Homogeneous Plurisubharmonic Polynomials in Higher Dimensions

Lars Simon[1] ⓘ

Abstract

We prove several results on homogeneous plurisubharmonic polynomials on $\mathbb{C}^n$, $n \in \mathbb{Z}_{\geq 2}$. Said results are relevant to the problem of constructing local bumpings at boundary points of pseudoconvex domains of finite D'Angelo 1-type in $\mathbb{C}^{n+1}$.

1 Introduction

Local bumpings at boundary points of certain bounded, smoothly bounded pseudoconvex domains of finite D'Angelo 1-type in $\mathbb{C}^{n+1}$, $n \in \mathbb{Z}_{\geq 1}$, have been used both in the construction of peak functions (e.g. [1,5,8]) and in the construction of integral kernels for solving the $\bar{\partial}$-equation (e.g. [4,9]).

As explained in [2,3], the problem of constructing such local bumpings naturally leads to the study of homogeneous plurisubharmonic polynomials on $\mathbb{C}^n$. Furthermore, in [7], results on homogeneous plurisubharmonic polynomials on $\mathbb{C}^2$ by Bharali, Stensønes [3], applied in combination with results from [6], played an important role in establishing sup-norm estimates for solutions to the $\bar{\partial}$-equation for a large class of pseudoconvex domains in $\mathbb{C}^3$. Specifically, the crucial results on homogeneous plurisubharmonic polynomials on $\mathbb{C}^2$ are the following:

Result ([3, Proposition 1]) *Let P be a homogeneous, plurisubharmonic, non-pluriharmonic polynomial on $\mathbb{C}^2$. Then there are at most finitely many complex lines through the origin in $\mathbb{C}^2$ along which P is harmonic.*

✉ Lars Simon
lars.simon@ntnu.no; larsimon@gmail.com

[1] Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway

Result ([3, Theorem 2]) *Let $P: \mathbb{C}^2 \to \mathbb{R}$ be a non-constant, homogeneous, plurisubharmonic polynomial without pluriharmonic terms. Assume that there exists a holomorphic function $g: \mathbb{C}^2 \to \mathbb{C}$, nonsingular on a non-empty open set $U \subseteq \mathbb{C}^2$, such that P is harmonic along every level set of $g|_U$.*
Then there exist a homogeneous, subharmonic polynomial $s: \mathbb{C} \to \mathbb{R}$ and a homogeneous holomorphic polynomial $h: \mathbb{C}^2 \to \mathbb{C}$, such that $P = s \circ h$ on $\mathbb{C}^2$.

Result ([3, Theorem 3]) *Let $P: \mathbb{C}^2 \to \mathbb{R}$ be a non-constant, homogeneous, plurisubharmonic polynomial without pluriharmonic terms and assume that P is homogeneous of degree $2d_1$ in $z_1, \overline{z_1}$ and homogeneous of degree $2d_2$ in $z_2, \overline{z_2}$, where $d_1, d_2 \in \mathbb{Z}_{\geq 1}$. Then, there exist a homogeneous, subharmonic polynomial $s: \mathbb{C} \to \mathbb{R}$ without harmonic terms and integers $j, l \in \mathbb{Z}_{\geq 1}$, such that*

$$P(z_1, z_2) = s(z_1{}^j z_2{}^l) \text{ for all } (z_1, z_2) \in \mathbb{C}^2.$$

When attempting to adapt the methods from [7] to higher dimensions, it is natural to ask for generalizations of the above-mentioned results on homogeneous plurisubharmonic polynomials on $\mathbb{C}^2$ to higher dimensions. Specifically, it is natural to ask the following questions:

Question A *Given a homogeneous, plurisubharmonic, non-pluriharmonic polynomial P on $\mathbb{C}^n$, $n \geq 2$, is it true that there are at most finitely many complex hyperplanes through the origin in $\mathbb{C}^n$ along which P is pluriharmonic?*

Question B *Let $P: \mathbb{C}^n \to \mathbb{R}$, $n \in \mathbb{Z}_{\geq 2}$, be a non-constant, homogeneous, plurisubharmonic polynomial without pluriharmonic terms. Assume that there exists a holomorphic map $G: \mathbb{C}^n \to \mathbb{C}^m$, $1 \leq m \leq n - 1$, nonsingular on a non-empty open set $U \subseteq \mathbb{C}^n$, such that P is pluriharmonic along every level set of $G|_U$.*
Are there necessarily a homogeneous, plurisubharmonic polynomial $Q: \mathbb{C}^m \to \mathbb{R}$ and holomorphic polynomials $F_1, \ldots, F_m: \mathbb{C}^n \to \mathbb{C}$, all homogeneous of the same degree, such that $P = Q \circ (F_1, \ldots, F_m)$ on $\mathbb{C}^n$?

Question C *Let $P: \mathbb{C}^n \to \mathbb{R}$, $n \in \mathbb{Z}_{\geq 2}$, be a non-constant, homogeneous, plurisubharmonic polynomial without pluriharmonic terms and assume P is homogeneous in l variables separately, $1 \leq l \leq n - 1$.*
Are there necessarily a homogeneous, plurisubharmonic polynomial $Q: \mathbb{C}^{n-l} \to \mathbb{R}$ and holomorphic polynomials $F_1, \ldots, F_{n-l}: \mathbb{C}^n \to \mathbb{C}$, all homogeneous of the same degree, such that $P = Q \circ (F_1, \ldots, F_{n-l})$ on $\mathbb{C}^n$?

The purpose of this paper is to provide a detailed answer to Questions A, B and C. A formal statement of the results can be found in Sect. 2.

The answer to Question A is "yes" (Proposition 2.1).

The answer to Question B is "no" in general, even if we additionally assume that the component functions of G are holomorphic polynomials which are all homogeneous of the same degree (Proposition 2.2). However, the answer is "yes" in the special case $m = 1$, i.e., when the polynomial is pluriharmonic along the level sets of a single holomorphic function (Theorem 2.3).

The answer to Question C is "no" in general (Proposition 2.2). However, in the special case where the polynomial is homogeneous in all n variables separately, the answer is "yes" (Corollary 2.5). Furthermore, we get that the answer is "almost yes", or "yes, up to certain singular holomorphic coordinate changes" in the general setting (Theorem 2.4). In many cases, the latter theorem can be used to get the desired bumping results. This is important, since Proposition 2.2 shows that the result by Bharali, Stensønes [3, Theorem 3] does not generalize in this setting.

2 Statement of Results

In this section, we state the results of this paper. All the proofs can be found in the later sections.

The answer to Question A is "yes". We have:

Proposition 2.1 *Let* $P \colon \mathbb{C}^n \to \mathbb{R}$, $n \in \mathbb{Z}_{\geq 2}$, *be a homogeneous, plurisubharmonic, non-pluriharmonic polynomial. Then there are at most finitely many complex hyperplanes through the origin in* $\mathbb{C}^n$ *along which* P *is pluriharmonic.*

The answer to Question B is "no", even if we additionally assume that the component functions of G are holomorphic polynomials which are all homogeneous of the same degree. The answer to Question C is "no" as well. All of this is implied by the following:

Proposition 2.2 *Let* $P \colon \mathbb{C}^3 \to \mathbb{R}$,

$$P(z, w_1, w_2) = |z|^2 \cdot (|w_1|^4 + |w_1^2 - w_1 w_2|^2 + |w_2|^4).$$

Then P *is a non-constant, homogeneous, plurisubharmonic polynomial without pluriharmonic terms and* P *is homogeneous in one variable separately (see Definition 3.3). Furthermore, away from the coordinate hyperplanes,* P *is pluriharmonic along the level sets of* $G \colon \mathbb{C}^3 \to \mathbb{C}^2$, $G(z, w_1, w_2) = (zw_1^2, zw_2^2)$.
However, there do not exist a homogeneous, plurisubharmonic polynomial $Q \colon \mathbb{C}^2 \to \mathbb{R}$ *and holomorphic polynomials* $F_1, F_2 \colon \mathbb{C}^3 \to \mathbb{C}$, *homogeneous of the same degree, such that* $P = Q \circ (F_1, F_2)$ *on* $\mathbb{C}^3$.

