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Eirik Berge

Geometric and Functional Analytic Aspects of Time-Frequency Analysis

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of
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Faculty of Information Technology and Electrical
Engineering
Department of Mathematical Sciences

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Abstract

This thesis revolves around both geometric and functional analytic aspects of time-frequency analysis. More specifically, the thesis deals with the following three related topics:

Decomposition Spaces: Both Paper A and Paper B study decomposition spaces through the lens of *large scale geometry*. Decomposition spaces include the modulation spaces and the Besov spaces as special cases. We develop a notion of geometric embeddings between different decomposition spaces in Paper A. In Paper B we advance the theory of decomposition spaces on nilpotent Lie groups. Our main result in this direction establishes that a large class of modulation spaces on nilpotent Lie groups is distinct from their Euclidean counterparts.

Wavelet Spaces: In Paper C we study wavelet spaces by utilizing techniques from reproducing kernel Hilbert spaces. A special case of wavelet spaces has been investigated in time-frequency analysis under the name *Gabor spaces*. We discover a connection between fully interpolating Gabor spaces and the HRT-conjecture in time-frequency analysis.

Quantum Harmonic Analysis: In Paper D and Paper E we develop quantum harmonic analysis on the affine group. This requires a careful examination of the affine Wigner distribution and the affine Weyl quantization. Of particular interest is the development of a notion of admissibility for operators in the affine setting. Many of our results are aimed at connecting the affine Weyl quantization with convolutions on the affine group.

As indicated by the descriptions above, the thesis is concerned with generalizing time-frequency analysis in various directions. Despite the general approach considered in the thesis, some of the developed results are new even in well-studied settings. A common conceptual theme across the papers is *distinctness*:

Paper A: We consider various properties from large scale geometry (e.g. asymptotic dimension and hyperbolicity) that allows us to distinguish different

decomposition spaces. These properties even determine whether different decomposition spaces can embed into one another while preserving geometric properties.

Paper B: We construct modulation spaces on certain nilpotent Lie groups. Of central importance is the question of whether these new function spaces are distinct from the classical Euclidean modulation spaces. We answer this question affirmatively.

Paper C: We investigate wavelet spaces and their properties as reproducing kernel Hilbert spaces. One side-effect is that we generalize previously known results regarding distinctness of wavelet spaces by using well-known tools from representation theory.

Paper D: We consider the affine Wigner distribution and its basic properties. Among the applications is a minimization problem for the affine Wigner distribution. Even for the standard Wigner distribution it is not clear how many distinct minimizers exist. We settle this question in both the Heisenberg and the affine setting.

Paper E: We develop a quantum harmonic analysis framework for the affine group. Our framework heavily uses the so-called *affine parity operator*. It turns out that our approach is distinct from previous affine quantizations in the literature. In particular, we can represent the quantization procedure as operator convolutions with the affine parity operator.

Sammendrag

Denne avhandlingen omhandler geometriske og funksjonalanalytiske aspekter ved tid-frekvensanalyse. Mer spesifikt drøfter avhandlingen de tre følgende delvis relaterte temaene:

Dekomponeringsrom: I artikkel A og artikkel B studerer vi dekomponeringsrom ved å bruke teknikker fra *geometri i stor skala*. Dekomponeringsrom er en klasse med funksjonsrom som inkluderer modulasjonsrommene og Besov-rommene. Vi utvikler et begrep om geometriske avbildninger mellom forskjellige dekomponeringsrom. I tillegg utbygger vi teorien om dekomponeringsrom på nilpotente liegrupper. Hovedresultatet vårt i denne retningen etablerer at en stor klasse med modulasjonsrom på nilpotente liegrupper er ulik fra de euklidske modulasjonsrommene.

Waveletrom: I artikkel C studerer vi waveletrom ved å bruke teknikker fra reproduserbar kjerne hilbertrom. Et spesialtilfelle av waveletrom har blitt undersøkt tidligere i tid-frekvensanalyse under navnet *Gabor-rom*. Vi oppdager en sammenheng mellom fullstendige interpolerende Gabor-rom og HRT-formodningen i tid-frekvensanalyse.

Kvanteharmonisk analyse: I artikkel D og artikkel E utvikler vi kvanteharmonisk analyse på den affine gruppen. Dette krever en grundig undersøkelse av den affine Wigner-distribusjonen. Av spesiell interesse er definisjonen av tillatelige operatorer. Mange av resultatene vi gir sikter mot å knytte sammen den affine Weyl-kvantiseringen med konvolusjoner på den affine gruppen.

Beskrivelsene ovenfor hentyder at avhandlingen setter søkelys på å generalisere tid-frekvensanalyse i forskjellige retninger. På tross av generaliteten i avhandlingen utvikler vi resultater som vi tror er av interesse for konkrete eksempler. En rød tråd gjennom avhandlingen er *unikhet*:

Artikkel A: Vi undersøker forskjellige egenskaper fra geometri i stor skala (f.eks. asymptotisk dimensjon og hyperbolskhet) som differensierer forskjellige

dekomponeringsrom. Slike egenskapene bestemmer hvorvidt forskjellige dekomponeringsrom kan bli kontinuerlig avbildet inn i hverandre slik at geometriske egenskaper blir bevart.

Artikkel B: Vi konstruerer modulasjonsrom på bestemte nilpotente liegrupper. Av særskilt interesse er spørsmålet om hvorvidt de nye funksjonsrommene er ulike fra de klassiske euklidske modulasjonsrommene. Vi besvarer dette spørsmålet bekreftende.

Artikkel C: Vi undersøker waveletrom og deres egenskaper som reproduserbar kjerne hilbertrom. En konsekvens er at vi utvider tidligere resultater om unikhet av waveletrom ved å bruke verktøy fra representasjonsteori.

Artikkel D: Vi betrakter den affine Wigner-distribusjonen og dens grunnleggende egenskaper. En av anvendelsene er et minimeringsproblem for den affine Wigner-distribusjonen. Selv for den tradisjonelle Wigner-distribusjonen er det uklart hvor mange unike minimerere som eksisterer. Vi besvarer dette spørsmålet for Wigner-distribusjonen og den affine Wigner-distribusjonen.

Artikkel E: Vi utvikler et kvanteharmonisk analyse rammeverk for den affine gruppen. Rammeverket benytter seg av en operator som kalles den *affine paritetsoperatoren*. Det viser seg at vår fremgangsmåte er unik fra tidligere fremgangsmåter i den forstand at vi kan representere kvantiseringen som operatorkonvolusjon med den affine paritetsoperatoren.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (PhD) in Mathematical Sciences at the Norwegian University of Science and Technology (NTNU). The research presented here was conducted at the Department of Mathematical Sciences at NTNU. The candidate was supervised by Franz Luef as the main supervisor, and Felix Voigtlaender as the secondary supervisor.

Structure of the Thesis

I have written a rapid introduction to the topics necessary to understand the different papers in Part I for the readers convenience. Chapter 1 presents an introduction to central topics in time-frequency analysis. The emphasis will be on viewing time-frequency analysis through the lens of representation theory. The material in Chapter 1 will be used heavily in all five papers in Part II. Chapter 2 gives background material on creating function spaces with a geometric flavor, namely *decomposition spaces*. The material in Chapter 2 is only used in Paper A and Paper B. Chapter 3 presents the basic ideas of quantum harmonic analysis. The material in Chapter 3 is only needed in Paper D and Paper E.

Chapter 4 gives a high-level overview of the five papers that constitute the main scientific contribution of the thesis. The remaining part of the thesis, namely Part II, contains the five papers [14, 16, 17, 18, 19] with minor modifications from their published/accepted counterparts. At the end of the thesis, there is a joint bibliography for all five papers as well as the introductory chapters.

Acknowledgments

One of my fondest academic memories in the last four years is sitting alone on a bench in Vienna and munching on a mediocre burger. Why? I had completed the proof of a theorem that I had been working on for weeks. The problem just would

not budge. Then finally, without any warning, everything fell into place. Even though I had finished two collaborative papers previously, I doubted that I could do original mathematics myself. When sitting there trying to shield my burger from the cold rain, I felt like a *mathematician*. Better yet, I felt part of a mathematical community that supports one another. This feeling has persisted since then, partly due to new discoveries, but for the most part due to the continuous assault of support I've gotten from every angle.

First and foremost, I am grateful for the insightful discussions I've had with my supervisors Franz Luef and Felix Voigtlaender. Although I have yet to talk to Felix in person, the knowledgeable suggestions he has given transcends space and faulty electronic equipment. The talks I've had with Franz have shaped most of my research, and I am grateful for his continued enthusiasm for my work. I hope that you take on many more PhD students throughout the years and give them the same quality supervision you have given me.

In the last four years I have had research stays at the University of Vienna and at the Leibniz University Hannover. Both times, I have been warmly welcomed into the research groups. A special thanks goes to David Rottensteiner and Wolfram Bauer for letting me visit the two wonderful cities and mature as a researcher. I've also visited my hometown Bergen several times during my PhD to give talks there. I'm grateful to my previous supervisors Erlend Grong and Irina Markina for continuously inviting me and letting me share my research.

I will surely miss the various colleagues I've gotten to know in both Trondheim and abroad. A special thanks goes to my office mates Are Austad, Eirik Skrettingland, Helge Jørgen Samuelsen, and Simon Halvdansson for making my time at NTNU especially enjoyable. I would also like to thank Stefano Decio, Ulrik Enstad, Robert Fulsche, Mathias Palmstrøm, and Jordy Timo van Velthoven for interesting mathematical discussions at dinner-tables around the world. I hope that I will be able to spend more time with all of you in the years to come.

As a PhD is sometimes solitary work, I'm lucky to have friends and family that support me. Without your encouragement the last four years would have been difficult even in the best of times. Finally, I would like to thank my wife Stine Marie Berge for not only being an excellent mathematical collaborator, but for inspiring me to pursue a PhD degree. When you praise my results it means more to me than coming from anyone else.

Eirik Berge

Eirik Berge
Trondheim, February 2021

Contents

Part I

Introduction

Chapter 1

A Biased Introduction to Time-Frequency Analysis

Time-frequency analysis is a relatively new mathematical discipline. If one were to trace the origin of the field from an engineering perspective, then the groundbreaking paper *Theory of communication* [72] would be the natural starting point. However, from the lens of theoretical physics it is undeniable that time-frequency analysis is heavily inspired by the work of von Neumann, Weyl, and Wigner in the 1930s. When it comes to the rigorous mathematical development of time-frequency analysis, this took off with the work of Janssen [108, 109, 110, 111]. In the coming years, time-frequency analysis grew in parallel with the theory of wavelets. The two topics have much in common, and large parts of modern time-frequency analysis is inspired by the work of Daubechies et al. [44, 45, 46].

Since the turn of the century, time-frequency analysis has firmly established itself as a central topic of interest in mathematical analysis. This is in no doubt due to the plethora of connections with other interesting areas of modern mathematics like harmonic analysis, complex analysis, representation theory, and compressed sensing. In this chapter, we introduce some of the key players of modern time-frequency analysis in a self-contained manner. This chapter serves as a foundation for not only the next two chapters, but for all the five papers in Part II of the thesis.

1.1 An Informal Prelude

Before delving into the details of time-frequency analysis, let us first give an informal prelude that will motivate the topics that follow. As the name suggests, time-frequency analysis deals with understanding both the temporal information and the frequency information of a signal. To put this on firm ground, consider for simplicity a continuous function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ of a single variable $t \in \mathbb{R}$. To borrow some terminology from engineering disciplines, we think of f as a signal (e.g. sound signal, electrical signal, or pressure signal). For $t_0 \in \mathbb{R}$ we refer to $f(t_0)$ as the *temporal information* of f at time t_0. Denote by $\mathcal{F}f$ the *Fourier transform* of f given explicitly by

$$\mathcal{F}f(\omega) := \int_{-\infty}^{\infty} f(t)e^{-2\pi i t\omega}\, dt, \qquad \omega \in \mathbb{R}.$$

We refer to $\mathcal{F}f(\omega_0)$ as the *frequency information* of f at the frequency $\omega_0 \in \mathbb{R}$. One can think of f and $\mathcal{F}f$ as two sides of the same coin; both contain equivalent information regarding f. However, some information about f would be very difficult to extract from $\mathcal{F}f$ and vice versa.

It is of interest to understand the temporal information and the frequency information of a signal f simultaneously. As such, one looks for a two-dimensional function $Vf(x, \omega)$ such that $Vf(x_0, \omega_0)$ represents the intensity of the frequency $\omega_0 \in \mathbb{R}$ precisely at the time $x_0 \in \mathbb{R}$. We call Vf a *time-frequency representation* of the signal f. Although intuitive, this is actually impossible to find for a general function $f \in L^2(\mathbb{R})$ by the classical uncertainty principle. We refer the reader to [81, Chapter 2] for a discussion on various uncertainty principles and their implications in time-frequency analysis.

Although a perfect time-frequency representation does not exist, it is still possible to find useful ones. The most popular time-frequency representation is the *short-time Fourier transform (STFT)* of f given by

$$V_g f(x, \omega) := \int_{-\infty}^{\infty} f(t)\overline{g(t-x)}e^{-2\pi i t\omega}\, dt,$$

for $x, \omega \in \mathbb{R}$ where g is a rapidly decaying smooth function. The idea is to think of $V_g f$ as a sliding window Fourier transform. It turns out that although $V_g f$ is not a perfect time-frequency representation, it still possesses impressive properties and is immensely useful. We will delve into the details of the short-time Fourier transform in Section 1.2.

A closely related transformation to the short-time Fourier transform is the Wigner distribution. Given $f, g \in L^2(\mathbb{R})$ we can define the *cross-Wigner transform* $W(f, g)$ on $\mathbb{R}^2$ given by

$$W(f,g)(x,\omega) := \int_{-\infty}^{\infty} f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)}e^{-2\pi i t\omega}\, dt.$$

When $f = g$ we refer to $Wf := W(f, f)$ as the *Wigner distribution*. Although the short-time Fourier transform and the cross-Wigner transform look superficially similar, there are major differences. As an example, the Wigner distribution satisfies for well-behaved f the *marginal properties*

$$\begin{aligned} \int_{-\infty}^{\infty} Wf(x,\omega)\,d\omega &= |f(x)|^2, \\ \int_{-\infty}^{\infty} Wf(x,\omega)\,dx &= |\mathcal{F}f(\omega)|^2. \end{aligned} \tag{1.1.1}$$

It nevertheless turns out that the Wigner distribution and the short-time Fourier transform are related by the formula

$$W(f,g)(x,\omega) = 2e^{4\pi i x\cdot\omega} V_{Pg}f(2x,2\omega),$$

where $Pg(x) := g(-x)$ is the *parity operator*. The operator P plays an important role in time-frequency analysis.

Both the short-time Fourier transform and the Wigner distribution will be heavily used in Part II of the thesis. The Wigner distribution has its origin in quantum mechanics [154] and we will go through more details in Section 1.3. One of the things that separates the Wigner distribution from other time-frequency representations is its connection to the *Weyl quantization*. We will go through this connection in Section 1.4.

An important question is whether the short-time Fourier transform is integrable. To tackle this question, let us for simplicity fix $g(x) := e^{-\pi x^2}$ and consider an arbitrary element $f \in L^2(\mathbb{R})$. It is straightforward to show that $V_g f \in L^2(\mathbb{R}^2)$. However, it is not necessarily true that $V_g f \in L^1(\mathbb{R}^2)$. This leads to the definition

$$\mathcal{S}_0(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : V_g f \in L^1(\mathbb{R}^2) \right\}.$$

The space $\mathcal{S}_0(\mathbb{R})$ is called *Feichtinger's algebra* named after Hans Georg Feichtinger. Elements in $\mathcal{S}_0(\mathbb{R})$ are continuous functions and we have the inclusions

$$\mathcal{S}(\mathbb{R}) \subset \mathcal{S}_0(\mathbb{R}) \subset L^2(\mathbb{R}),$$

where $\mathcal{S}(\mathbb{R})$ denotes the *Schwartz functions*. The Feichtinger algebra $\mathcal{S}_0(\mathbb{R})$ is part of a whole family of Banach spaces $M^p(\mathbb{R})$ for $1 \leq p \leq \infty$ called the *modulation spaces*. In this notation, we have $M^1(\mathbb{R}) = \mathcal{S}_0(\mathbb{R})$ and $M^2(\mathbb{R}) = L^2(\mathbb{R})$. It is a general consensus that the modulation spaces are the correct setting for theoretical time-frequency analysis, see e.g. [81]. The modulation spaces will play an important role in the thesis and we review them more carefully in Section 1.5.

One aspect of time-frequency analysis that has been exploited more in recent years is its relation to the Heisenberg group. We will discuss this connection further in Section 1.6. For now, it suffices to say that the short-time Fourier transform is intimately related to the representation theory of the Heisenberg group. This approach gives a new way of viewing many basic facts in time-frequency analysis.

The connection between time-frequency analysis and representation theory gives a link to reproducing kernel Hilbert spaces. Recall that a *reproducing kernel Hilbert space* is a Hilbert space $\mathcal{H}$ consisting of functions $f : X \to \mathbb{C}$ on a set X such that the functionals

$$f \longmapsto E_x(f) := f(x), \qquad x \in X,$$

is well-defined and bounded. A classical example is the Hardy space H^2 of analytic functions on the unit disc $\mathbb{D}$ satisfying

$$\sup_{0 \leqslant r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta} \right) \right|^2 \, d\theta \right)^{\frac{1}{2}} < \infty.$$

It turns out that for any $g \in L^2(\mathbb{R})$ the *Gabor space* $V_g(L^2(\mathbb{R})) \subset L^2(\mathbb{R}^2)$ is a reproducing kernel Hilbert space. This is the starting point for involving theory from reproducing kernel Hilbert spaces to the setting of time-frequency analysis. We will investigate Gabor spaces further in Section 1.7.

The reader should be aware that the topics we discuss in this chapter do not in any way give a comprehensive introduction to modern time-frequency analysis. In fact, there are several major omissions such as Gabor frames and applications to pseudo-differential operators. We have chosen to omit these interesting topics to focus on the parts of time-frequency analysis that are relevant for Part II of the thesis. The interested reader is encouraged to seek out the standard textbooks [81] and [41] for more topics in time-frequency analysis.

1.2 A Joint Time-Frequency Distribution – The STFT

Our journey into time-frequency analysis starts with the short-time Fourier transform. Before giving the definition let us define two simple, yet important, operators. For $x, \omega \in \mathbb{R}^n$ we write $x\omega := x \cdot \omega$ for the usual inner product on $\mathbb{R}^n$. For $f \in L^2(\mathbb{R}^n)$ we define the operators

$$T_x f(t) := f(t - x), \qquad M_\omega f(t) := e^{2\pi i t \omega} f(t).$$

The operator T_x is called the *time-shift* with respect to x, while M_ω is called the *frequency-shift* with respect to ω. The motivation for the name frequency-shift comes from the simple identity

$$\mathcal{F}(T_x f) = M_{-x}(\mathcal{F} f),$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}f(\omega) := \int_{\mathbb{R}^n} f(t)e^{-2\pi i t\omega}\, dt.$$

Of fundamental importance is the non-commutativity of the operators T_x and M_ω. Specifically, one has for $x, \omega \in \mathbb{R}^n$ the relation

$$T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x.$$

Definition 1.2.1. Let $f, g \in L^2(\mathbb{R}^n)$ and $x, \omega \in \mathbb{R}^n$. We define the *short-time Fourier transform (STFT)* of f with respect to g to be the function on $\mathbb{R}^{2n}$ given by

$$V_g f(x,\omega) := \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi i t\omega}\, dt = \langle f, M_\omega T_x g\rangle_{L^2(\mathbb{R}^n)}.$$

Even though we can pick $g \in L^2(\mathbb{R}^n)$ arbitrarily, applications are typically interested in functions g such that both g and $\mathcal{F}g$ are well-localized around the origin. If this is the case, then we can conceptually view the short-time Fourier transform as a sliding window over the Fourier transform of f.

It is straightforward to verify that $V_g f$ is a bounded and uniformly continuous function that satisfies

$$V_g\left(T_y M_\eta f\right)(x,\omega) = e^{-2\pi i y\omega} V_g f(x-y, \omega-\eta), \tag{1.2.1}$$

for $x, y, \omega, \eta \in \mathbb{R}^n$. Moreover, the function $V_g f$ is square integrable on $\mathbb{R}^{2n}$ and satisfies the orthogonality relation

$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle_{L^2(\mathbb{R}^{2n})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^n)}\overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R}^n)}, \tag{1.2.2}$$

for $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^n)$. Hence it should be possible to reconstruct f from the values of $V_g f$ as long as $g \not\equiv 0$. By normalizing g so that $\|g\|_{L^2(\mathbb{R}^n)} = 1$ we have (in the weak sense) the vector-valued integral inversion formula

$$f(t) = \int_{\mathbb{R}^{2n}} V_g f(x,\omega) M_\omega T_x g(t)\, dx\, d\omega. \tag{1.2.3}$$

In time-frequency analysis it is sometimes more convenient to work with the *symplectic Fourier transform* $\mathcal{F}_\sigma$ on $F \in L^2(\mathbb{R}^{2n})$ given by

$$\mathcal{F}_\sigma F(z) := \int_{\mathbb{R}^{2n}} F(z')e^{-2\pi i \sigma(z,z')}\, dz',$$

where $z, z' \in \mathbb{R}^{2n}$ and σ is the standard symplectic form on $\mathbb{R}^{2n}$ given by

$$\sigma(z, z') := z' \cdot Jz, \qquad J := \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}. \tag{1.2.4}$$

The reader should pay close attention to the fact that the symplectic Fourier transform only exists in even dimensions, i.e. for $2n$ where $n \in \mathbb{N}$. We have the elementary relation

$$\mathcal{F}_\sigma F(z) = \mathcal{F}F(Jz) = \mathcal{F}(F \circ J)(z), \qquad z \in \mathbb{R}^{2n}. \tag{1.2.5}$$

For $f, g \in L^2(\mathbb{R}^n)$ we use the notation $f \otimes g$ for the element in $L^2(\mathbb{R}^{2n})$ given by

$$(f \otimes g)(x, \omega) := f(x)g(\omega).$$

The following result shows how the (symplectic) Fourier transform interacts with the short-time Fourier transform.

Proposition 1.2.2.

- For $f, g \in L^2(\mathbb{R}^n)$ and $z := (x, \omega) \in \mathbb{R}^{2n}$ we have the formulas

$$V_g f(z) = e^{-2\pi i x\omega} V_{\mathcal{F}g}\mathcal{F} f(Jz), \tag{1.2.6}$$

$$\mathcal{F}_\sigma \left(V_g f\right)(z) = e^{-2\pi i x\omega}\left(f \otimes \overline{\mathcal{F}g}\right)(z). \tag{1.2.7}$$

- For $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^n)$ we have the relationship

$$\mathcal{F}_\sigma\left(V_{g_1} f_1 \cdot \overline{V_{g_2} f_2}\right) = V_{f_2} f_1 \cdot \overline{V_{g_2} g_1}. \tag{1.2.8}$$

For the proof of (1.2.6), (1.2.7), and (1.2.8) we refer the reader to the elementary results [81, Lemma 3.1.1], [41, Lemma 1.2.4], and [41, Proposition 1.2.13], respectively. Sometimes (1.2.6) is called the *fundamental identity of time-frequency analysis*. It is not uncommon to refer to the right-hand side of (1.2.7) as the *cross-Rihaczek distribution*. In applications, it is often of interest to consider the *spectrogram*

$$\mathrm{Spec}_g f(x, \omega) := |V_g f(x, \omega)|^2.$$

By looking at (1.2.8) we see that we have the elegant special case

$$\mathcal{F}_\sigma\left(\mathrm{Spec}_f f\right) = \mathrm{Spec}_f f.$$

In other words, for any $f \in L^2(\mathbb{R}^n)$ the continuous function $\mathrm{Spec}_f f$ is an eigenfunction for the symplectic Fourier transform.

Another interesting observation, coming this time from (1.2.7), is that for non-zero $f, g \in L^2(\mathbb{R}^n)$ and $1 \leq p \leq 2$ we have the implication

$$V_g f \in L^p(\mathbb{R}^{2n}) \implies f \otimes \mathcal{F}g \in L^q(\mathbb{R}^{2n}) \implies f, \mathcal{F}g \in L^q(\mathbb{R}^n),$$

where q is the conjugate exponent $p^{-1} + q^{-1} = 1$. Hence it is simple to come up with examples showing that e.g. $V_g f \notin L^{\frac{3}{2}}(\mathbb{R}^{2n})$. This lack of regularity will motivate the modulation spaces in Section 1.5.

Finally, we want to mention that we can use the short-time Fourier transform to create a whole class of interesting operators as follows: Motivated by the inversion formula (1.2.3) we can for $g \in L^2(\mathbb{R}^n) \setminus \{0\}$ and any weight function $m \colon \mathbb{R}^{2n} \to \mathbb{C}$ consider the *localization operator* $\mathcal{A}_m^g$ on $L^2(\mathbb{R}^n)$ defined by

$$\mathcal{A}_m^g f := \int_{\mathbb{R}^{2n}} m(x,\omega) \cdot V_g f(x,\omega) M_\omega T_x g \, dx \, d\omega. \tag{1.2.9}$$

If we pick $m \equiv 1$ and $\|g\|_{L^2(\mathbb{R}^n)} = 1$, then the localization operator $\mathcal{A}_m^g$ is the identity operator. Typically we are interested in weights m that decay reasonably fast, giving us a localized picture of the time-frequency information of f through g. Requiring that m is real-valued and in $L^p(\mathbb{R}^{2n})$ for some $1 \leq p < \infty$ ensures that the localization operator $\mathcal{A}_m^g$ is self-adjoint and compact, see [156, Proposition 13.3]. We will explore operators in the wavelet setting analogous to localization operators in Paper E.

1.3 The Wigner Distribution

In addition to the short-time Fourier transform, there is another time-frequency representation that is omnipresent in the literature; the *Wigner distribution*. With its origin in quantum mechanics [154], the Wigner distribution has been widely studied in both mathematics and physics. In this section, we will give an introduction to the Wigner distribution with the aim of contrasting it with the short-time Fourier transform.

Definition 1.3.1. We define the *cross-Wigner transform* of $f, g \in L^2(\mathbb{R}^n)$ to be the function on $\mathbb{R}^{2n}$ given by

$$W(f,g)(x,\omega) := \int_{\mathbb{R}^n} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i t\omega} \, dt.$$

If $g = f$ then we refer to $Wf := W(f,f)$ as the *Wigner distribution* of f.

There are many similarities between the cross-Wigner transform and the short-time Fourier transform: The cross-Wigner transform $W(f,g)$ of $f, g \in L^2(\mathbb{R}^n)$ is a continuous and bounded function on $\mathbb{R}^{2n}$. Moreover, the cross-Wigner transform satisfies similarly to (1.2.2) the orthogonality relation

$$\langle W(f_1, g_1), W(f_2, g_2)\rangle_{L^2(\mathbb{R}^{2n})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^n)}\overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R}^n)}, \tag{1.3.1}$$

for $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^n)$.

To deduce more properties of the cross-Wigner transform, one can utilize the simple connection with the short-time Fourier transform

$$(\mathcal{F}_\sigma W(f, g))\,(x, \omega) = e^{\pi i x \omega} V_g f(x, \omega). \tag{1.3.2}$$

In particular, we have the simple property

$$W(\mathcal{F} f, \mathcal{F} g)(z) = W(f, g)(-Jz), \qquad z := (x, \omega),$$

where J is given in (1.2.4). Similarly, one can deduce that the Wigner distribution is well-behaved under time-frequency shifts

$$W(T_y M_\eta f)(x, \omega) = Wf(x - y, \omega - \eta),$$

for $x, y, \omega, \eta \in \mathbb{R}^n$.

Another source of insight into the cross-Wigner transform is given by the *Grossmann-Royer operator*

$$R(x, \omega)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

for $x, \omega \in \mathbb{R}^n$ defined by the relationship

$$W(f, g)(x, \omega) = \langle R(x, \omega) f, g\rangle_{L^2(\mathbb{R}^n)}. \tag{1.3.3}$$

Rather than giving the exact formula for R, we point out that $R(0, 0) = -2^n P$ where P is the *parity operator* given by $Pf(x) := f(-x)$ for $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The factor of 2^n is essential; it implies that the cross-Wigner transform satisfies

$$\|W(f, g)\|_{L^\infty(\mathbb{R}^{2n})} \leq 2^n \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}. \tag{1.3.4}$$

For any odd function $f \in L^2(\mathbb{R}^n)$ the bound in (1.3.4) is an equality since

$$Wf(0, 0) = -2^n \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{1.3.5}$$

One can get a better understanding of the Wigner distribution by asking a simple question: *Why is the Wigner distribution Wf called a distribution?* It is simple to see that $W(f, g) = \overline{W(g, f)}$, so Wf is certainly real-valued. A step in the direction towards viewing Wf as a type of probability distribution is that we have the *marginal properties* given in (1.1.1). The only thing that is missing is that we need to find functions $f \in L^2(\mathbb{R}^n)$ so that Wf is non-negative. By looking at (1.3.5) we see that the non-negativity of Wf poses a serious restriction. The following result of Hudson [101] shows how serious this restriction really is.

Theorem 1.3.2. *Let $f \in L^2(\mathbb{R}^n)$. Then Wf is a non-negative function if and only if f is a generalized Gaussian function*

$$f(x) = ce^{-\pi x \cdot Ax + 2\pi bx},$$

where A is a symmetric invertible $n \times n$ matrix with positive definite real part, $b \in \mathbb{C}^n$, and $c \in \mathbb{C}$. If this is the case, then we even have $Wf(x, \omega) > 0$ for all $(x, \omega) \in \mathbb{R}^{2n}$ as long as f is not identically zero.

We have so far seen three time-frequency representations; the short-time Fourier transform $V_g f$, the cross-Wigner transform $W(f, g)$, and the spectrogram $\mathrm{Spec}_g f$ for $f, g \in L^2(\mathbb{R}^n)$. The spectrogram is the only one that is always non-negative. Moreover, we have the relationship

$$\mathrm{Spec}_g f = W(P(g)) * W(f), \tag{1.3.6}$$

where P is the parity operator and $*$ denotes the convolution on $\mathbb{R}^{2n}$. In Paper D we will extend (1.3.6) to the affine setting. Thus far, the reader might be tempted to conclude that the short-time Fourier transform and the spectrogram are more well-behaved than the Wigner distribution. However, as the following section will show, the Wigner distribution has a central place in the theory of quantization.

1.4 Weyl Quantization

There are few terms in modern science that possess the amount of ambiguity and vagueness as the term *quantization*. The reader should rest assured that quantization for us will mean something very concrete. We will view the term *quantization* as a process of associating with a function σ on $\mathbb{R}^{2n}$ an operator L_σ sending a function f on $\mathbb{R}^n$ to another function $L_\sigma f$ on $\mathbb{R}^n$. When looking for well-behaved quantizations, there are many choices. A particular choice of quantization is often called a *quantization scheme*. The interested reader should consult [92, Chapter 13] for an elementary introduction to quantization schemes. For us and many others, the most obvious choice is the Weyl quantization.

Definition 1.4.1. Consider a Schwartz function $\sigma \in \mathcal{S}(\mathbb{R}^{2n})$. We refer to the operator L_σ acting on $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$L_\sigma f := \int_{\mathbb{R}^{2n}} \mathcal{F}\sigma(\eta, y)\, e^{-\pi i y \eta}\, T_{-y} M_\eta f \, dy \, d\eta \tag{1.4.1}$$

as the *Weyl quantization* of σ. The function σ is called the *symbol* of L_σ.

Remarks.

- There is a massive amount of variation in the literature when it comes to terminology. For instance, [41] uses the notation $\mathrm{Op}_w(a)$ for $a \in \mathcal{S}(\mathbb{R}^{2n})$ and refer to $\mathrm{Op}_w(a)$ as *Weyl operators*. Other authors refer to the quantization $\sigma \mapsto L_\sigma$ as the *Weyl transform*.
- The reader should be aware that most functions on $\mathbb{R}^{2n}$ that are of interest to compute the Weyl quantization of are not in $\mathcal{S}(\mathbb{R}^{2n})$. An example of physical relevance is the coordinate functions $\sigma_x(x,\omega) := x$ and $\sigma_\omega(x,\omega) := \omega$ for $(x,\omega) \in \mathbb{R}^{2n}$. The reason one typically starts with quantifying elements in $\mathcal{S}(\mathbb{R}^{2n})$ is that one can then be completely sure that expressions such as (1.4.1) are well-defined.

An essential property of the Weyl quantization [81, Proposition 14.3.3] is that for $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\sigma \in \mathcal{S}(\mathbb{R}^{2n})$ we have

$$\langle L_\sigma f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \sigma, W(g,f)\rangle_{L^2(\mathbb{R}^{2n})},$$

where $W(g,f)$ is the cross-Wigner transform of g and f. Hence the cross-Wigner transform is intimately connected with the Weyl quantization.

