimal characters $\mathfrak{g}$ is a pre-Lie algebra with respect to $\lhd$. Moreover, it turns out that the relation between the logarithms $\mathcal{L}_\succ(\Phi)$ and $\log^*(\Phi)$ is given in terms of the inverse of the Magnus operator as in Proposition 2.3.26.

Theorem 4.2.9 ([EFP18, EFP19])**.** *Let Φ be a character on $T(T_+(\mathcal{A}))$ and let ρ, β and κ the infinitesimal characters such that $\Phi = \exp^*(\rho) = \mathcal{E}_\prec(\kappa) = \mathcal{E}_\succ(\beta)$. Then*

$$\beta = W(\rho), \quad \kappa = -W(-\rho) \tag{4.2.12}$$

and

$$\rho = \Omega(\beta) = -\Omega(-\kappa), \tag{4.2.13}$$

where $W : \mathfrak{g} \to \mathfrak{g}$ is the Agrachev-Gamkrelidze operator associated to the pre-Lie algebra $(\mathfrak{g}, \lhd)$ (Proposition 2.3.26) and $\Omega : \mathfrak{g} \to \mathfrak{g}$ is the Magnus operator (Proposition 2.3.28).

Proof. We will follow the argument in Section 5 of [EFP19]. Take $\rho \in \mathfrak{g}$. Observe that from the shuffle identities, one can show that for any $0 < t < 1$ we have that

$$\begin{aligned}
&\frac{d}{dt}\Big((\exp^*((1-t)\rho) - \epsilon) \prec \exp^*(t\rho)\Big) \\
&\quad = \big(\exp^*((1-t)\rho) - \epsilon\big) \prec (\rho * \exp^*(t\rho)) - (\exp^*((1-t)\rho) * \rho) \prec \exp^*(t\rho) \\
&\quad = \Big(\big(\exp^*((1-t)\rho) - \epsilon\big) \prec \rho\Big) \prec \exp^*(t\rho) - \Big(\big(\exp^*((1-t)\rho) - \epsilon\big) \prec \rho \\
&\qquad + \big(\exp^*((1-t)\rho)\big) \succ \rho\Big) \prec \exp^*(t\rho) \\
&\quad = -\exp^*((1-t)\rho) \succ \rho \prec \exp^*(t\rho).
\end{aligned}$$

Notice that we used the shuffle identities and the splitting of $*$ in the second equality. Now, integrating over $t \in (0,1)$ and writing the r and ℓ notation for the iterated half-shuffles, we obtain that

$$\begin{aligned}
\exp^*(\rho) - \epsilon &= \int_0^1 \Big(\exp^*(\rho) * \exp(-s\rho)\Big) \succ \rho \prec \exp^*(s\rho)\, ds \\
&= \exp^*(\rho) \succ \int_0^1 e^{-s\ell_{\rho\succ}} e^{sr_{\prec\rho}}(\rho)\, ds.
\end{aligned}$$

Note that, thanks to the identity $(x \succ y) \prec z = x \succ (y \prec z)$, the operators $e^{-s\ell_{\rho\succ}}$ and $e^{sr_{\prec\rho}}$ commute. Hence, we can express the integrand as

$$\begin{aligned}
\exp^*(\rho) - \epsilon &= \exp^*(\rho) \succ \int_0^1 e^{-s\ell_{\rho\succ} + sr_{\prec\rho}}(\rho)\, ds \\
&= \exp^*(\rho) \succ \int_0^1 e^{sr_{\lhd\rho}}(\rho)\, ds \\
&= \exp^*(\rho) \succ \frac{e^{r_{\lhd\rho}} - \mathrm{id}}{r_{\lhd\rho}}(\rho) \\
&= \exp^*(\rho) \succ W(\rho),
\end{aligned}$$

where $W(\rho)$ is the Agrachev-Gamkrelidze operator on $\mathfrak{g}$. Hence, $W(\rho)$ is the solution to the fixed point equation

$$\exp^*(\rho) = \epsilon + \exp^*(\rho) \succ \beta.$$

By uniqueness, we conclude that $\beta = \mathcal{L}_{\succ}(\exp^*(\rho)) = W(\rho)$. Finally, since the Magnus operator is the compositional inverse of W, we conclude that $\rho = \Omega(\beta)$. The respective relations between ρ and κ are proved analogously. □

The main consequence of the above theorem is that the relations between three logarithms in the shuffle algebra $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{K})$ can be understood purely from a pre-Lie theoretically point of view. For instance, take two infinitesimal characters $\rho_1, \rho_2 \in \mathfrak{g}$.

From (2.3.18), the Agrachev-Gamkrelidze group law on $\mathfrak{g}$ can be written as

$$\rho_1 \# \rho_2 = W(\mathrm{BCH}(\Omega(\rho_1), \Omega(\rho_2))). \tag{4.2.14}$$

From Theorem 4.2.9, we have that $\mathcal{L}_\succ \circ \exp^*(\rho) = W(\rho)$ for any $\rho \in \mathfrak{g}$. This implies that

$$\rho_1 \# \rho_2 = \mathcal{L}_\succ \circ \exp^*(\mathrm{BCH}(\Omega(\rho_1), \Omega(\rho_2))) = \mathcal{L}_\succ \left(\mathcal{E}_\succ(\rho_1) * \mathcal{E}_\succ(\rho_2)\right). \tag{4.2.15}$$

Moreover, one can show that

$$\mathcal{L}_\prec(\mathcal{E}_\prec(\rho_1) * \mathcal{E}_\prec(\rho_2)) = -(-\rho_2 \# -\rho_1). \tag{4.2.16}$$

Indeed, observe that

$$-(-\rho_2 \# -\rho_1) = -\mathcal{L}_\succ(\mathcal{E}_\succ(-\rho_2) * \mathcal{E}_\succ(-\rho_1)) = -\mathcal{L}_\succ\left((\mathcal{E}_\prec(\rho_1) * \mathcal{E}_\prec(\rho_2))^{*-1}\right),$$

where we used Lemma 4.1.6. Hence

$$\mathcal{E}_\prec(-(-\rho_2 \# -\rho_1)) = \mathcal{E}_\prec\left(-\mathcal{L}_\succ\left((\mathcal{E}_\prec(\rho_1) * \mathcal{E}_\prec(\rho_2))^{*-1}\right)\right) = \mathcal{E}_\prec(\rho_1) * \mathcal{E}_\prec(\rho_2).$$

Corollary 2.3.31 provides another expression for the Agrachev-Gamkrelidze law.

Proposition 4.2.10 ([EFP19, Lem. 28])**.** *Let $\mathfrak{g}$ be the pre-Lie algebra of infinitesimal characters on $T(T_+(\mathcal{A}))$. For $\alpha_1, \alpha_2 \in \mathfrak{g}$, we have that*

$$\alpha_1 \# \alpha_2 = \alpha_2 + \mathcal{E}_\succ^{*-1}(\alpha_2) \succ \alpha_1 \prec \mathcal{E}_\succ(\alpha_2). \tag{4.2.17}$$

Proof. Recall that the pre-Lie product on $\mathfrak{g}$ is defined as $r_{\lhd y}(x) = r_{\prec y}(x) - \ell_{y\succ}(x)$. Observe that $r_{\prec y}$ and $\ell_{y\succ}$ commute by the shuffle identity (4.1.2). Using this property and Corollary 2.3.31, we get

$$\alpha_1 \# \alpha_2 = \alpha_2 + e^{r_{\lhd \Omega(\alpha_2)}}(\alpha_1) = \alpha_2 + e^{r_{\prec \Omega(\alpha_2)}} e^{\ell_{-\Omega(\alpha_2)\succ}}(\alpha_1).$$

On the other hand, the shuffle identities imply that

$$r^{(n)}_{\prec \Omega(\alpha_2)}(\alpha_1) = \alpha_1 \prec \Omega(\alpha_2)^{*n} \quad \text{and} \quad \ell^{(n)}_{-\Omega(\alpha_2)\succ}(\alpha_1) = (-\Omega(\alpha_2))^{*n} \succ \alpha_1.$$

Hence, we have

$$\begin{aligned} \alpha_1 \# \alpha_2 &= \alpha_2 + \exp^*(-\Omega(\alpha_2)) \succ \alpha_1 \prec \exp^*(\Omega(\alpha_2)) \\ &= \alpha_2 + \mathcal{E}_\succ^{*-1}(\alpha_2) \succ \alpha_1 \prec \mathcal{E}_\succ(\alpha_2), \end{aligned}$$

where we used that $\alpha_2 = \mathcal{L}_\succ \circ \exp^*(\Omega(\alpha_2))$ from Theorem 4.2.9. $\square$

Recall that $\mathcal{E}_{\prec}(x) = \mathcal{E}_{\succ}^{*-1}(-x)$ in a unital shuffle algebra. Equation (4.2.17) can be equivalently written as

$$\alpha_1 \# \alpha_2 = \alpha_2 + \mathcal{E}_{\prec}(-\alpha_2) \succ \alpha_1 \prec \mathcal{E}_{\prec}^{*-1}(-\alpha_2). \tag{4.2.18}$$

Let us introduce a final notation for this section.

Definition 4.2.11. Let $(\overline{D}, \prec, \succ)$ be a unital unshuffle algebra. For $x, y \in D$, we define *left half-adjoint action* as the element

$$\theta_y(x) = \mathcal{E}_{\prec}^{*-1}(y) \succ x \prec \mathcal{E}_{\prec}(y).$$

Analogously, we define *right half-adjoint action* as the element

$$\theta^y(x) = \mathcal{E}_{\succ}(y) \succ x \prec \mathcal{E}_{\succ}^{*-1}(y).$$

Remark 4.2.12. The shuffle identities imply that $\theta_{(\cdot)}$ is indeed an action. More precisely, for $x, y, z \in D$ elements in a shuffle algebra, we have

$$\begin{aligned}
\theta_z\left(\theta_y(x)\right) &= \theta_z\left(\, \mathcal{E}_{\prec}^{*-1}(y) \succ x \prec \mathcal{E}_{\prec}(y)\right) \\
&= \mathcal{E}_{\prec}^{*-1}(z) \succ \left(\mathcal{E}_{\prec}^{*-1}(y) \succ x \prec \mathcal{E}_{\prec}(y)\right) \prec \mathcal{E}_{\prec}(z) \\
&= \mathcal{E}_{\prec}^{*-1}(z) * \mathcal{E}_{\prec}^{*-1}(y) \succ x \prec \mathcal{E}_{\prec}(y) * \mathcal{E}_{\prec}(z) \\
&= \theta_w(x),
\end{aligned}$$

where $w = \mathcal{L}_{\prec}(\mathcal{E}_{\prec}(y) * \mathcal{E}_{\prec}(z)) = -(-z\# - y)$.

Again, we can see that $\theta^y(x) = \theta_{-y}(x)$. Applying the adjoint actions to infinitesimal characters, we obtain the following result.

Proposition 4.2.13. *Let $\mathfrak{g}$ be the Lie algebra of infinitesimal characters on $T(T_+(\mathcal{A}))$. For $\alpha_1, \alpha_2 \in \mathfrak{g}$, we have that $\theta_{\alpha_1}(\alpha_2), \theta^{\alpha_1}(\alpha_2) \in \mathfrak{g}$.*

Proof. First, we can notice that $\mathcal{L}_{\succ} \circ \mathcal{E}_{\prec}(\alpha_2) \in \mathfrak{g}$ by Theorem 4.2.6. On the other hand, from Proposition 4.2.10, we have that $\gamma_1 \# \gamma_2 = \gamma_2 + \mathcal{E}_{\succ}^{*-1}(\gamma_2) \succ \gamma_1 \prec \mathcal{E}_{\succ}(\gamma_2) \in \mathfrak{g}$ for any $\gamma_1, \gamma_2 \in \mathfrak{g}$. By taking $\gamma_1 = \alpha_1$ and $\gamma_2 = \mathcal{L}_{\succ} \circ \mathcal{E}_{\prec}(\alpha_2)$, we have that

$$\begin{aligned}
\mathfrak{g} \ni \alpha_1 \# \big(\mathcal{L}_{\succ} \circ \mathcal{E}_{\prec}(\alpha_2)\big) - \mathcal{L}_{\succ} \circ \mathcal{E}_{\prec}(\alpha_2) &= \mathcal{E}_{\succ}^{*-1}(\mathcal{L}_{\succ} \circ \mathcal{E}_{\prec}(\alpha_2)) \succ \alpha_1 \prec \mathcal{E}_{\succ}(\mathcal{L}_{\succ} \circ \mathcal{E}_{\prec}(\alpha_2)) \\
&= \mathcal{E}_{\prec}^{*-1}(\alpha_2) \succ \alpha_1 \prec \mathcal{E}_{\prec}(\alpha_2) \\
&= \theta_{\alpha_2}(\alpha_1).
\end{aligned}$$

The proof follows the fact that $\theta^{\alpha_2}(\alpha_1) \in \mathfrak{g}$ is done in a similar fashion, or by using the relation $\theta^{\alpha_2}(\alpha_1) = \theta_{-\alpha_2}(\alpha_1)$. $\square$

The following result tells how to express the half-shuffle exponentials of the sum of infinitesimal characters as the convolution product of half-shuffle exponentials.

Theorem 4.2.14 ([EFP19, Thm. 31]). *Let $\mathfrak{g}$ be the pre-Lie algebra of infinitesimal characters on $T(T_+(\mathcal{A}))$. For $\alpha_1, \alpha_2 \in \mathfrak{g}$, we have that*

$$\mathcal{E}_\prec(\alpha_1) * \mathcal{E}_\prec(\theta_{\alpha_1}(\alpha_2)) = \mathcal{E}_\prec(\alpha_1 + \alpha_2) \tag{4.2.19}$$

and

$$\mathcal{E}_\succ(\theta^{\alpha_2}(\alpha_1)) * \mathcal{E}_\succ(\alpha_2) = \mathcal{E}_\succ(\alpha_1 + \alpha_2). \tag{4.2.20}$$

Proof. We will only prove the second equality since the proof of the first can be done analogously. Notice that from Theorem 4.2.9, we have that $\alpha = \mathcal{L}_\succ \circ \exp^*(\Omega(\alpha))$, for any $\alpha \in \mathfrak{g}$. Then, by using the definition of BCH, we obtain

$$\begin{aligned}
\mathcal{E}_\succ(\theta^{\alpha_2}(\alpha_1)) * \mathcal{E}_\succ(\alpha_2) &= \exp^*(\Omega(\theta^{\alpha_2}(\alpha_1))) * \exp^*(\Omega(\alpha_2)) \\
&= \exp^*\big(\mathrm{BCH}\big(\Omega(\theta^{\alpha_2}(\alpha_1)), \Omega(\alpha_2)\big)\big) \\
&= \exp^*\big(\Omega(\theta^{\alpha_2}(\alpha_1) \# \alpha_2)\big),
\end{aligned}$$

where we used (4.2.14). On the other hand, (4.2.17) in conjunction with the shuffle identities imply that

$$\begin{aligned}
\theta^{\alpha_2}(\alpha_1) \# \alpha_2 &= \alpha_2 + \mathcal{E}_\succ^{*-1}(\alpha_2) \succ \theta^{\alpha_2}(\alpha_1) \prec \mathcal{E}_\succ(\alpha_2) \\
&= \alpha_2 + \mathcal{E}_\succ^{*-1}(\alpha_2) \succ \Big(\mathcal{E}_\succ(\alpha_2) \succ \alpha_1 \prec \mathcal{E}_\succ^{*-1}(\alpha_2)\Big) \prec \mathcal{E}_\succ(\alpha_2) \\
&= \alpha_2 + \Big(\mathcal{E}_\succ^{*-1}(\alpha_2) * \mathcal{E}_\succ(\alpha_2)\Big) \succ \alpha_1 \prec \Big(\mathcal{E}_\succ^{*-1}(\alpha_2) * \mathcal{E}_\succ(\alpha_2)\Big) \\
&= \alpha_2 + \epsilon \succ \alpha_1 \prec \epsilon \\
&= \alpha_1 + \alpha_2.
\end{aligned}$$

Finally, we conclude that

$$\mathcal{E}_\succ\big(\theta^{\alpha_2}(\alpha_1)\big) * \mathcal{E}_\succ(\alpha_2) = \exp^*\big(\Omega(\theta^{\alpha_2}(\alpha_1) \# \alpha_2)\big) = \exp^*\big(\Omega(\alpha_1 + \alpha_2)\big) = \mathcal{E}_\succ(\alpha_1 + \alpha_2),$$

as we wanted to show. □

We conclude this section by remarking that Theorem 4.2.4 extends for the general connected graded case. More precisely, let $(\overline{B}, \Delta_\prec, \Delta_\succ)$ be a connected graded unshuffle bialgebra and (A, m_A) be a commutative algebra. It follows that $(\mathrm{Lin}(\overline{B}, A), \prec, \succ)$ is a unital shuffle algebra, where $\prec$ and $\succ$ are the convolution half-shuffle products on $\mathrm{Lin}(\overline{B}, A)$ dual to $\Delta_\prec$ and $\Delta_\succ$, defined as

$$\begin{aligned}
f \prec g &= m_A \circ (f \otimes g) \circ \Delta_\prec, \\
f \succ g &= m_A \circ (f \otimes g) \circ \Delta_\succ,
\end{aligned}$$

for any $f, g \in \mathrm{Lin}(B, A)$. Also, by Remark 2.1.23, the set of characters $G_A(\overline{B}) \subset \mathrm{Lin}(\overline{B}, A)$ is a group. Analogously, the set of infinitesimal characters $\mathfrak{g}_A(\overline{B}) \subset \mathrm{Lin}(\overline{B}, A)$ is a Lie algebra. By the results of [EFP19], Theorem 4.2.6 and Theorem 4.2.9 can be extended to find three exponentials from $\mathfrak{g}_A(\overline{B})$ to $G_A(\overline{B})$ which are bijections, and whose logarithms are related through the pre-Lie Magnus expansion.

4.3 Non-commutative probability via a shuffle algebra approach

In this section, we take the ground field $\mathbb{K} = \mathbb{C}$ and fix a non-commutative probability space $(\mathcal{A}, \varphi)$. In Chapter 3, we wrote a_1a_2 to denote the product of two random variables in $\mathcal{A}$. Since this comes into conflict with the word notation in the tensor algebra over $\mathcal{A}$, in this section, we write the product on $\mathcal{A}$ by $a_1 \cdot_{\mathcal{A}} a_2$. Hence $a_1 \cdot_{\mathcal{A}} \ldots \cdot_{\mathcal{A}} a_n$ stands for the element of $\mathcal{A}$ given by the product of $a_1, \ldots, a_n$, while the notation $a_1 \cdots a_n$ is reserved for a word in $T_+(\mathcal{A})$.

Our next objective is to show how the double tensor algebra is a useful framework for non-commutative probability. With this purpose, let us introduce φ into the picture.

Definition 4.3.1. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and consider the double tensor algebra over $\mathcal{A}$, $T(T_+(\mathcal{A}))$. The linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is extended to a character $\Phi : T(T_+(\mathcal{A})) \to \mathbb{C}$ by setting $\Phi(\mathbf{1}) = 1$ and, for a word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$,

$$\Phi(w) := \varphi_n(a_1, \ldots, a_n) = \varphi(a_1 \cdot_{\mathcal{A}} \ldots \cdot_{\mathcal{A}} a_n).$$

We call Φ the *lifting of* φ.

In the same way, if $\{f_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ is a family of multilinear functionals, we define the *infinitesimal lifting of* $\{f_n\}_{n\geq 1}$ to be the infinitesimal character $\alpha : T(T_+(\mathcal{A})) \to \mathbb{C}$ given by

$$\alpha(w) := f_n(a_1, \ldots, a_n)$$

for a word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$.

Theorem 4.3.2 ([EFP15, EFP18])**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and consider Φ the character on $T(T_+(\mathcal{A}))$ extending φ. Let (κ, β, ρ) be the triple of infinitesimal characters such that*

$$\Phi = \exp^*(\rho) = \mathcal{E}_{\prec}(\kappa) = \mathcal{E}_{\succ}(\beta).$$

Then for any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$ we have that

$$\begin{aligned}
\mathcal{E}_{\prec}(\kappa)(w) &= \sum_{\pi \in \mathrm{NC}(n)} \prod_{V \in \pi} \kappa(a_V), \\
\mathcal{E}_{\succ}(\beta)(w) &= \sum_{\pi \in \mathrm{Int}(n)} \prod_{V \in \pi} \beta(a_V), \\
\exp^*(\rho)(w) &= \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} \prod_{V \in \pi} \rho(a_V).
\end{aligned}$$

In other words, κ, β and ρ are the infinitesimal liftings of the free, Boolean and monotone cumulant functionals of $(\mathcal{A}, \varphi)$, respectively:

$$\kappa(w) = k_n(a_1, \ldots, a_n), \quad \beta(w) = b_n(a_1, \ldots, a_n), \quad \rho(w) = h_n(a_1, \ldots, a_n),$$

for any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$.

Thanks to the above theorem, we can refer to the logarithms κ, β and ρ as the free, Boolean and monotone cumulants of the character Φ.

Proof. We will first prove the free case. Fix a word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. By induction on n. Recall that the left half-shuffle exponential satisfies the fixed-point equation $\Phi = \epsilon + \kappa \prec \Phi$. If $n = 1$ observe that

$$\Phi(a_1) = \mathcal{E}_{\prec}(\kappa)(a_1) = (\kappa \prec \Phi)(a) = \kappa(a)\Phi(\mathbf{1}) = \kappa(a).$$

Now assume that the desired equation is valid for every $m = 1, \ldots, n-1$. Then

$$\begin{aligned}
\Phi(a_1 \cdots a_n) &= \mathcal{E}_{\prec}(a_1 \cdots a_n) \\
&= (\kappa \prec \Phi)(a_1 \cdots a_n) \\
&= m_{\mathbb{C}} \circ (\kappa \otimes \Phi) \circ \Delta_{\prec}(a_1 \cdots a_n) \\
&= \sum_{1 \in S \subseteq [n]} \kappa(a_S)\Phi(a_{J_1}|\cdots|a_{J_r}).
\end{aligned}$$

Since Φ is a character, thus it is multiplicative. By using the induction hypothesis on each $\Phi(a_{J_i})$ we obtain that

$$\begin{aligned}
\mathcal{E}_{\prec}(a_1 \cdots a_n) &= \sum_{1 \in S \subseteq [n]} \kappa(a_S)\Phi(a_{J_1}) \cdots \Phi(a_{J_r}) \\
&= \sum_{1 \in S \subseteq [n]} \kappa(a_S) \left(\sum_{\pi_1 \in \mathrm{NC}(J_1)} \prod_{V_1 \in \pi_1} \kappa(a_{V_1}) \right) \cdots \left(\sum_{\pi_r \in \mathrm{NC}(J_r)} \prod_{V_r \in \pi_r} \kappa(a_{V_r}) \right) \\
&= \sum_{1 \in S \subseteq [n]} \kappa(a_S) \prod_{V \in \mathrm{NC}([n] \setminus S)} \kappa(a_V)
\end{aligned}$$

$$= \sum_{\pi\in\mathrm{NC}(n)} \prod_{V\in\pi} \kappa(a_V),$$

where we used that we can write π as the union of a block $S \ni 1$ and a non-crossing partition $\pi' \in \mathrm{NC}([n]\backslash S)$. This concludes the proof for the free case. The Boolean case follows analogously by considering the fixed point equation $\Phi = \epsilon + \Phi \succ \beta$.

We will outline the ideas for the monotone case. Recall that $\rho = \log^*(\Phi)$ is an infinitesimal character. This says that $\rho(\mathbf{1}) = 0$ and $\rho(w_1|w_2) = 0$ for any words $w_1, w_2 \in T_+(\mathcal{A})$. Then, according to the definition of the coassociative coproduct Δ, the only terms of the k-th iterated coproduct that may produce a non-zero contribution for $\rho^{*k}(a_1 \cdots a_n)$ are given by

$$\sum_{\substack{\pi\in\mathcal{M}^k(n)\\ \pi=(V_1,\ldots,V_k)}} a_{V_1} \otimes \cdots \otimes a_{V_k}, \tag{4.3.1}$$

where $\mathcal{M}^k(n)$ stands for the set of monotone partitions on $[n]$ with k blocks. Therefore

$$\begin{aligned} \exp^*(\rho)(a_1\cdots a_n) &= \sum_{k\geq 0} \frac{1}{k!}\rho^{*k}(a_1\cdots a_n) \\ &= \sum_{k=1}^{n} \frac{1}{k!} \sum_{\pi\in\mathcal{M}^k(n)} \prod_{V\in\pi} \rho(a_V) \\ &= \sum_{\pi\in\mathcal{M}(n)} \frac{1}{|\pi|!} \prod_{V\in\pi} \rho(a_V), \end{aligned}$$

as we wanted to show. □

The main consequence of the previous theorem is that we can identify cumulant functionals as elements in the (pre-)Lie algebra of infinitesimal characters on an unshuffle bialgebra. Therefore, the double tensor algebra provides a framework for understanding non-commutative probability theory by looking into the shuffle-algebraic picture. The algebraic properties and notions, such as the shuffle identities, the pre-Lie product, and the pre-Lie Magnus expansion, have their corresponding interpretation in non-commutative probability. The next step is to exploit all the algebraic machinery to develop and obtain new results in non-commutative probability.

The first application of the shuffle algebra framework is that it provides a concise description of the combinatorial relations between the free and Boolean cumulants by noticing the fact that their corresponding infinitesimal liftings are two half-shuffle logarithms of the same character. In order to recover the mentioned combinatorial relations in terms of non-crossing partitions, we have the following lemma.

Lemma 4.3.3 ([EFP20, Lem. 15])**.** *Let $\mathcal{A}$ be an algebra and consider its double tensor algebra $T(T_+(\mathcal{A}))$. Let $\Phi \in G$ and consider $\kappa, \beta \in \mathfrak{g}$ such that $\Phi = \mathcal{E}_{\prec}(\kappa) = \mathcal{E}_{\succ}(\beta)$. Then,* *for any $\alpha \in \mathfrak{g}$ and a word $w = a_1 \cdots a_n \in T_+(\mathcal{A})$, we have that*

$$\theta_\kappa(\alpha)(w) = \sum_{1,n \in S \subseteq [n]} \alpha(a_S)\Phi(a_{J^S_{[n]}}) = \sum_{\substack{\pi \in \mathrm{NC}_{\mathrm{irr}}(n) \\ 1 \in V}} \alpha(a_V) \prod_{\substack{W \in \pi \\ W \neq V}} \kappa(a_W) \tag{4.3.2}$$

and

$$\theta^\beta(\alpha)(w) = \sum_{1,n \in S \subseteq [n]} \alpha(a_S)\Phi^{*-1}(a_{J^S_{[n]}}) = \sum_{\substack{\pi \in \mathrm{NC}_{\mathrm{irr}}(n) \\ 1 \in V}} (-1)^{|\pi|-1}\alpha(a_V) \prod_{\substack{W \in \pi \\ W \neq V}} \beta(a_W). \tag{4.3.3}$$

Proof. Recall that $\theta_\kappa(\alpha) = \Phi^{*-1} \succ \alpha \prec \Phi$ is an infinitesimal character. Then, by shuffle identities, $\Phi \succ \theta_\kappa(\alpha) = \alpha \prec \Phi$. Using the definition of the right half-shuffle product, for a word $w = a_1 \cdots a_n \in T_+(\mathcal{A})$ we have that

$$\begin{aligned}
\sum_{1 \in S \subseteq [n]} \alpha(a_S)\Phi(a_{J_1}|\cdots|a_{J_r}) &= \alpha \prec \Phi(w) \\
&= \Phi \succ \theta_\kappa(\alpha)(w) \\
&= \sum_{1 \notin S \subset [n]} \Phi(a_S)\theta_\kappa(\alpha)(a_{J^S_{[n]}}) \\
&= \sum_{j=1}^{n-1} \Phi(a_{j+1}\cdots a_n)\theta_\kappa(\alpha)(a_1\cdots a_j) + \theta_\kappa(\alpha)(a_1\cdots a_n),
\end{aligned}$$

where in the last equality, we used that $\theta_\kappa(\alpha)$ is an infinitesimal character. The above computation implies that

$$\theta_\kappa(\alpha)(w) = \sum_{1,n \in S \subseteq [n]} \alpha(a_S)\Phi(a_{J^S_{[n]}}) + \sum_{\substack{1 \in S \subset [n] \\ n \notin S}} \alpha(a_S)\Phi(a_{J^S_{[n]}}) - \sum_{j=1}^{n-1} \Phi(a_{j+1}\cdots a_n)\theta_\kappa(\alpha)(a_1\cdots a_j).$$

We now claim that

$$\sum_{\substack{1 \in S \subset [n] \\ n \notin S}} \alpha(a_S)\Phi(a_{J^S_{[n]}}) = \sum_{j=1}^{n-1} \Phi(a_{j+1}\cdots a_n)\theta_\kappa(\alpha)(a_1\cdots a_j).$$

Notice that the claim implies the first equality in (4.3.2). By induction on $n \geq 2$. The left-hand side of the above equation is equal to $\alpha(a_1)\Phi(a_2)$ while the right-hand side is equal to $\Phi(a_2)\theta_\kappa(\alpha)(a_1)$. It is easy to show that $\alpha(a_1) = \theta_\kappa(\alpha)(a_1)$, and thus the base case follows. Now, assume that the formula (4.3.2) holds for every integer $j < n$. Then we can write

$$\sum_{j=1}^{n-1} \Phi(a_{j+1}\cdots a_n)\theta_\kappa(\alpha)(a_1\cdots a_j) = \sum_{j=1}^{n-1} \Phi(a_{j+1}\cdots a_n)\left(\sum_{1,j \in T \subseteq [j]} \alpha(a_T)\Phi(a_{J^T_{[j]}})\right)$$

$$= \sum_{\substack{1\in S\subset[n]\\ n\notin S}} \alpha(a_S)\Phi(a_{J^S_{[n]}}),$$

since the fact that $n \notin S$ implies that the last connected component J_r of $[n]\backslash S$ is of the form $\{j+1, j+2, \dots, n\}$ where $j \in S$ is the maximal element of S. Therefore our claim is proved. We conclude then that

$$\begin{aligned}
\theta_\kappa(\alpha)(w) &= \sum_{1,n\in S\subseteq[n]} \alpha(a_S)\Phi(a_{J_1}|\cdots|a_{J_r}) \\
&= \sum_{1,n\in S\subseteq[n]} \alpha(a_S)\prod_{j=1}^{r}\left(\sum_{\pi_j\in \mathrm{NC}(J_j)}\prod_{W\in\pi_j}\kappa(a_W)\right) \\
&= \sum_{\substack{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)\\ 1\in V}} \alpha(a_V)\left(\prod_{\substack{W\in\pi\\ W\neq V}}\kappa(a_W)\right),
\end{aligned}$$

where we used Theorem 4.3.2 in the second equality, and that any irreducible non-crossing partition $\pi \in \mathrm{NC}_{\mathrm{irr}}(n)$ is characterized by its unique outer block $V \ni 1, n$ and the irreducible components $\pi_1, \dots, \pi_r$ of the non-crossing partition $\pi\backslash\{V\}$.

The proof of (4.3.3) follows from the previous part by recalling that $\theta^\beta(\alpha) = \theta_{-\beta}(\alpha)$. □

Corollary 4.3.4. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and consider Φ the character on $T(T_+(\mathcal{A}))$ extending φ. Let κ and β be the infinitesimal liftings of the free and Boolean cumulants, respectively. Then the relations $\beta = \theta_\kappa(\kappa)$ and $\kappa = \theta^\beta(\beta)$ encode the formulas between free and Boolean cumulants (3.4.14)-(3.4.15).*

Proof. Recall that, from Proposition 4.2.8, the half-shuffle logarithms of Φ satisfy that

$$\beta = \Phi^{*-1} \succ \kappa \prec \Phi = \theta_\kappa(\kappa)$$

and $\kappa = \theta^\beta(\beta)$. For a word $w = a_1 \cdots a_n \in T_+(\mathcal{A})$, Equation (4.3.2) in Lemma 4.3.3 implies that

$$\beta(w) = \theta_\kappa(\kappa)(w) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \prod_{V\in\pi}\kappa(a_V),$$

that is precisely (3.4.14) once that we identify the infinitesimal characters with the cumulant functionals of $(\mathcal{A}, \varphi)$. Similarly, from (4.3.3), it follows that

$$\kappa(w) = \theta^\beta(\beta)(w) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1}\prod_{V\in\pi}\beta(a_V),$$

that is precisely (3.4.15). □

Remark 4.3.5. Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, and consider Φ the character on $T(T_+(\mathcal{A}))$ extending φ as well as the infinitesimal characters κ,β,ρ from Theorem 4.3.2. By using similar ideas as in the proof of Lemma 4.3.3, one can show that the fixed point equation

$$\exp^*(\rho)-\epsilon=\Phi-\epsilon=\Phi\succ\beta=\exp^*(\rho)\succ\beta$$

encodes the formula that writes the Boolean cumulants in terms of the monotone cumulants (3.4.16). Similarly, since

$$\Phi^{*-1}=\mathcal{E}_{\prec}^{*-1}(\kappa)=\mathcal{E}_{\succ}(-\kappa)=\epsilon+\Phi^{*-1}\succ(-\kappa), \tag{4.3.4}$$

the equation

$$\exp^*(-\rho)-\epsilon=\exp^*(-\rho)\succ(-\kappa)$$

encodes the formula that writes the free cumulants in terms of the monotone cumulants (3.4.17). However, the same method does not work for the converse formulas. We recall that thanks to Theorem 4.2.9, the infinitesimal lifting of the monotone cumulant can be described in terms of the free and Boolean infinitesimal characters through the Magnus operator

$$\rho=\Omega(\beta)=-\Omega(-\kappa).$$

Nonetheless, the combinatorial formulas obtained by evaluating on a word $w=a_1\cdots a_n$ are not straightforward. The aim of Chapters 8 and 9 is to obtain the combinatorial formulas, in terms of irreducible non-crossing partitions, that write monotone cumulants in terms of free and Boolean cumulants, as well as to have a deeper understanding of the relations between cumulants from a pre-Lie algebraic point of view.

As another application of Lemma 4.3.3, we can interpret the conditionally free cumulants of a c-non-commutative probability space (Definition 3.6.3) as a particular infinitesimal character defined by the infinitesimal liftings of the free and Boolean cumulants.

Following [EFP20], given a c-non-commutative probability space $(\mathcal{A},\varphi,\psi)$, consider the characters on $T(T_+(\mathcal{A}))$, Φ and Ψ, extending φ and χ, respectively. Also, consider the half-shuffle logarithms $\kappa_1,\beta_1,\kappa_2,\beta_2$ such that $\Phi=\mathcal{E}_{\prec}(\kappa_1)=\mathcal{E}_{\succ}(\beta_1)$ and $\Psi=\mathcal{E}_{\prec}(\kappa_2)=\mathcal{E}_{\succ}(\beta_2)$. Now, define the infinitesimal character γ on $T(T_+(\mathcal{A}))$ by

$$\gamma=\Phi\succ\beta_2\prec\Phi^{-1}=\theta^{\beta_1}(\beta_2). \tag{4.3.5}$$

By Lemma 4.3.3, for a word $w=a_1\cdots a_n\in T_+(\mathcal{A})$ we have that

$$\gamma(w)=\sum_{\substack{\pi\in\mathrm{NC}_{\mathrm{irr}}(n)\\ 1\in V}}(-1)^{|\pi|-1}\beta_2(a_V)\prod_{\substack{W\in\pi\\ W\neq W}}\beta_1(a_W),$$

that is precisely (3.6.2) which expresses the c-free cumulants in terms of the Boolean cumulants of φ and ψ. Furthermore, we can recover the definition of c-free cumulants in terms of non-crossing partitions as in (3.6.1).

Proposition 4.3.6 ([EFP20, Prop. 17])**.** *Let $(\mathcal{A},\varphi,\psi)$ be a c-free non-commutative probability space and consider Φ and Ψ the character on $T(T_+(\mathcal{A}))$ extending φ and χ, respectively. If $\Phi=\mathcal{E}_\prec(\kappa)=\mathcal{E}_\succ(\beta_1)$, $\Psi=\mathcal{E}_\succ(\beta_2)$ and $\gamma=\theta^{\beta_1}(\beta_2)$, for a word $w=a_1\cdots a_n\in T_+(\mathcal{A})$ we have*

$$\Psi(w)=\sum_{\pi\in\mathrm{NC}(n)}\left(\prod_{\substack{V\in\pi\\ V\text{ outer}}}\gamma(a_V)\right)\left(\prod_{\substack{V\in\pi\\ V\text{ inner}}}\kappa(a_W)\right). \tag{4.3.6}$$

In other words, the infinitesimal character γ is the infinitesimal lifting of the family of c-free cumulants

$$\gamma(w)=k_n^c(a_1,\ldots,a_n),$$

for any word $w=a_1\cdots a_n\in T_+(\mathcal{A})$.

Proof. Since $\gamma=\theta^{\beta_1}(\beta_2)=\Phi\succ\beta_2\prec\Phi^{*-1}$. We can use the shuffle identities to invert the relation and obtain $\beta_2=\Phi^{-1}\succ\gamma\prec\Phi=\theta_\kappa(\gamma)$. By definition of $\beta_2=\mathcal{L}_\succ(\Psi)$ as the solution of the fixed point equation $\Psi=\epsilon+\Psi\succ\beta_2$, we have that

$$\begin{aligned}\Psi(w) &= \big(\epsilon+\Psi\succ\beta_2\big)(w)\\ &= \sum_{j=1}^{n-1}\theta_\kappa(\gamma)(a_1\cdots a_j)\Psi(a_{j+1}\cdots a_n)\\ &= \sum_{j=1}^{n-1}\left(\sum_{\substack{\pi\in\mathrm{NC}_{\mathrm{irr}}(j)\\ 1\in V}}\gamma(a_V)\prod_{\substack{W\in\pi\\ W\text{ inner}}}\kappa(a_W)\right)\Psi(a_{j+1}\cdots a_n).\end{aligned}$$

By doing the same computation on $\Psi(a_{j+1}\cdots a_n)$, we obtain an outer block $V'\ni j+1$ of a non-crossing partition of $\{j+1,\ldots,n\}$ indexing $\gamma(a_{V'})$, the inner blocks covered by such outer blocks indexing a product $\prod_{V'\leq W}\kappa(a_W)$, as well as a factor $\Psi(a_{j'+1}\cdots a_n)$ with $j<j'$. This process will generate all the non-crossing partitions and produce (4.3.6). □

As we mentioned at the beginning of the section, the series of articles [EFP15, EFP16, EFP18, EFP19, EFP20] of Ebrahimi-Fard and Patras started a group-theoretical framework for cumulants in non-commutative probability. As a manner of concluding this section, we mention some recent works, apart from the works leading to the results of this thesis, which are derived from the shuffle algebra framework for non-commutative probability.

• *Operads of partitions, interacting bialgebras, and moment-cumulants relations.* Given the combinatorial nature of the non-crossing partitions, there are two natural operad structures associated to them and studied in the work [EFFKP20]. Each operad provides a different approach for the moment-cumulant relations in non-commutative probability: the shuffle-algebraic approach of Ebrahimi-Fard and Patras and the Möbius inversion approach of Speicher. The authors of [EFFKP20] showed that the two operads interact through their associated bialgebras, which is compatible with the shuffle structure.

• *Operator-valued non-commutative probability via a shuffle algebra approach.* A fruitful extension of non-commutative probability is the notion of *operator-valued free probability* ([Spe98]). An *operator-valued probability space* is a triple $(\mathcal{A}, E, \mathcal{B})$ where $\mathcal{B}$ is a unital algebra over $\mathbb{C}$, $\mathcal{A}$ is a $\mathcal{B}$-bimodule and unital algebra over $\mathbb{C}$, and $E : \mathcal{A} \to \mathcal{B}$ is a $\mathcal{B}$-bimodule unital algebra morphism. Since, in general, $\mathcal{B}$ is non-commutative, a notion of *operator-valued free cumulants* with respect to E must be defined by taking into account the nesting structure of the blocks of a non-crossing partition. In particular, the shuffle algebra approach for cumulants cannot be used straightforwardly. In the work [Gil22], the author considered the operad that keeps track of the nesting structure of the blocks and, by using notions of higher category theory, he constructed an unshuffle bialgebra over a certain object in a monoidal category. By constructing the analogue fixed point equations for the corresponding shuffles, it is possible to recover the operator-valued moment-cumulants relations.

• *Bi-non-commutative probability via a shuffle algebra approach.* In the same spirit as the previous work, the authors of [DGG22] used higher category theory in order to study *two-faced non-commutative probability*, introduced first in the free case by Voiculescu [Voi14] with the aim of considering left and right operators at the same time. The idea of [DGG22] consists in constructing a Hopf monoid, based on certain categories of words, that allows identifying left or right random variables. The coproduct of the Hopf monoid can be split, producing the corresponding unshuffle and shuffle structure which can be used to recover the bifree, biBoolean and bimonotone cumulants as the solutions of certain fixed point equations.

• *Non-commutative Wick polynomials.* Wick polynomials and Wick products have been used in classical probability because of their close relationship with the Gaussian distribution and the Hermite polynomials. In non-commutative probability, the free and Boolean analogues have been introduced in the works of Anshelevich ([Ans04]). The machinery of the shuffle algebra framework for non-commutative probability is capable of encompassing the theory of non-commutative Wick polynomials. This is done in [EFPTZ21], where the authors defined the free Wick maps as an operator W on the double tensor algebra $T(T_+(\mathcal{A}))$ given in terms of the inverse (with respect to the convolution product) of the character Φ extending the linear functional $\varphi : \mathcal{A} \to \mathbb{C}$. Similarly, the definitions of Boolean and conditionally free Wick maps are provided, and relations between them are studied through the relations in the shuffle algebra approach.

• *Formal power series, non-commutative probability and pre-Lie algebras.* Recall that the moment-cumulant relations can be effectively described using multivariable generating functions as the R-transform or the η-transform. The authors of [EFPTZ22] defined a group law and a Lie product on certain subsets of formal power series and found a group isomorphism and a Lie algebra isomorphism from the latter subsets to the group of characters and the Lie algebra of infinitesimal characters, respectively, of the double tensor algebra over the alphabet $\mathbb{N}$. The isomorphisms allow identifying the infinitesimal characters associated to cumulants in a non-commutative probability space with their corresponding multivariable generating functions, establishing a dictionary between the shuffle algebraic approach of Ebrahimi-Fard and Patras and the classical approach to cumulants via non-commutative formal power series.

Chapter 5

Shuffle-algebraic Central Limit Theorems

Algebraic central limit theorems are versions of the central limit theorem stating convergence of moments not requiring any other assumption than the existence of moments. They have been studied in detail since the works of von Waldenfels [GvW78, Wal86], Speicher-von Waldenfels [SvW94] and Schürmann [Sch86], to name a few. This chapter aims to show how the central limit theorems enter into the shuffle-algebraic picture for non-commutative probability theory analyzed in Chapter 4 and consequently obtain new proofs of the non-commutative central limit theorems.

The first step to explaining central limit theorems in the shuffle picture is to have an interpretation of the respective additive convolutions for the several notions of non-commutative independence. Such interpretation is described in Section 5.1. In particular, Theorem 5.1.2 motivates the conjecture of the central limit theorems via shuffle algebra. The subsequent two sections provide the shuffle-algebraic versions of the central limit theorems allowing us to obtain a new proof of such theorems. More precisely, Section 5.2 is devoted to giving the precise statement and proof of the monotone central limit theorem. In contrast, Section 5.3 exhibits the analogous theorems for free and Boolean cumulants.

5.1 Algebraic interpretation of additive convolutions

Let $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ be non-commutative probability spaces. We know from Theorem 3.3.13 that we can construct the respective product space $(\mathcal{A}, \varphi)$ for any of the free, Boolean, monotone and anti-monotone products in such a way that $\mathcal{A}_1$ and $\mathcal{A}_2$ can be regarded as independent subalgebras of $\mathcal{A}$ with respect to φ.

Recall that $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ is the free product of the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. Following the shuffle-algebraic framework for non-commutative probability studied in Chapter 4, the construction of a product space corresponds, in the language of double tensor algebras, to produce a character Φ on $T(T_+(\mathcal{A}))$ from the liftings $\Phi_1 : T(T_+(\mathcal{A}_1)) \to \mathbb{C}$ and

$\Phi_2 : T(T_+(\mathcal{A}_2)) \to \mathbb{C}$ of φ_1 and φ_2, respectively. Observe that Φ_1 can be extended to a character on $T(T_+(\mathcal{A}))$ by setting $\Phi_1(w) = 0$ if w contains a letter not in $\mathcal{A}_1$. A similar situation applies for Φ_2 as well as its infinitesimal characters associated by Theorem 4.3.2.

Motivated by the fact that, in the free and Boolean case, independence is characterized by the vanishing of mixed cumulants, the candidates for the lifting of the free and Boolean product are given in the next definition.

Definition 5.1.1 ([EFP19, Def. 29]). Let $\mathcal{A}$ be an algebra and consider its double tensor algebra $T(T_+(\mathcal{A}))$. Moreover, let $\Phi_1, \Phi_2 \in G(T(T_+(\mathcal{A})))$ be two characters and consider the corresponding pairs of infinitesimal characters (κ_1, β_1) and (κ_2, β_2) such that $\Phi_1 = \mathcal{E}_{\prec}(\kappa_1) = \mathcal{E}_{\succ}(\beta_1)$ and $\Phi_2 = \mathcal{E}_{\prec}(\kappa_2) = \mathcal{E}_{\succ}(\beta_2)$. Then we define the *left commutative group law* as the map $(\Phi_1, \Phi_2) \mapsto \Phi_1 \boxplus \Phi_2$, where $\Phi_1 \boxplus \Phi_2$ is the character on $T(T_+(\mathcal{A}))$ given by

$$\Phi_1 \boxplus \Phi_2 := \mathcal{E}_{\prec}(\kappa_1 + \kappa_2), \tag{5.1.1}$$

and the *right commutative group law* as the map $(\Phi_1, \Phi_2) \mapsto \Phi_1 \uplus \Phi_2$, where $\Phi_1 \uplus \Phi_2$ is the character on $T(T_+(\mathcal{A}))$ given by

$$\Phi_1 \uplus \Phi_2 := \mathcal{E}_{\succ}(\beta_1 + \beta_2). \tag{5.1.2}$$

The key idea of the present chapter is the following result in [EFP19], which relates shuffle group laws on $G(T(T_+(\mathcal{A})))$ with the several constructions of product spaces exposed in Section 3.3.

Theorem 5.1.2 ([EFP19, Thm. 32]). *Let $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ be two non-commutative probability spaces and consider its free product $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ (units are identified in the free case). If Φ_1 and Φ_2 are the liftings of φ_1 and φ_2, respectively, to characters on $T(T_+(\mathcal{A}))$, we have that*

1. *$\Phi_1 \boxplus \Phi_2$ is the lifting of the free product $\varphi_1 * \varphi_2 : \mathcal{A} \to \mathbb{C}$;*
2. *$\Phi_1 \uplus \Phi_2$ is the lifting of the Boolean product $\varphi_1 \diamond \varphi_2 : \mathcal{A} \to \mathbb{C}$;*
3. *$\Phi_1 * \Phi_2$ is the lifting of the monotone product $\varphi_1 \blacktriangleright \varphi_2 : \mathcal{A} \to \mathbb{C}$;*
4. *$\Phi_2 * \Phi_1$ is the lifting of the anti-monotone product $\varphi_1 \blacktriangleleft \varphi_2 : \mathcal{A} \to \mathbb{C}$.*

Proof. We follow [EFP19] to prove the third statement. Let $w = a_1 a_2 \cdots a_n$ an alternating word, i.e. $a_j \in \mathcal{A}_{i_j}$ and $i_j \neq i_{j+1}$ for any $1 \leq j < n$. For simplicity, we also assume that $n = 2m$ and $a_1 \in \mathcal{A}_1$ since the other cases follow analogously. First, by definition of the half-shuffle products on $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{C})$, we write

$$\Phi_1 * \Phi_2 = \Phi_1 \prec \Phi_2 + \Phi_1 \succ \Phi_2.$$

The first term of the right-hand side evaluated on w gives

$$\begin{aligned}\Phi_1 \prec \Phi_2(a_1 \cdots a_{2m}) &= \sum_{1\in S\subseteq[2m]} \Phi_1(a_S)\Phi_2(a_{J_1}|\cdots|a_{J_r}) \\ &= \sum_{1\in S\subseteq[2m]} \Phi_1(a_S)\prod_{i=1}^{r}\Phi_2(a_{J_i}),\end{aligned}$$

where we used that Φ_2 is a character. Now, recall that $\Phi_1(a_S) = 0$ if a_S is a subword of w containing a letter in $\mathcal{A}_2$. Since w is an alternating word and $a_1 \in \mathcal{A}_1$, then $\Phi_1(a_S)$ may produce a non-zero contribution if and only if $S \subseteq \{1, 3, \ldots, 2m-1\}$. However, if S is a proper subset, then we have that there is a connected component J_l of $[2m]\backslash S$ containing an odd number. This implies that $\Phi_2(a_{J_l}) = 0$ since a_{J_l} contains a letter in $\mathcal{A}_1$. We conclude that the only S that may give a non-zero contribution is precisely $S = \{1, 3, \ldots, 2m-1\}$. Thus

$$\Phi_1 \prec \Phi_2(a_1 \cdots a_{2m}) = \Phi_1(a_1a_3\cdots a_{2m-1})\prod_{i=1}^{m}\Phi_2(a_{2i}).$$

On the other hand, observe that

$$\Phi_1 \succ \Phi_2(a_1 \cdots a_{2m}) = \sum_{1\notin S\subseteq[2m]} \Phi_1(a_S)\prod_{i=1}^{r}\Phi_2(a_{J_i}) = 0$$

as $\Phi_2(a_{J_1}) = 0$ for any $S \not\ni 1$. Therefore

$$\begin{aligned}\Phi_1 * \Phi_2(a_1 \cdots a_{2m}) &= \Phi_1(a_1a_3\cdots a_{2m-1})\prod_{i=1}^{m}\Phi_2(a_{2i}) \\ &= \varphi_1(a_1a_3\cdots a_{2m-1})\prod_{i=1}^{m}\varphi_2(a_{2i}),\end{aligned}$$

which coincides with the definition of the monotone product $\varphi_1 \blacktriangleright \varphi_2 : \mathcal{A} \to \mathbb{C}$ in (3.3.6). □

One can motivate the definition of the left and right commutative group law in terms of additive convolutions (Definition 3.5.1). More precisely, let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $a^{(1)}, a^{(2)} \in \mathcal{A}$ be two freely independent random variables. For $i = 1, 2$, denote $\mathcal{A}_i$ the unital algebra generated by $a^{(i)}$. Since $T(T_+(\mathcal{A}_i)) \subset T(T_+(\mathcal{A}))$, the distribution of $a^{(i)}$ can be lifted to a character Φ_i on $T(T_+(\mathcal{A}))$ given by

$$\Phi_i(w) = \begin{cases} \varphi(a_1 \cdot_{\mathcal{A}} \ldots \cdot_{\mathcal{A}} a_n) & \text{if } w = a_1 \cdots a_n \in T_+(\mathcal{A}_i) \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the freeness condition on $a^{(1)}$ and $a^{(2)}$ implies the vanishing of mixed cumulants, i.e. $k_n(a_1, \dots, a_n) = 0$ when $w = a_1 \cdots a_n$ is a word containing letters from both $\mathcal{A}_1$ and $\mathcal{A}_2$. On the other hand, for $i = 1, 2$, let κ_i be the infinitesimal character on $T(T_+(\mathcal{A}))$ which coincides with $\mathcal{L}_\prec(\Phi_i)$ on $T(T_+(\mathcal{A}_i))$ and with 0 on words that contain a letter not in $\mathcal{A}_i$. By Theorem 4.3.2, κ_i is the infinitesimal lifting of the free cumulants of $a^{(i)}$, for $i = 1, 2$. Then, it readily follows that

$$(\kappa_1 + \kappa_2)(w) = 0$$

when w is a word containing letters from both $\mathcal{A}_1$ and $\mathcal{A}_2$. We conclude that $\kappa_1 + \kappa_2$ is the infinitesimal lifting of the free cumulant functionals of the distribution of $a^{(1)} + a^{(2)}$, and thus $\mathcal{E}_\prec(\kappa_1 + \kappa_2) = \Phi_1 \boxplus \Phi_2$ is the lifting of the free additive convolution of the distributions of $a^{(1)}$ and $a^{(2)}$.