Nevertheless, the answer to Question B in the special case where $m = 1$ is "yes":

Theorem 2.3 *Let* $P \colon \mathbb{C}^n \to \mathbb{R}$, $n \in \mathbb{Z}_{\geq 2}$, *be a non-constant, homogeneous, plurisubharmonic polynomial without pluriharmonic terms. Assume that there exists a holomorphic function* $G \colon \mathbb{C}^n \to \mathbb{C}$, *nonsingular on a non-empty open set* $U \subseteq \mathbb{C}^n$, *such that* P *is pluriharmonic along every level set of* $G|_U$.
Then, there exist a homogeneous, subharmonic polynomial $s \colon \mathbb{C} \to \mathbb{R}$ *without harmonic terms and a homogeneous holomorphic polynomial* $h \colon \mathbb{C}^n \to \mathbb{C}$, *such that* $P = s \circ h$ *on* $\mathbb{C}^n$.

Note that Theorem 2.3 generalizes the upper mentioned result by Bharali, Stensønes [3, Theorem 2] to higher dimension.

Although the answer to Question C is "no", we get the following result, which says that the answer is "yes, up to singular holomorphic coordinate changes":

Theorem 2.4 *Let $P \colon \mathbb{C}^n \to \mathbb{R}$, $n \in \mathbb{Z}_{\geq 2}$, be a non-constant plurisubharmonic polynomial without pluriharmonic terms and assume that P is homogeneous of degree $2k$, $k \in \mathbb{Z}_{\geq 1}$. Let $1 \leq l \leq n - 1$ and assume that P is homogeneous of degree $2d_j$, $d_j \in \mathbb{Z}_{>0}$, in z_j, $\overline{z_j}$ for $j = 1, \ldots, l$ (see Definition 3.3). Assume furthermore that $k - D > 0$, where $D = d_1 + \cdots + d_l$. Write $d := \gcd(d_1, \ldots, d_l, k) \in \mathbb{Z}_{\geq 1}$.*
Then, there exists a plurisubharmonic polynomial $Q \colon \mathbb{C}^{n-l} \to \mathbb{R}$ without pluriharmonic terms, homogeneous of degree $2k - 2D$, with the property that both the holomorphic and the anti-holomorphic degree (see Sect. 3) of every term appearing in Q are divisible by the integer $(k - D)/d$, such that we have for all $(z_1, \ldots, z_n) \in \mathbb{C}^n$:

$$P(z_1, \ldots, z_n) = Q(\tau z_{l+1}, \ldots, \tau z_n),$$

for every solution $\tau \in \mathbb{C}$ of $\tau^{(k-D)/d} = z_1^{d_1/d} \cdots z_l^{d_l/d}$.

Alternatively, we can carry out a *singular* holomorphic coordinate change $\Phi \colon \mathbb{C}^n \to \mathbb{C}^n$, $(z_1, \ldots, z_n) \mapsto (z_1^{(k-D)/d}, \ldots, z_l^{(k-D)/d}, z_{l+1}, \ldots, z_n)$ and write

$$(P \circ \Phi)(z_1, \ldots, z_n) = Q(z_1^{d_1/d} \cdots z_l^{d_l/d} z_{l+1}, \ldots, z_1^{d_1/d} \cdots z_l^{d_l/d} z_n).$$

Note that, without the assumptions $d_1, \ldots, d_l, k - D > 0$ in Theorem 2.4, P is effectively a polynomial in fewer than n variables, hence we can ignore that case.

As a corollary (of the proof) of Theorem 2.4 we get that the answer to Question C in the special case where P is homogeneous in all n variables separately is "yes":

Corollary 2.5 *Let $P \colon \mathbb{C}^n \to \mathbb{R}$, $n \in \mathbb{Z}_{\geq 2}$, be a non-constant, homogeneous, plurisubharmonic polynomial without pluriharmonic terms and assume that P is homogeneous of degree $2d_j$, $d_j \in \mathbb{Z}_{>0}$, in z_j, $\overline{z_j}$ for $j = 1, \ldots, n$ (see Definition 3.3). Write $d := \gcd(d_1, \ldots, d_n) \in \mathbb{Z}_{\geq 1}$.*
Then, there exists a homogeneous, subharmonic polynomial $s \colon \mathbb{C} \to \mathbb{R}$ without harmonic terms, such that

$$P(z_1, \ldots, z_n) = s(z_1^{d_1/d} \cdots z_n^{d_n/d})$$

for all $(z_1, \ldots, z_n) \in \mathbb{C}^n$.

Note that Corollary 2.5 generalizes the upper mentioned result by Bharali, Stensønes [3, Theorem 3] to higher dimension.

3 Preliminaries

For the remainder of this section we fix an integer $n \geq 2$ and a non-constant polynomial $P \colon \mathbb{C}^n \to \mathbb{R}$ with the following properties:

- P is $\mathbb{R}$-homogeneous of degree $2k$, for some positive integer k,

- P is plurisubharmonic,
- P does not have any pluriharmonic terms (i.e., purely holomorphic or purely anti-holomorphic terms).

In particular, there exists a collection $(a_{\alpha,\beta})_{(\alpha,\beta)\in\mathcal{J}}$ of complex numbers where

- $\mathcal{J}$ is the set of all pairs $(\alpha, \beta) \in (\mathbb{Z}_{\geq 0})^n \times (\mathbb{Z}_{\geq 0})^n$ satisfying $|\alpha| > 0$, $|\beta| > 0$ and $|\alpha| + |\beta| = 2k$,
- $\overline{a_{\alpha,\beta}} = a_{\beta,\alpha}$ for all $(\alpha, \beta) \in \mathcal{J}$,

such that

$$P(z) = \sum_{(\alpha,\beta)\in\mathcal{J}} a_{\alpha,\beta} z^{\alpha} \overline{z}^{\beta}$$

for all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Here we are making use of the usual multi-index notation: $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $z^{\alpha} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ (and analogously for β and $\overline{z}^{\beta}$). If $a_{\alpha,\beta} \neq 0$, then we say that $|\alpha|$ (resp. $|\beta|$) is the holomorphic (resp. anti-holomorphic) degree of the term $a_{\alpha,\beta} z^{\alpha} \overline{z}^{\beta}$.

Furthermore, let $\mathcal{L}(P; p, V)$ denote the Levi form of P at the point $p \in \mathbb{C}^n$ in direction $V = (V_1, \ldots, V_n)^t \in \mathbb{C}^n$, i.e.,

$$\mathcal{L}(P; p, V) = (V_1, \ldots, V_n) \begin{pmatrix} \frac{\partial^2 P}{\partial z_1 \partial \overline{z_1}}(p) & \cdots & \frac{\partial^2 P}{\partial z_1 \partial \overline{z_n}}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 P}{\partial z_n \partial \overline{z_1}}(p) & \cdots & \frac{\partial^2 P}{\partial z_n \partial \overline{z_n}}(p) \end{pmatrix} \begin{pmatrix} \overline{V_1} \\ \vdots \\ \overline{V_n} \end{pmatrix}.$$

Lemma 3.1 *Let $\mathcal{A} = \{\alpha \in (\mathbb{Z}_{\geq 0})^n : |\alpha| = k \text{ and } a_{\alpha,\alpha} \neq 0\}$ and let*

$$\mathcal{C} = \left\{ (c_1, \ldots, c_n) \in \mathbb{C}^n : \sum_{j=1}^{n} \alpha_j c_j = 0 \text{ for all } \alpha \in \mathcal{A} \right\}.$$

Then we have for all $(c_1, \ldots, c_n) \in \mathcal{C}$ and for all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$:

$$\mathcal{L}(P; z, (c_1 z_1, \ldots, c_n z_n)^t) = 0.$$

Proof For all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $(c_1, \ldots, c_n) \in \mathbb{C}^n$ a straightforward calculation shows that

$$\mathcal{L}(P; z, (c_1 z_1, \ldots, c_n z_n)^t) = \sum_{(\alpha,\beta)\in\mathcal{J}} a_{\alpha,\beta} \cdot \left(\sum_{j=1}^{n} \alpha_j c_j\right) \cdot \overline{\left(\sum_{j=1}^{n} \beta_j c_j\right)} \cdot z^{\alpha} \overline{z}^{\beta}.$$

