Estimation of Extreme Roll Motion using the First Order Reliability Method

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Abstract. Extreme value statistics can often be based on the assumption that exceedance events of a high threshold level are statistically independent and identically distributed (i.i.d. process), which further implies the Poisson assumption to be valid. This makes it possible to express the extreme response statistics through the mean up-crossing rate. For non-linear processes, analytic expressions of the mean up-crossing rate do not in general exist. Reliable statistics of mean up-crossing rate based on the brute-force approach, e.g. Monte Carlo simulation (MCS) require long time domain simulations considering a number of different ensemble input. The associated computations can be very time consuming especially when a detailed physical (e.g. hydrodynamic) model is applied. The First Order Reliability Method (FORM) has previously been found efficient for estimation of extreme value prediction of stationary stochastic time domain processes, However, if the non-linearity in a response is significant, the accuracy of the FORM linearized mean up-crossing rate can be limited.

The present work attempts to improve the extreme value prediction for non-linear parametric roll motions of ships based on applications of the FORM approach and suggests a model for the mean up-crossing rate for strong non-linear response, validated by comparing with MCS results.

Keywords: Extreme value statistics, Poisson up-crossing, intact stability of ships

Introduction

The Poisson assumption is often used for extreme value prediction for stationary stochastic processes and requires only two parameters: the up-crossing rate and the time duration of the response. For a linear system, the up-crossing rate is a product of the mean zero up-crossing rate and an exponential term depending on the threshold level in terms of the reliability index. For non-linear systems, the determination of the up-crossing rate is more complicated, and moreover, the reliability index is not a linear function of the threshold level. The question is then to determine the reliability index and to establish an accurate model of the up-crossing rate for non-linear systems. The First Order Reliability Method (FORM) can be used to determine the reliability index and it has been shown in several papers that often a good agreement is found as compared to direct Monte Carlo Simulations (MCS), e.g. *Der Kiureghian* (2000), *Jensen* (2015), *Choi et al.* (2017) and *Jensen et al.* (2017). The reason is that the reliability index is calculated using the exact non-linear limit state function. The mean zero up-crossing rate can be estimated from the linearized FORM solution, *Jensen and Capul* (2006), *Fujimura and Der Kiureghian* (2007), *Garrè and Der Kiureghian* (2010). However, due to the linearization it might not be as accurate as the reliability index, especially for systems with very strong (or bifurcation type of) non-linearity.

This paper attempts to improve the estimation of the up-crossing rate for non-linear response based on the framework of the First Order Reliability Method. As a case study, large parametric roll motions of a ship are considered in long-crested head sea seas, which is mainly a non-linear bifurcation problem without any linear components. Two models for the up-crossing rate are examined, both making use of the reliability index from FORM. The results are in line with findings based on Monte Carlo Simulations.

First Order Reliability method (FORM)

Extreme value statistics can often be based on the assumption of independent peaks implying a Poisson model. For stationary processes the probability that the response $X(t)$ exceeds the level x_0 during the time T then becomes

$$P\left[\max_T X(t) > x_0\right] = 1 - \exp\left(-\nu^+(x_0)T\right) \tag{1}$$

For linear (Gaussian) systems, the mean up-crossing rate $\nu^+(x_0)$ takes the form

$$\nu^+(x_0) = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}}\exp\left(-\frac{\left(x_0 - x_{mean}\right)^2}{2m_0}\right) \tag{2}$$

Eq. (2) requires calculation of the spectral moments m_0, m_2 and therefore also a spectral formulation of the response. For non-linear systems, a linearization around the design point x_0 using FORM can often provide a useful estimate of the mean up-crossing rate, e.g. *Jensen and Capul* (2006), *Fujimura and Der Kiureghian* (2007), *Garrè and Der Kiureghian* (2010). FORM searches the solution $\underline{u}^*$ of the limit state problem:

$$G(\underline{u}) \equiv x_0 - X(0|\underline{u}) = 0$$
$$\underline{u}^* : \text{ Minimize } \underline{u}^T \underline{u}; \text{ Subject to } G(\underline{u}) = 0 \tag{3}$$

where $\underline{u}^T = [u_1, u_2, ,..., u_n]$ are uncorrelated standard normal distributed variables defining the stochastic variations of the input $Y(t|\underline{u})$, e.g. the wave elevation and gust wind speed. The time instance t=0, must be so far away from the initial conditions that these do not influence the response. For parametric roll 300s was found in *Jensen* (2007) to be sufficient.