The same discussion can be applied to the Boolean case to show that $\mathcal{E}_\succ(\beta_1 + \beta_2) = \Phi_1 \uplus \Phi_2$ is the lifting of the Boolean additive convolution of the distributions of two random variables. The critical result applied is the equivalence between independence and the vanishing mixed cumulants conditions for the free and Boolean case. Moreover, the commutativity of the left and right commutative group laws are explained by the fact that the free and Boolean products are commutative natural products as in Definition 3.3.7. Concerning the monotone independence case, one recalls that the above discussion does not apply. Nevertheless, we know by Remark 3.5.5 that monotone cumulants satisfy that $h_n(a+b) = h_n(a) + h_n(b)$ when a and b are two monotone independent random variables with the same distribution. This can be interpreted in the shuffle algebra framework as the fact that

$$\Phi * \Phi = \exp^*(\rho) * \exp^*(\rho) = \exp^*(\rho + \rho)$$

if $\Phi = \exp^*(\rho)$. In other words, if Φ is the lifting of distribution of a random variable a, then Φ^{*n} is the lifting of the distribution of the sum of n monotone identically distributed random variables $a_1, \dots, a_n$ such that each a_i has the same distribution as a. In general, if $\Psi = \exp^*(\rho_2)$, then the monotone cumulants of the monotone product $\Phi * \Psi$ is given by the Baker-Campbell-Hausdorff formula

$$\Phi * \Psi = \exp^*(\rho_1) * \exp^*(\rho_2) = \exp^*(\mathrm{BCH}(\rho_1, \rho_2)).$$

The previous interpretation allows us to formulate the non-commutative central limit theorems in the scope of the shuffle algebra framework of Ebrahimi-Fard and Patras, as such theorems deal with sums of independent and identically distributed random variables. In order to introduce the corresponding scaling, we define, for any $t \in \mathbb{R}$, the map $\Lambda_t : T(T_+(\mathcal{A})) \to T(T_+(\mathcal{A}))$ given by

$$\Lambda_t(w_1|\cdots|w_m) = t^{|w_1|+\cdots+|w_m|}(w_1|\cdots|w_m),$$

where $|w_i|$ stands for the degree of $w_i \in T_+(\mathcal{A})$, i.e. the length of the word w_i.

A straightforward consequence of the previous definition is stated in the following lemma.

Lemma 5.1.3 ([Sch86]). *Let $\mathcal{A}$ be an algebra and consider its double tensor algebra $T(T_+(\mathcal{A}))$. Then, for any $\Phi, \Psi \in \mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{C})$ and $t \in \mathbb{R}$, we have that*

$$(\Phi \circ \Lambda_t) * (\Psi \circ \Lambda_t) = (\Phi * \Psi) \circ \Lambda_t. \tag{5.1.3}$$

Proof. Let $w \in T_+(\mathcal{A})$. By definition of the convolution product and the Λ_t map, we have

$$\begin{aligned} (\Phi \circ \Lambda_t) * (\Psi \circ \Lambda_t)(w) &= m_{\mathbb{C}} \circ \big((\Phi \circ \Lambda_t) \otimes (\Psi \circ \Lambda_t)\big) \circ \Delta(w) \\ &= \sum \Phi(t^{|w_1|} w_1) \Psi(t^{|w_2|} w_2) \\ &= \sum t^{|w_1|+|w_2|} \Phi(w_1) \Psi(w_2) \\ &= t^{|w|} \sum \Phi(w_1) \Psi(w_2) \\ &= m_{\mathbb{C}} \circ (\Phi \otimes \Psi) \circ \Delta(t^{|w|} w) \\ &= (\Phi * \Psi) \circ \Lambda_t(w), \end{aligned}$$

where we used the linearity of the maps and the fact that Δ is graded. The general case $w_1 | \cdots | w_m$ follows from the fact that Δ is an algebra morphism. □

5.2 A proof for the monotone central limit theorem

The first ingredient to state a general monotone central limit theorem is to have a precise candidate for the limiting distribution of the random variables. The univariate central limit theorems studied in Theorem 3.5.10 share the feature that the limiting distribution is completely determined by only considering cumulants of order two in the corresponding moment-cumulants formulas. This motivates the following definition for the multivariate case.

Definition 5.2.1. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $(c_{ij})_{i,j \in I}$ be an array of complex numbers. We say that the family of random variables $\{b_i\}_{i \in I} \subset \mathcal{A}$ is an *arcsinal family of covariance* $(c_{ij})_{i,j \in I}$ if the joint distribution of the family is given by

$$\varphi(b_{i_1} \cdots b_{i_n}) = \sum_{\pi \in \mathcal{M}_2(n)} \frac{1}{|n/2|!} \prod_{\substack{\{p,q\} \in \pi \\ p<q}} c_{i_p i_q}, \tag{5.2.1}$$

for any $n \geq 1$ and $i_1, \ldots, i_n \in I$.

It is clear that when the family of random variables consists of a single element $\{b\}$ such that $\varphi(b) = 0$ and $\varphi(b^2) = 1$, then we obtain the moments of the arcsine distribution

(Definition 3.5.8). Following [NS06, Thm. 8.17] and taking the above definition of a candidate for the monotone central limit theorem, we can then state the precise statement for the multivariate case.

Theorem 5.2.2. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and consider the sequence of family of random variables $\{a_1^{(i)}\}_{i\in I}, \{a_2^{(i)}\}_{i\in I}, \ldots \subset \mathcal{A}$ such that the elements of the sequence are monotone independent sets and have the same joint distribution, i.e. the moment $\varphi\left(a_r^{(i_1)} \cdots a_r^{(i_n)}\right)$ does not depend of r, for any $n \geq 1$ and $i_1, \ldots, i_n \in I$. Assume that*

$$\varphi\big(a_r^{(i)}\big) = 0, \quad \text{for any } r \geq 1 \text{ and } i \in I.$$

Then we have that

$$\left(\frac{a_1^{(i)} + \cdots + a_n^{(i)}}{\sqrt{n}}\right)_{i\in I} \xrightarrow[n\to\infty]{d} (b_i)_{i\in I}, \tag{5.2.2}$$

where $(b_i)_{i\in I}$ is an arcsinal family of covariance given by $c_{ij} := \varphi\big(a_1^{(i)} a_1^{(j)}\big)$ for any $i, j \in I$.

The proof of the above theorem that we will discuss in the current chapter is based on the shuffle algebra framework for non-commutative probability. As we discussed in the previous section (Theorem 5.1.2), monotone additive convolutions of distributions can be interpreted as the convolution product of the respective characters on $T(T_+(\mathcal{A}))$ associated to the distribution. Moreover, the corresponding scaling by $n^{-1/2}$ can be treated using Lemma 5.1.3. Thereby, the shuffle-algebraic version of the monotone central limit theorem can be stated as in the following theorem.

Theorem 5.2.3 (Shuffle-algebraic monotone CLT)**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Let $\{a_i\}_{i\geq 1} \subset \mathcal{A}$ be a sequence of random variables whose distribution is extended to a character Φ on $T(T_+(\mathcal{A}))$. Also, assume that $\Phi(a_i) = 0$, for any $i \geq 1$. If ψ is the infinitesimal character on $T(T_+(\mathcal{A}))$ given by*

$$\psi(w) = \begin{cases} \Phi(a_i a_j) & \text{if } w = a_i a_j \text{ for some } i, j \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.2.3}$$

then

$$\lim_{m\to\infty} \Phi^{*m} \circ \Lambda_{m^{-1/2}} = \exp^*(\psi). \tag{5.2.4}$$

In particular, $\Phi^{*m} \circ \Lambda_{m^{-1/2}}$ converges, as $m \to \infty$, to the lifting of the distribution of an arcsinal family of covariance $\big(\Phi(a_i a_j)\big)_{i,j\geq 1}$.

We will provide two proofs of the above theorem. The first proof only relies on a combinatorial analysis of the iterations of the coproduct Δ.

Proof. For the sake of notational simplicity, instead of taking a general word $a_{i_1} \cdots a_{i_n}$, we can relabel and take the word $w = a_1 \cdots a_n \in T_+(\mathcal{A})$. Next, note that

$\Lambda_{m^{-1/2}}(w) = m^{-n/2}w$. Also, thanks to Theorem 4.3.2, we want to show that the limiting distribution has monotone cumulants given by the infinitesimal character ψ:

$$\lim_{m\to\infty} \Phi^{*m} \circ \Lambda_{m^{-1/2}}(w) = \exp^*(\psi)(w) = \sum_{\pi\in\mathcal{M}_2(n)} \frac{1}{|n/2|!} \prod_{\substack{\{p,q\}\in\pi \\ p<q}} \Phi(a_p a_q).$$

The first step is to analyze the terms of the iterated coproduct that will produce a non-zero contribution when m goes to ∞. We start by noticing that, for m large enough, we have that

$$\Delta^{[m]}(a_1\cdots a_n) = \sum a_{S_1}\otimes a_{S_2}\otimes\cdots\otimes a_{S_m}, \tag{5.2.5}$$

where, for each $1\le i\le m$, a_{S_i} is an empty word, a non-empty subword of w, or a product of two or more non-empty subwords of w, and the sum is restricted according to certain combinatorial considerations dictated by the definition of Δ. Also, every term in the right-hand side of (5.2.5) produce $\Phi(a_{S_1})\cdots\Phi(a_{S_m})$ when we compute Φ^{*m}.

First, recall the definition of the coproduct Δ:

$$\Delta(a_1\cdots a_n) = \sum_{S\subseteq[n]} a_S\otimes a_{J_1}|\cdots|a_{J_r},$$

where $J_1,\dots,J_r$ are the maximal subintervals in $[n]\backslash S$. Thus, deleting the empty blocks, we can consider the non-crossing partition $\{S, J_1,\dots,J_r\}\in \mathrm{NC}(n)$, where it is clear that, for any $1\le l\le r$, S and J_l are not comparable, or else $S<J_l$, i.e. J_l is nested in S. For the second iteration of the coproduct, we compute

$$\begin{aligned}\Delta^{[3]}(a_1\cdots a_n) &= (\Delta\otimes\mathrm{id})\circ\Delta(a_1\cdots a_n)\\ &= \sum_{S\subseteq U\subseteq[n]} a_S\otimes(a_{J_1^{U\backslash S}})|\cdots|(a_{J_{r_S}^{U\backslash S}})\otimes(a_{J_1^{[n]\backslash U}})|\cdots|(a_{J_{r_U}^{[n]\backslash U}}).\end{aligned}$$

By the same argument, deleting the empty blocks we have that

$$\pi := \{S\}\cup\{J_l^{U\backslash S}\}_l\cup\{J_h^{[n]\backslash U}\}_h\in\mathrm{NC}(n)$$

is a non-crossing partition such that every $J_l^{U\backslash S}$ is not comparable with $J_h^{[n]\backslash U}$ or $J_l^{U\backslash S} < J_h^{[n]\backslash U}$, for any l,h. Analogously, S is not comparable with V, or $S<V$, for any $V\in\pi\backslash\{S\}$.

Generalizing the above argument, notice that we can rearrange the terms that produce the same contribution when computing Φ^{*m} in the following way: if $a_{S_{i_1}},\dots,a_{S_{i_r}}$ are the components different from $\mathbf{1}$ in $a_{S_1}\otimes\cdots\otimes a_{S_m}$, and

$$a_{S_{i_j}} = a_{V_1^j}|\cdots|a_{V_{n_j}^j},$$

where each $V_l^j \subset [n]$ defines the subword $a_{V_l^j}$, then the collection of subsets $\{V_l^j\}_{1\leq l\leq n_j, 1\leq j\leq r}$ defines a non-crossing partition π of $[n]$. Even more, if $j < j'$ and V^j and $V^{j'}$ are subsets of $[n]$ defining a factor of S_{i_j} and $S_{i_{j'}}$, respectively, then V^j and $V^{j'}$ are not comparable in π, or $V^j < V^{j'}$ in π. Hence, we have a non-crossing partition $\pi \in \mathrm{NC}(n)$ together with a non-decreasing labelling $f : \pi \to [r]$ compatible with the partial order structure on π given by the nesting of the blocks. In particular, if each $a_{S_{i_j}}$ consists of exactly one factor, we obtain a monotone partition with r blocks.

Now, given a non-crossing partition with non-decreasing labelling obtained as described above, we can find $\binom{m}{r}$ terms in the right-hand side of (5.2.5) that produce such labelled non-crossing partition; this corresponds to determining which of the m components of the pure tensor will be the r products of non-empty subwords of w. Notice that these $\binom{m}{r}$ terms will produce the same value when we compute Φ^{*m}.

The next step is using the assumption that $\Phi(a_1) = 0$ for any $i \geq 1$. This assumption implies that terms on the right-hand side of (5.2.5), with a component consisting of a single word of length 1, will give a zero contribution when we compute Φ^{*m}. This clearly implies, by the pigeonhole principle, that $r \leq n/2$, and the equality is reached when each $a_{S_{i_j}}$ is a subword of length 2.

The final ingredient is the elementary limit, for any positive integers $r \leq k$:

$$\lim_{m\to\infty} \frac{\binom{m}{r}}{m^k} = \begin{cases} \dfrac{1}{k!} & \text{if } k = r, \\ \\ 0 & \text{if } r < k. \end{cases}$$

Thus, in the limit $m \to \infty$, we have the following cases:

- If there is a component $a_{S_{i_j}}$ that is the product of two or more non-empty subwords, then $r < n/2$ and the contribution of its associated labelled non-crossing partition in the limit of (5.2.4) is $\lim\limits_{m\to\infty} \binom{m}{r} m^{-n/2} = 0$. Then no term on the right-hand side of (5.2.5) with a component that is the product of at least two non-empty subwords gives a non-zero contribution.

- If $a_{S_{i_1}}, \dots, a_{S_{i_r}}$ are all subwords of w, and there is one subword $a_{S_{i_j}}$ of length 3 or more, then we have that $r < n/2$, and hence the contribution in the limit is $\lim\limits_{m\to\infty} \binom{m}{r} m^{-n/2} = 0$.

- The remaining case is when $a_{S_{i_1}}, \dots, a_{S_{i_r}}$ are all subwords of w of length 2. Notice that this can happen only when $n = 2k$ for some integer k. Then we have that $r = n/2 = k$, and the contribution of the labelled non-crossing partition, that is indeed a monotone pair partition, is equal to $\lim\limits_{m\to\infty} \binom{m}{k} m^{-k} = \frac{1}{k!}$.

The previous analysis allows us to conclude that

$$\lim_{m\to\infty} \Phi^{*m} \circ \Lambda_{m^{-1/2}}(a_1 \cdots a_n) = \begin{cases} \displaystyle\sum_{\pi \in \mathcal{M}_2(2k)} \frac{1}{k!} \prod_{\substack{\{p,q\}\in\pi \\ p<q}} \Phi(a_p a_q) & \text{if } n = 2k, \\ 0 & \text{otherwise,} \end{cases}$$

as we wanted to prove. □

For the second proof, we will exploit the fact that ρ commutes with itself, which implies that $\Phi^{*m} = \exp^*(m\rho)$ if $\Phi = \exp^*(\rho)$.

Alternative proof of Theorem 5.2.3. Once more, assume that $w = a_1 \cdots a_n \in T_+(\mathcal{A})$. Hence, if $\Phi = \exp^*(\rho)$, we have

$$\Phi^{*m} \circ \Lambda_{m^{-1/2}}(w) = \exp^*(m\rho) \circ \Lambda_{m^{-1/2}}(w) = \sum_{s=0}^{\infty} \frac{1}{s!} m^{s-n/2} \rho^{*s}(w). \tag{5.2.6}$$

From the proof of Theorem 4.3.2 (see also the idea of the proof of the monotone case in Proposition 6.2.6), we have that

$$\rho^{*s}(w) = \sum_{\pi \in \mathcal{M}^s(n)} \prod_{V \in \pi} \rho(a_V).$$

Now, let us analyze the terms in the exponential when $m \to \infty$. We have the following cases:

i) If $s < n/2$, then $s - n/2 < 0$, which implies that the terms indexed by such s goes to zero since $\lim\limits_{m\to\infty} m^{s-n/2} = 0$.

ii) If $s > n/2$ and $\pi \in \mathcal{M}^s(n)$, then there must be a block of π of size one, namely $V = \{a_j\}$. However, this block indexes the factor $\rho(a_V) = \rho(a_j) = \Phi(a_j) = 0$, so that
$$\rho_\pi(w) = \prod_{W \in \pi} \rho(a_W) = 0.$$
In this way, any $\pi \in \mathcal{M}^s(n)$ produces a zero contribution if $s > n/2$.

Hence, the only value of s such that $\pi \in \mathcal{M}^s(n)$ may produce a non-zero contribution to the sum in the right-hand side of (5.2.6) is when $s = n/2$, i.e. π is a pair monotone partition. Therefore, we have

$$\begin{aligned}
\Phi^{*m} \circ \Lambda_{m^{-1/2}}(w) &= \frac{1}{|n/2|!} \sum_{\pi \in \mathcal{M}_2(n)} \rho_\pi(w) \\
&= \frac{1}{|n/2|!} \sum_{\pi \in \mathcal{M}_2(n)} \prod_{\substack{\{p,q\}\in\pi \\ p<q}} \rho(a_p a_q) \\
&= \frac{1}{|n/2|!} \sum_{\pi \in \mathcal{M}_2(n)} \prod_{\substack{\{p,q\}\in\pi \\ p<q}} \Phi(a_p a_q),
\end{aligned}$$

where we used that $\rho(a_p a_q) = \Phi(a_p a_q) - \Phi(a_p)\Phi(a_q) = \Phi(a_p a_q)$, and the proof is now complete. □

Remark 5.2.4. The algebraic version of central limit theorems for coassociative coalgebras have appeared in the work of Schürmann [Sch86, Thm. 2]. The statement of Theorem 5.2.3 can be obtained then from Schürmann's result by specializing in the coalgebra $(\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{C}), \Delta, \epsilon)$. However, our deduction of the statement in Theorem 5.2.3 was independently motivated by the interpretation of the convolution product in terms of monotone convolution (Theorem 5.1.2).

5.3 A proof for the free and Boolean central limit theorems

The same idea of identifying convolutions with the group laws as stated in Theorem 5.1.2 also drives to the corresponding statements for the free and Boolean central limit theorems. In other words, we should expect that

$$\lim_{m\to\infty} \Phi^{\boxplus m} \circ \Lambda_{m^{-1/2}} = \mathcal{E}_{\prec}(\psi) \quad \text{and} \quad \lim_{m\to\infty} \Phi^{\uplus m} \circ \Lambda_{m^{-1/2}} = \mathcal{E}_{\succ}(\psi),$$

where ψ is the infinitesimal character defined in the statement of Theorem 5.2.3. In this section, we will show that the previous relations are true, providing the shuffle-algebraic version of the central limit theorems for free and Boolean cumulants.

First of all, we shall need the free and Boolean analogues of Definition 5.2.1. The free analogue can be found in the study of the classical and free central limit theorems exposed in [NS06, Lec. 8].

Definition 5.3.1. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $(c_{ij})_{i,j\in I}$ be an array of complex numbers. We say that the family of random variables $\{s_i\}_{i\in I} \subset \mathcal{A}$ is an *semicircular family of covariance* $(c_{ij})_{i,j\in I}$ if the joint distribution of the family is given by

$$\varphi(s_{i_1}\cdots s_{i_n}) = \sum_{\pi\in\mathrm{NC}_2(n)} \prod_{\substack{\{p,q\}\in\pi\\ p<q}} c_{i_p i_q}, \tag{5.3.1}$$

for any $n \geq 1$ and $i_1,\ldots,i_n \in I$.

For the Boolean case, notice that the set of pair interval partitions of $[n]$ is $\mathrm{Int}_2(n) = \emptyset$ if n is odd and $\mathrm{Int}_2(2k) = \big\{\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}\big\}$. The corresponding definition of the Boolean limiting distribution is the following.

Definition 5.3.2. Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and $(c_{ij})_{i,j\in I}$ be an array of complex numbers. We say that the family of random variables $\{d_i\}_{i\in I} \subset \mathcal{A}$ is an *Bernoulli family of covariance* $(c_{ij})_{i,j\in I}$ if the joint distribution of the family is given by

$$\varphi(d_{i_1}\cdots d_{i_n}) = \begin{cases} \displaystyle\prod_{j=1}^{k} c_{i_{2j-1}i_{2j}}, & \text{if } n = 2k,\\ 0 & \text{otherwise,}\end{cases} \tag{5.3.2}$$

for any $n \geq 1$ and $i_1,\ldots,i_n \in I$.

The next technical lemmas describe the iterated half-shuffle products of the infinitesimal liftings of the free and Boolean cumulants. They are indeed a refinement of Theorem 4.3.2 since they show that the term of order s in the half-shuffle exponentials yields the non-crossing partitions with s blocks in the corresponding moment-cumulant formulas.

Lemma 5.3.3. *Let $\mathcal{A}$ be an algebra. For a character $\Phi \in G(T(T_+(\mathcal{A})))$, consider the infinitesimal character κ such that $\Phi = \mathcal{E}_{\prec}(\kappa)$. Let $w = a_1\cdots a_n \in \mathcal{A}^{\otimes n}$, and take a non-empty subset $S \subseteq [n]$. Then, for $s \geq 1$*

$$\kappa^{\prec s}(a_{J_1}|\cdots|a_{J_r}) = \sum_{\substack{\pi_i\in\mathrm{NC}(J_i)\\ 1\leq i\leq r\\ |\pi_1|+\cdots+|\pi_r|=s}} \prod_{i=1}^{r} \kappa_{\pi_i}(a_{J_i}), \tag{5.3.3}$$

where $\{J_1,\ldots,J_r\}$ is an interval partition of S of r subintervals ordered increasingly according to their minimum element.

Proof. The proof will be done by induction on $s \geq 1$. The base case $s = 1$ is easy to see: since κ is an infinitesimal character then $\kappa^{\prec 1}(a_{J_1}|\cdots|a_{J_r}) = 0$ if $r \geq 2$ and the only partition of S with one block is precisely $\{S\}$, which yields $\kappa^{\prec 1}(a_S) = \sum_{\pi\in\mathrm{NC}(S),|\pi|=1}\kappa_\pi(a_S)$.

Now, we assume that the formula holds for s. We will prove the result for $s+1$. Indeed, let $S \subseteq [n]$ and take an interval partition $\{J_1,\ldots,J_r\}$ of S as described in the statement of the lemma. Recall that $\kappa^{\prec s+1} = \kappa \prec \kappa^{\prec s}$. Thus we can compute

$$\begin{aligned}\kappa^{\prec s+1}(a_{J_1}|\cdots|a_{J_r}) &= (\kappa \prec \kappa^{\prec s})(a_{J_1}|\cdots|a_{J_r})\\ &= m_{\mathbb{C}}\circ(\kappa\otimes\kappa^{\prec s})\circ\Delta_{\prec}(a_{J_1}|\cdots|a_{J_r}).\end{aligned}$$

Let us analyze $\Delta_\prec(a_{J_1}|\cdots|a_{J_r})$. First, according to the definition of the unshuffle bialgebra structure on $T(T_+(\mathcal{A}))$, we have that

$$\Delta_\prec(a_{J_1}|\cdots|a_{J_r}) = \Delta_\prec(a_{J_1})\Delta(a_{J_2})\cdots\Delta(a_{J_r}). \tag{5.3.4}$$

Now, recall that

$$\Delta_\prec(a_{J_1}) = \sum_{\substack{U\subseteq J_1\\ \min(J_1)\in U}} a_U \otimes a_{J_{J_1}^U},$$

where $a_{J_{J_1}^U}$ is the product of subwords if a_{J_1} indexed by the maximal subintervals of $J_1\backslash U$. In particular, the sum defining $\Delta_\prec(a_{J_1})$ contains the term $a_{J_1}\otimes \mathbf{1}$ but does not contain the term $\mathbf{1}\otimes a_{J_1}$. On the other hand, the definition of $\Delta(a_{J_l})$ involves all the subsets of J_l, so the expansion of $\Delta(a_{J_l})$ contains the term $\mathbf{1}\otimes a_{J_l}$.

Now, observe that the fact that κ is an infinitesimal character implies that a term in the expansion of $\Delta_\prec(a_{J_1})\Delta(a_{J_2})\cdots\Delta(a_{J_r})$, such that its first component is a product of two or more subwords, will produce a zero contribution when evaluating on $\kappa\otimes\kappa^{\prec s}$. Recalling that $\Delta_\prec(a_{J_1})$ gives a non-empty subword in the first component, the only terms that may produce a non-zero contribution are

$$\sum_{\substack{U\subseteq J_1\\ \min(J_1)\in U}} a_U \otimes a_{J_{J_1}^U}|a_{J_2}|\cdots|a_{J_r}.$$

Hence we have

$$\kappa^{\prec s+1}(a_{J_1}|\cdots|a_{J_r}) = \sum_{\substack{U\subseteq J_1\\ \min(J_1)\in U}} \kappa(a_U)\kappa^{\prec s}(a_{J_{J_1}^U}|a_{J_2}|\cdots|a_{J_r}).$$

Now, since the $J_{J_1}^U = \{X_1,\ldots,X_p\}$ stands for the collection of maximal subintervals of $J_1\backslash U$, then one can readily check that $J_{J_1}^U\cup\{J_2,\ldots,J_r\}$ is an interval partition of $S\backslash U$. Therefore, we can apply our induction hypothesis on $\kappa^{\prec s}(a_{J_{J_1}^U}|a_{J_2}|\cdots|a_{J_r})$ and obtain:

$$\kappa^{\prec s+1}(a_{J_1}|\cdots|a_{J_r}) = \sum_{\substack{U\subseteq J_1\\ \min(J_1)\in U}} \kappa(a_U)\left(\sum_{\substack{\pi_i\in\mathrm{NC}(J_i),2\le i\le r\\ \sigma_j\in\mathrm{NC}(X_j),1\le j\le p\\ |\sigma_1|+\cdots+|\sigma_p|+|\pi_2|+\cdots+|\pi_r|=s}} \prod_{i=2}^r \kappa_{\pi_i}(a_{J_i})\prod_{j=1}^p \kappa_{\sigma_j}(a_{X_j})\right).$$

Observe that, according to the definition of $\Delta_\prec$, every X_j is an interval in $J_1\backslash U$, so that $\{U\}\cup\sigma_1\cup\cdots\cup\sigma_p$ is a non-crossing partition of J_1 such that U is an outer block containing $\min(J_1)$. Finally, we observe that there is a bijection between $\mathrm{NC}(J_1)$ and the set of tuples $(U,\sigma_1,\ldots,\sigma_p)$, where $\min(J_1)\in U\subseteq J_1$ and if $X_1,\ldots,X_p$ are the maximal subintervals of $J_1\backslash U$, ordered increasingly according their minimal elements, then $\sigma_i\in\mathrm{NC}(X_i)$ for any $1\le i\le p$. This bijection implies that we can rearrange the above sum and conclude that

$$\kappa^{\prec s+1}(a_{J_1}|\cdots|a_{J_r}) = \sum_{\substack{\pi_i\in \mathrm{NC}(J_i)\\ 1\leq i\leq r\\ |\pi_1|+\cdots+|\pi_r|=s+1}} \prod_{i=1}^{r} \kappa_{\pi_i}(a_{J_i}),$$

as we wanted to prove. □

The Boolean counterpart of the previous lemma can be proved by doing a similar analysis as in the proof of the previous lemma. Nevertheless, we will notice that the computations are simpler than in the free case since the products of words $a_{J_1^U}$ are directly evaluated on the infinitesimal cumulant β.

Lemma 5.3.4. *Let $\mathcal{A}$ be an algebra. For a character $\Phi \in G(T(T_+(\mathcal{A})))$, consider the infinitesimal character β such that $\Phi = \mathcal{E}_\succ(\beta)$. Let $w = a_1\cdots a_n \in \mathcal{A}^{\otimes n}$, and take a non-empty subset $S \subseteq [n]$. Then, for $s \geq 1$*

$$\beta^{\succ s}(a_{J_1}|\cdots|a_{J_r}) = \sum_{\substack{\pi_i\in \mathrm{Int}(J_i)\\ 1\leq i\leq r\\ |\pi_1|+\cdots+|\pi_r|=s}} \prod_{i=1}^{r} \beta_{\pi_i}(a_{J_i}), \tag{5.3.5}$$

where $\{J_1,\ldots,J_r\}$ is an interval partition of S of r subintervals ordered increasingly according to their minimum element.

Proof. We proceed similarly to the proof of Lemma 5.3.3 by induction on $s \geq 1$. The base case $s = 1$ from the fact that β is an infinitesimal character as in the proof of the previous lemma.

Now, we assume that the formula holds for s. We will prove the result for $s+1$. Indeed, let $S \subseteq [n]$ and take an interval partition $\{J_1,\ldots,J_r\}$ of S as described in the statement of the lemma. Since $\beta^{\succ s+1} = \beta^{\succ s} \succ \beta$, we have

$$\begin{aligned}
\beta^{\prec s+1}(a_{J_1}|\cdots|a_{J_r}) &= (\beta^{\succ s} \succ \beta)(a_{J_1}|\cdots|a_{J_r})\\
&= m_{\mathbb{C}} \circ (\beta^{\succ s}\otimes\beta)\circ\Delta_\succ(a_{J_1}|\cdots|a_{J_r})\\
&= m_{\mathbb{C}} \circ (\beta^{\succ s}\otimes\beta)\circ\Delta_\succ(a_{J_1})\Delta(a_{J_2})\cdots\Delta(a_{J_r}).
\end{aligned}$$

By definition of $\Delta_\succ$, the expansion of $\Delta_\succ$ does not contain the term $a_{J_1}\otimes\mathbf{1}$. Then every pure tensor in the expansion of $\Delta_\succ$ will have a non-empty word in the second component. Since β is an infinitesimal character, the only terms that may produce a non-zero contribution are

$$\sum_{\substack{U\subsetneq J_1\\ \min(J_1)\notin U}} a_U|a_{J_2}|\cdots|a_{J_r}\otimes a_{J_1^U}.$$

Again, because β is an infinitesimal character, we will need that $a_{J_1^U}$ is a word for a non-zero contribution. Then, the above sum can be restricted even more as:

$$\sum_{j=1}^{m-1} a_{i_{j+1}} \cdots a_{i_m} | a_{J_2} | \cdots | a_{J_r} \otimes a_{i_1} \cdots a_{i_j},$$

where we have written $J_1 = \{i_1 < \cdots < i_m\}$. Hence we have

$$\beta^{\succ s+1}(a_{J_1} | \cdots | a_{J_r}) = \sum_{j=1}^{m-1} \beta^{\succ s}\big(a_{i_{j+1}} \cdots a_{i_m} | a_{J_2} | \cdots | a_{J_r}\big)\beta(a_{i_1} \cdots a_{i_j}).$$

Moreover, we can check that $\{\{i_{j+1}, \ldots, i_m\}, J_2, \ldots, J_r\}$ is an interval partition of $S \backslash \{i_1, \ldots, i_j\}$. Thus we can apply the induction hypothesis and conclude in the same way that in the free case:

$$\begin{aligned}
\beta^{\succ s+1}(a_{J_1} | \cdots | a_{J_r}) &= \sum_{j=1}^{m-1} \left(\sum_{\substack{\sigma \in \mathrm{Int}(\{i_{j+1}, \ldots, i_m\}) \\ \pi_i \in \mathrm{Int}(J_i), 2 \le i \le r \\ |\sigma| + |\pi_2| + \cdots + |\pi_r| = s}} \beta_\sigma(a_{i_{j+1}} \cdots a_{i_m}) \prod_{i=2}^{r} \beta_{\pi_i}(a_{J_i}) \right) \beta(a_{i_1} \cdots a_{i_j}) \\
&= \sum_{\substack{\pi_i \in \mathrm{Int}(J_i) \\ 1 \le i \le r \\ |\pi_1| + \cdots + |\pi_r| = s+1}} \prod_{i=1}^{r} \beta_{\pi_i}(a_{J_i}).
\end{aligned}$$

□

To prove our next result, notice that the statement in Lemma 5.1.3 is also true when we replace the convolution product $*$ by each of the non-associative half-shuffle products $\prec$ and $\succ$. The advertised shuffle-algebraic central limit theorems are then finally stated as follows.

Theorem 5.3.5 (Shuffle-algebraic free and Boolean CLT)**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Let $\{a_i\}_{i \ge 1} \subset \mathcal{A}$ be a sequence of random variables whose distribution is extended to a character Φ on $T(T_+(\mathcal{A}))$. Also, assume that $\Phi(a_i) = 0$, for any $i \ge 1$. If ψ is the infinitesimal character on $T(T_+(\mathcal{A}))$ given by*

$$\psi(w) = \begin{cases} \Phi(a_i a_j) & \text{if } w = a_i a_j \text{ for some } i, j \ge 1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.3.6}$$

then

$$\lim_{m \to \infty} \Phi^{\boxplus m} \circ \Lambda_{m^{-1/2}} = \mathcal{E}_\prec(\psi). \tag{5.3.7}$$

In particular, $\Phi^{\boxplus m} \circ \Lambda_{m^{-1/2}}$ converges, as $m \to \infty$, to the lifting of the distribution of a *semicircular family of covariance* $\big(\Phi(a_ia_j)\big)_{i,j\geq 1}$. *Moreover*

$$\lim_{m\to\infty} \Phi^{\uplus m} \circ \Lambda_{m^{-1/2}} = \mathcal{E}_{\succ}(\psi). \tag{5.3.8}$$

In particular, $\Phi^{\uplus m} \circ \Lambda_{m^{-1/2}}$ *converges, as* $m \to \infty$*, to the lifting of the distribution of a Bernoulli family of covariance* $\big(\Phi(a_ia_j)\big)_{i,j\geq 1}$.

Proof. We will only prove the free case since the Boolean case follows similarly. As in the alternative proof of Theorem 5.2.3, we can take the word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. Note that by Theorem 4.3.2, we have to show that

$$\lim_{m\to\infty} \Phi^{\boxplus m} \circ \Lambda_{m^{-1/2}}(w) = \mathcal{E}_{\prec}(\psi)(w) = \sum_{\pi\in \mathrm{NC}_2(n)} \prod_{\substack{\{p,q\}\in\pi \\ p<q}} \Phi(a_pa_q).$$

Since the group law $\boxplus$ on $G(T(T_+(\mathcal{A})))$ is commutative, if κ is the infinitesimal character such that $\Phi = \mathcal{E}_{\prec}(\kappa)$, then $\Phi^{\boxplus m} = \mathcal{E}_{\prec}(m\kappa)$, for any $m \geq 1$. On the other hand, the definition of the half-shuffle exponential yields

$$\mathcal{E}_{\prec}(m\kappa) = \epsilon + m\kappa + m^2\kappa \prec \kappa + m^3\kappa \prec (\kappa \prec \kappa) + \cdots = \epsilon + \sum_{s\geq 1} m^s\kappa^{\prec s},$$

where $\kappa^{\prec s+1} = \kappa \prec \kappa^{\prec s}$ for any $s \geq 1$. By evaluating in $\Lambda_{m^{-1/2}}(w)$ and using Lemma 5.3.3, we have the following computation, for any $m \geq 1$:

$$\begin{aligned}
\Phi^{\boxplus m} \circ \Lambda_{m^{-1/2}}(w) &= \mathcal{E}_{\prec}(m\kappa)(m^{-n/2}w) \\
&= \sum_{s\geq 1} m^{s-n/2}\kappa^{\prec s}(w) \\
&\overset{\text{(Lemma 5.3.3)}}{=} \sum_{s=1}^{n} m^{s-n/2} \sum_{\pi\in \mathrm{NC}^s(n)} \kappa_\pi(w)
\end{aligned}$$

where $\mathrm{NC}^s(n)$ stands for the set of non-crossing partitions on $[n]$ with exactly s blocks.

As in the alternative proof of Theorem 5.2.3, we have that the only index s such that $\pi \in \mathrm{NC}^s(n)$ may produce a non-zero contribution when $m \to \infty$ is $s = \frac{n}{2}$, i.e. π is a pair non-crossing partition. Therefore we have

$$\begin{aligned}
\Phi^{\boxplus m} \circ \Lambda_{m^{-1/2}}(w) &= \sum_{\pi\in \mathrm{NC}_2(n)} \kappa_\pi(w) \\
&= \sum_{\pi\in \mathrm{NC}_2(n)} \prod_{\substack{\{p,q\}\in\pi \\ p<q}} \kappa(a_pa_q) \\
&= \sum_{\pi\in \mathrm{NC}_2(n)} \prod_{\substack{\{p,q\}\in\pi \\ p<q}} \Phi(a_pa_q),
\end{aligned}$$

where we used that $\kappa(a_p a_q) = \Phi(a_p a_q) - \Phi(a_p)\Phi(a_q) = \Phi(a_p a_q)$. This completes the proof for the free case.

The proof of the Boolean central limit theorem follows in a similar way by replacing $\boxplus, \kappa, \prec$, NC and Lemma 5.3.3 by $\uplus, \beta, \succ$, Int and Lemma 5.3.4, respectively. $\square$

We finish this chapter by remarking that the shuffle-algebraic free and Boolean CLT proved in Theorem 5.3.5 do not follow directly from [Sch86, Thm. 2]. The reason is simple: the half-unshuffle coproducts $\Delta_\prec$ and $\Delta_\succ$ are not coassociative; instead, both coproducts satisfy the unshuffle relations stated in Definition 4.1.7.

Chapter 6

Infinitesimal Non-commutative Probability via Shuffle Algebra

Variations and extensions of free probability have arisen from theoretical and applied problems. In this chapter, we will study the extension known as *infinitesimal free probability*. Roughly speaking, infinitesimal free probability arises from considering an additional linear functional φ' on the space $(\mathcal{A}, \varphi)$ that can be thought of as the formal derivative of φ. Our objective in this chapter is to extend the shuffle algebra framework for non-commutative probability studied in Chapter 4 in order to encompass the infinitesimal analogue of non-commutative probability and, consequently, understand and obtain results in such theory.

More precisely, the present chapter is organized in the following way. In Section 6.1, we introduce the notions of infinitesimal non-commutative cumulants. We start the discussion with the infinitesimal free case studied in detail in [FN10]. Following the same idea, we present the corresponding definitions of infinitesimal Boolean and infinitesimal monotone cumulants, which correspond to a particular case of the differential cumulants introduced in [Has11]. The principal idea is to consider cumulants as linear maps to a particular commutative algebra $\mathbb{G}$. Then, in Section 6.2, we extend the shuffle-algebraic framework for non-commutative probability of Section 4.3 with the objective of encompassing the infinitesimal cumulants. In particular, we show that the infinitesimal cumulants are also infinitesimal characters, and we find several shuffle-algebraic formulas that characterize, in a non-trivial way, the infinitesimal moment-cumulant relations (Theorem 6.2.7). After that, in Section 6.3, we apply the aforementioned extension of the shuffle-algebraic framework to obtain combinatorial formulas that relate the different brands of infinitesimal cumulants. Finally, Section 6.4 is devoted to studying the infinitesimal analogue of the Boolean Bercovici-Pata bijection. In particular, we prove, via shuffle-algebraic techniques, that the algebraic version of the semigroup property of [BN08b] also holds in the infinitesimal case.

The main results described in this chapter, as well as the ideas for their proofs, are based on the work [CEFP21].

6.1 Infinitesimal non-commutative cumulants

Infinitesimal free independence is a concept introduced in the work of Belinschi and Shlyakhtenko [BS12] following the previous work of Biane, Nica and Goodman [BGN03]. The idea in [BGN03] consisted of creating a non-commutative probability theory where the role of the non-crossing partitions is taken by the non-crossing partitions associated to Coxeter groups of type B. More precisely, they defined a notion of free independence of type B and the corresponding notion of free cumulants of type B in such a way that free independence of type B is characterized by the vanishing mixed cumulants of type B. The subsequent work [BS12] consists of compressing the definitions in such a way that an additional linear functional on a non-commutative probability space $(\mathcal{A}, \varphi)$ can be regarded as the derivative of φ, arising then the notion of infinitesimal free independence. This theory has recently drawn attention due to its application in random matrix theory, more specifically, in the study of outliers in the spectra of random matrix models with finite rank perturbations [Shl18].

We start by formally stating the precise definition of an infinitesimal space.

Definition 6.1.1 (Infinitesimal non-commutative probability space)**.** We say that the triple $(\mathcal{A}, \varphi, \varphi')$ is an *infinitesimal non-commutative probability space* if $(\mathcal{A}, \varphi)$ is a non-commutative probability space and $\varphi' : \mathcal{A} \to \mathbb{C}$ is a linear functional satisfying $\varphi'(1_{\mathcal{A}}) = 0$.

The motivation of the previous object came from the notion of free independence *up to order t* with respect to a family of linear functionals $\{\varphi_t : \mathcal{A} \to \mathbb{C}\}_{t>0}$ from [BS12]. This notion can be translated by considering the limiting functionals $\varphi = \lim_{t\to 0} \varphi_t$ and $\varphi' = \lim_{t\to\infty}(\varphi_t - \varphi)/t$. In the subsequent work [FN10], the authors extract the concrete conditions on φ and φ' to state the following definition.

Definition 6.1.2 (Infinitesimal freeness ([FN10]))**.** Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space. Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. We say that $\mathcal{A}_1, \dots, \mathcal{A}_k$ are *infinitesimally free* if the following holds: for $n \geq 2$, any sequence of indices $i_1, \dots, i_n \in [k]$ such that $i_j \neq i_{j+1}$ for $1 \leq j < n$, and elements $a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n}$ such that $\varphi(a_j) = 0$ for $j = 1, \dots, n$, the following is satisfied:

1. $\varphi(a_1 \cdots a_n) = 0$,
2. $$\varphi'(a_1 \cdots a_n) = \begin{cases} \varphi'\left(a_{(n+1)/2}\right) \prod_{j=1}^{(n-1)/2} \varphi(a_j a_{n+1-j}) & \text{if } n \text{ is odd and } i_j = i_{n+1-j} \\ & \text{for } j = 1, \dots, (n-1)/2, \\ 0 & \text{otherwise.} \end{cases}$$

In the work [FN10], the authors introduced a suitable notion of cumulants so that infinitesimal freeness is characterized by the vanishing of mixed free cumulants and infinitesimal free cumulants, analogously to Theorem 3.4.20. Roughly speaking, infinitesimal free cumulants arise by taking formal derivatives of free cumulants.

Definition 6.1.3 (Infinitesimal free cumulant functionals)**.** Let $(\mathcal{A}, \varphi, \varphi')$ be a non-commutative probability space. The *infinitesimal free cumulants* form the family of multilinear functionals $\{k'_n : \mathcal{A} \to \mathbb{C}\}_{n\geq 1}$ recursively defined by the following formula:

$$\varphi'(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \sum_{V \in \pi} k'_{|V|}(a_1, \ldots, a_n|V) \prod_{\substack{W \in \pi \\ W \neq V}} k_{|W|}(a_1, \ldots, a_n|W), \tag{6.1.1}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$, where $\{k_n\}_{n\geq 1}$ stands for the family of free cumulant functionals of $(\mathcal{A}, \varphi)$.

Remark 6.1.4. It is possible to show that the infinitesimal free cumulant-moment relations can be written as a sum indexed by the so-called non-crossing partitions of type B ([BGN03]). Hence, in some sense, one can have an analogous free probability theory of type B.

The key idea of the work [FN10] is that the functionals φ, φ' of an infinitesimal non-commutative probability space can be jointly considered as a linear map $\tilde{\varphi}$ from $\mathcal{A}$ to the following commutative algebra.

Definition 6.1.5 (Grassmann algebra)**.** The *Grassmann algebra* is the vector space

$$\mathbb{G} := \{x + \hbar y \,:\, x, y \in \mathbb{C}\} \cong \mathbb{C} \oplus \mathbb{C}$$

endowed with the product

$$(x + \hbar y)(u + \hbar v) = xu + \hbar(xv + yu), \quad \forall\, x, y, u, v \in \mathbb{C}. \tag{6.1.2}$$

Remark 6.1.6. It is easy to see that $\mathbb{G}$ is isomorphic to $\mathbb{C}[[\hbar]]/(\hbar^2)$. In other words, the multiplication of $\mathbb{G}$ is the same as the multiplication on formal power series with parameter $\hbar$ under the restriction that $\hbar^2 = 0$.

Now, given an infinitesimal non-commutative probability space $(\mathcal{A}, \varphi, \varphi')$, we define the map

$$\tilde{\varphi} := \varphi + \hbar\varphi' : \mathcal{A} \to \mathbb{G}.$$

It is clear that $\varphi(1_{\mathcal{A}}) = \varphi(1_{\mathcal{A}}) + \hbar\varphi'(1_{\mathcal{A}}) = 1_{\mathbb{G}}$. Moreover, since $\mathbb{G}$ is a commutative algebra, we can think of the triple $(\mathcal{A}, \varphi, \varphi')$ as a space $(\mathcal{A}, \tilde{\varphi})$, where we can define the $\mathbb{G}$-*valued free cumulants* as the family of $\mathbb{C}$-linear maps $\{\tilde{k}_n : \mathcal{A} \to \mathbb{G}\}_{n\geq 1}$ implicitly defined by the relations

$$\tilde{\varphi}(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{V \in \pi} \tilde{k}_{|V|}(a_1, \ldots, a_n | V),$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$. Observe that the product on the right-hand side of the previous equation is the product in $\mathbb{G}$ given by (6.1.2). According to such product rule, it is easy to obtain:

Proposition 6.1.7 ([FN10, Prop. 4.5]). *Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space, and consider the associated $\mathbb{G}$-valued free cumulants $\{\tilde{k}_n\}_{n\geq 1}$. If $\tilde{k}_n^{(1)}, \tilde{k}_n^{(2)} : \mathcal{A}^n \to \mathbb{C}$ are the multilinear functionals such that $\tilde{k}_n = \tilde{k}_n^{(1)} + \hbar \tilde{k}_n^{(2)}$, then $\tilde{k}_n^{(1)} = k_n$ and $\tilde{k}_n^{(2)} = k'_n$, for any $n \geq 1$.*

Following the same philosophy of taking formal derivatives, we find motivation for infinitesimal cumulants for the other types of non-commutative independence.

Definition 6.1.8 (Infinitesimal Boolean cumulant functionals). Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space and let $\{b_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ be the Boolean cumulant functionals of $(\mathcal{A}, \varphi)$. The *infinitesimal Boolean cumulants* are the family of multilinear functionals $\{b'_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ recursively defined by

$$\varphi'(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{Int}(n)} \sum_{V \in \pi} b'_{|V|}(a_1, \ldots, a_n | V) \prod_{\substack{W \in \pi \\ W \neq V}} b_{|W|}(a_1, \ldots, a_n | W), \tag{6.1.3}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$.

As in the free case, we can consider the $\mathbb{G}$-valued Boolean cumulants as the family of $\mathbb{C}$-linear maps $\{\tilde{b}_n : \mathcal{A} \to \mathbb{G}\}$ implicitly defined by

$$\tilde{\varphi}(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{Int}(n)} \tilde{b}_\pi(a_1, \ldots, a_n),$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$. It turns out that $\tilde{b}_n = b_n + \hbar b'_n$, for any $n \geq 1$, where $\{b_n\}_{n\geq 1}$ and $\{b'_n\}_{n\geq 1}$ are the Boolean and the infinitesimal Boolean cumulants, respectively. In particular, taking the $\hbar$-coefficient in the formula for $\tilde{b}_n$ in terms of the moments given by Möbius inversion, we have

$$b'_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{Int}(n)} (-1)^{|\pi|-1} \sum_{V \in \pi} \varphi'_{|V|}(a_1, \ldots, a_n | V) \prod_{\substack{W \in \pi \\ W \neq V}} \varphi_{|W|}(a_1, \ldots, a_n | W), \tag{6.1.4}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$, where, as before, $\{\varphi'_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ is the family of multilinear functionals defined by $\varphi'_n(a_1, \ldots, a_n) := \varphi'(a_1 \cdots a_n)$.

Remark 6.1.9. One could use the definition of infinitesimal Boolean cumulants to obtain a candidate for a notion of infinitesimal Boolean independence in such a way that it is characterized by the vanishing mixed cumulants and infinitesimal cumulants condition. More precisely, let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space, and $\mathcal{A}_1, \dots, \mathcal{A}_m$ be (non-unital) subalgebras of $\mathcal{A}$. Consider also the $\mathbb{G}$-valued Boolean cumulants $\{\tilde{b}_n : \mathcal{A} \to \mathbb{G}\}_{n \geq 1}$. Then, we assume the vanishing mixed cumulants condition for the Boolean cumulants $\{b_n\}_{n\geq 1}$ and the infinitesimal Boolean cumulants $\{b'_n\}_{n\geq 1}$ holds, i.e.

$$b_n(a_1, \dots, a_n) = 0 \quad \text{and} \quad b'_n(a_1, \dots, a_n) = 0,$$

if there exists two indexes $1 \leq s < r \leq n$, with $n \geq 2$, such that $a_r \in \mathcal{A}_{i_r}, a_s \in \mathcal{A}_{i_s}$ and $i_r \neq i_s$.

We want to find some conditions for φ and φ', which are equivalent to the vanishing of mixed cumulants. Let $n \geq 1$ and take elements $a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n}$, where $i_1, \dots, i_n \in [m]$ and $i_j \neq i_{j+1}$ for $1 \leq j < n$. Since we are assuming the vanishing mixed cumulants condition for b_n and b'_n, the unique partition π which does not give a zero contribution in the $\mathbb{G}$-valued moment-Boolean cumulant formula

$$\tilde{\varphi}_n(a_1, \dots, a_n) = \sum_{\pi \in \mathrm{Int}(n)} \tilde{b}_\pi(a_1, \dots, a_n), \tag{6.1.5}$$

is the minimal partition $0_n = \{\{1\}, \{2\}, \dots, \{n\}\} \in \mathrm{Int}(n)$. Thus, by using that $\tilde{\varphi}(a) = \tilde{b}_1(a)$ for any $a \in \mathcal{A}$:

$$\begin{aligned} \tilde{\varphi}_n(a_1, \dots, a_n) &= \sum_{\pi \in \mathrm{Int}(n)} \tilde{b}_\pi(a_1, \dots, a_n) \\ &= b_{0_n}(a_1, \dots, a_n) \\ &= \tilde{b}_1(a_1) \cdots \tilde{b}_1(a_n) \\ &= \tilde{\varphi}(a_1) \cdots \tilde{\varphi}(a_n). \end{aligned}$$

Recalling that $\tilde{\varphi} = \varphi + \hbar\varphi'$, we have that $\tilde{\varphi}(a_1 \cdots a_n) = \tilde{\varphi}(a_1) \cdots \tilde{\varphi}(a_n)$ is equivalent to the conditions

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n), \tag{6.1.6}$$

$$\varphi'(a_1 \cdots a_n) = \sum_{j=1}^{n} \varphi'(a_j) \prod_{\substack{r=1 \\ r \neq j}}^{n} \varphi(a_r), \tag{6.1.7}$$

for any $n \geq 1$ and $a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n}$, with $i_j \neq i_{j+1}$ for $1 \leq j < n$. Furthermore, notice that condition (6.1.6) is precisely the definition of Boolean independence.

Definition 6.1.10. Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space. Consider $\mathcal{A}_1, \dots, \mathcal{A}_m$ subalgebras of $\mathcal{A}$. We say that $\mathcal{A}_1, \dots \mathcal{A}_m$ are *infinitesimally Boolean independent* if for each $n \geq 1$ and any sequence of indices $i_1, \dots, i_n \in [m]$ such that $i_j \neq i_{j+1}$ for $j = 1, \dots, n-1$, and elements $a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n}$, we have that (6.1.6) and (6.1.7) hold.

Remark 6.1.11. From Remark 6.1.9, we have that the vanishing mixed Boolean and infinitesimal Boolean cumulants conditions imply infinitesimal Boolean independence. The converse is also true. The proof follows similar ideas that the proof Thm. 11.16 in [NS06] for free independence. It is worth noticing that the *product as argument formula* for free cumulants ([NS06, Thm. 11.12]) also holds for the case of Boolean cumulants [FMNS20, Prop. 2.12].

The same considerations can be taken into account in order to define the infinitesimal analogue of monotone cumulants.

Definition 6.1.12 (Infinitesimal monotone cumulant functionals)**.** Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space, let $\{h_n\colon \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ be the corresponding monotone cumulant functionals. The *infinitesimal monotone cumulants* are the family of multilinear functionals $\{h'_n\colon \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ recursively defined by

$$\varphi'_n(a_1, \dots, a_n) = \sum_{\pi\in\mathcal{M}(n)} \frac{1}{|\pi|!} \sum_{V\in\pi} h'_{|V|}(a_1, \dots, a_n|V) \prod_{\substack{W\in\pi\\ W\neq V}} h_{|W|}(a_1, \dots, a_n|W), \tag{6.1.8}$$

for any $n \geq 1$ and $a_1, \dots, a_n \in \mathcal{A}$.

Remark 6.1.13. It is important to mention that the notions of infinitesimal Boolean cumulants, infinitesimal Boolean independence, and infinitesimal monotone cumulants were previously defined by Hasebe in his work [Has11]. In this paper, the author introduced the notion of *differential cumulants* for the natural independence notions, as well as a higher-order generalization. Roughly speaking, Hasebe considered formal power series valued linear mappings $\phi_t : A \to \mathbb{C}[[t]]$ and defined the notion of differential independence according to the usual recipes for natural independence in the context of power series. In the present chapter and [CEFP21], we only considered the first-order differential cumulants, i.e. the infinitesimal cumulants, in order to show an application of the shuffle algebra framework for non-commutative probability.