Assume for the sake of a contradiction that the claim is wrong. We then find some $(c_1, \ldots, c_n) \in \mathcal{C}$ and some $r_1, \ldots, r_n \in \mathbb{R}_{\geq 0}$, $\phi_1, \ldots, \phi_n \in [0, 2\pi)$, such that:

$$\mathcal{L}(P; (r_1 e^{i\phi_1}, \ldots, r_n e^{i\phi_n}), (c_1 r_1 e^{i\phi_1}, \ldots, c_n r_n e^{i\phi_n})^t) \neq 0.$$

By continuity, and since P is plurisubharmonic, we then get:

$$
\begin{aligned}
0 &< \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{L}(P; (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}), (c_1 r_1 e^{i\theta_1}, \ldots, c_n r_n e^{i\theta_n})^t) d\theta_1 \ldots d\theta_n \\
&= \sum_{(\alpha,\beta)\in\mathcal{J}} a_{\alpha,\beta} \cdot \left(\sum_{j=1}^n \alpha_j c_j \right) \cdot \overline{\left(\sum_{j=1}^n \beta_j c_j \right)} \cdot r_1^{\alpha_1+\beta_1} \ldots r_n^{\alpha_n+\beta_n} \\
&\quad \cdot \left(\int_0^{2\pi} e^{i(\alpha_1-\beta_1)\theta_1} d\theta_1 \right) \cdots \left(\int_0^{2\pi} e^{i(\alpha_n-\beta_n)\theta_n} d\theta_n \right) \\
&= (2\pi)^n \sum_{\alpha\in\mathcal{A}} a_{\alpha,\alpha} \cdot \left| \sum_{j=1}^n \alpha_j c_j \right|^2 \cdot r_1^{\alpha_1+\alpha_1} \ldots r_n^{\alpha_n+\alpha_n} \\
&= 0,
\end{aligned}
$$

where the last equality is due to the fact that $(c_1, \ldots, c_n) \in \mathcal{C}$. We have arrived at the desired contradiction; the claim follows. $\square$

For all $\beta \in (\mathbb{Z}_{\geq 0})^n$ with $1 \leq |\beta| \leq 2k-1$ we define a homogeneous holomorphic polynomial $P_\beta \colon \mathbb{C}^n \to \mathbb{C}$,

$$
P_\beta(z) = \sum_{\alpha\,:\,|\alpha|=2k-|\beta|} a_{\alpha,\beta} z^\alpha.
$$

In particular, we can write

$$
P(z) = \sum_{\beta\,:\,1\leq|\beta|\leq 2k-1} \bar{z}^\beta P_\beta(z).
$$

Lemma 3.2 *Assume that there exists a holomorphic map $G \colon \mathbb{C}^n \to \mathbb{C}^m$, $1 \leq m \leq n-1$, nonsingular on a non-empty open set $U \subseteq \mathbb{C}^n$, such that P is pluriharmonic along every level set of $G|_U$.*
Then, for all $i_1, \ldots, i_m, L \in \{1, \ldots, n\}$ (not necessarily pairwise distinct) and for all $\beta \in (\mathbb{Z}_{\geq 0})^n$ with $1 \leq |\beta| \leq 2k-1$, the following equality holds on $\mathbb{C}^n$:

$$
\det \begin{pmatrix} \frac{\partial G_1}{\partial z_{i_1}} & \cdots & \frac{\partial G_1}{\partial z_{i_m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial z_{i_1}} & \cdots & \frac{\partial G_m}{\partial z_{i_m}} \end{pmatrix} \cdot \frac{\partial P_\beta}{\partial z_L} = \sum_{j=1}^m \det \begin{pmatrix} \frac{\partial G_1}{\partial z_{i_1}} & \cdots & \frac{\partial G_1}{\partial z_{i_m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_{j-1}}{\partial z_{i_1}} & \cdots & \frac{\partial G_{j-1}}{\partial z_{i_m}} \\ \frac{\partial P_\beta}{\partial z_{i_1}} & \cdots & \frac{\partial P_\beta}{\partial z_{i_m}} \\ \frac{\partial G_{j+1}}{\partial z_{i_1}} & \cdots & \frac{\partial G_{j+1}}{\partial z_{i_m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial z_{i_1}} & \cdots & \frac{\partial G_m}{\partial z_{i_m}} \end{pmatrix} \cdot \frac{\partial G_j}{\partial z_L}.
$$

Proof Fix a point $p \in U$. Then there exist an open neighborhood $W \subseteq U$ of p and holomorphic maps $K_1, \ldots, K_{n-m} \colon W \to \mathbb{C}^n \setminus \{0\}$, such that $\{K_1(z), \ldots, K_{n-m}(z)\}$ is a basis for the null space of $G'(z) \in \mathbb{C}^{m \times n}$ for all $z \in W$. For $l = 1, \ldots, n - m$, denote the component functions of K_l as $K_l^{(1)}, \ldots, K_l^{(n)}$. Since P is pluriharmonic along every level set of $G|_U$, we get

$$
0 = (K_l^{(1)}, \ldots, K_l^{(n)}) \begin{pmatrix} \frac{\partial^2 P}{\partial z_1 \partial \overline{z_1}} & \cdots & \frac{\partial^2 P}{\partial z_1 \partial \overline{z_n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 P}{\partial z_n \partial \overline{z_1}} & \cdots & \frac{\partial^2 P}{\partial z_n \partial \overline{z_n}} \end{pmatrix} \begin{pmatrix} \overline{K_l^{(1)}} \\ \vdots \\ \overline{K_l^{(n)}} \end{pmatrix}
$$

on W for $l = 1, \ldots, n - m$; but since the Complex Hessian matrix of P is positive semidefinite, we even get

$$
(0, \ldots, 0) = (K_l^{(1)}, \ldots, K_l^{(n)}) \begin{pmatrix} \frac{\partial^2 P}{\partial z_1 \partial \overline{z_1}} & \cdots & \frac{\partial^2 P}{\partial z_1 \partial \overline{z_n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 P}{\partial z_n \partial \overline{z_1}} & \cdots & \frac{\partial^2 P}{\partial z_n \partial \overline{z_n}} \end{pmatrix}.
$$

Writing

$$
P(z) = \sum_{\beta \,\colon\, 1 \le |\beta| \le 2k-1} \overline{z}^{\beta} P_{\beta}(z),
$$

as above, we get for $I = 1, \ldots, n$:

$$
0 = \sum_{J=1}^{n} K_l^{(J)}(z) \cdot \frac{\partial^2 P}{\partial z_J \partial \overline{z_I}}(z)
$$

$$
= \sum_{\beta \,\colon\, 1 \le |\beta| \le 2k-1} \left(\beta_I \overline{z_1}^{\beta_1} \cdots \overline{z_I}^{\beta_I - 1} \cdots \overline{z_n}^{\beta_n} \cdot \sum_{J=1}^{n} K_l^{(J)}(z) \frac{\partial P_{\beta}}{\partial z_J}(z) \right)
$$

for all $z \in W$, $l = 1, \ldots, n - m$; hence

$$
0 = \sum_{I=1}^{n} \overline{z_I} \cdot 0
$$

$$
= \sum_{\beta \,\colon\, 1 \le |\beta| \le 2k-1} \left(|\beta| \cdot \overline{z}^{\beta} \cdot \sum_{J=1}^{n} K_l^{(J)}(z) \frac{\partial P_{\beta}}{\partial z_J}(z) \right).
$$

