Bounding the log-derivative of the zeta-function

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Abstract

Assuming the Riemann hypothesis we establish explicit bounds for the modulus of the log-derivative of Riemann's zeta-function in the critical strip.

Keywords Zeta-function · Riemann hypothesis · Critical strip · Beurling–Selberg extremal problem · Bandlimited functions · Exponential type

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1 Introduction

Let $\zeta(s)$ be the Riemann zeta-function. In this paper we are interested in its log-derivative

$$\frac{\zeta'}{\zeta}(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} \quad (\mathrm{Re}\, s > 1)$$

and its growth behaviour in the strip $1/2 < \mathrm{Re}\, s < 1$ (above $\Lambda(n)$ is the von Mangoldt function). Let ρ denote the zeros of $\zeta(s)$ in the critical strip. The Riemann hypothesis (RH) states that the zeros are aligned: $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$. Assuming RH, a classical estimate for the log-derivative of $\zeta(s)$ (see [12, Theorem 14.5]) establishes that

$$\frac{\zeta'}{\zeta}(\sigma + it) = O\left((\log t)^{2-2\sigma}\right),$$

uniformly in $\frac{1}{2} + \delta \leq \sigma \leq 1 - \delta$, for any fixed $\delta > 0$. The purpose of this paper is to establish this bound in explicit form.

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Theorem 1 *Assume RH. Then*

$$\left|\frac{\zeta'}{\zeta}(\sigma + it)\right| \le \frac{B_\sigma}{\sigma(1-\sigma)}(\log t)^{2-2\sigma} + O\left(\frac{(\log t)^{2-2\sigma}}{(\sigma - \frac{1}{2})(1-\sigma)^2 \log\log t}\right),$$

uniformly in the range

$$\frac{1}{2} + \frac{\lambda_0 + c}{\log\log t} \le \sigma \le 1 - \frac{c}{\sqrt{\log\log t}} \quad and \quad t \ge 3, \tag{1.1}$$

for any fixed small $c > 0$, where $\lambda_0 = 0.771\ldots$ is such that $2\lambda_0\tanh(\lambda_0) = 1$ and

$$B_\sigma = \sqrt{\frac{(3\sigma^4 - 17\sigma^3 + 19\sigma^2 + 4\sigma - 4)(-\sigma^2 + 3\sigma - 1)}{\sigma(2-\sigma)}}.$$

In particular

$$\left|\frac{\zeta'}{\zeta}(\sigma + it)\right| \le \left(\frac{B_\sigma}{\sigma(1-\sigma)} + o(1)\right)(\log t)^{2-2\sigma}, \quad for \quad \tfrac{1}{2} + \delta \le \sigma \le 1 - \delta.$$

We believe that λ_0 is simply a by-product of our proof, although it is curious that such a number appears. It turns out that when $(\sigma - 1/2)\log\log t$ is too small, our main technique delivers a bound of the form $A_\sigma(\log t)/\log\log t$, however the calculations are lengthy and convoluted, and this it not the purpose of this note. Moreover, a conjecture of Ki [10], related to the distribution of the zeros of $\zeta'(s)$, states that the bound $O((\log t)^{2-2\sigma})$ still holds in the range $\sigma \ge 1/2 + c/\log t$, but this lies outside of what this technique can accomplish. Theorem 1 is derived by combining Theorem 2 and estimates for the real part of the log-derivative of $\zeta(s)$ obtained in [4, Theorem 2]:

$$\left|\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma + it)\right| \le \left(\frac{-\sigma^2 + 3\sigma - 1}{\sigma(1-\sigma)}\right)(\log t)^{2-2\sigma} + O\left(\frac{(\log t)^{2-2\sigma}}{(\sigma - \frac{1}{2})(1-\sigma)^2 \log\log t}\right),$$

uniformly in the range (1.1) (in fact $\lambda_0 + c$ can be replaced by just c).

Theorem 2 *Assume RH. Then*

$$\left|\operatorname{Im}\frac{\zeta'}{\zeta}(\sigma + it)\right| \le \frac{C_\sigma}{\sigma(1-\sigma)}(\log t)^{2-2\sigma} + O\left(\frac{(\log t)^{2-2\sigma}}{(\sigma - \frac{1}{2})(1-\sigma)^2 \log\log t}\right),$$

uniformly in the range (1.1), *where*

$$C_\sigma = \sqrt{\frac{2(-\sigma^2 + 5\sigma - 2)(-\sigma^2 + 3\sigma - 1)(-\sigma^2 + \sigma + 1)}{\sigma(2-\sigma)}}.$$

Theorem 2 is obtained using a known interpolation technique [4, Section 6]. Essentially, to bound the asymptotic growth of $\operatorname{Im}\frac{\zeta'}{\zeta}(s)$ one can bound instead its primitive $\log|\zeta(s)|$ (see [1, Theorems 1 and 2]) and its derivative (Theorem 3).