In order to simplify the notation when writing the infinitesimal moment-cumulant relations, let us introduce the following notation following [Min19]. Consider two sequences of multilinear functionals $\{f_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ and $\{f'_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$. For any $n \geq 1$, $\pi \in \mathrm{NC}(n)$, and $a_1, \dots, a_n \in \mathcal{A}$, we denote

$$\partial f_\pi(a_1, \dots, a_n) := \sum_{V\in\pi} f'_{|V|}(a_1, \dots, a_n|V) \prod_{\substack{W\in\pi\\ W\neq V}} f_{|W|}(a_1, \dots, a_n|W). \tag{6.1.9}$$

In particular, according to the product rule in $\mathbb{G}$, if $\tilde{f}_n = f_n + \hbar f'_n$ for each $n \geq 1$, we have that

$$\tilde{f}_\pi(a_1, \dots, a_n) = f_\pi(a_1, \dots, a_n) + \hbar\, \partial f_\pi(a_1, \dots, a_n). \tag{6.1.10}$$

Therefore, the infinitesimal moment-cumulants relations (6.1.1), (6.1.3) and (6.1.8) simply write, for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$:

$$\varphi'(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \partial k_\pi(a_1, \ldots, a_n), \tag{6.1.11}$$

$$\varphi'(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{Int}(n)} \partial b_\pi(a_1, \ldots, a_n), \tag{6.1.12}$$

$$\varphi'(a_1 \cdots a_n) = \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} \partial h_\pi(a_1, \ldots, a_n). \tag{6.1.13}$$

6.2 Infinitesimal moment-cumulant relations

This section aims to develop the shuffle machinery to treat infinitesimal cumulants. We start by fixing an infinitesimal non-commutative probability space $(\mathcal{A}, \varphi, \varphi')$. We will consider the double tensor algebra extension of $(\mathcal{A}, \varphi)$, i.e. the Hopf algebra $T(T_+(\mathcal{A}))$ and the character $\Phi : T(T_+(\mathcal{A})) \to \mathbb{C}$ given by

$$\Phi(a_1 \cdots a_m) = \varphi(a_1 \cdots a_m),$$

for any word $a_1 \cdots a_m \in T_+(\mathcal{A})$ (recall that the argument in $\varphi(a_1 \cdots a_m)$ stands for an element in $\mathcal{A}$ originated by the product on the algebra $\mathcal{A}$ of the elements $a_1, \ldots, a_m \in \mathcal{A}$).

Motivated by the manner in which the infinitesimal cumulants arise, we would like to consider a $\mathbb{C}$-linear map $\tilde{\Phi} : T(T_+(\mathcal{A})) \to \mathbb{G}$ such that $\tilde{\Phi} = \Phi + \hbar\Phi'$, where $\Phi' : T(T_+(\mathcal{A})) \to \mathbb{C}$ is linear defined in terms of φ and φ'.

Recall that the Grassmann algebra $\mathbb{G}$ is commutative. In particular, by the discussion at the end of Section 4.2, the statements of the results in Section 4.2 are valid when considering a commutative algebra A instead of the ground field $\mathbb{C}$. Thus, denote by $G_{\mathbb{G}}$ the group (under the convolution product $*$) of characters in $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{G})$.

Now, consider a character $\tilde{\Psi} = \Psi + \hbar\Psi' \in G_{\mathbb{G}}$. If $w_1, w_2 \in T(T_+(\mathcal{A}))$, then we have

$$\begin{aligned} \Psi(w_1|w_2) + \hbar\Psi'(w_1|w_2) &= \tilde{\Psi}(w_1|w_2) \\ &= \tilde{\Psi}(w_1)\tilde{\Psi}(w_2) \\ &= (\Psi(w_1) + \hbar\Psi'(w_1))\,(\Psi(w_2) + \hbar\Psi'(w_2)) \\ &= \Psi(w_1)\Psi(w_2) + \hbar\Big(\Psi(w_1)\Psi'(w_2) + \Psi'(w_1)\Psi(w_2)\Big). \end{aligned}$$

The previous computation drives the following definition.

Definition 6.2.1 ([CEFP21]). Let $\Psi' \in \mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{G})$. We say that Ψ' has the

Leibniz-type property if $\Psi'(\mathbf{1}) = 0$ and

$$\Psi'(w_1|w_2|\cdots|w_m) = \sum_{i=1}^{m} \Psi'(w_i) \prod_{j\neq i} \Psi(w_j), \tag{6.2.1}$$

for any $w_1, \ldots, w_m \in T(T_+(\mathcal{A}))$.

It is clear then that $\Psi \in G_{\mathbb{G}}$ if and only if $\Psi \in G$ and Ψ' has the Leibniz-type property. In particular, by considering Φ to be the lifting of φ and $\Phi' : T(T_+(\mathcal{A})) \to \mathbb{C}$ is the linear functional defined by the recipe

$$\Phi'(a_1 \cdots a_m) := \varphi'_m(a_1, \ldots, a_m), \quad \forall\, a_1, \ldots, a_m \in \mathcal{A}, m \geq 1$$

and satisfying the Leibniz-type property, we arrive to a character $\tilde{\Phi} = \Phi + \hbar\Phi' \in G_{\mathbb{G}}$, that we will call the $\mathbb{G}$*-valued lifting of* φ *and* φ'.

We recall again the fact that the constructions in Section 4.2 work when we replace $\mathbb{C}$ by $\mathbb{G}$. In particular, we have the following products for $\tilde{f}, \tilde{g} \in \mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{G})$:

$$\begin{aligned} \tilde{f} * \tilde{g} &= m_{\mathbb{G}} \circ (\tilde{f} \otimes \tilde{g}) \circ \Delta, \\ \tilde{f} \prec \tilde{g} &= m_{\mathbb{G}} \circ (\tilde{f} \otimes \tilde{g}) \circ \Delta_{\prec}, \\ \tilde{f} \succ \tilde{g} &= m_{\mathbb{G}} \circ (\tilde{f} \otimes \tilde{g}) \circ \Delta_{\succ}, \end{aligned}$$

where $m_{\mathbb{G}}$ stands for the commutative product in $\mathbb{G}$ and $\Delta_{\prec}, \Delta_{\succ}$ are the half-unshuffle coproducts on $T(T_+(\mathcal{A}))$. The products $\prec, \succ$ endow $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{G})$ a unital shuffle algebra structure. In particular, the exponential $\exp^*$ and half-shuffle exponentials $\mathcal{E}_{\prec}$, $\mathcal{E}_{\succ}$ are also defined in the $\mathbb{G}$-valued context.

Remark 6.2.2. Notice that we are using the same notation for the half-shuffle products on the shuffle algebras $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{C})$ and $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{G})$. However, it will be clear from the context which half-shuffle products we are referring to.

Remark 6.2.3. Let $\tilde{f}, \tilde{g} \in \mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{G})$. If we denote $f, f', g, g' \in \mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{C})$ the unique linear functionals such that $\tilde{f} = f + \hbar f'$ and $\tilde{g} = g + \hbar g'$, then it is easy to see that

$$\tilde{f} * \tilde{g} = f * g + \hbar(f * g' + f' * g),$$

where, in the right-hand side, $*$ stands for the convolution product on $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{C})$. Furthermore, if we assume that $\tilde{f}$ is invertible with respect to $*$, then we can compute

$$\tilde{f}^{*-1} = f^{*-1} - \hbar(f^{*-1} * f' * f^{*-1}),$$

where f^{*-1} stands for the inverse of f with respect to the convolution product on $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{C})$.

Due to its importance in this section, we state the $\mathbb{G}$-analogue of Theorem 4.2.6. For this, recall that $G_{\mathbb{G}}$ is the group of characters on $\mathrm{Lin}(T(T_+(\mathcal{A})),\mathbb{G})$. In addition, denote by $\mathfrak{g}_{\mathbb{G}}$ the Lie algebra of infinitesimal characters on $\mathrm{Lin}(T(T_+(\mathcal{A})),\mathbb{G})$.

Proposition 6.2.4 ([CEFP21, Prop. 4.1])**.** *For $\tilde{\Phi} \in G_{\mathbb{G}}$ a character, there exists a unique triple of infinitesimal characters $(\tilde{\kappa},\tilde{\beta},\tilde{\rho}) \in \mathfrak{g}_{\mathbb{G}}^3$ such that*

$$\tilde{\Phi} = \exp^*(\tilde{\rho}) = \mathcal{E}_{\prec}(\tilde{\kappa}) = \mathcal{E}_{\succ}(\tilde{\beta}). \tag{6.2.2}$$

The infinitesimal characters $\tilde{\kappa}$ and $\tilde{\beta}$ are the unique solutions of the fixed point equations

$$\tilde{\Phi} = \tilde{\epsilon} + \tilde{\kappa} \prec \tilde{\Phi} \quad \text{and} \quad \tilde{\Phi} = \tilde{\epsilon} + \tilde{\Phi} \succ \tilde{\beta}, \tag{6.2.3}$$

respectively, where $\tilde{\epsilon} \in G_{\mathbb{G}}$ is such that $\tilde{\epsilon}(w) = 0$ for any $T_+(T_+(\mathcal{A}))$. Conversely, given $\tilde{\alpha} \in \mathfrak{g}_{\mathbb{G}}$ then $\exp^*(\tilde{\alpha}), \mathcal{E}_{\prec}(\tilde{\alpha}), \mathcal{E}_{\succ}(\tilde{\alpha}) \in G_{\mathbb{G}}$.

We can observe that the fixed point equations described in the above theorem can be equivalently written as pairs of fixed point equations on the $\mathbb{C}$-valued case. More precisely, (6.2.3) are equivalent to

$$\Phi = \epsilon + \kappa \prec \Phi, \qquad \Phi' = \kappa' \prec \Phi + \kappa \prec \Phi' \tag{6.2.4}$$

$$\Phi = \epsilon + \Phi \succ \beta, \qquad \Phi' = \Phi \succ \beta' + \Phi' \succ \beta, \tag{6.2.5}$$

where $\kappa,\beta,\kappa',\beta' \in \mathrm{Lin}(T(T_+(\mathcal{A})),\mathbb{C})$ are such that $\tilde{\kappa} = \kappa + \hbar\kappa'$ and $\tilde{\beta} = \beta + \hbar\tilde{\beta}$. Indeed, it is easy to check that if $\tilde{\kappa}$ is an infinitesimal character, then $\kappa,\kappa' \in \mathfrak{g}$. Moreover, notice that

$$\begin{aligned}
\Phi + \hbar\Phi' = \quad \tilde{\Phi} &= \tilde{\epsilon} + \tilde{\kappa} \prec \tilde{\Phi} \\
&= \epsilon + (\kappa + \hbar\kappa') \prec (\Phi + \hbar\Phi') \\
&= (\epsilon + \kappa \prec \Phi) + \hbar(\kappa \prec \Phi' + \kappa' \prec \Phi).
\end{aligned}$$

Comparing both components, we obtain (6.2.4). In particular, the fixed point solution of (6.2.3) is of the form $\tilde{\kappa} = \kappa + \hbar\kappa'$, where κ stands for the infinitesimal lifting of the free cumulants. We will then show that κ' is precisely the infinitesimal lifting of the infinitesimal free cumulants and the analogue results for the Boolean and monotone cases. This corresponds to the $\mathbb{G}$-valued analogue of Theorem 4.3.2

Remark 6.2.5. The reader should not be confused by the use of the adjective “infinitesimal” in this chapter since it appears in two different contexts: on the one hand, we have infinitesimal characters that are a certain type of linear forms on a double tensor algebra;

on the other hand, we have infinitesimal cumulants that are families of multilinear functionals on a non-commutative probability space with an additional functional φ'. Thus, it makes sense, for instance, to talk about the infinitesimal character of the infinitesimal free cumulants on $(\mathcal{A}, \varphi, \varphi')$.

Proposition 6.2.6 ([CEFP21, Prop. 4.2, 4.5]). *Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space and consider $\tilde{\Phi} \in G_{\mathbb{G}}$ the $\mathbb{G}$-valued lifting of φ and φ'. Let $(\tilde{\kappa}, \tilde{\beta}, \tilde{\rho})$ be the triple of infinitesimal characters in $\mathfrak{g}_{\mathbb{G}}$ given in Proposition 6.2.4, and write*

$$\tilde{\kappa} = \kappa + \hbar\kappa', \quad \tilde{\beta} = \beta + \hbar\beta', \quad \tilde{\rho} = \rho + \hbar\rho'.$$

Then κ', β' and ρ' are the infinitesimal liftings of the infinitesimal free, infinitesimal Boolean and infinitesimal monotone cumulant functionals of $(\mathcal{A}, \varphi, \varphi')$, respectively:

$$\kappa'(w) = k'_n(a_1, \dots, a_n), \quad \beta'(w) = b'_n(a_1, \dots, a_n), \quad \rho'(w) = h'_n(a_1, \dots, a_n),$$

for any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$.

Proof. Following the philosophy of [FN10], the results about infinitesimal cumulants can be obtained by proving the analogue result in $\mathbb{G}$-valued case instead of the $\mathbb{C}$-valued case and concluding by taking the $\hbar$-coefficient. Using the fact that the results of Section 4.2 are also valid in the $\mathbb{G}$-valued case, we can adapt the proof of Theorem 4.3.2 to obtain the proof for the infinitesimal case. Let us elaborate on the Boolean case since the free case can be obtained from the proof that we studied in Theorem 4.3.2.

The proof is by induction on n. The base case $n = 1$ trivially follows. For the inductive step, assume that the result is true for words of length smaller than n. From the definition of the right half-unshuffle coproduct $\Delta_{\succ}$ and (6.2.5), we obtain, for any word $w = a_1 \cdots a_n \in T_+(\mathcal{A})$:

$$\begin{aligned} \varphi'_n(a_1, \dots, a_n) &= \Phi'(w) \\ &= \Phi \succ \beta'(w) + \Phi' \succ \beta(w) \\ &= \sum_{1 \notin S \subseteq [n]} \Phi(a_S)\beta'(a_J^S) + \sum_{1 \notin S \subseteq [n]} \Phi'(a_S)\beta(a_J^S) \\ &= \sum_{m=1}^{n} \beta'(a_1 \cdots a_m)\Phi(a_{m+1} \cdots a_n) + \sum_{m=1}^{n-1} \beta(a_1 \cdots a_m)\Phi'(a_{m+1} \cdots a_n), \end{aligned}$$

where in the last equation we used that both β and β' are infinitesimal characters, and the fact that $\Phi'(\mathbf{1}) = 0$. Recalling that β identifies with the family of Boolean cumulants, we can use the Boolean moment-cumulant relation and the inductive hypothesis to obtain

$$\begin{aligned}\sum_{m=1}^{n}\beta'(a_1\cdots a_m)\Phi(a_{m+1}\cdots a_n) &= \beta'(a_1\cdots a_n)\\ &\quad+\sum_{m=1}^{n-1}b'_m(a_1,\dots,a_m)\sum_{\pi\in\mathrm{Int}(\{m+1,\dots,n\})}\prod_{V\in\pi}b_{|V|}(a_{m+1},\dots,a_n|V)\\ &= \beta'(a_1\cdots a_n)+\sum_{\substack{\pi\in\mathrm{Int}(n)\\ \pi\neq 1_n}}b'_{|V_1|}(a_1,\dots,a_n|V_1)\prod_{\substack{W\in\pi\\ W\neq V_1}}b_{|W|}(a_1,\dots,a_n|W),\end{aligned}$$

where V_1 denotes the block in π containing 1. On the other hand, we can use the inductive hypothesis to write

$$\begin{aligned}\sum_{m=1}^{n-1}\beta(a_1\cdots a_m)\Phi'(a_{m+1}\cdots a_n) &= \sum_{m=1}^{n-1}\beta(a_1\cdots a_m)\sum_{\pi\in\mathrm{Int}(\{m+1,\dots,n\})}\sum_{V\in\pi}\beta'(a_V)\prod_{\substack{W\in\pi\\ W\neq V}}\beta(a_W)\\ &= \sum_{\pi\in\mathrm{Int}(n)}\sum_{\substack{V\in\pi\\ V\neq V_1}}b'_{|V|}(a_1,\dots,a_n|V)\prod_{\substack{W\in\pi\\ W\neq V}}b_{|W|}(a_1,\dots,a_n|W).\end{aligned}$$

Combining the two sums above, we obtain

$$\varphi'_n(a_1,\dots,a_n)=\beta'(a_1\cdots a_n)+\sum_{\substack{\pi\in\mathrm{Int}(n)\\ \pi\neq 1_n}}\partial b_\pi(a_1,\dots,a_n). \tag{6.2.6}$$

Finally, by comparing with (6.1.12), we are able to deduce that $\beta'(w)=b'_n(a_1,\dots,a_n)$. By induction, we conclude that β' is the infinitesimal lifting of the infinitesimal Boolean cumulants $\{b'_n\}_{n\geq 1}$.

The next step is to prove the monotone case. Consider any word $w=a_1\cdots a_n\in T_+(\mathcal{A})$. By induction on n. Once more, the base case $n=1$ trivially follows. Now assume that the result is valid for any words of length smaller than a certain n. Using the definition of the convolution exponential map in the $\mathbb{G}$-valued case, we have

$$\begin{aligned}\tilde{\Phi}=\exp^*(\rho+\hbar\rho') &= \sum_{k=0}^{\infty}\frac{(\rho+\hbar\rho')^{*k}}{k!}\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\rho^{*k}+\hbar\left(\sum_{m=1}^{k}\rho^{*(m-1)}*\rho'*\rho^{*(k-m)}\right)\right).\end{aligned} \tag{6.2.7}$$

We will compute the evaluation on a word $w=a_1\cdots a_n\in T_+(\mathcal{A})$ of the right-hand side in the above equation. Using the strategy in [EFP18] for the $\mathbb{C}$-valued case, we can use the fact that ρ and ρ' are infinitesimal characters (i.e. $\kappa(w_1|w_2)=0$ for any non-empty elements $w_1,w_2\in T_+(T_+(\mathcal{A}))$) to deduce that the convolution product between them only requires the reduced linearized part of the coproduct, which is given by

$$\overline{\Delta}_{\mathrm{irr}}(a_1 \cdots a_n) := \sum_{\substack{a_{I_1} a_{I_2} a_{I_3} = w \\ I_1 \sqcup I_3, I_2 \neq \emptyset}} a_{I_1} a_{I_3} \otimes a_{I_2}. \tag{6.2.8}$$

Then we have, for instance, that

$$\rho * \rho'(w) = m_{\mathbb{C}} \circ (\rho \otimes \rho') \circ \Delta(w) = m_{\mathbb{C}} \circ (\rho \otimes \rho') \circ \overline{\Delta}_{\mathrm{irr}}(w).$$

From Lemma 3 of [EFP18], if $\overline{\Delta}_{\mathrm{irr}}^{[q]} : T_+(A) \to T_+(A)^{\otimes q}$ stands for the q-fold left iterated reduced linearized coproduct, then

$$\overline{\Delta}_{\mathrm{irr}}^{[q]}(a_1 \cdots a_n) = \sum_{\substack{\pi \in \mathcal{M}^q(n) \\ \pi = (V_1, \ldots, V_q)}} a_{V_1} \otimes \cdots \otimes a_{V_q}, \tag{6.2.9}$$

where $\mathcal{M}^q(n)$ denotes the set of monotone partitions of $[n]$ into q blocks.

With the above equation, we can now compute the $\hbar$-coefficient of the expression of $\tilde{\Phi}$ obtained on the right-hand side of (6.2.7). Thus, given a $k \geq 1$ and $1 \leq m \leq k$ we get

$$\begin{aligned} \frac{1}{k!} \left(\rho^{*(m-1)} * \rho' * \rho^{k-m} \right) (a_1 \cdots a_n) &= \frac{1}{k!} m_{\mathbb{C}}^{[k]} \circ (\rho \otimes \cdots \otimes \rho' \otimes \cdots \otimes \rho) \circ \overline{\Delta}_{\mathrm{irr}}^{[k]}(a_1 \cdots a_n) \\ &= \frac{1}{k!} \sum_{\substack{\pi \in \mathcal{M}^k(n) \\ \pi = (V_1, \ldots, V_k)}} \rho(a_{V_1}) \cdots \rho'(a_{V_m}) \cdots \rho(a_{V_k}). \end{aligned}$$

Observe that the index m indicates the block on which the infinitesimal character ρ' will be evaluated. In this way, varying over m is equivalent to varying on a specific block of the monotone partition. We can represent the sum over m and k in terms of non-crossing partitions and the notation used for infinitesimal cumulants. In other words, by adding over $1 \leq m \leq k$ and $1 \leq k \leq n$, we obtain

$$\begin{aligned} \Phi'(w) &= \sum_{k=1}^{n} \frac{1}{k!} \sum_{m=1}^{k} \rho^{*(m-1)} * \rho' * \rho^{k-m}(a_1 \cdots a_n) \\ &= \sum_{k=1}^{n} \frac{1}{k!} \sum_{\pi \in \mathcal{M}^k(n)} \sum_{V \in \pi} \rho'(a_V) \prod_{\substack{W \in \pi \\ W \neq V}} \rho(a_W), \\ &= \sum_{k=1}^{n} \frac{1}{k!} \sum_{\pi \in \mathrm{NC}^k(n)} m(\pi) \sum_{V \in \pi} \rho'(a_V) \prod_{\substack{W \in \pi \\ W \neq V}} \rho(a_W), \end{aligned}$$

where we recall that $\mathrm{NC}^k(n)$ stands for the set of non-crossing partitions of $[n]$ with k blocks and $m(\pi)$ denotes the number of monotone labellings of π. Also recalling that

$m(\pi) = |\pi|!/t(\pi)!$, we get that

$$\varphi_n'(a_1, \cdots, a_n) = \sum_{\pi \in \mathrm{NC}(n)} \frac{1}{t(\pi)!} \sum_{V \in \pi} \rho'(a_V) \prod_{\substack{W \in \pi \\ W \neq V}} \rho(a_W), \tag{6.2.10}$$

Finally, separating the partition $\pi = 1_n$ from the above sum, using the induction hypothesis and the fact that ρ is the infinitesimal lifting of the monotone cumulants, we obtain

$$\begin{aligned} \varphi_n'(a_1, \ldots, a_n) &= \rho'(a_1 \cdots a_n) \\ &\quad + \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi \neq 1_n}} \frac{1}{t(\pi)!} \sum_{V \in \pi} h_{|V|}'(a_1, \ldots, a_n | V) \prod_{\substack{W \in \pi \\ W \neq V}} h_{|W|}(a_1, \ldots, a_n | W) \\ &= \rho'(a_1 \cdots a_n) + \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi \neq 1_n}} \frac{1}{t(\pi)!} \partial h_\pi(a_1, \ldots, a_n). \end{aligned}$$

We conclude the proof of the theorem by comparing the above equation with the infinitesimal moment-monotone cumulant relation (6.1.13) and obtain $\rho'(a_1 \cdots a_n) = h_n'(a_1, \ldots, a_n)$, as we wanted to show. $\square$

The previous theorem shows how fixed point equations (6.2.4) and (6.2.5) effectively describe the free and Boolean cumulants in the infinitesimal extension, as well as the exponential nature of the infinitesimal monotone cumulants. We are now interested in showing an alternative description of the infinitesimal cumulants in terms of the convolution product.

First, let us recall the $\mathbb{G}$-valued version of Theorem 4.2.14: if $\alpha_1, \alpha_2 \in \mathfrak{g}_{\mathbb{G}}$, then we have that

$$\mathcal{E}_\prec(\alpha_1) * \mathcal{E}_\prec(\theta_{\alpha_1}(\alpha_2)) = \mathcal{E}_\prec(\alpha_1 + \alpha_2).$$

Now, given a character $\tilde{\Phi} \in G_{\mathbb{G}}$, take the infinitesimal character $\tilde{\kappa} \in \mathfrak{g}_{\mathbb{G}}$ such that $\tilde{\Phi} = \mathcal{E}_\prec(\tilde{\kappa})$. Also, consider the infinitesimal characters $\kappa, \kappa' \in \mathfrak{g}$ such that $\tilde{\kappa} = \kappa + \hbar\kappa'$. Since $\mathbb{C} \subset \mathbb{G}$, it is clear that we can consider κ and $\hbar\kappa'$ as elements in $\mathfrak{g}_{\mathbb{G}}$. Hence, if $\tilde{\Phi} = \Phi + \hbar\Phi'$, we have that

$$\begin{aligned} \tilde{\Phi} = \mathcal{E}_\prec(\tilde{\kappa}) &= \mathcal{E}_\prec(\kappa + \hbar\kappa') \\ &= \mathcal{E}_\prec(\kappa) * \mathcal{E}_\prec\left(\theta_\kappa(\hbar\kappa')\right) \\ &= \Phi * \mathcal{E}_\prec(\hbar\theta_\kappa(\kappa')) \\ &= \Phi * (\epsilon + \hbar\theta_\kappa(\kappa')) \\ &= \Phi + \hbar\, \Phi * \theta_\kappa(\kappa'). \end{aligned}$$

Here we used that $\Phi = \mathcal{E}_\prec(\kappa)$ and the fact that $\hbar^2 = 0$ in

$$\mathcal{E}_\prec(\hbar\theta_\kappa(\kappa')) = \epsilon + \hbar\theta_\kappa(\kappa') + O(\hbar^2).$$

Comparing the respective $\hbar$-coefficient, we conclude that

$$\Phi' = \Phi * \theta_\kappa(\kappa'). \tag{6.2.11}$$

Recall that Φ' is the solution of the fixed point equation $\Phi' = \kappa' \prec \Phi + \kappa \prec \Phi'$, which provides the infinitesimal free moment-cumulant relations. In particular, since $\theta_\kappa(\kappa') \in \mathfrak{g}$ (Proposition 4.2.13) and $\Phi \in G$ is invertible, (6.2.11) implies that $\Phi^{*-1} * \Phi' \in \mathfrak{g}$. Finally, we also have the following identity:

$$\kappa' = \Phi \succ (\Phi^{*-1} * \Phi') \prec \Phi^{*-1}.$$

Indeed, observe that $\theta_\kappa(\kappa') = \Phi^{*-1} * \Phi'$. Also, denote $\beta = \theta_\kappa(\kappa)$. By Remark 4.2.12, we have

$$\theta_{-\beta}(\theta_\kappa(\kappa')) = \theta_{\mathcal{L}_\prec(\mathcal{E}_\prec(\kappa)*\mathcal{E}_\prec(-\beta))}(\kappa').$$

By Theorem 4.2.14,

$$\mathcal{L}_\prec\big(\mathcal{E}_\prec(\kappa) * \mathcal{E}_\prec(-\beta)\big) = \mathcal{L}_\prec\big(\mathcal{E}_\prec(\kappa) * \mathcal{E}_\prec(\theta_\kappa(-\kappa))\big) = \mathcal{L}_\prec(\mathcal{E}_\prec(\kappa - \kappa)) = \mathcal{L}_\prec(\epsilon) = 0.$$

Thus

$$\theta_{-\beta}(\theta_\kappa(\kappa')) = \theta_0(\kappa') = \epsilon \succ \kappa' \prec \epsilon = \kappa',$$

and in this way

$$\begin{aligned}
\kappa' &= \theta_{-\beta}(\theta_\kappa(\kappa')) \\
&= \theta_{-\theta_\kappa(\kappa)}(\Phi^{*-1} * \Phi') \\
&= \theta^{\theta_\kappa(\kappa)}(\Phi^{*-1} * \Phi') \\
&= \mathcal{E}_\succ(\theta_\kappa(\kappa)) \succ (\Phi^{*-1} * \Phi') \prec \mathcal{E}_\succ^{*-1}(\theta_\kappa(\kappa)) \\
&= \Phi \succ (\Phi^{*-1} * \Phi') \prec \Phi^{*-1},
\end{aligned}$$

where it was used that $\mathcal{L}_\succ(\Phi) = \theta_\kappa(\kappa)$, i.e. the Boolean cumulant linear form of Φ is given by the action $\theta_\kappa(\kappa)$, where $\kappa = \mathcal{L}_\prec(\Phi)$ is the free cumulant linear form of Φ.

The analogue game can be applied to the Boolean cumulants by using the $\mathbb{G}$-valued identity for the right half-shuffle exponential in Theorem 4.2.14. In this case, we obtain

$$\begin{aligned}
\tilde{\Phi} = \mathcal{E}_\succ(\tilde{\beta}) &= \mathcal{E}_\succ(\beta + \hbar\beta') \\
&= \mathcal{E}_\succ(\theta_{-\beta}(\hbar\beta')) * \mathcal{E}_\succ(\beta)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{E}_{\succ}(\hbar\theta_{-\beta}(\beta')) * \Phi \\
&= (\epsilon + \hbar\theta_{-\beta}(\beta')) * \Phi \\
&= \Phi + \hbar\,\theta_{-\beta}(\beta') * \Phi.
\end{aligned}
$$

Therefore, comparing the $\hbar$-coefficients:

$$\Phi' = \theta_{-\beta}(\beta') * \Phi. \tag{6.2.12}$$

In an analogue way, the right-hand side of (6.2.12) is the solution of the fixed point equation (6.2.5) which provides the infinitesimal Boolean moment-cumulant relation.

The monotone case is slightly different to handle. For $\tilde{\Phi} \in G_{\mathbb{G}}$, take $\tilde{\rho} \in \mathfrak{g}_{\mathbb{G}}$ such that $\tilde{\Phi} = \exp^*(\tilde{\rho})$. Also, consider $\rho, \rho' \in \mathfrak{g}$ such that $\tilde{\rho} = \rho + \hbar\rho'$. Once again, we can consider ρ and ρ' as elements in $\mathfrak{g}_{\mathbb{G}}$ and write

$$\tilde{\Phi} = \exp^*(\rho + \hbar\rho').$$

We can take advantage of the dual of the Baker-Campbell-Hausdorff formula, known as the *Zassenhaus formula* ([Reu93, Chap. 4]), in order to write

$$\exp^*(\rho + \hbar\rho') = \exp^*(\rho) * \exp^*(F(\rho, \hbar\rho')),$$

where in general, the map $F(\alpha, \gamma)$ is defined by the formula

$$F(\alpha, \gamma) := \gamma + \sum_{n>0} \frac{(-1)^n}{(n+1)!} \mathrm{ad}_\alpha^{(n)}(\gamma).$$

where the $\mathrm{ad}^{(n)}$ stands for the n-th iterated commutator, i.e. $\mathrm{ad}_\alpha^{(0)}(\gamma) = \gamma$ and

$$\mathrm{ad}_\alpha^{(n)}(\gamma) = [\alpha, \mathrm{ad}_\alpha^{(n-1)}(\gamma)] = \alpha * \mathrm{ad}_\alpha^{(n-1)}(\gamma) - \mathrm{ad}_\alpha^{(n-1)}(\gamma) * \alpha,$$

for $n \geq 1$. Next, by setting $\alpha = \rho$ and $\gamma = \hbar\rho'$, we can write

$$F(\rho, \hbar\rho') = \frac{e^{-\mathrm{ad}_\rho} - 1}{-\mathrm{ad}_\rho}(\hbar\rho') =: W_{-\rho}(\hbar\rho').$$

using an analogue notation to (2.3.21). Using the definition of $\mathrm{ad}^{(n)}$ and the fact that $\hbar^2 = 0$, we get that

$$\exp^*(\rho) * \exp^*(F(\rho, \hbar\rho')) = \exp^*(\rho) * (\epsilon + \hbar W_{-\rho}(\rho')). \tag{6.2.13}$$

Recall that $\mathfrak{g}$ is a Lie algebra. Since $W_{-\rho}(\rho')$ is defined by using sum and commutators, one can check that $W_{-\rho}(\rho') \in \mathfrak{g}$. If we write $\exp^*(\tilde{\rho}) = \tilde{\Phi} = \Phi + \hbar\Phi'$ and compare the

$\hbar$-coefficient with (6.2.13), we conclude

$$\Phi' = \Phi * W_{-\rho}(\rho'), \tag{6.2.14}$$

or equivalently $W_{-\rho}(\rho') = \Phi^{*-1} * \Phi'$. Comparing with (6.2.11), we obtain that

$$W_{-\rho}(\rho') = \theta_\kappa(\kappa'),$$

we obtain the infinitesimal analogue of Theorem 4.2.9, the relation $-W(-\rho) = \kappa$ between monotone and free cumulants. The corresponding relation between the infinitesimal Boolean and infinitesimal monotone cumulants can be obtained similarly by considering the decomposition

$$\exp^*(\hbar\rho' + \rho) = \exp^*(\hbar\rho') * \exp^*(F(\hbar\rho', \rho)).$$

As a way to conclude this section, we collect the obtained equations (6.2.11), (6.2.12) and (6.2.14) in the following statement.

Theorem 6.2.7 ([CEFP21, Prop. 4.7]). *Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space, and let $\tilde{\Phi} = \Phi + \hbar\Phi'$ be the corresponding $\mathbb{G}$-valued lifting of φ and φ' to a $\mathbb{G}$-valued character on $T(T_+(\mathcal{A}))$. Consider the pairs of infinitesimal characters $(\kappa, \kappa'), (\beta, \beta')$ and (ρ, ρ') described in Proposition 6.2.6. Then we have*

$$\begin{aligned}\Phi' &= \Phi * \theta_\kappa(\kappa') \\ &= \theta_{-\beta}(\beta') * \Phi \\ &= \Phi * W_{-\rho}(\rho').\end{aligned}$$

6.3 Relations between infinitesimal cumulants

A natural question about the different brands of infinitesimal cumulants is what the infinitesimal analogue of Theorem 3.4.25, i.e. the existence of combinatorial formulas, in terms of irreducible non-crossing partitions, that allow us writing a brand of cumulants in terms of the other brands. As we studied in Chapter 4, the formulas in Theorem 3.4.25 are encoded in the shuffle equations from Corollary 4.3.4 and Remark 4.3.5. Following the strategy of the previous section, the same shuffle equations encode the combinatorial formulas in the $\mathbb{G}$-valued case.

The statement of the relations between infinitesimal cumulants is the following.

Theorem 6.3.1 ([CEFP21, Thm. 1.1]). *Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal non-commutative probability space, and let $\{k_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{h_n\}_{n\geq 1}$, $\{k'_n\}_{n\geq 1}$, $\{b'_n\}_{n\geq 1}$ and $\{h'_n\}_{n\geq 1}$ be the families of free cumulants, Boolean cumulants, monotone cumulants, infinitesimal* *free cumulants, infinitesimal Boolean cumulants, and infinitesimal monotone cumulants, respectively. Then, for any $n \geq 1$ and elements $a_1, \ldots, a_n \in \mathcal{A}$ we have*

$$b'_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \partial k_\pi(a_1, \ldots, a_n), \tag{6.3.1}$$

$$k'_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1} \partial b_\pi(a_1, \ldots, a_n), \tag{6.3.2}$$

$$b'_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!} \partial h_\pi(a_1, \ldots, a_n), \tag{6.3.3}$$

$$k'_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{(-1)^{|\pi|-1}}{t(\pi)!} \partial h_\pi(a_1, \ldots, a_n). \tag{6.3.4}$$

Proof. Let $n \geq 1$ and $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. The relations in and (6.3.1) and (6.3.2) follow from the $\mathbb{G}$-valued analogue of Proposition 4.2.8 and Lemma 4.3.3, and taking the $\hbar$-coefficient of the products $\tilde{\kappa}$ and $\tilde{\beta}$ in $\mathbb{G}$ (6.1.10).

Let us elaborate on the proof of (6.3.3) and (6.3.4). Let $\tilde{\Phi}$ be the $\mathbb{G}$-valued lifting of φ and φ', and consider the infinitesimal characters $\tilde{\kappa}, \tilde{\beta}$ and $\tilde{\rho}$ such that $\tilde{\Phi} = \exp^*(\tilde{\rho}) = \mathcal{E}_{\prec}(\tilde{\kappa}) = \mathcal{E}_{\succ}(\tilde{\beta})$. By Proposition 6.2.6, we know that $\tilde{\kappa}$ is the $\mathbb{G}$-valued infinitesimal lifting of the free and infinitesimal free cumulants, and analogously for the Boolean and monotone cumulants.

Now, recall that $\tilde{\beta}$ satisfies the fixed point equation $\tilde{\Phi} = \tilde{\epsilon} + \tilde{\Phi} \succ \beta$. This implies that $\exp^*(\tilde{\rho}) - \tilde{\epsilon} = \exp^*(\tilde{\rho}) \succ \tilde{\beta}$. Furthermore, we also have the $\mathbb{G}$-valued analogue of Theorem 4.3.2:

$$\tilde{\Phi}(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \frac{1}{t(\pi)!} \tilde{h}_\pi(a_1, \ldots, a_n).$$

We will use the previous equations to prove the $\mathbb{G}$-valued analogue of (3.4.16)

$$\tilde{b}_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!} \tilde{h}_\pi(a_1, \ldots, a_n) \tag{6.3.5}$$

by induction on n. The case $n = 1$ follows from $\tilde{h}_1(a_1) = \tilde{\varphi}(a_1) = \tilde{b}_1(a_1)$. For the induction hypothesis, assume that the formula holds for words of length smaller than n. Then, using the definition of $\Delta_{\succ}$ and the fact that $\tilde{\beta}$ is an infinitesimal character, we have that

$$\begin{aligned}
\sum_{\pi \in \mathrm{NC}(n)} \frac{1}{t(\pi)!} \tilde{h}_\pi(a_1, \ldots, a_n) &= (\exp^*(\tilde{\rho}) - \tilde{\epsilon})\,(a_1 \cdots a_n) \\
&= \left(\exp^*(\tilde{\rho}) \succ \tilde{\beta}\right)(a_1 \cdots a_n) \\
&= \tilde{\beta}(a_1 \cdots a_n) + \sum_{j=1}^{n-1} \tilde{\beta}(a_1 \cdots a_j) \exp^*(\tilde{\rho})(a_{j+1} \cdots a_n)
\end{aligned}$$

$$\begin{aligned}
&= \tilde{b}_n(a_1,\dots,a_n) + \sum_{j=1}^{n-1}\left(\sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(j)} \frac{1}{t(\pi)!}\tilde{h}_\pi(a_1,\dots,a_j)\right) \\
&\qquad \times \left(\sum_{\sigma\in \mathrm{NC}(n-j)} \frac{1}{t(\sigma)!}\tilde{h}_\sigma(a_{j+1},\dots,a_n)\right) \\
&= \tilde{b}_n(a_1,\dots,a_n) + \sum_{\pi\in \mathrm{NC}(n)\setminus \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!}\tilde{h}_\pi(a_1,\dots,a_n).
\end{aligned}$$

Observe that we used the induction hypothesis in the fourth equality. We also used the fact that any non-irreducible non-crossing partition π can be seen as the concatenation of an irreducible non-crossing partition π_1, that is, the irreducible component containing 1, and the non-crossing partition π_2 consisting of the remaining irreducible components. Notice that this decomposition is unique and satisfies that $\frac{1}{t(\pi_1)!}\frac{1}{t(\pi_2)!} = \frac{1}{t(\pi)!}$. We conclude the inductive step by subtracting the sum on the right-hand side of the above equation, and finally, we obtain (6.3.3) by taking the $\hbar$-coefficient.

We proceed with the proof of (6.3.4) analogously. The $\mathbb{G}$-valued version of (4.3.4) establishes that

$$\tilde{\Phi}^{*-1} - \tilde{\epsilon} = -\tilde{\Phi}^{*-1} \succ \tilde{\kappa}.$$

We will prove then that

$$\tilde{k}_n(a_1,\dots,a_n) = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{(-1)^{|\pi|-1}}{t(\pi)!}\tilde{h}_\pi(a_1,\dots,a_n). \tag{6.3.6}$$

By induction on n. The case $n=1$ easily follows from $\tilde{k}_1(a_1) = \tilde{\varphi}(a_1) = \tilde{h}_1(a_1)$. Now, assume that the above formula holds for words of length smaller than n. First, note that

$$\begin{aligned}
(\tilde{\Phi}^{*-1} - \tilde{\epsilon})(a_1\cdots a_n) &= \exp^*(\tilde{\rho})^{*-1}(a_1\cdots a_n) \\
&= \exp^*(-\tilde{\rho})(a_1\cdots a_n) \\
&= \sum_{\pi\in \mathrm{NC}(n)} \frac{1}{t(\pi)!}\prod_{V\in\pi}(-\rho)(a_V) \\
&= \sum_{\pi\in \mathrm{NC}(n)} \frac{(-1)^{|\pi|}}{t(\pi)!} h_\pi(a_1,\dots,a_n).
\end{aligned}$$

On the other hand, by using the fact that $\tilde{\kappa}$ is an infinitesimal character, we have that

$$\begin{aligned}
\sum_{\pi\in \mathrm{NC}(n)} \frac{(-1)^{|\pi|}}{t(\pi)!} h_\pi(a_1,\dots,a_n) &= (\tilde{\Phi}^{*-1} - \tilde{\epsilon})(a_1\cdots a_n) \\
&= -(\tilde{\Phi}^{*-1} \succ \tilde{\kappa})(a_1\cdots a_n) \\
&= -\tilde{\kappa}(a_1\cdots a_n) - \sum_{j=1}^{n-1} \tilde{\kappa}(a_1\cdots a_j)\exp^*(-\tilde{\rho})(a_{j+1}\cdots a_n)
\end{aligned}$$

$$\begin{aligned}
&= -\tilde{k}_n(a_1,\ldots,a_n) - \sum_{j=1}^{n-1}\left(\sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(j)} \frac{(-1)^{|\pi|-1}}{t(\pi)!}\tilde{h}_\pi(a_1,\ldots,a_j)\right) \\
&\qquad \times \left(\sum_{\sigma\in \mathrm{NC}(n-j)} \frac{(-1)^{|\sigma|}}{t(\sigma)!}\tilde{h}_\sigma(a_{j+1},\ldots,a_n)\right) \\
&= -\tilde{k}_n(a_1,\ldots,a_n) - \sum_{\pi\in \mathrm{NC}(n)\setminus \mathrm{NC}_{\mathrm{irr}}(n)} \frac{(-1)^{|\pi|-1}}{t(\pi)!}\tilde{h}_\pi(a_1,\ldots,a_n).
\end{aligned}$$

By the above computation and induction, we obtain that (6.3.6) is true. Therefore, we obtain (6.3.4) by taking the $\hbar$-coefficient of (6.3.6). □

Remark 6.3.2. By looking at the to-be-developed framework for the combinatorial relations between cumulants via pre-Lie Magnus expansion, we can obtain the formulas that write infinitesimal monotone cumulants in terms of infinitesimal free and infinitesimal Boolean cumulants via the $\mathbb{G}$-valued analogue of the pre-Lie Magnus expansion $\tilde{\rho} = \Omega(\tilde{\beta}) = -\Omega(-\tilde{\kappa})$:

$$\begin{aligned}
h'_n(a_1,\ldots,a_n) &= \sum_{\mathrm{NC}_{\mathrm{irr}}(n)} \omega(t(\pi))\partial b_\pi(a_1,\ldots,a_n), \\
h'_n(a_1,\ldots,a_n) &= \sum_{\mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1}\omega(t(\pi))\partial k_\pi(a_1,\ldots,a_n),
\end{aligned}$$

for any $n \geq 1$ and $a_1,\ldots,a_n \in \mathcal{A}$, where $\omega(t(\pi))$ is the Murua coefficient (Definition 7.3.2) of the nesting tree $t(\pi)$. The reader will find a study and proof of the monotone-to-free and monotone-to-Boolean formulas in Chapter 8 and Chapter 9.

6.4 Infinitesimal Boolean Bercovici-Pata bijection

We are now interested in understanding the infinitesimal analogue of the Bercovici-Pata bijection in non-commutative probability (Definition 3.5.18). It has been studied in the shuffle algebra framework in [EFP19] for (usual) free probability. We aim to extend the result by considering the Grassman algebra $\mathbb{G}$.

First, the concept of distribution generalizes the infinitesimal framework in the following way. Let $(\mathcal{A},\varphi,\varphi')$ be an infinitesimal non-commutative probability space, and consider a m-tuple of random variables $(a_1,\ldots,a_m) \in \mathcal{A}^m$. The *infinitesimal distribution of* $(a_1,\ldots,a_m)$ is the pair (μ,μ') of linear functionals $\mu,\mu' : \mathbb{C}\langle X_1,\ldots,X_m\rangle \to \mathbb{C}$ such that

$$\begin{aligned}
\mu(X_{i_1}\cdots X_{i_s}) &= \varphi(a_{i_1}\cdots a_{i_s}), \\
\mu'(X_{i_1}\cdots X_{i_s}) &= \varphi'(a_{i_1}\cdots a_{i_s}),
\end{aligned}$$

for any $i_1, \ldots, i_s \in [m]$ and $s \geq 1$. In particular μ is the distribution of $(a_1, \ldots, a_m)$ in the sense of Definition 3.2.3.

It is clear that if (μ, μ') is an infinitesimal distribution of a m-tuple of random variables, then $\mu(1) = 1$ and $\mu'(1) = 0$. In general, we denote

$$\tilde{\mathcal{D}}(m) := \{(\mu, \mu') \, : \, \mu, \mu' : \mathbb{C}\langle X_1, \ldots, X_m \rangle \to \mathbb{C} \text{ linear such that } \mu(1) = 1, \mu'(1) = 0\}.$$

In order to state the definition of the infinitesimal Bercovici-Pata map, first consider an infinitesimal distribution $(\mu, \mu') \in \tilde{\mathcal{D}}(m)$. Following the previous strategy, we can equivalently think the pair (μ, μ') as the $\mathbb{C}$-linear map $\tilde{\mu} := \mu + \hbar\mu' : \mathbb{C}\langle X_1, \ldots, X_m \rangle \to \mathbb{G}$. Recall that we can define the free cumulants of a distribution μ as the linear functional determined by the evaluation of the free cumulant functionals $\{k_n\}_{n\geq 1}$ on the m-tuple of random variables whose distribution is μ. Similarly, we define the *infinitesimal free cumulants of* (μ, μ') as the linear functionals determined by the evaluation of $\{k'_n\}_{n\geq 1}$ on the m-tuple of random variables. This procedure allows us to talk about the $\mathbb{G}$-valued free cumulants of the infinitesimal distribution $\tilde{\mu}$ as the $\mathbb{C}$-linear maps $\tilde{k}_n(\tilde{\mu}) = k_n(\tilde{\mu}) + \hbar k'_n(\tilde{\mu})$, for any $n \geq 1$. Defining analogously in the Boolean case $\tilde{b}_n(\tilde{\mu}) = b_n(\tilde{\mu}) + \hbar b'_n(\tilde{\mu})$, we arrive to the next definition.

Definition 6.4.1. Let $k \geq 1$. We define the *infinitesimal Boolean Bercovici-Pata map*, denoted by $\tilde{\mathbb{B}} : \tilde{\mathcal{D}}(m) \to \tilde{\mathcal{D}}(m)$, as the map $\tilde{\mu} \mapsto \tilde{\mathbb{B}}(\tilde{\mu})$ such that $\tilde{\mathbb{B}}(\tilde{\mu})$ is uniquely defined by the fact that

$$\tilde{k}_n\left(\tilde{\mathbb{B}}(\tilde{\mu})\right) = \tilde{b}_n(\tilde{\mu}), \quad \forall\, n \geq 1,$$

i.e. the $\mathbb{G}$-valued free cumulants of $\tilde{\mathbb{B}}(\tilde{\mu})$ are precisely the $\mathbb{G}$-valued Boolean cumulants of $\tilde{\mu}$.

Now recall the importance of the Boolean Bercovici-Pata map $\mathbb{B}$ in free probability: in the case of analytical distributions, $\mathbb{B}$ restricts to a bijection between the set of $\boxplus$-infinite divisible distributions and the set of $\uplus$-infinite divisible distributions. In our case of interest, we are not considering analytical distributions. Nevertheless, we can recover the infinitesimal analogue definition of convolutions in the algebraic sense by recalling that a family of cumulants uniquely determines the distribution of a tuple of random variables.

Definition 6.4.2. Let $\tilde{\mu}, \tilde{\nu} \in \tilde{\mathcal{D}}(m)$ be two infinitesimal distributions. The *infinitesimal free additive convolution of* $\tilde{\mu}$ *and* $\tilde{\nu}$ is the infinitesimal distribution $\tilde{\mu} \boxplus \tilde{\nu} \in \tilde{\mathcal{D}}(m)$ whose $\mathbb{G}$-valued free cumulants are given by

$$\tilde{k}_n(\tilde{\mu} \boxplus \tilde{\nu}) = \tilde{k}_n(\tilde{\mu}) + \tilde{k}_n(\tilde{\nu}), \quad \forall\, n \geq 1.$$

Analogously, the *infinitesimal Boolean additive convolution of* $\tilde{\mu}$ *and* $\tilde{\nu}$ is the infinitesimal distribution $\tilde{\mu} \uplus \tilde{\nu} \in \tilde{\mathcal{D}}(m)$ whose $\mathbb{G}$-valued Boolean cumulants are given by

$$\tilde{b}_n(\tilde{\mu} \uplus \tilde{\nu}) = \tilde{b}_n(\tilde{\mu}) + \tilde{b}_n(\tilde{\nu}), \quad \forall\, n \geq 1.$$

From the above definition, if $\tilde{\mu} \in \tilde{\mathcal{D}}(m)$, it is clear that $\tilde{k}_n(\tilde{\mu}^{\boxplus r}) = r\tilde{k}_n(\tilde{\mu})$, for any $r \in \mathbb{N}$. We can then define the *infinitesimal free power additive convolutions of* $\tilde{\mu}$, for any $s \geq 0$, as the infinitesimal distribution uniquely determined by

$$\tilde{k}_n\left(\tilde{\mu}^{\boxplus s}\right) = s\tilde{k}_n(\tilde{\mu}), \quad \forall\, n \geq 1.$$

Similarly, we define the *infinitesimal Boolean power additive convolutions of* $\tilde{\mu}$, for any $s \geq 0$, as the infinitesimal distribution uniquely determined by

$$\tilde{b}_n\left(\tilde{\mu}^{\boxplus s}\right) = s\tilde{b}_n(\tilde{\mu}), \quad \forall\, n \geq 1.$$

Let us now introduce the shuffle-algebraic point of view for the Boolean Bercovici-Pata map. Let $(\mu, \mu') \in \tilde{\mathbb{D}}(m)$ be an infinitesimal distribution of a m-tuple of random variables in an infinitesimal non-commutative probability space $(\mathcal{A}, \varphi, \varphi')$. For this space, consider the double tensor Hopf algebra $T(T_+(\mathcal{A}))$, $\Phi : T(T_+(\mathcal{A})) \to \mathbb{C}$ the character extending φ, and $\Phi' : T(T_+(\mathcal{A})) \to \mathbb{C}$ the linear map extending φ' and satisfying the Leibniz-type property. Following the idea of replacing cumulants, we can define the Boolean Bercovici-Pata map in the shuffle algebra framework.

Proposition 6.4.3. *Let (μ, μ') be an infinitesimal distribution and consider its double tensor algebra extension $(T(T_+(\mathcal{A})), \Phi, \Phi')$ as previously described. If $\tilde{\Phi} = \Phi + \hbar\Phi' \in G_{\mathbb{G}}$, then the character $\tilde{\mathbb{B}}(\tilde{\Phi}) \in G_{\mathbb{G}}$ extending the Boolean Bercovici-Pata map $\tilde{\mathbb{B}}(\tilde{\mu})$ is given by*

$$\tilde{\mathbb{B}}(\tilde{\Phi}) = \mathcal{E}_{\prec}(\mathcal{L}_{\succ}(\tilde{\Phi})).$$

Recall that, from the results of Theorem 5.1.2, the $\mathbb{G}$-valued analogue of (5.1.1) and (5.1.2) provide the characters in $G_{\mathbb{G}}$ that extend the infinitesimal free and infinitesimal Boolean additive convolutions, respectively. More precisely, if $\tilde{\Phi}$ and $\tilde{\Psi}$ are the characters in $G_{\mathbb{G}}$ extending the infinitesimal distributions $\tilde{\mu}$ and $\tilde{\nu}$, respectively, then $\tilde{\Phi} \boxplus \tilde{\Psi}$ and $\tilde{\Phi} \uplus \tilde{\Psi}$ are the characters in $G_{\mathbb{G}}$ extending $\tilde{\mu} \boxplus \tilde{\nu}$ and $\tilde{\mu} \uplus \tilde{\nu}$, respectively. Analogously, the characters in $G(\mathbb{G})$ extending the infinitesimal power convolutions are given in the following definition.

Definition 6.4.4. Let $\alpha \in \mathfrak{g}_{\mathbb{G}}$ and $t \geq 0$. We define the corresponding $\mathbb{G}$-*valued free and Boolean convolution powers* by

$$\begin{aligned} \mathcal{E}_{\prec}(\alpha)^{\boxplus t} &:= \mathcal{E}_{\prec}(t\alpha) \\ \mathcal{E}_{\succ}(\alpha)^{\uplus t} &:= \mathcal{E}_{\succ}(t\alpha), \end{aligned}$$

respectively.

We now have all the ingredients to prove the infinitesimal version of the main theorem of the article [BN08b] in the shuffle-algebraic framework for non-commutative probability.

Theorem 6.4.5. *Let $\tilde{\mu} \in \tilde{\mathcal{D}}(m)$ be an infinitesimal distribution and $t \geq 0$. Denote $\tilde{\mathbb{B}}_t(\tilde{\mu}) \in \tilde{\mathcal{D}}(m)$ the infinitesimal distribution given by*

$$\tilde{\mathbb{B}}_t(\tilde{\mu}) = \left(\tilde{\mu}^{\boxplus 1+t}\right)^{\uplus \frac{1}{1+t}}. \tag{6.4.1}$$

Then we have that the family of maps $\{\tilde{\mathbb{B}}_t\}_{t\geq 0}$ satisfy the semigroup property

$$\tilde{\mathbb{B}}_s \circ \tilde{\mathbb{B}}_t = \tilde{\mathbb{B}}_{s+t}, \quad \forall\, s,t \geq 0. \tag{6.4.2}$$

Furthermore, we have that $\tilde{k}_n(\tilde{\mathbb{B}}_1(\tilde{\mu})) = \tilde{b}_n(\tilde{\mu})$, i.e. $\tilde{\mathbb{B}}_1(\tilde{\mu}) = \tilde{\mathbb{B}}(\tilde{\mu})$ is the infinitesimal distribution obtained from the infinitesimal Boolean Bercovici-Pata map.

