THE BOUNDARY RIGIDITY FOR HOLOMORPHIC SELF-MAPS OF SOME FIBERED DOMAINS

JOHN ERIK FORNÆSS, FENG RONG

Abstract. We prove Burns-Krantz type boundary rigidity theorems for holomorphic self-maps of some fibered domains, including polydisks and eggs.

1. Introduction

The classical Schwarz lemma states that if a holomorphic self-map f of the unit disk Δ fixes the origin, then $|f'(0)| \leq 1$ and the equality holds if and only if f is a rotation. In particular, if $f'(0) = 1$ then $f(z) \equiv z$, which is often referred to as the "rigidity" part of the Schwarz lemma.

The rigidity part of the Schwarz lemma was extended to a boundary fixed point by Burns and Krantz [1] as follows:

Theorem 1. [1, Theorem 4.5] *Let Ω be a bounded strongly pseudoconvex domain with C^∞-smooth boundary and F a holomorphic self-map of Ω. Suppose that*

$$F(z) = z + O(\|z-p\|^4), \quad z \to p \in \partial\Omega.$$

Then $F(z) \equiv z$.

In [1, p. 663], Burns and Krantz remarked that $O(\|z-p\|^4)$ in the above theorem can be replaced by $o(\|z-p\|^3)$. Later in [6], Huang gave a "localized" version of the boundary rigidity as follows:

Theorem 2. [6, Theorem 2.5] *Let Ω be a bounded domain with a C^∞-smooth strongly pseudoconvex boundary point p and F a holomorphic self-map of Ω. Suppose that*

$$F(z) = z + o(\|z-p\|^3), \quad z \to p \in \partial\Omega.$$

Then $F(z) \equiv z$.

In this paper, we prove Burns-Krantz type boundary rigidity theorems for holomorphic self-maps of some fibered domains, without assuming pseudoconvexity or boundary smoothness.

Consider bounded domains in $\mathbb{C}^{n+1}$ of the form

$$\Omega = \bigcup_{z\in\Delta} \Omega_z, \tag{1}$$

where Ω_z's are bounded complete Reinhardt domains in $\mathbb{C}^n$.

2010 *Mathematics Subject Classification.* 32H99, 32A40.

Key words and phrases. boundary rigidity, fibered domains.

The first author is partially supported by the Norwegian Research Council (grant no. 240569). The second author is partially supported by the National Natural Science Foundation of China (grant no. 11871333).

Denote by Δ_r the disk of radius r in $\mathbb{C}$. Let $\delta : \Delta \to \mathbb{R}^+$ be a function such that $\Delta^n_{\delta(z)} \subset \Omega_z$ for any $z \in \Delta$. For any $(z,w) \in \Omega$, denote by $D_{z,w}$ the image of the linear mapping:

$$L_{z,w} : \Delta \to \mathbb{C}^{n+1}; \quad \tau \mapsto \left(\tau, \frac{w}{z-1}(\tau - 1)\right),$$

which is a graph of Δ through (z,w) and $(1,0)$. We define a *cone* with end $(1,0)$ (of size δ) as

$$C_\delta := \bigcup \{D_{z,w};\ z \in \Delta,\ w \in \Delta^n_{\delta(z)}\}. \tag{2}$$

We say that Ω is a *fibered domain satisfying the cone condition* if it is of the form (1) and contains a cone with end $(1,0)$. Note that polydisks are such special domains.

Theorem 3. *Let Ω be a fibered domain satisfying the cone condition and F a holomorphic self-map of Ω. For $(z,w) \in \Omega$, suppose that*

$$F(z,w) = (z,w) + o(\|(1-z,w)\|^3), \quad (z,w) \to (1,0).$$

Then $F(z,w) \equiv (z,w)$.

Next, consider Ω of the form (1), but with Ω_z's being just bounded and containing the origin. Let $\rho : \Delta \to \mathbb{R}^+$ be a function such that $\Omega_z \subset \Delta^n_{\rho(z)}$ for any $z \in \Delta$ and $\rho \in C^0(\bar{\Delta})$. We say that Ω is a *fibered domain with boundary size zero* if there exists a $\rho(z)$ with

$$\int_{-\pi}^{\pi} \log \rho(e^{i\theta}) d\theta = -\infty. \tag{3}$$

Note that balls and eggs are such special domains (with $\rho(e^{i\theta}) \equiv 0$). More interestingly, polydisks with the "diagonal" disk as its base are also such special domains.