A classical result of Pool [138] states that the Weyl quantization is well-defined as a map from $L^2(\mathbb{R}^{2n})$ to the Hilbert-Schmidt operators $\mathcal{HS} := \mathcal{HS}(L^2(\mathbb{R}^n))$. Recall that the Hilbert-Schmidt operators constitute a Hilbert space with the inner product

$$\langle A, B\rangle_{\mathcal{HS}} := \sum_{i \in I} \langle Ae_i, Be_i\rangle,$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. The result of Pool [138] actually reveals that the Weyl quantization $\sigma \mapsto L_\sigma$ is a bijective isometry from $L^2(\mathbb{R}^{2n})$ to $\mathcal{HS}$. Additionally, the relationship

$$\langle L_\sigma f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \sigma, W(g,f)\rangle_{L^2(\mathbb{R}^{2n})} \tag{1.4.2}$$

holds for all $\sigma \in L^2(\mathbb{R}^{2n})$ and $f, g \in L^2(\mathbb{R}^n)$.

Example 1.4.2. Let us compute the Weyl quantization of $W(f_1, f_2)$ for elements $f_1, f_2 \in L^2(\mathbb{R}^n)$. Then (1.4.2) and the orthogonality relation (1.3.1) shows that

$$\begin{aligned}\langle L_{W(f_1,f_2)} g_1, g_2\rangle_{L^2(\mathbb{R}^n)} &= \langle W(f_1,f_2), W(g_2,g_1)\rangle_{L^2(\mathbb{R}^{2n})} \\ &= \langle f_1, g_2\rangle_{L^2(\mathbb{R}^n)} \langle g_1, f_2\rangle_{L^2(\mathbb{R}^n)}.\end{aligned}$$

On the other hand, one also has the *rank-one operator* $f_1 \otimes f_2$ given by

$$(f_1 \otimes f_2)g := \langle g, f_2\rangle f_1,$$

for $f_1, f_2, g \in L^2(\mathbb{R}^n)$. It is easy to check that

$$\langle (f_1 \otimes f_2)g_1, g_2\rangle_{L^2(\mathbb{R}^n)} = \langle f_1, g_2\rangle_{L^2(\mathbb{R}^n)}\langle g_1, f_2\rangle_{L^2(\mathbb{R}^n)}.$$

As such, we can conclude that $L_{W(f_1,f_2)}$ is precisely the rank-one operator $f_1 \otimes f_2$.

From a conceptual standpoint, the rank-one operators are the simplest operators in $\mathcal{HS}$. The finite-rank operators are dense in $\mathcal{HS}$. Hence it follows from Example 1.4.2 that linear combinations of cross-Wigner transforms constitute a dense subspace of $L^2(\mathbb{R}^{2n})$. However, one can consider the closed proper subset

$$\mathfrak{W}(\mathbb{R}^{2n}) := \left\{Wf : f \in L^2(\mathbb{R}^n)\right\} \subset L^2(\mathbb{R}^{2n}).$$

In [12] the authors consider for a given $\sigma \in L^2(\mathbb{R}^{2n})$ the approximation problem

$$\inf_{f \in L^2(\mathbb{R}^n)} \|\sigma - Wf\|_{L^2(\mathbb{R}^{2n})}. \tag{1.4.3}$$

They show in [12, Theorem 3] that the solution to (1.4.3) is closely linked with an eigenvalue problem. We consider this problem in the wavelet setting in Paper D. Our approach relies more on the quantization picture, and we even derive new results in the Euclidean case regarding the number of minimizers to (1.4.3).

We will use the notation $\mathcal{S}'(\mathbb{R}^n)$ for the anti-linear continuous functionals on $\mathcal{S}(\mathbb{R}^n)$. One can, through the relation (1.4.2), extend the Weyl quantization for $\sigma \in \mathcal{S}'(\mathbb{R}^{2n})$ to be an operator $L_\sigma \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. To verify this one has to check that $W(f,g) \in \mathcal{S}(\mathbb{R}^{2n})$ whenever $f, g \in \mathcal{S}(\mathbb{R}^n)$, which is straightforward. One can now rigorously verify that the Weyl quantization of the coordinate functions L_{σ_x} and L_{σ_ω} are the well known *position operator* and *momentum operator* in quantum mechanics. Also, by using (1.4.2), it is clear that the Weyl quantization of the function $1(x,\omega) := 1$ is the identity operator. The following example is the most interesting one for us.

Example 1.4.3. Consider the *point measure* $\delta(x,\omega) \in \mathcal{S}'(\mathbb{R}^{2n})$ given by

$$\langle \delta(x,\omega), \sigma\rangle := \overline{\sigma(x,\omega)}, \qquad \sigma \in \mathcal{S}(\mathbb{R}^{2n}).$$

The Weyl quantization of $\delta(x,\omega)$ is given by

$$\langle L_{\delta(x,\omega)} f, g\rangle = \langle \delta(x,\omega), W(g,f)\rangle = W(f,g)(x,\omega), \tag{1.4.4}$$

where $f, g \in \mathcal{S}(\mathbb{R}^n)$. If (1.4.4) reminds you of something, that is because it is precisely equation (1.3.3) which defines the Grossmann-Royer operator. Hence $L_{\delta(x,\omega)} = R(x,\omega)$. In particular we have

$$L_{\delta(0,0)} = -2^n P,$$

where P denotes the parity operator.

Let us demonstrate how the Weyl quantization helps us to understand the cross-Wigner transform with the following elementary result.

Proposition 1.4.4. Consider nonzero elements $g_1, g_2, g_3 \in L^2(\mathbb{R}^n)$ with the requirement that $g_1 \notin \mathrm{span}\{g_2, g_3\}$. Then there is no $t \in [0,1]$ such that there exists a convex combination

$$Wg_1 = t \cdot Wg_2 + (1-t) \cdot Wg_3. \tag{1.4.5}$$

Proof. The result becomes almost trivial when viewed through the lens of the Weyl quantization. We obtain from (1.4.5) and Example 1.4.2 that the Weyl quantization becomes

$$g_1 \otimes g_1 = t \cdot g_2 \otimes g_2 + (1-t) \cdot g_3 \otimes g_3. \tag{1.4.6}$$

Pick $h \in L^2(\mathbb{R}^n)$ that is orthogonal to g_2 and g_3, but not to g_1. By evaluating both sides of (1.4.6) on h we obtain the result. □

While the Weyl quantization is immensely useful, there are also other ways of obtaining results such as Proposition 1.4.4. In Corollary C.4.6 we prove a generalization of Proposition 1.4.4 in a setting where the Weyl quantization is not available.

1.5 Classical Modulation Spaces

We saw in Section 1.2 that for $f, g \in L^2(\mathbb{R}^n)$ we have $V_g f \in L^2(\mathbb{R}^{2n})$. However, it is in general not true that $V_g f \in L^p(\mathbb{R}^{2n})$ for any $1 \leq p < 2$. As an example, one can show that for $g(x) = e^{-\pi x^2}$ and the characteristic function $f = \chi_{[0,1]^n}$ of the cube $[0,1]^n \subset \mathbb{R}^n$ we have $V_g f \notin L^1(\mathbb{R}^{2n})$. Motivated by this, it is of interest to investigate the functions f such that $V_g f \in L^p(\mathbb{R}^{2n})$ for $1 \leq p \leq \infty$.

Definition 1.5.1. Fix $g(x) = e^{-\pi x^2}$. Define the *modulation space* $M^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ to be the space of elements $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $V_g f \in L^p(\mathbb{R}^{2n})$. We equip the space $M^p(\mathbb{R}^n)$ with the norm

$$\|f\|_{M^p(\mathbb{R}^n)} := \|V_g f\|_{L^p(\mathbb{R}^{2n})}. \tag{1.5.1}$$

The modulation spaces $(M^p(\mathbb{R}^n), \|\cdot\|_{M^p(\mathbb{R}^n)})$ for $1 \leq p \leq \infty$ are all Banach spaces. Moreover, we have $M^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and the inclusions

$$M^p(\mathbb{R}^n) \subset M^q(\mathbb{R}^n), \qquad 1 \leq p \leq q \leq \infty.$$

The reader might wonder about the dependence on the element $g(x) = e^{-\pi x^2}$. It turns out that one obtain the same modulation spaces with equivalent norms if one picks any other non-zero $g \in \mathcal{S}(\mathbb{R}^n)$ in Definition 1.5.1. The choice $g(x) = e^{-\pi x^2}$ is convenient for explicit computations.

Remarks. Before moving on, let us make a few remarks regarding the generality of our definition.

- First of all, instead of measuring the short-time Fourier transform $V_g f$ in $L^p(\mathbb{R}^{2n})$ one could choose other function spaces. It is common in the literature to consider the mixed-norm spaces $L^{p,q}(\mathbb{R}^{2n})$ for $1 \leq p, q \leq \infty$ or weighted spaces $L^p_w(\mathbb{R}^{2n})$, see [41, Chapter 2.2]. Mixed and weighted spaces will be used in Paper B, although we believe that these extensions are mostly of a technical nature. Hence we omit them in the introduction to simplify the exposition.

- One can relax the condition that $p \geq 1$ and consider the modulation spaces $M^p(\mathbb{R}^n)$ for the whole range $0 < p \leq \infty$. However, when $p < 1$ we only obtain quasi-Banach spaces. We refer the interested reader to [148] and the comprehensive thesis [147].

- Finally, the choice of the space $\mathbb{R}^n$ can also be generalized. We will in Section 2.3 and in Paper B consider modulation spaces on certain nilpotent Lie groups. One can also consider modulation spaces $M^p(G)$ for any locally compact abelian group G. We will not consider this setting, and refer the reader to the original technical report [57] and the more recent version [58].

The modulation spaces $M^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ are by (1.2.1) both time-shift and frequency-shift invariant

$$\|T_x f\|_{M^p(\mathbb{R}^n)} = \|M_\omega f\|_{M^p(\mathbb{R}^n)} = \|f\|_{M^p(\mathbb{R}^n)},$$

for $f \in M^p(\mathbb{R}^n)$ and $x, \omega \in \mathbb{R}^n$. Moreover, the modulation spaces are also invariant under the Fourier transform $\mathcal{F}$ due to the fundamental identity of time-frequency analysis (1.2.6). In even dimensions, the equation (1.2.5) implies that

$$\|\mathcal{F}_\sigma f\|_{M^p(\mathbb{R}^{2n})} = \|f\|_{M^p(\mathbb{R}^{2n})}, \qquad f \in M^p(\mathbb{R}^{2n}).$$

Example 1.5.2. It is true that $\mathcal{S}(\mathbb{R}^n) \subset M^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$. Moreover, as long as $p < \infty$ we have that $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^p(\mathbb{R}^n)$. However, there are strict inclusions between $\mathcal{S}(\mathbb{R}^n) \subset M^p(\mathbb{R}^n)$ and $M^p(\mathbb{R}^n) \subset M^q(\mathbb{R}^n)$ for all $1 \leq p \leq q \leq \infty$. To see that the first inclusion is strict, the function

$$f(x) = \frac{1}{1 + x^2}$$

is in $M^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$, but is clearly not in $\mathcal{S}(\mathbb{R}^n)$. To exemplify the second claim, the characteristic function $\chi_{[0,1]^n}$ is in $M^p(\mathbb{R}^n)$ for all $p > 1$. However, $\chi_{[0,1]^n}$ is not in $M^1(\mathbb{R}^n)$. For the more general statement, we refer the reader to [41, Proposition 2.3.26].

Of particular interest is the space $\mathcal{S}_0(\mathbb{R}^n) := M^1(\mathbb{R}^n)$. This space is often called the *Feichtinger algebra*, in reference to Hans Georg Feichtinger. Elements in $\mathcal{S}_0(\mathbb{R}^n)$ are continuous and bounded. Moreover, $\mathcal{S}_0(\mathbb{R}^n)$ is closed under both pointwise products and convolutions. The Feichtinger algebra $\mathcal{S}_0(\mathbb{R}^n)$ is, in a way that can be made precise [81, Theorem 12.1.9], the minimal Banach space invariant under time-shifts and frequency-shifts.

It is the authors personal opinion that the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}^n)$ is the most important function space in time-frequency analysis. In Example 2.2.2 we will give an equivalent definition of the modulation spaces through decomposition spaces. In the next section we turn to representation theory to see how time-frequency analysis is related to the Heisenberg group.

1.6 Through the Lens of Representation Theory

The aim of this section is to consider time-frequency analysis from a more abstract perspective. To be precise, we will try to understand the short-time Fourier transform through the representation theory of the Heisenberg group. As such, we will first recall some notions from representation theory of locally compact groups for the readers convenience. For more on this general setting, we refer to a survey of the author [15] and the references within.

Definition 1.6.1. A *locally compact group* is a locally compact Hausdorff topological space G with a group structure such that the product $(x, y) \mapsto x \cdot y$ and the inversion $x \mapsto x^{-1}$ for $x, y \in G$ are continuous maps.

The most important fact about locally compact groups is the existence of the Haar measures: We say that a Borel measure μ on G is *left-invariant* if

$$\mu(x \cdot E) = \mu(E)$$

for all $x \in G$ and all Borel sets $E \subset G$. There is a unique (up to multiplication by a positive constant) left-invariant Radon measure μ_L on G called the *left Haar measure*. In the same way, there exists a unique right-invariant Radon measure μ_R on G. If $\mu_L = \mu_R$, then we simply write $\mu := \mu_L = \mu_R$ and refer to the group G as *unimodular*.

In practice, the preceding discussion implies that we have a well-defined measure theory setting on any locally compact group. In particular, we can for $1 \leq p \leq \infty$ consider the spaces $L^p(G)$ of measurable functions $f \colon G \to \mathbb{C}$ satisfying

$$\|f\|_{L^p(G)} := \left(\int_G |f(x)|^p \, d\mu_L(x) \right)^{\frac{1}{p}} < \infty.$$

For $p = \infty$ we make the obvious modification to mimic the Euclidean case $L^\infty(\mathbb{R}^n)$. Examples of locally compact groups are $\mathbb{R}^n$, discrete groups, and Lie groups. For time-frequency analysis the most important example is the Heisenberg group.

Example 1.6.2. Consider the topological space $\mathbb{H}^n := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the product

$$\left(x, \omega, t\right) \cdot \left(x', \omega', t'\right) := \left(x + x', \omega + \omega', t + t' + \frac{1}{2}(x'\omega - x\omega')\right).$$

We refer to $\mathbb{H}^n$ as the *(full) Heisenberg group* of dimension n. The group operation of the Heisenberg group simulates the fundamental commutation relations in quantum mechanics, see [92].

Often a different realization of the Heisenberg group is considered: We denote by $\mathbb{H}^n_r := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$ with the product

$$\left(x, \omega, e^{2\pi i \tau}\right) \cdot \left(x', \omega', e^{2\pi i \tau'}\right) := \left(x + x', \omega + \omega', e^{2\pi i(\tau + \tau')} e^{\pi i (x'\omega - x\omega')}\right),$$

for $x, x', \omega, \omega' \in \mathbb{R}^n$ and $\tau, \tau' \in \mathbb{R}$. We refer to $\mathbb{H}^n_r$ as the *(reduced) Heisenberg group*. Both groups $\mathbb{H}^n$ and $\mathbb{H}^n_r$ are unimodular with Haar measures $dx\, d\omega\, dt$ and $dx\, d\omega\, d\tau$, respectively. While the full Heisenberg group $\mathbb{H}^n$ is simply connected, the reduced Heisenberg group $\mathbb{H}^n_r$ is not. However, the reduced Heisenberg group has certain integrability advantages that we will see shortly.

We are interested in representing elements in locally compact groups as well-behaved linear transformations on some Hilbert space $\mathcal{H}$. We restrict ourselves to the group of unitary transformations $\mathcal{U}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$ and give the following definition.

Definition 1.6.3. Let G be a locally compact group and let $\mathcal{H}_\pi$ be a Hilbert space. A group homomorphism $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ is called a *unitary representation* if the maps

$$G \ni x \mapsto \mathcal{W}_g f(x) := \langle f, \pi(x) g \rangle_{\mathcal{H}_\pi}$$

are continuous for all $f, g \in \mathcal{H}_\pi$. For a fixed $g \in \mathcal{H}_\pi$ we refer to the map $f \mapsto \mathcal{W}_g f$ as the *wavelet transform* of f with respect to the window g.

Of particular importance are unitary representations $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ that are *irreducible*, meaning that there do not exist non-trivial closed subspaces $\mathcal{M}$ of $\mathcal{H}_\pi$ such that $\pi(x) f \in \mathcal{M}$ for all $x \in G$ and $f \in \mathcal{M}$. The irreducible unitary representations of a locally compact group serves as the fundamental building blocks for how the group can be represented as unitary linear operators. The following example completely determines the irreducible unitary representations of the reduced Heisenberg group.

Example 1.6.4. Given the reduced Heisenberg group $\mathbb{H}_r^n$ we can first consider the one-dimensional representations

$$\chi_{\alpha,\beta} \colon \mathbb{H}_r^n \to \mathcal{U}(\mathbb{C}) \simeq \mathbb{T}$$

for $\alpha, \beta \in \mathbb{R}$ given by

$$\chi_{\alpha,\beta}\left(x, \omega, e^{2\pi i \tau}\right) := e^{2\pi i(\alpha x + \beta\omega)}, \qquad \left(x, \omega, e^{2\pi i \tau}\right) \in \mathbb{H}_r^n.$$

One-dimensional representations are often called *characters*, and it is straightforward to check that any character of $\mathbb{H}_r^n$ is of the form $\chi_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$. In addition to the characters, we have the *Schrödinger representation*

$$\rho \colon \mathbb{H}_r^n \to \mathcal{U}(L^2(\mathbb{R}^n))$$

given by

$$\rho\left(x, \omega, e^{2\pi i \tau}\right) f := e^{2\pi i \tau} e^{\pi i x \omega} T_x M_\omega f, \qquad \left(x, \omega, e^{2\pi i \tau}\right) \in \mathbb{H}_r^n.$$

One can also consider the slight modifications $\rho_n \colon \mathbb{H}_r^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ for any $n \in \mathbb{Z} \setminus \{0\}$ given by

$$\rho_n\left(x, \omega, e^{2\pi i \tau}\right) f := e^{2\pi i n \tau} e^{\pi i n x \omega} T_{nx} M_\omega f.$$

The impressive Stone-von Neumann Theorem [81, Corollary 9.3.5] implies that any irreducible representation of $\mathbb{H}_r^n$ is equivalent to either $\chi_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$ or ρ_n for some $n \in \mathbb{Z} \setminus \{0\}$.

Consider a unitary representation $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ and fix $f, g \in \mathcal{H}_\pi$. Notice that we immediately get that $\mathcal{W}_g f \in L^\infty(G)$ by the elementary estimate

$$|\mathcal{W}_g f(x)| = |\langle f, \pi(x) g\rangle_{\mathcal{H}_\pi}| \leq \|f\|_{\mathcal{H}_\pi} \|\pi(x) g\|_{\mathcal{H}_\pi} = \|f\|_{\mathcal{H}_\pi} \|g\|_{\mathcal{H}_\pi},$$

for all $x \in G$. However, it is in general not true that $\mathcal{W}_g f \in L^2(G)$. As this will be central for many arguments, we consider the following definition.

Definition 1.6.5. Let $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be an irreducible unitary representation of a locally compact group G. We say that π is *square integrable* if there exists a non-zero $g \in \mathcal{H}_\pi$ such that $\mathcal{W}_g g \in L^2(G)$. In this case, the element $g \in \mathcal{H}_\pi$ is also referred to as *square integrable*.

Example 1.6.6. Consider again the Schrödinger representation ρ of the reduced Heisenberg group $\mathbb{H}_r^n$. For $f, g \in L^2(\mathbb{R}^n)$ the wavelet transform $\mathcal{W}_g f$ corresponding to ρ is given by

$$\begin{aligned}\mathcal{W}_g f\left(x, \omega, e^{2\pi i \tau}\right) &= \left\langle f, \rho\left(x, \omega, e^{2\pi i \tau}\right) g\right\rangle_{L^2(\mathbb{R}^n)} \\ &= e^{-2\pi i \tau} e^{-\pi i x \omega} \langle f, T_x M_\omega g\rangle_{L^2(\mathbb{R}^n)} \\ &= e^{-2\pi i \tau} e^{\pi i x \omega} V_g f(x, \omega).\end{aligned}$$

As one can see from the calculation above, the wavelet transform corresponding to the Schrödinger representation is essentially just the short-time Fourier transform. As such, we can immediately conclude from (1.2.2) that the Schrödinger representation is square integrable since

$$\begin{aligned}\|\mathcal{W}_g f\|_{L^2(\mathbb{H}_r^n)}^2 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{T}} |\mathcal{W}_g f(x, \omega, \tau)|^2 \, dx \, d\omega \, d\tau \\ &= \int_{\mathbb{R}^{2n}} |V_g f(x, \omega)|^2 \, dx \, d\omega \\ &= \|f\|_{L^2(\mathbb{R}^n)}^2 \|g\|_{L^2(\mathbb{R}^n)}^2.\end{aligned}$$

It is also possible to define a Schrödinger representation for the full Heisenberg group $\mathbb{H}^n$, see [81, Chapter 9] for details. However, it is only on the reduced Heisenberg group $\mathbb{H}_r^n$ that the Schrödinger representation is square integrable. This is the main reason we consider the reduced Heisenberg group $\mathbb{H}_r^n$ instead of the full Heisenberg group $\mathbb{H}^n$.

Assume that $\pi \colon G \longrightarrow \mathcal{U}(\mathcal{H}_\pi)$ is a square integrable representation where $g \in \mathcal{H}_\pi$ is a square integrable element. It follows from [15, Proposition 2.21] that $\mathcal{W}_g f \in L^2(G)$ for all $f \in \mathcal{H}_\pi$. As such, we can ask whether the map

$$\mathcal{H}_\pi \ni f \mapsto \mathcal{W}_g f \in L^2(G)$$

is an isometry. The following classical theorem of Duflo and Moore establishes that the answer is affirmative up to a constant.

Theorem 1.6.7 (Duflo-Moore Theorem)**.** *Consider a square integrable representation $\pi \colon G \longrightarrow \mathcal{U}(\mathcal{H}_\pi)$. There exists a unique self-adjoint, positive, densely defined operator $C_\pi \colon \mathcal{D}(C_\pi) \subset \mathcal{H}_\pi \to \mathcal{H}_\pi$ with a densely defined inverse such that:*

- *An element $g \in \mathcal{H}_\pi \setminus \{0\}$ is square integrable precisely when $g \in \mathcal{D}(C_\pi)$.*
- *For $g_1, g_2 \in \mathcal{D}(C_\pi)$ and $f_1, f_2 \in \mathcal{H}_\pi$ we have the orthogonality relation*

$$\langle \mathcal{W}_{g_1} f_1, \mathcal{W}_{g_2} f_2\rangle_{L^2(G)} = \langle f_1, f_2\rangle_{\mathcal{H}_\pi} \overline{\langle C_\pi g_1, C_\pi g_2\rangle_{\mathcal{H}_\pi}}. \tag{1.6.1}$$

A square integrable element $g \in \mathcal{H}_\pi$ is called *admissible*[1] if $\|C_\pi g\|_{\mathcal{H}_\pi} = 1$. For unimodular groups G the Duflo-Moore Theorem implies that square integrable representations $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ satisfy $C_\pi = c_\pi \cdot Id_\pi$ for some $c_\pi > 0$. In the case of the Schrödinger representation ρ of the reduced Heisenberg group $\mathbb{H}_r^n$ we have that $c_\rho = 1$. Hence (1.6.1) is really just the orthogonality relations (1.2.2) in disguise for the Schrödinger representation.

The name *wavelet transform* comes from wavelet analysis. It should therefore not come as a surprise that we can describe wavelet analysis in terms of representation theory. By doing this, we also emphasize the similarities between time-frequency analysis and wavelet analysis.

Example 1.6.8. Recall that the affine group $\mathrm{Aff} := \mathbb{R} \times \mathbb{R}_+$ for $\mathbb{R}_+ := (0, \infty)$ is the locally compact group with the product

$$(x, a) \cdot_{\mathrm{Aff}} (y, b) := (x + ay, ab), \quad (x, a),\ (y, b) \in \mathrm{Aff}.$$

We will discuss the affine group in more detail in Section 3.3.

An important representation of the affine group is the *wavelet representation* $\pi\colon \mathrm{Aff} \to \mathcal{U}(L^2(\mathbb{R}))$ given by

$$\pi(x, a) f(t) := \frac{1}{\sqrt{a}} f\left(\frac{t - x}{a}\right), \qquad (x, a) \in \mathrm{Aff}. \tag{1.6.2}$$

The wavelet representation is not irreducible, but consists of the two irreducible subspaces H_+ and H_- where

$$H_\pm := \{f \in L^2(\mathbb{R}) : \mathrm{supp}\,(\mathcal{F} f) \subset \mathbb{R}_\pm\}$$

Let us for simplicity consider the irreducible representation of π restricted to H_+. The wavelet transform corresponding to the wavelet representation is given by

$$\mathcal{W}_g f(x, a) := \langle f, \pi(x, a) g\rangle_{L^2(\mathbb{R})} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{g\left(\frac{t - x}{a}\right)}\, dt, \tag{1.6.3}$$

for $f, g \in H_+$. The reader surely recognizes $\mathcal{W}_g f$ as the *continuous wavelet transform* in classical wavelet analysis. In fact, the continuous wavelet transform is the prototypical example of a wavelet transform in the general setting.

The Duflo-Moore operator C_π corresponding to the wavelet representation π on H_+ is the Fourier multiplier given by

$$C_\pi g = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{a}} \mathcal{F} g(a)\right), \qquad g \in \mathcal{D}(C_\pi).$$

[1]The terminology is not always consistent in the literature. In fact, in Paper E we use the terminology *admissible* for what we here have called *square integrable*.

As such, an element $g \in H_+$ is square integrable if and only if

$$\int_0^\infty \frac{|\mathcal{F}g(a)|^2}{|a|}\, da < \infty.$$

1.7 Reproducing Kernel Hilbert Spaces

We have throughout this introductory chapter been interested in the short-time Fourier transform. We now consider the image space

$$V_g(L^2(\mathbb{R}^n)) := \{V_g f : f \in L^2(\mathbb{R}^n)\} \subset L^2(\mathbb{R}^{2n}), \tag{1.7.1}$$

for a non-zero element $g \in L^2(\mathbb{R}^n)$. We refer to $V_g(L^2(\mathbb{R}^n))$ as the *Gabor space* corresponding to the window function g.

More generally, we can for a square integrable representation $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ consider the *wavelet space*

$$\mathcal{W}_g(\mathcal{H}_\pi) := \{\mathcal{W}_g f : f \in \mathcal{H}_\pi\} \subset L^2(G), \tag{1.7.2}$$

where $g \in \mathcal{H}_\pi$ is square integrable. Sometimes it is convenient to require that g is additionally admissible as we do in Paper C. However, this is only for convenience as any non-zero scalar multiple of g will give the same wavelet space. Let us for the moment work with the general wavelet space, and then restrict to the Gabor spaces afterwards. We will be interested in the following property of wavelet spaces.

Definition 1.7.1. A *reproducing kernel Hilbert space* $\mathcal{H}$ is a Hilbert space of functions $f\colon X \to \mathbb{C}$ on a set X such that the *evaluation functionals*

$$E_x(f) := f(x), \qquad x \in X,$$

are well-defined and bounded. If the evaluation functions $\{E_x\}_{x\in X}$ are uniformly bounded, then we refer to $\mathcal{H}$ as *uniform*.

For a reproducing kernel Hilbert space $\mathcal{H}$ one can, by the Riesz Representation Theorem, for each $x \in X$ find an element $k_x \in \mathcal{H}$ such that

$$f(x) = \langle f, k_x\rangle_{\mathcal{H}}, \qquad f \in \mathcal{H}.$$

The element k_x is called the *point kernel for* $x \in X$. The function $K\colon X \times X \to \mathbb{C}$ given by

$$K(x, y) := \langle k_y, k_x\rangle_{\mathcal{H}} \tag{1.7.3}$$

is called the *reproducing kernel* for $\mathcal{H}$. Notice that if $f_n \to f$ in the norm on $\mathcal{H}$, then we automatically get pointwise convergence since

$$|f_n(x) - f(x)| = |\langle f_n - f, k_x\rangle| \le \|f_n - f\|_{\mathcal{H}}\|k_x\|_{\mathcal{H}} \to 0.$$

Example 1.7.2. The *Bargmann-Fock space* $\mathcal{F}^2(\mathbb{C}^n)$ is the Hilbert space consisting of all holomorphic functions F on $\mathbb{C}^n$ such that

$$\|F\|_{\mathcal{F}^2} := \sqrt{\int_{\mathbb{C}^n} |F(z)|^2 e^{-\pi|z|^2}\,dz} < \infty.$$

The inner product on $\mathcal{F}^2(\mathbb{C}^n)$ is given by

$$\langle F, G\rangle_{\mathcal{F}^2} := \int_{\mathbb{C}^n} F(z)\overline{G(z)}e^{-\pi|z|^2}\,dz.$$

The Bargmann-Fock space $\mathcal{F}^2(\mathbb{C}^n)$ is a reproducing kernel Hilbert space with the point kernel for $w \in \mathbb{C}^n$ given by

$$k_w(z) = e^{\pi\overline{w}z}, \qquad z \in \mathbb{C}^n.$$

By considering $F(z) = z_1$ for $z := (z_1, \ldots, z_n) \in \mathbb{C}^n$ it is clear that $\mathcal{F}^2(\mathbb{C}^n)$ is not uniform.

As the reader probably suspects, we have the following result.

Proposition 1.7.3. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation and fix a square integrable element $g \in \mathcal{H}_\pi$. The wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$ is a uniform reproducing kernel Hilbert spaces with reproducing kernel

$$K(x, y) = \mathcal{W}_g\left(\pi(y)g\right)(x), \qquad x, y \in G.$$

For a simple proof of Proposition 1.7.3 we refer to the proof of Proposition C.3.3. Finally, let us consider the question of whether the wavelet spaces have an interpolation property.

Definition 1.7.4. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on the set X with reproducing kernel K. We say that $\mathcal{H}$ is *fully interpolating* if for any $m \in \mathbb{N}$ and any set

$$\Omega := \{x_1, \ldots, x_m\} \subset X$$

with $|\Omega| = m$ the $m \times m$ matrix

$$K_\Omega := \left\{K(x_i, x_j)\right\}_{i,j=1}^m$$

is strictly positive definite.

The definition above probably requires some explanation. That a reproducing kernel Hilbert space $\mathcal{H}$ is fully interpolating is equivalent to the following condition: For finitely many points $\Omega \coloneqq \{x_1, \ldots, x_m\} \subset X$ and possibly non-distinct scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ there is a function $F \in \mathcal{H}$ such that $F(x_i) = \lambda_i$ for all $i = 1, \ldots, m$.

A relatively straightforward computation shows the Gabor space $V_g(L^2(\mathbb{R}^n))$ with $g(x) = 2^{\frac{n}{4}} e^{-\pi x^2}$ is fully interpolating. What about the other Gabor spaces? We show in Proposition C.6.1 that this question is equivalent to the well known *HRT-conjecture*:

Conjecture (Heil-Ramanathan-Topiwala)**.** Is the set

$$\{M_\omega T_x g\}_{(x,\omega)\in\mathbb{R}^{2n}}$$

linearly independent in $L^2(\mathbb{R}^n)$ for all non-zero $g \in L^2(\mathbb{R}^n)$?

Chapter 2

In the Business of Constructing Function Spaces

As described in Section 1.5, the modulation spaces $M^p(\mathbb{R}^n)$ on $\mathbb{R}^n$ for $1 \leq p \leq \infty$ play an important role in time-frequency analysis. From a more abstract perspective, the modulation spaces are a special case of *decomposition spaces*. The origin of decomposition spaces can be found in the papers [56, 59]. Decomposition spaces have a geometric flavor, and are general enough to encompass many interesting function spaces.

In this chapter, we aim to introduce decomposition spaces from the perspective of large scale geometry. By doing this, we lay the groundwork for Paper A and Paper B. In Section 2.1 we review some basic notions from large scale geometry for the readers convenience. In Section 2.2 we define and discuss decomposition spaces as a general class of function spaces. Finally, in Section 2.3 we define stratified Lie groups and develop some basic properties in preparation for Paper B.

2.1 Basic Facts From Large Scale Geometry

We begin by giving some elementary definitions from large scale geometry. Large scale geometry is interested in the geometry of spaces when they are viewed from "far away". We will utilize large scale geometry when discussing decomposition spaces in Section 2.2. The reader should consult the standard reference on large scale geometry [133] for a more thorough introduction to the topic.