Theorem 3 *Assume RH . Then*

$$\operatorname{Re}\left(\frac{\zeta'}{\zeta}\right)'(\sigma + it) \le \left(\frac{-2\sigma^2 + 2\sigma + 2}{\sigma(1-\sigma)}\right)\log\log t\,(\log t)^{2-2\sigma} + O\left(\frac{(\log t)^{2-2\sigma}}{(\sigma - \frac{1}{2})(1-\sigma)^2}\right),$$

and

$$\mathrm{Re}\left(\frac{\zeta'}{\zeta}\right)'(\sigma+it) \geq -\left(\frac{-2\sigma^2+6\sigma-2}{\sigma(1-\sigma)}\right)\log\log t\,(\log t)^{2-2\sigma}+O\left(\frac{(\log t)^{2-2\sigma}}{(\sigma-\frac{1}{2})(1-\sigma)^2}\right),$$

uniformly in the range

$$\frac{1}{2}+\frac{\lambda_0}{\log\log t} \leq \sigma \leq 1-\frac{c}{\sqrt{\log\log t}} \quad and \quad t \geq 3,$$

for any fixed $c > 0$.

The main technique to prove these theorems revolves in bounding a certain sum over the ordinates of zeta-zeros

$$\sum_{\gamma} f(\gamma - t),$$

where f is some explicit real function that varies according to the problem of study. The key idea is to replace f by explicit bandlimited majorants and minorants that are in turn admissible for the Guinand–Weil explicit formula (Proposition 5). From there estimating the sum is usually easier. This bandlimited approximation idea originates in the works of Beurling and Selberg (see [14, Introduction]), and was first employed in this form by Goldston and Gonek [8], and Chandee and Soundararajan [6], but many others after them (see [1–4,7] to name a few). In our specific case, $f = f_a$ as in (2.1), which has *zero mass* and therefore is not in the scope of the machinery developed in [5], nor its close relatives (the constructions in [5] are regarded as the most general thus far and have been used widely). Nevertheless, we are able to overcome this difficulty with a very simple optimal construction which, in the majorant case, requires some basic results in the theory of de Branges spaces.

We recall that, without assuming RH, explicit bounds for $\frac{\zeta'}{\zeta}(s)$ are given by Trudgian [13] in a zero-free region for $\zeta(s)$.

2 Lemmata

For a given $a > 0$ we let

$$f_a(x) = \frac{x^2 - a^2}{\left(x^2 + a^2\right)^2}. \tag{2.1}$$

Lemma 4 (Representation lemma) *Assume RH. We have*

$$\mathrm{Re}\left(\frac{\zeta'}{\zeta}\right)'(\sigma + it) = \sum_{\gamma} f_{\sigma-1/2}(\gamma - t) + O\left(\frac{1}{t^2}\right),$$

for $\frac{1}{2} < \sigma \leq 1$ and $t \geq 3$, where the above sum runs over the ordinates of the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof Let $s = \sigma + it$ and $t \geq 3$. From the partial fraction decomposition for $\zeta'(s)/\zeta(s)$ (cf. [12, Eq. 2.12.7]), we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)+B+\frac{1}{2}\log\pi-\frac{1}{s-1},$$

with $B = -\sum_\rho \text{Re}\,(1/\rho)$. Differentiating and taking its real part we get

$$\text{Re}\left(\frac{\zeta'}{\zeta}\right)'(\sigma + it) = \sum_\gamma f_{\sigma-1/2}(\gamma - t) - \frac{1}{4}\text{Re}\left(\frac{\Gamma'}{\Gamma}\right)'\left(\frac{\sigma}{2} + 1 + \frac{it}{2}\right) + O\left(\frac{1}{t^2}\right).$$

Using Stirling's formula, that guarantees the Γ term is $O(1/t^2)$, we conclude.

As always, the crucial tool to work with sums as in Lemma 4 is the Guinand–Weil explicit formula (see [4, Lemma 8]), which for even functions reads as follows.

Proposition 5 (Guinand–Weil explicit formula) *Let $h(s)$ be analytic in the strip $|\text{Im}\,s| \leq \frac{1}{2} + \varepsilon$, for some $\varepsilon > 0$, such that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$, for some $\delta > 0$. Assume further that h is even. Then*

$$\sum_\rho h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\text{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1 + 2iu}{4}\right)du + 2h\left(\frac{i}{2}\right) - \frac{\log\pi}{2\pi}\widehat{h}(0)$$

$$- \frac{1}{\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\widehat{h}\left(\frac{\log n}{2\pi}\right),$$

where $\rho = \beta + i\gamma$ are the non-trivial zeros of $\zeta(s)$ and

$$\widehat{h}(y) = \int_{-\infty}^{\infty} h(x)e^{-2\pi ixy}\,dx$$

is the Fourier transform[1] of h.

2.1 Bandlimited approximations

Lemma 6 (Minorant) *For $a, \Delta > 0$ let*

$$L_{a,\Delta}(z) = \frac{z^2 - a^2 - (Az^2 + Ba^2)\sin^2(\pi\Delta z)}{(z^2 + a^2)^2}$$

where

$$A = \frac{2\lambda\coth(\lambda) - 1}{\sinh^2(\lambda)}, \quad B = \frac{2\lambda\coth(\lambda) + 1}{\sinh^2(\lambda)},$$

and $\lambda = \pi a\Delta$. Then:

(1) *The inequality*

$$L_{a,\Delta}(x) \leq f_a(x)$$

holds for all real x, $L_{a,\Delta} \in L^1(\mathbb{R})$ and its Fourier transform is supported in $[-\Delta, \Delta]$ (i.e. $L_{a,\Delta}$ is of exponential type at most $2\pi\Delta$);

(2) *We have*

$$\widehat{L}_{a,\Delta}(0) = \frac{1}{\Delta}\sum_{n\in\mathbb{Z}} f_a(n/\Delta) = -\frac{\pi^2\Delta}{\sinh^2(\pi a\Delta)}, \tag{2.2}$$

and any other function $F \neq L_{a,\Delta}$ having the same properties as $L_{a,\Delta}$ in item (1) has integral strictly less than the integral of $L_{a,\Delta}$.