Owing to the fact that the $K_l^{(J)}$ and the P_β are holomorphic, we then get for all $\beta \in (\mathbb{Z}_{\geq 0})^n$ with $1 \leq |\beta| \leq 2k - 1$:

$$0 = \sum_{J=1}^{n} K_l^{(J)}(z) \frac{\partial P_\beta}{\partial z_J}(z) \text{ for all } z \in W, l \in \{1, \ldots, n - m\},$$

i.e., $K_l(z) = (K_l^{(1)}(z), \ldots, K_l^{(n)}(z))^t$ is in the null space of the matrix $P'_\beta(z) \in \mathbb{C}^{1 \times n}$. But this implies that the null space of the matrix

$$\begin{pmatrix} \frac{\partial G_1}{\partial z_1}(z) & \cdots & \frac{\partial G_1}{\partial z_n}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial z_1}(z) & \cdots & \frac{\partial G_m}{\partial z_n}(z) \\ \frac{\partial P_\beta}{\partial z_1}(z) & \cdots & \frac{\partial P_\beta}{\partial z_n}(z) \end{pmatrix} \in \mathbb{C}^{(m+1) \times n}, z \in W,$$

is $(n - m)$-dimensional for all β with $1 \leq |\beta| \leq 2k - 1$, so the rank of said matrix is m. Hence, given $i_1, \ldots, i_m, L \in \{1, \ldots, n\}$, we have

$$0 = \det \begin{pmatrix} \frac{\partial G_1}{\partial z_{i_1}} & \cdots & \frac{\partial G_1}{\partial z_{i_m}} & \frac{\partial G_1}{\partial z_L} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial G_m}{\partial z_{i_1}} & \cdots & \frac{\partial G_m}{\partial z_{i_m}} & \frac{\partial G_m}{\partial z_L} \\ \frac{\partial P_\beta}{\partial z_{i_1}} & \cdots & \frac{\partial P_\beta}{\partial z_{i_m}} & \frac{\partial P_\beta}{\partial z_L} \end{pmatrix}$$

on W for all β with $1 \leq |\beta| \leq 2k - 1$; noting that all the entries of the latter matrix are holomorphic on $\mathbb{C}^n$, the identity theorem gives that the determinant vanishes on all of $\mathbb{C}^n$. The claim follows by Laplace expanding by the last column and calculating. $\square$

Definition 3.3 Given $l \in \{1, \ldots, n\}$, we say that P is *homogeneous in l variables separately*, provided there exist integers $1 \leq i_1 < \cdots < i_l \leq n$ and integers $d_{i_1}, \ldots, d_{i_l} \geq 1$, such that for every $(\alpha, \beta) \in \mathcal{J}$ with $a_{\alpha, \beta} \neq 0$ we have $\alpha_{i_1} + \beta_{i_1} = 2d_{i_1}, \ldots, \alpha_{i_l} + \beta_{i_l} = 2d_{i_l}$. In this case we say that P is homogeneous of degree $2d_{i_j}$ in $z_{i_j}, \overline{z_{i_j}}$ for all $j \in \{1, \ldots, l\}$.

Note 3.4 We restrict attention to even degrees in Definition 3.3 due to the plurisubharmonicity requirement. Note furthermore that, in the case $l = n - 1$, the polynomial P is necessarily homogeneous in all n variables separately, since P is homogeneous.

Lemma 3.5 *Let $1 \leq l \leq n - 1$ and assume that P is homogeneous of degree $2d_j$, $d_j \in \mathbb{Z}_{>0}$, in $z_j, \overline{z_j}$ for $j = 1, \ldots, l$. Assume furthermore that $k - D > 0$, where $D = d_1 + \cdots + d_l$.*
Then, away from the coordinate hyperplanes, P is pluriharmonic along the level sets of $G \colon \mathbb{C}^n \to \mathbb{C}^{n-l}$,

$$G(z_1, \ldots, z_n) = z_1^{d_1} \cdots z_l^{d_l} \cdot \left(z_{l+1}^{k-D}, z_{l+2}^{k-D}, \ldots, z_n^{k-D} \right).$$

Proof G is nonsingular on $U := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_1 \neq 0, \ldots, z_n \neq 0\}$. Given $z \in U$, we have to show that $\mathcal{L}(P; z, V) = 0$ for all V in the null space of $G'(z) \in \mathbb{C}^{(n-l) \times n}$. But since the Complex Hessian matrix of P is positive semidefinite, it suffices to verify this for a *basis* of said null space. If $z \in U$, then the collection of vectors

$$
\begin{pmatrix}
0 \\
\vdots \\
0 \\
(k - D)z_j \\
0 \\
\vdots \\
0 \\
-d_j z_{l+1} \\
\vdots \\
-d_j z_n
\end{pmatrix}
\quad
\begin{matrix}
\\ \\ \\
j\text{th entry} \\ \\ \\ \\
(l+1)\text{th entry} \\ \\ \\
\end{matrix}
\qquad , \text{ where } j \in \{1, \ldots, l\}
$$

forms a basis for the null space of $G'(z) \in \mathbb{C}^{(n-l) \times n}$. Hence, with $\mathcal{A}$ and $\mathcal{C}$ as in Lemma 3.1, it suffices to show that $C_j \in \mathcal{C}$ for $j \in \{1, \ldots, l\}$, where

$$
C_j := \begin{pmatrix} 0, & \ldots, & 0, & \underset{j\text{th}}{k - D}, & 0, & \ldots, & 0, & \underset{(l+1)\text{th}}{-d_j}, & \ldots, & -d_j \end{pmatrix} \in \mathbb{C}^n.
$$

To this end, we consider some $j \in \{1, \ldots, l\}$ and some $\alpha \in \mathcal{A}$. Since $a_{\alpha,\alpha} \neq 0$ and by assumption on P we then have $\alpha_1 = d_1, \ldots, \alpha_l = d_l$, and $\alpha_{l+1} + \cdots + \alpha_n = k - D$. Writing $C_j =: (c_1^{(j)}, \ldots, c_n^{(j)})$, we then have:

$$
\sum_{s=1}^{n} \alpha_s c_s^{(j)} = (k - D)\alpha_j - d_j(\alpha_{l+1} + \cdots + \alpha_n) = 0,
$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

4 Proof of Proposition 2.1

Proof of Proposition 2.1 We proceed by induction on the dimension n. The 2-dimensional case was handled by Bharali, Stensønes [3], so let $n \geq 3$ and assume the claim holds in dimensions $2, \ldots, n - 1$. Let P be as in the statement of the theorem and assume for the sake of a contradiction that P is pluriharmonic along infinitely many complex hyperplanes through $0 \in \mathbb{C}^n$. We then find a sequence $(\mathcal{H}_j)_{j \in \mathbb{Z}_{\geq 1}}$ of *pairwise distinct* such hyperplanes. It is furthermore easy to see that there exists a complex hyperplane A through $0 \in \mathbb{C}^n$, such that P is *not* pluriharmonic along A. Since P is pluriharmonic along each $\mathcal{H}_j$, $j \in \mathbb{Z}_{\geq 1}$, we get that P is pluriharmonic

along $A \cap \mathcal{H}_j$ for all $j \in \mathbb{Z}_{\geq 1}$. Hence, by induction, the set $\{A \cap \mathcal{H}_j : j \in \mathbb{Z}_{\geq 1}\}$ is finite. So, there exists an $(n-2)$-dimensional complex vector subspace V of A, such that $A \cap \mathcal{H}_j = V$ for *infinitely many* $j \in \mathbb{Z}_{\geq 1}$. Thus, after deleting some members of the sequence if necessary, we can assume that

$$A \cap \mathcal{H}_j = V \text{ for all } j \in \mathbb{Z}_{\geq 1}.$$

It is easy to verify that there exists a complex hyperplane B through $0 \in \mathbb{C}^n$, such that P is *not* pluriharmonic along B and B does not contain V. By repeating the same argument and again deleting some members of the sequence if necessary, we find an $(n-2)$-dimensional complex vector subspace W of B, such that

$$B \cap \mathcal{H}_j = W \text{ for all } j \in \mathbb{Z}_{\geq 1}.$$

Hence every $\mathcal{H}_j$ contains $V + W$. However, B contains W but does not contain V, so we get that $V + W$ is at least $(n-1)$-dimensional. We conclude that $\mathcal{H}_j = V + W$ for all $j \in \mathbb{Z}_{\geq 1}$. Since the members of the sequence $(\mathcal{H}_j)_{j \in \mathbb{Z}_{\geq 1}}$ were chosen to be pairwise distinct, we have arrived at the desired contradiction. $\qquad\square$

5 Proof of Proposition 2.2

Let P and G be as in the statement of Proposition 2.2. It is obvious that P is indeed a non-constant, homogeneous, plurisubharmonic polynomial without pluriharmonic terms and that P is homogeneous of degree 2 in $z, \bar{z}$, so P is homogeneous in one variable separately. Furthermore, away from the coordinate hyperplanes, P is pluriharmonic along the level sets of G by Lemma 3.5.