Definition 2.1.1. Let (X, d_X) and (Z, d_Z) be two metric spaces. We say that a map $f\colon X \to Z$ is a *quasi-isometric embedding* if there exist constants $C, L > 0$

such that for all $x, y \in X$ we have

$$\frac{1}{L} d_X(x, y) - C \leq d_Z(f(x), f(y)) \leq L d_X(x, y) + C.$$

Remark. Notice that f does not need to be injective nor surjective to be a quasi-isometric embedding; a trivial example is any map between finite metric spaces. A quasi-isometric embedding does not even need to be continuous; the map sending $x \in \mathbb{R}^n$ to the nearest point on the integer lattice $\mathbb{Z}^n$ is a quasi-isometric embedding that is not continuous.

We say that $N \subset Z$ is a *net* in (Z, d_Z) if there exists an absolute constant $C > 0$ such that for any $y \in Z$ there exists an $n \in N$ with $d_Z(y, n) \leq C$. This leads to the following definition.

Definition 2.1.2. Let (X, d_X) and (Z, d_Z) be two metric spaces. A quasi-isometric embedding

$$f\colon (X, d_X) \to (Z, d_Z)$$

is said to be a *quasi-isometry* if the image $f(X) \subset Z$ is a net in (Z, d_Z). If a quasi-isometry between (X, d_X) and (Z, d_Z) exists, then (X, d_X) and (Z, d_Z) are called *quasi-isometric*.

The notion of being quasi-isometric is an equivalence class on the collection of all metric spaces. The following example illustrates that quasi-isometries can be radically different from traditional isometries.

Example 2.1.3. It is clear that the inclusion map $i\colon \mathbb{Z}^n \to \mathbb{R}^n$ is a quasi-isometry. Hence two metric spaces of different cardinalities can be quasi-isometric. The intuition is that $\mathbb{Z}^n$ looks more and more like $\mathbb{R}^n$ when zooming out.

We say that two maps $f, g\colon (X, d_X) \to (Z, d_Z)$ are *close* if there exists a constant $C > 0$ such that

$$d_Z(f(x), g(x)) < C,$$

for all $x \in X$. The following equivalent characterization of quasi-isometries is straightforward to show.

Proposition 2.1.4. Let $f\colon (X, d_X) \to (Z, d_Z)$ be a quasi-isometric embedding between two metric spaces. The map f is a quasi-isometry if and only if there exists a quasi-isometric embedding $g\colon (Z, d_Z) \to (X, d_X)$ such that $f \circ g$ and $g \circ f$ are close to the identity maps Id_X and Id_Z, respectively.

Remark. There are a lot of structural similarities between quasi-isometries and homotopy equivalences in algebraic topology. While homotopy equivalences relax the notion of homeomorphisms between topological spaces, quasi-isometries relax the notion of classical isometries between metric spaces. In this analogy, closeness of quasi-isometric embeddings corresponds to homotopic maps. Both notions of equivalence are not cardinality dependent, e.g. $\mathbb{R}^n$ is homotopy equivalent to a point. The motivations for quasi-isometries and homotopy equivalences are also similar, namely to study invariant properties of spaces. In the same way that homotopy equivalences preserve homology and cohomology groups, quasi-isometries preserve several metric space invariants such as *asymptotic dimension* and *hyperbolicity*. We will study these metric space invariants in Paper A.

In the next section we are interested in a special class of metric spaces that we now develop.

Definition 2.1.5. Let $Q := (Q_i)_{i\in I}$ be a collection of subsets of a non-empty set X such that $\cup_{i\in I} Q_i = X$. Define the *admissibility constant* N_Q as

$$N_Q := \sup_{i\in I} |i^*|, \qquad i^* := \left\{ j \in I : Q_i \cap Q_j \neq \emptyset \right\}.$$

We say that Q is an *admissible covering* if $N_Q < \infty$.

We refer to i^* as the *set of neighboring indexes* of $i \in I$. Conceptually, the requirement $N_Q < \infty$ ensures that admissible coverings are not too clustered.

Example 2.1.6. For examples of admissible coverings we refer the reader to Example A.2.4, Example A.2.8, and Example A.2.9. For a counterexample, consider $X := B(0,1) \setminus \{0\} \subset \mathbb{R}^2$ and start with the two sets in polar coordinates

$$Q_{1,1} := \left\{ (r,\phi) \in X : \frac{1}{2} \le r \le 1,\ 0 \le \phi \le \pi \right\},$$
$$Q_{1,2} := \left\{ (r,\phi) \in X : \frac{1}{2} \le r \le 1,\ \pi \le \phi \le 2\pi \right\}.$$

We refer to $\{Q_{1,1}, Q_{1,2}\}$ as the *first layer*. The second layer will have radius values $1/4 \le r \le 1/2$. More generally, the n'th layer will have radius values $1/2^n \le r \le 1/2^{n-1}$. As we go further inwards, we rapidly divide the length of the angles. More precisely, for the n'th layer we divide $[0, 2\pi]$ into $(n+1)!$ elements. Hence we have a collection $Q := (Q_{n,k})_{n\in\mathbb{N}}$ where $k = 1, \ldots, (n+1)!$ for each $n \in \mathbb{N}$. It is clear that Q is a covering of X. However, Q is not an admissible covering since

$$\left|\left\{ (n,k) : Q_{(m,0)} \cap Q_{(n,k)} \neq \emptyset \right\}\right| \ge \left|\left\{ k : Q_{(m,0)} \cap Q_{(m+1,k)} \neq \emptyset \right\}\right| = m+1.$$

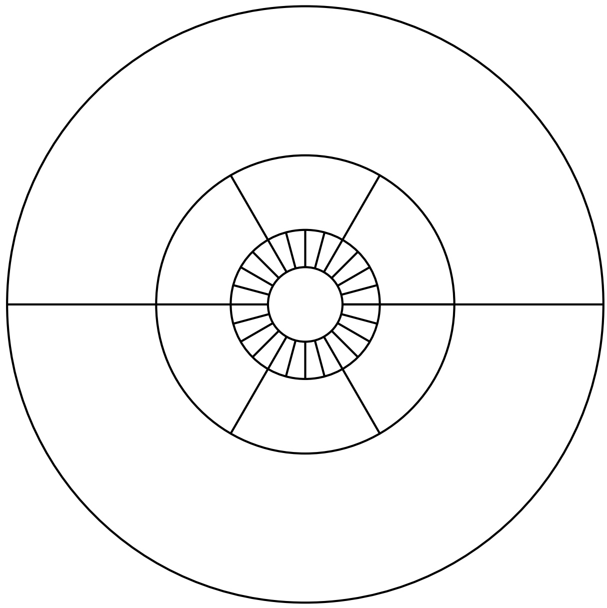

Figure 2.1: The first three layers of the non-admissible covering.

Let us connect the admissible coverings with concepts from large scale geometry. Firstly, we need to associate to any admissible covering a metric space. To do this, we use the terminology $\mathcal{Q}$-*chain* to refer to a sequence $Q_{i_1}, \ldots, Q_{i_k} \in \mathcal{Q}$ such that $Q_{i_l} \cap Q_{i_{l+1}} \neq \emptyset$ for every $1 \leq l \leq k-1$. If $x \in Q_{i_1}$ and $y \in Q_{i_k}$, then we call the sequence $Q_{i_1}, \ldots, Q_{i_k}$ a $\mathcal{Q}$-*chain of length* k between x and y. We use the notation $\mathcal{Q}(k, x, y)$ to denote the (possibly empty) set of all $\mathcal{Q}$-chains of length k between x and y. With this, we have the following definition.

Definition 2.1.7. Let $\mathcal{Q} := (Q_i)_{i \in I}$ be an admissible covering on X. Define $d_{\mathcal{Q}}$ to satisfy $d_{\mathcal{Q}}(x,x) := 0$ for $x \in X$ and

$$d_{\mathcal{Q}}(x,y) := \inf \{k : \mathcal{Q}(k,x,y) \neq \emptyset\}, \qquad x, y \in X,\ x \neq y,$$

where we use the convention that infimum over an empty set is infinity. We refer to $d_{\mathcal{Q}}$ as the *associated metric* to the covering $\mathcal{Q}$.

Despite the name, it is possible that $d_{\mathcal{Q}}(x,y) = \infty$ for some $x, y \in X$. As an example, consider $\mathcal{Q} := \{Q, Q^c\}$ where both Q and Q^c are non-empty. Then $d_{\mathcal{Q}}(x,y) = \infty$ whenever $x \in Q$ and $y \in Q^c$. To ensure that the associated metric $d_{\mathcal{Q}}$ is in fact a metric in the classical sense, we need to require that for all $x, y \in X$ there exists $k \in \mathbb{N}$ such that $\mathcal{Q}(k,x,y)$ is non-empty. We refer to admissible coverings that satisfy this criterion as *concatenations*.

Now that we have a metric space (X, d_Q) associated to any concatenation we can apply the theory from large scale geometry to coverings. This is investigated in detail in Paper A.

2.2 Decomposition Spaces

The goal of this section will be to give a conceptual overview of decomposition spaces. Moreover, we will use the notions from large scale geometry developed in Section 2.1 to discuss *geometric embeddings* between decomposition spaces. This is a novel concept that is investigated in Paper A and Paper B. We omit some details in the following definitions for brevity. Precise definitions are given in Section A.4.

Let $Q := (Q_i)_{i\in I}$ be an admissible covering on a locally compact space X. We say that a collection of non-negative continuous functions $\Phi := (\varphi_i)_{i\in I}$ is a *bounded admissible partition of unity* for Q if $\mathrm{supp}(\varphi_i) \subset Q_i$ and for every $x \in X$ we have

$$\sum_{i\in I} \varphi_i(x) = 1.$$

For simplicity, we refer to Φ as a Q-BAPU.

Definition 2.2.1. Let $Q := (Q_i)_{i\in I}$ be a concatenation on a locally compact space X with a Q-BAPU $\Phi := (\varphi_i)_{i\in I}$. We consider a Banach space $(B, \|\cdot\|_B)$, where B is a subspace of a set of distributions on X. Let $(Y, \|\cdot\|_Y)$ be a Banach space consisting of sequences on the index set I. The *decomposition space* $\mathcal{D}(Q, B, Y)$ consists of all distributions f such that

$$\|f\|_{\mathcal{D}(Q,B,Y)} := \left\| (\|f \cdot \varphi_i\|_B)_{i\in I} \right\|_Y < \infty. \tag{2.2.1}$$

We refer to B as the *local component* of $\mathcal{D}(Q, B, Y)$, while Y is called the *global component* of $\mathcal{D}(Q, B, Y)$.

The idea behind decomposition spaces is that we are looking for distributions that are locally well-behaved with respect to $(B, \|\cdot\|_B)$, while being globally well-behaved with respect to $(Y, \|\cdot\|_Y)$. Under suitable assumptions on Φ, B, and Y, the decomposition space $\mathcal{D}(Q, B, Y)$ does not depend on the choice of bounded admissibly partition of unity Φ. However, it heavily depends on the choice of the concatenation Q.

The decomposition space $\mathcal{D}(Q, B, Y)$ is a Banach space with the norm (2.2.1). Moreover, the dual space of $\mathcal{D}(Q, B, Y)$ is also a decomposition space and can be identified with

$$\mathcal{D}(Q, B, Y)^* \simeq \mathcal{D}(Q, B^*, Y^*).$$

In particular, if both B and Y are reflexive Banach spaces, then $\mathcal{D}(Q, B, Y)$ is a reflexive Banach space as well.

Example 2.2.2. Some of the most well known examples of decomposition spaces, like the modulation spaces and the Besov spaces, are not precisely of the form in Definition 2.2.1. They instead incorporate the Fourier transform in their definition. For the case of the modulation spaces $M^p(\mathbb{R}^n)$ we can use the covering

$$\mathcal{Q} := ([-1,1]^n + k)_{k\in\mathbb{Z}^n}$$

on $\mathbb{R}^n$ and the norm

$$\|f\|_{\mathcal{D}^{\mathcal{F}}(\mathcal{Q},L^p,l^p)} := \left\|\left(\left\|\mathcal{F}^{-1}\left(\mathcal{F}f\cdot\varphi_k\right)\right\|_{L^p(\mathbb{R}^n)}\right)_{k\in\mathbb{Z}^n}\right\|_{l^p(\mathbb{Z}^n)}, \tag{2.2.2}$$

where $\Phi := (\varphi_k)_{k\in\mathbb{Z}^n}$ is a $\mathcal{Q}$-BAPU. The norm (2.2.2) is equivalent to the norm on $M^p(\mathbb{R}^n)$ given in (1.5.1).

More generally than Example 2.2.2, we denote decomposition spaces on $\mathbb{R}^n$ that use the Fourier transform as in (2.2.2) by $\mathcal{D}^{\mathcal{F}}(\mathcal{Q},L^p,Y)$. Here Y is a general global component, $1 \le p \le \infty$, and $\mathcal{Q}$ is a concatenation on $\mathbb{R}^n$. The space $\mathcal{D}^{\mathcal{F}}(\mathcal{Q},L^p,Y)$ is called a *$\mathcal{F}$-type decomposition space*. For more details we refer the reader to Subsection A.4.1.

To weave together the material in Section 2.1 with decomposition spaces, we consider embeddings between decomposition spaces that play well with the large scale geometry of the underlying coverings. While we give a precise definition in Paper A for this, let us give an intuitive description here. As the definition we give does not depend on whether we consider decomposition spaces or $\mathcal{F}$-type decomposition spaces, we restrict ourselves to decompositions spaces for simplicity.

Let X and Z be locally compact spaces with concatenations $\mathcal{Q}$ and $\mathcal{P}$, respectively. We can consider two decomposition spaces $\mathcal{D}(\mathcal{Q},B_1,Y_1)$ and $\mathcal{D}(\mathcal{P},B_2,Y_2)$ and a Banach space embedding

$$F\colon \mathcal{D}(\mathcal{Q},B_1,Y_1) \to \mathcal{D}(\mathcal{P},B_2,Y_2).$$

The map F is called a *geometric embedding* if F induces a quasi-isometric embedding $F_*\colon (X,d_{\mathcal{Q}}) \to (Z,d_{\mathcal{P}})$ between the associated metric spaces $(X,d_{\mathcal{Q}})$ and $(Z,d_{\mathcal{P}})$. What we have described here is really a consequence of the more careful definition of a geometric embedding, see Definition A.4.5. However, it is really this consequence we are interested in. From a geometric standpoint, geometric embeddings between decomposition spaces are Banach space embeddings that respect the large scale geometric structure of the underlying coverings.

Parts of Paper A and Paper B are dedicated to determine the existence of geometric embeddings between decomposition spaces. To show non-existence, a strategy is to prove that there is no quasi-isometric embedding between the underlying metric spaces. As an example, we derive the following result in Theorem A.5.2 for the classical modulation spaces:

Theorem 2.2.3. *For any $1 \le p, q \le \infty$ there is a tower of geometric embeddings*

$$M^{p,q}(\mathbb{R}) \longrightarrow M^{p,q}(\mathbb{R}^2) \longrightarrow \cdots \longrightarrow M^{p,q}(\mathbb{R}^n) \longrightarrow \cdots,$$

while there are no geometric embeddings in the other direction. As a consequence, the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}) := M^{1,1}(\mathbb{R})$ embeds geometrically into any modulation space $M^{p,q}(\mathbb{R}^n)$ for $1 \le p, q \le \infty$ and $n \in \mathbb{N}$.

2.3 Modulation Spaces on Stratified Lie Groups

In Paper B we study α-modulation spaces on certain stratified Lie groups. As this is a technical topic, we go through modulation spaces on stratified Lie groups as a special case. We refer the reader to [27, 28, 30, 54, 91, 93, 94, 95] for more on α-modulation spaces in the Euclidean setting. Let us quickly recall some basic notions for stratified Lie groups.

Definition 2.3.1. Let G be a Lie group that is both connected and simply connected. We say that G is *stratified* if its Lie algebra $\mathfrak{g}$ can be given a *stratification*

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \quad [V_1, V_j] = \begin{cases} V_{j+1}, & \text{if } j = 1, \dots, s-1 \\ \{0\}, & \text{if } j = s \end{cases}.$$

The two numbers s and $k := \dim(V_1)$ are invariant under the choice of stratification and are called the *step* and *rank* of G, respectively. A stratified group is nilpotent, and hence also unimodular.

An important tool for stratified Lie groups is the fact that the exponential map $\exp_G : \mathfrak{g} \to G$ is a global diffeomorphism. As such, we can represent any stratified Lie group G as $(\mathbb{R}^n, *_G)$ for $n = \dim(G)$ and a product $*_G$. The product $*_G$ is polynomial and thus relatively concrete to work with. Under the identification with the exponential map we have that $L^2(G)$ is identified with $L^2(\mathbb{R}^n)$. Moreover, we can, due to the polynomial nature of the exponential map, define the Schwartz space $\mathcal{S}(G)$ on G as simply the usual space $\mathcal{S}(\mathbb{R}^n)$. Phrased another way, we have the following observation:

Neither the space $L^2(G)$ nor the space $\mathcal{S}(G)$ really sees the geometry of G. If we have another stratified group H with $\dim(G) = \dim(H) = n$, then we can identify $L^2(G) = L^2(H) = L^2(\mathbb{R}^n)$ and $\mathcal{S}(G) = \mathcal{S}(H) = \mathcal{S}(\mathbb{R}^n)$.

This leads to a natural question: *Is there a nice space (or class of spaces) on a stratified group that detects the geometry of the group?* We show in Paper B that the answer is affirmative for the α-modulation spaces. In this introductory section, we are content with defining the modulation spaces $M^{p,q}(G)$ for $1 \leq p, q \leq \infty$ that corresponds to the case $\alpha = 0$.

Recall that in Example 2.2.2 we used the covering $Q := ([-1,1]^n + k)_{k \in \mathbb{Z}^n}$ to define the modulation spaces $M^{p,q}(\mathbb{R}^n)$. To generalize this, we first need to pick an analogue to the "reference set" $[-1,1]^n$. A suitable way to do this for a general stratified Lie group $(\mathbb{R}^n, *_G)$ is to pick a *homogeneous quasi-norm* $\|\cdot\|$ and consider the reference set given by the unit ball

$$B^{\|\cdot\|}(0,1) := \{x \in \mathbb{R}^n : \|x\| < 1\}.$$

We refer to Definition B.2.2 for the definition of a homogeneous quasi-norm.

Another obstacle is that we do not necessarily have a natural analogue of $\mathbb{Z}^n$. In fact, not every stratified Lie group possess lattices. However, we can bypass this obstacle with the following construction: Consider the collection:

$$\left\{x *_G B^{\|\cdot\|}(0,1)\right\}_{x \in \mathbb{R}^n} = \left\{B^{\|\cdot\|}(x,1)\right\}_{x \in \mathbb{R}^n}. \tag{2.3.1}$$

It is obvious that the collection (2.3.1) is not an admissible covering. However, we show in Lemma B.3.4 that it is always possible to find elements $\{x_i\}_{i \in I} \subset \mathbb{R}^n$ such that the sub-collection

$$\mathcal{U}(G) := \left\{B^{\|\cdot\|}(x_i,1)\right\}_{i \in I}$$

is a concatenation.

To continue towards a definition of modulation spaces on $(\mathbb{R}^n, *_G)$ we need the existence of a $\mathcal{U}(G)$-BAPU. For this, we actually need to restrict to rational groups that have step less than or equal to two. We say that a stratified Lie group G is *rational* if there exists a lattice in G. Although we hope that this restriction can be lifted in the future, we need these conditions for now. We refer to a rational stratified Lie group of step less than or equal to two as an *admissible Lie group* for simplicity. With this in place, we can define the modulation spaces $M^{p,q}(G)$ as follows:

Definition 2.3.2. Let $(\mathbb{R}^n, *_G, \|\cdot\|)$ be an admissible Lie group with a homogeneous quasi-norm $\|\cdot\|$. Fix a $\mathcal{U}(G)$-BAPU $\Phi := (\varphi_i)_{i \in I}$ for $(\mathbb{R}^n, *_G, \|\cdot\|)$. The *generalized modulation space* $M^{p,q}(G)$ for $1 \leq p, q \leq \infty$ consists of appropriate distributions f on $\mathbb{R}^n$ that satisfy the condition

$$\|f\|_{M^{p,q}(G)} := \left\|\left(\|\mathcal{F}^{-1}\left(\varphi_i \cdot \mathcal{F}f\right)\|_{L^p}\right)_{i \in I}\right\|_{l^q(I)} < \infty.$$

We have been purposely vague with regard to what we mean by "appropriate distributions" to simplify the exposition. The reader should rest assured that all the technical details are carefully explained in Paper B. We also develop basic properties of the spaces $M^{p,q}(G)$ in Paper B. For $p = q = 1$ we obtain the space $\mathcal{S}_0(G)$ as a direct analogue to the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}^n)$. A special case of the main result of Paper B given in Theorem B.5.6 can be stated as follows:

Corollary 2.3.3. Let $(\mathbb{R}^n, *_G)$ be an admissible Lie group. Assume that the spaces $\mathcal{S}_0(G)$ and $\mathcal{S}_0(\mathbb{R}^n)$ coincide, that is,

$$\mathcal{S}_0(G) = \mathcal{S}_0(\mathbb{R}^n).$$

Then $(\mathbb{R}^n, *_G)$ is isomorphic to $(\mathbb{R}^n, +)$ where $+$ denotes the usual addition on $\mathbb{R}^n$. Hence the Feichtinger algebra can detect the geometry of the group.

Remark. Recently, there has been increased interest in extending classical time-frequency analysis to the nilpotent setting. We refer the reader to [83] where certain modulation spaces are constructed on nilpotent Lie groups from the perspective of coorbit theory.

Chapter 3

From Classical to Quantum Harmonic Analysis

In this chapter, the aim is to give a short introduction to quantum harmonic analysis. We have already started to lay the groundwork for this in Section 1.4. One can trace the origin of modern quantum harmonic analysis back to the paper [153]. While this is a rather specialized topic, we hope to convince the reader that the viewpoint of quantum harmonic analysis is both natural and valuable.

In Section 3.1 we will explain the *Fourier Wigner transform* and how this connects with the Weyl quantization and the symplectic Fourier transform. We will introduce *operator convolutions* in Section 3.2 and develop some basic properties. Finally, in Section 3.3 we will move towards the affine setting and set up the *affine Weyl quantization* and the *affine Wigner distribution*. This is the starting point for Paper D and Paper E.

3.1 The Fourier Wigner Transform

The aim of quantum harmonic analysis is to extend the basic objects in harmonic analysis, namely the Fourier transform and convolutions, to the operator setting. By doing this in the correct way, one obtains a framework able to represent objects such as the short-time Fourier transform and localization operators in an elegant manner.

In this section we will explain how one obtains an operator-theoretic version $\mathcal{F}_W$ of the symplectic Fourier transform $\mathcal{F}_\sigma$. The approach to do this is based on the representation-theoretic viewpoint we developed in Section 1.6. For a Hilbert space $\mathcal{H}$ we will denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$.

Definition 3.1.1. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a unitary representation of a locally compact group G. The *integrated representation* of π is the map

$$\tilde{\pi}\colon L^1(G) \to \mathcal{B}(\mathcal{H}_\pi)$$

given by

$$\tilde{\pi}(f)\psi := \int_G f(x) \cdot \pi(x)\psi \, d\mu_L(x), \qquad \psi \in \mathcal{H}_\pi.$$

The integrated representation satisfies the property

$$\tilde{\pi}(f *_G g) = \tilde{\pi}(f) \circ \tilde{\pi}(g),$$

where $*_G$ denotes the convolution product

$$(f *_G g)(x) := \int_G f(y) g(y^{-1}x) \, d\mu_L(y), \tag{3.1.1}$$

where $f, g \in L^1(G)$ and $x \in G$. Let us see how this construction manifests itself for the Schrödinger representation ρ of the reduced Heisenberg group.

Example 3.1.2. Consider the *integrated Schrödinger representation*

$$\tilde{\rho}\colon L^1(\mathbb{H}_r^n) \to \mathcal{B}(L^2(\mathbb{R}^n))$$

given by

$$\tilde{\rho}(f)\psi = \int_{\mathbb{R}^{2n}} \int_0^1 f(x, \omega, e^{2\pi i \tau}) e^{2\pi i \tau} e^{-\pi i x \omega} M_\omega T_x \psi \, d\tau \, dx \, d\omega.$$

Notice that any $f \in L^1(\mathbb{H}_r^n)$ of the form $f(x, \omega, e^{2\pi i \tau}) = f(x, \omega, 1)$ for all $\tau \in [0, 1]$ satisfies $\tilde{\rho}(f) = 0$. To avoid this, we restrict to the subspace $U \subset L^1(\mathbb{H}_r^n)$ of functions of the form

$$f(x, \omega, e^{2\pi i \tau}) = g(x, \omega) e^{-2\pi i \tau},$$

for some $g \in L^1(\mathbb{R}^{2n})$. Then the integrated Schrödinger representation for $f \in U$ becomes

$$\tilde{\rho}(f)\psi = \int_{\mathbb{R}^{2n}} g(x, \omega) e^{-\pi i x \omega} M_\omega T_x \psi \, dx \, d\omega. \tag{3.1.2}$$

Due to the clear correspondence $U \leftrightarrow L^1(\mathbb{R}^{2n})$ we will simply write $\tilde{\rho}(f)$ for $f \in L^1(\mathbb{R}^{2n})$ for simplicity when we consider (3.1.2).

An important observation is that the integrated Schrödinger representation $\tilde{\rho}$ is an isometry when viewed as a map

$$\tilde{\rho}\colon L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n}) \to \mathcal{HS}(L^2(\mathbb{R}^n)),$$

where the left-hand side is equipped with the L^2-norm. In fact, it follows from e.g. [68, Theorem 1.30] that $\tilde{\rho}$ extends to a unitary map

$$\tilde{\rho}\colon L^2(\mathbb{R}^{2n}) \to \mathcal{HS}(L^2(\mathbb{R}^n)).$$

Hence the following definition is well-defined.

Definition 3.1.3. For an operator $T \in \mathcal{HS}(L^2(\mathbb{R}^n))$ the function

$$\mathcal{F}_W(T) := \tilde{\rho}^{-1}(T) \in L^2(\mathbb{R}^{2n})$$

is called the *Fourier Wigner transform* of T.

The Fourier Wigner transform is thus a unitary operator

$$\mathcal{F}_W : \mathcal{HS}(L^2(\mathbb{R}^n)) \to L^2(\mathbb{R}^{2n}).$$

This should be viewed as the *Plancherel Theorem* in quantum harmonic analysis.

Definition 3.1.4. Let S be a compact operator on a separable Hilbert space $\mathcal{H}$ and let $1 \leq p \leq \infty$. We say that S is a *Schatten-class operator* of *order p* if the sequence of singular values $\{s_n(S)\}_{n\in\mathbb{N}}$ of S belongs to $l^p(\mathbb{N})$. For $p = 1$ we say that S is a *trace-class operator*. For $p = 2$ the operator S is precisely a Hilbert-Schmidt operator.

Recall that the symplectic Fourier transform $\mathcal{F}_\sigma$ maps functions in $L^1(\mathbb{R}^{2n})$ to continuous functions on $\mathbb{R}^{2n}$ that vanishes at infinity. The analogous statement for quantum harmonic analysis [125, Proposition 6.5] is that $\mathcal{F}_W(S)$ is continuous on $\mathbb{R}^{2n}$ and vanish at infinity whenever S is a trace-class operator.

Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Then for a trace-class operator S we can define the *trace* of S as the complex number

$$\operatorname{tr}(S) := \sum_{n\in\mathbb{N}} \langle Se_n, e_n\rangle_{\mathcal{H}}. \tag{3.1.3}$$

It should come as no surprise that the trace in (3.1.3) does not depend on the choice of orthonormal basis, see [67, Appendex A]. It is important to realize that any trace-class operator is also a Hilbert-Schmidt operator. For trace-class operators we have the following result, see [125, Proposition 6.2].

Proposition 3.1.5. For a trace-class operator S the Fourier Wigner transform $\mathcal{F}_W(S)$ is given by

$$\mathcal{F}_W(S)(x,\omega) = e^{-\pi i x\omega} \cdot \operatorname{tr}(M_{-\omega}T_{-x}S). \tag{3.1.4}$$

Example 3.1.6. Consider a rank-one operator $\varphi \otimes \eta$ for $\varphi, \eta \in L^2(\mathbb{R}^n)$. A simple computation based on (3.1.4) shows that

$$\mathcal{F}_W(\varphi \otimes \eta)(x,\omega) = e^{\pi i x\omega} V_\eta \varphi(x,\omega),$$

where $V_\eta\varphi$ is the short-time Fourier transform. More generally, consider a *finite-rank operator*

$$S \coloneqq \sum_{i=1}^{m} \varphi_i \otimes \eta_i$$

where $\varphi_i, \eta_i \in L^2(\mathbb{R}^n)$ for $i = 1, \dots, m$. The Fourier Wigner transform of S is given by

$$\mathcal{F}_W(S)(x,\omega) = \sum_{i=1}^{m} \mathcal{F}_W(\varphi_i \otimes \eta_i) = e^{\pi i x\omega} \sum_{i=1}^{m} V_{\eta_i}\varphi_i(x,\omega).$$

Recall from Section 1.4 that any Hilbert-Schmidt operator S on $L^2(\mathbb{R}^n)$ can be uniquely written as $S = L_\sigma$ for some $\sigma \in L^2(\mathbb{R}^{2n})$. For simplicity we write $S = S_\sigma$ when we want to emphasize the symbol σ of S. We end this section with the following elegant relationship between the Fourier Wigner transform and the Weyl quantization.

Theorem 3.1.7. *Let S_ρ be a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$ with symbol ρ. Then ρ is given by*

$$\rho = \mathcal{F}_\sigma \mathcal{F}_W(S_\rho), \tag{3.1.5}$$

where $\mathcal{F}_\sigma$ is the symplectic Fourier transform.

Proof. Recall that finite-rank operators are dense in the space of all Hilbert-Schmidt operators. Hence it suffices to prove (3.1.5) in the case of rank-one operators since both Fourier transforms $\mathcal{F}_W$ and $\mathcal{F}_\sigma$ are unitary transformations. Let $S \coloneqq \varphi \otimes \eta$ for $\varphi, \eta \in L^2(\mathbb{R}^n)$. Then, as we saw in Example 3.1.6, we have

$$\mathcal{F}_W(S)(x,\omega) = e^{\pi i x\omega} V_\eta \varphi(x,\omega)$$

for $(x,\omega) \in \mathbb{R}^{2n}$. On the other hand, it follows from Example 1.4.2 that the symbol of S is $W(\varphi, \eta)$, where W is the cross-Wigner transform. Hence the conclusion of Theorem 3.1.7 follows from (1.3.2). $\square$

There are many other properties of the Fourier Wigner transform that have been investigated. We refer the reader to [125, Proposition 6.6] for an operator-theoretic version of the Hausdorff-Young inequality for the Fourier Wigner transform. In the next section, we will see how the Fourier Wigner transform interacts with certain operator convolutions.

3.2 Operator Convolutions

Central to classical harmonic analysis is the notion of convolution. Recall that for a locally compact group G the space $L^1(G)$ is an algebra under the convolution product $*_G$ in (3.1.1). An important property is that $*_G$ detects whether or not the group G is commutative; the convolution product $*_G$ is commutative if and only if G is a commutative group, see [47, Theorem 1.6.4]. Moreover, in the case where $G = \mathbb{R}^{2n}$, the symplectic Fourier transform $\mathcal{F}_\sigma$ satisfies the formula

$$\mathcal{F}_\sigma(f *_{\mathbb{R}^{2n}} g) = \mathcal{F}_\sigma f \cdot \mathcal{F}_\sigma g \tag{3.2.1}$$

for $f, g \in L^1(\mathbb{R}^{2n})$. If there is no cause for confusion, we will simply write $f * g$ instead of $f *_{\mathbb{R}^{2n}} g$ for the convolution on the group $\mathbb{R}^{2n}$.

The goal of this section is to extend the convolution (3.1.1) to the setting of operators. Firstly, we would like to define a convolution between a function f and an operator S. To mimic (3.1.1) we need to be able to shift the operator S in an appropriate way.

Definition 3.2.1. Write $z := (x, \omega) \in \mathbb{R}^{2n}$ and $\pi(z) := M_\omega T_x$. Then we can for any bounded operator $A \in \mathcal{B}(L^2(\mathbb{R}^n))$ define the *shift* $\alpha_z(A)$ by z as the bounded operator given by the conjugation

$$\alpha_z(A) := \pi(z) A \pi(z)^*.$$

The reader can verify that we have the elementary property $\alpha_z \alpha_w = \alpha_{z+w}$. We can now define the convolution between a function and an operator as follows.