Proof. Consider the infinitesimal distribution (μ, μ') as well as its associated linear functionals $\Phi, \Phi' : T(T_+(\mathcal{A})) \to \mathbb{C}$ and $\tilde{\Phi} = \Phi + \hbar\Phi'$. We will show that if

$$\tilde{\mathbb{B}}_t(\tilde{\Phi}) = \left(\tilde{\Phi}^{\boxplus 1+t}\right)^{\uplus \frac{1}{1+t}},$$

then $\tilde{\mathbb{B}}_1 = \tilde{\mathbb{B}}$ and $\tilde{\mathbb{B}}_s \circ \tilde{\mathbb{B}}_t = \tilde{\mathbb{B}}_{s+t}$, for any $s,t \geq 0$. Let $\tilde{\kappa} = \mathcal{L}_{\prec}(\tilde{\Phi})$ be the $\mathbb{G}$-valued free infinitesimal cumulant character of $\tilde{\Phi}$. We can follow the proof of Lemma 42 in [EFP19] to obtain

$$\tilde{\mathbb{B}}_t(\tilde{\Phi}) = \mathcal{E}_{\prec}(t\theta_{\tilde{\kappa}}(\tilde{\kappa}))^{\uplus \frac{1}{t}} = \mathcal{E}_{\prec}\left(\theta_{t\tilde{\kappa}}(\tilde{\kappa})\right). \tag{6.4.3}$$

Indeed, take $t > 0$. By the $\mathbb{G}$-valued version of Theorem 4.2.14 we have that

$$\mathcal{E}_{\prec}((1+t)\tilde{\kappa}) = \mathcal{E}_{\prec}(\tilde{\kappa}) * \mathcal{E}_{\prec}(t\theta_{\tilde{\kappa}}(\tilde{\kappa})).$$

On the other hand, by using that $\mathcal{L}_{\succ}(\mathcal{E}_{\prec}(\tilde{\kappa})) = \theta_{\tilde{\kappa}}(\tilde{\kappa})$, we obtain by the shuffle identities

$$\begin{aligned}
\mathcal{E}_{\prec}((1+t)\tilde{\kappa}) &= \mathcal{E}_{\succ}\left(\theta_{(1+t)\tilde{\kappa}}((1+t)\tilde{\kappa})\right) \\
&= \mathcal{E}_{\succ}\left(\mathcal{E}_{\prec}^{*-1}(t\theta_{\tilde{\kappa}}(\tilde{\kappa})) * \mathcal{E}_{\prec}^{*-1}(\tilde{\kappa}) \succ (1+t)\tilde{\kappa} \prec \mathcal{E}_{\prec}(\tilde{\kappa}) * \mathcal{E}_{\prec}(t\theta_{\tilde{\kappa}}(\tilde{\kappa}))\right) \\
&= \mathcal{E}_{\succ}\left((1+t)\mathcal{E}_{\prec}^{*-1}(t\theta_{\tilde{\kappa}})(\tilde{\kappa}) \succ \theta_{\tilde{\kappa}}(\tilde{\kappa}) \prec \mathcal{E}_{\prec}(t\theta_{\tilde{\kappa}})(\tilde{\kappa})\right).
\end{aligned}$$

Then, by definition of $\mathbb{B}_t(\tilde{\Phi})$ and applying $\mathcal{E}_{\succ} \circ \mathcal{L}_{\succ} = \mathrm{id}$, we have

$$\begin{aligned}
\mathbb{B}_t(\tilde{\Phi}) &= \left(\mathcal{E}_{\prec}((1+t)\tilde{\kappa})\right)^{\uplus \frac{1}{1+t}} \\
&= \mathcal{E}_{\succ}\left(\frac{1}{1+t}(1+t)\mathcal{E}_{\prec}^{*-1}(t\theta_{\tilde{\kappa}})(\tilde{\kappa}) \succ \theta_{\tilde{\kappa}}(\tilde{\kappa}) \prec \mathcal{E}_{\prec}(t\theta_{\tilde{\kappa}})(\tilde{\kappa})\right) \\
&= \mathcal{E}_{\succ}\left(\frac{1}{t}\mathcal{E}_{\prec}^{*-1}(t\theta_{\tilde{\kappa}})(\tilde{\kappa}) \succ t\theta_{\tilde{\kappa}}(\tilde{\kappa}) \prec \mathcal{E}_{\prec}(t\theta_{\tilde{\kappa}})(\tilde{\kappa})\right)
\end{aligned}$$

$$= \mathcal{E}_{\succ}\left(\frac{1}{t}\theta_{t\theta_{\tilde{\kappa}}(\tilde{\kappa})}(t\theta_{\tilde{\kappa}}(\tilde{\kappa}))\right).$$

Recalling that $\theta_\alpha(\alpha) = \mathcal{L}_{\succ}(\mathcal{E}_{\prec}(\alpha))$ for any $\alpha \in \mathfrak{g}_{\mathbb{G}}$, we can take $\alpha = t\theta_{\tilde{\kappa}}(\tilde{\kappa})$ and get

$$\begin{aligned}\mathbb{B}_t(\tilde{\Phi}) &= \mathcal{E}_{\succ}\left(\frac{1}{t}\mathcal{L}_{\succ}(\mathcal{E}_{\prec}(t\theta_{\tilde{\kappa}}(\tilde{\kappa})))\right)\\ &= \mathcal{E}_{\prec}(t\theta_{\tilde{\kappa}}(\tilde{\kappa}))^{\uplus\frac{1}{t}},\end{aligned}$$

as we wanted to show. The second equality in (6.4.3) follows in a similar way by considering the factorization $\mathcal{E}_{\prec}((1+t)\tilde{\kappa}) = \mathcal{E}_{\prec}(t\tilde{\kappa}) * \mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(\tilde{\kappa}))$. Indeed, by using the identity $\mathcal{L}_{\succ}(\mathcal{E}_{\prec}(\tilde{\kappa})) = \theta_{\tilde{\kappa}}(\tilde{\kappa})$ and the shuffle identities, we have

$$\begin{aligned}\mathcal{E}_{\prec}((1+t)\tilde{\kappa}) &= \mathcal{E}_{\succ}\left(\theta_{(1+t)\tilde{\kappa}}((1+t)\tilde{\kappa})\right)\\ &= \mathcal{E}_{\succ}\left(\mathcal{E}_{\prec}^{*-1}(\theta_{t\tilde{\kappa}}(\tilde{\kappa})) * \mathcal{E}_{\prec}^{*-1}(t\tilde{\kappa}) \succ (1+t)\tilde{\kappa} \prec \mathcal{E}_{\prec}(t\tilde{\kappa}) * \mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(\tilde{\kappa}))\right)\\ &= \mathcal{E}_{\succ}\left((1+t)\mathcal{E}_{\prec}^{*-1}(\theta_{t\tilde{\kappa}}(\tilde{\kappa})) \succ \theta_{t\tilde{\kappa}}(\tilde{\kappa}) \prec \mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(\tilde{\kappa}))\right).\end{aligned}$$

Using the again the previous argument, we obtain

$$\begin{aligned}\tilde{\mathbb{B}}_t(\tilde{\Phi}) &= (\mathcal{E}_{\prec}((1+t)\tilde{\kappa}))^{\uplus\frac{1}{1+t}}\\ &= \mathcal{E}_{\succ}\left(\frac{1}{1+t}(1+t)\mathcal{E}_{\prec}^{*-1}(\theta_{t\tilde{\kappa}}(\tilde{\kappa})) \succ \theta_{t\tilde{\kappa}}(\tilde{\kappa}) \prec \mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(\tilde{\kappa}))\right)\\ &= \mathcal{E}_{\succ}\left(\mathcal{E}_{\prec}^{*-1}(\theta_{t\tilde{\kappa}}(\tilde{\kappa})) \succ \theta_{t\tilde{\kappa}}(\tilde{\kappa}) \prec \mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(\tilde{\kappa}))\right)\\ &= \mathcal{E}_{\succ}\left(\theta_{\theta_{t\tilde{\kappa}}(\tilde{\kappa})}(\theta_{t\tilde{\kappa}}(\tilde{\kappa}))\right).\end{aligned}$$

Taking $\alpha = \theta_{t\tilde{\kappa}}(\tilde{\kappa})$ in the identity $\theta_\alpha(\alpha) = \mathcal{L}_{\succ}(\mathcal{E}_{\prec}(\alpha))$, we conclude

$$\begin{aligned}\tilde{\mathbb{B}}_t(\tilde{\Phi}) &= \mathcal{E}_{\succ}\left(\mathcal{L}_{\succ}(\mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(\tilde{\kappa})))\right)\\ &= \mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(\tilde{\kappa})),\end{aligned}$$

as we wanted to show.

We can use the second equality in (6.4.3). In particular, taking $t = 1$ we get

$$\tilde{\mathbb{B}}_1(\tilde{\Phi}) = \mathcal{E}_{\prec}(\theta_{\tilde{\kappa}})(\tilde{\kappa}) = \mathcal{E}_{\prec}(\mathcal{L}_{\succ}(\tilde{\Phi})) = \tilde{\mathbb{B}}(\tilde{\Phi}),$$

and thus $\tilde{\mathbb{B}}_1 = \tilde{\mathbb{B}}$, as we wanted.

Now we will prove (6.4.2). By the second equality in (6.4.3), we have

$$\tilde{\mathbb{B}}_s \circ \tilde{\mathbb{B}}_t(\tilde{\Phi}) = \tilde{\mathbb{B}}_s\left(\mathcal{E}_{\prec}\left(\theta_{t\tilde{\kappa}}(\tilde{\kappa})\right)\right) = \mathcal{E}_{\prec}\left(\theta_{s\theta_{t\tilde{\kappa}}(\tilde{\kappa})}(\theta_{t\tilde{\kappa}}(\tilde{\kappa}))\right).$$

Looking at the argument of the last half-shuffle exponential, by the action property of θ

(Remark 4.2.12), we can get

$$\tilde{\gamma} := \theta_{s\theta_{t\tilde{\kappa}}(\tilde{\kappa})}(\theta_{t\tilde{\kappa}}(\tilde{\kappa})) = \theta_{\mathcal{L}_{\prec}(\mathcal{E}_{\prec}(t\tilde{\kappa})*\mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(s\tilde{\kappa})))}(\tilde{\kappa}).$$

We can apply the $\mathbb{G}$-valued version of Theorem 4.2.14 in the argument of $\mathcal{L}_{\prec}$ in order to obtain

$$\mathcal{E}_{\prec}(t\tilde{\kappa}) * \mathcal{E}_{\prec}(\theta_{t\tilde{\kappa}}(s\tilde{\kappa})) = \mathcal{E}_{\prec}(t\tilde{\kappa} + s\tilde{\kappa}) = \mathcal{E}_{\prec}((t+s)\tilde{\kappa}).$$

Therefore

$$\tilde{\mathbb{B}}_s \circ \tilde{\mathbb{B}}_t(\tilde{\Phi}) = \mathcal{E}_{\prec}(\tilde{\gamma}) = \mathcal{E}_{\prec}(\theta_{(t+s)\tilde{\kappa}}(\tilde{\kappa})) = \tilde{\mathbb{B}}_{t+s}(\Phi),$$

where we used (6.4.3) in the last equality. $\square$

Chapter 7

A Monotone Cumulant-Moment Formula via Schröder Trees

The present chapter is devoted to attacking the problem of finding a combinatorial formula, in terms of non-crossing partitions, that writes multivariate monotone cumulants of a sequence of random variables in terms of their moments. More precisely, we would like to find a description of the family of coefficients $\{\alpha(\pi)\}_{\pi\in\mathrm{NC}}$ such that the relation

$$h_n(a_1,\dots,a_n) = \sum_{\pi\in\mathrm{NC}(n)} \alpha(\pi)\varphi_\pi(a_1,\dots,a_n).$$

holds for any $n \geq 1$ and $a_1,\dots,a_n \in \mathcal{A}$. One could notice that this is the analogue of the formulas for free and Boolean cumulants (3.4.12) and (3.4.13), respectively, obtained by applying Möbius inversion on the lattices of non-crossing and interval partitions, respectively. However, this initial idea cannot be implemented in the monotone case due to the non-multiplicativity of the map $1/t(\pi)!$ with respect to the blocks of $\pi \in \mathrm{NC}(n)$.

Our approach for obtaining the desired formula combines the framework exposed in Chapter 4 and another Hopf algebra based on a particular type of planar rooted trees: the *Schröder trees*. This Hopf algebra, described in Section 7.1, provides another example of an unshuffle bialgebra ([JVMNT17]) which is nicely related to the double tensor algebra (Theorem 7.1.13). Afterwards, in Section 7.2, we exploit the relation above to describe the fixed point equations defining the half-shuffle exponentials in the context of the shuffle algebra of linear functionals over Schröder trees. This description leads us to look at the cumulant functionals as infinitesimal characters on another shuffle algebra. Finally, we specialize in the monotone case in Section 7.3. The shuffle equation $\rho = \log^*(\Phi)$ on the double tensor algebra does not directly imply a combinatorial monotone cumulant-to-moment formula. However, the corresponding equation on the Hopf algebra of Schröder trees has a nice combinatorial expression (Lemma 7.3.12). In the process, we identify and describe the corresponding coefficients $\alpha(\pi)$ in terms of certain coefficients closely related to the combinatorics of rooted trees and free pre-Lie algebras (Definition 7.3.2), which lead to the solution of our concern in Theorem 7.3.13. Our approach concludes that the Hopf algebra of trees leads to new cumulant-to-moment formulas with the feature of being indexed over Schröder trees (Theorem 7.3.17).

The main results described in this chapter, as well as the ideas for their proofs, are based on the joint work [AC21].

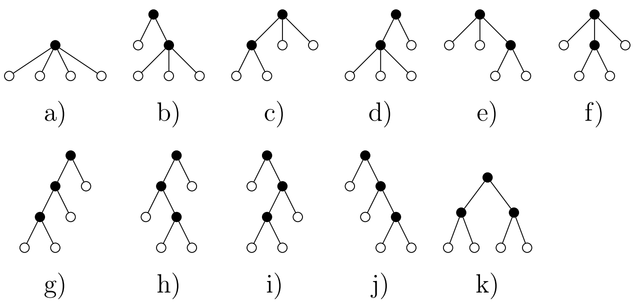
7.1 A Hopf algebra of Schröder trees

This section aims to present a Hopf algebra structure over certain planar rooted trees, denoted by $\mathcal{H}_{\mathcal{S}}$, that will be relevant throughout the chapter. We begin with the definition of a Schröder tree. The planarity drives us to consider a non-commutative polynomial algebra with a coproduct that can be regarded as a non-commutative version of the Connes-Kreimer coproduct. The main feature of $\mathcal{H}_{\mathcal{S}}$, which is important for our purposes, is that an unshuffle bialgebra structure can be defined on it, such that there exists an unshuffle bialgebra morphism from the double tensor algebra and a decorated version of $\mathcal{H}_{\mathcal{S}}$.

The precise definition of the previously mentioned Schröder trees is the following.

Definition 7.1.1. A *Schröder tree* is a planar rooted tree such that each of its internal vertices has at least two children.

Example 7.1.2. The Schröder trees with four leaves are depicted in the figure below. In this work, the leaves of a Schröder tree are depicted in white colour while the internal vertices are drawn in black colour.



In particular, notice that  is not a Schröder tree.

Let us fix some notation for this chapter. For any $n \geq 1$, we denote

$$\mathrm{ST}(n) = \{t \text{ is a Schröder tree with } n+1 \text{ leaves}\}.$$

We also set the convention that $\mathrm{ST}(0)$ is the set containing the *Schröder tree of a single vertex*, which will be denoted by $\circ$. Thus, the set of all Schröder trees is denoted by

$\mathrm{ST} := \bigcup_{n\geq 0} \mathrm{ST}(n)$. A well-known fact is that Schröder trees are counted by the sequence of *little Schröder numbers*, also known as *super-Catalan numbers*:

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, \ldots$$

(see sequence A001003 in the OIES).

The aim of this section is to present a Hopf algebra structure based on Schröder trees. Due to the planarity of the Schröder trees, it will be convenient to consider the *non-commutative* polynomial algebra over ST given by

$$\mathcal{H}_{\mathcal{S}} := \mathbb{K}\langle t \,:\, t \in \mathrm{ST}\rangle/(\circ - 1).$$

Note that we are identifying the unique element of ST(0) with the unit of $\mathbb{K}$. Moreover, as it is mentioned in the examples of Hopf algebras over trees in Section 2.2, it will be helpful to identify a non-commutative monomial of Schröder trees $t_1 \cdots t_n$ as an ordered forest consisting of the sequence $(t_1, \ldots, t_n)$. Under this identification, $\mathcal{H}_{\mathcal{S}}$ is the linear span of the set of Schröder forests.

The coproduct on $\mathcal{H}_{\mathcal{S}}$ can be described by using a slight variation of the notion of admissible cuts of t (Definition 2.2.12).

Definition 7.1.3. Let t be a Schröder tree. A $\mathcal{S}$*-admissible cut of* t is a subset c of the set of internal vertices of t such that for any path from the root to any leaf, there is at most one vertex of the path contained in c.

Remark 7.1.4. Notice that the main difference with the notion of admissible cut given in Definition 2.2.12 is that we do not consider the whole set of vertices of a Schröder tree t. In the case of regular rooted trees, we consider the whole set of edges.

Given c an $\mathcal{S}$-admissible cut of a Schröder tree t, we can naturally order the elements in c from left to right according to their position in t since it is planar. Hence, we can extend the notion of trunk $R_c(t)$ and pruning $P_c(t)$ associated to a $\mathcal{S}$-admissible cut c as follows.

Definition 7.1.5. Let t be a Schröder tree and c be an $\mathcal{S}$-admissible cut of t. The *pruning of* t associated to c is the ordered forest of Schröder trees $P_c(t)$ formed by the subtrees obtained by cutting the edge above each element of c. In addition, the *trunk of* t associated to c is the Schröder tree $R_c(t)$ obtained by replacing in t each subtree of $P_c(t)$ by a leaf.

In the context of the previous definition, we should notice that if c is the admissible cut only containing the root of a Schröder tree t, then we set $R_c(t) = \circ$ and $P_c(t) = t$.

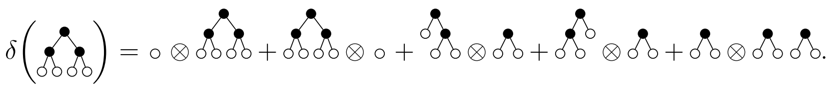
Remark 7.1.6. It is noticeable the difference between the definition of admissible cut of a non-planar tree (Definition 2.2.12) and the definition of a $\mathcal{S}$-admissible cut. While the former refers to a subset of the whole set of edges of the tree, the latter is a subset of the set of internal vertices of a Schröder tree. Also, in the construction of prunings and trunks associated to a usual admissible cut, we delete the corresponding edges in the cut, in contrast with Definition 7.1.5 where the vertices in the cut are not deleted, but they are the roots of the pruning and new leaves on the trunk.

Given a Schröder tree t, we denote $\mathrm{Adm}_{\mathcal{S}}(t)$ as the set of all $\mathcal{S}$-admissible cuts of t. With the previous notions, the coproduct of interest on $\mathcal{H}_{\mathcal{S}}$ can be defined as a non-commutative version of the Connes-Kreimer coproduct (2.2.5), i.e. we define $\delta_{\mathcal{S}}$ to be the algebra morphism $\delta_{\mathcal{S}} : \mathcal{H}_{\mathcal{S}} \to \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{S}}$ given by

$$\delta_{\mathcal{S}}(t) = \sum_{c \in \mathrm{Adm}_{\mathcal{S}}(t)} R_c(t) \otimes P_c(t) \tag{7.1.1}$$

for any Schröder tree t.

Example 7.1.7. We have the following computation for the coproduct $\delta_{\mathcal{S}}$:


(7.1.2)

Remark 7.1.8. One should notice that Definition 7.1.3 and Definition 7.1.5 can be directly extended to the case of Schröder forests, i.e. non-commutative monomials of Schröder trees. The multiplicativity of the coproduct $\delta_{\mathcal{S}}$ is then represented by the fact that if f is a Schröder forest, then $\delta_{\mathcal{S}}$ is defined with the same expression as (7.1.1) but now considering the corresponding definition of trunk and pruning of a Schröder forest f.

It is not difficult to associate a graded structure to $\mathcal{H}_{\mathcal{S}}$. However, it will not be the same as in the case of regular rooted trees. Indeed, if $t \in \mathrm{ST}(n)$, we define its degree by $\deg(t) = n$. More generally, for an ordered forest of Schröder trees $f = t_1 \cdots t_n$, we define

$$\deg(f) = \deg(t_1) + \cdots + \deg(t_n).$$

It is clear that the product is graded; hence $\mathcal{H}_{\mathcal{S}}$ is a graded algebra, where the homogeneous component $\mathcal{H}_{\mathcal{S}}(n)$ consists of the linear span of all the Schröder forests of degree n, for each $n \geq 1$. In particular, we have that $\mathcal{H}_{\mathcal{S}}(0) = \mathbb{K}\circ \cong \mathbb{K}$. The coproduct $\delta_{\mathcal{S}}$ is also graded. As we are in a connected graded case, the counit $\varepsilon : \mathcal{H}_{\mathcal{S}} \to \mathbb{K}$ is the algebra morphism given by $\varepsilon(\circ) = 1$ and $\varepsilon(t_1 \cdots t_n) = 0$ for any Schröder trees $t_1, \ldots, t_n$ and $n \geq 1$. Therefore, we have the following statement.

Theorem 7.1.9 ([Foi02, JVMNT17])**.** *The triple $(\mathcal{H}_{\mathcal{S}}, \cdot, \delta_{\mathcal{S}})$ is a connected graded non-commutative non-cocommutative Hopf algebra.*

The previous Hopf algebra was originally studied by Foissy in [Foi02], where the author also studied the decorated version of the Hopf algebra, which we will discuss in the next lines. Let $\mathcal{A}$ be a vector space, and define the algebra

$$\mathcal{H}_{\mathcal{S}}(\mathcal{A}) := \bigoplus_{n\geq 0} \big(\mathcal{H}_{\mathcal{S}}(n) \otimes \mathcal{A}^{\otimes n}\big), \tag{7.1.3}$$

where the product is induced by the non-commutative product on $\mathcal{H}_{\mathcal{S}}$ and the concatenation product on the tensor algebra $T(\mathcal{A})$. The idea for defining the coproduct in $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ is that pure tensor of the form $t \otimes a_1 \cdots a_n \in \mathrm{ST}(n) \otimes \mathcal{A}^{\otimes n}$ can be identified with a decorated Schröder tree, where the decoration is defined by a labelling of the n sectors between the $n+1$ leaves of t from left to right with the letters $a_1, \ldots, a_n$, as in the following figure:

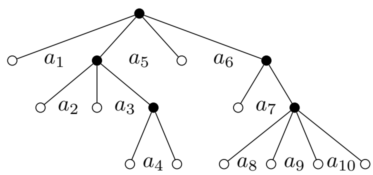


Figure 7.1: Decorated Schröder tree $t \otimes a_1 \cdots a_{10}$.

In the case of having a pure tensor of the form $f \otimes w \in \mathcal{H}_{\mathcal{S}}(n) \otimes \mathcal{A}^{\otimes n}$ with f being a Schröder forest, recall that a Schröder forest is a sequence of planar trees. Since $\deg(f) = n$, then the sum of the degrees of the corresponding trees in the sequence is equal to n, meaning that there are n sectors that can be ordered from left to right. Last, we decorate such n sectors with the letters of the word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$.

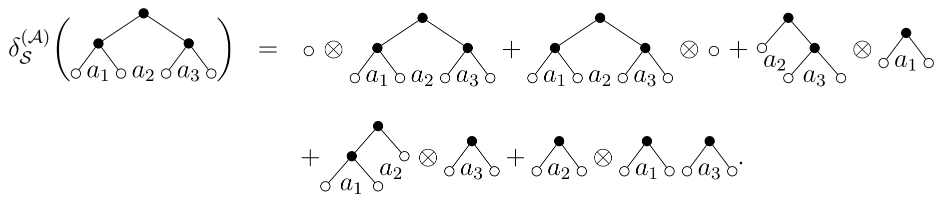
Now, under the identification of decorated Schröder trees, it is not so difficult to see that (7.1.1) also defines a coproduct $\delta_{\mathcal{S}}^{(\mathcal{A})}$ in $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ with the additional rule of carrying the decorations of the trunk and the pruning. An example of this is observed in the following picture.

In general, for $f \otimes w \in \mathcal{H}_{\mathcal{S}}(n) \otimes \mathcal{A}^{\otimes n}$ we will write

$$\delta_{\mathcal{S}}^{(\mathcal{A})}(t \otimes w) = \sum_{c \in \mathrm{Adm}_{\mathcal{S}}(f)} R_c(f, w) \otimes P_c(f, w), \tag{7.1.4}$$

where $R_c(f, w)$ and $P_c(f, w)$ are the trunk and pruning of f associated to $c \in \mathrm{Adm}_{\mathcal{S}}(f)$ decorated accordingly with w.

Example 7.1.10. Considering the decoration $a_1a_2a_3 \in \mathcal{A}^{\otimes 3}$ of the Schröder tree in Equation (7.1.2), we have the following computation of the coproduct in $H_{\mathcal{S}}(\mathcal{A})$:



Since $\deg(t) = n$ if $t \in \mathrm{ST}(n)$, i.e. t has $n+1$ leaves, we obtain again that $t \otimes w$ has degree 0 if and only if t has one leaf and w has length 0, and this happens if only if $t = \circ$ and $w = \mathbf{1}$. Thus the 0-th homogeneous component of $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ is $\mathcal{H}_{\mathcal{S}}(\mathcal{A})(0) \cong \mathbb{K}$ and so $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ is connected.

It is clear that $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ is graded by construction. In this setting, it follows that the counit $\varepsilon : \mathcal{H}_{\mathcal{S}}(\mathcal{A}) \to \mathbb{K}$ is provided by the algebra morphism such that

$$\varepsilon^{(\mathcal{A})}(t \otimes w) = \begin{cases} 1 & \text{if } t = \circ \text{ and } w = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$

With the above, we can state the following result.

Theorem 7.1.11 ([Foi02, JVMNT17]). *Let $\mathcal{A}$ be a vector space. The triple $(\mathcal{H}_{\mathcal{S}}(\mathcal{A}), \cdot, \delta_{\mathcal{S}}^{(\mathcal{A})})$ is a connected graded non-commutative non-cocommutative Hopf algebra.*

A key feature of the previous Hopf algebra of decorated Schröder trees is that, as well as the coproduct in the double tensor algebra, the coproduct can be split into two non-coassociative coproducts yielding an unshuffle bialgebra structure on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$.

Let us establish some notation. Let $\mathcal{H}_{\mathcal{S}}^{+}(\mathcal{A})$ be the augmentation ideal of $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$, i.e.

$$\mathcal{H}_{\mathcal{S}}^{+}(\mathcal{A}) = \bigoplus_{n \geq 1} \mathcal{H}_{\mathcal{S}}(n) \otimes \mathcal{A}^{\otimes n}.$$

Furthermore, define the following subset of $\mathcal{S}$-admissible cuts of a Schröder tree t:

$$\mathrm{Adm}_{\prec}(t) := \{c \in \mathrm{Adm}_{\mathcal{S}}(t) \;:\; R_c(t) \text{ contains the leftmost leaf of } t\}.$$

With the previous notations, we proceed to define the maps

$$\delta_{\prec}^{+}(t \otimes w) = \sum_{c \in \mathrm{Adm}_{\prec}(t)} R_c(t, w) \otimes P_c(t, w), \tag{7.1.5}$$

and

$$\delta_{\succ}^{+}(t \otimes w) = \delta_{\mathcal{S}}^{(\mathcal{A})}(t \otimes w) - \delta_{\prec}^{+}(t \otimes w), \tag{7.1.6}$$

for any $t \otimes w \in \mathrm{ST}(n) \otimes \mathcal{A}^{\otimes n}$.

Now, following the developments in Section 4.2, we define the linear maps

$$\delta_{\prec}^{+}, \delta_{\succ}^{+} : \mathcal{H}_{\mathcal{S}}^{+}(\mathcal{A}) \to \mathcal{H}_{\mathcal{S}}(\mathcal{A}) \otimes \mathcal{H}_{\mathcal{S}}(\mathcal{A})$$

by the recipe

$$\begin{aligned}
\delta_{\prec}^{+}((t_1 \otimes w_1)\cdots(t_m \otimes w_m)) &= \delta_{\prec}^{+}(t_1 \otimes w_1)\delta_{\mathcal{S}}^{(\mathcal{A})}\big((t_2 \otimes w_2)\cdots(t_m \otimes w_m)\big),\\
\delta_{\succ}^{+}((t_1 \otimes w_1)\cdots(t_m \otimes w_m)) &= \delta_{\succ}^{+}(t_1 \otimes w_1)\delta_{\mathcal{S}}^{(\mathcal{A})}\big((t_2 \otimes w_2)\cdots(t_m \otimes w_m)\big),
\end{aligned}$$

for any $t_1 \otimes w_1, \ldots, t_m \otimes w_m$ decorated Schröder trees. Finally, by setting

$$\delta_{\prec}(t \otimes w) = \delta_{\prec}^{+}(t \otimes w) - (t \otimes w)\bigotimes(\circ \otimes \mathbf{1}) \quad \text{and} \quad \delta_{\succ}(t \otimes w) = \delta_{\succ}^{+}(t \otimes w) - (\circ \otimes \mathbf{1})\bigotimes(t \otimes w)$$

for any $t \otimes w$ a decorated Schröder tree, we can state the following important result.

Theorem 7.1.12 ([JVMNT17, Thm. 7.1])**.** *Let $\mathcal{A}$ be a vector space and consider the Hopf algebra of decorated Schröder trees $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ as well as the splitting of the coproduct $\delta_{\mathcal{S}}^{(\mathcal{A})}$ defined by* (7.1.5) *and* (7.1.6)*. Then $(\mathcal{H}_{\mathcal{S}}(\mathcal{A}), \delta_{\prec}, \delta_{\succ})$ is an unshuffle bialgebra.*

The previous theorem provides another example of an unshuffle bialgebra so that its graded dual will give us a new example of a unital shuffle algebra. The other important example of unshuffle bialgebra in this work is the double tensor algebra $T(T_+(\mathcal{A}))$ over an algebra $\mathcal{A}$ studied in detail in Section 4.2. A natural question is then to find out if there is a relation between them. Gladly, we have the following description of the relationship.

Theorem 7.1.13 ([JVMNT17, Thm. 7.3])**.** *Let $\mathcal{A}$ be an algebra and consider the double tensor algebra $T(T_+(\mathcal{A}))$ as well as the Hopf algebra of decorated Schröder trees $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$. Let $\iota : T(T_+(\mathcal{A})) \to \mathcal{H}_{\mathcal{S}}(\mathcal{A})$ be the algebra morphism defined by*

$$\iota(w) = \sum_{t \in \mathrm{ST}(|w|)} t \otimes w, \tag{7.1.7}$$

for any word $w \in T_+(\mathcal{A})$. Then ι is a coalgebra morphism and an unshuffle bialgebra morphism.

7.2 Schröder trees and non-commutative probability

In this section, we will apply the algebraic machinery over Schröder trees, studied in the previous section, to describe the link between the shuffle-algebraic framework for non-commutative probability (Chapter 4) and the unshuffle bialgebra $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$. This connection will allow understanding cumulants via another shuffle algebra. In the process, we will obtain combinatorial relations between Boolean cumulants and moments in terms of Schröder trees by using Hopf algebra techniques.

The starting point is the work of Josuat-Vergès, Menous, Novelli and Thibon [JVMNT17]. In their work, the authors used Schröder trees extensively to describe the functional equation between the moment series and the R-transform from a point of view based on the so-called notion of *operad* (see [LV12] for a study of the notion of operad).

Remark 7.2.1. Besides the work of Jousat-Vergès et al. [JVMNT17], Schröder trees have recently appeared in the recent work [Bia21], where the author describes the fluctuations of a particular process in terms of certain polynomials given in terms of Schröder trees. The connection with free probability showed in [Bia21] is that such polynomials are the free cumulants of a family of commuting random variables.

Henceforth, we take $\mathbb{K} = \mathbb{C}$ and consider a non-commutative probability space $(\mathcal{A}, \varphi)$. From the results of Section 4.3, we have that the free and Boolean cumulant functionals of $(\mathcal{A}, \varphi)$ can be effectively described as infinitesimal characters κ and β on the double tensor algebra $T(T_+(\mathcal{A}))$ that satisfy the respective shuffle fixed point equation

$$\Phi = \epsilon + \kappa \prec \Phi \quad \text{and} \quad \Phi = \epsilon + \Phi \succ \beta,$$

where Φ is a character on $T(T_+(\mathcal{A}))$ extending the linear functional φ. Since we now have another unshuffle bialgebra, the Hopf algebra of decorated Schröder trees $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$, it makes sense to consider the above fixed-point equations in the shuffle algebra of linear functionals on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$.

More precisely, given the linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, we define its $\mathcal{S}$*-lifting* as the element $\hat{\Phi}$ in the group of characters of $\mathrm{Lin}(\mathcal{H}_{\mathcal{S}}(\mathcal{A}), \mathbb{C})$, denoted by $G(\mathcal{H}_{\mathcal{S}}(\mathcal{A}))$, given by the recipe $\hat{\Phi}(\circ) = 1$ and if $t \otimes a_1 \cdots a_n \in \mathrm{ST}(n) \otimes \mathcal{A}^{\otimes n}$, then

$$\hat{\Phi}(t \otimes a_1 \cdots a_n) = \begin{cases} \varphi(a_1 \cdots a_n) & \text{if } t \text{ is a corolla with } n+1 \text{ leaves}, \\ 0 & \text{otherwise}. \end{cases} \tag{7.2.1}$$

Recall that a corolla with n leaves is a rooted tree consisting of only one internal vertex, the root, and exactly n leaves.

Then, it makes sense to consider the following fixed point equations in $\mathrm{Lin}(\mathcal{H}_{\mathcal{S}}(\mathcal{A}), \mathbb{C})$:

$$\hat{\Phi} = \varepsilon^{(A)} + \hat{\kappa} \prec \hat{\Phi}, \tag{7.2.2}$$

$$\hat{\Phi} = \varepsilon^{(A)} + \hat{\Phi} \succ \hat{\beta}, \tag{7.2.3}$$

where the products $\prec, \succ$ and $*$ on $\mathrm{Lin}(\mathcal{H}_{\mathcal{S}}(\mathcal{A}), \mathbb{C})$ are defined by

$$f \prec g = m_{\mathbb{C}} \circ (f \otimes g) \circ \delta_{\prec}, \tag{7.2.4}$$

$$f \succ g = m_{\mathbb{C}} \circ (f \otimes g) \circ \delta_{\succ}, \tag{7.2.5}$$

$$f * g = m_{\mathbb{C}} \circ (f \otimes g) \circ \delta_{\mathcal{S}}^{(\mathcal{A})}, \tag{7.2.6}$$

for any $f, g \in \operatorname{Lin}(\mathcal{H}_\mathcal{S}(\mathcal{A}), \mathbb{C})$, making $(\operatorname{Lin}(\mathcal{H}_\mathcal{S}(\mathcal{A}), \mathbb{C}), \prec, \succ)$ a unital shuffle algebra. It turns out that the solutions of the above fixed-point equations are related to the solutions in the double tensor algebra case (Theorem 4.2.6) via the unshuffle bialgebra morphism ι declared in Theorem 7.1.13, and the fact that if Φ is the lifting of φ, then

$$\Phi = \hat{\Phi} \circ \iota. \tag{7.2.7}$$

Remark 7.2.2. Observe that we are using the same notation for the half-shuffle products on the shuffle algebras $\operatorname{Lin}(T(T_+(\mathcal{A})), \mathbb{C})$ and $\operatorname{Lin}(\mathcal{H}_\mathcal{S}(\mathcal{A}), \mathbb{C})$. Nevertheless, the context will make clear to which shuffle algebra we are referring.

Proposition 7.2.3. *Let $\hat{\Phi} \in G(\mathcal{H}_\mathcal{S}(\mathcal{A}))$ be the $\mathcal{S}$-lifting of φ. If $\hat{\kappa}$ and $\hat{\beta}$ stand for the solution of (7.2.2) and (7.2.3), respectively, then $\kappa := \hat{\kappa} \circ \iota$ and $\beta := \hat{\beta} \circ \iota$ are the infinitesimal liftings of the free, resp., Boolean cumulants of φ.*

Proof. First, observe that we can assume that $\hat{\kappa}$ and $\hat{\beta}$ exist and are infinitesimal characters, since the half-shuffle exponentials $\mathcal{E}_\prec$ and $\mathcal{E}_\succ$ are bijections between $G(\mathcal{H}_\mathcal{S}(\mathcal{A}))$ and $\mathfrak{g}(\mathcal{H}_\mathcal{S}(\mathcal{A}))$, the Lie algebra of infinitesimal characters on $\operatorname{Lin}(\mathcal{H}_\mathcal{S}(\mathcal{A}), \mathbb{C})$ (see [EFP19, Thm. 14], and also Theorem 4.2.6 and the discussion at the end of Section 4.2). Also, using that ι is an algebra morphism, we conclude that β is an infinitesimal character:

$$\beta(w_1|w_2) = \hat{\beta}(\iota(w_1|w_2)) = \hat{\beta}(\iota(w_1)\iota(w_2)) = 0,$$

for any $w_1, w_2 \in T(T_+(\mathcal{A}))$. We finish the proof by showing that $\beta = \hat{\beta} \circ \iota$ satisfies the respective fixed-point equation, for any $w \in T_+(T_+(\mathcal{A}))$:

$$\Phi(w) = (\hat{\Phi}\circ\iota)(w) = \hat{\Phi}(\iota(w)) = \big(\varepsilon^{(\mathcal{A})} + \hat{\Phi} \succ \hat{\beta}\big)(\iota(w)) = (\hat{\Phi}\circ\iota) \succ (\hat{\beta}\circ\iota)(w) = (\epsilon + \Phi \succ \beta)(w),$$

where we used that $\Phi = \hat{\Phi} \circ \iota$ and that ι is an unshuffle bialgebra morphism in the fourth equality. Thus $\Phi = \mathcal{E}_\succ(\beta)$ and, since Φ is the lifting of φ, we conclude by using Theorem 4.3.2. The free case is treated with similar arguments by using the left fixed-point equation. $\square$

The above proposition and the definition of ι allow expressing the infinitesimal characters associated to the free and Boolean cumulants, respectively as

$$\kappa(w) = \sum_{t \in \mathrm{ST}(|w|)} \hat{\kappa}(t \otimes w) \quad \text{and} \quad \beta(w) = \sum_{t \in \mathrm{ST}(|w|)} \hat{\beta}(t \otimes w),$$

for any word $w \in T_+(\mathcal{A})$. The authors of [JVMNT17] describe $\hat{\kappa}$ in terms of a subset of Schröder trees. More precisely, for each $n \geq 0$, we define the set

$$\mathrm{PST}(n) := \{t \in \mathrm{ST}(n) \,:\, \text{the leftmost subtree of } t \text{ is a leaf}\}.$$

In addition, we will need the following combinatorial procedure in which we can associate a non-crossing partition to a Schröder tree.

Definition 7.2.4. Let t be a Schröder tree with $n+1$ leaves. We label the sectors between the leaves from left to right from 1 to n. We will associate a non-crossing partition $\pi(t) \in \mathrm{NC}(n)$ as follows: for each internal vertex v of t, consider the corolla that has as root v and is a subtree of t. For this corolla, define the block V whose elements are the labels of the sectors that define the corolla.

Example 7.2.5. The following picture shows the non-crossing partitions associated to the Schröder tree depicted in Figure 7.1.

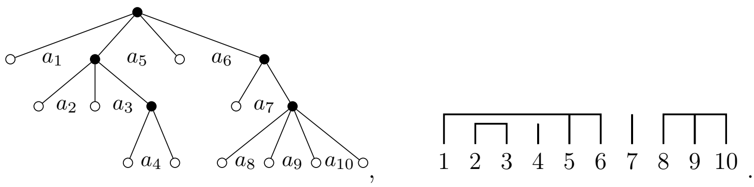


Figure 7.2: Decorated Schröder tree $t \otimes w$ and its associated non-crossing partition $\pi(t)$.

The above notions are needed to write the following result in [JVMNT17].

Proposition 7.2.6 ([JVMNT17, Thm. 7.2])**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Also, let $t \in \mathrm{ST}(n)$ and a word $w = a_1 \cdots a_n \in T_+(\mathcal{A})$, for $n \geq 1$. If $\hat{\Phi}$ is the $\mathcal{S}$-lifting of φ and $\hat{\kappa}$ is as defined in (7.2.2), then we have*

$$\hat{\kappa}(t \otimes w) = \begin{cases} (-1)^{|\pi(t)|-1} \varphi_{\pi(t)}(a_1, \ldots, a_n) & \text{if } t \in \mathrm{PST}(n), \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.8}$$

Remark 7.2.7. The above proposition combined with Proposition 7.2.3 implies that

$$k_n(a_1, \ldots, a_n) = \sum_{t \in \mathrm{PST}(n)} (-1)^{|\pi(t)|-1} \varphi_{\pi(t)}(a_1, \ldots, a_n). \tag{7.2.9}$$

Notice that (7.2.9) is rather different than the formula in terms of the Möbius inversion coefficients and non-crossing partitions, since for any non-crossing partition $\sigma \in \mathrm{NC}(n)$ there may be several $t \in \mathrm{PST}(n)$ such that $\pi(t) = \sigma$. Hence, in some sense, we can say (7.2.9) provides a finer description of free cumulants of random variables.

We are now interested in obtaining the analogue Boolean counterpart of Proposition 7.2.6. Our strategy consists of analyzing the combinatorial description of the fixed point equation (7.2.3). Recall that

$$\beta(w) = \sum_{t \in \mathrm{ST}(|w|)} \hat{\beta}(t \otimes w),$$

where $\beta \in \mathfrak{g}$ is the infinitesimal lifting on $T(T_+(\mathcal{A}))$ of the Boolean cumulants. To state the next result, we consider the subset of Schröder trees $\mathrm{BST}(n) \subset \mathrm{ST}(n)$ defined by the following condition: $t \in \mathrm{BST}(n)$ if, and only if, for any x an internal vertex of t with ordered set of children $(y_1, \ldots, y_m)$, the vertices $y_2, \ldots, y_m$ are leaves of t.

Example 7.2.8. The following picture illustrates the definition of the subsets $\mathrm{BST}(n)$. Graphically, $\mathrm{BST}(n)$ is the set of Schröder trees that lean to the right.

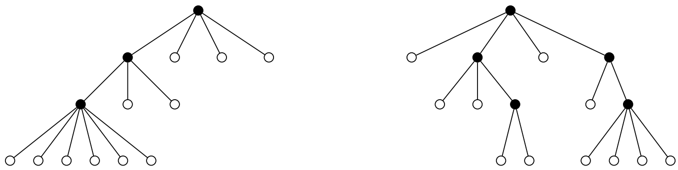


Figure 7.3: On the left, we have a drawing of an element in BST(10). On the right, we have an example of a Schröder tree in ST(10) not in BST(10).

The central proposition of the ongoing section is stated and proved as follows.

Proposition 7.2.9 ([AC21, Prop. 6.1])**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Also, let $t \in \mathrm{ST}(n)$ and a word $w = a_1 \cdots a_n \in T_+(\mathcal{A})$, for $n \geq 1$. If $\hat{\Phi}$ is the $\mathcal{S}$-lifting of φ and $\hat{\beta}$ is as defined in (7.2.3), then we have*

$$\hat{\beta}(t \otimes w) = \begin{cases} (-1)^{|\pi(t)|-1}\varphi_{\pi(t)}(a_1, \ldots, a_n) & \text{if } t \in \mathrm{BST}(n), \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.10}$$

Proof. Let $t \in \mathrm{ST}(n)$ and $a_1, \ldots, a_n \in \mathcal{A}$. Let us analyze the fixed point equation that defines $\hat{\beta}$. By definition, $\delta_{\succ}$ is a sum indexed over the set

$$\mathrm{Adm}(t) \backslash \mathrm{Adm}_{\prec}(t) = \{c \in \mathrm{Adm}(t) \,:\, c \text{ contains the leftmost leaf of } t\}.$$

Using that $\hat{\beta}$ is an infinitesimal character and the definition of $\delta_{\succ}$, we have that

$$\begin{aligned} \hat{\Phi}(t \otimes a_1 \cdots a_n) &= (\varepsilon^{(A)} + \hat{\Phi} \succ \hat{\beta})(t, a_1 \cdots a_n) \\ &= \sum_{c \in \mathrm{Adm}(t) \backslash \mathrm{Adm}_{\prec}(t)} \hat{\Phi}(R_c(t, a_1 \cdots a_n))\hat{\beta}(P_c(t, a_1 \cdots a_n)) \\ &= \sum_{\substack{c = \{v\} \\ v \text{ is an internal vertex belonging} \\ \text{the leftmost branch of } t}} \hat{\Phi}(R_c(t, a_1 \cdots a_n))\hat{\beta}(P_c(t, a_1 \cdots a_n)). \end{aligned}$$

Observe that the last equality above is due to the fact that $\hat{\beta}$ being an infinitesimal character implies that $P_c(t, a_1 \cdots a_n) = 0$ when $|c| \neq 1$. On the other hand, the definition of $\hat{\Phi}$ in (7.2.1) implies that $\hat{\Phi}(R_c(t, a_1 \cdots a_n))\hat{\beta}(P_c(t, a_1 \cdots a_n)) = 0$ when $R_c(t)$ is neither a corolla nor the single-vertex tree. Hence, the only possibility for a non-zero contribution in the above sum is that c is the set containing the root of t, or c is the set containing the leftmost child of the root. This implies that

$$\hat{\Phi}(t \otimes a_1 \cdots a_n) = \hat{\beta}(t \otimes a_1 \cdots a_n) + \hat{\Phi}(t'' \otimes a_{m+1} \cdots a_n)\hat{\beta}(t' \otimes a_1 \cdots a_m). \tag{7.2.11}$$

where $t' \otimes a_1 \cdots a_m$ is the leftmost decorated subtree of $t \otimes a_1 \cdots a_n$, and $t'' \otimes a_{m+1} \cdots a_n$ is the tree obtained from $t \otimes a_1 \cdots a_n$ when we delete $t' \otimes a_1 \cdots a_m$.

Now, we will proceed to prove (7.2.10) by induction on the number of internal vertices of t, denoted by $i(t)$. For the base case, let t be a decorated Schröder tree with $i(t) = 1$, i.e. t is a decorated corolla. Using (7.2.11) and the definition of $\hat{\Phi}$ (7.2.1) we have

$$\begin{aligned} (-1)^{i(t)-1}\varphi_{1_n}(a_1, \dots, a_n) &= \hat{\Phi}(t \otimes a_1 \cdots a_n) \\ &= \hat{\beta}(t \otimes a_1 \cdots a_n) + \hat{\beta}(\circ \otimes \mathbf{1})\hat{\Phi}(t \otimes a_1 \cdots a_n) \\ &= \hat{\beta}(t \otimes a_1 \cdots a_n) \end{aligned}$$

since $\hat{\beta}(\circ \otimes \mathbf{1}) = 0$, and this concludes the base step. For the inductive step, we assume that the result holds for any decorated Schröder tree with less than k internal vertices and take a Schröder tree $t \in \mathrm{ST}(n)$ such that $i(t) = k$. Again, by using (7.2.11) and the definition of $\hat{\Phi}$ we have the relation

$$0 = \hat{\Phi}(t \otimes a_1 \cdots a_n) = \hat{\beta}(t \otimes a_1 \cdots a_n) + \hat{\Phi}(t'' \otimes a_{m+1} \cdots a_n)\hat{\beta}(t' \otimes a_1 \cdots a_m),$$

where t' and t'' are the decorated Schröder subtrees described above. In particular, we can write

$$\hat{\beta}(t \otimes a_1 \cdots a_n) = -\hat{\Phi}(t'' \otimes a_{m+1} \cdots a_n)\hat{\beta}(t' \otimes a_1 \cdots a_m). \tag{7.2.12}$$

Observe that the condition $t \in \mathrm{BST}(n)$ is equivalent to the fact that $t' \in \mathrm{BST}(m)$ with $i(t)-1$ internal vertices, and t'' is a corolla. Hence, if $t \in \mathrm{BST}(n)$, we can use the inductive hypothesis on t' where we have

$$\begin{aligned} \hat{\beta}(t \otimes a_1 \cdots a_n) &= -\varphi(a_{m+1} \cdots a_n)(-1)^{i(t)-2}\varphi_{\pi(t')}(a_1, \dots, a_m) \\ &= (-1)^{|\pi(t)|-1}\varphi_{\pi(t)}(a_1, \dots, a_n), \end{aligned}$$

where we used that $|\pi(t)| = i(t)$. On the other hand, if $t \notin \mathrm{BST}(n)$, we can have that $t' \notin \mathrm{BST}(m)$ or t'' is not a corolla. In the first case, we use the inductive hypothesis to conclude that $\hat{\beta}(t' \otimes a_1 \cdots a_m) = 0$, while in the second case we have that $\hat{\Phi}(t'' \otimes a_{m+1} \cdots a_n) = 0$. In any case, from (7.2.12) we obtain that $\hat{\beta}(t \otimes a_1 \cdots a_n) = 0$ as we wanted to show. This concludes the inductive step and the proof of the proposition as well. □

Remark 7.2.10. The previous proposition, jointly with the relation $\beta = \hat{\beta} \circ \iota$, tells us that the Boolean cumulants of $(\mathcal{A}, \varphi)$ can be expressed by using the formula

$$b_n(a_1, \dots, a_n) = \sum_{t \in \mathrm{BST}(n)} (-1)^{|\pi(t)|-1} \varphi_{\pi(t)}(a_1, \dots, a_n),$$

for any $n \geq 1$ and $a_1, \dots, a_n \in \mathcal{A}$. In contrast with the result of [JVMNT17] regarding the free cumulants, where the formula in terms of Schröder trees is finer than the formula in terms of non-crossing partitions, the above formula is equivalent to applying the Möbius inversion formula in the moment-Boolean cumulant relation. The reason of this is that we have a straightforward bijection

$$\mathrm{BST}(n) \ni t \mapsto \pi(t) \in \mathrm{Int}(n).$$

The next natural question is to obtain the analogue result for the monotone cumulant. The next section is devoted to attacking this problem, obtaining in the process a sum over Schröder trees and a nice description of the coefficients of each tree.

7.3 Monotone cumulants in terms of moments via Schröder trees

In this section, we will prove the main result of the chapter: the formula that writes monotone cumulants in terms of moments by using Schröder trees. The strategy follows the same Hopf-algebraic techniques as in the previous section but now employs the coassociative coproduct on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$.

As before, we fix a non-commutative probability space $(\mathcal{A}, \varphi)$, and consider the double tensor algebra $T(T_+(\mathcal{A}))$ as well as the lifting $\Phi \in G$ of φ. We also take the Hopf algebra of decorated Schröder trees $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ as well as $\hat{\Phi} \in G(\mathcal{H}_{\mathcal{S}}(\mathcal{A}))$ the $\mathcal{S}$-lifting of φ defined in (7.2.1). Since $\hat{\Phi}$ is a character, we can take $\hat{\rho} \in \mathfrak{g}(\mathcal{H}_{\mathcal{S}}(\mathcal{A}))$ to be the infinitesimal character on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ such that $\hat{\Phi} = \exp^*(\hat{\rho})$. We then have the following proposition by the same argument in Proposition 7.2.3 but now using that ι is a bialgebra morphism.

Proposition 7.3.1 ([AC21, Prop. 5.1]). *Let $\hat{\Phi} \in G(\mathcal{H}_{\mathcal{S}}(\mathcal{A}))$ be the $\mathcal{S}$-lifting of φ. If $\hat{\rho} = \log^*(\hat{\Phi})$, then $\rho := \hat{\rho} \circ \iota$ is the infinitesimal lifting of the monotone cumulants of φ.*

Using the above proposition and the definition of ι, we have that the infinitesimal lifting of the monotone cumulants can be written as

$$\rho(w) = \sum_{t \in \mathrm{ST}(|w|)} \hat{\rho}(t \otimes w), \tag{7.3.1}$$

for any $w \in T_+(\mathcal{A})$. On the other hand, recall the expansion

$$\hat{\Phi} = \exp^*(\hat{\rho}) \Leftrightarrow \hat{\rho} = \log^*(\hat{\Phi}) = \log^*\left(\varepsilon^{(\mathcal{A})} + (\hat{\Phi} - \varepsilon^{(\mathcal{A})})\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \hat{\Phi}_0^{*k}, \qquad (7.3.2)$$

where $\hat{\Phi}_0 = \hat{\Phi} - \varepsilon^{(\mathcal{A})}$.

7.3.1 The Murua coefficient of a rooted tree

In order to find an expression for the k-fold convolution product $\hat{\Phi}_0^{*k}$, we need to describe the following coefficients that will be a fundamental piece in the results of Chapter 8.

Before stating the following definition, we should note the fact that any (non-planar or planar) rooted tree can be considered as a poset (Remark 2.2.9). Furthermore, in this context, a forest can be regarded as a poset formed by a disjoint union of the posets defined by its trees.