As shown in *Jensen* (2011) the spectral density $S_x(\omega)$ of the FORM linearized response becomes

$$S_x(\omega_i)d\omega = \left(\left. \frac{\partial X}{\partial u_i} \right|_{\underline{u}=\underline{u}^*} \right)^2 = e_i^2 \tag{4}$$

when each u_i is associated with a sinusoidal linearized response variation in time with frequency ω_i. Hence,

$$x_{mean} = x_0 - \underline{e}^T \underline{u}^*$$
$$m_0 = \int S_x(\omega)d\omega = \sum_{i=1}^{n} e_i^2 = \|e\|^2 \tag{5}$$
$$m_2 = \int \omega^2 S_x(\omega)d\omega = \sum_{i=1}^{n} \omega_i^2 e_i^2$$

It is noted that both m_0 and m_2 depend on the design point $\underline{u}^*$ and thereby on x_0. Eq. (2) can then be written, using Eq. (5)

$$v^+(x_0) = \frac{1}{2\pi\beta} \sqrt{\sum_{i=1}^{n} \omega_i^2 u_i^{*2}} \exp\left(-\frac{\beta^2}{2} \right) \tag{6}$$

where the reliability index β is defined as

$$\beta(x_0) = \sqrt{\underline{u}^{*T} \underline{u}^*} = \frac{\underline{e}^T \underline{u}^*}{\|e\|} \tag{7}$$

The autocorrelation function of the linearized FORM solution becomes

$$R(t|\underline{u}^*) = \int S_x(\omega)\cos\omega t d\omega = \sum_{i=1}^{n} e_i^2 \cos\omega_i t \tag{8}$$

The corresponding most probable linear response is then

$$X(t|\underline{u}^*)_{Linear} = \frac{x_0}{m_0} R(t|\underline{u}^*) \tag{9}$$

A comparison between $X(t|\underline{u}^*)_{Linear}$ and $X(t|\underline{u}^*)$ provides an indication of how close the FORM linearization is to the real limit state surface $G(\underline{u})$ around the design point $\underline{u}^*$.

Choi and Jensen (2019) have investigated the accuracy of Eq. (6) for two types of ship roll motions: dead ship conditions in beam sea and parametric rolling in long-crested head sea. For the dead ship case, comparison between Eq. (9) and $X(t|\underline{u}^*)$ shows good agreement, whereas the opposite was the case for parametric rolling, where a significant difference was found between the dominating period in the two results for the most probable response. Hence, Eq. (6) as based on the FORM linearization might not be accurate for highly non-linear responses and alternative expressions are proposed in the next section.

Parametric roll responses are obtained by solving 6 DOF equation of motion for a Post-Panamax container ship. Radiation forces are calculated by using the impulse response function approach, and the nonlinearities are considered in the damping, the Froude-Krylov and restoring forces. The waves are generated using JONSWAP spectrum with the significant wave height of 3.25m, and the zero up-crossing period of 13.1 sec. Further details can be found in *Choi and Jensen (2019)*. The probability Density Function (PDF) of the calculated roll response are shown in Fig.1. A strong non-linearity can be seen from the shape of PDFs as they do not follow the Gaussian distribution, and moreover different wave realizations produce different distributions.

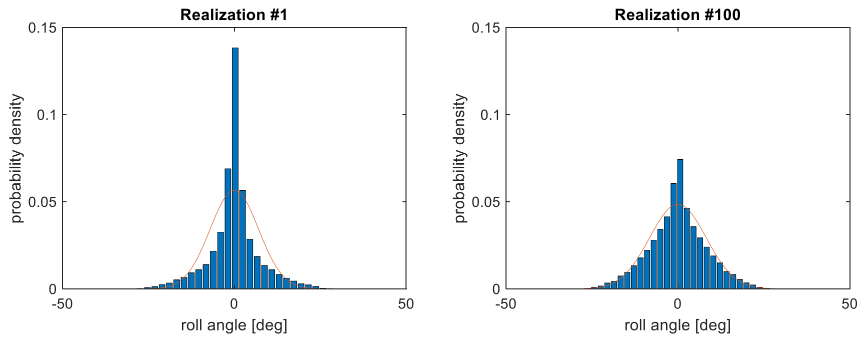

Fig.1. Probability density function of parametric roll angle. Red curve: Gaussian distribution