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[1] We shall use this definition of the Fourier transform throughout the paper.

Proof Note first that the constants A, B were chosen so the numerator of $L_{a,\Delta}$ vanishes doubly at $z = \pm ia$. We then see that $L_{a,\Delta}$ is entire, of exponential type at most $2\pi\Delta$ and belongs to $L^1(\mathbb{R})$. Therefore, the Paley-Wiener Theorem guarantees its Fourier transform is supported in $[-\Delta, \Delta]$. Since $B > A > 0$ we have $L_{a,\Delta}(x) \leq f_a(x)$ for all real x. This proves item (1). We now prove item (2). Suppose F is an $L^1(\mathbb{R})$-function, $F(x) \leq f_a(x)$ for all real x and $\widehat{F}$ is supported in $[-\Delta, \Delta]$. Poisson summation implies

$$\widehat{F}(0) = \frac{1}{\Delta}\sum_{n\in\mathbb{Z}}F(n/\Delta) \leq \frac{1}{\Delta}\sum_{n\in\mathbb{Z}}f_a(n/\Delta) = \frac{1}{\Delta}\sum_{n\in\mathbb{Z}}L_a(n/\Delta) = \widehat{L}_{a,\Delta}(0),$$

where the last identity is due to the fact that $L_{a,\Delta}$ interpolates (in second order) f_a in $\frac{1}{\Delta}\mathbb{Z}$. Equality is attained if and only if $F(x) = L_{a,\Delta}(x)$ in second order for all $x \in \frac{1}{\Delta}\mathbb{Z}$. However, this completely characterizes $F = L_{a,\Delta}$ (see [14, Theorem 9]). Finally, using that $\widehat{f_a}(y) = -2\pi^2|y|e^{-2\pi a|y|}$, identity (2.2) can easily be derived using Poisson summation over $\frac{1}{\Delta}\mathbb{Z}$.

It turns out that because $f_a(x)$ has a local maximum at $x = \sqrt{3}\,a$, the bandlimited majorant of f_a with minimal total mass will have to be adjusted when $\pi a\Delta$ is small. This adjustment will require some de Branges spaces theory.

Lemma 7 (Majorant) *For a, $\Delta > 0$ let*

$$U_{a,\Delta}(z) = \frac{z^2 - a^2 + (Cz^2 + Da^2)(\cos(\pi\Delta z) - E\pi\Delta z\sin(\pi\Delta z))^2}{(z^2 + a^2)^2},$$

where

$$(C, D, E) = \begin{cases} \left(\dfrac{2\lambda\tanh(\lambda) - 1}{\cosh^2(\lambda)}, \dfrac{2\lambda\tanh(\lambda) + 1}{\cosh^2(\lambda)}, 0\right) & \text{if } \lambda \geq \lambda_0, \\[12pt] \left(0, \dfrac{1}{2}\left(\dfrac{2\lambda + \tanh(\lambda)}{\sinh(\lambda) + \lambda\,\mathrm{sech}(\lambda)}\right)^2, \dfrac{1 - 2\lambda\tanh(\lambda)}{2\lambda^2 + \lambda\tanh(\lambda)}\right) & \text{if } \lambda < \lambda_0, \end{cases}$$

$\lambda = \pi a\Delta$ *and* $\lambda_0 = 0.771\ldots$ *is such that* $2\lambda_0\tanh(\lambda_0) = 1$. *Then:*

(1) *The inequality*

$$f_a(x) \leq U_{a,\Delta}(x)$$

holds for all real x, $U_{a,\Delta} \in L^1(\mathbb{R})$ and its Fourier transform is supported in $[-\Delta, \Delta]$ (i.e. $U_{a,\Delta}$ is of exponential type at most $2\pi\Delta$);

(2) *We have*

$$\widehat{U}_{a,\Delta}(0) = \begin{cases} \dfrac{\pi^2\Delta}{\cosh^2(\lambda)} & \text{if } \lambda \geq \lambda_0, \\[12pt] \dfrac{\pi^2\Delta}{\sinh^2(\lambda)}\left(\dfrac{2\lambda + \sinh(2\lambda)}{8\lambda}\left(\dfrac{2\lambda + \tanh(\lambda)}{\sinh(\lambda) + \lambda\,\mathrm{sech}(\lambda)}\right)^2 - 1\right) & \text{if } \lambda < \lambda_0. \end{cases}$$

Moreover, any other function $F \neq U_{a,\Delta}$ having the same properties as $U_{a,\Delta}$ in item (1) has integral strictly greater than the integral of $U_{a,\Delta}$.