Definition 3.2.2. Let $f \in L^1(\mathbb{R}^{2n})$ and let S be a trace-class operator on $L^2(\mathbb{R}^n)$. The *(function-operator) convolution* $f \star S$ is the trace-class operator on $L^2(\mathbb{R}^n)$ acting on $\varphi \in L^2(\mathbb{R}^n)$ by

$$(f \star S)\varphi := \int_{\mathbb{R}^{2n}} f(z) \alpha_z(S) \varphi \, dz. \tag{3.2.2}$$

We remark that (3.2.2) should be interpreted as a vector-valued integral, see [125, Section 2.3] for details. The fact that $f \star S$ is a trace-class operator follows from [125, Proposition 2.5]. If we have two elements $f, g \in L^1(\mathbb{R}^{2n})$, then a straightforward computation shows that the two expressions $(f * g) \star S$ and $f \star (g \star S)$ coincide.

Example 3.2.3. Let us consider the case when $S := \varphi \otimes \varphi$ for some $\varphi \in L^2(\mathbb{R}^n)$. Then for $f \in L^1(\mathbb{R}^{2n})$ we have

$$\begin{aligned}(f \star S)\psi &= \int_{\mathbb{R}^{2n}} f(z)\alpha_z(\varphi \otimes \varphi)\psi \, dz \\ &= \int_{\mathbb{R}^{2n}} f(z) V_\varphi \psi \pi(z)\varphi \, dz \\ &= \mathcal{A}_f^\varphi \psi,\end{aligned}$$

where $\mathcal{A}_f^\varphi$ is the localization operator given in (1.2.9). As such, localization operators are a special case of operators that can be investigated through operator convolutions.

We would also like to define the convolution between two operators S and T. To do this, we will first need to extend the parity operator P to the operator setting.

Definition 3.2.4. Given an operator $A \in \mathcal{B}(L^2(\mathbb{R}^n))$ we can define the *flipped operator* $\check{A}$ as

$$\check{A} := PAP^* = PAP,$$

where P is the parity operator $Pg(x) := g(-x)$ for $g \in L^2(\mathbb{R}^n)$.

It is clear that flipping an operator is an idempotent operation, i.e. $\check{\check{A}} = A$. The purpose of the parity operator is to make the convolution between two operators commutative.

Definition 3.2.5. Let S and T be two trace-class operators on $L^2(\mathbb{R}^n)$. The *(operator-operator) convolution* $S \star T$ is an integrable function on $\mathbb{R}^{2n}$ given by

$$(S \star T)(z) := \operatorname{tr}\left(S \circ \alpha_z\left(\check{T}\right)\right), \qquad z \in \mathbb{R}^{2n}.$$

We refer the reader to [125, Lemma 4.1] for a proof of the integrability of $S \star T$. The following result, which follows from a straightforward computation, shows that the new convolutions are compatible.

Theorem 3.2.6. *Let $f \in L^1(\mathbb{R}^{2n})$ and let S and T be two trace-class operators on $L^2(\mathbb{R}^n)$. Then*

$$(f \star S) \star T = f * (S \star T).$$

Moreover, we have $S \star T = T \star S$.

Central to the theory of quantum harmonic analysis is the following result, mimicking the classical relationship (3.2.1). For a proof we refer the reader to [125, Proposition 6.4].

Proposition 3.2.7. Let $f \in L^1(\mathbb{R}^{2n})$ and consider two trace-class operators S and T on $L^2(\mathbb{R}^n)$. We have the *decoupling equations*

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma f \cdot \mathcal{F}_W(S)$$
$$\mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S) \cdot \mathcal{F}_W(T).$$

Example 3.2.8. Assume that $S := \varphi \otimes \eta$ for $\varphi, \eta \in L^2(\mathbb{R}^n)$. What can we say about the function $S \star S^*$? By using Proposition 3.2.7 and Example 3.1.6 we have that

$$\begin{aligned}\mathcal{F}_\sigma(S \star S^*)(x, \omega) &= \mathcal{F}_W(\varphi \otimes \eta)(x, \omega) \cdot \mathcal{F}_W(\eta \otimes \varphi)(x, \omega) \\ &= e^{2\pi i x \omega} V_\eta \varphi(x, \omega) \cdot V_\varphi \eta(x, \omega) \\ &= V_\eta \varphi(x, \omega) \cdot \overline{V_\eta \varphi(-x, -\omega)}.\end{aligned}$$

If the pair (φ, η) is such that $V_\eta \varphi(-x, -\omega) = V_\eta \varphi(x, \omega)$ for all $(x, \omega) \in \mathbb{R}^{2n}$, then

$$\mathcal{F}_\sigma(S \star S^*) = \mathrm{Spec}_\eta \varphi.$$

In this case, it follows from (1.2.8) that

$$S \star S^* = V_\varphi \varphi \cdot \overline{V_\eta \eta}.$$

To exemplify, if $\varphi(x) = \eta(x) = 2^{\frac{n}{4}} e^{-\pi |x|^2}$ then we have that

$$S \star S^*(x, \omega) = e^{-\pi(|x|^2 + |\omega|^2)}.$$

3.3 Turning Towards the Affine Setting

In Paper D and Paper E we are concerned with an affine version of the Weyl quantization. In Paper E we also consider operator convolutions and an affine Fourier Wigner transform. The end-result is that we build a full-fledged affine quantum harmonic analysis framework. As a setup, we will go through some constructs on the affine group and define the affine Weyl quantization in a rigorous manner.

Recall the affine group Aff from Example 1.6.8. To elaborate, the affine group Aff and its Lie algebra $\mathfrak{aff}$ can be given the matrix representations

$$\mathrm{Aff} = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} : a > 0,\, x \in \mathbb{R} \right\}, \quad \mathfrak{aff} = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} : \, u, v \in \mathbb{R} \right\}.$$

The affine group is not unimodular; the left and right Haar measures are given by

$$\mu_L(x, a) = \frac{dx\, da}{a^2}, \qquad \mu_R(x, a) = \frac{dx\, da}{a}.$$

The wavelet representation $\pi\colon \mathrm{Aff} \to \mathcal{U}(L^2(\mathbb{R}))$ in (1.6.2) gives rise to the continuous wavelet transform $\mathcal{W}_g f$ for $f, g \in L^2(\mathbb{R})$ defined in (1.6.3). Rather than working with π, we work with a unitary representation defined as follows: The unitary representation $U\colon \mathrm{Aff} \to \mathcal{U}(L^2(\mathbb{R}_+))$ of the affine group Aff is given by

$$U(x,a)\psi(r) := e^{2\pi i x r}\psi(ar), \qquad \psi \in L^2(\mathbb{R}_+).$$

We remark that $L^2(\mathbb{R}_+)$ is the space of square integrable functions on the multiplicative group $\mathbb{R}_+ := (0,\infty)$ with the Haar measure $r^{-1}\,dr$. Using the representation U we can form the following operator.

Definition 3.3.1. Let $\lambda\colon \mathbb{R} \to \mathbb{R}_+$ denote the function

$$\lambda(u) := \frac{ue^u}{e^u - 1}, \qquad u \in \mathbb{R}.$$

We can for each point $(x,a) \in \mathrm{Aff}$ form the *Stratonovich-Weyl operator* $\Omega(x,a)$ as the densely defined operator given on $L^2(\mathbb{R}_+)$ by

$$\Omega(x,a)\psi(r) := a\int_{\mathbb{R}^2} e^{-2\pi i(xu+av)}U\left(\frac{ve^u}{\lambda(u)}, e^u\right)\psi(r)\,du\,dv.$$

To get a quantization scheme, we will need to consider the space $L^2_r(\mathrm{Aff})$ consisting of measurable functions $f\colon \mathrm{Aff} \to \mathbb{C}$ that satisfy

$$\|f\|^2_{L^2_r(\mathrm{Aff})} := \int_{\mathrm{Aff}} |f(x,a)|^2\,d\mu_R(x,a) = \int_{-\infty}^{\infty}\int_0^{\infty} |f(x,a)|^2\,\frac{da\,dx}{a} < \infty.$$

Using the Stratonovich-Weyl operator we can form the following correspondence: For $f \in L^2_r(\mathrm{Aff})$ we consider the operator $A_f : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ given by

$$A_f\psi(r) := \int_{-\infty}^{\infty}\int_0^{\infty} f(x,a)\Omega(x,a)\psi(r)\,\frac{da\,dx}{a}, \qquad \psi \in L^2(\mathbb{R}_+).$$

We refer to $f \mapsto A_f$ as the *affine Weyl quantization*. We have the following result from [73]:

Proposition 3.3.2. The affine Weyl quantization $f \mapsto A_f$ is a bijective isometry between $L^2_r(\mathrm{Aff})$ and the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$. Given a Hilbert-Schmidt operator A on $L^2(\mathbb{R}_+)$ we can calculate the inverse of the affine Weyl quantization as

$$f_A(x,a) = \int_{-\infty}^{\infty} A_K\left(a\lambda(u), a\lambda(-u)\right)e^{-2\pi i x u}\,du,$$

where $A_K \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ is the integral kernel of A. The element f_A is called the *affine Weyl symbol* of A.

Using the affine Weyl quantization, we can define a notion of an affine Wigner distribution motivated by the relation (1.4.2). More specifically, by considering the affine Weyl symbol of a rank-one operator we obtain the following definition.

Definition 3.3.3. For $\phi, \psi \in L^2(\mathbb{R}_+)$ we define the *affine (cross-)Wigner distribution* $W_{\mathrm{Aff}}^{\psi,\phi}$ by the formula

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) := \int_{-\infty}^{\infty} \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}e^{-2\pi i x u}\, du,$$

where $(x,a) \in \mathrm{Aff}$.

Remark. An important observation is that the affine Wigner distribution and the affine Weyl quantization are defined using the right Haar measure μ_R. On the other hand, the continuous wavelet transform $\mathcal{W}_g f$ of $f, g \in L^2(\mathbb{R})$ is heavily linked with the left Haar measure μ_L, see e.g. [44, Proposition 2.4.1]. This dichotomy is invisible in the classical Euclidean setting since the Heisenberg group is unimodular.

Much of Paper D is dedicated to understanding subtle features of the affine Wigner distribution. The results developed in Paper D are used in Paper E to develop a quantum harmonic analysis framework for the affine setting. We remark that the affine Wigner distribution has been present in the engineering literature for many years, see Section D.4 for a brief overview.

Chapter 4

Summary of Papers

The main scientific contribution of the thesis consists of the five papers presented in Part II. All five papers have been given minor modification from their published/accepted counterparts to ensure consistent notation throughout the thesis.

Before commencing, we give a succinct overview of each of the papers from Section 4.1 to Section 4.5. It is assumed that the reader has read the previous chapters of the thesis so that we can utilize the built up terminology. For a more in-depth overview of each paper, we refer the reader to the introductions of the respective papers. Finally, we outline open problems and further directions of study in Section 4.6.

4.1 Paper A—A Large Scale Approach to Decomposition Spaces [19]

In Paper A we investigate general decomposition spaces through the lens of large scale geometry. We build on the definitions and results presented in Section 2.1 and Section 2.2. One of the main goals of the paper is to identify obstructions to the existence of geometric embeddings between decomposition spaces defined on different underlying sets. While there is not a singular main result in Paper A, we would like to highlight Theorem A.4.9 describing the existence of *spatially implemented* geometric embeddings between decomposition spaces.

The reader has already seen a concrete example illustrating the general theory developed in Paper A in Theorem 2.2.3. Other examples include Besov spaces in Proposition A.4.10 and decomposition spaces on the Heisenberg group in Proposition A.5.4. Throughout the paper, we employ techniques from large scale geometry to derive the results. The paper is influenced by the works [59, 115].

4.2 Paper B—α-Modulation Spaces for Step Two Stratified Lie Groups [14]

In Paper B we investigate modulation spaces (and more generally α-modulation spaces) on a particular class of nilpotent Lie groups. The paper builds on the ideas presented in Paper A and Section 2.3. The main aim of Paper B is to determine whether the new function spaces on these nilpotent Lie groups are in fact distinct from their Euclidean counterparts. The paper has a clear main result, namely Theorem B.5.6. A special case of Theorem B.5.6 was presented in Corollary 2.3.3. We also emphasize throughout Paper B that α-modulation spaces on certain nilpotent Lie groups can be realized as distributions on Euclidean space. This makes them more concrete to work with and less intimidating for readers unfamiliar with nilpotent Lie groups.

Throughout the paper, tools from large scale geometry are used to prove results that would otherwise be difficult to approach. The notion of geometric embeddings introduced in Paper A is investigated in Section B.6 in the nilpotent setting. We emphasize that the new α-modulation spaces can be realized on Euclidean spaces, making them less obscure. The paper is influenced by the works [56, 64].

4.3 Paper C—Interpolation in Wavelet Spaces and the HRT-Conjecture [16]

In Paper C we attempt to extend the understanding of general wavelet spaces as defined in (1.7.2). Of particular interest are the Gabor spaces defined in (1.7.1). One of the main results is Theorem C.4.2, showing that wavelet spaces are either equal or completely distinct in the sense of having trivial intersection. A quick consequence is Corollary C.4.5, showing that differences between "wavelet functions" are never non-zero functions of positive type.

Another interesting point is the connection in Proposition C.6.1 between the HRT-Conjecture and whether the Gabor spaces are fully interpolating as reproducing kernel Hilbert spaces. While there has been much work towards the solution of the HRT-Conjecture in the last decades, we hope that involving reproducing kernel Hilbert space theory can shed some light on the problem. Many of the results in the paper are derived by using techniques from representation theory and reproducing kernel Hilbert spaces. The paper is influenced by the works [77, 124].

4.4 Paper D—The Affine Wigner Distribution [17]

In Paper D we investigate an affine Wigner distribution through the affine Weyl quantization described in Section 3.3. While we formalize certain results present in the engineering literature, we also develop new results that we believe are of interest. As an example, we show existence and uniqueness results regarding a minimization problem for the affine Wigner distribution in Theorem D.8.1. In fact, the precise number of minimizers of the minimization problem did not seem to be known even for the Euclidean minimization problem (1.4.3).

Another result of interest is Theorem D.5.1, showing the connection between the scalogram and the affine Wigner distribution. This result is the affine analogue of (1.3.6). In Proposition D.7.3 we show, as a side-effect, that the affine Wigner distribution can not produce any analytic functions. This is in stark contrast with the Euclidean Wigner distribution we considered in Section 1.3. We approach the affine Wigner distribution from a more functional analytic standpoint than many other sources. As such, proofs for various results emphasize arguments from functional analysis rather than concrete computations. The paper is influenced by the works [12, 73].

4.5 Paper E—Affine Quantum Harmonic Analysis [18]

In Paper E we extend the quantum harmonic analysis framework described in Section 3.1 and Section 3.2 to the affine setting. The goal is to represent concrete operators, e.g. *affine localization operators*, in the language of operator convolutions. One of the main results is Theorem E.3.20, showing that affine convolution can be used to represent the affine Weyl quantization. Another pleasant result is Theorem E.3.10 showing that coordinate functions are correctly quantized in the affine setting by our approach. This indicates that our choice of quantization on the affine group is the right one.

Quantum harmonic analysis in the affine setting is greatly affected by the non-unimodularity of the affine group. We develop a theory of admissible operators in the non-unimodular setting in Section E.4. This concept reduces to the traditional notion of square integrability when considering rank-one operators. Although we primarily use techniques from functional analysis to derive new results, we also employ representation theory in many parts of the paper. The paper is influenced by the works [76, 125, 153].

4.6 Open Problems for Further Research

The five papers in Part II generate several new problems and directions of interest. Let us, to intrigue the reader, mention a few of these:

Paper A: In Subsection A.5.3 we consider a decomposition space of *hyperbolic type*. Specifically, we consider a decomposition space on the special linear group $SL(2,\mathbb{R})$ where the underlying metric space is *quasi-hyperbolic*. Almost nothing is known about this space, or more general decomposition spaces of hyperbolic type. An interesting direction for further research is to find relations between decomposition spaces of hyperbolic type and other well-established mathematical objects.

Paper B: The obvious open problem is to extend the main result, namely Theorem B.5.6, to nilpotent Lie groups with higher step than two. Other than this, Section B.7 discusses many directions of interest to pursue. One particularly appealing direction is to investigate the Feichtinger algebra $\mathcal{S}_0(G)$ for an admissible Lie group G. Another problem is to determine whether the modulation spaces $M^{p,q}(G)$ in Definition 2.3.2 can be given a coorbit description. We refer the reader to the author's survey [15] for an introduction to coorbit theory.

Paper C: Except for the HRT-Conjecture, we consider the following problem in Section C.7: Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation and let $\mathcal{A}_\pi$ denote the equivalence classes of admissible vectors in $\mathcal{H}_\pi$ modulo rotations by elements of $\mathbb{T}$. We denote by $\widehat{G}_s$ the equivalence classes of square integrable representations of G. Can we characterize the groups G such that

$$\overline{\bigoplus_{\pi\in\widehat{G}_s} \operatorname*{span}_{g\in\mathcal{A}_\pi}\left\{\mathcal{W}_g f : f \in \mathcal{H}_\pi\right\}} = L^2(G)?$$

Groups G satisfying the above criterion are called *wavelet complete*. We refer the reader to Section C.7 for some remarks and examples regarding wavelet completeness.

Paper D: In Section D.9 we discuss two open questions. The most pressing one concerns the non-negativity of the affine Wigner distribution. Motivated by Theorem 1.3.2 the goal is to characterize the functions $\psi \in L^2(\mathbb{R}_+)$ such that the affine Wigner distribution W^{ψ}_{Aff} is non-negative on the affine group Aff. We conjecture that this is the case precisely when ψ is a *generalized Klauder wavelet* on the form

$$\psi(r) := Cr^{-i(x+ia)}e^{i(y+ib)r} \qquad C\in\mathbb{C},\ (x,a),(y,b)\in \mathrm{Aff}.$$

Paper E: The most prominent direction to work on would be to generalize admissible operators to the general non-unimodular setting. Moreover, since affine quantum harmonic analysis is so new, it would be beneficial to connect it with other mathematical objects. We refer the reader to Section E.6 for connections to affine localization operators, covariant integral quantizations, and affine Cohen class operators.

Part II

Research Papers

Paper A

A Large Scale Approach to Decomposition Spaces

Eirik Berge and Franz Luef

Accepted in *Studia Mathematica*

Paper A

A Large Scale Approach to Decomposition Spaces

Abstract

Decomposition spaces are a class of function spaces constructed out of "well-behaved" coverings and partitions of unity of a set. The structure of the covering determines the properties of the decomposition space. Besov spaces, shearlet spaces, and modulation spaces are well known decomposition spaces. In this paper, we focus on the geometric aspects of decomposition spaces and utilize that these are naturally captured by the large scale properties of a metric space associated to the covering. We demonstrate that decomposition spaces constructed out of quasi-isometric covered spaces have many geometric features in common.

The notion of geometric embedding is introduced to formalize the way one decomposition space can be embedded into another decomposition space while respecting the geometric features of the coverings. Some consequences of the large scale approach to decomposition spaces are (i) comparison of coverings of different sets, (ii) study of embeddings of decomposition spaces based on the geometric features and the symmetries of the coverings, and (iii) the use of notions from large scale geometry, such as asymptotic dimension or hyperbolicity, to study the properties of decomposition spaces.

A.1 Introduction

Large scale geometry has its origins in the seminal work of Gromov in [86, 88] and has led to substantial progress in group theory, operator algebras, and geometry. In this paper we add another item to the long list of applications of large scale geometry: The theory of function spaces, in particular the decomposition spaces of Feichtinger and Gröbner [56, 59]. The link between decomposition spaces and coarse geometry has also been pointed out in the Ph.D. thesis of Koch [115].

Several function spaces in time-frequency analysis and harmonic analysis possess a description through a geometric decomposition of the domain space. These spaces are referred to as *decomposition spaces* and contain among them Besov spaces and modulation spaces. Since the inception of decomposition spaces in [59], a fundamental question has been to decide whether one decomposition space embeds into another decomposition space. The first serious study of embeddings between decomposition spaces was in [80].

Embedding questions have mostly been considered when the two decomposition spaces in question consist of functions/distributions on the same underlying space, e.g. [25, 54, 93, 118]; an exception is the tour de force paper [148] where many results treat the case where the underlying spaces are different open subsets of the same ambient Euclidean space with non-empty intersection. We will investigate embeddings of a geometric nature between decomposition spaces defined on entirely different sets by utilizing tools from large scale geometry.

Let us briefly sketch the construction of decomposition spaces, see Section A.4 for details. Consider a well-behaved covering $Q := (Q_i)_{i\in I}$ on a set X and consider a partition of unity $\Phi := (\varphi_i)_{i\in I}$ subordinate to the covering Q. Decomposition spaces consist of functions $f: X \to \mathbb{C}$ that have nice local behavior with respect to the partition Φ measured in terms of a Banach space $(B, \|\cdot\|_B)$: This local information is encoded in the sequence

$$f_i := \|f \cdot \varphi_i\|_B, \qquad i \in I.$$

Furthermore we want to ensure global regularity of f, which we obtain by imposing the sequence $(f_i)_{i\in I}$ to be an element of a suitable sequence space $(Y, \|\cdot\|_Y)$ over the index set I. Hence, the *decomposition space* $\mathcal{D}(Q, B, Y)$ is the space of functions such that

$$\|f\|_{\mathcal{D}(Q,B,Y)} := \|(f_i)_{i\in I}\|_Y < \infty.$$

The way to relate decomposition spaces with large scale geometry is to associate to any well-behaved covering Q on X a metric space (X, d_Q). The metric $d_Q(x, y)$ essentially counts the minimum number of borders of the sets Q_i one needs to cross when going from x to y. The important features of the covering Q are encapsulated in the metric space structure of (X, d_Q).

Recall that a map $F: (X, d_X) \to (Z, d_Z)$ between metric spaces is called a *quasi-isometric embedding* if there exist constants $L, C > 0$ such that

$$\frac{1}{L} d_X(x, y) - C \leq d_Z(F(x), F(y)) \leq L d_X(x, y) + C,$$

for all $x, y \in X$. Quasi-isometric embeddings are generalizations of isometric embeddings that allow the spaces to be locally different as long as they have the same global behavior. There is a standard notion of *equivalence* between coverings $\mathcal{Q}, \mathcal{P}$ on the same space X present in the literature [36, 59, 118, 148]. We give a new proof of Proposition A.2.7 stating that the coverings $\mathcal{Q}$ and $\mathcal{P}$ are equivalent if and only if the identity map $Id_X\colon (X, d_{\mathcal{Q}}) \to (X, d_{\mathcal{P}})$ is a bijective quasi-isometric embedding. The statement goes back to the paper [59] and has been recently proved in a special case in the Ph.D. thesis of Koch [115] where its purpose was to compare decomposition spaces with coarse geometric methods. This framework provides a natural extension of equivalent coverings to coverings defined on different sets.

The functorial way of associating the metric space $(X, d_{\mathcal{Q}})$ to the space X equipped with the well-behaved covering $\mathcal{Q}$ allows us to consider quasi-isometric invariant properties of the covering $\mathcal{Q}$. In particular, we discuss the *asymptotic dimension*, *growth type*, and *quasi-hyperbolicity* of a well-behaved covering. These properties will be used time and time again in later sections to simplify arguments already present in the literature.

There is a canonical way of associating to a path-connected, locally compact group G a covering $\mathcal{U}(G)$ reflecting the group operation introduced in [56]. We will show in Theorem A.3.2 that we can reduce the problem of understanding the covering $\mathcal{U}(G)$ to the study of the asymptotic dimension, the growth type, or the hyperbolicity of certain finitely generated subgroups of G. This is explored in more detail for stratified Lie groups in Proposition A.3.4 and solvable groups in Proposition A.3.10, where the finitely generated subgroups are respectively nilpotent and strongly polycyclic.

For a stratified Lie group G and a lattice N in G, we establish in Theorem A.3.6 a correspondence between the growth type of the metric space $(N, d_{\mathcal{U}(G)})$ and the *homogeneous dimension* of the stratified Lie group G. We remark that in a subsequent paper [14], the first author constructs decomposition spaces (specifically α-modulation spaces) on a wide range of stratified Lie groups. Techniques developed in this paper are heavily used in [14] to deduce non-trivial results regarding the resulting decomposition spaces.

Consider two decomposition spaces $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$ related to the coverings $\mathcal{Q}$ and $\mathcal{P}$ on the locally compact spaces X and Z, respectively. We will investigate the existence of Banach space embeddings

$$F\colon \mathcal{D}(\mathcal{Q}, B_1, Y_1) \to \mathcal{D}(\mathcal{P}, B_2, Y_2)$$

that induce a quasi-isometric embedding between the metric spaces $(X, d_{\mathcal{Q}})$ and $(P, d_{\mathcal{P}})$. These embeddings are called *geometric embeddings* and are introduced in Subsection A.4.2. Two highlights are Proposition A.4.6, showing that geometric embeddings induce quasi-isometric embeddings of the underlying coverings, and Theorem A.4.9, showing when quasi-isometries between the metric spaces $(X, d_{\mathcal{Q}})$ and $(Z, d_{\mathcal{P}})$ can induce geometric embeddings between the decomposition spaces

$\mathcal{D}(Q, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$.

In Section A.5 we look at geometric embeddings between well known decomposition spaces such as the modulation spaces $M^{p,q}(\mathbb{R}^n)$. In Theorem A.5.2 we show that there is a tower of compatible geometric embeddings

$$M^{p,q}(\mathbb{R}) \xrightarrow{\Gamma_1^2} M^{p,q}(\mathbb{R}^2) \xrightarrow{\Gamma_2^3} \cdots \xrightarrow{\Gamma_{n-1}^n} M^{p,q}(\mathbb{R}^n) \xrightarrow{\Gamma_n^{n+1}} \cdots,$$

while there are no geometric embeddings in the other direction. Combining this result with [81, Theorem 12.2.2] shows that there exists a geometric embedding from the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}) := M^{1,1}(\mathbb{R})$ to any of the modulation spaces $M^{p,q}(\mathbb{R}^n)$.

Finally, we consider in Subsection A.5.3 the decomposition space

$$\mathcal{D}^{p,q}\,(SL(2,\mathbb{R})) := \mathcal{D}(\mathcal{U}(SL(2,\mathbb{R})), L^p, l^q)$$

on the semisimple Lie group $SL(2,\mathbb{R})$. The associated metric space $(SL(2,\mathbb{R}), d_{\mathcal{U}})$ is quasi-hyperbolic by Proposition A.3.11 and we show in Proposition A.5.7 that the decomposition space $\mathcal{D}^{p,q}\,(SL(2,\mathbb{R}))$ is radically different from the modulation spaces and Besov spaces.

In order to make this paper accessible for a broad audience we have included basic results and definitions from large scale geometry. These are given when they are needed rather than including them in an appendix since there are several excellent introductory texts available [121, 133].

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A.2 From Admissible Coverings to the Large Scale Setting

A.2.1 Covered Spaces and Associated Metric Spaces

The first order of business is to associate a metric space to any sufficiently nice covering. Let X be a non-empty set. A collection $Q := (Q_i)_{i\in I}$ of non-empty subsets of X is called an *admissible covering* if it is a covering of X such that

$$N_Q := \sup_{i\in I} |i^*| < \infty, \qquad i^* := \left\{ j \in I : Q_i \cap Q_j \neq \emptyset \right\}.$$

We will call the constant N_Q the *admissibility constant* of the covering, while i^* is called the *set of neighboring indices* of $i \in I$. The sets Q_j for $j \in i^*$ are called the *neighbors* of the set Q_i. This notion can be inductively extended by $i^{k*} := (i^{(k-1)*})^*$ for $k \geq 2$. Moreover, the abbreviations

$$Q_i^* := \bigcup_{j \in i^*} Q_j, \qquad Q^{k*} := \left(Q^{(k-1)*}\right)^*, \quad k \geq 2$$

will be used to ease the notation. Note that $i \in j^{k*}$ if and only if $j \in i^{k*}$ for all $k \geq 1$. For other properties of neighboring indices, see [59, Lemma 2.1] [1].

We call a sequence $Q_{i_1}, \dots, Q_{i_k} \in \mathcal{Q}$ with $x \in Q_{i_1}$ and $y \in Q_{i_k}$ a $\mathcal{Q}$-*chain* from x to y of *length* k whenever $Q_{i_l} \cap Q_{i_{l+1}} \neq \emptyset$ for every $1 \leq l \leq k-1$. The notation $\mathcal{Q}(k, x, y)$ will be used to denote all $\mathcal{Q}$-chains of length k from x to y. We will need one additional assumption on admissible coverings so that we can associate to them metric spaces in a natural manner.

Definition A.2.1. An admissible covering $\mathcal{Q}$ on a set X is called a *concatenation* if for every pair of points $x, y \in X$ there exists a positive number $k \in \mathbb{N}$ such that $\mathcal{Q}(k, x, y) \neq \emptyset$. We will refer to the pair $(X, \mathcal{Q})$ as a *covered space* whenever $\mathcal{Q}$ is a concatenation on X.

The notion of a concatenation first appeared in [59] and is equivalent to the requirement that

$$X = \bigcup_{k=1}^{\infty} Q_i^{k*},$$

for some (and hence all) $Q_i \in \mathcal{Q}$.

Definition A.2.2. Define the metric $d_{\mathcal{Q}}$ on the covered space $(X, \mathcal{Q})$ by the rule

$$d_{\mathcal{Q}}(x, x) = 0, \quad d_{\mathcal{Q}}(x, y) = \inf\{k : \mathcal{Q}(k, x, y) \neq \emptyset\}, \qquad x, y \in X, \ x \neq y.$$

The defining properties of a covered space ensure that $(X, d_{\mathcal{Q}})$ is a metric space. We will refer to $(X, d_{\mathcal{Q}})$ as the *associated metric space* to the covered space $(X, \mathcal{Q})$.

The metric space $(X, d_{\mathcal{Q}})$ was introduced in [59] together with a few basic properties. Notice that $(X, d_{\mathcal{Q}})$ is a uniformly discrete metric space since $d_{\mathcal{Q}}(x, y) \geq 1$ whenever x and y are distinct points.

A common way of comparing two coverings on the same space is as follows: Let X be a set equipped with two admissible coverings $\mathcal{Q} := (Q_i)_{i \in I}$ and $\mathcal{P} := (P_j)_{j \in J}$. We say that $\mathcal{Q}$ is *almost subordinate* to $\mathcal{P}$ and write $\mathcal{Q} \leq \mathcal{P}$ if there exists a $k \in \mathbb{N}$ such that for every $i \in I$ there is a $j \in J$ with $Q_i \subset P_j^{k*}$. The coverings $\mathcal{Q}$ and $\mathcal{P}$ are said to be *equivalent* if both $\mathcal{Q} \leq \mathcal{P}$ and $\mathcal{P} \leq \mathcal{Q}$ hold. It follows that any admissible covering $\mathcal{Q}$ on a set X is equivalent to the covering $\mathcal{Q}^{k*} := \{Q_i^{k*} \mid i \in I\}$ for any $k \geq 1$. So far in the study of decomposition spaces, only coverings on the same set have been compared in the literature.

[1] The reader should be aware that first statement in [59, Lemma 2.1] is false.

Definition A.2.3. A *(metric) net* in a metric space (X, d_X) is a subset N of X such that there exists a constant $M > 0$ with

$$\inf_{y \in N} d_X(x, y) \leq M,$$

for every $x \in X$. A map $F\colon (X, d_X) \to (Z, d_Z)$ between metric spaces is called a *quasi-isometric embedding* if there exist constants $L, C > 0$ such that

$$\frac{1}{L} d_X(x, y) - C \leq d_Z(F(x), F(y)) \leq L d_X(x, y) + C,$$

for all $x, y \in X$. The constants L, C are called the *parameters* of the quasi-isometric embedding. The map F will be called a *quasi-isometry* if it in addition satisfies that $F(X)$ is a net in Z.

The notation $(X, d_X) \simeq (Z, d_Z)$ indicates that there exists a quasi-isometry between the metric spaces (X, d_X) and (Z, d_Z). It is common to refer to the quasi-isometry class of a metric space as its *large scale geometry*. A quasi-isometric embedding can have discontinuities and need not be injective. We can always choose the parameters L and C of a quasi-isometric embedding to be integers by enlarging them.

Two maps $F, G\colon (X, d_X) \to (Z, d_Z)$ between metric spaces are said to be *close* if there exists a constant $C > 0$ such that

$$d_Z(F(x), G(x)) < C,$$

for every $x \in X$. It follows from [121, Proposition 5.1.10] that a quasi-isometric embedding $F\colon (X, d_X) \to (Z, d_Z)$ is a quasi-isometry if and only if there exists a quasi-isometric embedding $G\colon (Z, d_Z) \to (X, d_X)$ such that $G \circ F$ and $F \circ G$ are close to their respective identity maps.