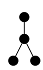
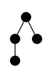
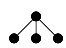
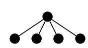
Definition 7.3.2 (Murua coefficients ([Mur06, Def. 12])). Let t be a (non-planar or planar) rooted tree seen as a poset with $n = |t|$ vertices. For any integer $0 < k < n+1$, we denote by $\omega_k(t)$ the number of surjective functions $f : t \to \{1, \ldots, k\}$ such that for any two vertices v, w in t with $v < w$, we have that $f(v) < f(w)$. We then define the *Murua coefficient of* t, denoted by $\omega(t)$, as the quantity given by the formula

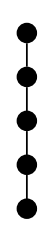
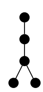
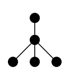
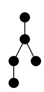
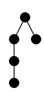
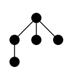
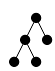
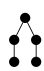
$$\omega(t) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \omega_k(t). \qquad (7.3.3)$$

If f is a forest consisting of trees $t_1, \ldots, t_m$, we extend the definition multiplicatively by

$$\omega(f) = \omega(t_1) \cdots \omega(t_m).$$

Example 7.3.3. The following tables show some values of Murua coefficients that can be computed by hand.

t									
$\omega(t)$	1	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{12}$	0	$-\frac{1}{30}$

t								
$\omega(t)$	$\frac{1}{5}$	$\frac{3}{20}$	$\frac{1}{30}$	$\frac{1}{10}$	$\frac{1}{20}$	$-\frac{1}{60}$	$\frac{1}{60}$	$\frac{1}{30}$

More generally, if ℓ_n stands for the *ladder tree of* n *vertices*, i.e. a tree with n vertices such that every internal vertex has exactly one child, then $\omega(\ell_n) = (-1)^{n-1}/n$. Furthermore, it is possible to show that if $\mathfrak{c}_n$ stands for a corolla with n leaves, then $\omega(\mathfrak{c}_n) = B_n$, where B_n stands for the n-th *Bernoulli number* ([Mur06, Ex. 9]). In particular, $\omega(\mathfrak{c}_{2n+1}) = 0$ for any $n \geq 1$.

Murua coefficients appeared in the work of Murua [Mur06] on the Baker-Campbell-Hausdorff (BCH) formula, where the author explicitly computed the expansion of the logarithm $\log(\exp(x_1)\cdots\exp(x_n))$ in a Hall basis. It is also noticeable that Murua coefficients have also appeared in the work of P. Chartier, E. Hairer and G. Vilmart in the context of the numerical analysis of PDEs [Mur06, Rmk. 12], and more recently, in the context of non-commutative probability [CEFPP21, AC21, CP22].

For the moment, we will not discuss the relation of Murua coefficients with the Magnus expansion. However, let us state a crucial recursive identity that will be useful in the subsequent chapter. Recall that, for a rooted tree t, $B^-(t)$ stands for the forest obtained by deleting the root of tree t (Definition 2.2.10). Also, $\{B_n\}_{n\geq 0}$ stands for the sequence of Bernoulli numbers.

Proposition 7.3.4 ([Mur06, Rmk. 11])**.** *For any rooted tree t with $|t| > 1$, we have that*

$$\omega(t) = \sum_{s\in K(B^-(t))} \frac{B_{|s|}}{s!}\omega(C^s(B^-(t))), \tag{7.3.4}$$

where, for f a forest, $K(f)$ stands for the multiset of subforests of f that contain all the roots of the trees of f, and where, if $s \in K(f)$, $C^s(f)$ stands for the forest obtained from f by removing the edges that connect the vertices of s with their parents.

Remark 7.3.5. It is fundamental to observe that in Proposition 7.3.4, the forests f and trees t are viewed as posets, as explained in Remark 2.2.9. A subforest of f or t is a subset of f or t equipped with the induced order. Notice that $C^s(f)$ has the same set of vertices as f since, in the definition of $C^s(f)$, edges are removed but not vertices.

7.3.2 Proof of the formula for $\hat{\rho}$

The starting point of our proof strategy is the development in (7.3.2). Fix $t \in \mathrm{ST}(n)$, elements $a_1, \dots, a_n \in \mathcal{A}$, and $k \in [n]$. Our aim is to compute $\hat{\Phi}_0^{*k}(t \otimes a_1 \cdots a_n)$. First, observe that, by definition of the convolution product, we have

$$\hat{\Phi}_0^{*k}(t \otimes w) = m_{\mathbb{C}}^{[k]} \circ \hat{\Phi}_0^{\otimes k} \circ (\delta_{\mathcal{S}}^{(\mathcal{A})})^{[k]}(t \otimes w), \tag{7.3.5}$$

where $(\delta_{\mathcal{S}}^{(\mathcal{A})})^{[k]}$ is the k-fold iterated coproduct

$$(\delta_{\mathcal{S}}^{(\mathcal{A})})^{[k]} = (\delta_{\mathcal{S}}^{(\mathcal{A})} \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{S}}(\mathcal{A})}^{\otimes k-2}) \circ (\delta_{\mathcal{S}}^{(\mathcal{A})})^{[k-1]},$$

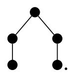
with $(\delta_{\mathcal{S}}^{(\mathcal{A})})^{[2]} = \delta_{\mathcal{S}}^{(\mathcal{A})}$, $(\delta_{\mathcal{S}}^{(\mathcal{A})})^{[1]} = \mathrm{id}_{\mathcal{H}_{\mathcal{S}}(\mathcal{A})}$, and, similarly, $m_{\mathbb{C}}^{[k]}$ stands for the multiplication of complex numbers $m_{\mathbb{C}}^{[k]}(z_1 \otimes \cdots \otimes z_k) = z_1 \cdots z_k$.

Let us introduce the following combinatorial notion associated to a Schröder tree which will allow us to relate Schröder trees to regular rooted trees.

Definition 7.3.6 ([MNT17, Thm. 11.3])**.** Let t be a Schröder tree with m internal vertices. We define the *skeleton of* t, denoted by $\mathrm{sk}(t)$, to be the planar rooted tree with m vertices each of them in correspondence with a unique internal vertex of t, and such that there is an arrow from v to w in $\mathrm{sk}(t)$ if and only if, the corresponding vertex to w in t is a child of the corresponding vertex to v in t.

In other words, the skeleton of a Schröder tree t is a subtree of t induced by its set of internal vertices.

Example 7.3.7. Let t be the underlying Schröder tree depicted in Figure 7.1. Its skeleton $\mathrm{sk}(t)$ is given by the tree



The usefulness of the notion of skeleton of t a Schröder tree is that $\mathcal{S}$-admissible cuts of t corresponds bijectively to (usual) admissible cuts of $\mathrm{sk}(t)$ (Definition 2.2.12), with the convention that the $\mathcal{S}$-admissible cut $c = \{\mathrm{rt}(t)\}$, where $\mathrm{rt}(t)$ stands for the root of t, corresponds to the admissible cut of $\mathrm{sk}(t)$ only containing the *stem of* $\mathrm{sk}(t)$, i.e. an invisible upward edge from the root of $\mathrm{sk}(t)$. Notice that this admissible cut will produce the term $\mathbf{1} \otimes t$ in (2.2.5).

Remark 7.3.8. For the following proofs, we will consider a usual admissible cut of a rooted tree t as a subset of vertices of t instead of a subset of edges. Indeed, if c is an admissible cut of t, the bijective correspondence for each edge $e \in c$ is given by the respective end vertex of e. In particular, the vertex associated to the stem of t is precisely the root of t.

Next, we will introduce some notation that will help to write the iterated coproduct $(\delta_{\mathcal{S}}^{(\mathcal{A})})^{[k]}$. Let t be a Schröder tree with l internal vertices. For each $k \in [l]$, define the set $\widetilde{\mathrm{Adm}}_k(t)$ of sequences $(c_1, \dots, c_{k-1})$ whose elements satisfying the following conditions:

i) c_1 is an admissible cut of $\mathcal{R}_0(\mathrm{sk}(t)) := \mathrm{sk}(t)$. Then we set $\mathcal{R}_1(\mathrm{sk}(t)) = R_{c_1}\big(\mathcal{R}_0(\mathrm{sk}(t))\big)$ and $\mathcal{P}_1(\mathrm{sk}(t)) = P_{c_1}\big(\mathcal{R}_0(\mathrm{sk}(t))\big)$.

ii) Inductively, if $1 \le j \le k-1$, c_j is an admissible cut of $\mathcal{R}_{j-1}(\mathrm{sk}(t))$. Afterwards, we set $\mathcal{R}_j(\mathrm{sk}(t)) = R_{c_j}\big(\mathcal{R}_{j-1}(\mathrm{sk}(t))\big)$ and $\mathcal{P}_j(\mathrm{sk}(t)) = P_{c_j}\big(\mathcal{R}_{j-1}(\mathrm{sk}(t))\big)$.

iii) $\mathcal{R}_{k-1}(\mathrm{sk}(t)) = \{\mathrm{rt}(\mathrm{sk}(t))\}$ and $\mathcal{P}_j(\mathrm{sk}(t))$ is a non-empty forest of single-vertex trees, for every $1 \le j \le k-1$.

Remark 7.3.9. Let t be a Schröder tree and $\tilde{c} = (c_1, \ldots, c_{k-1}) \in \widetilde{\mathrm{Adm}}_k(\mathrm{sk}(t))$. By Remark 7.3.8, the sequence $\tilde{c}$ is considered as a sequence of subsets of $V(t)$. Also, by definition of $\widetilde{\mathrm{Adm}}_k(t)$, we have that $\mathcal{R}_{k-1}(\mathrm{sk}(t)) = \{\mathrm{rt}(\mathrm{sk}(t))\}$. The latter implies that $\tilde{c}$ is an ordered partition of $V(\mathrm{sk}(t))\backslash\{\mathrm{rt}(\mathrm{sk}(t))\}$. Furthermore, since any $\mathcal{P}_j(\mathrm{sk}(t))$ is a non-empty forest of single-vertex trees, it readily follows that c_j is a non-empty subset of the set of leaves of $\mathcal{R}_{j-1}(\mathrm{sk}(t))$ and $\mathcal{P}_j(\mathrm{sk}(t)) = c_j$, for any $j \in [k-1]$.

The first lemma in this section shows how the particular definition of $\widetilde{\mathrm{Adm}}_k(t)$ is nicely related to Murua coefficients.

Lemma 7.3.10 ([AC21, Lem. 5.10])**.** *Let t be a Schröder tree and $k \geq 1$. Then there is a bijection between $\widetilde{\mathrm{Adm}}_k(t)$ and the set of strictly order-preserving surjective functions $f : \mathrm{sk}(t) \to [k]$.*

Proof. Take a sequence $\tilde{c} = (c_1, \ldots, c_{k-1}) \in \widetilde{\mathrm{Adm}}_k(t)$ and consider $\mathrm{sk}(t)$ as a poset. We define the function $f : \mathrm{sk}(t) \to [k]$ by $f(\mathrm{rt}(\mathrm{sk}(t))) = 1$, and $f(v) = k - j + 1$ if $v \in c_j$. Recall that, for any $j \in [k-1]$, $c_j \subset V(\mathrm{sk}(t))$ (Remark 7.3.8).

First, we show that the previous map is well-defined. Indeed, note that $\big(c_1, \ldots, c_k, \{\mathrm{rt}(\mathrm{sk}(t))\}\big)$ is an ordered partition of the poset $\mathrm{sk}(t)$. Moreover, since every c_j is a non-empty set, we have that f is surjective. On the other hand, take v, w vertices of $\mathrm{sk}(t)$ such that $v < w$, i.e. there is a downward directed path from v to w in $\mathrm{sk}(t)$. The elements of the subsequence $(c_1, \ldots, c_j)$ are consecutive prunings of $\mathrm{sk}(t)$ and these are taken from the leaves of $\mathcal{R}_{j-1}(\mathrm{sk}(t))$ upward the root as j increases. Then we have that $v \in c_i, w \in c_j$ and $j < i$, and in particular $f(v) = k - i + 1 < k - j + 1 = f(w)$. Hence f is strictly order-preserving.

Now, we will prove that the aforementioned correspondence $\tilde{c} \mapsto f$ is bijective. It is clear that two different elements in $\widetilde{\mathrm{Adm}}_k(t)$ will produce two different functions. It remains to show that our correspondence is surjective. To this end, for any $f : \mathrm{sk}(t) \to [k]$ strictly order-preserving surjective function, define the sequence of sets $(c_1, \ldots, c_{k-1})$ by

$$c_j = \{v \in \mathrm{sk}(t) \, : \, f(v) = k - j + 1\},$$

for any $1 \leq j < k$. Now, observe that c_1 is an admissible cut of $\mathrm{sk}(t)$ since c_1 is a subset of the set of leaves of $\mathrm{sk}(t)$. Actually, if c_1 contains a vertex v that is not a leaf, there exists a descendant of v, namely w such that $k = f(v) < f(w)$ and this is a contradiction. Hence c_1 is an admissible cut such that $\mathcal{P}_1(\mathrm{sk}(t)) = P_{c_1}(\mathrm{sk}(t)) = c_1$, i.e. $\mathcal{P}_1(\mathrm{sk}(t))$ is a non-empty forest of single-vertex trees. Observe that the same argument can be applied to the tree $\mathcal{R}_1(\mathrm{sk}(t))$ in order to show that c_2 is an admissible cut such that $\mathcal{P}_2(\mathrm{sk}(t)) = c_2$ is a non-empty forest of single-vertex trees, and in general, c_j is an admissible cut such that $\mathcal{P}_j(\mathrm{sk}(t)) = c_j$ is a non-empty forest of single-vertex trees, for any $j \in [k-1]$. Since f is strictly order-preserving, we have that $f^{-1}(1) = \{\mathrm{rt}(\mathrm{sk}(t))\}$. In addition, since f is surjective, we have that $(c_1, \ldots, c_{k-1}, \{\mathrm{rt}(\mathrm{sk}(t))\})$ is an ordered partition of the set of vertices of $\mathrm{sk}(t)$ with $\mathcal{R}_{k-1}(\mathrm{sk}(t)) = \{\mathrm{rt}(\mathrm{sk}(t))\}$. The above argument shows that $(c_1, \ldots, c_{k-1}) \in \widetilde{\mathrm{Adm}}_k(t)$ is a tuple such that their corresponding strictly order-preserving surjective function is f. Hence the map $\tilde{c} \mapsto f$ is surjective, and the proof is complete. □

As a consequence, we immediately have:

Corollary 7.3.11. *Let t be a Schröder tree and $k \geq 1$. Then $|\widetilde{\mathrm{Adm}}_k(t)| = \omega_k(t)$ (Definition 7.3.2).*

The main lemma of this section is stated below. It establishes the connection between Murua coefficients and the moments of the random variables $a_1, \ldots, a_n \in \mathcal{A}$.

Lemma 7.3.12. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Also, let $t \in \mathrm{ST}(n)$ and a word $w = a_1 \cdots a_n \in T_+(\mathcal{A})$, for $n \geq 1$. If $\hat{\Phi} \in G(\mathcal{H}_{\mathcal{S}}(\mathcal{A}))$ is the $\mathcal{S}$-lifting of φ defined in (7.2.1) and $\hat{\rho} = \log^*(\hat{\Phi})$, then we have*

$$\hat{\rho}(t \otimes w) = \omega(\mathrm{sk}(t))\varphi_{\pi(t)}(a_1, \ldots, a_n). \tag{7.3.6}$$

Proof. Recall that

$$\hat{\rho}(t \otimes w) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \hat{\Phi}_0^{*k}(t \otimes a_1 \cdots a_n),$$

where $\hat{\Phi}_0^{*k}$ is defined in (7.3.5). By the definition of Murua coefficients, it is enough to show that

$$\hat{\Phi}_0^{*k}(t \otimes a_1 \cdots a_n) = \omega_k(\mathrm{sk}(t))\varphi_{\pi(t)}(a_1, \ldots, a_n), \qquad \text{for any } k \geq 1. \tag{7.3.7}$$

Let $k \geq 1$. First, we observe that according to the definition of $\delta_{\mathcal{S}}^{(\mathcal{A})}$, the k-fold iterated coproduct is of the form

$$(\delta_{\mathcal{S}}^{(\mathcal{A})})^{[k]}(t \otimes a_1 \cdots a_n) = \sum_{\text{iterated admissible cuts of } t} f_1 \otimes \cdots \otimes f_k, \tag{7.3.8}$$

where f_1 is a decorated subtree of t, each $f_2, \ldots, f_k$ is a forest of decorated subtrees of t, and the sum is over iterated admissible cuts, i.e. doing admissible cuts on each iteration of the coproduct. Now observe that for any decorated tree, its evaluation on $\hat{\Phi}_0 = \hat{\Phi} - \varepsilon^{(\mathcal{A})}$ is equal to zero if the tree is empty or is not a corolla. In particular, a term $f_1 \otimes \cdots \otimes f_k$ in the sum (7.3.8) will produce a zero contribution on $\hat{\Phi}_0^{*k}$ if any of $f_1, \ldots, f_k$ is the empty forest or is a forest which contains a tree that is not a corolla. Hence, the only terms that can produce a non-zero contribution are such that f_1 is a corolla, and each of $f_2, \ldots, f_k$ is a forest of corollas. On the other hand, since corollas are associated to single-vertex trees via the skeleton map, by Remark 7.3.8 the sum in (7.3.8) is done precisely over the set $\widetilde{\mathrm{Adm}}_k(t)$.

In addition, if for any v internal vertex of t we denote by c_v the decorated corolla given by the decorated subtree of t consisting of v and its children, the fact that $\hat{\Phi}_0$ is multiplicative implies that

$$\hat{\Phi}_0(f_1)\cdots\hat{\Phi}_0(f_k) = \prod_{v \text{ internal vertex of } t} \hat{\Phi}(c_v) = \prod_{V\in\pi(t)} \varphi_{|V|}(a_1,\dots,a_n|V) = \varphi_{\pi(t)}(a_1,\dots,a_n),$$

where $\pi(t)$ stands for the non-crossing partition associated to the Schröder tree t (Definition 7.2.4). Finally Lemma 7.3.10 implies that the k-fold convolution product is equal to

$$\begin{aligned}\hat{\Phi}_0^{*k}(t\otimes a_1\cdots a_n) &= \sum_{\text{iterated admissible cuts of } t} \hat{\Phi}_0(f_1)\cdots\hat{\Phi}_0(f_k)\\ &= \sum_{\tilde{c}\in\widetilde{\mathrm{Adm}}_k(t)} \varphi_{\pi(t)}(a_1,\dots,a_n)\\ &= \omega_k(\mathrm{sk}(t))\varphi_{\pi(t)}(a_1,\dots,a_n),\end{aligned}$$

since $\varphi_{\pi(t)}(a_1,\dots,a_n)$ does not depend of $\tilde{c}\in\widetilde{\mathrm{Adm}}_k(t)$. □

The previous lemma is the bulk of the developments in this chapter since the formula for monotone cumulants, in terms of moments and Schröder trees, is now straightforward.

Theorem 7.3.13. *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and let $\{h_n:\mathcal{A}^n\to\mathbb{C}\}_{n\geq 1}$ be the family of monotone cumulants of $(\mathcal{A},\varphi)$. Then we have that*

$$h_n(a_1,\dots,a_n) = \sum_{t\in\mathrm{ST}(n)} \omega(\mathrm{sk}(t))\varphi_{\pi(t)}(a_1,\dots,a_n), \tag{7.3.9}$$

for any $n\geq 1$ and $a_1,\dots,a_n\in\mathcal{A}$.

Proof. Let ρ and $\hat{\rho}$ be the infinitesimal liftings of the monotone cumulants $\{h_n\}_{n\geq 1}$ on $T(T_+(\mathcal{A}))$ and $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$, respectively. By the relation $\rho=\hat{\rho}\circ\iota$ and Lemma 7.3.12 we conclude:

$$\begin{aligned}h_n(a_1,\dots,a_n) &= \rho(a_1\cdots a_n)\\ &= \sum_{t\in\mathrm{ST}(n)} \hat{\rho}(t\otimes a_1\cdots a_n)\\ &= \sum_{t\in\mathrm{ST}(n)} \omega(\mathrm{sk}(t))\varphi_{\pi(t)}(a_1,\dots,a_n).\end{aligned}$$

□

The desired combinatorial formula for writing monotone cumulants in terms of moments via non-crossing partitions is now an obvious corollary from the formula in terms of Schröder trees.

Corollary 7.3.14. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ be the family of monotone cumulants of $(\mathcal{A}, \varphi)$. Then we have that*

$$h_n(a_1, \dots, a_n) = \sum_{\pi \in \mathrm{NC}(n)} \alpha(\pi)\varphi_\pi(a_1, \dots, a_n), \tag{7.3.10}$$

for any $n \geq 1$ and $a_1, \dots, a_n \in \mathcal{A}$, where

$$\alpha(\pi) = \sum_{\substack{t \in \mathrm{ST}(n) \\ \pi(t) = \pi}} \omega(\mathrm{sk}(t)), \qquad \forall\, \pi \in \mathrm{NC}(n). \tag{7.3.11}$$

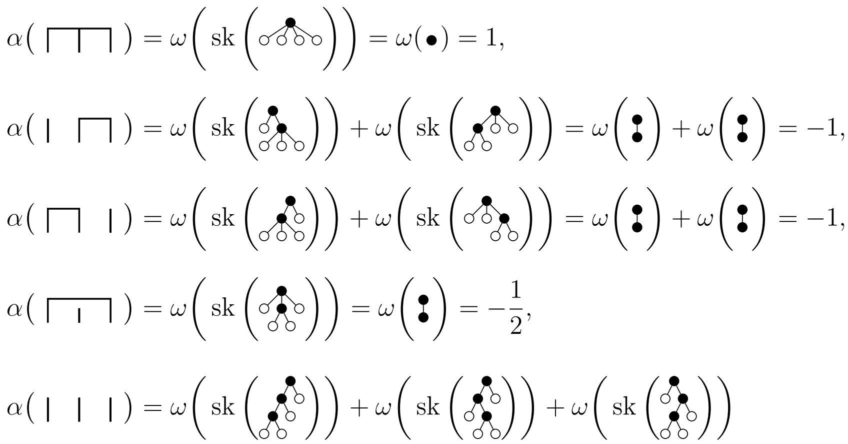
Example 7.3.15. We will verify the formula provided by Corollary 7.3.14 for the case $n = 3$. From the monotone moment-cumulant formula (3.4.10), one can get the first cumulants in terms of moments

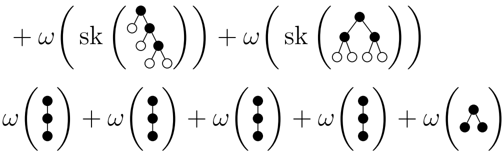
$$\begin{aligned}
\varphi_1(a_1) &= h_1(a_1), \\
\varphi_2(a_1, a_2) &= h_2(a_1, a_2) + h_1(a_1)h_1(a_2), \\
\varphi_3(a_1, a_2, a_3) &= h_3(a_1, a_2, a_3) + h_2(a_1, a_2)h_1(a_3) + h_2(a_2, a_3)h_1(a_1) \\
&\quad + \frac{1}{2}h_2(a_1, a_3)h_1(a_2) + h_1(a_1)h_1(a_2)h_1(a_3).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
h_3(a_1, a_2, a_3) &= \varphi_3(a_1, a_2, a_3) - \varphi_1(a_1)\varphi_2(a_2, a_3) - \varphi_1(a_3)\varphi_2(a_2, a_3) \\
&\quad - \frac{1}{2}\varphi_1(a_2)\varphi_2(a_1, a_3) + \frac{3}{2}\varphi_1(a_1)\varphi_1(a_2)\varphi_1(a_3).
\end{aligned} \tag{7.3.12}$$

On the other hand, by using Corollary 7.3.14, the Schröder trees listed in Example 7.1.2 and the table with the values of ω, we have that the respective coefficients α are given by

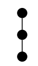
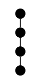
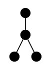
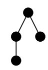
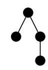




$$= \frac{3}{2}.$$

We can observe that the obtained α coefficients match the coefficients appearing in (7.3.12).

Example 7.3.16. In this example, we will verify Corollary 7.3.14 for the univariate monotone cumulant-moment formula of order $n = 4$. One can check that Schröder trees with 5 leaves can have the following skeleton and associated Murua coefficients:

	$a)$	$b)$	$c)$	$d)$	$e)$	$f)$	$g)$	$h)$
t								
$\omega(t)$	1	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{12}$	$-\frac{1}{12}$

We can use these values to compute $\rho(t \otimes a^4)$. Denote $m_n = \varphi(a^n)$ and $h_n = h_n(a)$, for each $n \geq 1$. By using the monotone moment-cumulant formula in terms of non-crossing partitions, we have that

$$m_4 = h_4 + 3h_2h_1^2 + \frac{3}{2}h_2^2 + \frac{13}{3}h_2h_1^2 + h_1^4,$$

where we can get

$$h_4 = m_4 - 3m_3m_1 - \frac{3}{2}m_2^2 + \frac{37}{6}m_2m_1^2 - \frac{8}{3}m_1^4.$$

Now we compute the right-hand side of the formula in terms of Schröder trees. It is known that there are 45 Schröder trees with five leaves. The summary of the complete list is given below:

- 1 tree with skeleton $a)$: it corresponds to the corolla with five leaves, $\mathfrak{c}_5$, and the associated moment is m_4;
- 9 trees with skeleton $b)$: six of those trees have associated moment m_1m_3, and the remaining three have associated moment m_2^2;
- 16 trees with skeleton $c)$: they have associated the moment $m_1^2m_2$;
- 5 with skeleton $d)$: they have associated the moment $m_1^2m_2$;

The remaining 14 Schröder trees whose skeletons are of the form $e)$–$h)$ are precisely the 14 planar binary trees with 4 internal vertices. Every of these Schröder trees has associated the moment m_1^4. The precise description of the skeletons of the planar binary trees with 4 internal vertices is the following:

- 8 with skeleton e);
- 2 with skeleton f);
- 2 with skeleton g);
- 2 with skeleton h).

Finally, we have that

$$\begin{aligned}\rho(a^4) &= \sum_{t\in \mathrm{ST}(4)} \hat{\rho}(t\otimes a^4)\\ &= m_4 - \frac{1}{2}(6m_1m_3+3m_2^2) + \frac{16}{3}m_1^2m_2 + \frac{5}{6}m_1^2m_2\\ &\quad -\frac{8}{4}m_1^4 - \frac{2}{6}m_1^4 - \frac{2}{12}m_1^4 - \frac{2}{12}m_1^4\\ &= m_4 - 3m_1m_3 - \frac{3}{2}m_2^2 + \frac{37}{6}m_1^2m_2 - \frac{8}{3}m_1^4,\end{aligned}$$

which agrees with the previous computation.

As a manner to conclude the current chapter, we collect the results obtained in Proposition 7.2.9 and Theorem 7.3.13 together with the result from [JVMNT17] for free cumulants. Thereby, the next theorem provides new cumulant-to-moment formulas indexed by Schröder trees instead of non-crossing partitions.

Theorem 7.3.17 ([AC21, Prop. 6.3])**.** *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and consider $\hat{\Phi}:\mathcal{H}_{\mathcal{S}}(\mathcal{A})\to\mathbb{C}$ the $\mathcal{S}$-lifting of φ defined in (7.2.1). Let $(\hat{\kappa},\hat{\beta},\hat{\rho})$ be the triple of infinitesimal characters on $\mathcal{H}_{\mathcal{S}}(\mathcal{A})$ satisfying the identities*

$$\hat{\Phi} = \mathcal{E}_{\prec}(\hat{\kappa}) = \mathcal{E}_{\succ}(\hat{\beta}) = \exp^*(\hat{\rho}).$$

Then, for any $t\in\mathrm{ST}(n)$ and $a_1,\dots,a_n\in\mathcal{A}$, the triple of infinitesimal characters is given by

$$\begin{aligned}\hat{\kappa}(t\otimes a_1\cdots a_n) &= \begin{cases}(-1)^{|\pi(t)|-1}\varphi_{\pi(t)}(a_1,\dots,a_n) & \text{if } t\in\mathrm{PST}(n),\\ 0 & \text{otherwise,}\end{cases}\\ \hat{\beta}(t\otimes a_1\cdots a_n) &= \begin{cases}(-1)^{|\pi(t)|-1}\varphi_{\pi(t)}(a_1,\dots,a_n) & \text{if } t\in\mathrm{BST}(n)\\ 0 & \text{otherwise,}\end{cases}\\ \hat{\rho}(t\otimes a_1\cdots a_n) &= \omega(\mathrm{sk}(t))\varphi_{\pi(t)}(a_1,\dots,a_n).\end{aligned}$$

Moreover, the evaluations of $\hat{\kappa}\circ\iota$, $\hat{\beta}\circ\iota$ and $\hat{\rho}\circ\iota$ on a word $w=a_1\cdots a_n\in\mathcal{A}^{\otimes n}$ coincide *with the free, Boolean and monotone cumulants of* $a_1, \ldots, a_n$*, respectively:*

$$(\hat{\kappa} \circ \iota)(w) = k_n(a_1, \ldots, a_n), \quad (\hat{\beta} \circ \iota)(w) = b_n(a_1, \ldots, a_n), \quad (\hat{\rho} \circ \iota)(w) = h_n(a_1, \ldots, a_n),$$

where ι *is the coalgebra morphism given in Theorem 7.1.13.*

Chapter 8

Monotone Cumulant-Cumulant Formulas from Pre-Lie Magnus Expansion

The present chapter is devoted to solving a relevant problem in non-commutative probability theory: to find combinatorial formulas, in terms of non-crossing partitions, that write the multivariate monotone cumulants of a sequence of random variables in terms of their free and Boolean cumulants. This problem was attacked initially by the authors in [AHLV15] to complement the cumulant-cumulant formulas in Theorem 3.4.25 in the attempt to have a complete picture of the combinatorial relations between cumulants in non-commutative probability. Indeed, the authors in [AHLV15] provided a combinatorial formula for writing univariate monotone cumulants in terms of univariate free cumulants. However, the multivariate case was left open.

The exposition of this chapter is based on the work [CEFPP21]. In the former, the formulas for the multivariate case were stated and proved via the Hopf-algebraic framework for non-commutative probability of Ebrahimi-Fard and Patras presented in Chapter 4 of this manuscript. Our strategy to attack the problem of interest is described in Section 8.1, where the starting point is the pre-Lie-algebraic relation $\rho = \Omega(\beta) = -\Omega(-\kappa)$ between the infinitesimal characters associated to the monotone, Boolean and free cumulants (Theorem 4.2.9). To this end, we give a concrete combinatorial description of the pre-Lie product on the pre-Lie algebra of infinitesimal characters on the double tensor algebra $T(T_+(\mathcal{A}))$. Moreover, we mention how this pre-Lie algebra can be identified with a certain pre-Lie algebra of words. Later, in Section 8.2, the description of the pre-Lie product generalizes to a combinatorial description of the iterated right pre-Lie products $r^{(n)}_{\triangleleft\rho}(\beta)$, required to compute the pre-Lie Magnus expansion $\Omega(\beta)$. Finally, Section 8.3 is devoted to finding and proving the desired combinatorial formulas between cumulants. The main task is identifying and describing the corresponding coefficients that define the transition from free and Boolean cumulants to monotone cumulants. It will turn out that these coefficients have already appeared in Chapter 7, in the context of expressing monotone cumulants of random variables in terms of their moments.

The main results described in this chapter, as well as the ideas for their proofs, are based on the work [CEFPP21].

8.1 The pre-Lie algebra of infinitesimal characters $\mathfrak{g}$

The objective of this section is to make explicit the pre-Lie algebra structure underlying in the space of infinitesimal characters, denoted by $\mathfrak{g}$, on a double tensor algebra $T(T_+(\mathcal{A}))$. Having a concise formula for the pre-Lie product will facilitate the subsequent job of computing the iterations of such a product, which are required to calculate the pre-Lie Magnus expansion of an infinitesimal character. Thereafter, we will consider an equivalent pre-Lie algebra of words that will help to have a deeper understanding of the computations done in the first sections of this chapter.

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. By the theory studied in Chapter 4, in particular Theorem 4.2.6, Theorem 4.2.9 and Theorem 4.3.2, we have that

$$\beta = W(\rho), \quad \kappa = -W(-\rho)$$

and

$$\rho = \Omega(\beta) = -\Omega(-\kappa),$$

where κ, β and ρ are the infinitesimal liftings of the free, Boolean and monotone cumulants, respectively, of $(\mathcal{A}, \varphi)$, and the maps $W, \Omega : \mathfrak{g} \to \mathfrak{g}$ are the Agrachev-Gamkrelidze operator (Definition 2.3.25) and the pre-Lie Magnus operator (Definition 2.3.27) on $\mathfrak{g}$, respectively.

The formulas (3.4.16) and (3.4.17), originally proved in [AHLV15], that write Boolean and free cumulants in terms of the monotone cumulants can be approached in the shuffle algebra framework, for instance, in a similar fashion that is done in Theorem 6.3.1. However, the same method cannot be applied to the Magnus operator. Nevertheless, the pre-Lie algebraic relation $\rho = \Omega(\beta)$ can be alternatively approached by the recursive definition of the Magnus operator (Proposition 2.3.28)

$$\rho = \sum_{n \geq 0} \frac{B_n}{n!} r^{(n)}_{\lhd \rho}(\beta). \tag{8.1.1}$$

Therefore, our plan of action to obtain the sought formula is as follows:

1. Describe the pre-Lie product $\lhd$ explicitly on the pre-Lie algebra of infinitesimal characters on $T(T_+(\mathcal{A}))$.

2. Find a combinatorial formula, in terms of non-crossing partitions, of the iterations of the pre-Lie product.

3. Propose the corresponding coefficient for any non-crossing partition from the relation between the Bernoulli numbers and the iterated pre-Lie products on (8.1.1).

We start by proposing a formula for the pre-Lie product on $\mathfrak{g}$. To this end, take two infinitesimal characters $\alpha, \gamma \in \mathfrak{g}$. By Proposition 4.1.4, the shuffle algebra structure on $\mathrm{Lin}(T(T_+(\mathcal{A})), \mathbb{C})$ yields a right pre-Lie algebra structure on $\mathfrak{g}$ by the formula

$$\alpha \lhd \gamma = \alpha \prec \gamma - \gamma \succ \alpha.$$

We can obtain a description of the pre-Lie product by using the definitions of the half-shuffle products $\prec$ and $\succ$. Actually, if $w = a_1 \cdots a_n \in T_+(\mathcal{A})$, we have that

$$(\alpha \lhd \gamma)(w) = \sum_{1 \in S \subseteq [n]} \alpha(a_S)\gamma(a_{J^S_{[n]}}) - \sum_{1 \notin S \subset [n]} \gamma(a_S)\alpha(a_{J^S_{[n]}}). \tag{8.1.2}$$

Recall that α and γ are infinitesimal characters, so $\alpha(J_1|\ldots|J_r) = 0 = \gamma(J_1|\cdots|J_r)$, for $r \neq 1$.

Let us analyze the first sum in (8.1.2). Notice that, in order to produce a non-zero contribution, S should be of the form $\{1, 2, \ldots, j\}$ for $j < n$ or

$$\{1, \ldots, r, r+s+1, r+s+2, \ldots, n\}$$

for some integers $r, s \geq 1$. In the former case, the complement of S is the non-empty interval $\{j+1, \ldots, n\}$, while in the latter, the complement is the non-empty interval $\{r+1, r+2, \ldots, r+s\}$. Thus we can write

$$\sum_{1 \in S \subseteq [n]} \alpha(a_S)\gamma(a_{J^S_{[n]}}) = \sum_{j=1}^{n-1} \alpha(a_1 \cdots a_j)\gamma(a_{j+1} \cdots a_n) + \sum_{\substack{S \neq \emptyset \text{ interval of } [n] \\ 1, n \notin S}} \alpha(a_{[n] \setminus S})\gamma(a_S).$$

On the other hand, notice that the second sum in the above equation coincides with the second sum on the right-hand side of (8.1.2). Therefore we conclude that

$$(\alpha \lhd \gamma)(w) = \sum_{\substack{w_1 w_2 w_3 = w \\ |w_i| > 0}} \alpha(w_1 w_3)\gamma(w_2), \tag{8.1.3}$$

where the notation $w = w_1 w_2 w_3$ means that w is the concatenation of the words w_1, w_2 and w_3.

Observe our objects of interest will be linear functionals on $T(T_+(\mathcal{A}))$ associated to the cumulant functionals on $(\mathcal{A}, \varphi)$. Since we know that such functionals are infinitesimal characters on $T(T_+(\mathcal{A}))$, we can identify them with linear functionals on $T_+(\mathcal{A})$, and thereby we have $\mathfrak{g} \cong \mathrm{Lin}(T_+(\mathcal{A}), \mathbb{C}) =: L_{\mathfrak{g}}$. For the sake of simplicity, if $\alpha \in \mathfrak{g}$, then its associated element in $L_{\mathfrak{g}}$ is also denoted by α.

Remark 8.1.1. Recall that L is a *locally finite connected graded vector space* if $L = \bigoplus_{n\geq 0} L_n$, where $L_0 = 0$ and each L_n is a finite-dimensional vector space, for $n \geq 1$. In this context, we can identify L with its graded dual $L^* = \bigoplus_{n\geq 0} L_n^*$, where we denote $V^* := \mathrm{Lin}(V, \mathbb{C})$ for any vector space V. In particular, for X a *finite alphabet*, we define L_X as the linear span of the set of non-empty words, denoted by $X^\star$, on the alphabet X. It is clear that L_X is locally finite, connected and graded, where $(L_X)_n$ is the linear span of the set of words of length n, for any $n \geq 0$. Thus, by defining the scalar product making $X^\star$ an orthonormal basis of L_X:

$$\langle \alpha | w \rangle = \begin{cases} 1 & \text{if } w = \alpha \\ 0 & \text{otherwise} \end{cases}, \qquad \forall\, \alpha, w \in X^\star, \tag{8.1.4}$$

we can identify L_X with its graded dual L_X^*. Notice that when $\mathcal{A}$ is finite-dimensional with basis $X_\mathcal{A}$, then $L_\mathfrak{g} = L_{X_\mathcal{A}}$. This observation will be relevant in the next chapter.

Proposition 8.1.2 ([CEFPP21, Prop. 1])**.** *Let $\mathcal{A}$ be a vector space and $L_\mathfrak{g}$ be the dual linear space of $T_+(\mathcal{A})$. Then $(L, \lhd)$ is a right pre-Lie algebra, with $\lhd$ given in (8.1.3).*

Proof. The development yielding (8.1.3) is actually a proof of the formula. Nevertheless, we will prove the proposition directly by verifying the right pre-Lie identity (2.3.3). With this purpose, take $\alpha, \gamma, \xi \in L_\mathfrak{g}$. Then, for a word $w = a_1 \cdots a_n$, we have that

$$\begin{aligned} ((\alpha \lhd \gamma) \lhd \xi)(w) &= \sum_{\substack{w_1 w_2 w_3 = w \\ |w_i|>0}} (\alpha \lhd \gamma)(w_1 w_3)\xi(w_2) \\ &= \sum_{\substack{w_1 w_2 w_3 = w \\ |w_i|>0}} \left(\sum_{\substack{w_{11} w_{12} w_{13} = w_1 \\ |w_{11}|,|w_{12}|>0}} \alpha(w_{11} w_{13} w_3)\gamma(w_{12})\xi(w_2) \right. \\ &\quad \left. + \sum_{\substack{w_{11} w_{12} = w_1 \\ w_{32} w_{33} = w_3 \\ |w_{11}|,|w_{33}|>0 \\ |w_{12}|,|w_{32}|>0}} \alpha(w_{11} w_{33})\gamma(w_{12} w_{32})\xi(w_2) + \sum_{\substack{w_{31} w_{32} w_{33} = w_3 \\ |w_{32}|,|w_{33}|>0}} \alpha(w_1 w_{31} w_{33})\gamma(w_{32})\xi(w_2) \right). \end{aligned}$$

On the other hand

$$\begin{aligned} (\alpha \lhd (\gamma \lhd \xi))(w) &= \sum_{\substack{w_1 w_2 w_3 = w \\ |w_i|>0}} \alpha(w_1 w_3)(\gamma \lhd \xi)(w_2) \\ &= \sum_{\substack{w_1 w_2 w_3 = w \\ |w_i|>0}} \sum_{\substack{w_{12} w_{22} w_{32} = w_2 \\ |w_{i2}|>0}} \alpha(w_1 w_3)\gamma(w_{12} w_{32})\xi(w_{22}). \end{aligned}$$

Therefore

$$\big((\alpha\lhd\gamma)\lhd\xi-\alpha\lhd(\gamma\lhd\xi)\big)(w)=\sum_{\substack{w_1w_2w_3w_4w_5=w\\|w_1|,|w_2|,|w_4|,|w_5|>0}}\Big(\alpha(w_1w_3w_5)\gamma(w_2)\xi(w_4)+\alpha(w_1w_3w_5)\gamma(w_4)\xi(w_2)\Big).$$

As the expression on the right-hand side is symmetric in γ and ξ, we obtain that it also computes

$$\big((\alpha\lhd\xi)\lhd\gamma-\alpha\lhd(\xi\lhd\gamma)\big)(w).$$

from which we conclude that the right pre-Lie identity holds. □

Remark 8.1.3. It is easy to see that the conditions over the sum defining $\lhd$ allow us to write (8.1.3) in terms of non-crossing partitions. More precisely, we have

$$(\alpha\lhd\gamma)(w)=\sum_{\substack{\pi\in\mathrm{NC}^2_{\mathrm{irr}}(n)\\\pi=\{V,W\},V\leq W}}\alpha(a_V)\gamma(a_W),\tag{8.1.5}$$

where we recall that $V\leq W$ means that the block W is nested in V, and $\mathrm{NC}^2_{\mathrm{irr}}(n)$ stands for the irreducible non-crossing partitions of $[n]$ with exactly two blocks.

The reader can notice that the proof of Proposition 8.1.2 is based on a specific way in which the word w is deconcatenated. By considering the discussion of Remark 8.1.1 as well as the duality defined by (8.1.4), we obtain the following result.

Proposition 8.1.4 ([CP22, Prop. 3.1]). *Let X be an alphabet and L_X be the vector space defined in Remark 8.1.1. Then $(L_X,\lhd)$ is a graded pre-Lie algebra, with $\lhd$ given by*

$$\alpha\lhd\gamma=\sum_{\substack{\alpha_1\alpha_2=\alpha\\\alpha_1,\alpha_2\neq\emptyset}}\alpha_1\gamma\alpha_2,\qquad\forall\,\alpha,\gamma\in X^\star.\tag{8.1.6}$$

The previous proposition prompts us to give a proper name to the vector space L_X.

Definition 8.1.5. Let X be an alphabet. The pre-Lie algebra L_X defined in Remark 8.1.1 is called the *pre-Lie algebra of words over X*.

8.2 Iterated pre-Lie products on $L_{\mathfrak{g}}$

Continuing with our project to obtain a formula that writes monotone cumulants in terms of free and Boolean cumulants, we have to find a combinatorial formula for the right iteration of the pre-Lie product on $L_{\mathfrak{g}}=\mathrm{Lin}(T_+(A),\mathbb{C})$.

We start by noticing that non-crossing partitions have already appeared in the writing of the pre-Lie product given in (8.1.5). Let us look into the second right iteration of the pre-Lie product. From the computations in the proof of Proposition 8.1.2, we can extract

$$
\begin{aligned}
((\gamma_3 \lhd \gamma_2) \lhd \gamma_1)(w) \;=\; & \sum_{\substack{w_1w_2w_3=w\\ |w_i|>0}} \left(\sum_{\substack{w_{11}w_{12}w_{13}=w_1\\ |w_{11}|,|w_{12}|>0}} \gamma_3(w_{11}w_{13}w_3)\gamma_2(w_{12})\gamma_1(w_2) \right.\\
& \left. + \sum_{\substack{w_{11}w_{12}=w_1\\ w_{32}w_{33}=w_3\\ |w_{11}|,|w_{33}|>0\\ |w_{12}|,|w_{32}|>0}} \gamma_3(w_{11}w_{33})\gamma_2(w_{12}w_{32})\gamma_1(w_2) + \sum_{\substack{w_{31}w_{32}w_{33}=w_3\\ |w_{32}|,|w_{33}|>0}} \gamma_3(w_1w_{31}w_{33})\gamma_2(w_{32})\gamma_1(w_2) \right),
\end{aligned}
$$

for any $\gamma_1, \gamma_2, \gamma_3 \in L_{\mathfrak{g}}$ and a word $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$. Analyzing how the deconcatenation of the word w is done on each of the sums in the above equation, we observe that such sums can be written in terms of irreducible non-crossing partitions with exactly three blocks as follows:

$$
\begin{aligned}
((\gamma_3 \lhd \gamma_2) \lhd \gamma_1)(w) = & \sum_{\substack{\pi\in \mathrm{NC}^3_{\mathrm{irr}}(m)\\ \pi=\{V_1,V_2,V_3\}\\ 1\in V_1, \max(V_2)<\min(V_3)}} \gamma_3(w_{V_1})\gamma_2(w_{V_2})\gamma_1(w_{V_3})\\
& + \sum_{\substack{\pi\in \mathrm{NC}^3_{\mathrm{irr}}(m)\\ \pi=\{V_1,V_2,V_3\}\\ 1\in V_1, V_2<V_3}} \gamma_3(w_{V_1})\gamma_2(w_{V_2})\gamma_1(w_{V_3}) + \sum_{\substack{\pi\in \mathrm{NC}^3_{\mathrm{irr}}(m)\\ \pi=\{V_1,V_2,V_3\}\\ 1\in V_1, \max(V_3)<\min(V_2)}} \gamma_3(w_{V_1})\gamma_2(w_{V_2})\gamma_1(w_{V_3}).
\end{aligned}
$$

Observe that the above sums can be equivalently described by considering the nesting tree $t(\pi)$ of an irreducible partition π. Indeed, the partitions indexing the first and the third sum above have a nesting tree of the form $t(\pi) =$ , and the second sum is indexed by partitions with $t(\pi) =$ . Furthermore, the fact that there is a distinction between the first and the third sum can be explained by considering the monotone labellings of the nesting tree. Actually, if $\pi = \{V_1, V_2, V_3\}$ with $1 \in V_1$, every monotone labelling of $t(\pi)$ will produce the respective factors $\gamma_3(w_{V_1})\gamma_2(w_{V_2})\gamma_1(w_{V_3})$ and $\gamma_3(w_{V_1})\gamma_2(w_{V_3})\gamma_1(w_{V_2})$.

As the previous analysis sheds light on the general form of the right iterated pre-Lie product, we can state the following proposition.

Proposition 8.2.1 ([CEFPP21, Prop. 4])**.** *Let $\gamma_1, \ldots, \gamma_{n+1} \in L_{\mathfrak{g}}$ and $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$ such that $|w| = m \geq n+2$, with $n > 0$. Then the right iteration of the pre-Lie product acts on w by*

$$
\big((\cdots(\gamma_{n+1} \lhd \gamma_n) \lhd \cdots) \lhd \gamma_1\big)(w) = \sum_{\substack{\pi\in \mathcal{M}^{n+1}_{\mathrm{irr}}(m)\\ \pi=(V_1,\ldots,V_{n+1})}} \gamma_{n+1}(w_{V_1}) \cdots \gamma_1(w_{V_{n+1}}), \tag{8.2.1}
$$

where $\mathcal{M}^{n+1}_{\mathrm{irr}}(m)$ stands for the set of monotone irreducible non-crossing partitions with *exactly $n+1$ blocks.*

Proof. First, note that if $|w| \leq n+1$, there will be at least one γ_j that will evaluate an empty word, so the iterated pre-Lie product on w will be equal to 0. Thereby, we can assume all the words are long enough to produce an evaluation different from 0. Now, we start the proof by induction on n. The base case $n = 1$ is precisely (8.1.5).

For the inductive step, assume that (8.2.1) holds for any n elements in $L_{\mathfrak{g}}$. We will prove that (8.2.1) holds for any $\gamma_1, \ldots, \gamma_{n+1} \in L_{\mathfrak{g}}$. To this end, consider the known fact that π is a non-crossing partition if and only if there exists an interval $V \in \pi$ such that $\pi' = \pi \backslash \{V\}$ is also a non-crossing partition. In particular, if $\pi \in \mathrm{NC}(m)$ is irreducible and different from the one-block partition 1_m, then the interval block V does not contain 1 nor m. Even more, if (π, λ) is a monotone partition, we can take the interval block as the block $V \in \pi$ such that $|\pi| = \lambda(V) > \lambda(W)$ for any other block $W \neq V$ in π. Observe that a block V that satisfies the previous condition is necessarily an interval since otherwise, it would contradict the fact that λ is a monotone labelling. Therefore, if we denote $I(m)$ the set of non-empty intervals of $[m]$ that do neither contain 1 nor m, we have a bijection

$$\{\pi \in \mathcal{M}_{\mathrm{irr}}^{n+1}(m)\} \leftrightarrow \{(\pi', V) : \pi' \in \mathcal{M}_{\mathrm{irr}}^{n}([m] \backslash V) \text{ and } V \in I(m)\}. \tag{8.2.2}$$

Then, by definition of $\lhd$, we have

$$\begin{aligned} &\big((\cdots(\gamma_{n+1} \lhd \gamma_n) \lhd \cdots) \lhd \gamma_1\big)(w) \\ &\quad = \sum_{V_{n+1} \in I(m)} \big((\cdots(\gamma_{n+1} \lhd \gamma_n) \lhd \cdots) \lhd \gamma_2\big)(w_{[m] \backslash V_{n+1}}) \gamma_1(w_{V_{n+1}}). \end{aligned} \tag{8.2.3}$$

On the other hand, the induction hypothesis tells us that

$$\begin{aligned} &\big((\cdots(\gamma_{n+1} \lhd \gamma_n) \lhd \cdots) \lhd \gamma_2\big)(w_{[m] \backslash V_{n+1}}) \\ &\quad = \sum_{\substack{\pi' \in \mathcal{M}_{\mathrm{irr}}^{n}([m] \backslash V_{n+1}) \\ \pi' = (V_1, \ldots, V_n)}} \gamma_{n+1}(w_{V_1}) \cdots \gamma_2(w_{V_n}), \end{aligned}$$

We conclude the inductive step by rearranging (8.2.3) by using the bijection stated in (8.2.2). □

Remark 8.2.2. The formula (8.2.1) for the right iteration of the pre-Lie product resembles the formula obtained in (6.2.8) for the action of the k-th convolution power of an infinitesimal character ρ^{*k}. Observe that in the former formula, the sum is done over irreducible monotone partitions with k blocks, while in the latter formula, the sum is over all the monotone partitions with k blocks.

Due to its importance in the recursive expansion of the Magnus operator, we state the particular expression of $r_{\lhd\rho}^{(n)}(\beta)$ as a corollary.

Corollary 8.2.3 ([CEFPP21, Cor. 3])**.** *Let $\beta, \rho \in L_{\mathfrak{g}}$ and $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$ be a word such that $|w| = m \geq n+2$, with $n > 0$. Then*

$$r^{(n)}_{\triangleleft\rho}(\beta)(w) = \sum_{\substack{\pi \in \mathrm{NC}^{n+1}_{\mathrm{irr}}(m) \\ 1,m \in V_1 \in \pi}} m(\pi)\beta(w_{V_1})\rho_{\pi\setminus\{V_1\}}(w_{[n]\setminus V_1}), \tag{8.2.4}$$

where $\rho_{\pi\setminus\{V_1\}}(w_{[n]\setminus V_1}) := \prod_{\substack{W\in\pi \\ W\neq V_1}} \rho(w_W)$, and in general $\rho_\sigma(w) = \prod_{W\in\sigma} \rho(w_W)$, for any $\sigma \in \mathrm{NC}(|w|)$.

Proof. Using Proposition 8.2.1 with the linear functionals $\gamma_{n+1} = \beta$ and $\gamma_1 = \cdots = \gamma_n = \rho$, we obtain

$$r^{(n)}_{\triangleleft\rho}(\beta)(w) = \sum_{\substack{\pi \in \mathcal{M}^{n+1}_{\mathrm{irr}}(m) \\ \pi=(V_1,\ldots,V_{n+1})}} \beta(w_{V_1})\rho_{\pi\setminus\{V_1\}}(w_{[n]\setminus V_1}).$$

Since the unique outer block is always labelled with 1, one can observe that any term in the sum of the right-hand side of the previous equation does not depend on the labelling of $\pi \in \mathrm{NC}^{n+1}_{\mathrm{irr}}(m)$. Since there are exactly $m(\pi) = \frac{|\pi|!}{t(\pi)!}$ many different monotone partitions associated to π, we obtain the conclusion in (8.2.4). $\square$

Remark 8.2.4. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and consider ρ, β and κ the infinitesimal liftings of the monotone, Boolean and free cumulants, respectively. Proposition 2.3.26 establishes that

$$W(\rho) = \sum_{n\geq 0} \frac{1}{(n+1)!} r^{(n)}_{\triangleleft\rho}(\rho).$$

Since $\beta = W(\rho)$, Corollary 8.2.3 provides a new purely pre-Lie algebraic proof of the relation between cumulants in (3.4.16). Indeed, if $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$ we have

$$\begin{aligned} \beta(w) &= \sum_{n\geq 0} \frac{1}{(n+1)!} r^{(n)}_{\triangleleft\rho}(\rho)(w) \\ &= \sum_{n=1}^{m-1} \frac{1}{n!} \sum_{\pi\in\mathrm{NC}^n_{\mathrm{irr}}(m)} m(\pi)\rho_\pi(w) \\ &= \sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}(m)} \frac{1}{t(\pi)!}\rho_\pi(w), \end{aligned}$$

where we used the notation of Corollary 8.2.3 for ρ_π and the fact that $m(\pi) = |\pi|!/t(\pi)!$. Notice that we can proceed analogously to obtain a pre-Lie algebraic proof of (3.4.17) by using the pre-Lie relation $\kappa = -W(-\rho)$.