Proof Note that the constants (C, D, E) are chosen so that $U_{a,\Delta}$ is entire, that is, its numerator vanishes doubly at $z = \pm ia$. Since $U_{a,\Delta}$ is visibly of exponential type at most $2\pi\Delta$ and

belongs to $L^1(\mathbb{R})$, the Paley–Wiener Theorem guarantees its Fourier transform is supported in $[-\Delta, \Delta]$. Noting that $C, D \geq 0$ we have $f_a(x) \leq U_{a,\Delta}(x)$ for all real x, and this proves item (1). We now show item (2). Suppose F is an $L^1(\mathbb{R})$-function, $F(x) \geq f_a(x)$ for all real x and $\widehat{F}$ is supported in $[-\Delta, \Delta]$. We now apply the generalized Poisson summation formula of Littmann for bandlimited functions [11, Theorem 2.1] for $\gamma = (\pi E)^{-1}$ with $E > 0$. It translates to

$$
\begin{aligned}
\widehat{F}(0) &= \frac{1}{\Delta} \sum_{\mathcal{B}(t)=0} \left(1 - \frac{\pi E}{\pi(\pi^2 E^2 t^2 + 1) + \pi E}\right) F(t/\Delta) \\
&\geq \frac{1}{\Delta} \sum_{\mathcal{B}(t)=0} \left(1 - \frac{\pi E}{\pi(\pi^2 E^2 t^2 + 1) + \pi E}\right) f_a(t/\Delta) \\
&= \frac{1}{\Delta} \sum_{\mathcal{B}(t)=0} \left(1 - \frac{\pi E}{\pi(\pi^2 E^2 t^2 + 1) + \pi E}\right) U_{a,\Delta}(t/\Delta) \\
&= \widehat{U}_{a,\Delta}(0),
\end{aligned}
$$

where $\mathcal{B}(z) = \cos(\pi z) - E\pi z \sin(\pi z)$. Note when $E = 0$, that is, $\lambda \geq \lambda_0$, this is the classical Poisson summation over $\frac{1}{\Delta}(\frac{1}{2}+\mathbb{Z})$. Equality is attained if and only if $F(t/\Delta) = U_{a,\Delta}(t/\Delta)$ in second order for all real t with $\mathcal{B}(t) = 0$. We claim this completely characterizes $F = U_{a,\Delta}$. The trick is to use the theory of de Branges spaces and the interpolation formula [9, Theorem A] (the introduction of [9] gives a solid short background on the necessary de Branges spaces theory which we will use here without much explanation). First we note that the function $\mathcal{E}(z) = (i + \pi Ez)e^{-\pi i z}$ is of Hermite–Biehler class (i.e. $|\mathcal{E}(\bar{z})| < |\mathcal{E}(z)|$ for all z with $\text{Im } z > 0$) and therefore the de Branges space $\mathcal{H}(\mathcal{E}^2)$ exists, and it consists of all entire functions of exponential type at most 2π belonging to $L^2(\mathbb{R}, dx/(1 + E^2\pi^2 x^2))$. Note also that $\mathcal{B}(z) = i(\overline{\mathcal{E}(\bar{z})} - \mathcal{E}(z))/2$. Moreover, it is not hard to show that all conditions of [9, Theorem A] are satisfied by $\mathcal{E}(z)$, and thus we conclude that any function $G \in \mathcal{H}(\mathcal{E}^2)$ is completely characterized by its values $G(t)$ and $G'(t)$ for all real t with $\mathcal{B}(t) = 0$. Now it is simply a matter to note that $(i + \pi Ez)^2 F(z/\Delta)$ and $(i + \pi Ez)^2 U_{a,\Delta}(z/\Delta)$ both belong to $\mathcal{H}(\mathcal{E}^2)$, and so they must be equal.[2]

Finally, in the case $\lambda \geq \lambda_0$ one can use Poisson summation over $\frac{1}{\Delta}(\frac{1}{2} + \mathbb{Z})$ to evaluate the integral of $U_{a,\Delta}$ and obtain

$$
\widehat{U}_{a,\Delta}(0) = \frac{\pi^2 \Delta}{\cosh^2(\pi a \Delta)}.
$$

If $\lambda < \lambda_0$ then we can use Poisson summation over $\frac{1}{\Delta}\mathbb{Z}$ to obtain

$$
\begin{aligned}
\widehat{U}_{a,\Delta}(0) &= \frac{1}{\Delta} \sum_{n \in \mathbb{Z}} \left(f_a(n/\Delta) + \frac{Da^2}{(n^2/\Delta^2 + a^2)^2} \right) \\
&= -2\pi^2 \Delta \sum_{n \in \mathbb{Z}} |n| e^{-2\lambda|n|} + D\pi^2 \Delta \sum_{n \in \mathbb{Z}} (|n| + \tfrac{1}{2\lambda}) e^{-2\lambda|n|} \\
&= -\frac{\pi^2 \Delta}{\sinh^2(\lambda)} + D\pi^2 \Delta \frac{2\lambda + \sinh(2\lambda)}{4\lambda \sinh^2(\lambda)} \\
&= \frac{\pi^2 \Delta}{\sinh^2(\lambda)} \left(\frac{2\lambda + \sinh(2\lambda)}{8\lambda} \left(\frac{2\lambda + \tanh(\lambda)}{\sinh(\lambda) + \lambda \, \text{sech}(\lambda)} \right)^2 - 1 \right).
\end{aligned}
$$

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[2] Note when $E = 0$ this argument reduces to classical Paley-Wiener space theory and Poisson summation.