Example A.2.4. It is illustrative to see that the class of metric spaces that can be obtained as the associated metric space of a covered space is rather large. Let G be a finitely generated group with a symmetric generating set Σ that contains the identity element of G. We obtain a left-invariant metric d_G on G by defining

$$d_G(g, h) := \min \left\{ n : g^{-1}h = \sigma_1 \cdots \sigma_n,\ \sigma_i \in \Sigma \right\}. \tag{A.2.1}$$

Consider the covering $Q := (g\Sigma)_{g \in G}$ on G. The admissibility condition is satisfied due the cardinality of the generating set Σ.

To see that Q is a concatenation it suffices to connect the identity to an arbitrary element $g := \sigma_1 \cdots \sigma_k$ where $\sigma_i \in \Sigma$ for $i = 1, \ldots, k$. The chain

$$\Sigma, \sigma_1\Sigma, \sigma_1\sigma_2\Sigma, \ldots, g\Sigma$$

connects the identity to g and we have

$$\sigma_1 \cdots \sigma_{s+1} \in (\sigma_1 \cdots \sigma_s \Sigma) \cap (\sigma_1 \cdots \sigma_{s+1} \Sigma)\,,$$

for every $1 \leq s \leq k-1$. Hence $(G, \mathcal{Q})$ is a covered space where the identity $Id_G\colon (G, d_{\mathcal{Q}}) \to (G, d_G)$ is a quasi-isometry. Moreover, it follows from the result [133, Theorem 1.3.12] that any other choice of finite generating set than Σ in (A.2.1) would give a quasi-isometric metric space.

Proposition A.2.5. Let $(X, \mathcal{Q})$ and $(Z, \mathcal{P})$ be covered spaces. Then a map

$$F\colon (X, d_{\mathcal{Q}}) \to (Z, d_{\mathcal{P}})$$

is a quasi-isometric embedding if and only if there exist constants $L, C \in \mathbb{N}$ such that

$$\mathcal{Q}(L(k+C), x, y) \neq \emptyset, \qquad \mathcal{Q}\left(\left\lfloor \frac{k-C}{L} \right\rfloor, x, y\right) = \emptyset, \tag{A.2.2}$$

for every $x, y \in X$, where k is the smallest natural number such that

$$\mathcal{P}(k, F(x), F(y)) \neq \emptyset.$$

Proof. Let $F\colon (X, \mathcal{Q}) \to (Z, \mathcal{P})$ be a map that satisfies (A.2.2). Fix $x, y \in X$ and choose the smallest $k \in \mathbb{N}$ such that $d_{\mathcal{P}}(F(x), F(y)) \leq k$. Then we have $\mathcal{P}(k, F(x), F(y)) \neq \emptyset$ and it follows that

$$\mathcal{Q}(L(k+C), x, y) \neq \emptyset.$$

Hence $d_{\mathcal{Q}}(x, y) \leq L(k+C)$. The upper bound in the definition of a quasi-isometric embedding is verified similarly. Conversely, let $F\colon (X, d_{\mathcal{Q}}) \to (Z, d_{\mathcal{P}})$ be a quasi-isometric embedding with integer parameters $L, C > 0$. Fix $x, y \in X$ and let $k := d_{\mathcal{P}}(F(x), F(y))$. Then we have

$$L(k+C) \geq d_{\mathcal{Q}}(x, y) \geq \frac{k-C}{L}.$$

These inequalities imply that F satisfies (A.2.2) by the definition of the distance function $d_{\mathcal{Q}}$. $\square$

A metric space (X, d) is *coarsely connected* if there exists a constant $c > 0$ such that for any two points $x, y \in X$ there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) \leq c$ for $i = 0, \ldots, n-1$. Coarse connectedness is a property that is invariant under quasi-isometries. It is clear from the construction that the associated metric space $(X, d_{\mathcal{Q}})$ of a covered space $(X, \mathcal{Q})$ is coarsely connected.

Example A.2.6. Consider $\mathbb{N}_0$ with the metric

$$d(n, m) := \max\{n, m\}, \quad \text{when } n \neq m$$

and $d(n, n) := 0$ for any $n, m \in \mathbb{N}_0$. Clearly $(\mathbb{N}_0, d)$ is a uniformly discrete metric space. However, for $m > 1$ we have $d(1, m) = m$ and $d(n, m) \geq m$ for every $n \in \mathbb{N}_0$. Since we can pick m arbitrary large the metric space $(\mathbb{N}_0, d)$ is not coarsely connected. Therefore, the metric space $(\mathbb{N}_0, d)$ is not quasi-isometric to any associated metric space of a covered space.

Remark. Let $(X, \mathcal{Q})$ be a covered space with associated metric space $(X, d_{\mathcal{Q}})$. It is often more convenient to work with a smaller metric space; we do this by considering a net N in $(X, d_{\mathcal{Q}})$. The inclusion $N \hookrightarrow X$ is then a quasi-isometry when we restrict the metric $d_{\mathcal{Q}}$ to the set N. We will usually consider nets in X with *bounded geometry*, that is, nets N such that

$$|B_N(x, r)| \leq \psi(r), \quad r > 0,$$

for some function ψ that does not depend on the point $x \in N$. One option for a bounded geometry net N in $(X, d_{\mathcal{Q}})$ is picking a uniformly finite number k of points in each $Q_i \in \mathcal{Q}$. Then we have

$$|B_N(x, r)| \leq k N_{\mathcal{Q}}^r, \quad x \in N, \, r > 0.$$

The following proposition originates in the paper [59, Proposition 3.8 C)] where it was formulated in terms of bi-Lipschitz equivalences. The fact that any bijective quasi-isometry on a uniformly discrete, bounded geometry metric space is a bi-Lipschitz equivalence [121, Proposition 9.4.2] gives the transition between their statement and the one below. Prior to our investigations, a special case of the result [59, Proposition 3.8 C)] was proved in the Ph.D. thesis [115, Theorem 5.2.6] containing more details than the original source. We will give a new proof since a detailed proof of the general version of the statement is lacking in the literature.

Proposition A.2.7. Let $(X, \mathcal{Q})$ and $(X, \mathcal{P})$ be covered spaces. Then $\mathcal{Q} \leq \mathcal{P}$ if and only if the identity map $Id_X : (X, d_{\mathcal{Q}}) \to (X, d_{\mathcal{P}})$ is Lipschitz continuous. Hence the coverings $\mathcal{Q}$ and $\mathcal{P}$ are equivalent if and only if the identity map Id_X is a quasi-isometry between $(X, d_{\mathcal{Q}})$ and $(X, d_{\mathcal{P}})$.

Proof. We start by assuming that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$. For two distinct points $x, y \in X$, there exists a number $M \in \mathbb{N}$ such that $\mathcal{Q}(M, x, y) \neq \emptyset$. Pick a $\mathcal{Q}$-chain $Q_{i_1}, Q_{i_2}, \ldots, Q_{i_M}$ from x to y of length M. Then there exists a $k \in \mathbb{N}$ such that for each $l = 1, \ldots, M$ we can find $P_{j(l)} \in \mathcal{P}$ such that

$$Q_{i_l} \subset P_{j(l)}^{k*}.$$

Since $P_{j(1)}^{k*}$ has non-empty intersection with $P_{j(2)}^{k*}$ we know that

$$\operatorname{diam}_{\mathcal{P}}\left(P_{j(1)}^{k*} \cup P_{j(2)}^{k*}\right) \leq 2k.$$

Continuing this, we obtain by iteration that

$$\operatorname{diam}_{\mathcal{P}}\left(\bigcup_{l=1}^{M} P_{j(l)}^{k*}\right) \leq Mk.$$

Hence we can find a $\mathcal{P}$-chain between x and y with length at most Mk. This shows that

$$d_{\mathcal{P}}(x, y) \leq k d_{Q}(x, y),$$

and hence the identity map $Id_X : (X, d_Q) \to (X, d_{\mathcal{P}})$ is Lipschitz continuous.

Conversely, assume that $d_{\mathcal{P}}(x, y) \leq M d_Q(x, y)$ for every $x, y \in X$ and some $M > 0$. We can assume that M is an integer by enlarging it. Fix $x_0 \in X$ and choose $Q_i \in Q$ and $P_j \in \mathcal{P}$ such that $x_0 \in Q_i \cap P_j$. Then for any $y \in Q_i$ we have $d_Q(x_0, y) \leq 1$ and thus $d_{\mathcal{P}}(x, y) \leq M$. Hence there is a $\mathcal{P}$-chain

$$P_j = P_{j_1}, P_{j_2}, \ldots, P_{j_M}$$

from x_0 to y. This shows that $y \in P_j^{M*}$ for any $y \in Q_i$ and so $Q_i \subset P_j^{M*}$. Since the constant M does not depend on x and y we have that Q is almost subordinate to $\mathcal{P}$. □

The notion of quasi-isometries between associated metric spaces of covered spaces is more flexible than the notion of equivalent coverings since we can compare coverings on different sets. This will allow us to consider quasi-isometric invariant properties of covered spaces through the associated metric space in Subsection A.2.3. The motivation for considering this is to show that certain decomposition spaces can not embed nicely into other decomposition spaces in Section A.4 and Section A.5.

Example A.2.8. Consider the *uniform covering*

$$\mathcal{U} := (Q_{n_1,\ldots,n_k})_{n_1,\ldots,n_k \in \mathbb{Z}}, \qquad Q_{n_1,\ldots,n_k} := [0,1]^k + (n_1, \ldots, n_k),$$

on $\mathbb{R}^k$. It is straightforward to check that $\mathcal{U}$ is a concatenation. We will call the resulting metric space $(\mathbb{R}^k, d_{\mathcal{U}})$ the *uniform metric space* on $\mathbb{R}^k$. The set $\mathbb{Z}^k$ is a net in $(\mathbb{R}^k, d_{\mathcal{U}})$ and we have

$$d_{\mathcal{U}}((n_1, \ldots, n_k), (m_1, \ldots, m_k)) = \max\{|m_1 - n_1|, \ldots, |m_k - n_k|\},$$

where $(n_1, \ldots, n_k), (m_1, \ldots, m_k) \in \mathbb{Z}^k$. In this example a special feature emerges; the integer lattice $\mathbb{Z}^k$ is also a group that acts on itself by isometries when equipped with the metric $d_{\mathcal{U}}$. Hence the symmetries of the uniform covering $\mathcal{U}$ on $\mathbb{R}^k$ is incorporated in the metric $d_{\mathcal{U}}$ through being left (and right) invariant under the action of $\mathbb{Z}^k$.

Example A.2.9. Consider the *dyadic covering* $\mathcal{B} := \mathcal{B}(\mathbb{R}^n) := (D_m)_{m\in\mathbb{N}_0}$ on $\mathbb{R}^n$ given by the *dyadic intervals* $D_0 := \{x \in \mathbb{R}^n : \|x\|_2 \leq 2\}$ and

$$D_m := \left\{x \in \mathbb{R}^n : 2^{m-1} \leq \|x\|_2 \leq 2^{m+1}\right\}, \quad m \in \mathbb{N},$$

where $\|\cdot\|_2$ denotes the Euclidean norm. As only the magnitude of elements in $\mathbb{R}^n$ determines which dyadic interval they are in, the covering is inherently one-dimensional. Hence by picking the net

$$N := \{(2^n, \ldots, 0) : n \in \mathbb{N}_0\},$$

we have that $(\mathbb{R}^n, d_{\mathcal{B}(\mathbb{R}^n)}) \simeq (N, d_{\mathcal{B}(\mathbb{R}^n)})$ is quasi-isometric to $\mathbb{N}_0$ with its usual metric. In particular, the metric spaces $(\mathbb{R}^n, d_{\mathcal{B}(\mathbb{R}^n)})$ and $(\mathbb{R}^m, d_{\mathcal{B}(\mathbb{R}^m)})$ are quasi-isometric for all $n, m \geq 1$.

A.2.2 Incorporating the Symmetries of a Covering

We take a closer look into the symmetries of a covering implemented by group actions, as seen in Example A.2.8. First we have to introduce some terminology to describe the setting. Let G be a finitely generated group acting on a metric space (X, d_X) by isometries. For $x \in X$ and $R > 0$, the *R-stabilizer* $\mathrm{Stab}_R(x)$ is the set

$$\mathrm{Stab}_R(x) := \{g \in G : d_X(gx, x) \leq R\}.$$

We will call the action of G on (X, d_X) *large scale stable* if any non-identity element $g \in G$ satisfies

$$0 < \sup_{x\in X} d_X(gx, x) < \infty.$$

Note that a large scale stable action is actually *effective* due to the lower bound, that is, $gx = x$ for every $x \in X$ implies that g is the identity element of G.

We call a point $x_0 \in X$ *almost transitive* if for every $x \in X$ there exists a $g \in G$ such that

$$d_X(gx_0, x) \leq C,$$

where $C > 0$ does not depend on the point $x \in X$. This is a large scale analogue of a transitive action where one allows for some uniform error. Finally, recall that a finitely generated group N is *nilpotent* if its *lower central series* terminates; there should exist $n \in \mathbb{N}_0$ such that

$$N = C_0(N) \rhd C_1(N) \rhd \cdots \rhd C_n(N) = \{e\},$$

where

$$C_i(N) := [N, C_{i-1}(N)], \qquad i = 1, \dots, n.$$

Theorem A.2.10. *Let (X, Q) be a covered space with associated metric space (X, d_Q). Assume there is a large scale stable action of a finitely generated group G on (X, d_Q).*

(a) The function

$$d_G(g, h) := \sup_{x \in X} d_Q(gx, hx), \quad g, h \in G,$$

defines a left-invariant metric on G.

(b) Assume that there exists an almost transitive point $x_0 \in X$ such that

$$\sup_{x \in X} d_Q(gx, x) \le L d_Q(gx_0, x_0) + C \tag{A.2.3}$$

holds for arbitrary $g \in G$ and uniform constants $L, C > 0$. Then (G, d_G) is quasi-isometric to (X, d_Q).

(c) Assume that we have the bound

$$|Stab_n(x)| \le p(n) \tag{A.2.4}$$

for every $x \in X$ and $n \in \mathbb{N}$, where p is a polynomial with integer coefficients. Then G is quasi-isometric to a finitely generated nilpotent group.

Proof.

(a) The function d_G is well-defined by the upper bound in the definition of a large scale stable action. If $d_G(g, h) = 0$, then we have $d_Q(gx, hx) = 0$ for every $x \in X$ and the positivity of d_Q implies that $h^{-1}gx = x$ for every $x \in X$. Since the action is effective we conclude that $g = h$. The left-invariance of the metric d_G is a reformulation of the fact the G acts by isometries on X.

(b) Assume there exists an almost transitive point $x_0 \in X$ such that (A.2.3) is satisfied and consider the map $\phi\colon G \to X$ defined by $\phi(g) := gx_0$. We want to show that ϕ is a quasi-isometry between (G, d_G) and (X, d_Q). It is tautological that

$$d_Q(\phi(g), \phi(h)) \leq d_G(g, h).$$

Moreover, the estimate (A.2.3) is a simplification of the lower-bound estimate for a quasi-isometric embedding with parameters $L, C > 0$ where the isometry property is incorporated. Finally, the image of ϕ is a net because x_0 is a transitive point.

(c) The n-stabilizer bound (A.2.4) implies in particular that the metric d_G is *proper*, that is,

$$|B_G(e, n)| < \infty, \qquad \text{for every } n \in \mathbb{N}.$$

It follows from [133, Theorem 1.3.12] that all proper, left-invariant metrics on G give quasi-isometric metric spaces. Moreover, Gromov's celebrated Polynomial Growth Theorem [87] implies that the bound (A.2.4) is equivalent with G being *virtually nilpotent*, that is, possessing a nilpotent subgroup $N \subset G$ with finite index. The result follows from [121, Corollary 5.4.5] stating that finite index subgroups of finitely generated groups are nets. □

Example A.2.11. Let $\mathcal{P} := (P_{n,m,l})_{n,m,l \in \mathbb{Z}}$ be the concatenation on $\mathbb{R}^3$ given by

$$P_{n,m,l} := (n, m, l) * [0, 1]^3,$$

where

$$(n, m, l) * (n', m', l') := (n + n', m + m', l + l' + nm').$$

This is almost the same as the uniform covering $\mathcal{U}$ on $\mathbb{R}^3$ introduced in Example A.2.8, except for the intertwining in the third component. It is straightforward to check that the *discrete Heisenberg group* $\mathbb{H}_3(\mathbb{Z}) := (\mathbb{Z}^3, *)$ acts on the metric space $(\mathbb{R}^3, d_{\mathcal{P}})$ by isometries. It satisfies all the assumptions in Theorem A.2.10 (b) and we deduce that the associated metric space $(\mathbb{R}^3, d_{\mathcal{P}})$ is quasi-isometric to the discrete Heisenberg group with any proper, left-invariant metric. We will see after Example A.3.5 that the concatenation $\mathcal{P}$ on $\mathbb{R}^3$ is not equivalent to the uniform covering $\mathcal{U}$ on $\mathbb{R}^3$.

A.2.3 Large Scale Invariants of a Covered Space

Let P denote a quasi-isometric invariant property of a metric space. We say that the covered space (X, Q) has *property* P if the associated metric space (X, d_Q) has property P. The first property we will introduce for covered spaces is a variant of topological dimension adapted to the quasi-isometric setting.

Asymptotic Dimension

Definition A.2.12. Let $\mathcal{U}$ be a covering of a metric space (X, d_X). The *R-multiplicity* of $\mathcal{U}$ for $R > 0$ is the smallest integer n such that each ball $B(x, R)$ intersects at most n elements of $\mathcal{U}$ for all $x \in X$. The *asymptotic dimension* of X is the smallest number $n \in \mathbb{N}_0$ such that for each $R > 0$ there exists a covering $\mathcal{U} := (U_i)_{i \in I}$ with uniformly bounded diameters and with R-multiplicity $n + 1$. If no $n \in \mathbb{N}_0$ satisfies the condition, then the metric space (X, d_X) is said to have *infinite asymptotic dimension*. We use the notation $\operatorname{asdim}(X, d_X)$ or simply $\operatorname{asdim}(X)$ if the metric is clear from the context.

The asymptotic dimension is invariant under quasi-isometries, for details see [133, Theorem 2.2.5]. In particular, if $\mathcal{Q}$ and $\mathcal{P}$ are two concatenations on a set X such that $\operatorname{asdim}(X, d_{\mathcal{Q}}) \neq \operatorname{asdim}(X, d_{\mathcal{P}})$, then Proposition A.2.7 implies that $\mathcal{Q}$ and $\mathcal{P}$ are not equivalent coverings.

Example A.2.13. As an illustration we will show that a covered space has asymptotic dimension zero if and only if it is quasi-isometric to a point. Let $(X, \mathcal{Q})$ be a covered space with asymptotic dimension zero. Consider a net $N \subset X$ formed by picking one element $x_i \in Q_i$ for each $i \in I$. It suffices to consider $(N, \mathcal{Q})$ since asymptotic dimension is invariant under quasi-isometries. For $R = 2$ there exists a covering $\mathcal{U} := (U_j)_{j \in J}$ with uniformly bounded diameters such that $B(x_i, 2)$ only intersects one of the U_i's for $x_i \in N$. Since $\mathcal{U}$ is a covering it follows that $B(x_i, 2) \subset U_j$ for some $j \in J$. If $x_k \in N$ with $d_{\mathcal{Q}}(x_k, x_i) = 1$, then $B(x_k, 2)$ also has to be contained in the same U_j. Continuing this way shows that $N \subset U_j$ since $\mathcal{Q}$ is a concatenation. Since U_j is bounded it follows that $(N, d_{\mathcal{Q}})$, and hence $(X, d_{\mathcal{Q}})$, is quasi-isometric to a point. Conversely, any bounded metric space clearly has asymptotic dimension zero.

We emphasize that the argument in Example A.2.13 relies on that $(X, d_{\mathcal{Q}})$ is coarsely connected. The set of p-adic numbers $\mathbb{Q}_p$ for a prime p has asymptotic dimension zero as a consequence of the inequality

$$d_{\mathbb{Q}_p}(x, z) \leq \max\left\{d_{\mathbb{Q}_p}(x, y), d_{\mathbb{Q}_p}(y, z)\right\}, \quad x, y, z \in \mathbb{Q}_p,$$

without being bounded as a metric space.

Proposition A.2.14. The uniform metric spaces $(\mathbb{R}^n, d_{\mathcal{U}})$ and $(\mathbb{R}^m, d_{\mathcal{U}})$ considered in Example A.2.8 are quasi-isometric only when $n = m$. Moreover, there exists a quasi-isometric embedding from $(\mathbb{R}^n, d_{\mathcal{U}})$ to $(\mathbb{R}^m, d_{\mathcal{U}})$ precisely when $n \leq m$.

Proof. We have already established in Example A.2.8 that $(\mathbb{R}^n, d_{\mathcal{U}})$ is quasi-isometric to the integer lattice $\mathbb{Z}^n$ with its usual left-invariant metric. A standard fact in large scale geometry [133, Example 2.2.6] states that the asymptotic dimension of $\mathbb{Z}^n$ is n. Hence the first statement follows from the quasi-isometric invariance of asymptotic dimension.

For the second statement, assume that there is a quasi-isometric embedding $\phi\colon \mathbb{Z}^n \to \mathbb{Z}^m$. The subspace $\phi(\mathbb{Z}^n) \subset \mathbb{Z}^m$ has to have asymptotic dimension less than m by restricting any covering fulfilling the definition of asymptotic dimension. Hence $\mathbb{Z}^n \simeq \phi(\mathbb{Z}^n)$ implies the necessity of $n \leq m$. If $n \leq m$, the inclusion from $\mathbb{Z}^n$ into the first n coordinates of $\mathbb{Z}^m$ is clearly a quasi-isometric embedding. □

Example A.2.15. The associated metric space of $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, the dyadic covered space considered in Example A.2.9, is quasi-isomorphic to $\mathbb{N}_0$ with its usual metric. Since $\mathbb{N}_0 \subset \mathbb{Z}$ and $\mathbb{N}_0$ is not bounded, we can conclude from Example A.2.13 that the asymptotic dimension of $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is one. Hence the dyadic covering $\mathcal{B}(\mathbb{R}^n)$ and the uniform covering $\mathcal{U}(\mathbb{R}^n)$ considered in Example A.2.8 are not equivalent as coverings unless possibly when $n = 1$. However, it follows from a straightforward calculation that there are no quasi-isometries between $\mathbb{N}_0$ and $\mathbb{Z}$ with their usual metrics. Hence the associated metric spaces $(\mathbb{R}^n, d_{\mathcal{U}})$ and $(\mathbb{R}^l, d_{\mathcal{B}(\mathbb{R}^l)})$ are not quasi-isometric for any values $n, l \geq 1$. Although this is rather straightforward to show directly as well, it showcases the potential of the large scale approach.

We showed in Example A.2.4 that every finitely generated group may be considered as the associated metric space of a covered space. There are examples of finitely generated groups that do not have finite asymptotic dimension, such as the *wreath product* $\mathbb{Z} \wr \mathbb{Z}$. We refer the reader to [133, Proposition 2.6.3] for the definition of wreath product and the calculation giving that $\mathbb{Z} \wr \mathbb{Z}$ has infinite asymptotic dimension.

Representations as Graphs

We will associate a graph to any covered space and demonstrate how this makes certain properties of covered spaces more apparent. Consider a covered space (X, Q) and form a net $N := (x_i)_{i \in I} \subset X$ where $x_i \in Q_i$ for each $i \in I$. We can consider the graph $G(N)$ whose vertices are indexed by the points in N. We declare that there is an edge between the vertices x_i and x_j if and only if $d_Q(x_i, x_j) \leq 2$. Then the metric space (N, d_Q) is quasi-isometric to the usual graph metric on the vertices of $G(N)$, see [133, Example 1.1.10]. Moreover, we can extend the graph metric to the edges by identifying each edge $e = x_i x_j$ with the interval $[0, 1]$. The resulting metric space $(G(N), d_G)$ is quasi-isometric to (X, d_Q).

Definition A.2.16. A metric space (X, d_X) is said to be *(quasi-)geodesic* if there exist constants $L, C > 0$ such that for every two points $x, y \in X$ we can find a (quasi-)isometric embedding $\gamma\colon [0, d_X(x, y)] \to X$ with parameters L, C where $\gamma(0) = x$ and $\gamma(d_X(x, y)) = y$.

Since $(G(N), d_G)$ is a geodesic metric space it follows that (X, d_Q) is a quasi-geodesic metric space. The relationship between covered spaces and graph theory is more than superficial, and there is parallel terminology in the two subjects. Recall that the *degree* of a vertex in a graph is the number of neighboring vertices. A connected graph is said to have *bounded geometry* if the degrees of the vertices are uniformly bounded. Hence the associated metric space of any covered space is quasi-isometric to a connected graph with bounded geometry. This allows us to borrow results from the well-established theory of graphs, a connection that to our knowledge has not been made before. In particular, we have the following result from [155, Example 3.8].

Proposition A.2.17. Let (X, Q) be any covered space with admissibility constant $N_Q \geq 3$. Then (X, d_Q) is quasi-isometric to a connected graph (G, d) equipped with the graph metric and with degrees bounded above by 3.

If the number of elements in each Q_i is larger than N_Q, then it is clear from the construction in [155, Example 3.8] that we can take the vertices of G to be elements in X in Proposition A.2.17. The number 3 is clearly sharp, as any concatenation Q with $N_Q = 2$ can only have two elements.

Remark. There is a more general notion than quasi-isometries present in the large scale literature known as *coarse equivalences*, see [133, Definition 1.4.1]. The reason we consider quasi-isometries rather than coarse equivalences follows from the fact that the two definitions coincide between quasi-geodesic metric spaces by [133, Theorem 1.4.13].

Hyperbolicity

There is a notion of hyperbolicity of a quasi-geodesic metric space that we will use as an invariant of a covered space. First of all, a (L, C) *quasi-geodesic triangle* in a metric space (X, d_X) is a triple $(\gamma_1, \gamma_2, \gamma_3)$ of quasi-isometric embeddings $\gamma_i\colon [0, L_i] \to X$ with parameters $L, C > 0$ such that

$$\gamma_1(L_1) = \gamma_2(0), \quad \gamma_2(L_2) = \gamma_3(0), \quad \gamma_3(L_3) = \gamma_1(0).$$

We call such a quasi-geodesic triangle *δ-slim* if there exists $\delta > 0$ such that

$$\mathrm{Im}(\gamma_i) \subset \bigcup_{x \in \mathrm{Im}(\gamma_j) \cup \mathrm{Im}(\gamma_k)} B(x, \delta),$$

where $i, j, k \in \{1, 2, 3\}$ are all distinct.

Definition A.2.18. Let (X, d_X) be a quasi-geodesic metric space. We say that (X, d_X) is *quasi-hyperbolic* if there exist constants $L, C, \delta > 0$ such that every (L', C') quasi-geodesic triangle in (X, d_X) is δ-slim for all $L' \geq L$ and $C' \geq C$.

Quasi-hyperbolicity is a quasi-isometric invariant by [121, Proposition 7.2.9]. Hence we can declare a covered space (X, Q) to be quasi-hyperbolic if the associated metric space (X, d_Q) is quasi-hyperbolic. If a finitely generated group G is quasi-hyperbolic with any (hence all) proper, left-invariant metric, it is common in the literature to simply call it a *hyperbolic group* and we will follow this convention. We will now present basic results regarding quasi-hyperbolic metric space assembled from [121, Chapter 7] that will be used in Subsection A.3.3 and Subsection A.5.3.

Lemma A.2.19. *(a) Let (X, d_X) be a quasi-geodesic metric space and let (Z, d_Z) be a quasi-hyperbolic metric space. Then the existence of a quasi-isometric embedding $\phi\colon (X, d_X) \to (Z, d_Z)$ implies that (X, d_X) is also quasi-hyperbolic.*

(b) The hyperbolic plane $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with its usual hyperbolic metric is quasi-hyperbolic.

(c) Among the groups $\mathbb{Z}^n$ for $n \geq 1$, only the group $\mathbb{Z}$ is hyperbolic. Moreover, if G is any hyperbolic group and $g \in G$ has infinite order, then the map

$$\psi\colon (\mathbb{Z}, d_{\mathbb{Z}}) \longrightarrow (G, d_G), \qquad n \longmapsto g^n$$

is a quasi-isometric embedding.

(d) Any group that contain a subgroup isomorphic to $\mathbb{Z}^2$ is not hyperbolic.

A.3 Uniform Metric Spaces on Locally Compact Groups

In this section we investigate coverings on path-connected, locally compact groups that reflect the group structure. While starting generally, we quickly focus in on stratified Lie groups and solvable Lie groups to obtain concrete examples. Finally, we examine a hyperbolic covering on the special linear group $SL(2, \mathbb{R})$. In Section A.4 we will start to build decomposition spaces on top of these coverings. The metric space machinery developed in Section A.2 together with results in this section will be used in Subsection A.4.2 and Section A.5 to show that certain embeddings between different decomposition spaces are impossible.

A.3.1 Uniform Metric Spaces

We begin by recalling some basic definitions related to locally compact groups. A *locally compact group* G is a locally compact Hausdorff space with a group structure such that the multiplication and inversion are continuous maps. A subset $A \subset G$ is called *symmetric* if $A^{-1} = A$, where

$$A^{-1} \coloneqq \{y^{-1} : y \in A\}.$$

One can always find a symmetric and precompact neighborhood of the identity on a locally compact group G by considering $U^{-1}U$, where U is a precompact neighborhood of the identity.

On any locally compact group G there exists a unique *left Haar measure* μ up to scaling, that is, a non-zero Radon measure satisfying $\mu(gE) = \mu(E)$ for any Borel set $E \subset G$ and $g \in G$. The analogous statement also holds true for a *right Haar measure*. Locally compact groups where the right and left Haar measure coincide are called *unimodular*. We will later consider the spaces $L^p(G) \coloneqq L^p(G, \mathcal{B}, \mu)$ for $p \in [1, \infty)$, where $\mathcal{B}$ is the Borel sigma-algebra and μ is a fixed left Haar measure.

In Subsection A.3.2 and Subsection A.3.3 we will consider *lattices* in locally compact groups G; they are discrete subgroups Γ in G such that there exists a G-invariant Borel measure $\mu_{G/\Gamma}$ on the quotient G/Γ with $\mu_{G/\Gamma}(G/\Gamma) < \infty$. The prototypical example to have in mind is the lattice $\mathbb{Z}^n$ inside the locally compact group $\mathbb{R}^n$. The concrete examples considered in Section A.5 will all be Lie groups. We refer the readers to [67] and [152] for basic material about locally compact groups and Lie groups, respectively.

Let G be a locally compact group that is path-connected and fix a Haar measure μ on G. We will associate to G a metric space that will reflect the group structure. Pick a precompact and symmetric set $Q_0 \subset G$ with non-void interior called a *reference set* and consider the *continuous covering* $\{gQ_0\}_{g\in G}$ in the language of [56]. The precompactness of Q_0 insures that $\mu(Q_0) < \infty$ while the non-void interior guarantees that $0 < \mu(Q_0)$. It follows from the symmetry of Q_0 and the result [56, Theorem 4.1 (A)] that there exist elements $\{g_i\}_{i\in I}$ in G such that

$$\mathcal{U} \coloneqq \mathcal{U}(G) \coloneqq \{g_i Q_0\}_{i\in I}$$

defines an admissible covering on G.

We simplify the notation $Q_i \coloneqq g_i Q_0$ and assume without loss of generality that $g_0 = e$ to make the notation compatible with the one already in place for the reference set Q_0. Furthermore, we have from [56, Theorem 4.1 (B)] that any other family $\{P_j\}_{j\in J}$ in G with the same property defines an equivalent covering. Moreover, the specific choice of the reference set Q_0 is easily seen to be irrelevant.

Hence we can always choose Q_0, and hence Q_i, to be open if we so desire. We refer to $\mathcal{U}(G)$ as the *uniform covering* of the path-connected, locally compact group G. Notice that this notation is compatible with Example A.2.8 since $\mathcal{U}(\mathbb{R}^n)$ is the uniform covering on $\mathbb{R}^n$.