Theorem 4. *Let Ω be a fibered domain with boundary size zero and F a holomorphic self-map of Ω. Suppose that*

$$F(z,w) = (z,w) + o(\|(1-z,w)\|^3), \quad (z,w) \to (1,0).$$

Then $F(z,w) \equiv (z,w)$.

In section 2, we prove Theorem 3 for the bidisk. In section 3, we prove Theorem 3 for general Ω. The proof of Theorem 4 is given in section 4. More general versions of Theorem 3 are given in section 5.

2. Proof of Theorem 3 for the bidisk

To illustrate the key ideas, we first prove Theorem 3 for the bidisk.

For any $z \in \Delta$ and small $\epsilon > 0$, set

$$\Delta_{z,\epsilon} = \{(z,w) \in \Delta^2 :\ |w| < \epsilon(1 - |z|)\}.$$

For any $(z,w) \in \Delta_{z,\epsilon}$, denote by $D_{z,w}$ the intersection of Δ^2 with the complex line through (z,w) and $(1,0)$. Denote by π the vertical projection from Δ^2 to $D_{z,w}$. Write $F(z,w) = (F_0(z,w), F_1(z,w))$.

Parametrize $D_{z,w}$ as

$$D_{z,w} = \left\{\tau \in \Delta :\ \left(\tau, \frac{w}{z-1}(\tau - 1)\right)\right\}.$$

Consider the map $f(\tau) = \pi \circ F|_{D_{z,w}}$.

Then, one readily checks using the assumptions that f is a holomorphic self-map of $D_{z,w}$ with

$$f(\tau) = \tau + o(|1-\tau|^3), \quad \tau \to 1. \tag{4}$$

Thus, by Burns-Krantz rigidity, we have $f(\tau) \equiv \tau$. In particular, for $\tau = z$, we have

$$F_0(z,w) = z.$$

Since this holds for every $(z,w) \in \Delta_{z,\epsilon}$, it follows from the identity theorem that $F_0(z,w) \equiv z$.

Now, write $F_1(z,w)$ as

$$F_1(z,w) = a_0(z) + a_1(z)w + a_2(z)w^2 + a_3(z)w^3 + \cdots .$$

Denote $g_z(w) = F_1(z,w)$. Then g_z is a holomorphic self-map of Δ and $g_z'(0) = a_1(z)$. By Schwarz-Pick lemma, it follows that

$$|a_1(z)| \leq 1 - |a_0(z)|^2 \leq 1.$$

Thus a_1 is a holomorphic self-map of Δ.

At the point $(z,w) = (z, 1-z)$, we have

$$\begin{aligned}&F(z,w) - (z,w)\\ =&(0, a_0(z) + (1-z)(a_1(z)-1) + (1-z)^2 a_2(z) + (1-z)^3 a_3(z) + o(|1-z|^3)).\end{aligned}$$

At the point $(z,w) = (z, z-1)$, we have

$$\begin{aligned}&F(z,w) - (z,w)\\ =&(0, a_0(z) - (1-z)(a_1(z)-1) + (1-z)^2 a_2(z) - (1-z)^3 a_3(z) + o(|1-z|^3)).\end{aligned}$$

At the point $(z,w) = (z, \frac{1}{2}(1-z))$, we have

$$\begin{aligned}&F(z,w) - (z,w)\\ =&(0, a_0(z) + \frac{1}{2}(1-z)(a_1(z)-1) + \frac{1}{4}(1-z)^2 a_2(z) + \frac{1}{8}(1-z)^3 a_3(z) + o(|1-z|^3)).\end{aligned}$$

At the point $(z,w) = (z, \frac{1}{2}(z-1))$, we have

$$\begin{aligned}&F(z,w) - (z,w)\\ =&(0, a_0(z) - \frac{1}{2}(1-z)(a_1(z)-1) + \frac{1}{4}(1-z)^2 a_2(z) - \frac{1}{8}(1-z)^3 a_3(z) + o(|1-z|^3)).\end{aligned}$$

From the above four expressions and the assumption of Theorem 3, we get

$$|a_1(z) - 1| = o(|1-z|^2), \quad z \to 1.$$

Consider $h(z) = za_1(z)$ as a holomorphic self-map of Δ with $h(0) = 0$ and

$$|h(z) - z| = o(|1-z|^2), \quad z \to 1. \tag{5}$$

Then, it follows from [6, Corollary 1.5] that $h(z) \equiv z$, i.e. $a_1(z) \equiv 1$. Again, using Schwarz-Pick lemma, we get $F_1(z,w) \equiv w$.