One may wonder about a combinatorial expression of the left iteration of the pre-Lie product. As we can observe in the proof of Proposition 8.2.1, the second left iteration can be easily written in terms of irreducible non-crossing partitions of three blocks whose nesting tree is . The general form of the left iteration is given in the next proposition. For the proof, we make the convention that if $\pi \in \mathrm{NC}_{\mathrm{irr}}^{n+1}(m)$ and $t(\pi) = \ell_{n+1}$ is the ladder tree, then we write the blocks as $\pi = \{V_1, V_2, \ldots, V_{n+1}\}$, where $V_1 < V_2 < \cdots < V_{n+1}$.

Proposition 8.2.5 ([CEFPP21, Prop. 3])**.** *Let $\gamma_1, \ldots, \gamma_{n+1} \in L_{\mathfrak{g}}$ and $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$ such that $|w| = m \geq 2n+1$, with $n > 0$. Then the left iteration of the pre-Lie product acts on w by*

$$\big(\gamma_{n+1} \lhd (\cdots \lhd (\gamma_2 \lhd \gamma_1) \cdots)\big)(w) = \sum_{\substack{\pi \in \mathrm{NC}_{\mathrm{irr}}^{n+1}(m) \\ t(\pi)=\ell_{n+1} \\ \pi=\{V_1,\ldots,V_{n+1}\}}} \gamma_{n+1}(w_{V_1}) \cdots \gamma_1(w_{V_{n+1}}). \tag{8.2.5}$$

Proof. Similarly to the proof of Proposition 8.2.1, we proceed by induction on n. The base case $n = 1$ is described in (8.1.5). Now for the inductive step, assume that (8.2.5) holds for any n elements in $L_{\mathfrak{g}}$, and take $\gamma_1, \ldots, \gamma_{n+1} \in L_{\mathfrak{g}}$. Then, for a word $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$ we compute

$$\begin{aligned}
\big(\gamma_{n+1} \lhd (\cdots \lhd (\gamma_2 \lhd \gamma_1) \cdots)\big)(w) &= \sum_{\substack{1,m \in V_1 \subsetneq [m] \\ [m]\backslash V_1 \text{ is an interval}}} \gamma_{n+1}(w_{V_1}) \big(\gamma_n \lhd (\cdots \lhd (\gamma_2 \lhd \gamma_1) \cdots)\big)(w_{[m]\backslash V_1}) \\
&= \sum_{\substack{1,m \in V_1 \subsetneq [m] \\ [m]\backslash V_1 \text{ is an interval}}} \sum_{\substack{\pi \in \mathrm{NC}_{\mathrm{irr}}^{n}([m]\backslash V_1) \\ t(\pi)=\ell_n \\ \pi=\{V_2,\ldots,V_{n+1}\}}} \gamma_{n+1}(w_{V_1}) \gamma_n(w_{V_2}) \cdots \gamma_1(w_{V_{n+1}}),
\end{aligned}$$

where in the second equality, we used the induction hypothesis. We conclude then by noticing that, in a similar way as the bijection used in the proof of Proposition 8.2.1, we can find a bijection

$$\{\sigma \in \mathrm{NC}_{\mathrm{irr}}^{n+1}(m) \,:\, t(\sigma) = \ell_{n+1}\} \leftrightarrow$$

$$\{(\pi, V_1) \,:\, 1, m \in V_1 \subsetneq [m],\ \ [m]\backslash V_1 \text{ is an interval},\ \ \pi \in \mathrm{NC}_{\mathrm{irr}}^{n}([m]\backslash V_1) \text{ with } t(\pi) = \ell_n\}.$$

The inductive step finalizes by using this bijection to write the double sum in the above equation in order to obtain (8.2.5). □

8.3 Monotone cumulants via pre-Lie Magnus expansion

The third section of the present chapter is dedicated to the third and last step of our plan: to find a formula, in terms of non-crossing partitions, that writes monotone cumulants in terms of free and Boolean cumulants. The strategy is, by using the developments in the previous sections, to identify and propose the corresponding coefficients in the cumulant-cumulant formula.

Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, with associated Boolean and monotone cumulants $\{b_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$, respectively. By looking at the cumulant-cumulant formulas appearing in Theorem 3.4.25, we would expect a formula in terms of irreducible non-crossing partitions:

$$h_n(a_1,\dots,a_m) = \sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}(m)} \alpha(\pi) b_\pi(a_1,\dots,a_m),$$

where $\alpha(\pi)$ is an scalar. Even more, the coefficients appearing in the formulas of Theorem 3.4.25 only depend on the nesting tree of the irreducible partitions. In this way, we should expect that the same phenomenon will occur in the case of the α coefficients.

Now, consider the infinitesimal liftings ρ and β of $\{b_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$, respectively. The fundamental relation is given by the pre-Lie Magnus operator (Theorem 4.2.9):

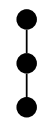
$$\rho(w) = \Omega(\beta)(w) = \sum_{n\geq 0} \frac{B_n}{n!} r^{(n)}_{\triangleleft\Omega(\beta)}(\beta)(w),$$

for any word $w\in T_+(\mathcal{A})$. Notice that if $w = a_1\cdots a_m \in \mathcal{A}^{\otimes}$, then the n-th term in the above sum will produce a zero contribution for any $n > m-1$ because of the degree conditions on w stated in Proposition 8.2.1. The following example shows the computations for some small values of m.

Example 8.3.1. Let $a_1,\dots,a_5\in\mathcal{A}$. By using (8.1.1) and Corollary 8.2.3, we can compute the monotone cumulants of words of length at most 5:

$$\begin{aligned}
\rho(a_1) &= \beta(a_1),\\
\rho(a_1a_2) &= \beta(a_1a_2),\\
\rho(a_1a_2a_3) &= \beta(a_1a_2a_3) - \frac{1}{2}\beta(a_1a_3)\beta(a_2),\\
\rho(a_1a_2a_3a_4) &= \beta(a_1a_2a_3a_4) - \frac{1}{2}\beta(a_1a_2a_4)\beta(a_3) - \frac{1}{2}\beta(a_1a_3a_4)\beta(a_2)\\
&\quad -\frac{1}{2}\beta(a_1a_4)\beta(a_2a_3) + \frac{1}{6}\beta(a_1a_4)\beta(a_2)\beta(a_3),\\
\rho(a_1a_2a_3a_4a_5) &= \beta(a_1a_2a_3a_4a_5) - \frac{1}{2}\beta(a_1a_2a_3a_5)\beta(a_4) - \frac{1}{2}\beta(a_1a_2a_4a_5)\beta(a_3)\\
&\quad -\frac{1}{2}\beta(a_1a_3a_4a_5)\beta(a_2) - \frac{1}{2}\beta(a_1a_5)\beta(a_2a_3a_4) - \frac{1}{2}\beta(a_1a_4a_5)\beta(a_2a_3)\\
&\quad -\frac{1}{2}\beta(a_1a_2a_5)\beta(a_3a_4) + \frac{1}{6}\beta(a_1a_2a_5)\beta(a_3)\beta(a_4) + \frac{1}{6}\beta(a_1a_4a_5)\beta(a_2)\beta(a_3)\\
&\quad +\frac{1}{6}\beta(a_1a_3a_5)\beta(a_2)\beta(a_4) + \frac{1}{6}\beta(a_1a_5)\beta(a_2a_3)\beta(a_4)\\
&\quad +\frac{1}{6}\beta(a_1a_5)\beta(a_2)\beta(a_3a_4) + \frac{1}{3}\beta(a_1a_5)\beta(a_2a_4)\beta(a_3).
\end{aligned}$$

As expected, at least for the words of length at most 5, the corresponding coefficients $\alpha(\pi)$ of the irreducible non-crossing partitions π indexing the above sums only depend on the nesting tree of the partition. We can summarize the coefficients in the following table.

t				
$\alpha(t)$	1	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Remarkably, the α coefficients coincide with Murua coefficients $\omega(t)$ from Definition 7.3.2. Therefore, our conjecture is then

$$\rho(w) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(|w|)} \omega(t(\pi))\beta_\pi(w).$$

In order to prove this conjecture, we want to relate the fundamental formula (8.1.1), Proposition 8.2.1 and Murua coefficients. Looking at the Bernoulli numbers, we can recall Proposition 7.3.4. Since we are dealing with trees that are trees of nestings of irreducible non-crossing partitions, it will be convenient to translate the objects $K(f)$ and $C^s(f)$ from the statement of Proposition 7.3.4 in terms of non-crossing partitions.

For the next construction, we start with a non-crossing partition with k exactly blocks, namely $\pi = \{V_1, \dots, V_k\} \in \mathrm{NC}(m)$. Considering π as a poset, define the collection

$$S(\pi) := \{S \subseteq \pi \,:\, S \text{ contains all the minimal elements of } \pi\}.$$

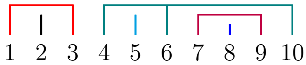
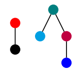
Notice that $S \in S(\pi)$ if and only if $S \subseteq \pi$ contains all the outer blocks of π. Moreover, we can consider any $S \in S(\pi)$ itself as a non-crossing partition of $\cup_{V \in S} V$. The non-crossing partition associated to S will be denoted as π_S. Observe that $S(\pi)$ is in clear bijection with the multiset $K(t(\pi))$ in the statement of Proposition 7.3.4, where $S \in S(\pi)$ is mapped to the forest $s(S)$ (as poset) induced by the vertices associated to S.

Now, take $S = \{W_1, \dots, W_l\} \in S(\pi)$. Define the *$S$-connected components* of π as the collection of sets $\pi^S = \{\pi_1^S, \dots, \pi_l^S\}$ where, for any $1 \le j \le l$, we set

$$\pi_j^S = \{V \in \pi \,:\, V \ge W_j \text{ and } \nexists\, j' \le l \text{ such that } V \ge W_{j'} > W_j\}. \tag{8.3.1}$$

In other words, π_j^S is the collection of blocks of π that are nested in W_j and that are not nested in another $W_{j'}$ that is also nested in W_j. By construction, each π_j^S as an irreducible non-crossing partition on the set $X_j^S := \cup_{V \in \pi_j^S} V$ whose unique outer block is W_j. Moreover, observe that $X^S = \{X_1^S, \dots, X_l^S\}$ is a non-crossing partition coarser than π. Notice that the construction π^S is clearly equivalent to the construction $C^{s(S)}(t(\pi))$, where $s(S)$ is the subforest of $t(\pi)$ given by $S \in S(\pi)$.

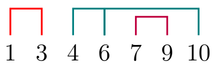
Example 8.3.2. Consider the non-crossing partition together with its nesting forest

$$\pi \quad = $$  $$\in \mathrm{NC}(10), \qquad t(\pi) \quad = $$  ,

where we have coloured the blocks for the sake of clarity. Notice that an element of $S(\pi)$ must contain the red and the green block. Thus we can take

$$S = \Bigg\{$$  $$\Bigg\} \in S(\pi),$$

with associated non-crossing partition

$$\pi_S = $$  $$\in \mathrm{NC}(\{1,3,4,6,7,9,10\}).$$

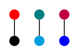
To this subset of blocks of π, we have the list of irreducible non-crossing partitions

$$\pi^S = \Bigg\{$$  $$\Bigg\}$$

and the coarser non-crossing partition

$$X^S = $$  $$\in \mathrm{NC}(10).$$

In the language of Proposition 7.3.4, we have that the equivalent objects are

$$s \quad = $$  $$\in K(t(\pi)), \qquad C^s(t(\pi)) \quad = $$  .

Remark 8.3.3. It is noticeable that the above construction provides a bijection between the pairs $(\pi, S) \in \mathrm{NC}(m) \times S(\pi)$ and pairs $(\sigma, (\tau_1, \dots, \tau_l))$, where $\sigma = \{W_1, \dots, W_l\} \in \mathrm{NC}^l(m)$ and τ_j is an irreducible non-crossing partition of W_j, for any $j \in [l]$. The bijection is given precisely by

$$W_j = X_j^S, \qquad \tau_j = \pi_j^S, \quad \forall\, j \in [l],$$

i.e. $\sigma = X^S$. It is not difficult to see that the inverse process provides

$$\pi = \bigcup_{j=1}^{l} \tau_j, \qquad S = \{B_1, \dots, B_j\},$$

where B_j stands for the unique outer block of τ_j, for any $j \in [l]$.

Under this construction, it is easy to translate Proposition 7.3.4

Proposition 8.3.4 ([CEFPP21, Prop. 5])**.** *For any irreducible partition* $\pi \in \mathrm{NC}_{\mathrm{irr}}(m)$

such that $\pi = \{V_1, \ldots, V_n\}$ *with* $n \geq 2$ *and* $1 \in V_1$*, denote by* π' *the non-crossing partition* $\{V_2, \ldots, V_n\}$*. Then*

$$\omega(t(\pi)) = \sum_{S \in S(\pi')} \frac{B_{|S|}}{t(\pi_S)!} \omega\left(t((\pi')^S)\right). \tag{8.3.2}$$

Remark 8.3.5. The above construction and the previous proposition can be also stated in terms of another partial order defined in $\mathrm{NC}(m)$ known as the *min-max order* ([BN08a]), denoted by $\ll$. More precisely, we say that $\pi \ll \sigma$ if and only if, $\pi \leq \sigma$ in the reverse refinement order $\mathrm{NC}(m)$ and, in addition, for any block $W \in \sigma$, there exists a block $V_W \in \pi$ such that $\min(W), \max(W) \in V_W$. Now, take two non-crossing partitions such that $\pi \ll \sigma$ with $\sigma = \{W_1, \ldots, W_l\}$. Let $V(\sigma)$ be the subset of π given by the blocks of π that contains the extremal points of the blocks of σ, i.e. $V(\sigma) = \{V_{W_1}, \ldots, V_{W_l}\} \subset \pi$. Notice that $V(\sigma)$ can be considered as a non-crossing partition. Also, by the definition of $\ll$, it is clear that $V(\sigma)$ contain all the outer blocks of π. Furthermore, also by definition of $\ll$ we have that each restriction $\pi|_{W_j}$ is an irreducible non-crossing partition. Hence, it is clear that we have a bijection

$$\{V(\sigma) \,:\, \pi \ll \sigma\} \leftrightarrow S(\pi)$$

such that $\pi_j^{V(\sigma)} = \pi|_{W_j}$, for any $j \in [l]$. If we denote $W(\pi) := \{\pi|_{W_1}, \ldots, \pi|_{W_l}\}$, we have that (8.3.2) can be written as

$$\omega(t(\pi)) = \sum_{\sigma \gg \pi} \frac{B_{|V(\sigma)|}}{t(V(\sigma))!} \omega\big(t\big(W(\pi)\big)\big).$$

After the previous propositions and remarks, we are now ready to attack the problem of finding the desired cumulant-cumulant formulas. The statement and the proof are finally as follows.

Theorem 8.3.6 ([CEFPP21, Thm. 3])**.** *Let* $(\mathcal{A}, \varphi)$ *be a non-commutative probability space, and consider* $\rho, \beta, \kappa \in L_{\mathfrak{g}}$ *the infinitesimal liftings of the monotone, Boolean and free cumulants on* $(\mathcal{A}, \varphi)$*, respectively. Then we have*

$$\begin{aligned} \rho(w) &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(m)} \omega(t(\pi)) \prod_{V \in \pi} \beta(w_V), \\ \rho(w) &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(m)} (-1)^{|\pi|-1} \omega(t(\pi)) \prod_{V \in \pi} \kappa(w_V), \end{aligned}$$

for any word $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$.

Proof. We only prove the monotone-Boolean case since the monotone-free case can be proved similarly. The proof is done by induction on m, the length of w. The base case $m = 1$ follows from the fact that $\omega(\bullet) = 1$ and the computation $\rho(a_1) = \beta(a_1)$

in Example 8.3.1. Now, assume that cumulant-cumulant is valid for words of length smaller than m. By the developments carried out throughout the chapter, in particular Corollary 8.2.3, we have for a word $w = a_1 \cdots a_m \in \mathcal{A}^{\otimes m}$:

$$\begin{aligned}\rho(w) = \Omega(\beta)(w) &= \sum_{n\geq 0}^{m-1} \frac{B_n}{n!} r^{(n)}_{\lhd\Omega(\beta)}(\beta)(w)\\ &= \sum_{n=0}^{m-1} \frac{B_n}{n!} \sum_{\substack{\pi\in \mathrm{NC}^{n+1}_{\mathrm{irr}}(m)\\ \pi=\{V_1,\ldots,V_{n+1}\}\\ 1,m\in V_1}} m(\pi)\beta(w_{V_1}) \prod_{i=2}^{n+1} \rho(w_{V_i}),\end{aligned}$$

where we used Corollary 8.2.3 in the third equality above. Recall that $m(\pi) = \frac{|\pi|!}{t(\pi)!}$, where clearly $|\pi| = n+1$ if $\pi \in \mathrm{NC}^{n+1}_{\mathrm{irr}}(m)$. Writing $\pi' = \pi\backslash\{V_1\}$ to denote the non-crossing partition obtained by removing the block $V_1 \in 1, m$ from π, we have

$$m(\pi) = \frac{|\pi|!}{t(\pi)!} = \frac{|\pi|!}{|\pi| t(\pi')!} = \frac{n!}{t(\pi')!}.$$

Then

$$\begin{aligned}\rho(w) &= \sum_{n=0}^{m-1} \frac{B_n}{n!} \sum_{\substack{\pi\in \mathrm{NC}^{n+1}_{\mathrm{irr}}(m)\\ \pi=\{V_1,\ldots,V_{n+1}\}\\ 1,m\in V_1}} \frac{n!}{t(\pi')!}\beta(w_{V_1}) \prod_{i=2}^{n+1} \rho(w_{V_i})\\ &= \sum_{n=0}^{m-1} \sum_{\substack{\pi\in \mathrm{NC}^{n+1}_{\mathrm{irr}}(m)\\ \pi=\{V_1,\ldots,V_{n+1}\}\\ 1,m\in V_1}} \frac{B_{|\pi'|}}{t(\pi')!}\beta(w_{V_1}) \prod_{i=2}^{n+1} \left(\sum_{\sigma_i\in \mathrm{NC}_{\mathrm{irr}}(V_i)} \omega(t(\sigma_i))\beta_{\sigma_i}(w_{V_i}) \right),\end{aligned}$$

where we used the induction hypothesis in the last equality above. In order to rearrange the above sum as desired, we will use the bijection described in Remark 8.3.3. More precisely, we set

$$\mu := \bigsqcup_{i=2}^{n+1} \sigma_i \in \mathrm{NC}([m]\backslash V_1), \qquad S = \{V_1^{\sigma_2}, \ldots, V_1^{\sigma_{n+1}}\}, \tag{8.3.3}$$

where $V_1^{\sigma_i}$ stands for the unique outer block of σ_i, for $2 \leq i \leq n+1$. We also set $\mu' := \mu \cup \{V_1\}$, and notice that $\mu' \in \mathrm{NC}_{\mathrm{irr}}(m)$. For the purpose of making a correct interpretation of the indexes, we notice:

i) By definition, $|S| = n = |\pi'|$.

ii) μ_S is the non-crossing partition of $\bigcup_{i=2}^{n+1} V_1^{\sigma_i}$ whose blocks are the elements of S as defined in (8.3.3). Since σ_i is an irreducible non-crossing partition of V_i for any

$2 \leq i \leq |\pi'|$, we conclude that the nesting structure of the blocks of π' is encoded in the nesting structure provided by the set of outer blocks of the irreducible non-crossing partition $\sigma_2, \ldots, \sigma_{n+1}$. Hence $t(\pi') = t(\mu_S)$.

iii) By the definition of the S-connected components of μ (8.3.1), we have that $\mu_i^S = \sigma_i$ for any $2 \leq i \leq n+1$. Thus $\mu^S = \{\sigma_2, \ldots, \sigma_{n+1}\}$ and

$$\omega\big(t\big(\mu^S\big)\big) = \prod_{i=2}^{n+1} \omega(t(\sigma_i)).$$

Finally, since the above sum is indexed by all the pairs $(\pi', (\sigma_i)_{2\leq i\leq |\pi'|+1})$, we can rearrange it as a sum over all the pairs (μ, S):

$$\begin{aligned}\rho(w) &= \sum_{\mu' \in \mathrm{NC}_{\mathrm{irr}}(m)} \sum_{S \in S(\mu)} \frac{B_{|S|}}{t(\mu_S)!} \beta(w_{V_1}) \beta_\mu(w_{[m]\setminus V_1}) \omega(t(\mu^S)) \\ &= \sum_{\mu' \in \mathrm{NC}_{\mathrm{irr}}(m)} \beta_{\mu'}(w) \left(\sum_{S \in S(\mu)} \frac{B_{|S|}}{t(\mu_S)!} \omega\big(t\big(\mu^S\big)\big) \right).\end{aligned}$$

We finally conclude by applying Proposition 8.3.4 to the above equation

$$\rho(w) = \sum_{\mu' \in \mathrm{NC}_{\mathrm{irr}}(m)} \omega(t(\mu')) \beta_{\mu'}(w),$$

and therefore, the proof is now complete. □

For the sake of completeness, we state the previous theorem in the language of the cumulant functionals.

Corollary 8.3.7. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and consider $\{h_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ and $\{k_n\}_{n\geq 1}$ to be the monotone, Boolean and free cumulants on $(\mathcal{A}, \varphi)$, respectively. Then we have*

$$h_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \omega(t(\pi)) b_\pi(a_1, \ldots, a_n), \tag{8.3.4}$$

$$h_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1} \omega(t(\pi)) k_\pi(a_1, \ldots, a_n), \tag{8.3.5}$$

for any $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$.

Remark 8.3.8. As we have mentioned, the formulas that write monotone cumulants in terms of the free and Boolean cumulants have already appeared in the work [AHLV15] only for the univariate case. The approach in [AHLV15, Thm. 1.2] differs from our approach presented here. Actually, the main idea of [AHLV15] is to use the combinatorics of the notion of *matricial free independence* ([Len10]), which relates monotone to free and Boolean independence as well. The description of the coefficient $\omega(t(\pi))$ appearing in [AHLV15] is given in terms of free cumulants k_n times the linear coefficient of $P_\pi(k)$, where $P_\pi(k)$ counts the number of non-decreasing labellings of the blocks of the non-crossing partition π. The reader should notice the difference with $\omega(t(\pi))$: the description of ω_k is the number of *strictly* order-preserving functions, not only non-decreasing.

Remark 8.3.9. We can use the formulas we have just proved to provide a different approach to the main question posed in Chapter 7, i.e. to give a formula that writes monotone cumulants in terms of moments. Actually, combining (8.3.4) and the formula (3.4.13) that writes Boolean cumulants in terms of moments, we obtain for any $n \geq 1$ and $a_1, \dots, a_n \in \mathcal{A}$:

$$\begin{aligned} h_n(a_1,\dots,a_n) &= \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \omega(t(\pi)) \prod_{V\in\pi} b_{|V|}(a_1,\dots,a_n|V) \\ &= \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \omega(t(\pi)) \prod_{V\in\pi} \sum_{\sigma\in \mathrm{Int}(V)} (-1)^{|\sigma|-1}\varphi_\sigma(a_1,\dots,a_n|V). \end{aligned}$$

Observe that if $\pi \in \mathrm{NC}_{\mathrm{irr}}(n)$ and for each $V \in \pi$, we take any $\sigma_V \in \mathrm{Int}(V)$, when we consider the disjoint union $\sigma := \bigsqcup_{V\in\pi} \sigma_V$, we will obtain a non-crossing partition such that $\sigma \leq \pi$ and such that the restriction $\sigma|_V$ on each $V \in \pi$ is an interval partition. If we denote by $\pi_1 \sqsubseteq \pi_2$ the fact π_1 and π_2 are non-crossing partitions satisfying the two previous conditions, we have that $\sqsubseteq$ is indeed a partial order on $\mathrm{NC}(n)$ (see [JV15]). With this notation, we have

$$\begin{aligned} h_n(a_1,\dots,a_n) &= \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} \omega(t(\pi)) \sum_{\sigma\sqsubseteq\pi} (-1)^{|\sigma|-|\pi|}\varphi_\sigma(a_1,\dots,a_n) \\ &= \sum_{\sigma\in \mathrm{NC}(n)} \varphi_\sigma(a_1,\dots,a_n) \sum_{\substack{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)\\ \sigma\sqsubseteq\pi}} (-1)^{|\sigma|-|\pi|}\omega(t(\pi)). \end{aligned}$$

By Corollary 7.3.14, we should have that

$$\sum_{\substack{t\in \mathrm{ST}(n)\\ \pi(t)=\pi}} \omega(\mathrm{sk}(t)) = \alpha(\pi) = \sum_{\substack{\sigma\in \mathrm{NC}_{\mathrm{irr}}(n)\\ \pi\sqsubseteq\sigma}} (-1)^{|\pi|-|\sigma|}\omega(t(\sigma)). \tag{8.3.6}$$

Observe that, in general, the ω map evaluated on the nesting tree of σ is not the same that the evaluation of ω on a Schröder tree whose associated partition is σ. Therefore, it is of interest to clarify the relation between these two different types of trees associated to a non-crossing partition.

The monotone-free and monotone-Boolean cumulant formulas obtained in Theorem 8.3.6 correspond to a particular computation of the pre-Lie Magnus expansion in the pre-Lie algebra of infinitesimal characters $\mathfrak{g}$. We close the ongoing chapter by remarking that such formulas are indeed particular cases of a more general (and effective) method of computation of the pre-Lie Magnus expansion in any locally finite connected graded pre-Lie algebra L, or even more generally, the calculation of iterated brace products and the associative product on the enveloping algebra of L (Section 2.3). The next chapter is dedicated to developing this method of computations in such general pre-Lie algebras.

Chapter 9

Pre-Lie Magnus Expansion via Forest Formulas

The results obtained in Chapter 8 elucidate a more general phenomenon in the context of pre-Lie algebras of words. In particular, the structure of the computations motivates us to have a more systematic understanding of the iterated pre-Lie products and the pre-Lie Magnus expansion computed in the previous chapter. Looking for a more general framework leads us to consider the notions of symmetric braces algebras and right-hand polynomial Hopf algebras, described in Section 2.3. The aim of this chapter is to understand how the iterations of the pre-Lie and brace products can be computed by looking at the dual Hopf algebra, in particular the coproduct, originated by a specific type of pre-Lie algebras. In the process of computing iterations of coproducts, we will generalize Zimmermann's forest-type formula for computing the antipode of a Hopf algebra. The contribution is then a new effective method for computation of iterated braces that, in particular, will allow us to compute the Magnus operator in a pre-Lie algebra of words.

We start Section 9.1 by defining the particular type of pre-Lie algebras to be considered, namely, locally finite connected graded pre-Lie algebras. Moreover, we construct the coproduct dual to the braces and show how the duality implies that the calculation of iterated pre-Lie and brace products is equivalent to the calculation of iterated coproducts appropriately restricted. The section finishes by stating and proving the particular expressions for the braces and the dual coproduct in the case of pre-Lie algebras of words. Our next objective is achieved in Section 9.2, where we explain the aforementioned forest formulas for the coproduct, which will provide a method to compute the Magnus operator.

The methods developed in Section 9.1 and Section 9.2 bear fruit in Section 9.3, where we can apply our new approach to compute the pre-Lie exponential and the Magnus operator on pre-Lie algebras of words. In other words, we can obtain the formulas that relate monotone cumulants with the free and Boolean cumulants. Furthermore, in order to show that our new method does not only apply in the previous case, we also explain in Section 9.4 how to compute the pre-Lie exponential and the Magnus operator of the generator of the free pre-Lie algebra of rooted trees. One feature of the corresponding proofs is that they are based on techniques that purely rely on pre-Lie algebra theory. As we have mentioned in the Introduction, the tree-series expansion of the pre-Lie exponential and Magnus operator are known. Nevertheless, our approach sheds light on further connections between the combinatorics of rooted trees and free pre-Lie algebras.

The main results of the present chapter, as well as the ideas for their proofs, are based on the work [CP22].

9.1 Iterated brace products

In this first section, we will consider the general problem of computing iterated pre-Lie products and iterated braces in a particular class of pre-Lie algebras. Solving this problem is the first step in developing a computational method to obtain the pre-Lie exponential and Magnus operator. Then we will study the pre-Lie algebra of words in detail, having in mind our ultimate objective of understanding the cumulant-cumulant formulas in non-commutative probability.

Throughout this section, assume that $(L, \triangleleft)$ is a locally finite connected graded pre-Lie algebra, i.e. $L = \bigoplus_{n \geq 0} L_n$ with $L_0 = 0$, each L_n is a finite-dimensional vector space for $n \geq 0$ and the pre-Lie product $\triangleleft$ on L is graded, i.e. $\alpha \triangleleft \gamma \in L_{m+n}$ whenever $\alpha \in L_m$ and $\gamma \in L_n$. If L^* stands for the graded dual of L, then the duality pairing $\langle \cdot | \cdot \rangle : L \otimes L^* \to \mathbb{K}$ extends to a pairing $\langle \cdot | \cdot \rangle : \mathbb{K}[L] \otimes \mathbb{K}[L^*] \to \mathbb{K}$ by the formula

$$\langle \alpha_1 \cdots \alpha_n | w_1 \cdots w_m \rangle = \begin{cases} \displaystyle\sum_{\sigma \in S_n} \prod_{i=1}^{n} \langle \alpha_{\sigma(i)} | w_i \rangle & \text{if } n = m, \\ 0 & \text{otherwise,} \end{cases} \tag{9.1.1}$$

for any $m, n \geq 1$, $\alpha_1, \ldots, \alpha_n \in L$, $w_1, \ldots, w_m \in L^*$ and S_n stands for the group of permutations on $[n]$. Observe that the fact that L is locally finite, connected and graded allows us to identify $\mathbb{K}[L^*]$ as the graded dual of $\mathbb{K}[L]$.

Remark 9.1.1. In the case that L is a locally finite connected graded pre-Lie algebra, observe that we have two graduations on $\mathbb{K}[L]$: the *polynomial degree* which is given by the length of a monomial $\alpha_1 \cdots \alpha_n \in \mathbb{K}[L]$, and the *full degree* which is given by $\deg(\alpha_1) + \cdots + \deg(\alpha_n)$, where we set $\deg(\gamma) = m$ if $\gamma \in L_m$.

By the results of Section 2.3, more specifically Theorem 2.3.20, the enveloping algebra of L is the Hopf algebra given by $(\mathbb{K}[L], *, \Delta_u)$, where Δ_u is the coproduct in the polynomial Hopf algebra (Example 2.2.4) and the associative product $*$ is defined by the formula (2.3.9) through the symmetric brace product $\{\,,\} : L \otimes \mathbb{K}[L] \to L$ defined in Lemma 2.3.14 extending $\triangleleft$. The dual Hopf algebra is of the form $(\mathbb{K}[L^*], \cdot, \delta)$, where $\cdot$ stands for the (commutative) polynomial product on $\mathbb{K}[L^*]$ and δ is the coproduct dual to $*$.

The fact that $*$ is defined through the symmetric braces permits us to describe δ also in terms of the braces. More precisely, in the case that L is connected graded and locally finite, for each $n \geq 1$ we define the map $\delta_n : L^* \to L^* \otimes \mathbb{K}[L^*]_n$ as the dual of the restriction of the brace product to a map $L \otimes \mathbb{K}[L]_n \to L$, which means

$$\langle \alpha\{\gamma_1, \dots, \gamma_n\} | w\rangle = \langle \alpha \otimes \gamma_1 \cdots \gamma_n | \delta_n(w)\rangle, \tag{9.1.2}$$

for any $\alpha, \gamma_1, \dots, \gamma_n \in L$ and $w \in L^*$. Here $\mathbb{K}[L]_n$ for the linear span of the monomials in $\mathbb{K}[L]$ of polynomial degree n, and similarly for $\mathbb{K}[L^*]_n$. Hence, we proceed to define

$$\overline{\delta} = \sum_{n \geq 1} \delta_n : L^* \to L^* \otimes \mathbb{K}[L^*]. \tag{9.1.3}$$

Observe that if $w \in (L_i)^* \cong L_i^*$, the fact that the pre-Lie product $\triangleleft$ is graded implies that $\delta_n(w) = 0$ for any $n \geq i$ since $\deg(\alpha\{\gamma_1, \dots \gamma_n\}) \geq n+1$. Thus $\overline{\delta}(w)$ is actually a finite sum. Finally, by setting $\delta : \mathbb{K}[L^*] \to \mathbb{K}[L^*] \otimes \mathbb{K}[L^*]$ to be the algebra morphism given by $\delta(w) = \overline{\delta}(w) + w \otimes 1 + 1 \otimes w$, Lemma 2.3.17 implies that δ is precisely the dual coproduct of $*$.

Theorem 9.1.2 ([CP22, Thm. 2.4])**.** *Let L be a locally finite connected graded pre-Lie algebra. Then $(\mathbb{K}[L^*], \cdot, \delta)$ defines the Hopf algebra structure dual to the Hopf algebra $(\mathbb{K}[L], *, \Delta_u)$, where the corresponding irreducible coproduct $\overline{\delta}$ is defined by* (9.1.3).

One of the main features of the symmetric brace product associated to a pre-Lie product is that it is a natural framework to understand the computations of iterations of the pre-Lie product. This statement is exemplified in the lemmas below, and it will be a crucial observation in the calculations of the forthcoming sections. Before stating the lemmas, let us introduce a particular restriction of the dual coproduct δ.

Definition 9.1.3 (Irreducible coproduct)**.** Let $(L, \triangleleft)$ be a locally finite connected graded pre-Lie algebra and $(\mathbb{K}[L^*], \cdot, \delta)$ be the dual of the enveloping algebra of L. We define the *irreducible coproduct* δ_{irr} as the linear map given by $\delta_{\mathrm{irr}} := \delta_{\mathrm{irr}}^{[2]} := \delta_1$, where δ_1 is given by (9.1.2). More generally, for $n \geq 2$ we define the map $\delta_{\mathrm{irr}}^{[n]}$ to be the restriction of the iterated coproduct $\delta^{[n]}$ to a map from L^* to $(L^*)^{\otimes n}$ by restricting the domain to $L^* \subset \mathbb{K}[L^*]$ and composing with the orthogonal projection from $\mathbb{K}[L^*]^{\otimes n}$ to $(L^*)^{\otimes n}$.

Remark 9.1.4. The fact that δ_1 identifies with the restriction of δ to a map from L^* to $(L^*)^{\otimes 2}$ is a consequence of the fact that the braces determine the associative product $*$ (Lemma 2.3.17).

The irreducible coproduct allows computing the action of the pre-Lie product as stated in the following lemma.

Lemma 9.1.5. *Let $(L, \lhd)$ be a locally finite connected graded pre-Lie algebra. Then*

$$\langle \alpha \lhd \gamma | w \rangle = \langle \alpha \otimes \gamma | \delta_{\mathrm{irr}}(w) \rangle,$$

for any $\alpha, \gamma \in L$ and $w \in L^*$.

Proof. Let $\alpha, \gamma \in L$ and $w \in L^*$. Using duality and the fact that the symmetric braces extend the pre-Lie product, we have that

$$\begin{aligned} \langle \alpha \otimes \gamma | \delta_{\mathrm{irr}}(w) \rangle &= \langle \alpha \otimes \gamma | \delta_1(w) \rangle \\ &= \langle \alpha\{\gamma\} | w \rangle \\ &= \langle \alpha \lhd \gamma | w \rangle, \end{aligned}$$

which proves the lemma. □

Remark 9.1.6. One can interpret the formula in the above lemma in terms of the reduced coproduct $\overline{\delta}$. More precisely, by considering any $\alpha \in L$ as an element in $(L^*)^*$, we define $\hat{\alpha}$ as the infinitesimal character on $\mathbb{K}[L^*]$ that coincides with α on L^*. Then, we can show that

$$\langle \alpha \lhd \gamma | w \rangle = \langle \hat{\alpha} \otimes \hat{\gamma} \, | \, \overline{\delta}(w) \rangle,$$

for any $\alpha, \gamma \in L$ and $w \in L^*$. Indeed, by Lemma 9.1.5 and the fact that $\overline{\delta} = \delta_{\mathrm{irr}} + \sum_{n \geq 2} \delta_n$ with $\delta_n(w) \in L^* \otimes \mathbb{K}[L^*]_n$, we get that

$$\begin{aligned} \langle \hat{\alpha} \otimes \hat{\gamma} | \overline{\delta}(w) \rangle &= \langle \hat{\alpha} \otimes \hat{\gamma} | \delta_{\mathrm{irr}}(w) + \sum_{n \geq 2} \delta_n(w) \rangle \\ &= \langle \hat{\alpha} \otimes \hat{\gamma} | \delta_{\mathrm{irr}}(w) \rangle + 0 \\ &= \langle \alpha \otimes \gamma | \delta_{\mathrm{irr}}(w) \rangle, \end{aligned}$$

where in the last equality, we used that $\hat{\alpha}$ and $\hat{\gamma}$ coincide with α and γ on L^*, respectively.

The following generalization of Lemma 9.1.5 will allow us to compute the iterated pre-Lie products via the iteration of the irreducible coproduct.

Lemma 9.1.7. *Let $(L, \lhd)$ be a locally finite connected graded pre-Lie algebra. Then*

$$\langle (\cdots(\alpha_1 \lhd \alpha_2) \lhd \cdots) \lhd \alpha_n | w \rangle = \langle \alpha_1 \otimes \cdots \otimes \alpha_n | \delta_{\mathrm{irr}}^{[n]}(w) \rangle = \langle \hat{\alpha}_1 \otimes \cdots \otimes \hat{\alpha}_n | \overline{\delta}^{[n]}(w) \rangle, \quad (9.1.4)$$

for any $n \geq 2$, $\alpha_1, \ldots, \alpha_n \in L$ and $w \in L^*$.

Proof. We prove the lemma by induction on n. The case $n = 2$ was actually proved in Lemma 9.1.5. Now, assume that the lemma holds for $n - 1$ and define

$$\gamma := (\cdots(\alpha_1 \lhd \alpha_2) \lhd \cdots) \lhd \alpha_{n-1} \in L.$$

Also, introduce the Sweedler-type notation $\delta_{\mathrm{irr}}(w) = w'^{(1)} \otimes w'^{(2)}$. Hence

$$\begin{aligned}
\langle \gamma \lhd \alpha_n | w \rangle &= \langle \gamma \otimes \alpha_n | w'^{(1)} \otimes w'^{(2)} \rangle \\
&= \langle \gamma | {w'}^{(1)} \rangle \langle \alpha_n | {w'}^{(2)} \rangle \\
&= \langle \alpha_1 \otimes \cdots \otimes \alpha_{n-1} | \delta_{\mathrm{irr}}^{[n-1]}({w'}^{(1)}) \rangle \langle \alpha_n | w'^{(2)} \rangle
\end{aligned}$$

where we used the induction hypothesis in the last equality. Now, we claim that

$$\delta_{\mathrm{irr}}^{[n]} = \left(\delta_{\mathrm{irr}}^{[n-1]} \otimes \mathrm{id} \right) \circ \delta_{\mathrm{irr}}.$$

Indeed, the coassociativity of δ implies that $\delta^{[n]} = \left(\delta^{[n-1]} \otimes \mathrm{id} \right) \circ \delta$. In addition, $\delta_{\mathrm{irr}}^{[n]}$ is defined by the restriction of $\delta^{[n]}$ to a map $L^* \to (L^*)^{\otimes n}$. Combining both statements, we obtain a proof of the claim. Hence, we can write

$$\begin{aligned}
\langle \gamma \lhd \alpha_n | w \rangle &= \langle \alpha_1 \otimes \cdots \otimes \alpha_n | \delta_{\mathrm{irr}}^{[n-1]}({w'}^{(1)}) \otimes {w'}^{(2)} \rangle \\
&= \langle \alpha_1 \otimes \cdots \otimes \alpha_n | \delta_{\mathrm{irr}}^{[n]}(w) \rangle,
\end{aligned}$$

and thus, the first equality in the lemma is proved. The proof of the second equality in (9.1.4) follows similarly by induction from Lemma 9.1.5 and the coassociativity of the reduced coproduct $\overline{\delta}$. □

The following lemma states that we can compute iterated symmetric brace products via iterations of the reduced coproduct, analogously to the result with the pre-Lie product and the irreducible coproduct.

Lemma 9.1.8. *Let $(L, \lhd)$ be a locally finite connected graded pre-Lie algebra. Then*

$$\begin{aligned}
&\Big\langle (\cdots(\alpha_{1,1}\{\alpha_{1,2}, \ldots \alpha_{m_2,2}\}) \cdots)\{\alpha_{1,n}, \ldots, \alpha_{m_n,n}\} \mid w \Big\rangle \\
&\qquad = \Big\langle \alpha_{1,1} \otimes \alpha_{1,2} \cdots \alpha_{m_2,2} \otimes \cdots \otimes \alpha_{1,n} \cdots \alpha_{m_n,n} \mid \overline{\delta}^{[n]}(w) \Big\rangle \qquad (9.1.5)
\end{aligned}$$

for any $n \geq 2$, $\alpha_{i,j} \in L$ for $1 \leq i \leq n$, $1 \leq j \leq n_i$ $(n_1 = 1)$, and $w \in L^*$.

The proof of the previous lemma can be obtained similarly to the proof of Lemma 9.1.7. In a few words, $(\cdots(\alpha_{1,1}\{\alpha_{1,2}, \ldots, \alpha_{m_2,2}\}) \cdots)\{\alpha_{1,n}, \ldots, \alpha_{m_n,n}\} \in L$ is the projection on L orthogonally to the other graded components of $\mathbb{K}[L]$ of the element

$$\alpha_{1,1} * (\alpha_{1,2} \cdots \alpha_{m_2,2}) * \cdots * (\alpha_{1,n} \cdots \alpha_{m_n,n}) \in \mathbb{K}[L].$$

The lemma follows as the $*$ product in $\mathbb{K}[L]$ is dual to the coproduct δ in $\mathbb{K}[L^*]$.

Now, observe that under the assumptions of Lemma 9.1.7, we can compute the Agrachev-Gamkrelidze and the pre-Lie Magnus operator on a pre-Lie algebra L by duality, as long as we have a concrete computation of the irreducible coproduct $\delta_{\mathrm{irr}}^{[n]}$. The work done in Chapter 8 motivates us to investigate such formulas for the particular case of pre-Lie algebras of words.

Henceforth, let X be a finite alphabet. Recall that we have a pre-Lie algebra structure on the linear span of non-empty words on X given by

$$\alpha \lhd \gamma = \sum_{\substack{\alpha_1\alpha_2=\alpha \\ \alpha_1,\alpha_2\neq\emptyset}} \alpha_1\gamma\alpha_2, \tag{9.1.6}$$

for any α, γ non-empty words on X, where the notation $\alpha_1\alpha_2$ means the concatenation of the words α_1 and α_2. It readily follows that L_X is a locally finite connected graded pre-Lie algebra, where the n-th homogeneous component $(L_X)_n$ is given by the linear span of the words of length n, for any $n \geq 1$. The first step to make in order to particularize the lemmas in this section, for the specific case of a pre-Lie algebra of words, is to obtain a detailed description of the symmetric braces and the dual coproduct associated to the pre-Lie product defined in (8.1.6). The description of the symmetric brace product is provided in the following proposition.

Proposition 9.1.9. *Let L_X be the pre-Lie algebra of words over X. The symmetric brace map on L_X given by Lemma 2.3.14 is given by*

$$\alpha\{\gamma_1, \dots, \gamma_n\} = \sum_{\sigma\in S_n} \sum_{\substack{\alpha_1\cdots\alpha_{n+1}=\alpha \\ \alpha_1,\alpha_{n+1}\neq\emptyset}} \alpha_1\gamma_{\sigma(1)}\alpha_2\gamma_{\sigma(2)}\cdots\alpha_n\gamma_{\sigma(n)}\alpha_{n+1}, \tag{9.1.7}$$

for any $\alpha, \gamma_1, \dots, \gamma_n \in X^\star$, $n \geq 1$, where $X^\star$ stands for the set of non-empty words on X.

Proof. We will prove the statement by induction on n. The base case $n = 1$ is straightforward from $\alpha\{\gamma_1\} = \alpha \lhd \gamma_1$ and the definition of $\lhd$ given in (9.1.6). Now assume that the formula (9.1.7) holds for a positive integer $n = k \geq 1$. We aim to prove that the formula (9.1.7) also holds for $n = k + 1$. In order to do this, we take $\alpha, \gamma_1, \dots, \gamma_{k+1} \in X^\star$. By Lemma 2.3.17, the braces are recursively defined by the formula

$$\alpha\{\gamma_1, \dots, \gamma_{k+1}\} \;=\; (\alpha\{\gamma_1, \dots, \gamma_k\})\{\gamma_{k+1}\} - \sum_{i=1}^{k} \alpha\{\gamma_1, \dots, \gamma_i\{\gamma_{k+1}\}, \dots, \gamma_k\}.$$

Using the inductive hypothesis, the first term of the right-hand side of the above equation is

$$(\alpha\{\gamma_1, \dots, \gamma_k\})\{\gamma_{k+1}\} = \left(\sum_{\sigma\in S_k} \sum_{\substack{\alpha_1\cdots\alpha_{k+1}=\alpha \\ \alpha_1,\alpha_{k+1}\neq\emptyset}} \alpha_1\gamma_{\sigma(1)}\alpha_2\gamma_{\sigma(2)}\cdots\alpha_k\gamma_{\sigma(k)}\alpha_{k+1} \right) \{\gamma_{k+1}\}.$$

According to the definition of the pre-Lie product, the term $\alpha_1\gamma_{\sigma(1)}\cdots\alpha_k\gamma_{\sigma(k)}\alpha_{k+1} \lhd \gamma_{k+1}$ is a sum indexed by the subwords generated by inserting γ_{k+1} in $\alpha_1\gamma_{\sigma(1)}\cdots\alpha_k\gamma_{\sigma(k)}\alpha_{k+1}$, with the condition that the leftmost and rightmost generated subwords are non-empty. This process induces a partition of every word α_i and γ_j into two subwords, with the only one restriction that the left subword of α_1 and the right subword of α_{k+1} are non-empty. This can be expressed as

$$\begin{aligned}(\alpha\{\gamma_1,\dots,\gamma_k\})\{\gamma_{k+1}\} \;=\; & \sum_{\sigma\in S_k}\sum_{\substack{\alpha_1\cdots\alpha_{k+1}=\alpha\\ \alpha_1,\alpha_{k+1}\neq\emptyset}}\left(\sum_{\substack{\alpha_{1,1}\alpha_{1,2}=\alpha_1\\ \alpha_{1,1}\neq\emptyset}}\alpha_{1,1}\gamma_{k+1}\alpha_{1,2}\gamma_{\sigma(1)}\cdots\gamma_{\sigma(k)}\alpha_{k+1}\right.\\ & +\sum_{i=2}^{k}\sum_{\alpha_{i,1}\alpha_{i,2}=\alpha_i}\alpha_1\gamma_{\sigma(1)}\cdots\alpha_{i,1}\gamma_{k+1}\alpha_{i,2}\cdots\gamma_{\sigma(k)}\alpha_{k+1}\\ & +\sum_{\substack{\alpha_{k+1,1}\alpha_{k+1,2}=\alpha_{k+1}\\ \alpha_{k+1,2}\neq\emptyset}}\alpha_1\gamma_{\sigma(1)}\cdots\alpha_{k+1,1}\gamma_{k+1}\alpha_{k+1,2}\\ & \left.+\sum_{i=1}^{k}\sum_{\substack{\gamma_{i,1}\gamma_{i,2}=\gamma_{\sigma(i)}\\ \gamma_{i,1},\gamma_{i,2}\neq\emptyset}}\alpha_1\cdots\alpha_i\gamma_{i,1}\gamma_{k+1}\gamma_{i,2}\alpha_{i+1}\cdots\alpha_{k+1}\right).\end{aligned}$$

Notice that in the last sum above, we add the condition $\gamma_{i,1},\gamma_{i,2}\neq\emptyset$. The latter is because the cases in which one of such subwords is empty correspond to the cases in which the neighbour subword $\alpha_{i,j}=\emptyset$ for some $j\in\{1,2\}$, which is already considered in the other sums.

On the other hand, the remaining sum of k braces can be computed as

$$\begin{aligned}\sum_{i=1}^{k}\alpha\{\gamma_1,\dots,\gamma_i\{\gamma_{k+1}\},\dots,\gamma_k\} \;=\; & \sum_{i=1}^{k}\sum_{\substack{\gamma_{i,1}\gamma_{i,2}=\gamma_i\\ \gamma_{i,1},\gamma_{i,2}\neq\emptyset}}\alpha\{\gamma_1,\dots,\gamma_{i,1}\gamma_{k+1}\gamma_{i,2},\dots,\gamma_k\}\\ = & \sum_{i=1}^{k}\sum_{\substack{\gamma_{i,1}\gamma_{i,2}=\gamma_i\\ \gamma_{i,1},\gamma_{i,2}\neq\emptyset}}\sum_{\sigma\in S_k}\sum_{\substack{\alpha_1\cdots\alpha_{k+1}=\alpha\\ \alpha_1,\alpha_{k+1}\neq\emptyset}}\alpha_1\gamma'_{\sigma(1)}\alpha_2\cdots\gamma'_{\sigma(2)}\cdots\alpha_k\gamma'_{\sigma(k)}\alpha_{k+1},\end{aligned}$$

where in the second equality, we used the inductive hypothesis and set

$$\gamma'_{\sigma(j)} := \begin{cases}\gamma_{\sigma(j)} & \text{if } \sigma(j)\neq i,\\ \gamma_{i,1}\gamma_{k+1}\gamma_{i,2} & \text{if } \sigma(j)=i.\end{cases}$$

By subtracting the above equation from the previous one, we obtain

$$\alpha\{\gamma_1,\dots,\gamma_{k+1}\} = \sum_{\sigma\in S_k}\sum_{\substack{\alpha_1\cdots\alpha_{k+1}=\alpha\\ \alpha_1,\alpha_{k+1}\neq\emptyset}}\sum_{i=1}^{k+1}\sum_{\substack{\alpha_{i,1}\alpha_{i,2}=\alpha_i\\ \alpha_{1,1},\alpha_{k+1,2}\neq\emptyset}}\alpha_1\gamma_{\sigma(1)}\cdots\alpha_{i,1}\gamma_{k+1}\alpha_{i,2}\cdots\gamma_{\sigma(k)}\alpha_{k+1}. \tag{9.1.8}$$

Now, given any $\sigma \in S_k$ and $i \in \{1, \dots, k+1\}$, we define $\tau \in S_{k+1}$ by

$$\tau(j) = \begin{cases} \sigma(j) & \text{if } 1 \le j < i, \\ k+1 & \text{if } j = i, \\ \sigma(j-1) & \text{if } i < j \le k+1. \end{cases}$$

This map defines a bijection between $S_k \times \{1, \dots, k+1\}$ and S_{k+1}. Hence, Equation (9.1.8) can be written as

$$\alpha\{\gamma_1, \dots, \gamma_{k+1}\} = \sum_{\tau \in S_{k+1}} \sum_{\substack{\alpha_1 \cdots \alpha_{k+2} = \alpha \\ \alpha_1, \alpha_{k+2} \neq \emptyset}} \alpha_1 \gamma_{\tau(1)} \alpha_2 \gamma_{\tau(2)} \cdots \alpha_{k+1} \gamma_{\tau(k+1)} \alpha_{k+2},$$

as we wanted to show. □

For the next proposition, recall that the scalar product making $X^\star$ an orthonormal basis of L_X

$$\langle \alpha | w \rangle = \begin{cases} 1 & \text{if } w = \alpha, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall\, \alpha, w \in X^\star, \tag{9.1.9}$$

allows us to identify L_X with its graded dual L_X^*. In this way, the formula for the dual coproduct associated to $\triangleleft$ via (9.1.3) is stated as follows.

Proposition 9.1.10. *Let L_X be the pre-Lie algebra of words over X and consider $(\mathbb{K}[L_X^*], \cdot, \delta)$ the Hopf algebra dual to the enveloping algebra of L_X. Then the restriction of reduced coproduct $\overline{\delta} : L_X^* \to L_X^* \otimes \mathbb{K}[L_X^*]$ is given by*

$$\overline{\delta}(w) = \sum_{m=1}^{\infty} \sum_{\substack{w_1 \cdots w_{2m+1} = w \\ w_1, w_{2m+1}, w_{2i} \neq \emptyset, \forall\, i}} w_1 w_3 \cdots w_{2m+1} \otimes w_2 \cdot w_4 \cdot \ \cdots \ \cdot w_{2m}, \qquad \forall\, w \in X^\star. \tag{9.1.10}$$

Remark 9.1.11. Observe that on the right-hand side of (9.1.10), the first component of each summand is given by concatenating the subwords $w_1 w_3 \cdots w_{2m+1}$. On the other hand, the second component of each summand consists of a monomial provided by the product in the polynomial algebra $\mathbb{K}[L_X^*]$ of the elements $w_2, w_4 \dots, w_{2m}$.