Lemma A.3.1. *The uniform covering of any path-connected, locally compact group G is a concatenation.*

Proof. Fix $g, h \in G$ and let $\gamma\colon [0,1] \to G$ denote a continuous path such that $\gamma(0) = g$ and $\gamma(1) = h$. We choose Q_0 to be open and consider the sets

$$U_i := \gamma^{-1}(Q_i \cap \operatorname{Im}(\gamma)), \quad i \in I.$$

The collection $(U_i)_{i\in I}$ forms an open covering of $[0,1]$ and the compactness of the interval $[0,1]$ implies that there exists a finite sub-covering $U_{i_1}, \ldots, U_{i_n}$. Thus

$$\operatorname{Im}(\gamma) \subset \bigcup_{l=1}^{n} Q_{i_l},$$

and we have that $\mathcal{U}(x, y, n) \neq \emptyset$. The necessity of requiring that G is path-connected follows from considering $G = \mathbb{Z}_2$. □

In our language, we obtain that $(G, \mathcal{U})$ is a covered space such that the quasi-isometry class of $(G, d_{\mathcal{U}})$ does not depend on the construction. We will call the resulting metric space $(G, d_{\mathcal{U}})$ the *uniform metric space* on the path-connected, locally compact group G. We make the convention that a covering $\mathcal{U}$ on a path-connected, locally compact group G is assumed to be the uniform covering unless stated otherwise.

Remark. Uniform metric spaces have also been considered by René Koch in his exceptional Ph.D. thesis [115] through a slightly different construction: The author defines a metric d_W on any locally compact group G by fixing a symmetric and precompact unit neighborhood W and defining the distance $d_W(x, y)$ between two distinct points $x, y \in G$ to be the minimal number m such that $yx^{-1} \in W^m$. This description is convenient and makes it obvious that the resulting metric d_W on G is left-invariant. The reader should be aware that [115] allows the metric to take infinite values as he also consider locally compact groups that are not necessarily path-connected.

A metric d on a set X is said to be *proper* if the balls induced by d are precompact. This coincides with our use of the term proper in the proof of Theorem A.2.10 and in Example A.2.11. The following result shows that we can sometimes understand the uniform metric space on path-connected, locally compact groups by understanding the large scale geometry of a finitely generated subgroup.

Theorem A.3.2. *Let G be a path-connected, locally compact group and let d be a proper, left-invariant metric on G that is compatible with the topology on G. Assume N is a finitely generated subgroup of G that is a net in G and that d restricts to a locally finite metric on N. Then the uniform metric space $(G, d_{\mathcal{U}})$ is quasi-isometric to the space (N, d).*

Proof. Since N is a net in G we can find a constant $M > 0$ such that

$$\mathcal{U} := \{nB(e, M)\}_{n\in N} = \{B(n, M)\}_{n\in N}$$

is a covering on G. By picking a left Haar measure μ on G it follows that $0 < \mu(B(n, M)) < \infty$ since the balls $B(n, M)$ for $n \in N$ are precompact due to the properness of the metric. If we can show that $\mathcal{U}$ is a concatenation, then it follows that $\mathcal{U}$ is the uniform covering on G.

Since d restricts to a locally finite left-invariant metric on N we have

$$|B(n, R) \cap N| = |B(e, R) \cap N| < \infty$$

for every $R \geq 0$. Assume that $B(n, M) \cap B(m, M) \neq \emptyset$ for $n, m \in N$. Then the triangle inequality implies that $m \in B(n, 2M)$ and we have the bound

$$N_{\mathcal{U}} \leq |B(e, 2M) \cap N| < \infty,$$

where $N_{\mathcal{U}}$ is the admissibility constant of the covering $\mathcal{U}$. Hence $\mathcal{U}$ is admissible and it is straightforward to see that $\mathcal{U}$ is a concatenation since

$$B(e, kM) \subset B(e, M)^{k*}, \qquad \bigcup_{k=1}^{\infty} B(e, kM) = G.$$

By picking $n \in B(n, M)$ we conclude that $(N, d_{\mathcal{U}})$ is quasi-isometric to the uniform metric space $(G, d_{\mathcal{U}})$. Moreover, it is clear that $d_{\mathcal{U}}$ is a left-invariant metric on N by construction. The result follows since the quasi-isometry class of a finitely generated group does not depend on the choice of the proper, left-invariant metric. □

Note that the uniform metric space $(G, d_{\mathcal{U}})$ is also quasi-isometric to (G, d) since N was a net in G. However, two left-invariant and compatible metrics on G are not necessarily quasi-isometric. While the uniform metric space is quasi-isometric to N with *any* proper, left-invariant metric, this does not hold for G.

Although the number of assumptions in Theorem A.3.2 might look overwhelming at first, there are many examples fitting into settings of this type. In particular, any left-invariant Riemannian metric on a connected Lie group G induces a left-invariant and proper metric d on G. Notice that any two left-invariant Riemannian metrics on a Lie group induce quasi-isometric distances.

It is important to keep in mind that an arbitrary locally compact group might not have a proper, left-invariant metric compatible with its topology. In fact, a classical result of Struble [145] gives that the existence of a compatible, proper, and left-invariant metric on G is equivalent to G being second countable. Therefore, we restrict our attention to second countable and path-connected locally compact groups to avoid pathological examples.

A.3.2 Stratified Lie Groups

We will now investigate a large class of examples within nilpotent Lie groups called *stratified Lie groups*. In this setting, we will obtain stronger statements in Proposition A.3.4 and Theorem A.3.6 than what was possible for general path-connected, locally compact groups.

Definition A.3.3. A *stratified Lie group G* is a connected and simply connected Lie group such that its Lie algebra $\mathfrak{g}$ has a *stratification*

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \qquad [V_1, V_j] = V_{j+1}, \quad j = 1, \ldots, s-1, \qquad [V_1, V_s] = 0.$$

The *homogeneous dimension* of a stratified Lie algebra is defined to be

$$Q := \sum_{j=1}^{s} j \cdot \dim_{\mathbb{R}}(V_j).$$

The homogeneous dimension of a stratified Lie group is by definition the homogeneous dimension of its Lie algebra and is independent of the chosen stratification of the Lie algebra by [48, Proposition 1.17]. The Lie group exponential map from $\mathfrak{g}$ to G is a diffeomorphism for stratified Lie groups. Moreover, the Haar measure on G is simply the push-forward of the Lebesgue measure on $\mathfrak{g}$ under the exponential map. In particular, every stratified Lie group is unimodular and diffeomorphic to Euclidean space.

On stratified Lie groups there is a class of metrics that are intimately tied with the stratification of the Lie algebra: Fix an inner product $\langle \cdot, \cdot \rangle$ on V_1 and left translate this to obtain a Riemannian metric g on G that is only defined on the subbundle

$$\mathcal{H} \subset TM, \qquad \mathcal{H}_x := dL_x V_1, \quad x \in G.$$

The metric g is called a *sub-Riemannian metric* on G. An absolutely continuous curve $\gamma \colon [a, b] \to G$ is called *horizontal* if

$$dL_{\gamma(t)}^{-1}(\gamma(t)) \in V_1 \subset \mathfrak{g},$$

for almost every $t \in [a,b]$. This gives a left-invariant distance function d_{CC} by considering the infimum over horizontal curves: For $x, y \in G$ we define

$$d_{CC}(x,y) := \inf_{\gamma} \int_a^b |\dot{\gamma}(t)|\, dt,$$

where the infimum is taken over all horizontal curves such that $\gamma(a) = x$ and $\gamma(b) = y$. The distance function d_{CC} is called the *Carnot-Carathéodory distance* on G. The completeness of the metric space (G, d_{CC}) follows from Chow's Theorem [131, Chapter 2] in sub-Riemannian geometry. It is also common to refer to a stratified Lie group G together with the data $(\mathcal{H}, g)$ as a *Carnot group* in the sub-Riemannian literature.

Finally, recall that if $X_1, \dots, X_n$ is a basis for $\mathfrak{g}$ then $\{c_{ij}^k\}$ defined by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k, \qquad i, j, k = 1, \dots, n,$$

are called the *structure constants* of the Lie algebra $\mathfrak{g}$ in the basis $X_1, \dots, X_n$. We call a Lie group *realizable over the rationals* if there exists a basis for its Lie algebra such that the resulting structure constants are rational numbers.

Proposition A.3.4. Let G be a stratified Lie group that is realizable over the rationals and let $N \subset G$ be any lattice in G. Then the uniform metric space $(G, d_{\mathcal{U}})$ is quasi-isometric to (N, d), where d is any proper, left-invariant metric.

Proof. Fix a stratification $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$ for the Lie algebra $\mathfrak{g}$ of G. The existence of a lattice N in G is equivalent to the requirement that G is realizable over the rationals by [139, Theorem 2.12]. Moreover, every lattice in a stratified Lie group is a finitely generated nilpotent group [139, Theorem 2.10] that is additionally *uniform* [139, Theorem 2.1], that is, the quotient space G/N is compact.

Fix a Carnot-Carathéodory distance d_{CC} on G arising from an inner product on V_1 and notice that N is then a net since we can write

$$G = \bigcup_{n \in N} nC,$$

where C is some compact subset. Moreover, it follows from [121, Corollary 5.5.9] that the quasi-isometry class of (N, d_{CC}) does not depend on the choice of the lattice. The inclusion

$$\overline{B_{d_{CC}}(e,R) \cap N} \subset \overline{B_{d_{CC}}(e,R)}$$

together with the properness of d_{CC} implies that $\overline{B_{d_{CC}}(e,R) \cap N}$ is finite due to the discreteness of N. Hence the metric d_{CC} restricted to N is locally finite and the result follows from Theorem A.3.2. $\square$

Remark. Whenever the dimension of the Lie group is less than seven, the assumption that the Lie group is realizable over the rationals is automatically satisfied. This follows from the classification of real nilpotent Lie algebras with low dimension given in [79].

Another useful invariant of a finitely generated group is its growth type. We will not go into the explicit definition of this since it slightly cumbersome and is well explained in [121, Chapter 6]. The idea is that the number of elements in $B(e,n)$ for a finitely generated group N with proper, left-invariant metric is not a quasi-isometric invariant. However, the *growth type* (e.g. if it grows linearly, quadratically, or exponentially) is a quasi-isometric invariant of the group. We will illustrate how this can be used in the following example.

Example A.3.5. For $n \in \mathbb{N}$ we consider the Heisenberg group $(\mathbb{H}_{2n+1}, *)$ consisting of all matrices on the form

$$\left\{ \begin{pmatrix} 1 & \mathbf{a} & c \\ 0 & I_{n\times n} & \mathbf{b} \\ 0 & 0 & 1 \end{pmatrix} : \mathbf{a}, \mathbf{b} \in \mathbb{R}^n,\ c \in \mathbb{R} \right\},$$

where the operation $*$ denotes the usual matrix multiplication. It is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}_{2n+1}$ can be identified as a vector space with $\mathbb{R}^{2n+1} = \mathbb{R}^{2n} \oplus \mathbb{R}$. If $e_1, \ldots, e_{2n+1}$ is the standard basis for $\mathbb{R}^{2n+1}$ then the Lie bracket satisfies

$$[e_i, e_j] = \delta_{i+n,j} e_{2n+1}, \quad i \le j < 2n+1, \qquad [e_i, e_{2n+1}] = 0.$$

Fix an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n} \subset \mathfrak{g}_{2n+1}$ making the basis $e_1, \ldots, e_{2n}$ orthonormal. We can equip $(\mathbb{H}_{2n+1}, *)$ with a sub-Riemannian metric g by left translating $\langle \cdot, \cdot \rangle$. The subset $\mathbb{Z}^{2n+1} \subset \mathbb{H}_{2n+1}$ is a finitely generated subgroup. The metric d_{CC} restricts to a locally finite metric on $\mathbb{Z}^{2n+1}$ such that $\mathbb{Z}^{2n+1}$ is a net in $\mathbb{H}_{2n+1}$ due to the reasons pointed out in the proof of Proposition A.3.4. Hence by Theorem A.3.2 it follows that the uniform metric space $(\mathbb{H}_{2n+1}, d_{\mathcal{U}})$ is quasi-isometric to $(\mathbb{Z}^{2n+1}, *)$ with any proper, left-invariant metric. A tedious but straightforward computation shows that the group $(\mathbb{Z}^{2n+1}, *)$ has polynomial growth of order $2n+2$, while $(\mathbb{Z}^k, +)$ has polynomial growth of order k. Since growth type is a quasi-isometric invariant by [121, Corollary 6.2.6] we have that

$$(\mathbb{H}_{2n+1}, d_{\mathcal{U}}) \not\simeq (\mathbb{H}_{2m+1}, d_{\mathcal{U}}),\ m \neq n, \qquad (\mathbb{H}_{2s+1}, d_{\mathcal{U}}) \not\simeq (\mathbb{R}^k, d_{\mathcal{U}}),\ k \neq 2s+2.$$

However, since $(\mathbb{Z}^{2n+1}, *)$ and $(\mathbb{Z}^{2n+2}, +)$ have the same polynomial growth we need a different approach to show that $(\mathbb{H}_{2n+1}, d_{\mathcal{U}})$ is not quasi-isometric to $(\mathbb{R}^{2n+2}, d_{\mathcal{U}})$.

Assume by contradiction that $(\mathbb{H}_{2n+1}, d_{\mathcal{U}}) \simeq (\mathbb{R}^{2n+2}, d_{\mathcal{U}})$. Then it follows from [121, Corollary 6.3.16] that $(\mathbb{Z}^{2n+1}, *)$ then would have a finite index subgroup isomorphic to $(\mathbb{Z}^{2n+2}, +)$. By intersecting all the conjugates of $(\mathbb{Z}^{2n+2}, +)$ in $(\mathbb{Z}^{2n+1}, *)$ one can assure that there exists a normal abelian subgroup of $(\mathbb{Z}^{2n+1}, *)$ with finite index. The reason the intersection still has finite index is due to the easily verifiable formula

$$|G : B \cap C| \leq |G : B| \cdot |G : C|,$$

when B, C are subgroups of G with finite index. However, since $(\mathbb{Z}^{2n+1}, *)$ is nilpotent and torsion free, it follows from [114, Lemma 3.1] that this forces $(\mathbb{Z}^{2n+1}, *)$ to be abelian. Since this is not the case the claim follows.

Remark. The uniform covering $\mathcal{U}(\mathbb{H}_3)$ is can be considered on $\mathbb{R}^3$ since $\mathbb{H}_3$ is diffeomorphic to $\mathbb{R}^3$. There, it is precisely the covering $\mathcal{P}$ introduced in Example A.2.11. It thus follows from Example A.3.5 that the two coverings $\mathcal{P}$ and $\mathcal{U}$ in Example A.2.11 are not equivalent coverings.

Given a stratified Lie group G with Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$, we call the multi-index

$$\mathfrak{G}(G) \coloneqq (n_1, \ldots, n_s),$$

the *growth vector* of G, where $n_i \coloneqq \dim_{\mathbb{R}}(V_i)$ for $i = 1, \ldots, s$. The argument we used in the last part of Example A.3.5 does not generalize easily. We remedy this by proving a stronger statement about when two uniform metric spaces on different stratified Lie groups can not be quasi-isometric. The first statement in the following theorem is folklore, but we provide a proof as we could not find a complete reference.

Theorem A.3.6. *Let G be a stratified Lie group and assume that $N \subset G$ is a lattice in G. Then N has polynomial growth type of order equal to the homogeneous dimension of G. Let H be another stratified Lie group that is realizable over the rationals such that the uniform metric spaces $(G, d_{\mathcal{U}})$ and $(H, d_{\mathcal{U}})$ are quasi-isometric. Then their growth vectors $\mathfrak{G}(G)$ and $\mathfrak{G}(H)$ have to be equal.*

Proof. We will build a correspondence between the lower central series of N and the stratification on the Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$. Consider the commutator subgroup $[G, G] \subset G$. Then [152, Theorem 3.50] implies that $[G, G]$ is a Lie subgroup of G whose corresponding Lie algebra is isomorphic to $[\mathfrak{g}, \mathfrak{g}] = V_2 \oplus \cdots \oplus V_s$. Denote the projection onto the quotient by $\pi \colon G \to G/[G, G]$.

It is straightforward to check that $G/[G, G]$ is isomorphic as a Lie group to Euclidean space and $\pi(N)$ is a lattice in $G/[G, G]$. However, lattices in Euclidean spaces are finitely generated abelian groups whose rank is equal to the dimension of the ambient Euclidean space. Hence it follows that $\pi(N)$ is generated by $\dim(G/[G,G]) = \dim(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) = \dim(V_1)$ elements. This gives

$$\mathrm{rank}_{\mathbb{Z}} C_0(N)/C_1(N) = \dim_{\mathbb{R}}(V_1),$$

where $C_i(N)$ denotes the i'th term in the lower central series of N. We can proceed inductively to obtain that

$$\mathrm{rank}_{\mathbb{Z}}\left(C_i(N)/C_{i+1}(N)\right) = \dim_{\mathbb{R}}(V_{i+1}), \quad i = 0, \ldots, s-1. \tag{A.3.1}$$

The first statement of the theorem now follows from the Bass - Guivarc'h formula [10, Theorem 2], stating that the polynomial growth of a finitely generated nilpotent group N is precisely

$$\sum_{k=1}^{n} k \cdot \mathrm{rank}_{\mathbb{Z}}\left(C_{k-1}(N)/C_k(N)\right).$$

Let H be another stratified Lie group that is realizable over the rationals and pick a lattice M in H. A quasi-isometry between the uniform spaces on G and H induce a quasi-isometry between (N, d_N) and (M, d_M), where d_N and d_M are any proper, left-invariant metrics. Since the rank of the the quotients in the lower central series of a finitely generated nilpotent group are quasi-isometric invariants, we have that

$$\mathrm{rank}_{\mathbb{Z}}\left(C_i(N)/C_{i+1}(N)\right) = \mathrm{rank}_{\mathbb{Z}}\left(C_i(M)/C_{i+1}(M)\right).$$

The correspondence (A.3.1) gives that the growth vectors $\mathfrak{G}(G)$ and $\mathfrak{G}(H)$ are the same. □

Example A.3.7. Let $\mathfrak{g}$ be the nilpotent Lie algebra spanned by the elements X_1, X_2, X_3, X_4 with non-trivial bracket relations

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4.$$

We call $\mathfrak{g}$ the *Engel algebra* and it is has a stratification given by

$$\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{X_1, X_2\} \oplus \mathrm{span}_{\mathbb{R}}\{X_3\} \oplus \mathrm{span}_{\mathbb{R}}\{X_4\}.$$

The connected and simply connected Lie group G corresponding to $\mathfrak{g}$ is called the *Engel group* and appears for instance in [49].

Since G is diffeomorphic to $\mathbb{R}^4$ through the exponential map, we can consider the two coverings $\mathcal{U}(G)$ and $\mathcal{U}\left(\mathbb{R}^4\right)$ on $\mathbb{R}^4$. To check that the two coverings are not equivalent is not a complete triviality from a computational perspective. However, their uniform metric spaces are not quasi-isometric by Theorem A.3.6 since their growth vectors are different. Hence the coverings they induce on $\mathbb{R}^4$ are non-equivalent by Proposition A.2.7. This illustrates the novelty of the large scale approach, even when the coverings are on the same space.

A.3.3 More Examples

Solvable Groups

We will now consider the more general class of solvable Lie groups and we begin by recalling the definition of an (abstract) solvable group. The *derived series* of a group N is defined by

$$N^{(0)} := N, \quad N^{(i)} := [N^{(i-1)}, N^{(i-1)}],$$

for $i \geq 1$. A group N is said to be *solvable* if its derived series eventually reaches the trivial group. Every nilpotent group is solvable, although the converse is false. A group N is called *virtually solvable* if it contains a solvable subgroup of finite index.

To see that virtually solvable groups play a prominent role in the setting of uniform metric spaces on Lie groups, consider a connected Lie subgroup G of $GL(n, \mathbb{R})$ for $n \geq 1$. Assume that d is a proper, left-invariant metric on G and that N is a finitely generated subgroup of G such that d restricts to a locally finite metric on N. Then Theorem A.3.2 shows that $(G, d_{\mathcal{U}}) \simeq (N, d_N)$, where d_N is any proper, left-invariant metric on N. Since N is a finitely generated subgroup of $GL(n, \mathbb{R})$ we can apply the famous Tits Alternative [121, Theorem 4.4.7] in group theory to conclude that N is either virtually solvable or has a free subgroup of rank two as a finite index subgroup. Motivated by this, we examine the uniform metric spaces on solvable Lie groups more closely.

Definition A.3.8. A *solvable* Lie group is a connected Lie group such that its Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{g}^n = \{0\}$ for some $n \in \mathbb{N}_0$, where

$$\mathfrak{g}^0 := \mathfrak{g}, \qquad \mathfrak{g}^i := [\mathfrak{g}^{i-1}, \mathfrak{g}^{i-1}], \quad i \geq 1.$$

An example of a solvable Lie group is all upper-triangular $n \times n$ matrices with positive determinant. As we will be interested in lattices in solvable Lie groups so that we can apply Theorem A.3.2, let us remark that the existence of lattices in solvable Lie groups are more complicated that in the nilpotent case. Unlike a stratified Lie group, a solvable Lie group does not need to be unimodular, that is, the right and left Haar measures might be different. There are no lattices in a non-unimodular locally compact group by [139, Remark 1.9]. In particular, the *affine group* (also known as the $Ax + b$ group) given by

$$\mathrm{Aff} := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\ b \in \mathbb{R} \right\}$$

does not admit lattices even though it is solvable. We will relate the uniform metric spaces on solvable Lie groups admitting lattices to the following subclass of finitely generated solvable groups.

Definition A.3.9. A group Γ is *polycyclic* if it admits a chain of subgroups

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_k = \{e\},$$

where each term in the chain is a normal subgroup of the previous term and the quotients Γ_{i-1}/Γ_i are cyclic groups for $i = 1, \ldots, k$. It is called *strongly polycyclic* if it admits such a chain where each quotient Γ_{i-1}/Γ_i is infinitely cyclic.

Proposition A.3.10. Let G be a connected and simply connected solvable Lie group and assume there exists a lattice Γ in G. Then the uniform metric space $(G, d_{\mathcal{U}})$ is quasi-isometric to (Γ, d) where d is any proper, left-invariant metric on Γ. Moreover, Γ is strongly polycyclic.

Proof. Any lattice in a solvable Lie group is uniform by [139, Theorem 3.1]. It follows from [139, Proposition 3.7] that any lattice in a simply connected solvable Lie group is strongly polycyclic and hence finitely generated. By fixing a Riemannian metric g on G by left translating an inner product on the Lie algebra, it is clear that all the conditions in Theorem A.3.2 are satisfied and the result follows. □

The Special Linear Group and the Hyperbolic Plane

We will illustrate a uniform metric space that has fundamentally different properties than those built on solvable Lie groups. Consider the Lie group $SL(2, \mathbb{R})$ of 2×2 matrices with real coefficients and unit determinant. It is related to the hyperbolic plane $\mathbb{H}^2$ with the usual hyperbolic distance by the fact that $SL(2, \mathbb{R})$ acts on $\mathbb{H}^2$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}, \qquad A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),\ z \in \mathbb{H}^2.$$

Notice that both A and $-A$ induce the same transformation.

An action of a discrete group G on a topological space X is said to be *properly discontinuous* if every point $x \in X$ has a neighborhood U such that $(g \cdot U) \cap U = \emptyset$ for every non-identity element $g \in G$. Finally, recall that a group action is said to be free if $g \cdot x = x$ for some $x \in X$ and $g \in G$ implies that g is the identity element of the group G.

Theorem A.3.11. *The uniform metric space $(SL(2, \mathbb{R}), d_{\mathcal{U}})$ is quasi-isometric to the fundamental group of any compact Riemann surface of genus $g \geq 2$. Moreover, this is again quasi-isometric to the hyperbolic space $\mathbb{H}^2$ with its usual hyperbolic distance. In particular, the uniform metric space $(SL(2, \mathbb{R}), d_{\mathcal{U}})$ is quasi-hyperbolic.*

Proof. Fix an inner product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ consisting of 2×2 matrices with real coefficients and zero trace. Left translate this to obtain a Riemannian metric on $SL(2,\mathbb{R})$ and consider the Carnot-Carathéodory metric d_{CC} associated to it. Then $(SL(2,\mathbb{R}), d_{CC})$ satisfies all the initial assumptions in Theorem A.3.2.

Let X be a compact Riemann surface of genus $g \geq 2$. The fundamental group $\pi_1(X)$ of X can be realized as a uniform and torsion free discrete subgroup Γ of $SL(2,\mathbb{R})$. Conversely, any uniform and torsion free discrete subgroup Γ of $SL(2,\mathbb{R})$ acts on $\mathbb{H}^2$ freely and properly discontinuously such that the orbit space $\mathbb{H}^2/\Gamma$ is a compact Riemann surface. These observations are built from several standard results about compact Riemannian surfaces and they can all be found in the lecture notes [71].

Fix such a uniform and torsion free discrete subgroup Γ of $SL(2,\mathbb{R})$. Then Γ acts on $SL(2,\mathbb{R})$ by left translations and it follows from the Milnor-Švarc lemma [133, Proposition 1.3.13] that Γ is finitely generated. The fact that Γ is uniform implies that it is a net in $(SL(2,\mathbb{R}), d_{CC})$. The discreteness of Γ implies that d_{CC} is locally finite on Γ. We can conclude by Theorem A.3.2 that the uniform metric space $(SL(2,\mathbb{R}), d_{\mathcal{U}})$ is quasi-isometric to Γ with any proper, left-invariant metric. The choice of Γ does not matter since [121, Corollary 5.5.9] implies that any two uniform, discrete subgroups of $SL(2,\mathbb{R})$ are quasi-isometric. The quasi-isometry between the fundamental group $\pi_1(X)$ and the hyperbolic plane $\mathbb{H}^2$ is well known and can be found in [121, Corollary 5.4.10]. The final statement follows from Lemma A.2.19 (b). $\square$

Remark. In the proof of Theorem A.3.11 it is tempting to consider the lattice $SL(2,\mathbb{Z})$ in $SL(2,\mathbb{R})$ instead of Γ. However, $SL(2,\mathbb{Z})$ has a free group of rank two as a finite index subgroup as shown in [121, Example 4.4.1]. This implies together with [26, Theorem 1] that $SL(2,\mathbb{Z})$ is not quasi-isometric to $\mathbb{H}^2$. The reason for this failure lies with the non-compactness of the homogeneous space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.

Proposition A.3.12. The metric space $(SL(2,\mathbb{R}), d_{\mathcal{U}})$ is not quasi-isometric to $(\mathbb{H}_{2n+1}, d_{\mathcal{U}})$ or $(\mathbb{R}^k, d_{\mathcal{U}})$ for any $n, k \in \mathbb{N}$. In fact, there are no quasi-isometric embeddings

$$(\mathbb{R}^k, d_{\mathcal{U}}) \longrightarrow (SL(2,\mathbb{R}), d_{\mathcal{U}}), \qquad (\mathbb{H}_{2n+1}, d_{\mathcal{U}}) \longrightarrow (SL(2,\mathbb{R}), d_{\mathcal{U}}),$$

unless $k = 1$.

Proof. Consider the elements

$$A = \begin{pmatrix} 1 & e_1 & 0 \\ 0 & I_{n\times n} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & I_{n\times n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}_{2n+1},$$

where $e_1 = (1, 0, \ldots, 0)$. The subgroup $\langle A, B\rangle$ generated by A and B is commutative and the mapping

$$\begin{aligned} \phi\colon \langle A, B\rangle &\longrightarrow \mathbb{Z}^2 \\ A^r B^s &\longmapsto (r, s) \end{aligned}$$

gives an isomorphism between $\langle A, B\rangle$ and $\mathbb{Z}^2$. Hence Lemma A.2.19 (d) implies that the Heisenberg groups are not hyperbolic. We mentioned in Lemma A.2.19 (c) that $\mathbb{Z}^k$ is not a hyperbolic group unless $k = 1$. Hence neither of the quasi-isometric embeddings

$$(\mathbb{R}^k, d_{\mathcal{U}}) \longrightarrow (SL(2, \mathbb{R}), d_{\mathcal{U}}), \qquad (\mathbb{H}_{2n+1}, d_{\mathcal{U}}) \longrightarrow (SL(2, \mathbb{R}), d_{\mathcal{U}})$$

are possible due to Lemma A.2.19 (a) for $n \in N$ and $k \geq 2$.

For $k = 1$ one obtain several quasi-isometric embeddings

$$(\mathbb{R}, d_{\mathcal{U}}) \longrightarrow (SL(2, \mathbb{R}), d_{\mathcal{U}})$$

from Lemma A.2.19 (c). We can not use hyperbolicity to conclude that $(\mathbb{R}, d_{\mathcal{U}})$ is not quasi-isometric to $(SL(2, \mathbb{R}), d_{\mathcal{U}})$. However, we can consider their asymptotic dimensions together with Theorem A.3.11 to derive

$$\operatorname{asdim}(SL(2, \mathbb{R}), d_{\mathcal{U}}) = \operatorname{asdim}(\mathbb{H}^2) = 2 \neq 1 = \operatorname{asdim}(\mathbb{Z}) = \operatorname{asdim}(\mathbb{R}, d_{\mathcal{U}}).$$

Here we have used that the asymptotic dimension of $\mathbb{H}^2$ is equal to two, a result going back to Gromov [86]. Hence the claim follows from the quasi-isometric invariance of asymptotic dimension. □

Notice that we used both asymptotic dimension and hyperbolicity in the proof of Proposition A.3.12. Arguments such as these are our main motivation for considering invariants from large scale geometry. For another class of examples, we refer the reader interested in shearlet groups to the Ph.D. thesis of René Koch [115, Section 5.4] and the subsequent paper [70] where novel results are derived.

A.4 Decomposition Spaces and Geometric Embeddings

This section is devoted to introducing embeddings between decomposition spaces that induce quasi-isometric embeddings between the underlying coverings called *geometric embeddings*. In Subsection A.4.3 we will give some criteria for when quasi-isometries between the underlying coverings can induce geometric embeddings between decomposition spaces.

A.4.1 Definitions and Basic Properties

We will start by reviewing basic definitions and results regarding decomposition spaces given in [59]. This is done to make our exposition complete as well as to fix notation and settle our conventions. Throughout this section, we let X denote an arbitrary locally compact topological space and denote by $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ a subspace of $C_b(X, \mathbb{C})$ with a norm $\|\cdot\|_{\mathcal{A}}$ making it into a Banach algebra under pointwise multiplication. Moreover, we additionally stipulate that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is closed under complex conjugation and that it is *regular*, that is, $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is sufficiently large to separate points from closed sets by continuous functions.

A *partition of unity* $\Phi := (\varphi_i)_{i\in I}$ on X subordinate to an admissible covering $\mathcal{Q} := (Q_i)_{i\in I}$ is a collection of non-negative continuous functions such that for all $x \in X$ one has

$$\operatorname{supp}(\varphi_i) \subset Q_i, \quad \sum_{i\in I} \varphi_i(x) = 1. \tag{A.4.1}$$

Since the covering $\mathcal{Q}$ is assumed to be admissible, there is no convergence issue in the sum (A.4.1).

Definition A.4.1. Let $\mathcal{Q} := (Q_i)_{i\in I}$ be an admissible covering on X. A *bounded admissible partition of unity* (BAPU) in $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ subordinate to $\mathcal{Q}$ is a partition of unity $\Phi := (\varphi_i)_{i\in I}$ subordinate to $\mathcal{Q}$ where $\varphi_i \in \mathcal{A}$ for every $i \in I$ and

$$\sup_{i\in I} \|\varphi_i\|_{\mathcal{A}} < \infty. \tag{A.4.2}$$

It is common to refer to Φ as a $\mathcal{Q}$-BAPU to emphasize the covering $\mathcal{Q}$ in question.

We denote by $\mathcal{A}_c$ the elements of $\mathcal{A}$ that have compact support. When forming the decomposition space $\mathcal{D}(\mathcal{Q}, B, Y)$ in Definition A.4.2, we need some weak assumptions on the Banach spaces $(B, \|\cdot\|_B)$ and $(Y, \|\cdot\|_Y)$ to deduce nice properties of the decomposition space $\mathcal{D}(\mathcal{Q}, B, Y)$.

Our standing assumptions are that B is continuously embedded into the dual $\mathcal{A}_c^*$, that $\mathcal{A}_c$ is densely embedded into B, and that B is a Banach module over $\mathcal{A}$ under pointwise operations. We assume that $(Y, \|\cdot\|_Y)$ is a Banach space consisting of sequences on the index set I. Moreover, we assume *minimality* of $(Y, \|\cdot\|_Y)$, meaning that the finitely supported sequences are required to form a dense subspace of $(Y, \|\cdot\|_Y)$. Define the *clustering map* $\Gamma_{\mathcal{Q}} \colon Y \longrightarrow Y$ by

$$(a_i)_{i\in I} \longmapsto \left(\sum_{j\in i^*} a_j\right)_{i\in I}.$$

It will henceforth be assumed that the clustering map $\Gamma_{\mathcal{Q}}$ is well-defined and bounded on Y. Finally, we impose that Y should be *solid*, meaning that if $x = (x_i)_{i\in I}$

is a sequence in Y and $y = (y_i)_{i\in I}$ is a sequence in $\mathbb{C}^I$ such that $|y_i| \leq |x_i|$ for every $i \in I$, then $y \in Y$ with $\|y\|_Y \leq \|x\|_Y$. We refer the reader to [59, Section 2] for a more thorough discussion of these assumptions.