3. Proof of Theorem 3

For any $z \in \Delta$ and $0 < \epsilon < \delta(z)$, set

$$\Delta^n_{z,\epsilon} = \{(z, w) \in \Omega : \ w \in \Delta^n_\epsilon\}.$$

For any $(z, w) \in \Delta^n_{z,\epsilon}$, denote by $D_{z,w}$ the intersection of Ω with the complex line through (z, w) and $(1, 0)$. Denote by π the vertical projection from Ω to $D_{z,w}$. Write $F(z, w) = (F_0(z, w), F_1(z, w), \cdots , F_n(z, w))$.

A similar argument as in the previous section shows that $F_0(z, w) \equiv z$.

Now, for $1 \le j \le n$, write $F_j(z, w)$ as

$$F_j(z, w) = \sum_\alpha a_{j,\alpha}(z) w^\alpha,$$

where $\alpha = (\alpha_1, \cdots , \alpha_n) \in \mathbb{N}^n$ and $w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n}$.

For each $1 \le j \le n$, consider $g_{j,z}(w_j) = F_j(z, 0, \cdots , 0, w_j, 0, \cdots , 0)$. Then a similar argument as in the previous section shows that

$$a_{j,e_j}(z) \equiv 1, \quad a_{j,le_j}(z) \equiv 0, \ l \ne 1,$$

where e_j denotes the j-th unit vector.

Therefore, we can write $F_j(z, w)$ as

$$F_j(z, w) = w_j + \sum_{1\le k\ne j\le n} a_{j,e_k}(z) w_k + \sum_{|\alpha|\ge 2} a_{j,\alpha}(z) w^\alpha,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For any fixed $z \in \Delta$, consider the holomorphic self-map $G_z(w)$ of Ω_z,

$$G_z(w) = (F_1(z, w), \cdots , F_n(z, w)).$$

By a linear change of coordinates, we can suppose that the Jacobian $JG_z(0)$ is in upper triangular form with all diagonal entries equal to 1. Consider the iterations G_z^k and apply Cauchy's estimates, we get that all the off-diagonal entries must be identically zero, i.e. $JG_z(0) = id$. Then, it follows from Cartan's uniqueness theorem that $G_z(w) \equiv w$. This proves Theorem 3.

Similar to [6, Corollary 1.5, Corollary 2.7] (see also [5, Corollary 3]), we have the following

Theorem 5. *Let Ω be as in Theorem 3 and F a holomorphic self-map of Ω. Suppose that*

$$F(z, w) = (z, w) + o(\|(1 - z, w)\|^2), \quad (z, w) \to (1, 0).$$

Assume further that the fixed point set Γ of F satisfies

$$\Gamma \cap D_{z,w} \ne \emptyset, \ \forall D_{z,w} \subset C_\delta.$$

Then $F(z, w) \equiv (z, w)$.

Proof. The difference here is that in (4) we have $o(|1-\tau|^2)$ instead of $o(|1-\tau|^3)$, and in (5) we have $o(|1 - z|)$ instead of $o(|1 - z|^2)$. But with the additional assumption on the fixed point set Γ, [6, Corollary 1.5] still applies. The rest of the proof is exactly the same. □

4. Proof of Theorem 4

Consider the z-disk

$$\Delta_z = \{(z,w) \in \Omega : \ w = 0\}.$$

Arguing as in section 2, we get $\pi \circ F|_{\Delta_z}(\tau) \equiv \tau$, i.e. $F|_{\Delta_z}$ is a graph over Δ_z. Thus, there exists a holomorphic mapping $f : \Delta_z \to \mathbb{C}^n$ such that $F|_{\Delta_z} = (\tau, f(\tau))$.

Write $f = (f_1, \cdots, f_n)$. Then for any $1 \le j \le n$ and $\tau \in \Delta_z$, $|f_j(\tau)| < \rho(\tau)$. Since $\log|f_j(\tau)|$ is subharmonic, we get from (3) that

$$\log|f_j(\tau)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi-\theta)\log\rho(e^{i\theta})d\theta = -\infty, \quad \tau = re^{i\varphi}.$$

Here P_r is the Poisson kernel. Thus we must have $f_j(\tau) \equiv 0$, i.e. F fixes the z-disk Δ_z.