Proof. We check that $\overline{\delta}$ defined in (9.1.10) is the dual coproduct of the symmetric brace operation, i.e. (9.1.2) holds for any $\alpha, \gamma_1, \dots, \gamma_n \in L_X$, $w \in L_X^*$, $n \ge 1$. Since the duality pairing defined in (9.1.9) extends to a pairing $\mathbb{K}[L_X] \otimes \mathbb{K}[L_X^*] \to \mathbb{K}$ by using (9.1.1), we have

$$\begin{aligned} \langle \alpha \otimes \gamma_1 \cdot \dots \cdot \gamma_n | \overline{\delta}(w) \rangle &= \sum_{m=1}^{\infty} \sum_{\substack{w_1 \cdots w_{2m+1} = w \\ w_1, w_{2m+1}, w_{2i} \neq \emptyset, \forall\, i}} \langle \alpha | w_1 w_3 \cdots w_{2m+1} \rangle \langle \gamma_1 \cdot \dots \cdot \gamma_n | w_2 \cdot \dots \cdot w_{2m} \rangle \\ &= \sum_{\substack{w_1 \cdots w_{2n+1} = w \\ w_1, w_{2n+1}, w_{2i} \neq \emptyset, \forall\, i}} \sum_{\sigma \in S_n} \langle \alpha | w_1 w_3 \cdots w_{2n+1} \rangle \langle \gamma_{\sigma(1)} | w_2 \rangle \cdots \langle \gamma_{\sigma(n)} | w_{2n} \rangle, \end{aligned}$$

Now fix a permutation $\sigma \in S_n$. Observe that the non-vanishing terms in the double sum above are such that

$$\langle \alpha | w_1 w_3 \cdots w_{2n+1} \rangle \langle \gamma_{\sigma(1)} | w_2 \rangle \cdots \langle \gamma_{\sigma(n)} | w_{2n} \rangle = 1 \tag{9.1.11}$$

which happens, for $w = w_1 \cdots w_{2n+1}$, $w_1, w_{2n+1}, w_{2i} \neq \emptyset$, for every $1 \leq i \leq n$, when we have that $\alpha = w_1 w_3 \cdots w_{2n+1}$ and $w_{2i} = \gamma_{\sigma(i)}$, for every $1 \leq i \leq n$.

On the other hand, recalling Proposition 9.1.9 and using the same notation as in the proposition, we have for the non-vanishing terms arising in the expansion of $\langle \alpha\{\gamma_1, \ldots, \gamma_n\} | w \rangle$:

$$\langle \alpha_1 \gamma_{\sigma(1)} \cdots \alpha_n \gamma_{\sigma(n)} \alpha_{n+1} | w \rangle = 1 \tag{9.1.12}$$

which happens when there exist a permutation σ and $2n+1$ subwords of w, namely $w_1, \ldots, w_{2n+1}$, such that $w = w_1 \cdots w_{2n+1}$, $w_{2i-1} = \alpha_i$, $w_{2i} = \gamma_{\sigma(i)}$ for every $1 \leq i \leq n$, with $\alpha_1, \alpha_{n+1} \neq \emptyset$ and $\alpha = \alpha_1 \cdots \alpha_{n+1}$.

We conclude by noticing that the terms in both (9.1.11) and (9.1.12) are clearly in bijection, and so we obtain (9.1.10). $\square$

We conclude this section by recalling that the initial setup of Chapter 8 can be understood in the framework of this section, in the sense that the cumulants can be considered as elements of a specific pre-Lie algebra of words $L_{\mathfrak{g}}$. Thereby, the formulas obtained in Lemma 9.1.7 and Lemma 9.1.8 will provide a method to compute a formula for the iterated pre-Lie products as in Proposition 8.2.1. The combinatorial formulas will then appear by having combinatorial formulas for the iterated irreducible coproduct $\delta_{\mathrm{irr}}^{[n]}$ and reduced coproduct $\overline{\delta}^{[n]}$. Obtaining such formulas will be the work in the next section.

9.2 Forest formulas for iterated coproducts

As explained at the end of the previous section, this section aims to obtain our main tool to evaluate the pre-Lie exponential and the Magnus operator: the forest-type formulas for iterated coproducts introduced in [MP18]. More specifically, we describe the required notation and graphical objects. We also give a different proof of the aforementioned forest formulas by introducing a symmetry factor not considered in the original formulas of [MP18].

Forest formulas for iterated coproducts and antipodes have their origins in quantum field theory. The motivation of such formulas is to be computationally effective in the sense that they drastically reduce the number of terms in the expression of the antipode.

Let us fix the framework for this section. Let L be a locally finite connected graded pre-Lie algebra and consider $(\mathbb{K}[L^*], \cdot, \delta)$ the Hopf algebra of Theorem 9.1.2. We will assume that L^* has a countable basis $\mathcal{B} = \{b_i\}_{i=0}^{\infty}$.

The (commutative) product of elements in the basis will be represented by the notation $b_I := \prod_{i \in I} b_i$, where I is a *multiset* of $\mathbb{N}_0$, where $\mathbb{N}_0$ stands for the set of non-negative integers. In particular, for any two multisets $I, J \subset \mathbb{N}_0$, we have that $b_I \cdot b_J = b_{I \cup J}$, where $I \cup J$ stands for the union of the multisets I and J. For instance, if $I = \{1,2,2\}$ and $J = \{1,2,3,3\}$, then $I \cup J = \{1,1,2,2,2,3,3\}$. Also, we set $b_\emptyset := 1$.

Now, let us recall that $(\mathbb{K}[L^*], \cdot, \delta)$ is the Hopf algebra dual to the enveloping algebra of L. By the construction given in (9.1.3), the reduced coproduct $\overline{\delta}$ satisfies that $\overline{\delta}(L^*) \subset L^* \otimes \mathbb{K}[L^*]$. In particular, we have that

$$\overline{\delta}(b_i) = \sum_{i_0, I \neq \emptyset} \lambda_I^{i;i_0} b_{i_0} \otimes b_I, \quad \forall\, b_i \in \mathcal{B}, \tag{9.2.1}$$

where the coefficients $\lambda_I^{i;i_0} \in \mathbb{K}$ are indexed by $i_0 \in \mathbb{N}_0$ and a non-empty multiset $I \subseteq \mathbb{N}_0$. The important observation is that the family of coefficients $\{\lambda_I^{i;i_0}\}_{i,i_0,I}$ completely determines the coproduct and its action on products, as well as the action of the iterated coproducts.

A key remark in the work [MP18] is that (9.2.1) can be written as a sum indexed by *decorated non-planar rooted trees*. More precisely, we have

$$\overline{\delta}(b_i) = \sum \lambda\Big(\underset{i_1 \qquad i_k}{\overset{(i;i_0)}{\bigwedge}} \Big) b_{i_0} \otimes b_{i_1} \dots b_{i_k}, \tag{9.2.2}$$

where the decoration of the root of the tree is given by a pair of non-negative integers $(i; i_0)$, and the leaves are decorated with non-negative integers $i_1, \dots, i_k$. Observe that we are considering non-planar trees because of the commutativity of the polynomial algebra $\mathbb{K}[L^*]$.

Example 9.2.1. The following example from [MP18, Sec. 3] sheds light on how general decorated non-planar trees can be used to describe the iterations of the reduced coproduct $\overline{\delta}$. For this purpose, let us consider a single term in the sum (9.2.2) for the reduced coproduct such that the indexing multiset is of the form $I = \{i_1, i_2\}$, namely

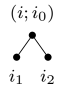
$$\lambda\Big(\underset{i_1 \ \ i_2}{\overset{(i;i_0)}{\wedge}} \Big) b_{i_0} \otimes b_{i_1} b_{i_2}. \tag{9.2.3}$$

Since in general $\delta(w) = \overline{\delta}(w) + 1 \otimes w + w \otimes 1$ for any $w \in \mathbb{K}[L^*]$, we can write

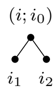
$$\overline{\delta}(b_{i_1} b_{i_2}) = (1 \otimes b_{i_1} + b_{i_1} \otimes 1 + \overline{\delta}(b_{i_1}))(1 \otimes b_{i_2} + b_{i_2} \otimes 1 + \overline{\delta}(b_{i_2})) - 1 \otimes b_{i_1} b_{i_2} - b_{i_1} b_{i_2} \otimes 1.$$

Observe that the contribution of (9.2.3) to the second iteration $\overline{\delta}^{[3]}(b_i) = (\mathrm{id} \otimes \overline{\delta})(\overline{\delta}(b_i))$ will split in four terms, whose complexity is encoded by the appearance of products of coefficients $\lambda_J^{j;j_0}$. Let us describe each of these four kinds of terms.

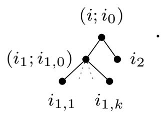
i) The element $b_{i_1} \otimes b_{i_2} + b_{i_2} \otimes b_{i_1}$ produces a term with no more complexity than in $\overline{\delta}(b_i)$:

$\lambda\Big($  $\Big) b_{i_0} \otimes (b_{i_1} \otimes b_{i_2} + b_{i_2} \otimes b_{i_1}).$

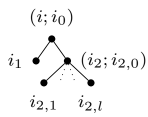
ii) The product $(1 \otimes b_{i_2} + b_{i_2} \otimes 1)\overline{\delta}(b_{i_1})$ produces a term where only the reduced coproduct of b_{i_1} occurs:

$\lambda\Big($  $\Big) \Big(\sum \lambda^{i_1;i_{1,0}}_{i_{1,1},\ldots,i_{1,k}} b_{i_0} \otimes (b_{i_{1,0}} \otimes b_{i_2} b_{\{i_{1,1},\ldots,i_{1,k}\}} + b_{i_2} b_{i_{1,0}} \otimes b_{\{i_{1,1},\ldots,i_{1,k}\}})\Big).$

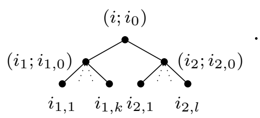
This is naturally indexed by trees of the form



iii) In a similar fashion, there is a contribution, corresponding to $(1 \otimes b_{i_1} + b_{i_1} \otimes 1)\overline{\delta}(b_{i_2})$, indexed by the trees



iv) Finally the terms in relation with $\overline{\delta}(b_{i_1})\overline{\delta}(b_{i_2})$ are indexed by trees



With the purpose of presenting a general formula indexed by decorated trees for the iterated coproducts, we will need the following notation.

Definition 9.2.2 (Decorated trees [MP18], [CP22, Def. 5.2])**.** A *decorated tree* T is a finite non-planar rooted tree whose internal vertices are decorated by a pair $p = (p_1; p_2)$ of non-negative integers, and the leaves are decorated by non-negative integers. In the case of T being a single-vertex tree, the vertex is considered a leaf.

1. We denote by $\mathrm{Intl}(T)$ and $\mathrm{Leaf}(T)$ the sets of internal vertices of T and leaves of T, respectively. For any $x \in \mathrm{Intl}(T)$, we denote $d(x) = (d_1(x); d_2(x))$ its decoration, and, if $x \in \mathrm{Leaf}(T)$, we denote for convenience its decoration $d(x) = d_1(x) = d_2(x)$. For any $x \in \mathrm{Intl}(T)$, we also set $\mathrm{succ}(x)$ to be the set of children of x. The root of T is denoted by $\mathrm{rt}(T)$.

2. If the root of T is decorated by i or $(i; i_0)$, we say that the tree is *associated to* b_i. We denote by $\mathcal{T}_i$ the set of decorated trees associated to b_i.

3. For a pair p of integers and decorated trees $T_1, \ldots, T_s$, we denote by $B_p^+(T_1 \cdots T_s)$ the tree obtained by adding a common root decorated by p to the trees $T_1, \ldots, T_s$. In a similar way if $T = B_p^+(T_1 \cdots T_s)$, we denote by $B^-(T)$ the multiset of decorated trees $T_1 \cdots T_s$.

It will also be convenient to consider the decorated trees as a finite poset (Remark 2.2.9). In order to distinguish different vertices regarded as elements of such a poset, we need to consider the symmetry coefficient of the tree.

Definition 9.2.3 (Symmetry coefficient)**.** Let F be a multiset of decorated trees so that we write F as

$$F = \{T_{1,i_1}^{k_{1,1}}, \ldots, T_{s_1,i_1}^{k_{s_1,1}}\} \cup \cdots \cup \{T_{1,i_p}^{k_{1,p}}, \ldots, T_{s_p,i_p}^{k_{s_p,p}}\} \tag{9.2.4}$$

such that

- the tree T_{j,i_q} is associated to i_q, for every $1 \leq q \leq p$;
- the trees T_{j,i_q} are all distinct, for every $1 \leq j \leq s_q$;
- the notation $T_{j,i_q}^{k_{j,q}}$ means that the tree T_{j,i_q} appears with multiplicity $k_{j,q}$ in the multiset F, for every $1 \leq j \leq s_q$ and $1 \leq q \leq p$.

The *symmetry coefficient of* F, denoted by $\mathrm{sym}(F)$ is defined by the formula

$$\mathrm{sym}(F) := \prod_{j=1}^{p} \binom{k_{1,j} + \cdots + k_{s_j,j}}{k_{1,j}, \ldots, k_{s_j,j}}.$$

According to the computations in Example 9.2.1, one may conjecture that each term in the expression of the k-fold iterated reduced coproduct will have a coefficient λ that is the product of at most k coefficients of the form $\lambda_I^{i;i_0}$, where each index i, i_0, I is given by the decoration of the tree. With this motivation and the previous notions on decorated trees, we define the following coefficients.

Definition 9.2.4. Let T be a decorated tree. We define the associated coefficient $\lambda(T)$ as follows: if $\bullet_i$ stands for the single-vertex tree with decoration i, then $\lambda(\bullet_i) := 1$. More generally, if $T = B_{(i;i_0)}^+(T_1 \cdots T_s)$, then we set

$$\lambda(T) := \lambda_{i_1,\ldots,i_s}^{i;i_0} \cdot \mathrm{sym}(F) \cdot \lambda(T_1) \cdots \lambda(T_s) \tag{9.2.5}$$

when $T_1, \ldots, T_s$ are trees respectively associated to $b_{i_1}, \ldots, b_{i_s}$, and where F is the multiset of decorated trees $\{T_1, \ldots, T_s\}$. In other words, if for $x \in V(T)$, T^x stands for the decorated subtree of T consisting of x and all its descendants, then

$$\lambda(T) = \prod_{x \in \mathrm{Intl}(T)} \lambda^{d(x)}_{d_1(\mathrm{succ}(x))} \operatorname{sym}(B^-(T^x)), \tag{9.2.6}$$

where $d_1(\mathrm{succ}(x)) = \{d_1(y) \,:\, y \in \mathrm{succ}(x)\}$.

The next ingredient describes all the tensors of length k that appear in the expression of $\overline{\delta}^{[k]}$.

Definition 9.2.5 (k-linearization of a poset [CP22, Def. 5.3])**.** Let P be a finite poset of cardinality n.

i) A *linearization* of P is a bijective, strictly order-preserving map $f : P \to [n]$.

ii) A *k-linearization* of P is a surjective, strictly order-preserving map $f : P \to [k]$.

iii) A *weak k-linearization* of P is a strictly order-preserving map $f : P \to [k]$.

We denote by $\mathrm{lin}(P)$, resp. k-$\mathrm{lin}(P)$, resp. w-k-$\mathrm{lin}(P)$, the set of linearizations, resp. k-linearizations of P, resp. weak k-linearizations of P.

It is clear that a k-linearization is a surjective weak k-linearization, and a linearization a bijective k-linearization.

Remark 9.2.6. Let P be a finite poset. In the case that P is a disjoint union of posets $P_1 \sqcup \cdots \sqcup P_q$, with no relations between the elements of two distinct components, then weak k-linearizations of P are in bijection with p-tuples of weak k-linearizations of the components $P_1, \ldots, P_q$. This observation does not hold for k-linearizations.

Recall that any tree can be considered as a poset. Then we have the following definition.

Definition 9.2.7 ([CP22, Def. 5.4])**.** Let T be a decorated tree with decoration $d = (d_1; d_2)$. If f is a linearization of T, a k-linearization of T, or a weak k-linearization of T, we write

$$C(f) := \left(\prod_{x_1 \in f^{-1}(\{1\})} b_{x_1} \right) \otimes \cdots \otimes \left(\prod_{x_k \in f^{-1}(\{k\})} b_{x_k} \right),$$

where b_{x_i} stands for $b_{d_2(x_i)}$. For notational convenience, we set $\prod_{x_i \in f^{-1}(\{i\})} b_{x_i} := 1$ if $f^{-1}(\{i\}) = \emptyset$. Notice that this can happen only with weak k-linearizations.

The following theorem finally states the forest formulas for the iterated coproduct and the iterated reduced coproduct. The formula for the iterated reduced coproduct originally appeared in the work [MP18]. However, the definition of $\lambda(T)$ provided by the authors did not take into account symmetry factors associated with each step of the calculation of λ. We propose a different approach that allows us to obtain a proof also for the forest formula for the iterated coproduct.

Theorem 9.2.8 (Forest formulas for iterated coproducts [MP18, Lem. 12],[CP22, Thm. 5.5])**.** *For any element $b_i \in \mathcal{B}$ in the basis of L^*, we have for the action of the k-fold iterated reduced coproduct, respectively the k-fold iterated coproduct, on $\mathbb{K}[L^*]$:*

$$\overline{\delta}^{[k]}(b_i) = \sum_{T\in\mathcal{T}_i} \sum_{f\in k\text{-lin}(T)} \lambda(T)C(f). \tag{9.2.7}$$

$$\delta^{[k]}(b_i) = \sum_{T\in\mathcal{T}_i} \sum_{f\in \mathrm{w}-k\text{-lin}(T)} \lambda(T)C(f). \tag{9.2.8}$$

Proof. We start the proof by finding formulas that permit us to write the iterated coproduct in terms of the iterated reduced coproduct and vice versa. Such formulas will imply that the above formulas (9.2.7) and (9.2.8) are equivalent.

In general, let $(C, \Delta, \varepsilon, \eta)$ be a coaugmented coalgebra over $\mathbb{K}$ (Remark 2.1.20) and take the map $\nu := \eta \circ \varepsilon : C \to C$. Recall that the reduced coproduct is a map defined on $\ker \varepsilon$. In addition, the map $\mathrm{id} - \nu$ is the orthogonal projection $C \to \ker \varepsilon$. Thus the iterated coproduct and the iterated reduced coproduct are then related by the identity:

$$\overline{\Delta}^{[k]} = (\mathrm{id} - \nu)^{\otimes k} \circ \Delta^{[k]}, \tag{9.2.9}$$

for any $k \geq 2$. On the other hand, to express $\Delta^{[k]}$ in terms of iterated reduced coproducts, we notice that

$$\Delta^{[k]} = (\nu + (\mathrm{id} - \nu))^{\otimes k} \circ \Delta^{[k]}$$
$$= \left(\sum_{l=1}^{k} \sum_{1\leq i_1<\cdots<i_l\leq k} (\mathrm{id} - \nu) \otimes \cdots \otimes \nu \otimes \cdots \otimes \nu \otimes \cdots \otimes (\mathrm{id} - \nu) \right) \circ \Delta^{[k]},$$

where the term in the last summation formula contains l copies of ν in positions $i_1, \ldots, i_l$. Now, for $n \geq k$, take $f : [k] \hookrightarrow [n]$ be an increasing injection and denote $\hat{f}$ the map from $C^{\otimes k}$ to $C^{\otimes n}$ defined by

$$\hat{f}(c_1 \otimes \cdots \otimes c_k) := d_1 \otimes \cdots \otimes d_n$$

with $d_{f(i)} := c_i$ for $1 \leq i \leq k$, and $d_j := 1$ if j is not in the image of f. Since Δ is counital, the above expression of $\Delta^{[k]}$ can be written as

$$\Delta^{[k]} = \sum_{l\leq k} \sum_{f:[k-l]\hookrightarrow[k]} \hat{f} \circ \overline{\Delta}^{[k-l]}.$$

On the other hand, it is easy to see that any weak k-linearization can be uniquely obtained as the composition of a l-linearization, with $l \leq k$, and an injection f from $[l]$

into $[k]$. Notice that this corresponds to the usual decomposition of a map between two finite sets as a surjection followed by an injection. This remark implies the equivalence between (9.2.7) and (9.2.8).

We are now ready to prove the formulas jointly by induction on $k \geq 2$. Writing

$$\overline{\delta}(b_i) = \sum_{i_0, i_1, \ldots, i_n} \lambda^{i;i_0}_{i_1,\ldots,i_n} b_{i_0} \otimes b_{i_1} \cdots b_{i_n},$$

we note that the base case $k = 2$ for (9.2.7) follows directly from (9.2.2). Let us take $k > 2$ and assume that the forest formulas are valid for $k-1$. By coassociativity, the k-fold iterated reduced coproduct can be written by

$$\overline{\delta}^{[k]}(b_i) = \sum_{i_0, i_1, \ldots, i_n} \lambda^{i;i_0}_{i_1,\ldots,i_n} b_{i_0} \otimes \overline{\delta}^{[k-1]}(b_{i_1} \cdots b_{i_n}).$$

By (9.2.9), we have that

$$\overline{\delta}^{[k-1]}(b_{i_1} \cdots b_{i_n}) = (\mathrm{id} - \nu)^{\otimes k-1} \circ \delta^{[k-1]}(b_{i_1} \cdots b_{i_n}).$$

Recalling that the coproduct is a morphism of algebras, we have

$$\delta^{[k-1]}(b_{i_1} \cdots b_{i_n}) = \delta^{[k-1]}(b_{i_1}) \cdots \delta^{[k-1]}(b_{i_n}).$$

Observe that we can use the induction hypothesis on each $\delta^{[k-1]}(b_{i_j})$ for $1 \leq j \leq n$ and obtain

$$\delta^{[k-1]}(b_{i_j}) = \sum_{T \in \mathcal{T}_{i_j}} \sum_{f \in \text{w-}(k-1)\text{-lin}(T)} \lambda(T) C(f),$$

so that

$$\begin{aligned} \delta^{[k-1]}(b_{i_1}) \cdots \delta^{[k-1]}(b_{i_n}) &= \prod_{j=1}^{n} \left(\sum_{T \in \mathcal{T}_{i_j}} \sum_{f \in \text{w-}(k-1)\text{-lin}(T)} \lambda(T) C(f) \right) \\ &= \sum_{\substack{T_j \in \mathcal{T}_{i_j} \\ 1 \leq j \leq n}} \lambda'(T) \left(\sum_{f \in (k-1)\text{-lin}(T)} C(f) \right), \end{aligned} \tag{9.2.10}$$

where in the last equality, T is the multiset of decorated trees $T_1, \ldots, T_n$. That is, as a poset, T is the disjoint union of the T_i. Also, we have set

$$\lambda'(T) := \lambda(T_1) \cdots \lambda(T_n).$$

Moreover, in (9.2.10) we used the fact that weak $(k-1)$-linearizations of T are in bijection with families of $(k-1)$-linearizations of the posets $T_1, \ldots, T_n$ in order to obtain

$$\begin{aligned}\overline{\delta}^{[k-1]}(b_{i_1}\cdots b_{i_n}) &= (\mathrm{id}-\nu)^{\otimes k-1}\circ \delta^{[k-1]}(b_{i_1}\cdots b_{i_n}) \\ &= \sum_{\substack{T_j\in\mathcal{T}_{i_j}\\ 1\le j\le n}} \lambda'(T)\left(\sum_{f\in(k-1)\text{-lin}(T)} C(f)\right). \end{aligned}\tag{9.2.11}$$

Now, define $\mathcal{T}_{i_1,\ldots,i_n}$ to be the set of forests of decorated trees $T=\{T_1,\ldots,T_n\}$ such that there exists a permutation σ of $[n]$ satisfying that each tree T_j is associated to a $b_{i_{\sigma(j)}}$. Hence we can rewrite (9.2.11) as

$$\overline{\delta}^{[k-1]}(b_{i_1}\cdots b_{i_n}) = \sum_{T\in\mathcal{T}_{i_1,\ldots,i_n}} \lambda'(T)\gamma(T)\left(\sum_{f\in(k-1)\text{-lin}(T)} C(f)\right), \tag{9.2.12}$$

for a certain coefficient $\gamma(T)$ to be determined. Observe that the difference between (9.2.11) and (9.2.12) is that in the former, we are considering an ordering of the trees associated to the multiset of decorated trees, while in the latter, the forest T is not ordered.

Let us rewrite now $b_{i_1}\cdots b_{i_n}$ as a monomial $b_{j_1}^{p_1}\cdots b_{j_m}^{p_m}$, with the b_{j_h} pairwise distinct. Viewing now $T\in\mathcal{T}_{i_1,\ldots,i_n}$ as a multiset of decorated trees, we use the same notation as in Definition 9.2.3 and express T as

$$T=\{T_{1,j_1}^{k_{1,1}},\ldots,T_{s_1,j_1}^{k_{s_1,1}}\}\cup\cdots\cup\{T_{1,j_m}^{k_{1,m}},\ldots,T_{s_m,j_m}^{k_{s_m,m}}\}.$$

We then have $k_{1,q}+\cdots+k_{s_q,q}=p_q$ for every $1\le q\le m$ and $p_1+\cdots+p_m=n$. The multiplicity of T in the right-hand side of the expansion of $\overline{\delta}^{[k-1]}(b_{i_1}\cdots b_{i_n})$ in (9.2.12) is then obtained as the product over $1\le j\le m$ of the number of ordered partitions of a set X of cardinal p_j into a disjoint union $X_1\sqcup\cdots\sqcup X_{s_j}$ with $|X_l|=k_{l,j}$ for $1\le l\le s_j$. This is precisely the symmetry coefficient of T, $\mathrm{sym}(T)$, that is $\gamma(T)=\mathrm{sym}(T)$ and hence

$$\overline{\delta}^{[k-1]}(b_{i_1}\cdots b_{i_n}) = \sum_{T\in\mathcal{T}_{i_1,\ldots,i_n}} \lambda(T)\,\mathrm{sym}(T)\left(\sum_{f\in(k-1)\text{-lin}(T)} C(f)\right).$$

Therefore (9.2.7) as well as the equivalent formula (9.2.8) follow as we wanted to prove. □

We finish this section by obtaining a forest-type formula for the iterated restricted coproduct $\delta_{\mathrm{irr}}^{[n]}$ on L^*, which follows as a direct corollary of the forest formulas (9.2.7) and (9.2.8).

Theorem 9.2.9 (Forest formula for the irreducible coproduct [CP22, Thm. 5.6])**.** *For* *any $b_i \in \mathcal{B}$, we have for the action of $\delta^{[k]}_{\mathrm{irr}}$:*

$$\delta^{[k]}_{\mathrm{irr}}(b_i) = \sum_{T \in \mathcal{T}_i} \sum_{f \in \mathrm{lin}(T)} \lambda(T) C(f). \tag{9.2.13}$$

Proof. Recall that $\delta^{[k]}_{\mathrm{irr}}$ is the restriction of the iterated coproduct $\delta^{[k]} : \mathbb{K}[L^*] \to \mathbb{K}[L^*]^{\otimes k}$ to a map $L^* \to (L^*)^{\otimes k}$. That is, $\delta^{[k]}(b_i)$ can be split into two terms, namely $\delta^{[k]}_{\mathrm{irr}}(b_i)$ and a linear combination of terms $b_{I_1} \otimes \cdots \otimes b_{I_k}$ such that at least one of the b_{I_j} is equal to the unit 1 or is such that $|I_j| \geq 2$. This second term is projected to 0 when we restrict $\delta^{[k]}$ in order to obtain $\delta^{[k]}_{\mathrm{irr}}$.

Now, consider a term $C(f)$ in the forest formula (9.2.8) for δ, given in Theorem 9.2.8. If f is not a surjective map, then there exists an index j such that $b_{I_j} = 1$. Hence when we restrict, such f would have a zero contribution in the expression of $\delta^{[k]}_{\mathrm{irr}}$. On the other hand, if f is a surjective map, then $|T| \geq k$. In the case that f is not a bijection, i.e. $|T| > k$, there exists $j \in [k]$ such that $f^{-1}(j)$ contains at least two elements. In this case we would have that $\prod_{x_j \in f^{-1}(j)} b_{x_j} \notin L^*$ and thus $C(f)$ is projected to 0 when we consider $\delta^{[k]}_{\mathrm{irr}}$.

Finally, take $C(f)$ in (9.2.8) such that f is a bijection. It easily follows that $C(f) \in (L^*)^{\otimes k}$ and hence this term appears in $\delta^{[k]}_{\mathrm{irr}}(b_i)$. The conclusion is that $\delta^{[k]}_{\mathrm{irr}}$ also has a forest-type formula obtained from (9.2.8) by only considering k-linearizations of T that are bijections, i.e. linearizations of T. Therefore we obtain (9.2.13). □

9.3 A new proof of the monotone cumulant-cumulant formulas

The forest formulas proved in the last section, together with the duality formulas stated in Lemma 9.1.7 and Lemma 9.1.8, provide an effective method to compute the pre-Lie exponential and the pre-Lie Magnus expansion on any locally finite connected graded pre-Lie algebra L, for instance:

$$\langle W(\alpha)|w\rangle = \sum_{n\geq 0} \frac{1}{(n+1)!} \langle r^{(n)}_{\triangleleft\alpha}(\alpha) \,|\, w\rangle = \sum_{n\geq 1} \frac{1}{n!} \langle \alpha^{\otimes n} \,|\, \delta^{[n]}_{\mathrm{irr}}(w)\rangle,$$

for any $\alpha \in L$, $w \in L^*$. In this section, we apply this novel approach to the pre-Lie algebra of words L_X. When taking the particular case of the pre-Lie algebra of infinitesimal characters associated to a non-commutative probability space, we will obtain a new and more conceptual proof of the formulas that write monotone cumulants in terms of the free and Boolean cumulants.

We start by taking X to be a finite alphabet and considering the pre-Lie algebras of words L_X given in Proposition 8.1.4. First, observe that L_X falls in the framework of the forest formulas in Section 9.2. Indeed, L_X is a locally finite connected graded pre-Lie algebra where we can identify L_X with its graded dual L_X^*, and the set of non-empty words on X, denoted by $X^\star$, is a countable basis of L_X^*. For our purposes, let us fix an ordering of the basis $X^\star = \{w_i\}_{i\in\mathbb{N}_0}$.

The first step to make in order to apply the forest formulas for the iterated coproducts is to find a description of the $\lambda_I^{i;i_0}$ coefficients defining the dual reduced coproduct

$$\overline{\delta}(w_i) = \sum_{i_0, I\neq\emptyset} \lambda_I^{i;i_0} w_{i_0} \otimes w_I,$$

where $\overline{\delta}$ is given by Proposition 9.1.10. This description is provided in the next lemma.

Lemma 9.3.1 ([CP22, Lem. 9.7])**.** *Let $\{w_i\}_{i\in\mathbb{N}_0}$ be the set of non-empty words on X. For any indexes $i, i_0, i_1, \ldots, i_s$, the coefficient $\lambda_{i_1,\ldots,i_s}^{i;i_0}$ is the number of ways in which it is possible to select subwords $v_1, v_2, \ldots, v_{s+1}$ of w_i with $v_1, v_{s+1} \neq \emptyset$, such that $w_{i_0} = v_1 v_2 \cdots v_{s+1}$, and there exists a permutation $\sigma \in S_s$ such that $w_i = v_1 w_{i_{\sigma(1)}} v_2 w_{i_{\sigma(2)}} \cdots v_s w_{i_{\sigma(s)}} v_{s+1}$.*

Recall that a fundamental combinatorial object in the cumulant-cumulant formulas is the set of irreducible non-crossing partitions $\mathrm{NC}_{\mathrm{irr}}(n)$. In the forest formulas, the combinatorial role is taken by the set of decorated trees. The next lemma shows that both notions are nicely related in the context of pre-Lie algebras of words.

Lemma 9.3.2 ([CP22, Lem. 9.8])**.** *Let w_i be a non-empty word on X and $k \geq 1$. Denote by $\mathcal{T}_i'^{,k}$ the subset of decorated trees associated to w_i, $T \in \mathcal{T}_i$, such that $\lambda(T) \neq 0$ and $|T| = k$. Then, for any $k \geq 1$, there is a surjection $G : \mathrm{NC}_{\mathrm{irr}}^k(|w_i|) \to \mathcal{T}_i'^{,k}$ such that $|G^{-1}(T)| = \lambda(T)$, for any $T \in \mathcal{T}_i'^{,k}$.*

Proof. We start by describing the surjective map G. For any $\pi \in \mathrm{NC}_{\mathrm{irr}}^k(|w_i|)$, we consider the non-planar rooted tree given by nesting tree of π, $t(\pi)$ (Remark 3.4.9). It is clear that $|t(\pi)| = k$. Now assume that $w_i = a_1 \cdots a_n$ with $a_1, \ldots, a_n \in X$. We regard $t(\pi)$ a as decorated tree T_π as follows: for any $x \in V(t(\pi))$, if x is associated to the block $V = \{j_1 < j_2 < \cdots < j_\ell\} \in \pi$, then $d_1(x)$ is the index in the list of words $X^\star = \{w_j\}_{j\in\mathbb{N}_0}$ such that

$$w_{d_1(x)} = a_{j_1} a_{j_1+1} \cdots a_{j_2} a_{j_2+1} \cdots a_{j_\ell},$$

and $d_2(x)$ is the index such that

$$w_{d_2(x)} = a_{j_1} a_{j_2} \cdots a_{j_\ell}.$$

Since clearly $d_1(\mathrm{rt}(T_\pi)) = w_i$ and $\lambda_{d_1(\mathrm{succ}(x))}^{d_1(x);d_2(x)} \neq 0$ for any $x \in \mathrm{Intl}(T_\pi)$, then (9.2.6) implies that $\lambda(T_\pi) \neq 0$, i.e. $T_\pi \in \mathcal{T}_i'^{,k}$. We can then define $G : \mathrm{NC}_{\mathrm{irr}}^k(|w_i|) \to \mathcal{T}_i'^{,k}$ by $G(\pi) = T_\pi$, for any $\pi \in \mathrm{NC}_{\mathrm{irr}}^k(|w_i|)$.

It remains to prove that the map G is surjective. Take $T \in \mathcal{T}_i'^{,k}$. If T is a single-vertex tree, i.e. $k = 1$, we consider the single-block partition $1_{|w_i|} \in \mathrm{NC}_{\mathrm{irr}}^1(|w_i|)$. Otherwise, assume that $T = B^+_{(i;i_0)}(T_1, \dots, T_s)$, where T_j is associated to w_{i_j}, for any $1 \le j \le s$. Since $\lambda(T) \neq 0$, by definition we have that $\lambda^{i;i_0}_{i_1,\dots,i_s} = m > 0$. By the description of the $\lambda^{i;i_0}_I$ coefficients given in Lemma 9.3.1, there are m ways to take $s+1$ subwords $v_1, \dots, v_{s+1}$ of $w_i = a_1 \cdots a_n$ such that $v_1, v_{s+1} \neq \emptyset$, $w_{i_0} = v_1 \cdots v_{s+1}$, and

$$w_i = v_1 w_{i_{\sigma(1)}} v_2 w_{i_{\sigma(2)}} \cdots v_s w_{i_{\sigma(s)}} v_{s+1}$$

for a permutation $\sigma \in S_s$. Each of the m possible selection of the subwords $v_1, \dots, v_{s+1}$ corresponds to a selection of pairwise disjoint subsets $C_1, \dots, C_{s+1} \subseteq [n]$ such that $1 \in C_1$ and $n \in C_{s+1}$. We now construct the block $V_0 := C_1 \cup \cdots \cup C_{s+1}$. The remaining indexes $[n] \backslash V_0$ can be grouped into s non-empty pairwise disjoint intervals $J_1, \dots, J_s$ such that $J_l = \{r_l, r_l+1, \dots, r_l+q\}$ is the subset of indexes such that $w_{i_{\sigma(l)}} = a_{r_l} a_{r_l+1} \cdots a_{r_l+q}$, for any $1 \le l \le s$. Due to the possible repetitions in the subwords w_{i_j}, there are $\mathrm{sym}(B^-(T))$ ways to allocate the decorated trees $T_1, \dots, T_s$ to the subsets $J_1, \dots, J_s \in [n] \backslash V_0$.

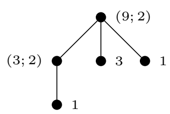
Finally, we proceed by induction on the number of blocks k. Indeed, since for any $1 \le l \le s$, T_l is associated to an interval J_p and $|T_l| < k$, we can find $\lambda(T_l)$ different irreducible non-crossing partitions $\pi_l \in \mathrm{NC}^{|T_l|}_{\mathrm{irr}}(J_p)$ such that their corresponding decorated nesting tree is T_l. Therefore, we can construct $\lambda^{i;i_0}_{i_1,\dots,i_s} \mathrm{sym}(B^-(T)) \lambda(T_1) \cdots \lambda(T_s) = \lambda(T)$ irreducible non-crossing partitions of the form

$$\pi = \{V_0\} \cup \pi_1 \cup \cdots \cup \pi_s \in \mathrm{NC}_{\mathrm{irr}}(n)$$

such that $1, n \in V_0$, $|\pi| = 1 + |T_1| + \cdots + |T_s| = |T| = k$ and $G(\pi) = T$. This completes the proof. $\square$

Notice that the surjective map G described in the above proposition is actually a bijection in the case that $w = a_1 \cdots a_n$ consists of pairwise different letters, i.e. $a_i \neq a_j$ for any $i \neq j$. In this case, it is easy to see that every $\lambda^{i;i_0}_I \in \{0, 1\}$ and also the corresponding symmetric factors are equal to 1.

Example 9.3.3. The following example provides a decorated tree T whose inverse image contains more than one element. Consider the alphabet $X = \{a\}$ and the associated pre-Lie algebra L_X. Moreover, consider the ordering of the basis $X^\star = \{a^i\}_{i \ge 1}$, where a^i stands for the concatenation of i letters a. Now, let T be the decorated tree

$T =$ 

.

Notice that

$$\lambda(T) = \lambda^{9;2}_{3,3,1} \mathrm{sym}(B^-(T)) \lambda\left(\begin{smallmatrix}\bullet^{(3,2)}\\ \vert \\ \bullet_{1}\end{smallmatrix}\right) \lambda(\bullet_3) \lambda(\bullet_1).$$

It is easy to see that $\mathrm{sym}(B^-(T)) = 2$ and $\lambda\big(\substack{\bullet\,^{(3,2)}\\ \bullet_{1}}\big) = \lambda(\bullet_3) = \lambda(\bullet_1) = 1$. Also, a simple computation shows that $\lambda^{9;2}_{3,3,1} = 3$, i.e. there are 3 terms of the form

$$a^2 \otimes a^3 \cdot a^3 \cdot a$$

in the computation of $\overline{\delta}(a^9)$. Thus we have that $\lambda(T) = 6$. The six elements in $\mathrm{NC}^5_{\mathrm{irr}}(9)$ are listed in the following picture.

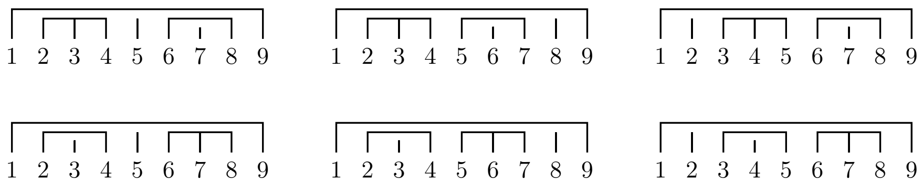


Figure 9.1: The six elements in $G^{-1}(T)$.

The previous lemma will be essential in applying the forest formulas to compute the pre-Lie exponential in the pre-Lie algebra of words, shown in the following theorem.

Theorem 9.3.4 ([CP22, Thm. 9.9])**.** *Let L_X be the pre-Lie algebra of words over an alphabet X. Then, for $\alpha \in L_X$ and a word $w_i \in X^\star \subset L_X^*$ such that $w_i = a_1 \cdots a_n$ with $a_1, \ldots, a_n \in X$, the action of the pre-Lie exponential (Agrachev-Gamkrelidze) operator W is given by*

$$\langle W(\alpha)|w_i\rangle = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!} \alpha_\pi(w_i),$$

where $\alpha_\pi(w_i) := \prod_{V \in \pi} \langle \alpha | w_V \rangle$ for any $\pi \in \mathrm{NC}(n)$, with $w_V = a_{j_1} a_{j_2} \cdots a_{j_r}$ is the subword of w_i determined by the block $V = \{j_1 < j_2 < \cdots < j_r\}$.

Proof. By the definition of the pre-Lie exponential W on L_X, Lemma 9.1.7 and the forest formula (9.2.13), we have that $\langle W(\alpha)|w_i\rangle = \sum_{k \geq 1} \frac{1}{k!} \langle r^{(k-1)}_{\triangleleft\alpha}(\alpha)|w_i\rangle$. Consequently, for any $k \geq 1$, we compute

$$\begin{aligned}
\langle r_{\lhd\alpha}^{(k-1)}(\alpha)|w_i\rangle &= \langle \alpha\otimes\cdots\otimes\alpha|\delta_{\mathrm{irr}}^{[k]}(w_i)\rangle \\
&= \left\langle \alpha\otimes\cdots\otimes\alpha \,\middle|\, \sum_{T\in\mathcal{T}_i}\sum_{f\in\mathrm{lin}(T)} \lambda(T)C(f) \right\rangle \\
&= \sum_{\substack{T\in\mathcal{T}_i' \\ |T|=k}} \lambda(T) \sum_{f\in\mathrm{lin}(T)} \prod_{j=1}^{k} \left\langle \alpha|w_{d_2(f^{-1}(j))}\right\rangle \\
&= \sum_{\substack{T\in\mathcal{T}_i' \\ |T|=k}} \lambda(T)m(T) \prod_{x\in V(T)} \langle \alpha|w_{d_2(x)}\rangle,
\end{aligned}$$

where $\mathcal{T}_i'$ stands for the subset of decorated trees associated to w_i, $T\in\mathcal{T}_i$, such that $\lambda(T)\neq 0$, and $m(T)=|\mathrm{lin}(T)|$. Also, we used that the product $\prod_{j=1}^{k}\langle\alpha|w_{d_2(f^{-1}(j))}\rangle$ does not depend of the linearization $f\in\mathrm{lin}(T)$. Thus, we can apply Lemma 9.3.2 to rearrange the sum on the right-hand side of the last equation as a sum indexed in terms of irreducible non-crossing partitions as:

$$\langle \alpha\otimes\cdots\otimes\alpha|\delta_{\mathrm{irr}}^{[k]}(w_i)\rangle = \sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}^{k}(n)} m(t(\pi)) \prod_{V\in\pi}\langle\alpha|w_V\rangle. \tag{9.3.1}$$

The above equation says, in particular, that k cannot be greater or equal than n since an irreducible non-crossing partition $\pi\in\mathrm{NC}_{\mathrm{irr}}(n)$ has at most $n-1$ blocks. Hence the pre-Lie exponential can be written in the following way:

$$\begin{aligned}
\langle W(\alpha)|w_i\rangle &= \sum_{k=1}^{\infty}\frac{1}{k!}\langle \alpha\otimes\cdots\otimes\alpha|\delta_{\mathrm{irr}}^{[k]}(w_i)\rangle \\
&= \sum_{k=1}^{n-1}\frac{1}{k!}\sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}^{k}(n)} m(t(\pi))\alpha_\pi(w_i) \\
&= \sum_{k=1}^{n-1}\sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}^{k}(n)} \frac{m(t(\pi))}{|\pi|!}\alpha_\pi(w_i) \\
&= \sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!}\alpha_\pi(w_i),
\end{aligned}$$

where in the last equality we used that $m(t(\pi))=\frac{|\pi|!}{t(\pi)!}$. This concludes the proof of the theorem. □

Remark 9.3.5. Let $(\mathcal{A},\varphi)$ be a non-commutative probability space. By the final observation of Remark 8.1.1, we know that the pre-Lie algebra of infinitesimal characters of $T(T_+(\mathcal{A}))$, denoted by $\mathfrak{g}$, can be identified with a pre-Lie algebra of linear forms

$L_{\mathfrak{g}} \cong T_+(\mathcal{A})^*$, and at the same time, $L_{\mathfrak{g}}$ can be identified with a pre-Lie algebra of words L_X when X is a finite basis of $\mathcal{A}$ as vector space. It is important to notice that taking a finite set X is enough since we are interested in computing formulas for cumulants of finite sets of random variables $\{a_1, \dots, a_n\} \subset \mathcal{A}$. This discussion implies that Theorem 9.3.4 can be used to obtain the formula that writes Boolean and free cumulants in terms of the monotone cumulants. Indeed, with the notations in Theorem 4.3.2 and Theorem 4.2.9, taking a word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$ and $\alpha = \rho$, the relation $\beta = W(\rho)$ implies

$$b_n(a_1, \dots, a_n) = \beta(w) = \langle W(\rho)|w\rangle = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!}\rho_\pi(w) = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!} h_\pi(a_1, \dots, a_n).$$

Analogously, taking $\alpha = -\rho$ in Theorem 9.3.4, the relation $\kappa = -W(-\rho)$ implies

$$\begin{aligned} k_n(a_1, \dots, a_n) &= -\langle W(-\rho)|w\rangle \\ &= -\sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{t(\pi)!}(-\rho)_\pi(w) \\ &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{(-1)^{|\pi|-1}}{t(\pi)!} h_\pi(a_1, \dots, a_n). \end{aligned}$$

We have therefore recovered the formulas (3.4.16) and (3.4.17) from [AHLV15].

Remark 9.3.6. The ideas in the proof of Theorem 9.3.4 also allow us to obtain a proof of Proposition 8.2.1. Indeed, if $\alpha_1, \dots, \alpha_{n+1} \in L_X$ and $w_i = a_1 \cdots a_m \in X^\star$, Lemma 9.1.7 and the forest formula (9.2.13) imply

$$\begin{aligned} \big\langle (\cdots(\alpha_{n+1} \lhd \alpha_n) \lhd \cdots) \lhd \alpha_1 | w_i \big\rangle &= \langle \alpha_{n+1} \otimes \cdots \otimes \alpha_1 | \delta_{\mathrm{irr}}^{[n+1]}(w_i)\rangle \\ &= \left\langle \alpha_{n+1} \otimes \cdots \otimes \alpha_1 \,\middle|\, \sum_{T \in \mathcal{T}_i} \sum_{f \in \mathrm{lin}(T)} \lambda(T) C(f) \right\rangle \\ &= \sum_{\substack{T \in \mathcal{T}_i' \\ |T| = n+1}} \lambda(T) \sum_{f \in \mathrm{lin}(T)} \prod_{j=1}^{n+1} \big\langle \alpha_{n+2-j} | w_{d_2(f^{-1}(j))} \big\rangle . \end{aligned}$$

Using Lemma 9.3.2, we can write

$$\begin{aligned} \big\langle (\cdots(\alpha_{n+1} \lhd \alpha_n) \lhd \cdots) \lhd \alpha_1 | w_i \big\rangle &= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}^{n+1}(m)} \sum_{f \in \mathrm{lin}(t(\pi))} \prod_{j=1}^{n+1} \big\langle \alpha_{n+2-j} | w_{d_2(f^{-1}(j))} \big\rangle \\ &= \sum_{\pi \in \mathcal{M}_{\mathrm{irr}}^{n+1}(m)} \prod_{j=1}^{n+1} \big\langle \alpha_{n+2-j} | w_{d_2(f^{-1}(j))} \big\rangle , \end{aligned}$$

which is precisely the formula (8.2.1) in Proposition 8.2.1. Notice that to write the last sum above, we used the clear bijection $\mathcal{M}_{\mathrm{irr}}^{n+1}(m) \leftrightarrow \{(\pi, f) : \pi \in \mathrm{NC}_{\mathrm{irr}}^{n+1}(m),\ f \in \mathrm{lin}(t(\pi))\}$.

With the motivation of proving the converse relations with the machinery developed in this chapter, we turn to the problem of computing the pre-Lie Magnus expansion of a word α in the pre-Lie algebra of words L_X. In a similar way that in Chapter 8, a first approach could consist of using the recursive definition of Ω (Proposition 7.3.4) which involves iterations of the pre-Lie product $r^{(n)}_{\triangleleft\Omega(\alpha)}(\alpha)$ and doing a similar analysis that in the proof of Theorem 9.3.4. Nevertheless, the richer structure provided by the symmetric braces and the associative product on the enveloping algebra of L_X allows us to tackle the problem more elegantly and directly.

Recall that Theorem 2.3.33 is another description of the pre-Lie Magnus expansion in terms of the map

$$\mathrm{sol}_1(\alpha_1\cdots\alpha_n) := \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\sum_{\substack{\pi\in\mathcal{OP}^k(n)\\ \pi=(I_1,\dots,I_k)}}\alpha_{I_1}*\cdots*\alpha_{I_k},\quad \forall\,\alpha_1,\dots,\alpha_n\in L_X.$$

It is important to notice that the above formula resembles the expression (7.3.3) defining Murua coefficients that, in Section 7.3, were obtained during the computation of the logarithm with respect to the convolution product on $G(T(T_+(\mathcal{A})))$.

The main theorem of this section allows us to recover the multivariate formulas for the free-to-monotone and Boolean-to-monotone cumulant relations proved in Theorem 8.3.6. The result is proved by combinatorial analysis in the computation of Ω and using the corresponding forest formula to compute iterations of symmetric braces.

Theorem 9.3.7 ([CP22, Thm. 9.11]). *Let L_X be the pre-Lie algebra of words over an alphabet X. Then, for $\alpha\in L_X$ and a word $w_i\in X^\star\subset L_X^*$ such that $w_i=a_1\cdots a_n$ with $a_1,\dots,a_n\in X$, the action of the pre-Lie Magnus operator Ω is given by*

$$\langle\Omega(\alpha)|w_i\rangle=\sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}(n)}\omega(t(\pi))\alpha_\pi(w_i). \tag{9.3.2}$$

Proof. By Theorem 2.3.33 and the definition of sol_1, we start the computation for $\alpha\in L_X$ and $w_i=a_1\cdots a_n\in L_X^*$ as follows:

$$\begin{aligned}\langle\Omega(\alpha)|w_i\rangle &= \big\langle\,\mathrm{sol}_1(\exp^{\cdot}(\alpha))|w_i\big\rangle\\ &= \sum_{m\geq 0}\frac{1}{m!}\langle\mathrm{sol}_1(\alpha^{\cdot m})|w_i\rangle\\ &= \sum_{m\geq 0}\frac{1}{m!}\sum_{k=1}^{m}\frac{(-1)^{k-1}}{k}\sum_{\substack{j_1,j_2,\dots,j_k\geq 1\\ j_1+j_2+\cdots+j_k=m}}\binom{m}{j_1,j_2,\dots,j_k}\langle\alpha^{j_1}*\alpha^{j_2}*\cdots*\alpha^{j_k}|w_i\rangle\end{aligned} \tag{9.3.3}$$

Now we recall how the associative product is constructed from the braces in (2.3.9): for any symmetric brace algebra V, we have that

$$(a_1 \cdots a_l) * (b_1 \cdots b_m) = \sum_f B_0(a_1\{B_1\}) \cdots (a_l\{B_l\}),$$

for any $a_1, \ldots, a_l, b_1, \ldots, b_m \in V$, where the sum is over all the maps $f : \{1, \ldots, m\} \to \{0, \ldots, l\}$. Returning to our case of interest, if in the expression $\alpha^{j_1} * \left(\alpha^{j_2} * \cdots * \alpha^{j_k}\right)$ we have that $j_1 \geq 2$, then $\alpha^{j_1} * \left(\alpha^{j_2} * \cdots * \alpha^{j_k}\right)$ is a monomial in $\mathbb{K}[L_X]$ of polynomial degree greater than one. Since w_i has polynomial degree one, then we have that only the case $j_1 = 1$ may produce a non-zero contribution in the last sum of the right-hand side of (9.3.3). Furthermore, the only term in the right-hand side of (2.3.9) for the product $\alpha * \alpha^{j_2}$ that may produces a non-zero contribution in (9.3.3) is given when $B_0 = 1$, and this term is precisely

$$\alpha\{\underbrace{\alpha, \ldots, \alpha}_{j_2 \text{ times}}\} =: \alpha\{\alpha^{j_2}\} \in L_X.$$

Since $\alpha\{\alpha^{j_2}\}$ is again an element in L_X, we will have that the only one term in the iterated product $(\cdots((\alpha * \alpha^{j_2}) * \alpha^{j_3}) * \cdots) * \alpha^{j_k}$ that may produce a non-zero contribution in the last sum of the right-hand side of (9.3.3) is given by the iterated brace products

$$(\cdots((\alpha\{\alpha^{j_2}\})\{\alpha^{j_3}\})\cdots)\{\alpha^{j_k}\} \in L_X.$$

We have then proved that the iterated $*$ products can be replaced by iterated brace products. Then we can use Lemma 9.1.8 to obtain

$$\langle \Omega(\alpha) | w_i \rangle = \sum_{m \geq 0} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{\substack{j_2, \ldots, j_k \geq 1 \\ 1 + j_2 + \cdots + j_k = m}} \frac{1}{j_2! \cdots j_k!} \langle \alpha \otimes \alpha^{j_2} \otimes \cdots \otimes \alpha^{j_k} \,|\, \overline{\delta}^{[k]}(w_i) \rangle.$$

Observe that, by the definition of the reduced coproduct, a value $m > n$ will produce a zero contribution in $\langle \Omega(\alpha) | w_i \rangle$. Hence, by the forest formula for the iterated reduced coproduct (9.2.7) in Theorem 9.2.8, we have

$$\langle \Omega(\alpha) | w_i \rangle = \sum_{m=1}^{n} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{T \in \mathcal{T}_i'} \sum_{f \in k\text{-lin}(T)} \lambda(T) \sum_{\substack{j_2, \ldots, j_k \geq 1 \\ 1 + j_2 + \cdots + j_k = m}} \frac{1}{j_2! \cdots j_k!} \langle \alpha \otimes \alpha^{j_2} \otimes \cdots \otimes \alpha^{j_k} \,|\, C(f) \rangle,$$

where $\mathcal{T}_i'$ stands for the subset of decorated trees $T \in \mathcal{T}_i$ associated to w_i such that $\lambda(T) \neq 0$. Now, given $1 \leq m \leq n$, $1 \leq k \leq m$, and a tuple $(j_2, \ldots, j_k)$ such that $1 + j_2 + \cdots + j_k = m$, the only decorated trees $T \in \mathcal{T}_i'$ such that there exists a $f \in k\text{-lin}(T)$ which produces a non-zero contribution for $\langle \alpha \otimes \alpha^{j_2} \otimes \cdots \otimes \alpha^{j_k} \,|\, C(f) \rangle$ must satisfy that $|T| = m$. On the other hand, by definition of $C(f)$ (Definition 9.2.7), for $T \in \mathcal{T}_i'$ and

$f \in k\text{-lin}(T)$ we have

$$\begin{aligned}
\langle \alpha \otimes \alpha^{j_2} \otimes \cdots \otimes \alpha^{j_k} | C(f) \rangle &= \langle \alpha | w_{d_2(f^{-1}(1))} \rangle \prod_{l=2}^{k} \langle \alpha^{j_l} | w_{d_2(f^{-1}(l))} \rangle \\
&= \langle \alpha | w_{d_2(f^{-1}(1))} \rangle \prod_{l=2}^{k} \left(\sum_{\sigma \in S_{j_l}} \prod_{h \in f^{-1}(l)} \langle \alpha | w_h \rangle \right) \\
&= j_2! \cdots j_k! \prod_{x \in V(T)} \langle \alpha | w_{d_2(x)} \rangle,
\end{aligned}$$

where in the second equality, we used the duality described in (9.1.1). Combining this with the fact that, given $f \in k\text{-lin}(T)$, there is exactly one tuple $(j_2', \dots, j_k')$ of positive integers such that $\langle \alpha \otimes \alpha^{j_2'} \otimes \cdots \otimes \alpha^{j_k'} | C(f) \rangle \neq 0$, and that it is given by $j_l' = |f^{-1}(l)|$ for $2 \leq l \leq k$, it follows that

$$\begin{aligned}
\langle \Omega(\alpha) | w_i \rangle &= \sum_{m=1}^{n} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{\substack{T \in \mathcal{T}_i' \\ |T|=m}} \lambda(T) \sum_{f \in k\text{-lin}(T)} \prod_{x \in V(T)} \langle \alpha | w_{d_2(x)} \rangle \\
&= \sum_{m=1}^{n} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{\substack{T \in \mathcal{T}_i' \\ |T|=m}} \lambda(T) \prod_{x \in V(T)} \langle \alpha | w_{d_2(x)} \rangle \sum_{f \in k\text{-lin}(T)} 1 \\
&= \sum_{m=1}^{n} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{\substack{T \in \mathcal{T}_i' \\ |T|=m}} \lambda(T) \prod_{x \in V(T)} \langle \alpha | w_{d_2(x)} \rangle \omega_k(T),
\end{aligned}$$

where we recall that $\omega_k(T) = |k\text{-lin}(T)|$ counts the number of k-linearizations of T. As in the proof of Theorem 9.3.4, we use Lemma 9.3.2 to rewrite the above equation in terms of non-crossing partitions as follows:

$$\begin{aligned}
\langle \Omega(\alpha) | w_i \rangle &= \sum_{m=1}^{n} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}^{m}(n)} \omega_k(t(\pi)) \prod_{V \in \pi} \langle \alpha | w_V \rangle \\
&= \sum_{m=1}^{n} \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}^{m}(n)} \alpha_\pi(w_i) \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \omega_k(t(\pi)) \\
&= \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \omega(t(\pi)) \alpha_\pi(w_i),
\end{aligned}$$

which concludes the proof. □

Remark 9.3.8. In the context of Remark 9.3.5, we can use Theorem 9.3.7 to obtain the cumulant-cumulant formulas derived in Theorem 8.3.6. Indeed, by taking $\alpha = \beta$, from $\rho = \Omega(\beta)$ we obtain Boolean-to-monotone cumulant relation (8.3.4) in Corollary 8.3.7. On the other hand, by considering $\alpha = -\kappa$, from $\rho = -\Omega(-\kappa)$ we obtain the free-to-monotone cumulant relation (8.3.5).