Above we used that $\widehat{f_a}(y) = -2\pi^2|y|e^{-2\pi a|y|}$ and the Fourier transform of $\frac{a^2}{(x^2+a^2)^2}$ is $\pi^2\left(|y| + \frac{1}{2\pi a}\right)e^{-2\pi a|y|}$.

Lemma 8 *The functions defined in Lemmas 6 and 7 satisfy the following inequalities for* $-\Delta < y < \Delta$:

$$\widehat{L}_{a,\Delta}(y) < 0$$

and, if $\pi a\Delta \geq \lambda_0$,

$$\widehat{U}_{a,\Delta}(y) > \widehat{f_a}(y).$$

Proof First we deal with the minorant. Using that $\widehat{f_a}(y) = -2\pi^2|y|e^{-2\pi a|y|}$ and the Fourier transforms of $\frac{1}{x^2+a^2}$ and $\frac{a^2}{(x^2+a^2)^2}$ are

$$\frac{\pi}{a}e^{-2\pi a|y|} \quad \text{and} \quad \pi^2\left(|y| + \frac{1}{2\pi a}\right)e^{-2\pi a|y|},$$

respectively, we obtain

$$\widehat{L}_{a,\Delta}(y) = -2\pi^2|y|e^{-2\pi a|y|} - \frac{2\,\text{Id} - T_\Delta - T_{-\Delta}}{4}\left[\pi^2\left(\frac{B+A}{2\pi a} + (B-A)|y|\right)e^{-2\pi a|y|}\right],$$

where T_h is the operator of translation by h and Id is the identity operator. These operators come from the (distributional) Fourier transform of $\sin^2(\pi\Delta x)$. We claim that the function $e^{2\pi ay}\widehat{L}_{a,\Delta}(y)$ is convex in the range $0 < y < \Delta$, which would show that $\widehat{L}_{a,\Delta}(y)$ is negative in the same range since it is negative at $y = 0$ and vanishes at $y = \Delta$. For $0 < y < \Delta$ we have

$$\frac{d^2}{dy^2}\left[e^{2\pi ay}\widehat{L}_{a,\Delta}(y)\right] = \frac{d^2}{dy^2}\left[\frac{\pi^2}{4}\left(\frac{B+A}{2\pi a} + (B-A)(\Delta-y)\right)e^{2\pi a(2y-\Delta)} + \text{linear}\right]$$

$$= (A + \pi a(B-A)(\Delta-y))\,4a\pi^3 e^{2\pi a(2y-\Delta)}$$

$$> 0,$$

because $B > A > 0$. The majorant case is simpler, since if $\lambda = \pi a\Delta \geq \lambda_0$ a similar computation leads to

$$\widehat{U}_{a,\Delta}(y) = -2\pi^2|y|e^{-2\pi a|y|} + \frac{2\,\text{Id} + T_\Delta + T_{-\Delta}}{4}\left[\pi^2\left(\frac{C+D}{2\pi a} + (D-C)|y|\right)e^{-2\pi a|y|}\right],$$

and so the desired inequality follows because $D > C \geq 0$.

3 Proof of Theorem 3

Let $\frac{1}{2} < \sigma < 1$ and $\Delta > 0$. Throughout the rest of the paper we set $a = \sigma - \frac{1}{2}$ and $\lambda = \pi a\Delta$. Using Lemma 4 and the evenness of the zeta-zeros we obtain

$$\text{Re}\left(\frac{\zeta'}{\zeta}\right)'(\sigma + it) = \sum_\gamma \left(\tfrac{1}{2}f_a(\gamma - t) + \tfrac{1}{2}f_a(\gamma + t) + f_a(\gamma)\right) + O(1),$$

as $t \to \infty$, where we have used that $f_a(x) = O(1/x^2)$ uniformly for $|x| \geq 1$ and $0 < a < 1/2$, hence $\sum_\gamma f_a(\gamma) = O(1)$. We then apply Lemmas 6 and 7 to get

$$\sum_\gamma M_t L_{a,\Delta}(\gamma) + O(1) \leq \mathrm{Re}\left(\frac{\zeta'}{\zeta}\right)'(\sigma + it) \leq \sum_\gamma M_t U_{a,\Delta}(\gamma) + O(1), \qquad (3.1)$$

where $M_t = \frac{1}{2}T_t + \frac{1}{2}T_{-t} + \mathrm{Id}$. Note that for each $t \geq 0$ the functions $M_t L_{a,\Delta}$ and $M_t U_{a,\Delta}$ are even and admissible for the Guinand–Weil explicit formula (Proposition 5). We use the operator M_t because its Fourier transform is the operator that multiplies by $2\cos^2(\pi t x)$, which is nonnegative. This will allow us to simply discard (or easily bound) the sum over primes in the explicit formula.