Assume for the sake of a contradiction that there exist a homogeneous, plurisubharmonic polynomial $Q : \mathbb{C}^2 \to \mathbb{R}$ and holomorphic polynomials $F_1, F_2 : \mathbb{C}^3 \to \mathbb{C}$, homogeneous of the same degree, such that $P = Q \circ (F_1, F_2)$ on $\mathbb{C}^3$. As in Sect. 3 we write

$$P(z, w_1, w_2) = \bar{z} \cdot \overline{w_1}^2 \cdot \left(2zw_1^2 - zw_1w_2\right) + \bar{z} \cdot \overline{w_1} \cdot \overline{w_2} \cdot \left(zw_1w_2 - zw_1^2\right)$$
$$+ \bar{z} \cdot \overline{w_2}^2 \cdot \left(zw_2^2\right).$$

Even though the holomorphic map $(F_1, F_2) : \mathbb{C}^3 \to \mathbb{C}^2$ is (a priori) not necessarily non-singular, an argument analogous to the proof of Lemma 3.2 gives that the matrix

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial z} & \dfrac{\partial F_1}{\partial w_1} & \dfrac{\partial F_1}{\partial w_2} \\[2mm] \dfrac{\partial F_2}{\partial z} & \dfrac{\partial F_2}{\partial w_1} & \dfrac{\partial F_2}{\partial w_2} \\[2mm] 2w_1^2 - w_1w_2 & 4zw_1 - zw_2 & -zw_1 \\[1mm] w_1w_2 - w_1^2 & -2zw_1 + zw_2 & zw_1 \\[1mm] w_2^2 & 0 & 2zw_2 \end{pmatrix}$$

has the same rank as the matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial w_1} & \frac{\partial F_1}{\partial w_2} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial w_1} & \frac{\partial F_2}{\partial w_2} \end{pmatrix}$$

at every point of some non-empty open subset U of $\mathbb{C}^3$, which does not meet the coordinate hyperplanes. In particular, said rank is 2 and (F_1, F_2) is non-singular on U; hence F_1 and F_2 are both non-constant and homogeneous of degree $d := \deg F_1 = \deg F_2 \geq 1$. After adding the fourth row (of the former matrix) to the third row and applying the identity theorem, we get that the following holds on $\mathbb{C}^3$ for $j \in \{1, 2\}$:

$$0 = \det \begin{pmatrix} \frac{\partial F_j}{\partial z} & \frac{\partial F_j}{\partial w_1} & \frac{\partial F_j}{\partial w_2} \\ w_1{}^2 & 2zw_1 & 0 \\ w_2{}^2 & 0 & 2zw_2 \end{pmatrix},$$

and, using that F_j is homogeneous, a calculation then gives

$$3z \frac{\partial F_j}{\partial z} = z \frac{\partial F_j}{\partial z} + w_1 \frac{\partial F_j}{\partial w_1} + w_2 \frac{\partial F_j}{\partial w_2} = d \cdot F_j.$$

From this we readily deduce that $d/3$ is a positive integer and that there exist holomorphic polynomials $0 \not\equiv f_1, f_2 : \mathbb{C}^2 \to \mathbb{C}$, homogeneous of degree $2d/3$, such that

$$F_j(z, w_1, w_2) = z^{\frac{d}{3}} \cdot f_j(w_1, w_2) \text{ on } \mathbb{C}^3$$

for $j \in \{1, 2\}$. Since $Q : \mathbb{C}^2 \to \mathbb{R}$ is a homogeneous, plurisubharmonic (and clearly also non-pluriharmonic) polynomial, its degree $\deg Q$ is even. But $d/3$ is an integer and $6 = \deg P = d \cdot \deg Q$, so we necessarily have $\deg Q = 2$ and $d = 3$. In particular, f_j is homogeneous of degree 2 and $F_j(z, w_1, w_2) = z \cdot f_j(w_1, w_2)$ on $\mathbb{C}^3$ for $j \in \{1, 2\}$. Since P does not have any pluriharmonic terms, we can assume that Q does not have any pluriharmonic terms either. So there exist $a, c \in \mathbb{R}, b \in \mathbb{C}$, such that we have for all $(x, y) \in \mathbb{C}^2$:

$$Q(x, y) = a \cdot |x|^2 + b \cdot x\overline{y} + \overline{b} \cdot \overline{x}y + c \cdot |y|^2.$$

For $j \in \{1, 2\}$ we furthermore find $\sigma_j, \rho_j, \mu_j \in \mathbb{C}$, such that we have for all $(w_1, w_2) \in \mathbb{C}^2$:

$$f_j(w_1, w_2) = \sigma_j w_1{}^2 + \rho_j w_1 w_2 + \mu_j w_2{}^2.$$

A calculation then shows that

$$Q(z \cdot f_1(w_1, w_2), z \cdot f_2(w_1, w_2))$$
$$= |z|^2 \cdot \left(\overline{w_1}^2 \cdot g_1(w_1, w_2) + \overline{w_1 w_2} \cdot h(w_1, w_2) + \overline{w_2}^2 \cdot g_2(w_1, w_2) \right)$$

for some $g_1, h, g_2 \colon \mathbb{C}^2 \to \mathbb{C}$ contained in the $\mathbb{C}$-vector space $\mathcal{V}$ spanned by f_1 and f_2; we trivially have $\dim_{\mathbb{C}} \mathcal{V} \leq 2$. Recalling that $P = Q \circ (F_1, F_2)$, we necessarily have

$$g_1(w_1, w_2) = 2w_1^2 - w_1 w_2, \quad h(w_1, w_2) = w_1 w_2 - w_1^2, \quad g_2(w_1, w_2) = w_2^2$$

which implies that $\dim_{\mathbb{C}} \mathcal{V} \geq 3$. We have arrived at the desired contradiction.

6 Proof of Theorem 2.3

In this section we will (without further comment) identify holomorphic polynomials $\mathbb{C}^n \to \mathbb{C}$ with elements of the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ in the obvious way.

Notation 6.1 Let $0 \neq g \in \mathbb{C}[z_1, \ldots, z_n]$ be a homogeneous polynomial of positive degree. Then the set

$$\{m \in \mathbb{Z}_{>0} \colon g = \widetilde{g}^m \text{ for some homogeneous } \widetilde{g} \in \mathbb{C}[z_1, \ldots, z_n]\}$$

is clearly non-empty and bounded from above. Hence it has a maximum, which we denote as $M_g \in \mathbb{Z}_{>0}$.

Lemma 6.2 *Let $0 \neq g \in \mathbb{C}[z_1, \ldots, z_n]$ be a homogeneous polynomial of positive degree and let $h \in \mathbb{C}[z_1, \ldots, z_n]$ be any homogeneous polynomial with $g = h^{M_g}$ (see Notation 6.1). If $0 \neq f \in \mathbb{C}[z_1, \ldots, z_n]$ is a homogeneous polynomial of positive degree satisfying*

$$(\deg g) \cdot g \cdot \frac{\partial f}{\partial z_l} = (\deg f) \cdot f \cdot \frac{\partial g}{\partial z_l} \text{ for all } l \in \{1, \ldots, n\},$$

then $M_g \cdot (\deg f)/(\deg g)$ is a positive integer and there exists a $c \in \mathbb{C} \setminus \{0\}$, such that $f = c \cdot h^{M_g \cdot \frac{\deg f}{\deg g}}$.