Notice that we can take the recursive definition of the Magnus operator instead to give a proof of Theorem 9.3.7 as we did in Theorem 8.3.6. The strategy consists of using the recursion (7.3.4) for Murua coefficients. This alternative proof is included for completeness, as it provides further insights into the combinatorics of free probability.

Alternative proof of Theorem 9.3.7. By definition of the recursive of the pre-Lie Magnus operator given in Proposition 2.3.28, Lemma 9.1.7, and the forest formula (9.2.13), we have that

$$\begin{aligned}
\langle \Omega(\alpha)|w_i\rangle &= \sum_{m\geq 0}\frac{B_m}{m!}\langle r^{(m)}_{\Omega(\alpha)}(\alpha)|w_i\rangle\\
&= \sum_{m\geq 0}\frac{B_m}{m!}\langle \alpha\otimes\Omega(\alpha)\otimes\cdots\otimes\Omega(\alpha)|\delta^{[m+1]}_{\mathrm{irr}}(w_i)\rangle\\
&= \sum_{m=0}^{n-1}\frac{B_m}{m!}\sum_{\substack{T\in\mathcal{T}'_i\\|T|=m+1}}\sum_{f\in\mathrm{lin}(T)}\lambda(T)\langle \alpha\otimes\Omega(\alpha)\otimes\cdots\otimes\Omega(\alpha)|C(f)\rangle,
\end{aligned}$$

where we recall that $\mathcal{T}'_i$ is the subset of decorated trees associated to w_i, $T\in\mathcal{T}_i$, such that $\lambda(T)\neq 0$. Also, observe that the sum in the last equation above is bounded by $n-1$ because of degree reasons in $\delta^{[m+1]}_{\mathrm{irr}}(w_i)$. Moreover, the same argument that in the proof of Theorem 9.3.4, the term $\langle \alpha\otimes\cdots\otimes\Omega(\alpha)|C(f)\rangle$ does not depend of f. Hence, by using Lemma 9.3.2, we get that

$$\begin{aligned}
\langle \Omega(\alpha)|w_i\rangle &= \sum_{T\in\mathcal{T}'_i}\frac{B_{|T|-1}}{(|T|-1)!}m(T)\lambda(T)\langle\alpha|w_{d_2(\mathrm{rt}(T))}\rangle\prod_{x\in V(T)\setminus\{\mathrm{rt}(T)\}}\langle\Omega(\alpha)|w_{d_2(x)}\rangle\\
&= \sum_{\substack{\pi\in\mathrm{NC}_{\mathrm{irr}}(n)\\1,n\in V_0}}\frac{B_{|\pi|-1}}{t(\pi\backslash\{V_0\})!}\langle\alpha|w_{V_0}\rangle\prod_{\substack{W\in\pi\\W\neq V_0}}\langle\Omega(\alpha)|w_W\rangle.
\end{aligned}$$

Since $|w_W|<n$ for any $W\in\pi$, we can conclude by induction in the same way that in the proof of Theorem 8.3.6 and obtain the following by the recursive definition of Murua coefficients (Proposition 7.3.4):

$$\begin{aligned}
\langle \Omega(\alpha)|w_i\rangle &= \sum_{\substack{\mu\in\mathrm{NC}_{\mathrm{irr}}(n)\\1,n\in V_0}}\sum_{S\in K(t(\mu'))}\frac{B_{|S|}}{S!}\langle\alpha|w_{V_0}\rangle\left(\prod_{W\in\mu'}\langle\alpha|w_W\rangle\right)\omega(C^S(t(\mu')))\\
&= \sum_{\mu\in\mathrm{NC}_{\mathrm{irr}}(n)}\omega(t(\mu))\alpha_\mu(w_i).
\end{aligned}$$

□

Remark 9.3.9. The proof of Theorem 8.3.6 ([CEFPP21, Thm. 3]) uses fundamentally Proposition 8.2.1 ([CEFPP21, Prop. 4]), which gives a combinatorial expression in terms of monotone non-crossing partitions for the iterated pre-Lie product. In the approach explained in the last proof of the present section, the role of Proposition 8.2.1 is taken by the forest formula (9.2.13).

9.4 Appendix: An application to the pre-Lie algebra of rooted trees

As we advertised at the beginning of the present chapter, the methods developed throughout the previous sections provide a framework to compute effectively the pre-Lie exponential and the pre-Lie Magnus operator on a locally finite connected graded pre-Lie algebra L. In order to show that our machinery not only works on pre-Lie algebras of words, we will also compute the W and Ω operators on the generator of the free pre-Lie algebra of rooted trees. Although these results are known, our method provides new insights into the coefficients appearing in the tree-series expansion of W and Ω.

We start by recalling that the linear span of the set of non-planar rooted trees $\mathcal{T}$ is a pre-Lie algebra, where, for any $t, t' \in \mathcal{T}$, the pre-Lie product $t \lhd t'$ is given by the sum of the trees obtained by grafting the root of t' to the vertices of t:

$$t \lhd t' = \sum_{v \in V(t)} t \leftarrow_v t'.$$

Even more, $(L^{\mathcal{T}}, \lhd)$ is the free pre-Lie algebra over the generator $\bullet$ (Proposition 2.3.21). Also, it can be easily checked that $L^{\mathcal{T}}$ is a locally finite connected graded pre-Lie algebra, where the n-th homogeneous component $L^{\mathcal{T}}_n$ is given by the linear span of the trees with exactly n vertices. In other words, the grading is given by $\deg(t) = |t| = |V(t)|$, for any $t \in \mathcal{T}$.

The previous discussion implies that the formulas for the iterated pre-Lie product and the iterated symmetric braces in Lemma 9.1.7 and Lemma 9.1.8 can be applied in the setting of the pre-Lie algebra $L^{\mathcal{T}}$. Furthermore, Proposition 2.3.22 establishes that the dual of the enveloping algebra of $L^{\mathcal{T}}$ is precisely the Connes-Kreimer Hopf algebra. Observe that the duality pairing described in (2.2.8) allows to identify $L^{\mathcal{T}}$ with its graded dual $(L^{\mathcal{T}})^*$, where $\mathcal{T}$ is a countable basis of $(L^{\mathcal{T}})^*$. Therefore, we are also in the setting of the forest formulas for iterated coproducts. For the following computations, we set an ordering $\mathcal{T} = \{t_i\}_{i \geq 0}$ and $t_0 := \bullet$, the single-vertex tree.

Remark 9.4.1. Let $t \in \mathcal{T}$. Then the duality pairing described in (2.2.8) reads

$$\langle t \,|\, t \rangle = \sigma(B^+(t)) = \sigma(t). \tag{9.4.1}$$

The last equality follows from the definition of the internal symmetry factor $\sigma(t) = |\operatorname{Aut}(t)|$.

The following proposition shows how the forest formulas are applied in $L^{\mathcal{T}}$ to compute the pre-Lie exponential of $\bullet$.

Proposition 9.4.2. *For the generator $\bullet$ of the free pre-Lie algebra $L^{\mathcal{T}}$, we have that*

$$W(\bullet) = \sum_{t\in\mathcal{T}} \frac{|t|!}{\sigma(t)t!}t. \tag{9.4.2}$$

Proof. Let us expand the pre-Lie exponential in the basis of rooted trees as

$$W(\bullet) = \sum_{t\in\mathcal{T}} c(t)t.$$

Our objective is to describe the coefficients $c(t)$ for all $t \in \mathcal{T}$. For this purpose, consider a tree $t_i \in (L^{\mathcal{T}})^*$ with $|t_i| = k$ vertices. Observe that, by definition of the pre-Lie product, t_i only appears in the expansion of $\frac{r^{(k-1)}_{\triangleleft\bullet}(\bullet)}{k!}$. Actually, it is easy to see that $r^{(k-1)}_{\triangleleft\bullet}(\bullet)$ is a linear combination of the set of non-planar rooted trees with exactly k vertices. Now, by the characterization of W as a series of iterated pre-Lie products, we have

$$\langle W(\bullet)|t_i\rangle = \frac{1}{k!}\langle r^{(k-1)}_{\triangleleft\bullet}(\bullet)|t_i\rangle = \frac{1}{k!}\langle c(t_i)t_i|t_i\rangle = \frac{1}{k!}c(t_i)\sigma(t_i), \tag{9.4.3}$$

where in the last equality, we used (9.4.1). On the other hand, by using Lemma 9.1.7 and the forest formula (9.2.13) with the basis $\mathcal{T} = \{t_j\}_{j\geq 0}$ of $(L^{\mathcal{T}})^*$, we obtain that

$$\begin{aligned} \langle r^{(k-1)}_{\triangleleft\bullet}(\bullet)|t_i\rangle &= \langle \bullet\otimes\cdots\otimes\bullet|\delta^{[k]}_{\mathrm{irr}}(t_i)\rangle \\ &= \sum_{\substack{T\in\mathcal{T}_i\\|T|=k}} \sum_{f\in\mathrm{lin}(T)} \lambda(T)\langle \bullet\otimes\cdots\otimes\bullet|C(f)\rangle. \end{aligned} \tag{9.4.4}$$

By definition of the duality pairing, the non-vanishing terms in the above sum correspond to $T \in \mathcal{T}_i$ such that $\lambda(T) \neq 0$ and

$$C(f) = \bullet\otimes\cdots\otimes\bullet$$

for any $f \in \mathrm{lin(T)}$. Note that this happens when $T \in \mathcal{T}_i$ is a decorated tree such that $d_2(x) = 0$ for any vertex x of T, where we recall that $t_0 = \bullet$. Let us show by induction on the number of vertices of t_i that there is only one tree $T \in \mathcal{T}_i$ that provides a non-zero contribution in (9.4.4): the decorated tree denoted $T_i^{\bullet}$, which is the tree t_i where each vertex v is decorated by the pair $(i_v; 0)$, where t_{i_v} is the maximal subtree of t_i whose root is v.

Write $t_i = B^+(t_{i_1}\cdots t_{i_n})$ as (usual) non-planar rooted trees. Recall that $\delta^{[k]}_{\mathrm{irr}}$ is computed by iterating the coproduct $\delta^{[k]}$ and then taking the corresponding restrictions in the domain and codomain. Now, the calculation of $\overline{\delta}^{[k]}(t_i)$ in the proof of Theorem 9.2.8 says that only the index $i_0 = 0$ satisfies that $C(f) = \bullet\otimes\cdots\otimes\bullet$, and the only term to be kept is $\bullet \otimes t_{i_1} \dots t_{i_n}$, by the definition of the Connes-Kreimer coproduct. Thereafter, we need to compute

$$\delta^{[k-1]}(t_{i_1} \cdots t_{i_n}) = \delta^{[k-1]}(t_{i_1}) \cdots \delta^{[k-1]}(t_{i_n}).$$

Observe that $|t_{i_j}| < |t_i|$ for each $1 \leq j \leq n$, and then which we can apply the induction hypothesis: the only tree in $\mathcal{T}_{i_j}$ that contributes to the calculation of $\langle r_\bullet^{(k-1)}(\bullet)|t_i\rangle$ is $T_{i_j}^\bullet$, for each $1 \leq j \leq n$. Therefore, we can conclude that the only decorated tree that contributes to the calculation of $\langle r_\bullet^{(k-1)}(\bullet)|t_i\rangle$ is obtained by grafting $T_{i_1}^\bullet, \dots, T_{i_n}^\bullet$ to a common root $\bullet$ decorated by $(i;0)$, and this is precisely $T_i^\bullet$, as we wanted.

Now, if $T_i^\bullet = B_{(i;0)}^+(T_{i_1}^\bullet \cdots T_{i_n}^\bullet)$, then we can rewrite the forest $F = T_{i_1}^\bullet \cdots T_{i_n}^\bullet$ as in the definition of the symmetry coefficient (9.2.4) as

$$\{(T_{a_1}^\bullet)^{k_1}\} \cup \cdots \cup \{(T_{a_p}^\bullet)^{k_p}\},$$

where the a_i are different indexes. This implies $\mathrm{sym}(F) = 1$. Also by the definition of the Connes-Kreimer coproduct, it is clear that $\lambda_{i_1,\dots,i_n}^{i;0} = 1$. By induction on the subtrees of $T_i^\bullet$, to which the same argument applies, we conclude that

$$\lambda(T_i^\bullet) = \lambda_{i_1,\dots,i_n}^{i;0}\, \mathrm{sym}(F)\lambda(T_{i_1}^\bullet) \cdots \lambda(T_{i_n}^\bullet) = 1.$$

Therefore

$$\begin{aligned}
\sum_{\substack{T\in\mathcal{T}_i\\|T|=k}} \sum_{f\in\mathrm{lin}(T)} \lambda(T)\langle \bullet \otimes \cdots \otimes \bullet | C(f)\rangle &= \sum_{f\in\mathrm{lin}(T_i^\bullet)} \langle \bullet \otimes \cdots \otimes \bullet | C(f)\rangle \\
&= \sum_{f\in\mathrm{lin}(t_i)} \prod_{i=1}^{k} \langle \bullet | \bullet \rangle \\
&= \sum_{f\in\mathrm{lin}(t_i)} 1 \\
&= m(t_i),
\end{aligned}$$

where $m(t_i) = |\,\mathrm{lin}(t_i)|$ is the number of linearizations of t_i. Thus, we can compare it with (9.4.3) and conclude

$$c(t)\sigma(t) = m(t).$$

Finally, since $m(t) = \frac{|t|!}{t!}$, it follows that

$$c(t) = \frac{m(t)}{\sigma(t)} = \frac{|t|!}{\sigma(t)t!},$$

as we wanted to show. $\square$

Remark 9.4.3. The coefficients

$$\mathrm{CM}(t) := \frac{|t|!}{\sigma(t)t!}$$

for $t \in \mathcal{T}$ are known as the *Connes-Moscovici coefficients* and have appeared in the context of the Butcher series of Runge-Kutta methods ([But87]). The reader may see [Bro00] for a direct method to compute the CM coefficients.

Using our forest formula machinery, we now compute the pre-Lie Magnus operator of $\bullet$. The proof is similar to the proof in Theorem 9.3.7. In particular, we shall use that iterated brace products can compute the action of the sol_1 map.

Proposition 9.4.4. *For the generator* $\bullet$ *of the free pre-Lie algebra* $L^{\mathcal{T}}$*, we have that*

$$\Omega(\bullet) = \sum_{t\in\mathcal{T}} \frac{\omega(t)}{\sigma(t)} t. \tag{9.4.5}$$

Proof. The pre-Lie Magnus operator action on the generator of the free pre-Lie algebra can be written as a tree-series as

$$\Omega := \Omega(\bullet) = \sum_{t\in\mathcal{T}} d(t)t,$$

for some coefficients $d(t) \in \mathbb{K}$ to be determined. Let $t_i \in \mathcal{T}$. We will show that $d(t_i) = \omega(t_i)/\sigma(t_i)$. On one hand, (2.2.8) implies that

$$\langle \Omega | t_i \rangle = d(t_i)\sigma(t_i).$$

On the other hand, by Theorem 2.3.33 we also have that

$$\langle \mathrm{sol}_1(\exp^{\cdot}(\bullet)) | t_i \rangle = d(t_i)\sigma(t_i). \tag{9.4.6}$$

We analyze the left-hand side of the previous equation. We know that $\exp^{\cdot}(\bullet) = \sum_{n\geq 0} \frac{\bullet^n}{n!}$, where $\bullet^n$ stands for the monomial given by the polynomial product of n single-vertex trees. Thus, we can write

$$\mathrm{sol}_1(\bullet^n) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \sum_{\substack{j_1, j_2 \dots, j_k \geq 1 \\ j_1+j_2+\dots+j_k=n}} \binom{n}{j_1, j_2, \dots, j_k} \bullet^{j_1} * \bullet^{j_2} * \dots * \bullet^{j_k}. \tag{9.4.7}$$

By the definition of the product $*$, the term $\bullet^{j_1} * \dots * \bullet^{j_k}$ in the equation above is a forest with $j_1 + \dots + j_k = n$ vertices. Hence, if $|t_i| = n$ we have that

$$\langle \mathrm{sol}_1(\exp(\bullet)) | t_i \rangle = \frac{1}{n!} \langle \mathrm{sol}_1(\bullet^n) | t_i \rangle. \tag{9.4.8}$$

By the same argument that in the proof of Theorem 9.3.7, we have that only $j_1 = 1$ may produce a non-zero contribution in (9.4.7), so that the only term in the expansion of the iterated product $(\cdots((\bullet * \bullet^{j_2}) * \bullet^{j_3}) * \cdots) * \bullet^{j_k}$ that produces a non-zero contribution in (9.4.8) is precisely

$$(\cdots((\bullet\{\bullet^{j_2}\})\{\bullet^{j_3}\})\cdots)\{\bullet^{j_k}\},$$

where the braces are associated to the pre-Lie product $\triangleleft$ by Lemma 2.3.14 and we set

$$\bullet\{\bullet^j\} := \bullet\{\underbrace{\bullet, \dots, \bullet}_{j\ \text{times}}\} \in L^{\mathcal{T}}.$$

Thus, from (9.4.7) and Lemma 9.1.8, we can compute the action of the iterated brace products and obtain

$$\begin{aligned}
\frac{1}{n!}\langle \mathrm{sol}_1(\bullet^n)|t_i\rangle &= \frac{1}{n!}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\sum_{\substack{j_2\dots,j_k\geq 1\\ 1+j_2+\cdots+j_k=n}}\binom{n}{1, j_2, \dots, j_k}\langle(\cdots(\bullet\{\bullet^{j_2}\})\cdots)\{\bullet^{j_k}\}|t_i\rangle \\
&= \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\sum_{\substack{j_2\dots,j_k\geq 1\\ 1+j_2+\cdots+j_k=n}}\frac{1}{j_2!\cdots j_k!}\big\langle \bullet\otimes\bullet^{j_2}\otimes\cdots\otimes\bullet^{j_k}|\overline{\delta}^{[k]}(t_i)\big\rangle. \qquad (9.4.9)
\end{aligned}$$

Now, by using the forest formula (9.2.7), it follows that

$$\Big\langle \bullet\otimes\bullet^{j_2}\otimes\cdots\otimes\bullet^{j_k}\Big|\, \overline{\delta}^{[k]}(t_i)\Big\rangle = \sum_{T\in\mathcal{T}_i}\sum_{f\in k\text{-lin}(T)}\lambda(T)\langle\bullet\otimes\bullet^{j_2}\otimes\cdots\otimes\bullet^{j_k}|C(f)\rangle.$$

Observe that the decorated trees $T\in\mathcal{T}_i$ that produce a non-zero contribution in the above sum must satisfy that $|T| = 1 + j_2 + \cdots + j_k = n$ and $d_2(x) = 0$ for any $x \in V(T)$. Thus, as in the proof of Proposition 9.4.2, the only tree $T\in\mathcal{T}_i$ that may produce a non-zero contribution in the above equation is $T_i^\bullet = t_i$ with $d(x) = (i_x; 0)$ for any $x \in V(T_i^\bullet)$, where t_{i_x} is the maximal subtree of t_i whose root is x.

Recall that we also have that $\lambda(T_i^\bullet) = 1$. Then, we get

$$\begin{aligned}
&\sum_{\substack{j_2\dots,j_k\geq 1\\ 1+j_2+\cdots+j_k=n}}\frac{1}{j_2!\cdots j_k!}\Big\langle\bullet\otimes\bullet^{j_2}\otimes\cdots\otimes\bullet^{j_k}\Big|\,\overline{\delta}^{[k]}(t_i)\Big\rangle \\
&= \sum_{f\in k\text{-lin}(T_i^\bullet)}\sum_{\substack{j_2\dots,j_k\geq 1\\ 1+j_2+\cdots+j_k=n}}\frac{1}{j_2!\cdots j_k!}\langle\bullet\otimes\bullet^{j_2}\otimes\cdots\otimes\bullet^{j_k}|C(f)\rangle.
\end{aligned}$$

Notice that, given a k-linearization $f \in k\text{-lin}(T_i^\bullet) = k\text{-lin}(t_i)$, there is exactly one tuple $(j_2, \dots, j_k)$ such that $\langle\bullet\otimes\bullet^{j_2}\otimes\cdots\otimes\bullet^{j_k}|C(f)\rangle \neq 0$. This tuple is given by $j_m = |f^{-1}(m)| =: j'_m$, for any $2 \leq m \leq k$. Hence, the right-hand side of the above equation is equal to

$$\begin{aligned}
\sum_{f\in k\text{-lin}(t_i)} \frac{1}{j_2'!\cdots j_k'!}\langle \bullet\otimes \bullet^{j_2'}\otimes\cdots\otimes \bullet^{j_k'}|C(f)\rangle &= \sum_{f\in k\text{-lin}(t_i)} \frac{1}{j_2'!\cdots j_k'!}\prod_{m=2}^{k}\langle \bullet^{j_m'}|\bullet^{j_m'}\rangle \\
&= \sum_{f\in k\text{-lin}(t_i)} \frac{1}{j_2'!\cdots j_k'!}\prod_{m=2}^{k}\sigma(B^+(\bullet^{j_m'})) \\
&= \sum_{f\in k\text{-lin}(t_i)} \frac{1}{j_2'!\cdots j_k'!}\prod_{m=2}^{k} j_m'! \\
&= \sum_{f\in k\text{-lin}(t_i)} 1 \\
&= |k\text{-lin}(t_i)|,
\end{aligned}$$

where in the second equality, we used the duality pairing given in Theorem 2.2.21. Finally, recalling that $\omega_k(t_i) = |k\text{-lin}(t_i)|$ as in the definition of Murua coefficients, combining the previous development with (9.4.9), we obtain that

$$\frac{1}{n!}\langle \mathrm{sol}_1(\bullet^n)|t_i\rangle = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\omega_k(t_i) = \omega(t_i).$$

Therefore, using (9.4.6) we conclude that

$$d(t_i) = \frac{\omega(t_i)}{\sigma(t_i)},$$

and the proof is now complete. □

Remark 9.4.5. The approach we have developed in the previous proposition appeared in [CP22, Prop. 7.3] as a new way to understand the Magnus operator. This new approach relies on the forest formula for iterated coproducts and the particular form taken by the canonical projection from the enveloping algebra of a pre-Lie algebra to the underlying pre-Lie algebra.

In the same way that we did for Theorem 9.3.7, we can also give a proof of Proposition 9.4.4 based on the recursive definition of the Magnus operator. Although the arguments are a bit more tricky than in the former proof, the alternative proof provides new insight into the connections with [Mur06], the combinatorics of trees and free pre-Lie algebras.

From Proposition 7.3.4, for any non-planar rooted tree with $|t| > 1$ we have that

$$\omega(t) = \sum_{s\in K(B^-(t))}\frac{B_{|s|}}{s!}\omega\big(C^s(B^-(t))\big).$$

The technical part of the advertised alternative proof is to describe the objects $K(f)$ and

$C^s(f)$ in terms of trees T decorated by non-decorated trees t and the forest formulas for iterated coproducts.

Lemma 9.4.6 ([CP22, Lem. 8.3])**.** *Let $t_i \in \mathcal{T}$ with $i \neq 0$ and denote by $\mathcal{T}_i^r$ the subset of decorated trees $T \in \mathcal{T}_i$ such that $\lambda(T) \neq 0$ and $d_2(\mathrm{rt}(T)) = 0$. There exists then a surjective map $B : K(B^-(t_i)) \to \mathcal{T}_i^r$ such that*

$$|\{s \in K(B^-(t_i)) \,:\, B(s) = T\}| = \lambda(T),$$

for any $T \in \mathcal{T}_i^r$.

Proof. We start by describing the map B. To this end, assume that $t_i = B^+(t_{i_1} \cdots t_{i_k})$ and consider an element $s \in K(B^-(t_i))$ such that $s = t_{j_1} \cdots t_{j_k}$, where for each $1 \leq m \leq k$, t_{j_m} is a subtree of t_{i_m}, with the same root. Then, we define the decorated tree $B(s) := B^+_{(i;0)}(t_{j_1} \cdots t_{j_k})$, where, for each $1 \leq m \leq k$, the decoration of $x \in V(t_{j_m})$ is given by:

- $d_1(x)$ is the index associated to the subtree of t defined by x and all its descendants,
- $d_2(x)$ is the index associated to the tree obtained from $t_{d_1(x)}$ once all the subtrees determined by the elements of $V(t_{j_m})\backslash\{x\}$ and their descendants have been removed. Observe that we also delete the edges connecting these elements with their parents in $t_{d_1(x)}$.

This map is well-defined since by construction and the definition of the Connes-Kreimer coproduct, $\lambda(B(s)) \neq 0$, so that $B(s) \in \mathcal{T}_i^r$.

With the purpose of showing that B is a surjective map, we take a decorated tree $T \in \mathcal{T}_i^r$. By the proof of Proposition 9.4.2, we can write $T = B^+_{(i;0)}(T_1 \cdots T_k)$, where for $1 \leq m \leq k$, T_m is a decorated tree associated to t_{i_m}. Also, the condition $\lambda(T) \neq 0$ implies that $\lambda^{d(x)}_{d_1(\mathrm{succ}(x))} \neq 0$ for any $x \in V(T)$. Recall that by the definition of the Connes-Kreimer coproduct, $\lambda^{d(x)}_{d_1(\mathrm{succ}(x))}$ is the number of admissible cuts c of $t_{d_1(x)}$ such that $R_c(t_{d_1(x)}) = t_{d_2(x)}$ and $P_c(t_{d_1(x)}) = t_{d_1(y_1)} \cdots t_{d_1(y_r)}$ with $\mathrm{succ}(x) = \{y_1, \ldots, y_r\}$.

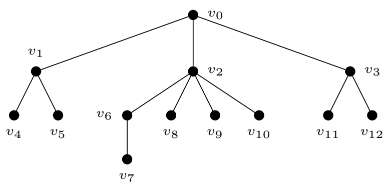
Now, observe that, for $1 \leq m \leq k$, the process of construction of T_m from t_{i_m} corresponds to a certain contraction of t_{i_m}. To be more precise, every $x \in V(T_m)$ can be seen as a vertex in t_{i_m}. Then, $x \in V(T_m)$ is obtained by collapsing to a single vertex the subtree $t_{d_2(x)}$ in t_{i_m} which has x as root. We denote $\lambda'(T_m)$ the number of ways to obtain T_m from t_{i_m} from the above procedure. Since $V(T_m)$ must contain the root of t_{i_m}, it is clear that if $|T_m| = 1$, then T_m is the single-vertex tree associated to t_{i_m}. Thus we have that $\lambda'(T_m) = 1$ in this case.

Now assume that $|T_m| > 1$, so that $B^-(T_m) = T_{m_1} \cdots T_{m_\ell}$ is a non-empty forest, where T_{m_j} is associated to a $t_{i_{m_j}}$, for each $1 \leq j \leq \ell$. Observe that the contraction process can be equivalently considered as making admissible cuts. More precisely, if x stands for the root of T_m, there are $\lambda^{d(x)}_{i_{m_1},\ldots,i_{m_\ell}}$ ways to obtain $t_{d_2(x)}$ as a subtree of t_{i_m}, with x as root, in a such a way that when we collapse $t_{d_2(x)}$ to a vertex, the remaining children of x are the roots of $t_{i_{m_1}},\ldots,t_{i_{m_k}}$. In addition, a symmetry factor appears: there are $\mathrm{sym}(B^-(T_m))$ ways to allocate the decorated trees of the decorated forest $T_{m_1}\cdots T_{m_\ell}$ to the subtrees $\{t_{i_{m_1}},\ldots,t_{i_{m_\ell}}\}$. Since the roots of $T_{m_1},\ldots,T_{m_\ell}$ are elements in $V(T_m)$, we can proceed inductively to deduce that the number of ways of obtaining T_m from t_{i_m} by the contraction process is given by

$$\lambda'(T_m) = \lambda^{d(x)}_{i_{m_1},\ldots,i_{m_\ell}}\,\mathrm{sym}(B^-(T_m))\lambda'(T_{m_1})\cdots\lambda'(T_{m_\ell}).$$

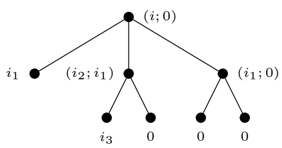
Hence, we get that $\lambda'(T_m)=\lambda(T_m)$, for any $1\le m\le k$. From this we conclude that the number $\lambda(T) = \lambda^{i;0}_{i_1,\ldots,i_k}\,\mathrm{sym}(B^-(T))\lambda(T_1)\cdots\lambda(T_k)$ counts the number of ways in which we can contract the subtrees $t_{i_1},\ldots,t_{i_k}$ in order to obtain the trees $T_1,\ldots,T_k$. In other words, there are $\lambda(T)$ elements $s\in K(B^-(t_i))$ such that $B(s)=T$, as we wanted to prove. □

Example 9.4.7. Consider the tree

$t_i =$ 

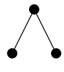
$\in \mathcal{T}$,

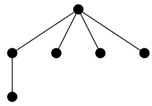
and consider the decorated tree

$T =$ 

$\in \mathcal{T}_i$,

where

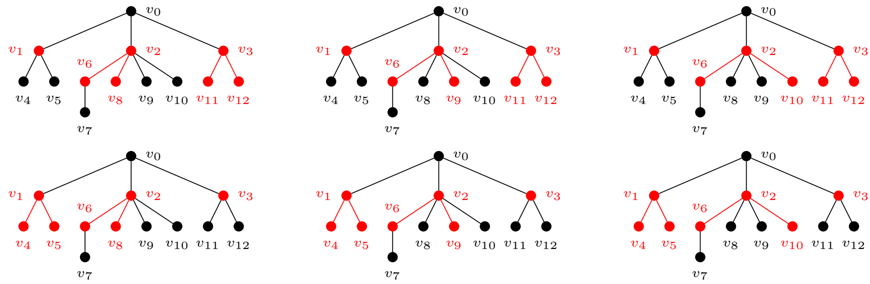
$t_{i_1} =$  ,

$t_{i_2} =$  ,

$t_{i_3} =$  .

Observe that we have labelled the vertices in t_i since the definition of $K(B^-(t_i))$ requires considering subposets of t_i and not just isomorphism classes of subposets.

One can easily compute that $\lambda(T) = \mathrm{sym}(B^-(T))\lambda^{i_2;i_1}_{i_3,0} = 2\cdot 3 = 6$. The six elements in $K(B^{-}(t_i))$ are depicted in red as subposets of t_i as follows:



The red-coloured edges are the edges cut in the selected admissible cuts to construct out T from t_i.

Remark 9.4.8. Let $t_i \in \mathcal{T}$ and $s \in K(B^{-}(t_i))$. From the definition of $C^s(B^{-}(t_i))$ and the construction of $B(s) = T$ in the proof of Lemma 9.4.6, one can readily check that

$$C^s(B^{-}(t_i)) = \big\{t_{d_2(x)} \,:\, x \in V(T)\backslash\{\mathrm{rt}(T)\}\big\}.$$

After the above technical observations, the advertised alternative proof of Proposition 9.4.4 is the following:

Alternative proof of Proposition 9.4.4. We start by writing

$$\Omega := \Omega(\bullet) = \sum_{t\in\mathcal{T}} d(t)t$$

for the pre-Lie Magnus operator. We show that $d(t) = \omega(t)/\sigma(t)$, for every $t \in \mathcal{T}$, by using the recursive definitions of Murua coefficients and the Magnus operator. On one hand, (2.2.8) implies that

$$\langle \Omega | t\rangle = d(t)\sigma(t) =: d'(t). \tag{9.4.10}$$

for any $t \in \mathcal{T}$. On the other hand, for an element $t \in \mathcal{T}$ we have

$$\begin{aligned}\langle \Omega \,|\, t\rangle &= \sum_{m\geq 0} \frac{B_m}{m!}\langle r_{\Omega}^{(m)}(\bullet)\,|\,t\rangle \\ &= \sum_{m=0}^{|t|-1} \frac{B_m}{m!}\big\langle \bullet \otimes \Omega \otimes \cdots \otimes \Omega | \delta_{\mathrm{irr}}^{[m+1]}(t)\big\rangle,\end{aligned}$$

where to get the upper limit of the sum in the last equality, we used that if s, t are trees such that $|s| > |t|$, then $\langle s|t\rangle = 0$. Notice that if $t = \bullet$, then $\langle \Omega \,|\, t\rangle = 1$. If $t \neq \bullet$, by using the forest formula (9.2.13), the previous equation can be rewritten (with $t = t_i$) as

$$\langle \Omega | t_i \rangle = \sum_{m=0}^{|t|-1} \frac{B_m}{m!} \sum_{\substack{T \in \mathcal{T}_i \\ |T|=m+1}} \sum_{f \in \mathrm{lin}(T)} \lambda(T) \langle \bullet \otimes \Omega \otimes \cdots \otimes \Omega | C(f) \rangle, \tag{9.4.11}$$

Observe that, if $T \in \mathcal{T}_i$ is such that $d_2(\mathrm{rt}(T)) \neq 0$, then $\langle \bullet \otimes \Omega \otimes \cdots \otimes \Omega | C(f) \rangle = 0$ for any $f \in \mathrm{lin}(T)$. Moreover, since $f^{-1}(1) = \mathrm{rt}(T)$ for any $f \in \mathrm{lin}(T)$, then $\langle \bullet \otimes \Omega \otimes \cdots \otimes \Omega | C(f) \rangle$ does not depend of f. Hence the above sum can be restricted to the set $\mathcal{T}_i^r$ defined in Lemma 9.4.6, and the right-hand side of (9.4.11) can be rearranged as

$$\langle \Omega \,|\, t_i \rangle = \sum_{T \in \mathcal{T}_t^i} \frac{B_{|T|-1}}{(|T|-1)!} m(T) \lambda(T) \prod_{\substack{x \in V(T) \\ x \neq \mathrm{rt}(T)}} \langle \Omega | t_{d_2(x)} \rangle. \tag{9.4.12}$$

Thus, by Lemma 9.4.6, Remark 9.4.8, and noticing that for $s \in K(B^-(t_i))$ such that $B(s) = T$ we have

$$m(T) = \frac{|T|!}{T!} = \frac{|T|!}{|T| B^-(T)!} = \frac{(|s|+1)!}{(|s|+1)s!}$$

by definition of tree factorial, we conclude that the right-hand side of (9.4.12) can be expressed as follows:

$$\sum_{s \in K(B^-(t_i))} \frac{B_{|s|}}{|s|!} \frac{(|s|+1)!}{(|s|+1)s!} \prod_{S \in C^s(B^-(t_i))} d'(S) = \sum_{s \in K(B^-(t_i))} \frac{B_{|s|}}{s!} \prod_{S \in C^s(B^-(t_i))} d'(S).$$

We have shown that $d'(t)$ satisfies the recursion (7.3.4) with the initial condition $d'(\bullet) = 1$. Therefore $\omega(t) = d'(t) = \sigma(t) d(t)$, which implies that $d(t) = \omega(t)/\sigma(t)$ for any $t \in \mathcal{T}$, as we wanted to show. □

Bibliography

[AC21] Octavio Arizmendi and Adrian Celestino. Monotone cumulant-moment formula and Schröder trees. *arXiv preprint arXiv:2111.02179. Accepted in SIGMA*, 2021.

[AG81] Andrei A. Agrachev and Revaz V. Gamkrelidze. Chronological algebras and nonstationary vector fields. *Journal of Soviet Mathematics*, 17(1):1650–1675, 1981.

[AHLV15] Octavio Arizmendi, Takahiro Hasebe, Franz Lehner, and Carlos Vargas. Relations between cumulants in noncommutative probability. *Advances in Mathematics*, 282:56–92, 2015.

[Akh20] Naum I. Akhiezer. *The classical moment problem and some related questions in analysis*. SIAM, 2020.

[Ans04] Michael Anshelevich. Appell polynomials and their relatives. *International Mathematics Research Notices*, 2004(65):3469–3531, 2004.

[BGN03] Philippe Biane, Frederick Goodman, and Alexandru Nica. Non-crossing cumulants of type B. *Transactions of the American Mathematical Society*, 355(6):2263–2303, 2003.

[Bia98] Philippe Biane. Processes with free increments. *Mathematische Zeitschrift*, 227(1):143–174, 1998.

[Bia21] Philippe Biane. Combinatorics of the Quantum Symmetric Simple Exclusion Process, associahedra and free cumulants. *arXiv preprint arXiv:2111.12403*, 2021.

[BLS96] Marek Bożejko, Michael Leinert, and Roland Speicher. Convolution and limit theorems for conditionally free random variables. *Pacific Journal of Mathematics*, 175(2):357–388, 1996.

[BN08a] Serban T. Belinschi and Alexandru Nica. η-series and a Boolean Bercovici–Pata bijection for bounded k-tuples. *Advances in Mathematics*, 217(1):1–41, 2008.

[BN08b] Serban T. Belinschi and Alexandru Nica. On a remarkable semigroup of homomorphisms with respect to free multiplicative convolution. *Indiana University Mathematics Journal*, 57(4):1679–1713, 2008.

[BN09] Serban T. Belinschi and Alexandru Nica. Free Brownian motion and evolution towards $\boxplus$-infinite divisibility for k-tuples. *International Journal of Mathematics*, 20(03):309–338, 2009.

[BPB99] Hari Bercovici, Vittorino Pata, and Philippe Biane. Stable laws and domains of attraction in free probability theory. *Annals of Mathematics*, 149(3):1023–1060, 1999.

[Bro00] Christian Brouder. Runge-Kutta methods and renormalization. *The European Physical Journal C-Particles and Fields*, 12(3):521–534, 2000.

[BS91] Marek Bozejko and Roland Speicher. ψ-independent and symmetrized white noises. *Quantum Probability and Related Fields*, VI:219–236, 1991.

[BS12] Serban T. Belinschi and Dimitri Shlyakhtenko. Free probability of type B: analytic interpretation and applications. *American Journal of Mathematics*, 134(1):193–234, 2012.

[But87] John Charles Butcher. *The numerical analysis of ordinary differential equations: Runge-Kutta and general linear methods.* Wiley-Interscience, 1987.

[BV93] Hari Bercovici and Dan Voiculescu. Free convolution of measures with unbounded support. *Indiana University Mathematics Journal*, 42(3):733–773, 1993.

[CEFP21] Adrian Celestino, Kurusch Ebrahimi-Fard, and Daniel Perales. Relations between infinitesimal non-commutative cumulants. *Documenta Mathematica*, 26:1145–1185, 2021.

[CEFPP21] Adrian Celestino, Kurusch Ebrahimi-Fard, Frédéric Patras, and Daniel Perales. Cumulant–cumulant relations in free probability theory from Magnus' expansion. *Foundations of Computational Mathematics*, 22(3):1–23, 2021.

[CHS18] Benoit Collins, Takahiro Hasebe, and Noriyoshi Sakuma. Free probability for purely discrete eigenvalues of random matrices. *Journal of the Mathematical Society of Japan*, 70(3):1111–1150, 2018.

[CL01] Frédéric Chapoton and Muriel Livernet. Pre-Lie algebras and the rooted trees operad. *International Mathematics Research Notices*, 2001(8):395–408, 2001.

[CP13] Frédéric Chapoton and Frédéric Patras. Enveloping algebras of preLie algebras, Solomon idempotents and the Magnus formula. *International Journal of Algebra and Computation*, 23(04):853–861, 2013.

[CP21] Pierre Cartier and Frédéric Patras. *Classical Hopf Algebras and Their Applications*. Springer, 2021.

[CP22] Adrian Celestino and Frédéric Patras. A forest formula for pre-Lie exponentials, Magnus' operator and cumulant-cumulant relations. *arXiv preprint arXiv:2203.11968. Submitted*, 2022.

[DCPT15] Gabriel C. Drummond-Cole, Jae-Suk Park, and John Terilla. Homotopy probability theory I. *Journal of Homotopy and Related Structures*, 10(3):425–435, 2015.

[DGG22] Joscha Diehl, Malte Gerhold, and Nicolas Gilliers. Dendriform algebras and noncommutative probability for pairs of faces. *arXiv preprint arXiv:2201.11747*, 2022.

[EFFKP20] Kurusch Ebrahimi-Fard, Loïc Foissy, Joachim Kock, and Frédéric Patras. Operads of (noncrossing) partitions, interacting bialgebras, and moment-cumulant relations. *Advances in Mathematics*, 369:107170, 2020.

[EFP15] Kurusch Ebrahimi-Fard and Frédéric Patras. Cumulants, free cumulants and half-shuffles. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2176):20140843, 2015.

[EFP16] Kurusch Ebrahimi-Fard and Frédéric Patras. The splitting process in free probability theory. *International Mathematics Research Notices*, 2016(9):2647–2676, 2016.

[EFP18] Kurusch Ebrahimi-Fard and Frédéric Patras. Monotone, free, and Boolean cumulants: a shuffle algebra approach. *Advances in Mathematics*, 328:112–132, 2018.

[EFP19] Kurusch Ebrahimi-Fard and Frédéric Patras. Shuffle group laws: applications in free probability. *Proceedings of the London Mathematical Society*, 119(3):814–840, 2019.

[EFP20] Kurusch Ebrahimi-Fard and Frédéric Patras. A group-theoretical approach to conditionally free cumulants. *"Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA) Vol. 1", IRMA Lectures in Mathematics and Theoretical Physics*, 31, 2020.

[EFPTZ21] Kurusch Ebrahimi-Fard, Frédéric Patras, Nikolas Tapia, and Lorenzo Zambotti. Wick polynomials in noncommutative probability: a group-theoretical approach. *Canadian Journal of Mathematics*, First View:1–27, 2021.

[EFPTZ22] Kurusch Ebrahimi-Fard, Frédéric Patras, Nikolas Tapia, and Lorenzo Zambotti. Shifted substitution in non-commutative multivariate power series with a view toward free probability. *arXiv preprint arXiv:2204.01445*, 2022.

[Fév12] Maxime Février. Higher order infinitesimal freeness. *Indiana University Mathematics Journal*, 61(1):249–295, 2012.

[FM15] Roland M. Friedrich and John McKay. Homogeneous Lie groups and quantum probability. *arXiv preprint arXiv:1506.07089*, 2015.

[FMNS19] Maxime Février, Mitja Mastnak, Alexandru Nica, and Kamil Szpojankowski. A construction which relates c-freeness to infinitesimal freeness. *Advances in Applied Mathematics*, 110:299–341, 2019.

[FMNS20] Maxime Février, Mitja Mastnak, Alexandru Nica, and Kamil Szpojankowski. Using Boolean cumulants to study multiplication and anti-commutators of free random variables. *Transactions of the American Mathematical Society*, 373(10):7167–7205, 2020.

[FN10] Maxime Février and Alexandru Nica. Infinitesimal non-crossing cumulants and free probability of type B. *Journal of Functional Analysis*, 258(9):2983–3023, 2010.

[Foi02] Loïc Foissy. Les algebres de Hopf des arbres enracinés décorés, I. *Bulletin des sciences mathematiques*, 126(3):193–239, 2002.

[Foi07] Loïc Foissy. Bidendriform bialgebras, trees, and free quasi-symmetric functions. *Journal of Pure and Applied Algebra*, 209(2):439–459, 2007.

[Gil22] Nicolas Gilliers. A shuffle algebra point of view on operator-valued probability theory. *Advances in Mathematics*, 408:108614, 2022.

[GR20] Darij Grinberg and Victor Reiner. *Hopf algebras in combinatorics*. Mathematisches Forschungsinstitut Oberwolfach gGmbH, 2020.

[GS02] Anis Ben Ghorbal and Michael Schürmann. Non-commutative notions of stochastic independence. *Mathematical Proceedings of the Cambridge Philosophical Society*, 133(3):531–561, 2002.

[GvW78] Narasimhan Giri and Wilhelm von Waldenfels. An algebraic version of the central limit theorem. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 42(2):129–134, 1978.

[Has11] Takahiro Hasebe. Differential independence via an associative product of infinitely many linear functionals. *Colloquium Mathematicum*, 124(1):79–94, 2011.

[Hof03] Michael Hoffman. Combinatorics of rooted trees and Hopf algebras. *Transactions of the American Mathematical Society*, 355(9):3795–3811, 2003.

[HS11a] Takahiro Hasebe and Hayato Saigo. Joint cumulants for natural independence. *Electronic Communications in Probability*, 16:491–506, 2011.

[HS11b] Takahiro Hasebe and Hayato Saigo. The monotone cumulants. *Annales de l'IHP Probabilités et statistiques*, 47(4):1160–1170, 2011.

[JV15] Matthieu Josuat-Vergès. Refined enumeration of noncrossing chains and hook formulas. *Annals of Combinatorics*, 19(3):443–460, 2015.

[JVMNT17] Matthieu Josuat-Vergès, Frédéric Menous, Jean-Christophe Novelli, and Jean-Yves Thibon. Free cumulants, Schröder trees, and operads. *Advances in Applied Mathematics*, 88:92–119, 2017.

[Leh02] Franz Lehner. Free cumulants and enumeration of connected partitions. *European Journal of Combinatorics*, 23(8):1025–1031, 2002.

[Len10] Romuald Lenczewski. Matricially free random variables. *Journal of Functional Analysis*, 258(12):4075–4121, 2010.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346. Springer, 2012.

[Maa92] Hans Maassen. Addition of freely independent random variables. *Journal of functional analysis*, 106(2):409–438, 1992.

[Min19] James A. Mingo. Non-crossing annular pairings and the infinitesimal distribution of the GOE. *Journal of the London Mathematical Society*, 100(3):987–1012, 2019.

[MN10] Mitja Mastnak and Alexandru Nica. Hopf algebras and the logarithm of the s-transform in free probability. *Transactions of the American Mathematical Society*, 362(7):3705–3743, 2010.

[MNT17] Frédéric Menous, Jean-Christophe Novelli, and Jean-Yves Thibon. Combinatorics of Poincaré's and Schröder's equations. *Resurgence, Physics and Numbers*, 20:329–378, 2017.

[MP18] Frédéric Menous and Frédéric Patras. Right-handed Hopf algebras and the preLie forest formula. *Annales de l'Institut Henri Poincaré D*, 5(1):103–125, 2018.

[MS17a] Sarah Manzel and Michael Schürmann. Non-commutative stochastic independence and cumulants. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 20(02):1750010, 2017.

[MS17b] James A. Mingo and Roland Speicher. *Free Probability and Random Matrices*. Fields Institute Monographs. Springer New York, 2017.

[Mur00] Naofumi Muraki. Monotonic convolution and monotonic Lévy-Hincin formula. *preprint*, 2000.

[Mur03] Naofumi Muraki. The five independences as natural products. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 6(03):337–371, 2003.

[Mur06] Ander Murua. The Hopf algebra of rooted trees, free Lie algebras, and Lie series. *Foundations of Computational Mathematics*, 6(4):387–426, 2006.

[Nic09] Alexandru Nica. Multi-variable subordination distributions for free additive convolution. *Journal of Functional Analysis*, 257(2):428–463, 2009.

[NS06] Alexandru Nica and Roland Speicher. *Lectures on the combinatorics of free probability*, volume 13. Cambridge University Press, 2006.

[OG08] Jean-Michel Oudom and Daniel Guin. On the Lie enveloping algebra of a pre-Lie algebra. *Journal of K-theory*, 2(1):147–167, 2008.

[PW11] Mihai Popa and Jiun-Chau Wang. On multiplicative conditionally free convolution. *Transactions of the American Mathematical Society*, 363(12):6309–6335, 2011.

[Rei97] Victor Reiner. Non-crossing partitions for classical reflection groups. *Discrete Mathematics*, 177(1-3):195–222, 1997.

[Reu93] Christophe Reutenauer. *Free Lie algebras*. Oxford University Press, 1993.

[Sch86] Michael Schürmann. A central limit theorem for coalgebras. *Probability Measures on Groups*, VIII:153–157, 1986.

[Sch95] Michael Schürmann. Direct sums of tensor products and non-commutative independence. *Journal of Functional Analysis*, 133(1):1–9, 1995.

[Shl18] Dimitri Shlyakhtenko. Free probability of type-B and asymptotics of finite-rank perturbations of random matrices. *Indiana University Mathematics Journal*, 67(2):971–991, 2018.

[Spe83] Terence P. Speed. Cumulants and partition lattices 1. *Australian Journal of Statistics*, 25(2):378–388, 1983.

[Spe94] Roland Speicher. Multiplicative functions on the lattice of non-crossing partitions and free convolution. *Mathematische Annalen*, 298(1):611–628, 1994.

[Spe97] Roland Speicher. On universal products. *The Fields Institute Communications*, 12:257–266, 1997.

[Spe98] Roland Speicher. *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, volume 627. American Mathematical Society, 1998.

[SvW94] Roland Speicher and Wilhelm von Waldenfels. A general central limit theorem and invariance principle. *Quantum Probability And Related Topics: QP-PQ*, IX:371–387, 1994.

[SW97] Roland Speicher and Reza Woroudi. Boolean convolutions. Free probability theory. *The Fields Institute Communications*, 12:267–279, 1997.

[Swe69] Moss E. Sweedler. *Hopf Algebras.* Mathematics lecture note series. W. A. Benjamin, 1969.

[VDN92] Dan Voiculescu, Ken J. Dykema, and Alexandru Nica. *Free random variables.* CRM monograph series. American Mathematical Society, 1992.

[Voi85] Dan Voiculescu. Symmetries of some reduced free product C^*-algebras. *Operator algebras and their connections with topology and ergodic theory. Lecture Notes in Mathematics*, 1132:556–588, 1985.

[Voi93] Dan Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory, I. *Communications in Mathematical Physics*, 155(1):71–92, 1993.

[Voi95] Dan Voiculescu. Free probability theory: random matrices and von Neumann algebras. *Proceedings of the International Congress of Mathematicians*, 1:227–242, 1995.

[Voi14] Dan Voiculescu. Free probability for pairs of faces I. *Communications in Mathematical Physics*, 332(3):955–980, 2014.

[Wal86] Wilhelm von Waldenfels. Non-commutative algebraic central limit theorems. *Probability Measures on Groups*, VIII:174–202, 1986.

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