Up-crossing rate

For a stochastic process the up-crossing rate $\nu^+(x)$ can in general be calculated as

$$\nu^+(x) = \int_{\dot{x}=0}^{\infty} p(x,\dot{x})\dot{x}d\dot{x} \tag{10}$$

For stationary processes, the covariance of x and $\dot{x}$ is zero. For some processes, this implies that x and $\dot{x}$ are (nearly) uncorrelated. In that case, the joint and marginal density functions of x and $\dot{x}$ are related by $p(x,\dot{x}) = p_x(x)p_{\dot{x}}(\dot{x})$ and the up-crossing rate becomes

$$\nu^+(x) = \nu^+(0)\frac{p_x(x)}{p_x(0)} \tag{11}$$

The reliability index β is defined through the standard normal distribution function Φ

$$\int_x^{\infty} p_x(u)du \equiv \Phi(-\beta) = \int_{\beta(x)}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)du \tag{12}$$

Thus, by differentiation

$$p_x(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\beta(x)^2\right)\frac{d\beta}{dx}$$

Hence, with $\beta' \equiv d\beta/dx$

$$\nu^+(x) = \nu^+(0)\exp\left(-\frac{1}{2}(\beta(x)^2 - \beta(0)^2\right)\frac{\beta'(x)}{\beta'(0)}$$

For symmetric processes, e.g. parametric roll, $\beta(0) = 0$ yielding

$$\nu^+(x) = \nu^+(0)\frac{\beta'(x)}{\beta'(0)}\exp\left(-\frac{1}{2}\beta(x)^2\right) \tag{13}$$

Eq. (13) depends only on the assumption of statistical independence between $x, \dot{x}$. For Gaussian processes β varies linearly with x and the usual formula

$$\nu^{+}(x)=\nu^{+}(0)\exp\left(-\frac{1}{2}\beta(x)^{2}\right)$$

is obtained. For non-linear processes the effective up-crossing rate of the mean level in Eq. (13)

$$\nu^{+}{}_{mean}(x)=\nu^{+}(0)\frac{\beta'(x)}{\beta'(0)} \tag{14}$$

depends solely on x in contrast to the true mean up-crossing rate, i.e. Eq. (10).

Without the assumptions of statistical independence, Eq. (14) is replaced by

$$\nu^{+}{}_{mean}(x)=\nu^{+}(0)\frac{\beta'(x)f(x)}{\beta'(0)f(0)} \tag{15}$$

where

$$f(x)=\int_{0}^{\infty}p_{\dot{x}}(\dot{x}|x)\dot{x}d\dot{x} \tag{16}$$

It could be noted that the FORM result, Eq. (6), for $\nu^{+}(x)$ does not assume statistical independence of $x,\dot{x}$.

Based on Eqs (11)-(16) two semi-empirical formulas for the up-crossing rate are considered. The first is

$$\text{Model-A: } \nu^{+}(x)=\nu_{0}^{+}\exp\left(-\frac{1}{2}\beta(x)^{2}\right) \tag{17}$$

The differences between in Eq.(17) and Eq. (2) are in the introduction of ν_0 and β. The up-crossing rate relative to the mean level ν_0 can be obtained either from a small number of simulations or estimated from the roll natural frequency ω_{roll}, i.e. $\nu_0 = \omega_{roll}/2\pi$. In Eq. (17), β is used due to the mapping of the non-linear process to the standard normal distribution, Eq.(12). From Monte Carlo Simulations the reliability index can be calculated as

$$\beta_{MCS}(x_i)=-\Phi^{-1}\left(1-\frac{i-1}{M}\right);\quad 1=1,2,\ldots,M \tag{18}$$

where x_i are the ordered ensemble samples of roll response taken at a fixed time instance: $x_i \le x_{i+1}; i = 1,2,\ldots,M$., (e.g. *Jensen et al.* 2017).

The second model is based on Eq. (15), and for better estimation of extreme values, a reference point x_{ref} introduced

$$\text{Model-B: } \nu^+(x_0) = \nu^+(x_{ref}) \frac{\beta'(x_0) f(x)}{\beta'(x_{ref}) f(x_{ref})} \exp\left(-\frac{1}{2}\left(\beta(x_0)^2 - \beta(x_{ref})^2\right)\right) \quad (19)$$

where, x_{ref} is selected as large as possible within the range where the MCS up-crossing rate, $\nu^+(x_{ref})$ is accurate. The derivative $\beta'(x)$ of the reliability index is calculated numerically from $\beta(x)$.