Proof Let $g, h, f \in \mathbb{C}[z_1, \ldots, z_n]$ be as in the statement of the lemma. Write

$$g = u \, p_1^{\alpha_1} \cdots p_m^{\alpha_m},$$

where m is a positive integer (since $\deg g > 0$), $u \in \mathbb{C} \setminus \{0\}$ is a unit, $p_1, \ldots, p_m \in \mathbb{C}[z_1, \ldots, z_n]$ are pairwise non-associate primes, and $\alpha_1, \ldots, \alpha_m$ are positive integers. Since $\deg f > 0$, we get that g divides $f \cdot (\partial g)/(\partial z_l)$ for $l = 1, \ldots, n$. Hence, considering any $s \in \{1, \ldots, m\}$, we get that $p_s^{\alpha_s}$ divides

$$f \cdot \frac{\partial g}{\partial z_l} = f \cdot u \cdot \sum_{j=1}^{m} p_1^{\alpha_1} \cdots \widehat{p_j^{\alpha_j}} \cdots p_m^{\alpha_m} \cdot \alpha_j \cdot p_j^{\alpha_j - 1} \cdot \frac{\partial p_j}{\partial z_l}.$$

If $j \neq s$, then the corresponding summand is trivially divisible by $p_s{}^{\alpha_s}$. But this implies that $p_s{}^{\alpha_s}$ divides

$$f \cdot u \cdot p_1{}^{\alpha_1} \cdots \widehat{p_s{}^{\alpha_s}} \cdots p_m{}^{\alpha_m} \cdot \alpha_s \cdot p_s{}^{\alpha_s - 1} \cdot \frac{\partial p_s}{\partial z_l}.$$

Hence

$$p_s \left| \; f \cdot p_1{}^{\alpha_1} \cdots \widehat{p_s{}^{\alpha_s}} \cdots p_m{}^{\alpha_m} \cdot \frac{\partial p_s}{\partial z_l}. \right.$$

Since p_s is prime, it divides one of the factors. Since the primes $p_1, \ldots, p_m$ are pairwise non-associate, we get for all $l \in \{1, \ldots, n\}$:

$$p_s \mid f \text{ or } p_s \left| \; \frac{\partial p_s}{\partial z_l}. \right.$$

But since p_s is prime and hence $(\partial p_s)/(\partial z_{l_s}) \neq 0$ for some $l_s \in \{1, \ldots, n\}$, we get $p_s \mid f$. We have shown that every prime factor of g divides f. But by reversing the roles of f and g and repeating the same argument, we also get that every prime factor of f divides g. We conclude that there exist a unit $v \in \mathbb{C} \setminus \{0\}$ and positive integers $\beta_1, \ldots, \beta_m$, such that

$$f = v p_1{}^{\beta_1} \cdots p_m{}^{\beta_m}.$$

By assumption we then have for all $l \in \{1, \ldots, n\}$:

$$(\deg g) \cdot u p_1{}^{\alpha_1} \cdots p_m{}^{\alpha_m} \cdot v \cdot \sum_{j=1}^{m} p_1{}^{\beta_1} \cdots \widehat{p_j{}^{\beta_j}} \cdots p_m{}^{\beta_m} \cdot \beta_j \cdot p_j{}^{\beta_j - 1} \cdot \frac{\partial p_j}{\partial z_l}$$

$$= (\deg f) \cdot v p_1{}^{\beta_1} \cdots p_m{}^{\beta_m} \cdot u \cdot \sum_{j=1}^{m} p_1{}^{\alpha_1} \cdots \widehat{p_j{}^{\alpha_j}} \cdots p_m{}^{\alpha_m} \cdot \alpha_j \cdot p_j{}^{\alpha_j - 1} \cdot \frac{\partial p_j}{\partial z_l},$$

and hence

$$0 = \sum_{j=1}^{m} p_1 \cdots \widehat{p_j} \cdots p_m \cdot \frac{\partial p_j}{\partial z_l} \cdot (\beta_j (\deg g) - \alpha_j (\deg f)).$$

Considering any $s \in \{1, \ldots, m\}$, we note that p_s obviously divides the jth summand for $j \neq s$, and hence

$$p_s \left| \; p_1 \cdots \widehat{p_s} \cdots p_m \cdot \frac{\partial p_s}{\partial z_l} \cdot (\beta_s (\deg g) - \alpha_s (\deg f)). \right.$$

Using again that the primes $p_1, \ldots, p_m$ are pairwise non-associate and considering some $l_s \in \{1, \ldots, n\}$ with $(\partial p_s)/(\partial z_{l_s}) \neq 0$, we get

$$p_s \mid (\beta_s(\deg g) - \alpha_s(\deg f)),$$

i.e., $\beta_s(\deg g) - \alpha_s(\deg f) = 0$. So, since s was chosen arbitrarily, we have

$$\frac{\beta_j}{\alpha_j} = \frac{\deg f}{\deg g} \text{ for all } j \in \{1, \ldots, m\}.$$

Since $g = h^{M_g}$, we can write

$$h = w \, p_1{}^{\gamma_1} \cdots p_m{}^{\gamma_m},$$

where $w \in \mathbb{C} \setminus \{0\}$, $w^{M_g} = u$, and $\gamma_j = \alpha_j / M_g$ is a positive integer for $j = 1, \ldots, m$. Due to the defining properties of M_g (see Notation 6.1) we furthermore have $\gcd(\gamma_1, \ldots, \gamma_m) = 1$, i.e., there exist $d_1, \ldots, d_m \in \mathbb{Z}$, such that

$$1 = \sum_{j=1}^{m} d_j \gamma_j.$$

Hence

$$M_g \cdot \frac{\deg f}{\deg g} = \sum_{j=1}^{m} d_j \cdot \gamma_j \cdot M_g \cdot \frac{\deg f}{\deg g} = \sum_{j=1}^{m} d_j \cdot \frac{\alpha_j}{M_g} \cdot M_g \cdot \frac{\beta_j}{\alpha_j} = \sum_{j=1}^{m} d_j \beta_j$$

is a (positive) integer, as desired. Finally, we compute

$$
\begin{aligned}
h^{M_g \cdot \frac{\deg f}{\deg g}} &= w^{M_g \cdot \frac{\deg f}{\deg g}} \cdot \prod_{j=1}^{m} \left(p_j{}^{\gamma_j} \right)^{M_g \cdot \frac{\beta_j}{\alpha_j}} \\
&= w^{M_g \cdot \frac{\deg f}{\deg g}} \cdot \prod_{j=1}^{m} \left(p_j^{\frac{\alpha_j}{M_g}} \right)^{M_g \cdot \frac{\beta_j}{\alpha_j}} \\
&= w^{M_g \cdot \frac{\deg f}{\deg g}} \, p_1{}^{\beta_1} \cdots p_m{}^{\beta_m} \\
&= \frac{w^{M_g \cdot \frac{\deg f}{\deg g}}}{v} \cdot f,
\end{aligned}
$$

and the claim follows.   □

Armed with Lemma 6.2, we can provide a proof for Theorem 2.3.