Theorem 4 then follows from [6, Theorem 2.2] (with $\epsilon = 1$ and $\mu = 1$).

Remark 1. Theorem 4 is a generalization of [6, Theorem 0.3], in which the domains are fibered domains with $\rho(e^{i\theta}) \equiv 0$.

Similar to Theorem 5, we also have the following

Theorem 6. *Let Ω be as in Theorem 4 and F a holomorphic self-map of Ω. Suppose that $F(z_0, 0) = (z_0, 0)$ for some $z_0 \in \Delta$ and*

$$F(z,w) = (z,w) + o(\|(1-z,w)\|^2), \quad (z,w) \to (1,0).$$

Then $F(z,w) \equiv (z,w)$.

Proof. The difference here is that in (4) we have $o(|1-\tau|^2)$ instead of $o(|1-\tau|^3)$, but with $f(z_0) = z_0$. Thus, [6, Corollary 1.5] applies, as does [6, Theorem 2.2]. □

5. More general domains

In Theorem 3, we can allow the base of the domain Ω to be any bounded domain with a strongly pseudoconvex boundary point p.

More precisely, let Ω be a bounded domain of the form

$$\Omega = \bigcup_{z \in D} \Omega_z, \tag{6}$$

where D is a bounded domain in $\mathbb{C}^m$ with a C^∞-smooth strongly pseudoconvex boundary point p, and Ω_z's are bounded complete Reinhardt domains in $\mathbb{C}^n$.

Let $\delta : \Delta \to \mathbb{R}^+$ be a function such that $\Delta^n_{\delta(z)} \subset \Omega_z$ for any $z \in \Delta$. For any $(z,w) \in \Omega$, denote by $D_{z,w}$ the linear graph over D through (z,w) and $(p,0)$. We then define a cone with end $(p,0)$ (of size δ) as in (2).

We again call Ω a *fibered domain satisfying the cone condition* if it is of the form (6) and contains a cone with end $(p,0)$.

Theorem 7. *Let Ω be a fibered domain satisfying the cone condition and F a holomorphic self-map of Ω. For $(z,w) \in \mathbb{C}^m \times \mathbb{C}^n$, suppose that*

$$F(z,w) = (z,w) + o(\|(p-z,w)\|^3), \quad (z,w) \to (p,0).$$

Then $F(z,w) \equiv (z,w)$.

Proof. For simplicity, assume $n = 1$. Write $F(z, w) = (F_0(z, w), F_1(z, w))$. Then, arguing as in section 2 using Huang's rigidity theorem on bounded domains with a C^∞-smooth strongly pseudoconvex boundary point, we get that $F_0(z, w) \equiv z$.

Write $F_1(z, w) = a_0(z) + a_1(z)w + a_2(z)w^2 + a_3(z)w^3 + \cdots$. Then, by Schwarz-Pick lemma, we get that $|a_1(z)| \leq 1$. And a similar argument as in section 2 (using $\|p - z\|$ instead of $|1 - z|$) shows that

$$|a_1(z) - 1| = o(\|p - z\|^2), \quad z \to p.$$

Let ψ be a change of coordinates in a neighborhood U of p such that $V := \psi(U \cap D)$ is strongly convex near $q := \psi(p)$. Then, there exists a small cone C with end q contained in V, which is the union of one-dimensional simply-connected domains with q on the boundary. For each such a one-dimensional simply-connected domain W, let ϕ be a Riemann mapping from W to the unit disk with $\lim_{k\to\infty} \phi(\zeta_k) = 1$ for some $\zeta_k \to q$.

Set $\tilde{a}_1 := a_1 \circ \psi^{-1}|_W \circ \phi^{-1}$. Since W has C^2-smooth boundary, it follows from Lemma 8 below that

$$|1 - \tau| \sim |q - \phi^{-1}(\tau)|, \quad \tau \to 1.$$

Thus, $\tilde{a}_1$ is a holomorphic self-map of the unit disk satisfying

$$|\tilde{a}_1(\tau) - 1| = o(|1 - \tau|^2), \quad \tau \to 1.$$

Then, arguing exactly as in section 2, we get that $a_1 \equiv 1$ and $F_1(z, w) \equiv w$. $\square$

The following lemma is probably known. We give a proof for completeness.