Definition A.4.2. Let B and Y be Banach spaces satisfying the standing assumptions above. Moreover, let $\Phi := (\varphi_i)_{i\in I}$ be a Q-BAPU in $\mathcal{A}$ corresponding to an admissible covering Q on X. The *decomposition (function) space* $\mathcal{D}(Q, B, Y)$ consists of all elements $f \in \mathcal{A}_c^*$ such that

$$\|f\|_{\mathcal{D}(Q,B,Y)} := \left\| (\|f \cdot \varphi_i\|_B)_{i\in I} \right\|_Y < \infty. \tag{A.4.3}$$

We call B the *local component* and Y the *global component* of the decomposition space $\mathcal{D}(Q, B, Y)$.

Equipping $\mathcal{D}(Q, B, Y)$ with the norm given by (A.4.3) gives us a Banach space by [59, Theorem 2.2 A]. The observant reader will have noticed that we have excluded the Q-BAPU $\Phi := (\varphi_i)_{i\in I}$ from the notation $\mathcal{D}(Q, B, Y)$. This is because [59, Theorem 2.3] implies that different Q-BAPU's give rise to the same spaces with equivalent norms. We summarize some well known properties of decomposition spaces in Proposition A.4.3 below. The last statement of Proposition A.4.3 is a straightforward extension of [59, Corollary 2.6].

Proposition A.4.3. The (continuous) dual space of $\mathcal{D}(Q, B, Y)$ can be identified with the decomposition space $\mathcal{D}(Q, B^*, Y^*)$. In particular, reflexivity of the local and global components gives reflexivity of the corresponding decomposition space. Moreover, we have the norm convergence

$$f = \sum_{i\in I} f \cdot \varphi_i,$$

in $\mathcal{D}(Q, B, Y)$ where $\Phi := (\varphi_i)_{i\in I}$ is any Q-BAPU for $\mathcal{D}(Q, B, Y)$. Finally, a function f belongs to $\mathcal{D}(Q, B, Y)$ if and only if there exist $k \in \mathbb{N}$ and $f_i \in B$ with $\mathrm{supp}(f_i) \subset Q_i^{k*}$ such that $\{\|f_i\|_B\}_{i\in I} \in Y$ and $f = \sum_{i\in I} f_i$ in $\mathcal{A}_c^*$.

Remark. The requirement that $\mathcal{A}_c$ is dense in B is only needed for the duality statement in Proposition A.4.3, while the requirement that the finite sequences are dense in Y is required for both the duality statement and the norm convergence statement in Proposition A.4.3. The reader interested in cases where these requirements do not hold, such as $Y = l^\infty(I)$, can safely use all subsequent results that do not invoke these properties.

Many of the decomposition spaces appearing in the literature such as modulation spaces and Besov spaces are built on open subsets of some Euclidean space. However, they are not precisely decomposition spaces as defined in [59], but rather a variation that incorporates the Fourier transform. We briefly outline this distinction and refer the reader to the paper [148] for more details.

Let $\mathcal{Q} := (Q_i)_{i\in I}$ be an admissible covering for the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^k$ with a $\mathcal{Q}$-BAPU $\Phi := (\varphi_i)_{i\in I}$. Moreover, let B and Y be Banach spaces satisfying the standing assumptions where $\mathcal{A} := \mathcal{F}L^1$ is the Fourier transform of all integrable functions. Then the decomposition space $\mathcal{D}(\mathcal{Q}, \mathcal{F}L^p, Y)$ consists of all elements $f \in \mathcal{A}_c^*$ such that

$$\left\| \left(\left\| f \cdot \varphi_i \right\|_{\mathcal{F}L^p} \right)_{i\in I} \right\|_Y = \left\| \left(\left\| \mathcal{F}^{-1} \left(f \cdot \varphi_i \right) \right\|_{L^p} \right)_{i\in I} \right\|_Y < \infty. \tag{A.4.4}$$

The local component $\mathcal{F}L^p$ is a Banach module under pointwise multiplication over $\mathcal{A}$ since

$$\|f \cdot a\|_{\mathcal{F}L^p} = \|\mathcal{F}^{-1}\left(f \cdot a\right)\|_{L^p} = \|\mathcal{F}^{-1}(f) * \mathcal{F}^{-1}(a)\|_{L^p} \leq \|f\|_{\mathcal{F}L^p} \cdot \|a\|_{\mathcal{A}},$$

for $a \in \mathcal{A}$ and $f \in \mathcal{F}L^p$. The expression (A.4.4) is well-defined by the uniform bound (A.4.2).

Definition A.4.4. The *$\mathcal{F}$-type decomposition space* $\mathcal{D}^{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is defined by

$$\mathcal{D}^{\mathcal{F}}(\mathcal{Q}, L^p, Y) := \mathcal{F}^{-1}\left(\mathcal{D}(\mathcal{Q}, \mathcal{F}L^p, Y)\right).$$

For $f \in \mathcal{D}^{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ we are interested in the natural norm

$$\|f\|_{\mathcal{D}^{\mathcal{F}}(\mathcal{Q},L^p,Y)} := \left\| \left(\left\| \mathcal{F}^{-1}\left(\mathcal{F}(f) \cdot \varphi_i\right) \right\|_{L^p} \right)_{i\in I} \right\|_Y. \tag{A.4.5}$$

If we want to indicate that a decomposition space in a statement can be either a $\mathcal{F}$-type decomposition space or a standard decomposition space, we refer to it as a ($\mathcal{F}$*-type*) *decomposition space*.

Remark. One avenue that we have not pursued is to consider the quasi-Banach setting, that is, where the local component $(B, \|\cdot\|_B)$ and the global component $(Y, \|\cdot\|_Y)$ of ($\mathcal{F}$-type) decomposition spaces are quasi-Banach spaces. The most common examples are $B = L^p$ and $Y = l^q$ for $0 < p, q < 1$. Although these have received increased interest in the last few years, we will avoid this more technical case since the underlying geometry of the coverings are not affected by this extension. We refer the interested reader to [148] for the most comprehensive exposition on decomposition spaces with quasi-Banach spaces as local and global components.

A.4.2 Geometric Embeddings

We now take up the question of whether one ($\mathcal{F}$-type) decomposition space embeds nicely into another ($\mathcal{F}$-type) decomposition space. As the ($\mathcal{F}$-type) decomposition spaces are Banach spaces, they can embed into each other as *Banach spaces* without this actually reflecting the underlying geometry of the coverings. Moreover, the embedding may then be artificial and not readily available. Hence we will consider a refined notion of embeddings between ($\mathcal{F}$-type) decomposition spaces that incorporates the underlying coverings.

Recall that an *embedding* between Banach spaces $(B_1, \|\cdot\|_{B_1})$ and $(B_2, \|\cdot\|_{B_2})$ is an injective linear map $F\colon B_1 \to B_2$ such that $\|F(f)\|_{B_2} \leq A\|f\|_{B_1}$ for some constant $A > 0$ not depending on $f \in B_1$. Let $(X, \mathcal{Q})$ be a covered space and consider a decomposition space $\mathcal{D}(\mathcal{Q}, B, Y)$. We define the *adapted support* of an element $f \in \mathcal{D}(\mathcal{Q}, B, Y)$ with respect to the $\mathcal{Q}$-BAPU $\Phi := (\varphi_i)_{i\in I}$ to be

$$C[f] := \bigcup_{i\in I} \{Q_i : \|f \cdot \varphi_i\|_B \neq 0\}.$$

Notice that $f_i := \sum_{j\in i^*} \varphi_i \in \mathcal{D}(\mathcal{Q}, B, Y)$ is a non-zero function that satisfies $C[f_i] \subset Q_i^{2*}$.

If we are considering $\mathcal{F}$-type decomposition spaces, then the *adapted spectrum* of $f \in \mathcal{D}^{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ with respect to the $\mathcal{Q}$-BAPU $\Phi := (\varphi_i)_{i\in I}$ is defined to be

$$C[f] := \bigcup_{i\in I} \{Q_i : \|\mathcal{F}(f) \cdot \varphi_i\|_{\mathcal{F}L^p} \neq 0\}.$$

Notice that the adapted support and adapted spectrum might depend on the choice of $\mathcal{Q}$-BAPU. However, it will be clear in Definition A.4.5 that the choice of $\mathcal{Q}$-BAPU is irrelevant.

Definition A.4.5. Let $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$ be ($\mathcal{F}$-type) decomposition spaces with underlying covered spaces $(X, \mathcal{Q})$ and $(Z, \mathcal{P})$.

- We say that a map

$$F\colon \mathcal{D}(\mathcal{Q}, B_1, Y_1) \to \mathcal{D}(\mathcal{P}, B_2, Y_2)$$

  is a *geometric embedding* of decomposition spaces if it is an embedding of Banach spaces with the following additional requirement: There should exist constants $L, C > 0$ such that for any $k \in \mathbb{N}_0$ and any $f, g \in \mathcal{D}(\mathcal{Q}, B_1, Y_1)$ with $C[f] \subset Q_i^{k*}$ and $C[g] \subset Q_j^{k*}$, we have

$$\frac{1}{L} d_{\mathcal{Q}}(x, y) - C \leq d_{\mathcal{P}}(z, w) \leq L d_{\mathcal{Q}}(x, y) + C, \tag{A.4.6}$$

  where $x \in Q_i^{k*}$, $y \in Q_j^{k*}$, $z \in C[F(f)]$ and $w \in C[F(g)]$ are arbitrary.

- Two decomposition spaces $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$ are said to be *geometrically isomorphic* if there exists an invertible geometric embedding from $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$ to $\mathcal{D}(\mathcal{P}, B_2, Y_2)$ whose inverse is also a geometric embedding.

Although it would seem more convenient to require (A.4.6) only for $f = \chi_{Q_i}$ and $g = \chi_{Q_j}$, this is often not sufficient for the simple reason that χ_{Q_i} might not be in $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$. An example where this happens is the modulation space $M^1(\mathbb{R}^n)$ defined in Subsection A.5.1 since every element in $M^1(\mathbb{R}^n)$ is continuous. Moreover, we will give an example at the end of Subsection A.5.2 showing that two decomposition spaces $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$ can be equal as Banach spaces without being geometrically isomorphic.

To see why the definition of geometric embeddings encodes the geometry of the decomposition space, we consider the case where $X = Z$. Assume that the identity mapping

$$\mathcal{D}(\mathcal{Q}, B_1, Y_1) \ni f \longmapsto f \in \mathcal{D}(\mathcal{P}, B_2, Y_2)$$

is a geometric isomorphism. Then the identity map from $(X, d_{\mathcal{Q}})$ to $(X, d_{\mathcal{P}})$ is a quasi-isometry by (A.4.6). Hence it follows from Proposition A.2.7 that the coverings $\mathcal{Q}$ and $\mathcal{P}$ are equivalent. Conversely, if the identity map from $(X, d_{\mathcal{Q}})$ to $(X, d_{\mathcal{P}})$ is a quasi-isometry, then the identity map acting on functions $f\colon X \to \mathbb{C}$ satisfies the estimate (A.4.6). However, it is not guaranteed that the identity map $f \mapsto f$ embeds $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$ continuously into $\mathcal{D}(\mathcal{P}, B_2, Y_2)$.

Proposition A.4.6. Let $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$ be ($\mathcal{F}$-type) decomposition spaces with underlying covered spaces $(X, \mathcal{Q})$ and $(Z, \mathcal{P})$. If

$$F\colon \mathcal{D}(\mathcal{Q}, B_1, Y_1) \to \mathcal{D}(\mathcal{P}, B_2, Y_2)$$

is a geometric embedding, then F induces a quasi-isometric embedding between the metric spaces $(X, d_{\mathcal{Q}})$ and $(Z, d_{\mathcal{P}})$. In particular, the decomposition spaces $\mathcal{D}(\mathcal{Q}, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$ can be geometrically isomorphic only when the associated metric spaces $(X, d_{\mathcal{Q}})$ and $(Z, d_{\mathcal{P}})$ are quasi-isometric.

Proof. Assume that $F\colon \mathcal{D}(\mathcal{Q}, B_1, Y_1) \to \mathcal{D}(\mathcal{P}, B_2, Y_2)$ is a geometric embedding. We define a map $\eta\colon (X, d_{\mathcal{Q}}) \to (Z, d_{\mathcal{P}})$ as follows: For $x \in X$ we have $x \in Q_i$ for some $i \in I$. Choose a non-zero function $f \in \mathcal{D}(\mathcal{Q}, B_1, Y_1)$ with $C[f] \subset Q_i^{k*}$ for some $k \in \mathbb{N}_0$. Since F is injective there exists an element $y \in C\,[F(f)]$. Define $\eta(x) = y$. The estimate (A.4.6) gives that η is a quasi-isometric embedding. □

We can now use results we have developed for covered spaces to deduce obstructions about geometric embeddings between ($\mathcal{F}$-type) decomposition spaces.

Whenever we consider the uniform covering $\mathcal{U}(G)$ on a path-connected, locally compact group G, we use the simplified notation

$$\mathcal{D}(G, B, Y) := \mathcal{D}(\mathcal{U}(G), B, Y), \qquad \mathcal{D}^{\mathcal{F}}(G, B, Y) := \mathcal{D}^{\mathcal{F}}(\mathcal{U}(G), B, Y).$$

Proposition A.4.7. There are no geometric embeddings

$$\begin{aligned} \mathcal{D}(\mathbb{R}^k, B_1, Y_1) &\longrightarrow \mathcal{D}(\mathbb{R}^l, B_2, Y_2), && l < k, \\ \mathcal{D}(\mathbb{H}_{2m+1}, B_3, Y_3) &\longrightarrow \mathcal{D}(\mathbb{H}_{2n+1}, B_4, Y_4), && n < m, \\ \mathcal{D}(\mathbb{R}^k, B_1, Y_1) &\longrightarrow \mathcal{D}(\mathbb{H}_{2n+1}, B_4, Y_4), && 2n+1 < k, \\ \mathcal{D}(\mathbb{H}_{2m+1}, B_3, Y_3) &\longrightarrow \mathcal{D}(\mathbb{R}^l, B_2, Y_2), && l < 2m+1, \end{aligned}$$

where $B_1, \ldots, B_4$ and $Y_1, \ldots, Y_4$ are arbitrary Banach spaces satisfying the standing assumptions. The decomposition spaces $\mathcal{D}(\mathbb{R}^k, B_1, Y_1)$ and $\mathcal{D}(\mathbb{H}_{2n+1}, B_4, Y_4)$ are not geometrically isomorphic for any $n, k \in \mathbb{N}$.

Proof. It follows from Proposition A.4.6 that it suffices to show that there are no quasi-isometric embeddings between the underlying uniform metric spaces. Assume by contradiction that there exists a quasi-isometric embedding

$$F : (\mathbb{H}_{2m+1}, d_{\mathcal{U}}) \to (\mathbb{H}_{2n+1}, d_{\mathcal{U}}), \qquad n < m.$$

Then

$$\operatorname{asdim}(\mathbb{H}_{2m+1}, d_{\mathcal{U}}) \leq \operatorname{asdim}(\mathbb{H}_{2n+1}, d_{\mathcal{U}}).$$

However, this contradicts [33, Theorem 3.5] stating that the asymptotic dimension of the net $(\mathbb{Z}^{2n+1}, *)$ in $\mathbb{H}_{2n+1}$ is equal to $2n+1$. Since we know that

$$\operatorname{asdim}(\mathbb{R}^k, d_{\mathcal{U}}) = \operatorname{asdim}(\mathbb{Z}^k, +) = k,$$

the other statements follows. The last claim follows from Example A.3.5. □

Remark A.4.8. Since $\mathbb{H}_{2m+1}$ is diffeomorphic to $\mathbb{R}^{2m+1}$ we can consider the uniform covering $\mathcal{U}$ on $\mathbb{H}_{2m+1}$ as a covering on $\mathbb{R}^{2m+1}$. Hence $\mathcal{D}^{\mathcal{F}}(\mathbb{H}_{2m+1}, L^q, Y_2)$ is well-defined. The statements in Proposition A.4.7 also hold if we consider the $\mathcal{F}$-type decomposition spaces $\mathcal{D}^{\mathcal{F}}(\mathbb{R}^k, L^p, Y_1)$ and $\mathcal{D}^{\mathcal{F}}(\mathbb{H}_{2m+1}, L^q, Y_2)$ for $1 \leq p, q < \infty$.

A.4.3 Spatially Implemented Geometric Embeddings

In Proposition A.4.6 we showed that geometric embeddings

$$F : \mathcal{D}(\mathcal{Q}, B_1, Y_1) \to \mathcal{D}(\mathcal{P}, B_2, Y_2)$$

between ($\mathcal{F}$-type) decomposition spaces induce quasi-isometric embeddings between the associated metric spaces (X, d_Q) and $(Z, d_\mathcal{P})$ of the underlying coverings. A question that naturally arises is whether the opposite might be true in certain situations: Does a quasi-isometric embedding between (X, d_Q) and $(Z, d_\mathcal{P})$ induce a geometric embedding between the ($\mathcal{F}$-type) decomposition spaces $\mathcal{D}(Q, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$? Although the answer in general is no, we present criteria for when this holds and examine an illustrative example.

Firstly, we need to examine how a quasi-isometric embedding affects the global components of decomposition spaces. Let Y be a sequence space on the countable index set I satisfying the standard assumptions given in Subsection A.4.1. Consider two admissible coverings $Q := (Q_i)_{i\in I}$ and $\mathcal{P} := (P_j)_{j\in J}$ on the sets X and Z, respectively. Assume that $\phi\colon (Z, d_\mathcal{P}) \to (X, d_Q)$ is a surjective quasi-isometric embedding. For each $j \in J$ we pick $i \in I$, denoted by $\phi(j)$, such that

$$\phi(P_j) \cap Q_i \neq \emptyset.$$

If this selection can be performed such that each $i \in I$ is picked precisely once, then we say that ϕ *induces a bijection between index sets*. If this is so, we define the normed sequence space $(Y_\phi, \|\cdot\|_{Y_\phi})$ by

$$Y_\phi := \left\{(x_j)_{j\in J} \in \mathbb{C}^J : \left(x_{\phi^{-1}(i)}\right)_{i\in I} \in Y\right\},$$

with norm

$$\left\|(x_j)_{j\in J}\right\|_{Y_\phi} := \left\|\left(x_{\phi^{-1}(i)}\right)_{i\in I}\right\|_Y .$$

Let us see why the sequence space Y_ϕ does not depend on the precise choice of bijection that ϕ induces: Consider two induced bijections $\phi_1, \phi_0\colon J \to I$ and let $i \in I$ be arbitrary. Then for $j := \phi_1^{-1}(i)$ and $l = \phi_0^{-1}(i)$ we have $\phi_1(P_j) \cap Q_i \neq \emptyset$ and $\phi_0(P_l) \cap Q_i \neq \emptyset$. For $x \in \phi_1(P_j) \cap Q_i$ and $y \in \phi_0(P_l) \cap Q_i$ we use that ϕ is a quasi-isometric embedding to obtain

$$d_\mathcal{P}(z_x, z_y) \leq L + C, \quad z_x \in \phi^{-1}(x),\ z_y \in \phi^{-1}(y).$$

Hence there exists a $k = k(L, C) \in \mathbb{N}$ such that $j \in l^{k*}$. The fact that the clustering map Γ_Q is bounded on Y ensures the required independence. It is straightforward to check that all properties required of the global component of a decomposition space are satisfied for Y_ϕ if they are satisfied for Y.

Theorem A.4.9. *Let $\phi\colon (Z, d_\mathcal{P}) \to (X, d_Q)$ be a surjective quasi-isometric embedding between the associated metric space of two covered spaces (X, Q) and $(Z, \mathcal{P})$ that induces a bijection between index sets. Consider two ($\mathcal{F}$-type) decomposition spaces $\mathcal{D}(Q, B_1, Y)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_\phi)$ where the local components B_1 and B_2 consist of functions on X and Z, respectively. Assume that the mapping*

$$\phi^* f(y) := f(\phi(y))$$

between B_1 and B_2 is bounded. Then ϕ induces a geometric embedding from $\mathcal{D}(Q, B_1, Y)$ to $\mathcal{D}(\mathcal{P}, B_2, Y_\phi)$ on the form

$$\phi^* f := \sum_{i \in I} \phi^*(f \cdot \varphi_i),$$

where $\Phi := (\varphi_i)_{i \in I}$ is any choice of Q-BAPU.

Proof. Let us fix a Q-BAPU $\Phi := (\varphi_i)_{i\in I}$ and write $f = \sum_{i\in I} f \cdot \varphi_i$ for each $f \in \mathcal{D}(Q, B_1, Y)$ by Proposition A.4.3. Then using that ϕ induces a bijection between the index sets allows us to write

$$\phi^* f = \sum_{i \in I} \phi^*(f \cdot \varphi_i) = \sum_{j \in J} \phi^*(f \cdot \varphi_{\phi(j)}),$$

where $\phi^*(f \cdot \varphi_{\phi(j)}) \in B_2$ by the boundedness of ϕ^*. We want to apply the last statement Proposition A.4.3 to conclude that $\phi^* f \in \mathcal{D}(\mathcal{P}, B_2, Y_\phi)$. To do this, we need to first check that the support condition is satisfied.

We denote as usual the quasi-isometric parameters of ϕ by $L, C > 0$. Let $j \in J$ be arbitrary and set $i := \phi(j)$. Since $\phi(P_j) \cap Q_i \neq \emptyset$ we can find $y_i \in P_j \subset Z$ such that $\phi(y_i) \in Q_i$. Then the constraint $d_{\mathcal{P}}(y, y_i) > L(C + 1)$ on $y \in Z$ ensures that $\phi(y) \notin Q_i$ since we have

$$d_Q(\phi(y), \phi(y_i)) \geq \frac{1}{L} d_{\mathcal{P}}(y, y_i) - C > 1.$$

Hence

$$\begin{aligned} \operatorname{supp}\left(\phi^*(f \cdot \varphi_{\phi(j)})\right) &= \operatorname{supp}\left(\phi^*(f \cdot \varphi_i)\right) \\ &\subset \{y \in Z : d_{\mathcal{P}}(y, y_i) \leq L(C+1)\} \\ &\subset P_j^{k*}, \end{aligned}$$

for some fixed $k = k(C, L) \in \mathbb{N}$. The equivalence

$$\left(\|\phi^*(f \cdot \varphi_{\phi(j)})\|_{B_2}\right)_{j \in J} \in Y_\phi \Leftrightarrow \left(\|\phi^*(f \cdot \varphi_i)\|_{B_2}\right)_{i \in I} \in Y$$

together with the boundedness of $\phi^* \colon B_1 \to B_2$ ensure that we can apply the last statement of Proposition A.4.3 to obtain $\phi^* f \in \mathcal{D}(\mathcal{P}, B_2, Y_\phi)$. Moreover, the boundedness of ϕ implies that there exists a constant $S > 0$ such that

$$\|\phi^* f\|_{\mathcal{D}(\mathcal{P}, B_2, Y_\phi)} \leq S \|f\|_{\mathcal{D}(Q, B_1, Y)}.$$

To show injectivity of ϕ^* we make the following observation: For an element $f \in \mathcal{D}(Q, B_1, Y)$ we have $f \cdot \varphi_i \in B_1$ as a genuine function. Since $\sum_{i \in I} f \cdot \varphi_i = f$

in the norm of $\mathcal{D}(Q, B_1, Y)$ by Proposition A.4.3, we can make sense of f as a function on X. Assume that $\phi^* f = 0$. Then

$$0 = \sum_{i \in I} \phi^*(f \cdot \varphi_i) = \sum_{i \in I} (f \circ \phi) \cdot (\varphi_i \circ \phi).$$

Since $(\varphi_i \circ \phi)_{i \in I}$ is a partition of unity on Z we have that $f \circ \phi$ is the zero function on Z. Thus the surjectivity of ϕ implies that f is the zero function on X. Hence $f = 0$ in B_1 and injectivity follows. □

Remark. There are several ways of modifying the statement in Theorem A.4.9 to obtain useful variants. To illustrate this, let us consider $B_1 = L^p$ and $B_2 = L^q$ for $1 \leq p, q < \infty$ on the spaces $X = \mathbb{R}^n$ and $Z = \mathbb{R}^m$. Since the spaces B_1 and B_2 consist of equivalence classes of functions and not functions themselves, we can not apply Theorem A.4.9 in this setting. A closer look at the proof of injectivity of ϕ^* above shows that we the only thing we can conclude from the statement $\phi^* f = 0$ in $B_2 = L^q$ is that $\phi^* f$ is zero almost everywhere as a function on $\mathbb{R}^m$. If we add the assumption that $\phi\colon Z = \mathbb{R}^m \to X = \mathbb{R}^n$ should map sets with measure zero to sets with measure zero (with respect to the respective Lebesgue measures), then the following argument carries through: If $\phi^* f = 0$ in B_2 then $f \circ \phi$ is zero on a set $Z \setminus N \subset Z$ where N has measure zero. Then $X = \phi(Z \setminus N) \cup \phi(N)$ due to the surjectivity of ϕ and $\phi(N)$ has measure zero. Hence f is zero almost everywhere and hence represents the equivalence class of the zero function in L^p. Therefore ϕ^* is injective. The assumption that ϕ should preserve sets with Lebesgue measure zero is easily satisfied in concrete situations.

We will refer to the geometric embeddings in Theorem A.4.9 as being *spatially implemented*. It should be remarked that not all geometric embeddings need to be spatially implemented, see Theorem A.5.2. Since surjective quasi-isometric embeddings are quasi-isometries, we can only hope to find spatially implemented geometric embeddings between decomposition spaces $\mathcal{D}(Q, B_1, Y_1)$ and $\mathcal{D}(\mathcal{P}, B_2, Y_2)$ whenever $(X, d_Q) \simeq (Z, d_{\mathcal{P}})$. Looking back at Example A.2.9 gives an obvious candidate that we now examine.

Consider the decomposition space

$$\mathbb{B}^s_{p,q}(\mathbb{R}^n) := \mathcal{D}\left(\mathcal{B}, L^p, l^q_{\omega(s)}\right), \tag{A.4.7}$$

for $1 \leq p, q < \infty$ where $\mathcal{B}(\mathbb{R}^n)$ is the dyadic covering on $\mathbb{R}^n$ given in Example A.2.9 and $\omega(s)$ is the weight $\omega(s)(j) := 2^{js}$ for $j \in \mathbb{N}_0$. We denote by $\mathbb{B}^s_{p,q}(\mathbb{R}_+)$ the decomposition space whose underlying covered space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ is the positive line with the restricted dyadic covering and the local and global components are the same as in (A.4.7). The notation $\mathbb{B}^s_{p,q}(\mathbb{R}^n)$ is motivated by the fact that the *(inhomogeneous) Besov spaces* $B^s_{p,q}(\mathbb{R}^n)$ appearing in classical harmonic analysis have the $\mathcal{F}$-type decomposition space description

$$B^s_{p,q}(\mathbb{R}^n) = \mathcal{D}^{\mathcal{F}}\left(\mathcal{B}, L^p, l^q_{\omega(s)}\right).$$

The reason we consider $\mathbb{B}^s_{p,q}(\mathbb{R}^n)$ instead of the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ is because we can then use Theorem A.4.9 to obtain a spatially implemented geometric embedding.

Proposition A.4.10. There is a spatially implemented geometric embedding from $\mathbb{B}^s_{p,q}(\mathbb{R}_+)$ to $\mathbb{B}^{ns}_{p,q}(\mathbb{R}^n)$ for any $n \geq 1$.

Proof. To invoke Theorem A.4.9 we define a map

$$\begin{aligned} \phi\colon \mathbb{R}^n &\longrightarrow \mathbb{R}_+ \\ x = (x_1, \dots, x_n) &\longmapsto (x_1^2 + \cdots + x_n^2)^{\frac{n}{2}} = \|x\|_2^n. \end{aligned}$$

The first step is to show that ϕ is a quasi-isometry. Associate to any $x \in \mathbb{R}^n$ the smallest number $m(x) \in \mathbb{N}_0$ such that $\|x\|_2 \leq 2^{m(x)}$. It is clear that the distance $d_{\mathcal{B}(\mathbb{R}^n)}(x, y)$ between two points $x, y \in \mathbb{R}^n$ satisfies

$$d_{\mathcal{B}(\mathbb{R}^n)}(x, y) = d_{\mathcal{B}(\mathbb{R}^n)}((2^{m(x)}, \dots, 0), (2^{m(y)}, \dots, 0)) + \alpha = |m(x) - m(y)| + \alpha,$$

where α will denote a constant that is either one or zero (consider when x and y are in the same dyadic interval to see the necessity of α). Then we have

$$d_{\mathcal{B}(\mathbb{R}_+)}(\phi(x), \phi(y)) = d_{\mathcal{B}(\mathbb{R}_+)}\left(2^{m(x)n}, 2^{m(y)n}\right) + \alpha = n|m(x) - m(y)| + \alpha.$$

This is clearly a quasi-isometric embedding with parameters $L = n$ and $C = 1$. It is also clear that $\phi(\mathbb{R}^n)$ is all of $\mathbb{R}_+$ by considering the image of any line through the origin. Hence ϕ is a surjective quasi-isometry.

However, ϕ induces the map $\mathbb{N}_0 \ni m \mapsto nm \in \mathbb{N}_0$ between the index sets. Since this is not a bijection (unless $n = 1$) we need to make the following modification: Scale the dyadic covering on $\mathbb{R}^n$ so that the dyadic intervals have the form

$$\widetilde{D_0} := \left\{x \in \mathbb{R}^n : \|x\|_2 \leq 2^{\frac{1}{n}}\right\}, \qquad \widetilde{D_m} := \left\{x \in \mathbb{R}^n : 2^{\frac{m-1}{n}} \leq \|x\|_2 \leq 2^{\frac{m+1}{n}}\right\}.$$

The scaled dyadic covering still defines the same decomposition space $B^s_{p,q}(\mathbb{R}^n)$ and the map ϕ satisfies $\phi(\widetilde{D_l}) = D_l$ for all $l \in \mathbb{N}_0$. Hence we obtain that ϕ induces a bijection between index sets and the correct sequence space on $\mathbb{R}^n$ is

$$\left(l^q_{\omega(s)}\right)_\phi = l^q_{\omega(ns)}.$$

We can apply Theorem A.4.9 as longs as we can show that the mapping $\phi^* f(y) = f(\phi(y))$ between $L^p(\mathbb{R}_+)$ and $L^p(\mathbb{R}^n)$ is both bounded above and below. A computation using spherical coordinates gives that

$$\begin{aligned}
\|\phi^*(f)\|_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \left| f\left(\left(x_1^2 + \cdots + x_n^2\right)^{\frac{n}{2}}\right)\right|^p dx_1 \cdots dx_n\right)^{\frac{1}{p}} \\
&= \left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}\right)^{\frac{1}{p}} \left(\int_0^\infty |f(r^n)|^p r^{n-1} dr\right)^{\frac{1}{p}} \\
&= \left(\frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}\right)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}_+)},
\end{aligned}$$

where Γ denotes the Gamma function. Hence Theorem A.4.9 implies that ϕ^* is a spatially implemented geometric embedding from $\mathbb{B}^s_{p,q}(\mathbb{R}_+)$ to $\mathbb{B}^{ns}_{p,q}(\mathbb{R}^n)$. □

A.5 Examples

In this final section we will put our developed machinery to the test in concrete settings. We will consider the modulation spaces, both on $\mathbb{R}^n$ and on the Heisenberg group $\mathbb{H}_{2n+1}$; the latter case was recently considered in [64]. Finally, we describe a class of decomposition spaces in Subsection A.5.3 where the underlying covering is quasi-hyperbolic.

A.5.1 Euclidean Modulation Spaces

Modulation spaces are a class of function spaces in time-frequency analysis that have been extensively studied in the last decades. They were introduced by Hans Georg Feichtinger and is widely recognized as the correct setting for theoretical time-frequency analysis after its appearance in the standard reference on the topic [81]. The original description was given by Feichtinger in the language of decomposition spaces, while the modern approach is usually through integrability of the short-time Fourier transform. We will begin by giving a brief review of the modern approach. In Theorem A.5.2 we show that geometric embeddings between modulation spaces in different dimensions can only exist when the dimension is increasing.

The two fundamental operators in time-frequency analysis are the *time-shift* operator T_x and the *frequency-shift* operator M_ω. They act on $f \in L^2(\mathbb{R}^n)$ by

$$T_x f(t) := f(t-x), \quad M_\omega f(t) := e^{2\pi i t\cdot\omega} f(t), \quad x, \omega \in \mathbb{R}^n.$$

Given two functions $f, g \in L^2(\mathbb{R}^n)$ where $g \neq 0$ we define the *short-time Fourier transform* (STFT) of f with respect to g to be

$$V_g f(x,\omega) := \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\, dt = \langle f, M_\omega T_x g\rangle_{L^2(\mathbb{R}^n)}. \tag{A.5.1}$$

This gives us localized frequency information about f by looking through the "window" g. It is clear from the inner product interpretation in (A.5.1) that we can extend the STFT to the setting where $f \in \mathcal{S}'(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$ by duality.