Proof of Theorem 2.3 As in Sect. 3 we write

$$P(z) = \sum_{\beta:\, 1 \le |\beta| \le 2k-1} \bar{z}^{\beta} P_{\beta}(z).$$

We then apply Lemma 3.2 and get for all $\beta \in (\mathbb{Z}_{\ge 0})^n$ with $1 \le |\beta| \le 2k - 1$ and for all $j, l \in \{1, \ldots, n\}$:

$$\frac{\partial P_{\beta}}{\partial z_l} \frac{\partial G}{\partial z_j} = \frac{\partial P_{\beta}}{\partial z_j} \frac{\partial G}{\partial z_l} \quad \text{on } \mathbb{C}^n.$$

Writing $G = G(0) + q + R$, where $q \colon \mathbb{C}^n \to \mathbb{C}$ is a non-constant, homogeneous, holomorphic polynomial and $R \colon \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function whose Taylor series at 0 does not involve any terms of degree $\le \deg q$, we get, owing to the fact that the P_{β} are homogeneous:

$$\frac{\partial P_{\beta}}{\partial z_l} \frac{\partial q}{\partial z_j} = \frac{\partial P_{\beta}}{\partial z_j} \frac{\partial q}{\partial z_l} \quad \text{on } \mathbb{C}^n$$

for all $\beta \in (\mathbb{Z}_{\ge 0})^n$ with $1 \le |\beta| \le 2k - 1$ and for all $j, l \in \{1, \ldots, n\}$. By multiplying with z_j and then summing over j, we get, using that both P_{β} and q are homogeneous:

$$(\deg P_{\beta}) \cdot P_{\beta} \cdot \frac{\partial q}{\partial z_l} = (\deg q) \cdot q \cdot \frac{\partial P_{\beta}}{\partial z_l} \quad \text{on } \mathbb{C}^n$$

for all β, l. Let $h \colon \mathbb{C}^n \to \mathbb{C}$ be any homogeneous, holomorphic polynomial with $q = h^{M_q}$ (see Notation 6.1). Lemma 6.2 then implies that, for a given β, the following is true:

- if $M_q \cdot \frac{\deg P_{\beta}}{\deg q} = M_q \cdot \frac{\deg P_{\beta}}{M_q \cdot \deg h} = \frac{2k - |\beta|}{\deg h}$ is not a positive integer, then $P_{\beta} \equiv 0$,
- if $\frac{2k - |\beta|}{\deg h}$ is a positive integer, then there exists a $c_{\beta} \in \mathbb{C}$, such that

$$P_{\beta} = c_{\beta} \cdot h^{M_q \cdot \frac{\deg P_{\beta}}{\deg q}} = c_{\beta} \cdot h^{\frac{2k - |\beta|}{\deg h}}$$

(if $P_{\beta} \equiv 0$, take $c_{\beta} = 0$, otherwise apply Lemma 6.2).

Since $P \not\equiv 0$, there exists a positive integer L, such that

$$\{1, \ldots, L\} = \left\{ \frac{2k - |\beta|}{\deg h} : \beta \in (\mathbb{Z}_{\ge 0})^n, 1 \le |\beta| \le 2k - 1, \frac{2k - |\beta|}{\deg h} \in \mathbb{Z}_{>0} \right\}.$$

Hence we can write on $\mathbb{C}^n$:

$$P(z) = \sum_{l=1}^{L} \sum_{\beta:\, |\beta| = 2k - l \cdot \deg h} \bar{z}^{\beta} \cdot c_{\beta} \cdot (h(z))^l$$

$$= \sum_{l=1}^{L} (h(z))^l \cdot \sum_{\beta\,:\; |\beta|=2k-l\cdot\deg h} c_\beta \cdot \bar{z}^\beta$$

$$= \sum_{l=1}^{L} \overline{h(z)}^l \cdot \sum_{\beta\,:\; |\beta|=2k-l\cdot\deg h} \overline{c_\beta} \cdot z^\beta,$$

where the last equality is due to the fact that P is real-valued. Since $h^l \not\equiv 0$, we can find an $\alpha_l \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha_l| = l \cdot \deg h$, such that z^{α_l} appears with coefficient $\gamma_l \neq 0$ in the Taylor expansion of h^l at 0 (note that α_l is not uniquely determined in general). Recalling that

$$P(z) = \sum_{\beta\,:\; 1\leq|\beta|\leq 2k-1} \bar{z}^\beta P_\beta(z),$$

we see that we necessarily have (recall that h is homogeneous):

$$P_{\alpha_l}(z) = \overline{\gamma_l} \cdot \sum_{\beta\,:\; |\beta|=2k-l\cdot\deg h} \overline{c_\beta} \cdot z^\beta,$$

and hence

$$P(z) = \sum_{l=1}^{L} \overline{h(z)}^l \cdot \frac{1}{\overline{\gamma_l}} \cdot P_{\alpha_l}(z).$$

Assume for the sake of a contradiction that $2k/(\deg h)$ is not an integer. Then we have for all $l = 1, \ldots, L$ that

$$\frac{2k - |\alpha_l|}{\deg h} = \frac{2k}{\deg h} - l \notin \mathbb{Z},$$

which implies that $P_{\alpha_l} \equiv 0$ (see above). But then $P \equiv 0$ and we arrive at the desired contradiction. Hence $2k/(\deg h) \in \mathbb{Z}$. But then we have that $\frac{2k-|\alpha_l|}{\deg h}$ is a positive integer for $l = 1, \ldots, L$, which, using the above, implies that

$$P_{\alpha_l} = c_{\alpha_l} \cdot h^{\frac{2k-|\alpha_l|}{\deg h}} = c_{\alpha_l} \cdot h^{\frac{2k}{\deg h}-l}.$$

We now define $s \colon \mathbb{C} \to \mathbb{R}$,

$$\tau \mapsto \sum_{l=1}^{L} \frac{c_{\alpha_l}}{\overline{\gamma_l}} \cdot \overline{\tau}^l \cdot \tau^{\frac{2k}{\deg h}-l}.$$

Note that s is indeed real-valued. It is now easy to see that s and h have all the desired properties.   □

7 Proof of Theorem 2.4 and Corollary 2.5

In this section we will provide a proof for Theorem 2.4. We will not provide a separate proof for Corollary 2.5, since it will be obvious from the proof of Theorem 2.4.

Proof of Theorem 2.4 We adapt the notation from Sect. 3. In particular we write

$$P(z) = \sum_{\beta\,:\,1\leq|\beta|\leq 2k-1} \bar{z}^\beta P_\beta(z) = \sum_{\beta\in B} \bar{z}^\beta P_\beta(z),$$

where $B = \{\beta \in (\mathbb{Z}_{\geq 0})^n : 1 \leq |\beta| \leq 2k - 1 \text{ and } P_\beta \not\equiv 0\}$. By assumption on P we find, for every $\beta \in B$, a holomorphic polynomial $q_\beta \colon \mathbb{C}^{n-l} \to \mathbb{C}$, homogeneous of degree $2k - (2d_1 + \cdots + 2d_l + \beta_{l+1} + \cdots + \beta_n)$, such that

$$P_\beta(z_1, \ldots, z_n) = z_1{}^{2d_1-\beta_1} \cdots z_l{}^{2d_l-\beta_l} q_\beta(z_{l+1}, \ldots, z_n).$$

For ease of notation we write $m = n - l$ and $(w_1, \ldots, w_m) = (z_{l+1}, \ldots, z_n)$. We will switch back and forth between notations whenever convenient. By Lemma 3.5, away from the coordinate hyperplanes, P is pluriharmonic along the level sets of $G \colon \mathbb{C}^n \to \mathbb{C}^m$,

$$G(z_1, \ldots, z_l, w_1, \ldots, w_m) = z_1{}^{d_1} \cdots z_l{}^{d_l} \cdot \left(w_1{}^{k-D}, \ldots, w_m{}^{k-D}\right).$$

Lemma 3.2 then gives that the following holds on $\mathbb{C}^n$ for all $\beta \in B$ and for all $\nu \in \{1, \ldots, n\}$:

$$\det\begin{pmatrix} \frac{\partial G_1}{\partial w_1} & \cdots & \frac{\partial G_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial w_1} & \cdots & \frac{\partial G_m}{\partial w_m} \end{pmatrix} \cdot \frac{\partial P_\beta}{\partial z_\nu} = \sum_{j=1}^{m} \det\begin{pmatrix} \frac{\partial G_1}{\partial w_1} & \cdots & \frac{\partial G_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_{j-1}}{\partial w_1} & \cdots & \frac{\partial G_{j-1}}{\partial w_m} \\ \frac{\partial P_\beta}{\partial w_1} & \cdots & \frac{\partial P_\beta}{\partial w_m} \\ \frac{\partial G_{j+1}}{\partial w_1} & \cdots & \frac{\partial G_{j+1}}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial w_1} & \cdots & \frac{\partial G_m}{\partial w_m} \end{pmatrix} \cdot \frac{\partial G_j}{\partial z_\nu}.$$