Lemma 8. *Let W be a simply-connected domain in $\mathbb{C}$ with C^2-smooth boundary near $q \in \partial W$. Let ϕ be a Riemann mapping from W to Δ with $\lim_{k\to\infty} \phi(\zeta_k) = 1$ for some $\zeta_k \to q$. Then ϕ extends to be bi-Lipschitz near $q \in \overline{W}$.*

Proof. First of all, by [3, Theorem 1.1], ϕ extends to be homeomorphic near $q \in \overline{W}$.

Set $\tau = \phi(\zeta)$. Denote by $d(\zeta, \partial W)$ the distance between $\zeta \in W$ and the boundary ∂W. Since W has C^2-smooth boundary near q, it follows from the Hopf lemma (see e.g. [2]) that

$$d(\tau, \partial\Delta) \sim d(\zeta, \partial W), \quad \tau \to \partial\Delta \text{ near } 1.$$

Denote by $K_W(\zeta; \xi)$ the Kobayashi metric at $\zeta \in W$ with unit vector ξ. Set $\xi' = \phi'(\zeta)\xi/|\phi'(\zeta)|$. Then from the definition of the Kobayashi metric, it follows that

$$K_\Delta(\tau; \xi') \leq K_W(\zeta; \xi)/|\phi'(\zeta)|.$$

Thus,

$$|\phi'(\zeta)| \leq \frac{K_W(\zeta; \xi)}{K_\Delta(\tau; \xi')} \lesssim \frac{d(\tau, \partial\Delta)}{d(\zeta, \partial W)} \sim 1, \quad \zeta \to q.$$

Here, for the second inequality (with a constant factor), we used the fact that $K_W(\zeta; \xi) \sim d(\zeta, \partial W)^{-1}$ (see e.g. [4]).

Since the same argument applies to ϕ^{-1}, we actually get that $|\phi'(\zeta)| \sim 1$ as $\zeta \to q$, which proves the lemma. $\square$

Remark 2. It is clear from the above proof that the base D of the fibered domain Ω can be even more general domains, which satisfies the following conditions:
i) The Burns-Krantz rigidity theorem holds on D at a boundary fixed point p;
ii) D is C^2-smooth near p and (after a change of coordinates centered at p if

necessary) there exists a small cone C with end p contained in D, which is the union of one-dimensional simply-connected domains with p on the boundary.

Remark 3. Theorem 5 can also be generalized to fibered domains with more general base D, which satisfies condition ii) above and the following condition:
i') The Burns-Krantz rigidity theorem with interior fixed point holds on D (i.e. $F(z_0) = z_0$ for some $z_0 \in D$ and $F(z) = z + o(\|z - p\|^2)$ implying $F(z) \equiv z$).
For instance, D can be chosen to be those in [5, Corollary 3].

References

[1] D. Burns, S. Krantz; *Rigidity for holomorphic mappings and a new Schwarz lemma at the boundary*, J. Amer. Math. Soc. **7** (1994), 661-676.

[2] J.E. Fornæss, B. Stensønes; Lectures on Counterexamples in Several Complex Variables, Reprint of the 1987 original, AMS Chelsea Publishing, Providence, RI, 2007.

[3] F. Forstneric, J.-P. Rosay; *Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann. **279** (1987), 239-252.

[4] I. Graham; *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary*, Trans. Amer. Math. Soc. **207** (1975), 219-240.

[5] X. Huang; *A preservation principle of extremal mappings near a strongly pseudoconvex point and its applications*, Illinois J. Math. **38** (1994), 283-302.

[6] X. Huang; *A boundary rigidity problem for holomorphic mappings on some weakly pseudoconvex domains*, Canad. J. Math. **47** (1995), 405-420.

[7] S. Krantz; *The Schwarz lemma at the boundary*, Complex Var. Elliptic Equ. **56** (2011), 455-468.

Department of Mathematical Sciences, Norwegian University of Science and Technology, Sentralbygg 2, Alfred Getz vei 1, 7491 Trondheim, Norway

Email address: `johnefo@ntnu.no`

Department of Mathematics, School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dong Chuan Road, Shanghai, 200240, P.R. China

Email address: `frong@sjtu.edu.cn`