3.1 Proof of the lower bound

Applying Proposition 5 and Lemmas 6 and 8 we obtain

$$\sum_\gamma M_t L_{a,\Delta}(\gamma) = \frac{1}{2\pi}\int_{-\infty}^\infty M_t L_{a,\Delta}(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1+2iu}{4}\right)du + 2M_t L_{a,\Delta}\left(\frac{i}{2}\right)$$

$$- \frac{2}{\pi}\sum_{n \geq 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{L}_{a,\Delta}\left(\frac{\log n}{2\pi}\right)\cos^2(\tfrac{1}{2}t\log n) - \frac{\log\pi}{\pi}\,\widehat{L}_{a,\Delta}(0)$$

$$\geq \frac{1}{2\pi}\int_{-\infty}^\infty M_t L_{a,\Delta}(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1+2iu}{4}\right)du + 2M_t L_{a,\Delta}\left(\frac{i}{2}\right).$$

In this part we assume that $\lambda \geq c$ for some given fixed $c > 0$. We now analyze the terms on the right-hand side above. The function $L_{a,\Delta}$ depends on the parameters A and B, but both behave like (since $\lambda \geq c$)

$$8\lambda e^{-2\lambda} + O(e^{-2\lambda}).$$

Hence $|L_{a,\Delta}(x)| \leq K(x^2 + a^2)^{-1}$ for some $K > 0$. Since $(s^2 + a^2)L_{a,\Delta}(s)$ has exponential type $2\pi\Delta$ and it is bounded on the real line, a routine application of the Phragmén–Lindelöf principle implies that

$$|L_{a,\Delta}(s)| \leq K\frac{e^{2\pi\Delta|\mathrm{Im}\,s|}}{|s^2 + a^2|}, \quad s \in \mathbb{C} \qquad (3.2)$$

(alternatively, one could derive such bound by direct computation). Using the bounds for A and B it follows that

$$2M_t L_{a,\Delta}\left(\frac{i}{2}\right) = \frac{4\pi a\Delta e^{(1-2a)\pi\Delta}}{a^2 - \frac{1}{4}} + O\left(\frac{e^{(1-2a)\pi\Delta}}{(a^2 - \frac{1}{4})^2} + \frac{e^{\pi\Delta}}{t^2}\right).$$

Using that M_t is self-adjoint and applying Stirling's approximation to obtain

$$M_t\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1+2iu}{4}\right) = \log t + O(\log(2 + |u|)),$$

we deduce that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} M_t L_{a,\Delta}(u) \, \mathrm{Re} \, \frac{\Gamma'}{\Gamma} \left(\frac{1 + 2iu}{4} \right) du = \frac{\pi \Delta \log t}{2 \sinh^2(\pi a \Delta)} + O \left(\frac{1}{a} \right)$$

$$= -2\pi \Delta e^{-2\pi a \Delta} \log t + O \left(\frac{1}{a} + \Delta e^{-4\pi a \Delta} \log t \right)$$

$$= -2\pi \Delta e^{-2\pi a \Delta} \log t + O \left(\frac{1 + e^{-2\pi a \Delta} \log t}{a} \right).$$

Combining the above bounds we obtain

$$\sum_{\gamma} M_t L_{a,\Delta}(\gamma - t) \geq -2\pi \Delta e^{-2\pi a \Delta} \log t + \frac{4\pi a \Delta e^{(1-2a)\pi \Delta}}{a^2 - \frac{1}{4}}$$

$$+ O \left(\frac{e^{(1-2a)\pi \Delta}}{(a^2 - \frac{1}{4})^2} + \frac{e^{\pi \Delta}}{t^2} + \frac{1}{a} \left(1 + e^{-2\pi a \Delta} \log t \right) \right).$$

Choosing $\pi \Delta = \log \log t$ (which is the optimal choice) and using (3.1) we obtain

$$\mathrm{Re} \left(\frac{\zeta'}{\zeta} \right)' (\sigma + it) \geq - \left(\frac{-2\sigma^2 + 6\sigma - 2}{\sigma(1 - \sigma)} \right) \log \log t \, (\log t)^{2 - 2\sigma} + O_c \left(\frac{(\log t)^{2 - 2\sigma}}{(\sigma - \frac{1}{2})(1 - \sigma)^2} \right)$$

for $\pi(\sigma - 1/2) \log \log t \geq c$. This proves the desired result.

3.2 Proof of the upper bound

Using Proposition 5 and Lemma 7 we obtain

$$\sum_{\gamma} M_t U_{a,\Delta}(\gamma) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} M_t U_{a,\Delta}(u) \, \mathrm{Re} \, \frac{\Gamma'}{\Gamma} \left(\frac{1 + 2iu}{4} \right) du + 2 M_t U_{a,\Delta} \left(\frac{i}{2} \right)$$

$$- \frac{2}{\pi} \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \, \widehat{U}_{a,\Delta} \left(\frac{\log n}{2\pi} \right) \cos^2(\tfrac{1}{2} t \log n).$$

When $\lambda \geq \lambda_0$ the computations are very similar to the lower bound and we just indicate them here. We still have both C and D behaving like $8\lambda e^{-2\lambda} + O(e^{-2\lambda})$, and a bound similar to (3.2) holds. Using Stirling's formula and Lemma 7 we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} M_t U_{a,\Delta}(u) \, \mathrm{Re} \, \frac{\Gamma'}{\Gamma} \left(\frac{1 + 2iu}{4} \right) du = \frac{\pi \Delta \log t}{2 \cosh^2(\pi a \Delta)} + O \left(\frac{1}{a} \right)$$

$$= 2\pi \Delta e^{-2\pi a \Delta} \log t + O \left(\frac{1 + e^{-2\pi a \Delta} \log t}{a} \right).$$