An analytic expression for the conditional partial probability density function $p_{\dot{x}}(\dot{x}|x)$ does not exist in general for non-linear processes. Therefore, $f(x_0)$ in Eq. (16) is calculated from the time histories by taking average of the velocities $\dot{x}$ at given threshold levels. If $\dot{x}$ and x are independent, then $f(x_0)/f(x_{ref}) = 1$.

The remaining problem is how to calculate an accurate reliability index β for extreme roll angles outside the range that can be covered by MCS. Here, FORM provides β values for extreme roll angles efficiently. For the present parametric roll example, the results in Fig. 2, taken from *Choi and Jensen (2019)*, will be used in the validation process of Model-A and Model-B. The figure shows that β_{FORM} is slightly smaller than β_{MCS} when large roll angles are considered, implying that the estimated results then will be slightly conservative using β_{FORM} instead of β_{MCS}. The reliability index $\beta_{FORM}(x_0)$ for $x_0 \geq x_{ref} = 28$ deg will thus be used in the following.

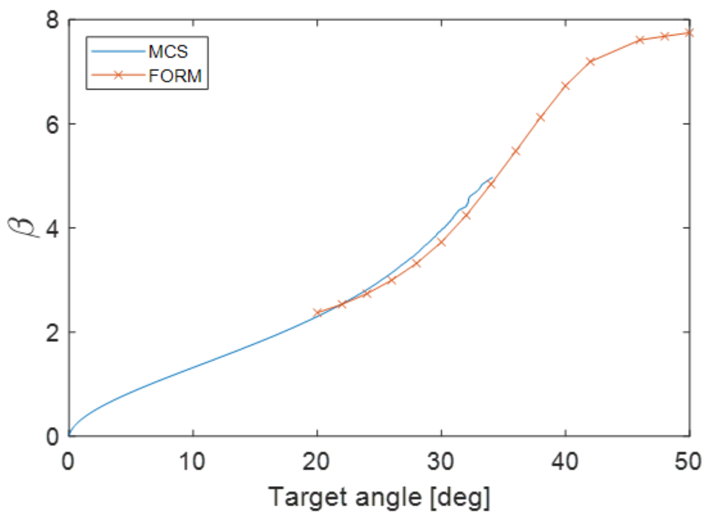

Fig. 2. Reliability index β from FORM and MCS, *Choi and Jensen (2019)*

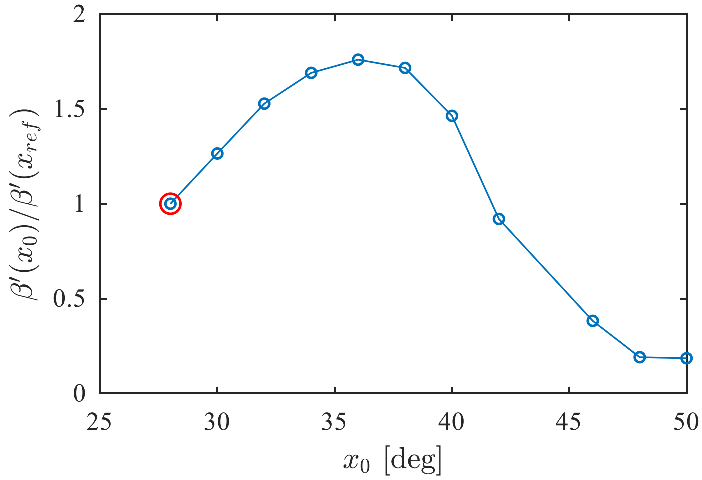

Fig. 3. $\beta'(x_0)/\beta'(x_{ref})$ from β_{FORM} for parametric rolling (x_{ref}=28 deg.)

The corresponding $\beta'(x_0)/\beta'(x_{ref})$ curve is shown in Fig. 3. For use in Model-B, $f(x_0)$, Eq.(16), is extrapolated for $x_0 \geq x_{ref} = 28$ deg from the Monte Carlo simulation and shown in Fig. 4. The dotted lines denote 95% of confidence interval of the $\beta'(x_0)/\beta'(x_{ref})$. The extrapolation chosen is linear fit based on the mean values in the tail regions, which inevitably leads to a large uncertainty. However, $f(x_0)$ is not that important for the up-crossing rate, which is dominated by exponential term.