A calculation then shows that the following holds on $\mathbb{C}^n$ for all $\beta \in B$, $\nu \in \{1, \ldots, l\}$:

$$(k - D)(2d_\nu - \beta_\nu)q_\beta(w_1, \ldots, w_m) = d_\nu \sum_{j=1}^{m} w_j \frac{\partial q_\beta}{\partial w_j}(w_1, \ldots, w_m),$$

but since $q_\beta \not\equiv 0$ is homogeneous, this simplifies to

$$(k - D)(2d_\nu - \beta_\nu)q_\beta(w_1, \ldots, w_m)$$
$$= d_\nu \cdot (2k - (2d_1 + \cdots + 2d_l + \beta_{l+1} + \cdots + \beta_n)) \cdot q_\beta(w_1, \ldots, w_m).$$

For $\beta \in B$ we have $q_\beta \not\equiv 0$, so we get for all $\beta \in B$, $\nu \in \{1, \ldots, l\}$:

$$(k - D)(2d_\nu - \beta_\nu) = d_\nu \cdot (2k - (2d_1 + \cdots + 2d_l + \beta_{l+1} + \cdots + \beta_n))$$
$$= d_\nu \cdot M_\beta,$$

where M_β is defined in the obvious way. Since $d = \gcd(d_1, \ldots, d_l, k)$, there exist $c, c_1, \ldots, c_l \in \mathbb{Z}$, such that $d = c \cdot (k - D) + \sum_{\nu=1}^{l} c_\nu \cdot d_\nu$. But then we have for all $\beta \in B$:

$$\frac{d \cdot M_\beta}{k - D} = c \cdot M_\beta + \sum_{\nu=1}^{l} c_\nu d_\nu \frac{M_\beta}{k - D} = c \cdot M_\beta + \sum_{\nu=1}^{l} c_\nu (2d\nu - \beta_\nu),$$

so $d \cdot M_\beta / (k - D)$ is an integer for all $\beta \in B$. If M_β was 0 for some $\beta \in B$, then $2d_\nu - \beta_\nu$ would be 0 for $\nu \in \{1, \ldots, l\}$, implying that $|\beta| = 2k$, in contradiction to $\beta \in B$. Hence $d \cdot M_\beta / (k - D)$ is a *positive* integer for all $\beta \in B$. This implies that for all $\beta \in B$ we have

$$P_\beta(z_1, \ldots, z_n) = \left(z_1^{\frac{d_1}{d}} \cdots z_l^{\frac{d_l}{d}} \right)^{\frac{d \cdot M_\beta}{k - D}} \cdot q_\beta(z_{l+1}, \ldots, z_n),$$

where all occurring exponents are positive integers. If $\beta \in B$, then $M_\beta \leq 2(k - D)$. If this was an equality, then, similarly to above, we would get $|\beta| = 0$, in contradiction to $\beta \in B$. Hence we have for all $\beta \in B$:

$$\frac{d \cdot M_\beta}{k - D}, 2d - \frac{d \cdot M_\beta}{k - D} \in \{1, 2, \ldots, 2d - 1\}.$$

For $\beta \in B$, $\nu \in \{1, \ldots, l\}$ we have

$$\beta_\nu = 2d_\nu - \frac{d_\nu \cdot M_\beta}{k - D} = \frac{d_\nu}{d} \cdot \left(2d - \frac{d \cdot M_\beta}{k - D} \right),$$

and both factors in the latter equality are positive integers. We now calculate, noting that all occurring exponents are positive integers:

$$P(z) = \sum_{\beta \in B} \bar{z}^\beta P_\beta(z)$$

$$= \sum_{\beta \in B} \left(z_1^{\frac{d_1}{d}} \cdots z_l^{\frac{d_l}{d}} \right)^{\frac{d \cdot M_\beta}{k - D}} \cdot \overline{\left(z_1^{\frac{d_1}{d}} \cdots z_l^{\frac{d_l}{d}} \right)}^{2d - \frac{d \cdot M_\beta}{k - D}}$$

$$\cdot \bar{z}_{l+1}^{\beta_{l+1}} \cdots \bar{z}_n^{\beta_n} q_\beta(z_{l+1}, \ldots, z_n)$$

$$= \sum_{j=1}^{2d-1} \left(z_1^{\frac{d_1}{d}} \cdots z_l^{\frac{d_l}{d}} \right)^j \cdot \overline{\left(z_1^{\frac{d_1}{d}} \cdots z_l^{\frac{d_l}{d}} \right)}^{2d - j}$$

$$\cdot \sum_{\beta \in B \,:\, \frac{d \cdot M_\beta}{k-D}=j} \overline{z_{l+1}}^{\beta_{l+1}} \cdots \overline{z_n}^{\beta_n} q_\beta(z_{l+1}, \ldots, z_n).$$

Now if $\beta \in B$ and $j \in \{1, \ldots, 2d-1\}$ satisfy $d \cdot M_\beta/(k-D) = j$, then every term occurring in $\overline{z_{l+1}}^{\beta_{l+1}} \cdots \overline{z_n}^{\beta_n} q_\beta(z_{l+1}, \ldots, z_n)$ has holomorphic degree $M_\beta = j \cdot (k-D)/d$ and anti-holomorphic degree $\beta_{l+1} + \cdots + \beta_n = (2d-j) \cdot (k-D)/d$. This implies that the polynomial $Q \colon \mathbb{C}^m \to \mathbb{R}$,

$$Q(w_1, \ldots, w_m) = \sum_{j=1}^{2d-1} \sum_{\beta \in B \,:\, \frac{d \cdot M_\beta}{k-D}=j} \overline{w_1}^{\beta_{1+l}} \cdots \overline{w_m}^{\beta_{m+l}} q_\beta(w_1, \ldots, w_m),$$

has the property that both the holomorphic and the anti-holomorphic degree of every term appearing in Q are divisible by the integer $(k-D)/d$. Furthermore $Q(w_1, \ldots, w_m) = P(1, \ldots, 1, w_1, \ldots, w_m)$, so Q is (indeed real-valued and) plurisubharmonic. It is now easy to see that Q has all the other desired properties. $\square$

References

1. Bedford, E., Fornæss, J.E.: A construction of peak functions on weakly pseudoconvex domains. Ann. Math. (2) **107**(3), 555–568 (1978)
2. Bharali, G.: Model pseudoconvex domains and bumping. Int. Math. Res. Not. IMRN **21**, 4924–4965 (2012)
3. Bharali, G., Stensønes, B.: Plurisubharmonic polynomials and bumping. Math. Z. **261**(1), 39–63 (2009)
4. Fornæss, J.E.: Sup-norm estimates for $\bar{\partial}$ in $\mathbb{C}^2$. Ann. Math. (2) **123**(2), 335–345 (1986)
5. Fornæss, J.E., Sibony, N.: Construction of P.S.H. functions on weakly pseudoconvex domains. Duke Math. J. **58**(3), 633–655 (1989)
6. Fornæss, J.E., Stensønes, B.: Maximally tangent complex curves for germs of finite type $\mathcal{C}^\infty$ pseudoconvex domains in $\mathbb{C}^3$. Math. Ann. **347**(4), 979–991 (2010)
7. Grundmeier, D., Simon, L., Stensønes, B.: Sup-norm estimates for $\bar{\partial}$ in $\mathbb{C}^3$ (2020). arXiv:1909.04080
8. Noell, A.: Peak Functions for Pseudoconvex Domains in $\mathbb{C}^n$, Several Complex Variables (Stockholm, 1987/1988), vol. 38, pp. 529–541. Princeton University Press, Princeton (1993)
9. Range, R.M.: Integral kernels and Hölder estimates for $\bar{\partial}$ on pseudoconvex domains of finite type in $\mathbb{C}^2$. Math. Ann. **288**(1), 63–74 (1990)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.