Using the estimates for C and D it follows that

$$2 M_t U_{a,\Delta} \left(\frac{i}{2} \right) = \frac{4\pi a \Delta e^{(1-2a)\pi \Delta}}{a^2 - \frac{1}{4}} + O \left(\frac{e^{(1-2a)\pi \Delta}}{(a^2 - \frac{1}{4})^2} \right) + O \left(\frac{e^{\pi \Delta}}{t^2} \right).$$

Since $\widehat{U}_{a,\Delta}$ is supported in $[-\Delta, \Delta]$, we estimate the sum over primes (which we cannot discard as before) using Lemma 8 and that $\widehat{f}_a(y) = -2\pi^2 |y| e^{-2\pi a |y|}$ to get

$$-\frac{2}{\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{U}_{a,\Delta}\left(\frac{\log n}{2\pi}\right)\cos^2(\tfrac{1}{2}t\log n)\leq 2\sum_{2\leq n\leq e^{2\pi\Delta}}\frac{\Lambda(n)}{n^{a+1/2}}\log n$$

$$=\frac{4\pi\Delta e^{(1-2a)\pi\Delta}}{\frac{1}{2}-a}+O\left(\frac{e^{(1-2a)\pi\Delta}}{\left(\frac{1}{2}-a\right)^2}+\frac{\Delta^3}{a}\right).$$

The above estimate follows from the prime number theorem (see [4, Eq. (B.2)]). Choosing $\pi\Delta=\log\log t$ and using (3.1) we obtain

$$\mathrm{Re}\left(\frac{\zeta'}{\zeta}\right)'(\sigma+it)\leq\left(\frac{-2\sigma^2+2\sigma+2}{\sigma(1-\sigma)}\right)\log\log t\,(\log t)^{2-2\sigma}+O_c\left(\frac{(\log t)^{2-2\sigma}}{(\sigma-\frac{1}{2})(1-\sigma)^2}\right)$$

in the range $(\sigma-1/2)\log\log t\geq\lambda_0$ and $(1-\sigma)\sqrt{\log\log t}\geq c$ for some fixed $c>0$; note that $\Delta^3=O_c\big((1/2-a)^{-2}(\log t)^{1-2a}\big)$. This finishes the proof. □

4 Proof of Theorem 2

To obtain the bounds for the imaginary part of the log-derivative $\zeta(s)$ we will employ the interpolation technique of [4, Section 6] for functions with slow growth, which we conveniently state in the form of a lemma.

Lemma 9 (Interpolation) *Let* $\varphi:(t_0,\infty)\to\mathbb{R}$ *be twice differentiable,* $t_0>0$, *and assume that*

$$-\beta_0(t)\leq\varphi(t)\leq\alpha_0(t)\quad and\quad -\beta_2(t)\leq\varphi''(t)\leq\alpha_2(t),$$

for some differentiable functions $\alpha_0,\beta_0,\alpha_2,\beta_2:(t_0,\infty)\to(0,\infty)$. *Suppose the numbers*

$$L=\sup_{t>t_0}\frac{2(\alpha_2(t)+\beta_2(t))(\alpha_0(t)+\beta_0(t))}{3\alpha_2(t)\beta_2(t)},$$

$$M_i=\sup_{t>t_0}|\alpha_i'(t)|\quad and\quad N_i=\sup_{t>t_0}|\beta_i'(t)|\quad(i=0,2)$$

are finite. Then, for $t>t_0+\sqrt{3L}$ *we have*

$$|\varphi'(t)|\leq\sqrt{\frac{2\alpha_2(t)\beta_2(t)(\alpha_0(t)+\beta_0(t))}{\alpha_2(t)+\beta_2(t)}}+M_0+N_0+(M_2+N_2)L.$$

Proof Since the bound is symmetric when we interchange α_0,β_0 and α_2,β_2 (i.e. we change φ by $-\varphi$), it is enough to prove that $\varphi'(t)$ is bounded above by the desired bound. An application of the mean value theorem easily gives that[3]

$$\varphi'(t)-\varphi'(t-h)=\varphi''(t^*)h$$
$$\leq h_+\alpha_2(t^*)+(-h)_+\beta_2(t^*)$$
$$\leq h_+\alpha_2(t)+(-h)_+\beta_2(t)+(M_2+N_2)|h|^2.\text{'}$$

Averaging in h in the interval $[-\nu(1-A),\nu A]$, for some $\nu>0$ (with $t-\nu>t_0$) and $0<A<1$, we obtain

$$\varphi'(t)$$

---

[3] The notation h_+ means $\max(h,0)$.