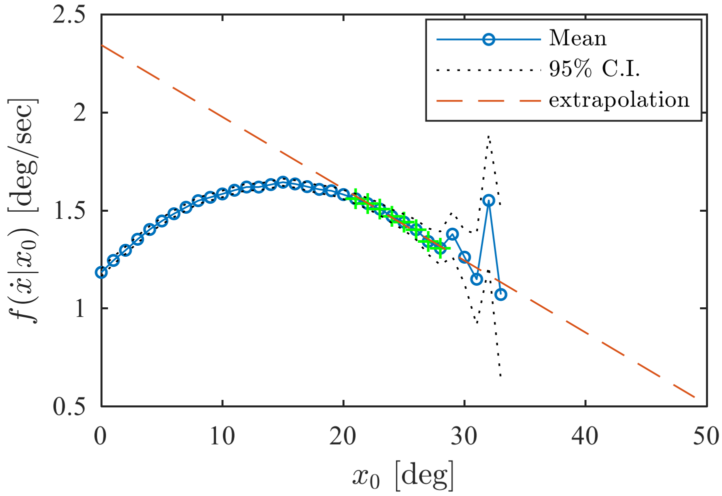

Fig. 4. The extrapolated $f(x_0)$ for parametric rolling obtained from MCS

Fig. 5 shows the calculated up-crossing rates from the two different models. In general, no significant differences are found. One attractive feature seen from the figure is that both methods A and B can capture the complex probability behavior at extreme levels ($x_0 > 40$ deg.) where a strong non-linearity (capsize) causes different response characteristics. This is difficult to obtain by an extrapolation technique and is one of the advantages of the FORM application for extreme value predictions. For larger extreme angles i.e. x_0>40 deg, the discrepancies between Model-A and Model-B seem not to be important since the probability levels are very small. It is interesting that Model-A can be used for extreme value predictions with good accuracy as it can be calculated purely from FORM to obtain β_{FORM} and by replacing ν_0 with $\omega_{roll}/2\pi$.

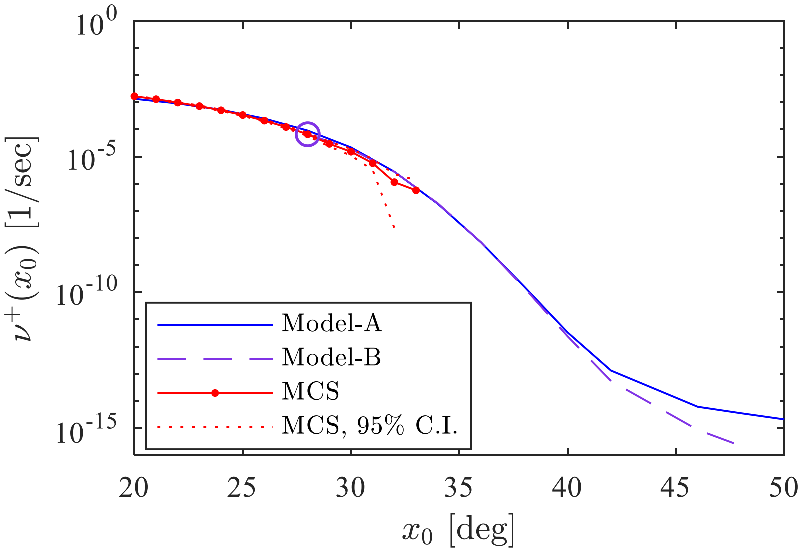

Fig. 5. Estimated up-crossing rates from the present methods

Conclusion

The purpose of the paper is to investigate the applicability of the Poisson extreme value model for stationary stochastic processes as applied to non-linear roll motion of ships. The probabilistic framework is the First Order Reliability Method (FORM), where previous studies have shown that the reliability index β obtained by FORM is very close to the results from direct Monte Carlo Simulations (MCS).

The paper focuses on the up-crossing rate. It was found in *Choi and Jensen (2019)* that for bifurcation type non-Gaussian processes such as parametric rolling of ships, a significant discrepancy between a Gaussian estimation and Monte Carlo Simulations is observed. Therefore, alternative models for mean up-crossing rate are evaluated in the present study through Monte Carlo Simulations, and it is found that a simple model (Model-A) provides a reasonably good estimation.

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