$$\leq \frac{1}{\nu}\left(\varphi(t + (1 - A)\nu) - \varphi(t - A\nu)\right) + \frac{\nu}{2}\left(A^2\alpha_2(t) + (1 - A)^2\beta_2(t)\right) + \frac{\nu^2}{3}(M_2 + N_2)$$

$$\leq \frac{1}{\nu}\left(\alpha_0(t + (1 - A)\nu) + \beta_0(t - A\nu)\right) + \frac{\nu}{2}\left(A^2\alpha_2(t) + (1 - A)^2\beta_2(t)\right) + \frac{\nu^2}{3}(M_2 + N_2)$$

$$\leq \frac{1}{\nu}\left(\alpha_0(t) + \beta_0(t)\right) + \frac{\nu}{2}\left(A^2\alpha_2(t) + (1 - A)^2\beta_2(t)\right) + M_0 + N_0 + \frac{\nu^2}{3}(M_2 + N_2).$$

Minimizing the main term above as a function of ν and A, we must set

$$\nu = \sqrt{\frac{2(\alpha_2(t) + \beta_2(t))(\alpha_0(t) + \beta_0(t))}{\alpha_2(t)\beta_2(t)}} \quad \text{and} \quad A = \frac{\beta_2(t)}{\alpha_2(t) + \beta_2(t)},$$

which gives, for $t > t_0 + \sqrt{3L}$, that

$$\varphi'(t) \leq \sqrt{\frac{2\alpha_2(t)\beta_2(t)(\alpha_0(t) + \beta_0(t))}{\alpha_2(t) + \beta_2(t)}} + M_0 + N_0 + (M_2 + N_2)L.$$

The lemma follows.

We will apply this lemma for

$$\varphi(t) = -\log|\zeta(\sigma + it)|$$

noting that

$$\varphi'(t) = \operatorname{Im}\frac{\zeta'}{\zeta}(\sigma + it) \quad \text{and} \quad \varphi''(t) = \operatorname{Re}\left(\frac{\zeta'}{\zeta}\right)'(\sigma + it).$$

Theorem 3 and [1, Theorems 1 and 2] establish respectively that

$$-\beta_0(t) \leq \varphi(t) \leq \alpha_0(t) \quad \text{and} \quad -\beta_2(t) \leq \varphi''(t) \leq \alpha_2(t),$$

in the range

$$\frac{1}{2} + \frac{\lambda_0}{\log\log t} \leq \sigma \leq 1 - \frac{c\sqrt{\lambda_0/(\lambda_0 + c)}}{\sqrt{\log\log t}} \quad \text{and} \quad t \geq 3, \tag{4.1}$$

where $c > 0$,

$$\alpha_2(t) = \frac{-2\sigma^2 + 2\sigma + 2}{\sigma(1 - \sigma)}\ell_{-1,\sigma}(t) + O_c\left(\frac{\ell_{0,\sigma}(t)}{(\sigma - \frac{1}{2})(1 - \sigma)^2}\right),$$

$$\beta_2(t) = \frac{-2\sigma^2 + 6\sigma - 2}{\sigma(1 - \sigma)}\ell_{-1,\sigma}(t) + O_c\left(\frac{\ell_{0,\sigma}(t)}{(\sigma - \frac{1}{2})(1 - \sigma)^2}\right),$$

$$\alpha_0(t) = \beta_0(t) = \frac{-\sigma^2 + 5\sigma - 2}{2\sigma(1 - \sigma)}\ell_{1,\sigma}(t) + O_c\left(\frac{\ell_{2,\sigma}(t)}{(1 - \sigma)^2}\right)$$

and $\ell_{n,\sigma}(t) = (\log t)^{2-2\sigma}(\log\log t)^{-n}$. We can then apply Lemma 9 with $t_0 = t_0(\sigma, c)$ equals to the smallest t such that (4.1) is not vacuous. A routine computation shows that $\sqrt{3L} = O_c(1)$ and that M_0, M_2, N_0, N_2 are $O_c((\sigma - \frac{1}{2})^{-1}(1 - \sigma)^{-2})$. We obtain

$$\left| \operatorname{Im} \frac{\zeta'}{\zeta}(\sigma + it) \right| \le \sqrt{\frac{2\alpha_2(t)\beta_2(t)(\alpha_0(t) + \beta_0(t))}{\alpha_2(t) + \beta_2(t)}} + O_c\left(\frac{1}{(\sigma - \frac{1}{2})(1-\sigma)^2} \right)$$

$$= \sqrt{\frac{2(-\sigma^2 + 5\sigma - 2)(-\sigma^2 + 3\sigma - 1)(-\sigma^2 + \sigma + 1)}{\sigma^3(1-\sigma)^2(2-\sigma)}} \, \ell_{0,\sigma}(t)$$

$$+ O_c\left(\frac{\ell_{1,\sigma}(t)}{(\sigma - \frac{1}{2})(1-\sigma)^2} \right)$$

if $t' = t - \sqrt{3L} \ge t_0$. Letting $t_1(c)$ be such that $\frac{\log\log t'}{\log\log t} \ge \frac{\lambda_0}{\lambda_0 + c}$ if $t \ge t_1(c)$, we conclude that the above estimate holds in the range

$$\frac{1}{2} + \frac{\lambda_0 + c}{\log\log t} \le \sigma \le 1 - \frac{c}{\sqrt{\log\log t}} \quad \text{and} \quad t \ge t_2(c),$$

where $t_2(c) = \max(t_1(c), 3 + \sqrt{3L})$. To finish the proof we note that if $3 \le t \le t_2(c)$ then a simple compactness argument gives the full desired range. □

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