Definition A.5.1. Fix $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and constants $1 \leq p, q < \infty$. We define the *(non-weighted) modulation space* $M^{p,q}(\mathbb{R}^n)$ to be all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ that satisfies $\|f\|_{M^{p,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{M^{p,q}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_g f(x,\omega)|^p\, dx\right)^{\frac{q}{p}} d\omega\right)^{\frac{1}{q}}.$$

It follows from [81, Proposition 11.3.2] that different choices of functions $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ yield equivalent norms. Moreover, the spaces $M^{p,q}(\mathbb{R}^n)$ are Banach spaces where the time-shift operators and the frequency-shift operators act by isometries [81, Theorem 11.3.5].

The modulation spaces have, in addition to their STFT-description, a presentation as $\mathcal{F}$-type decomposition spaces

$$M^{p,q}(\mathbb{R}^n) \simeq \mathcal{D}^{\mathcal{F}}(\mathbb{R}^n, L^p, l^q). \tag{A.5.2}$$

One refers to the description of $M^{p,q}(\mathbb{R}^n)$ given in Definition A.5.1 as the *coorbit description* of $M^{p,q}(\mathbb{R}^n)$, while (A.5.2) is referred to as the *decomposition description* of $M^{p,q}(\mathbb{R}^n)$.

Theorem A.5.2. *There is a tower of compatible geometric embeddings*

$$M^{p,q}(\mathbb{R}) \xrightarrow{\Gamma_1^2} M^{p,q}(\mathbb{R}^2) \xrightarrow{\Gamma_2^3} \cdots \xrightarrow{\Gamma_{n-1}^n} M^{p,q}(\mathbb{R}^n) \xrightarrow{\Gamma_n^{n+1}} \cdots,$$

where there are no geometric embeddings in the other direction.

Proof. It follows from Proposition A.4.7 and Remark A.4.8 that there are no geometric embeddings from $M^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^m)$ whenever $n > m$. We will now show that $M^{p,q}(\mathbb{R}^n)$ can be geometrically embedded into $M^{p,q}(\mathbb{R}^m)$ as long as $n \leq m$.

Define the map

$$\Gamma_n^m : \mathcal{S}(\mathbb{R}^n) \subset M^{p,q}(\mathbb{R}^n) \longrightarrow M^{p,q}(\mathbb{R}^m)$$

given by

$$f \longmapsto \Gamma_n^m(f)(\xi_1, \ldots, \xi_m) \coloneqq \mathcal{F}_m^{-1}\left(\mathcal{F}_n(f)(\xi_1, \ldots, \xi_n)\eta(\xi_{n+1}) \cdots \eta(\xi_m)\right),$$

where $0 \neq \eta \in C_c^\infty(\mathbb{R})$ and $\mathcal{F}_n$ denotes the n-dimensional Fourier transform. It is clear that the condition (A.4.6) is satisfied. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}(\mathbb{R}^n)$ by [81, Theorem 12.2.2] it suffices to show boundedness of Γ_n^m. To show this, we utilize the coorbit description of $M^{p,q}(\mathbb{R}^n)$. Since the Fourier transform interchanges time-shift operators and frequency-shift operators, it follows that $\mathcal{F}_n$ is a bounded operator from $M^{p,q}(\mathbb{R}^n)$ to $M^{q,p}(\mathbb{R}^n)$. Hence it suffices to show that the map $f \mapsto f \otimes \eta$ is a bounded map from $\mathcal{S}(\mathbb{R}^n) \subset M^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^{n+1})$ whenever $0 \neq \eta \in C_c^\infty(\mathbb{R})$ and $1 \leq p, q < \infty$.

The standard Gaussian $g_{n+1}(x) \coloneqq e^{-\pi x^2}$ on $\mathbb{R}^{n+1}$ splits as

$$g_{n+1}(x) = (g_n \otimes g_1)(x) \coloneqq g_n(\bar{x})g_1(x_{n+1}),$$

where $x = (x_1, \ldots, x_{n+1})$ and $\bar{x} = (x_1, \ldots, x_n)$. Hence

$$V_{g_{n+1}}(f \otimes \eta)(x, \omega) = V_{g_n \otimes g_1}(f \otimes \eta)(x, \omega) = V_{g_n} f(\bar{x}, \bar{\omega}) \cdot V_{g_1}\eta(x_{n+1}, \omega_{n+1}).$$

A straightforward calculation gives that

$$\|f \otimes \eta\|_{M^{p,q}(\mathbb{R}^{n+1})} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|V_{g_1}\eta(x_{n+1}, \omega_{n+1})\right|^p \, dx_{n+1}\right)^{\frac{q}{p}} \, d\omega_{n+1}\right)^{\frac{1}{q}} \cdot \|f\|_{M^{p,q}(\mathbb{R}^n)}.$$

Since $0 \neq \eta \in C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \subset M^{p,q}(\mathbb{R})$ it follows that Γ_n^m is a bounded map from $M^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^m)$.

The reason Γ is injective when viewed as a mapping from $M^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^m)$ is because the Fourier transform is an injective map from $M^{p,q}(\mathbb{R}^n)$ to $M^{q,p}(\mathbb{R}^n)$ and that $\eta \neq 0$. Hence Γ_n^m extends to a geometric embedding from $M^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^m)$ for $n \leq m$. Finally, the embeddings we constructed respect composition $\Gamma_m^l \circ \Gamma_n^m = \Gamma_n^l$ for all $l \geq m \geq n \geq 1$. □

We can say even more by allowing the indices $1 \leq p, q < \infty$ to vary. It follows from [81, Theorem 12.2.2] that we have the estimate

$$\|f\|_{M^{p_2,q_2}(\mathbb{R}^k)} \leq A\|f\|_{M^{p_1,q_1}(\mathbb{R}^k)}$$

for some $A > 0$ whenever $p_1 \leq p_2$ and $q_1 \leq q_2$.

Corollary A.5.3. Whenever $p_1 \leq p_2$, $q_1 \leq q_2$, and $n \leq m$ there exists a geometric embedding from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^m)$. In particular, there exists a geometric embedding from the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}) := M^{1,1}(\mathbb{R})$ to any modulation space $M^{p,q}(\mathbb{R}^n)$.

Hence the Feichtinger algebra is universal in the class of (non-weighted) modulation spaces on Euclidean spaces. Therefore, any ($\mathcal{F}$-type) decomposition space that embeds geometrically into $\mathcal{S}_0(\mathbb{R})$ does in fact embed geometrically into all the modulation spaces $M^{p,q}(\mathbb{R}^n)$.

A.5.2 Heisenberg Modulation Spaces

The STFT introduced in (A.5.1) is intimately related to the Heisenberg group $\mathbb{H}_{2n+1}$ in the following way: Define the *Schrödinger representation*

$$\rho\colon \mathbb{H}_{2n+1} \to \mathcal{U}(L^2(\mathbb{R}^n))$$

by

$$\rho(x, \omega, t) := e^{\pi i(2t + x\cdot\omega)} T_x M_\omega,$$

where $x, \omega \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\mathcal{U}(L^2(\mathbb{R}^n))$ denotes the unitary operators on $L^2(\mathbb{R}^n)$. Then a short computations shows that the matrix coefficients of the Schrödinger representation are (up to a phase factor) the STFT. The Stone-von Neumann theorem [81, Theorem 9.3.1] emphasizes the importance of the Schrödinger representation as it is essentially the only interesting irreducible unitary representation of the Heisenberg group.

Although it is clear from Definition A.5.1 that the Heisenberg group $\mathbb{H}_{2n+1}$ play a role in the traditional modulation spaces, the underlying covering of $M^{p,q}(\mathbb{R}^n)$ has $\mathbb{Z}^n$ as its associated metric space and not the discrete Heisenberg groups. Recently, decomposition spaces originating from a coorbit description of a certain nilpotent Lie group have been investigated in [64]. These decomposition spaces are truly related to the large scale geometry of the Heisenberg group. We outline their construction and extend one of their main results [64, Theorem 7.6] to geometric embeddings in Proposition A.5.4 since all the hard work has already been done in Section A.3 and Section A.4. We believe that our approach can make arguments clearer and emphasize the importance of viewing coverings from a metric perspective. Thus we are able to approach some of the novel results in [64] from a different angle because of our large scale machinery.

The *(abstract) Dynin-Folland Lie algebra* $\mathfrak{h}_{n,2}$ is the nilpotent Lie algebra with basis

$$\langle X_{u_1}, \ldots, X_{u_n}, X_{v_1}, \ldots, X_{v_n}, X_w, X_{x_1}, \ldots, X_{x_n}, X_{y_1}, \ldots, X_{y_n}, X_z, X_s \rangle, \quad \text{(A.5.3)}$$

and with non-vanishing commutation relations

$$[X_{u_j}, X_{v_k}]_{\mathfrak{h}_{n,2}} = \delta_{j,k} X_w, \quad [X_{u_j}, X_{x_k}]_{\mathfrak{h}_{n,2}} = \delta_{j,k} X_s, \quad [X_{u_j}, X_z]_{\mathfrak{h}_{n,2}} = -\frac{1}{2} X_{y_j},$$
$$[X_{v_j}, X_{y_k}]_{\mathfrak{h}_{n,2}} = \delta_{j,k} X_s, \quad [X_{v_j}, X_z]_{\mathfrak{h}_{n,2}} = \frac{1}{2} X_{x_j}, \qquad [X_w, X_z]_{\mathfrak{h}_{n,2}} = X_s,$$

where $j, k = 1, \ldots, n$. The first $2n+1$ basis vectors in (A.5.3) generate a subalgebra that is isomorphic to the Lie algebra of the Heisenberg group $\mathbb{H}_{2n+1}$. We denote by $\mathbf{H}_{n,2}$ the connected and simply connected Lie group corresponding to $\mathfrak{h}_{n,2}$ called the *Dynin-Folland group*.

In [64, Theorem 4.5 and Corollary 4.7] the authors classify all the irreducible and projective representations of the Dynin-Folland group by using Kirillov's orbit method. One of these projective representations is used to define the *Heisenberg modulation spaces* similarly to how the Schrödinger representation is used to define the modulation spaces $M^{p,q}(\mathbb{R}^n)$. We refer the reader to [64] for the explicit description as we will only need the decomposition space description of the Heisenberg modulation spaces.

In [64] they consider the lattice in $\mathbb{H}_{2n+1} \simeq \mathbb{R}^{2n+1}$ defined by

$$\Gamma := \left\{(a, b, c) \in \mathbb{R}^{2n+1} : a, b \in (2\mathbb{Z})^n,\ c \in 2\mathbb{Z}\right\}.$$

From this a covering $\mathcal{P}$ on $\mathbb{H}_{2n+1} \simeq \mathbb{R}^{2n+1}$ is induced by defining

$$\mathcal{P} := \left\{P * \gamma : P = (-\epsilon, 2+\epsilon)^{2n+1},\ \gamma \in \Gamma\right\},$$

where $\epsilon \in (0, \frac{1}{2})$ and the multiplication $P * \gamma$ is with the Heisenberg group structure. Define the $\mathcal{F}$-type decomposition spaces

$$E^{p,q}\left(\mathbb{H}_{2n+1}\right) := \mathcal{D}^{\mathcal{F}}(\mathcal{P}, L^p, l^q),$$

where $1 \leq p, q < \infty$ and the reservoir is the tempered distributions.

We remark that [64] consider the spaces with weights derived from the *homogeneous Cygan-Koranyi norm*

$$(p, q, t) \longmapsto \left((|p|^2 + |q|^2)^2 + 16t^2\right)^{\frac{1}{4}}.$$

We omit this extension as all the geometric features are already present in the case without weights. Moreover, we refer the reader to [64, Theorem 7.3] where the authors show that the spaces $E^{p,q}\left(\mathbb{H}_{2n+1}\right)$ coincide with the Heisenberg modulation spaces arising from the projective representations of the Dynin-Folland group $\mathbf{H}_{n,2}$.

Proposition A.5.4. None of the spaces $E^{p,q}\left(\mathbb{H}_{2n+1}\right)$, $M^{p,q}(\mathbb{R}^k)$, and $B^s_{p,q}(\mathbb{R}^l)$ are geometrically isomorphic for any values $n, k, l \geq 1$, $p, q \in [1, \infty)$.

Proof. It is clear from the results in Section A.3 that the covering $\mathcal{P}$ is the uniform covering on $\mathbb{H}_{2n+1}$. Since any lattice in a stratified Lie group is uniform, the lattice Γ is a net in $(\mathbb{H}_{2n+1}, d_{\mathcal{P}})$. Thus Proposition A.3.4 implies that the uniform metric space $(\mathbb{H}_{2n+1}, d_{\mathcal{P}})$ is quasi-isometric to Γ equipped with any proper, left-invariant metric.

The fact that $E^{p,q}(\mathbb{H}_{2n+1})$ and $M^{p,q}(\mathbb{R}^k)$ are not geometrically isomorphic follows from Proposition A.4.7. The order of the polynomial growth of Γ is $2n + 2$ by Theorem A.3.6, while the growth of the underlying covering of the Besov space $B^{p,q}(\mathbb{R}^l)$ is linear. Hence the spaces $E^{p,q}(\mathbb{H}_{2n+1})$ and $B^s_{p,q}(\mathbb{R}^l)$ are not geometrically isomorphic by Proposition A.4.6 since growth type is a quasi-isometric invariant. The modulation spaces $M^{p,q}(\mathbb{R}^k)$ and Besov spaces $B^s_{p,q}(\mathbb{R}^l)$ are not geometrically isomorphic by Proposition A.4.6 and Example A.2.15. $\square$

Notice that for $p = q = 2$, all three spaces $M^{2,2}(\mathbb{R}^{2n+1})$, $B_{2,2}(\mathbb{R}^{2n+1})$ and $E^{2,2}(\mathbb{H}_{2n+1})$ are all simply $L^2(\mathbb{R}^{2n+1})$ as Banach spaces by [148, Lemma 6.10]. However, the identity map

$$Id\colon M^{2,2}(\mathbb{R}^{2n+1}) \longrightarrow E^{2,2}(\mathbb{H}_{2n+1})$$

is not a geometric isomorphism between $\mathcal{F}$-type decomposition spaces since the associated metric spaces of the underlying coverings are not quasi-isometric. Hence geometric isomorphisms incorporate the coverings and thus treat decomposition spaces as more than Banach spaces.

A.5.3 A Decomposition Space of Hyperbolic Type

So far, we have looked at several examples of decomposition spaces that have already been present in the literature. We conclude by examining a new decomposition space having an underlying covering whose associated metric space is quasi-hyperbolic (and not infinite cyclic).

Definition A.5.5. We call the space

$$\mathcal{D}^{p,q}(SL(2,\mathbb{R})) := \mathcal{D}(SL(2,\mathbb{R}), L^p, l^q)$$

the *hyperbolic decomposition space* with parameters $1 \leq p, q < \infty$. Here L^p denotes the (equivalence classes of) p'th integrable functions on $SL(2,\mathbb{R})$ with respect to the Haar measure on $SL(2,\mathbb{R})$. Whenever $p = q = 1$, we call $\mathcal{D}(SL(2,\mathbb{R})) := \mathcal{D}^{1,1}(SL(2,\mathbb{R}))$ the *standard hyperbolic decomposition space*.

Since the group $SL(2,\mathbb{R})$ is unimodular we do not need to distinguish between the left and right Haar measure on $SL(2,\mathbb{R})$. The terminology *hyperbolic decomposition space* is motivated by Theorem A.3.11. We can take the reservoir to be

$\mathcal{A} = C_b(SL(2,\mathbb{R}),\mathbb{C})$ as this is of minor importance by [55, Theorem 1 (ii)]. It follows from Proposition A.4.3 that $\mathcal{D}^{p,q}(SL(2,\mathbb{R}))$ is reflexive as a Banach space whenever $1 < p, q < \infty$.

Example A.5.6. Let us, for the sake of concreteness, give an example of an element in the standard hyperbolic decomposition space $\mathcal{D}(SL(2,\mathbb{R}))$. Every element $\alpha \in SL(2,\mathbb{R})$ has an *Iwasawa decomposition*

$$\alpha = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} y & x \\ 0 & \frac{1}{y} \end{pmatrix},$$

for $0 \le \theta < 2\pi$, $x \in \mathbb{R}$, and $y > 0$ [31, Chapter 26]. We will write elements in $SL(2,\mathbb{R})$ as (θ, x, y) according to their Iwasawa decomposition. In these coordinates, the Haar measure on $SL(2,\mathbb{R})$ is given by $y^{-2}dx\,dy\,d\theta$.

Consider the function $f\colon SL(2,\mathbb{R}) \to \mathbb{R}_+$ given by

$$f(\theta, x, y) = y^3 e^{-y-x^2}.$$

Then a short computation shows that

$$\|f\|_{L^1} = \int_{SL(2,\mathbb{R})} f(z)d\mu(z) = \int_0^{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} y^3 e^{-y-x^2}\frac{dx\,dy\,d\theta}{y^2} = 2\pi^{\frac{3}{2}},$$

by utilizing the value of the Gamma function at zero. Since f is positive, we have the trivial estimate

$$\|f\|_{\mathcal{D}(SL(2,\mathbb{R}))} \le N_{\mathcal{U}} 2\pi^{\frac{3}{2}},$$

where $N_{\mathcal{U}}$ is the admissibility constant of the uniform covering $\mathcal{U}$.

We will now show that the hyperbolic decomposition space $\mathcal{D}^{p,q}(SL(2,\mathbb{R}))$ is fundamentally different from the decomposition spaces we previously examined.

Proposition A.5.7. There are no geometric embeddings

$$\begin{aligned} \phi_n\colon M^{p,q}(\mathbb{R}^n) &\longrightarrow \mathcal{D}^{p,q}(SL(2,\mathbb{R})), \\ \psi_m\colon E^{p,q}(\mathbb{H}_{2m+1}) &\longrightarrow \mathcal{D}^{p,q}(SL(2,\mathbb{R})), \\ \theta_d\colon B^s_{p,q}(\mathbb{R}^d) &\longrightarrow \mathcal{D}^{p,q}(SL(2,\mathbb{R})), \\ \eta_k\colon \mathcal{D}^{p,q}(SL(2,\mathbb{R})) &\longrightarrow M^{p,q}(\mathbb{R}^k), \\ \tau_l\colon \mathcal{D}^{p,q}(SL(2,\mathbb{R})) &\longrightarrow E^{p,q}(\mathbb{H}_{2l+1}), \\ \sigma_r\colon \mathcal{D}^{p,q}(SL(2,\mathbb{R})) &\longrightarrow B^s_{p,q}(\mathbb{R}^r), \end{aligned}$$

for $n \ge 2$ and $m, d, k, l, r \ge 1$. However, for $n = 1$ the Feichtinger algebra $\mathcal{S}_0(\mathbb{R})$ embeds geometrically into $\mathcal{D}(SL(2,\mathbb{R}))$.

Proof. The fact that ϕ_n, ψ_m, and θ_d can not be geometric embeddings for $n > 1$ and $m, d \geq 1$ follows from the hyperbolicity of $(SL(2,\mathbb{R}), d_{\mathcal{U}})$ together with Proposition A.4.6 and Lemma A.2.19 (a).

If we assume that η_k is a geometric embedding, then Proposition A.4.6 and Theorem A.3.11 imply that there is a quasi-isometric embedding between the hyperbolic plane $\mathbb{H}^2$ and $\mathbb{R}^k$. Since $\mathbb{R}^k$ is quasi-isometric to $\mathbb{Z}^k$ and $\mathbb{H}^2$ is quasi-isometric to $\pi_1(X)$ by Proposition A.3.11, where X is a compact Riemann surface of genus $g \geq 2$, we then have a quasi-isometric embedding

$$\widetilde{\eta}_k : \pi_1(X) \longrightarrow \mathbb{Z}^k.$$

Any hyperbolic group that is not finite or contain $\mathbb{Z}$ as a finite index subgroup does contains the free group on two generators as a subgroup [88]. The free group is easily seen to have exponential growth type. Hence it follows that $\pi_1(X)$ also has exponential growth type since any finitely generated group can have at most exponential growth type. On the other hand, the growth type of $\mathbb{Z}^k$ is, as we have mentioned previously, polynomial. Hence the impossibility of $\widetilde{\eta}_k$ follows from the basic result [121, Proposition 6.2.4]. The same argument works for τ_l and σ_r since the growth types of $\mathbb{H}_{2l+1}(\mathbb{Z})$ and $\mathbb{N}_0$ are both polynomial.

For the case $n = p = q = 1$, we can define a map

$$\phi_1 : \mathcal{S}_0(\mathbb{R}) \longrightarrow \mathcal{D}\left(SL(2,\mathbb{R})\right)$$

given by

$$\phi_1(f)(\alpha) = \phi_1(f)\left(\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix} y & x\\ 0 & \frac{1}{y}\end{pmatrix}\right) := f(x)\eta(y), \tag{A.5.4}$$

where $0 \neq \eta \in C_c^\infty(\mathbb{R})$ is supported in $\left[\frac{1}{2}, 1\right]$ and we have used the Iwasawa decomposition of $\alpha \in SL(2,\mathbb{R})$. The pointwise evaluation in (A.5.4) is well-defined since every function in $\mathcal{S}_0(\mathbb{R})$ is continuous [81, Proposition 12.1.4]. Let $\Phi := (\varphi_i)_{i\in I}$ be a $\mathcal{U}$-BAPU for the uniform covering $\mathcal{U}$ on $SL(2,\mathbb{R})$. Then a computation similar to Example A.5.6 shows that $\phi_1(f) \cdot \varphi_i \in L^1\left(SL(2,\mathbb{R})\right)$ for every $i \in I$ and

$$\left\{\|\phi_1(f) \cdot \varphi_i\|_{L^1(SL(2,\mathbb{R}))}\right\}_{i\in I} \in l^1, \quad \sum_{i\in I} \phi_1(f) \cdot \varphi_i = \phi_1(f).$$

Hence we can conclude from Proposition A.4.3 that $\phi_1(f) \in \mathcal{D}\left(SL(2,\mathbb{R})\right)$ and

$$\|\phi_1(f)\|_{\mathcal{D}(SL(2,\mathbb{R}))} \leq A\|f\|_{\mathcal{S}_0(\mathbb{R})},$$

where the constant $A > 0$ does not depend on $f \in \mathcal{S}_0(\mathbb{R})$. If $f \in \mathcal{S}_0(\mathbb{R})$ with $C[f] \subset [n-k, n+k]$ for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ then

$$C\left[\phi_1(f)\right] \subset (0, 2\pi) \times [n-k, n+k] \times \left[\frac{1}{2}, 1\right],$$

with respect to the Iwasawa decomposition. Hence ϕ_1 satisfies (A.4.6) since

$$\mathbb{Z} \ni n \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \in SL(2, \mathbb{R})$$

is a quasi-isometric embedding by Proposition A.2.19 (c).

Finally, the map ϕ_1 is injective since the bump function η is assumed to be non-zero. Thus ϕ_1 is a geometric embedding. It is not a geometric isomorphism since the image of ϕ_1 does not contain any function that depends on the variable θ with respect to the Iwasawa decomposition. Moreover, ϕ_1 is not a spatially implemented geometric embedding since $\mathbb{Z}$ is not quasi-isometric to $\mathbb{H}^2$ as they have different asymptotic dimension. □

Since every stratified Lie group G is diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ we can identify the uniform covering $\mathcal{U}(G)$ with a covering on $\mathbb{R}^n$ where the Fourier transform makes sense. Hence we can consider the decomposition space $\mathcal{D}^{\mathcal{F}}(G, L^p, l^q)$. Both $M^{p,q}(\mathbb{R}^n)$ and $E^{p,q}(\mathbb{H}_{2m+1})$ are particular examples in this class, and one might refer to them as *$\mathcal{F}$-type stratified decomposition spaces*.

In the case where the stratified Lie group is realizable over the rationals, we know from Theorem A.3.6 that the uniform metric space $(G, d_{\mathcal{U}})$ is quasi-isometric to a finitely generated group N with polynomial growth type. Hence the argument used in the first part of the proof of Proposition A.5.7 carries through to show that N is not hyperbolic unless N is quasi-isometric to $\mathbb{Z}$. This is only possible for $G = \mathbb{R}$, so $M^{p,q}(\mathbb{R})$ is the only $\mathcal{F}$-type stratified decomposition space built on a quasi-hyperbolic covering.

Thus a straightforward extension of Proposition A.5.7 shows that there are no geometric embeddings from $\mathcal{D}^{\mathcal{F}}(G, L^p, l^q)$ to $\mathcal{D}^{p,q}(SL(2, \mathbb{R}))$ or vice versa when G is a stratified Lie group realizable over the rationals that is not $\mathbb{R}$. In particular, this holds for the $\mathcal{F}$-type stratified decomposition space where the stratified Lie group is the Engel group introduced in Example A.3.7. Showing statements such as these without using invariants from large scale geometry seems highly non-trivial and highlights the usefulness of our approach.

Paper B

α-Modulation Spaces for Step Two Stratified Lie Groups

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Paper B

α-Modulation Spaces for Step Two Stratified Lie Groups

Abstract

We define and investigate α-modulation spaces $M^{s,\alpha}_{p,q}(G)$ associated to a step two stratified Lie group G with rational structure constants. This is an extension of the Euclidean α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ that act as intermediate spaces between the modulation spaces ($\alpha = 0$) in time-frequency analysis and the Besov spaces ($\alpha = 1$) in harmonic analysis. We will illustrate that the group structure and dilation structure on G affect the boundary cases $\alpha = 0, 1$ where the spaces $M^{s}_{p,q}(G)$ and $\mathcal{B}^{s}_{p,q}(G)$ have non-standard translation and dilation symmetries. Moreover, we show that the spaces $M^{s,\alpha}_{p,q}(G)$ are non-trivial and generally distinct from their Euclidean counterparts.

Finally, we examine how the metric geometry of the coverings $Q(G)$ underlying the $\alpha = 0$ case $M^{s}_{p,q}(G)$ allows for the existence of geometric embeddings

$$F\colon M^{s}_{p,q}(\mathbb{R}^k) \longrightarrow M^{s}_{p,q}(G),$$

as long as k (that only depends on G) is small enough. Our approach naturally gives rise to several open problems that is further elaborated at the end of the paper.

B.1 Introduction

The modulation spaces $M^{s}_{p,q}(\mathbb{R}^n)$ in time-frequency analysis and the (inhomogeneous) Besov spaces $\mathcal{B}^{s}_{p,q}(\mathbb{R}^n)$ in harmonic analysis are invaluable in their own fields. They are connected by the existence of a one-parameter family of Banach spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ where $0 \leq \alpha \leq 1$ such that the aforementioned spaces are the boundary cases $\alpha = 0$ and $\alpha = 1$. It was in the Ph.D. thesis [80] that the *α-modulation spaces* $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ were first introduced and they have subsequently been investigated for a plethora of reasons: The α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ are suitable spaces for studying diverse questions such as boundedness of pseudo-differential operators [28], embedding questions [54, 148], and Banach frame expansions [27]. Moreover, the spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ have found applications in non-linear approximation theory [30] and for studying the Cauchy problem for nonlinear Schrödinger equations [94, 95].

The modulation spaces $M^{s}_{p,q}(\mathbb{R}^n)$ are built out of a uniform covering $\mathcal{U}(\mathbb{R}^n)$ on $\mathbb{R}^n$, while the Besov spaces $\mathcal{B}^{s}_{p,q}(\mathbb{R}^n)$ have a dyadic covering $\mathcal{B}(\mathbb{R}^n)$ associated to them. The intermediate spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ have associated coverings $\mathcal{Q}^{\alpha}(\mathbb{R}^n)$ that interpolate between the extreme cases $\mathcal{U}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$. It is advantageous for several of the applications mentioned above to extend the α-modulation spaces to a setting that include non-uniform translation and dilation symmetries.

Modulation spaces can be defined on locally compact abelian groups [58], while the (homogeneous) Besov spaces have been generalized to stratified Lie groups in [38] through integrability properties of the sub-Laplacian. Recently, the paper [83] has extended certain modulation spaces to the nilpotent setting through a coorbit theory viewpoint. We aim to extend all the α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ to the setting of stratified Lie groups through a more geometric approach that emphasizes the underlying coverings mentioned above.

The choice to extend the α-modulation spaces to stratified Lie groups is motivated by the desire to obtain the following two properties for the resulting spaces $M^{s,\alpha}_{p,q}(G)$:

(i) We can realize all the elements in $M^{s,\alpha}_{p,q}(G)$ as distributions on $\mathbb{R}^n$ where $n = \dim(G)$. This will allow us to use the Euclidean Fourier transform in the description of the spaces $M^{s,\alpha}_{p,q}(G)$.

(ii) The fact that any stratified Lie group possesses dilations and a (typically non-abelian) group structure is needed for a satisfying definition of the boundary cases $\alpha = 0, 1$.

For a stratified Lie group G it is possible to identify G with $(\mathbb{R}^n, *_G)$ where $n = \dim(G)$ and $*_G$ is a product that is polynomial in each component. The initiated reader should have the Heisenberg groups $\mathbb{H}_n$ in mind. The special case $M^{s,0}_{p,q}(\mathbb{H}_n)$ has already been investigated in [64] with the help of representation theory.

We are able to define the spaces $M^{s,\alpha}_{p,q}(G)$ for any stratified Lie group G. However, we can only assure that the definition is not vacuous whenever the step of G is less than or equal two. The reason for this will be explained and discussed further in Subsection B.3.3. Although we expect the *generalized α-modulation* *spaces* $M_{p,q}^{s,\alpha}(G)$ to be well-defined for all stratified Lie groups G, we are not able to show this with current methods. Moreover, for the most part we need to restrict to the stratified Lie groups G being *rational*, meaning that there exists a lattice $N \subset G$. This is a mild condition and is easily checked in practice. Whenever possible, we will state results for arbitrary stratified Lie groups in the hope that the restriction to rational stratified Lie groups with step less than or equal two will be removed in the future.

The two properties (i) and (ii) above can be considered as necessary conditions for studying the spaces $M_{p,q}^{s,\alpha}(G)$. However, two generalizations are not equally rewarding and the reader should be skeptical whether this initial outset yields satisfying results. Except for expecting the spaces $M_{p,q}^{s,\alpha}(G)$ to satisfy basic results regarding completeness, duality and so on, the following five questions seem appropriate to answer:

1) Are there coverings $Q^{\alpha}(G)$ on $\mathbb{R}^n$ associated to the spaces $M_{p,q}^{s,\alpha}(G)$ in the same manner as in the Euclidean setting? Moreover, do these coverings reflect some geometric property of the stratified Lie group G in the uniform case $\alpha = 0$?

2) Can one use the spaces $M_{p,q}^{s,\alpha}(G)$ for an application without extensive knowledge of stratified Lie groups? That is, can someone with a time-frequency analysis or harmonic analysis background effectively work with the spaces $M_{p,q}^{s,\alpha}(G)$?

3) Have any of the spaces $M_{p,q}^{s,\alpha}(G)$ previously appeared in the literature? Are the spaces $M_{p,q}^{s,\alpha}(G)$ interesting whenever G is not isomorphic to $(\mathbb{R}^n, +)$ as a Lie group?

4) Is the extension from $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ to $M_{p,q}^{s,\alpha}(G)$ uninteresting in the sense that the definitions need only be trivially modified to obtain spaces with analogous properties? Do all the techniques used when studying the Euclidean α-modulation spaces $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ extend in an obvious way to solve the same problems for the spaces $M_{p,q}^{s,\alpha}(G)$?

5) Are the new spaces $M_{p,q}^{s,\alpha}(G)$ distinct from their Euclidean counterparts $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$? More precisely, is it possible that

$$M_{p_1,q_1}^{s_1,\alpha_1}(G) = M_{p_2,q_2}^{s_2,\alpha_2}(\mathbb{R}^n),$$

for some parameters $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$, and $0 \leq \alpha_1, \alpha_2 \leq 1$?

We will not attempt to address the first four questions in the introduction, but will answer them throughout the paper and return to them again in Section B.7.

The fifth question turns out to be the most challenging and the answer given in Theorem B.5.6 can be seen as the main technical achievement of the paper. Our result will extend the known result for the modulation spaces on the Heisenberg group given in [64, Theorem 7.6]. We say that the parameters p, q, s, α with $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$ are *non-trivial* if $(p, q, s) \neq (2, 2, 0)$. Question 5) above has the following complete answer.

Theorem. *(Main Theorem) Let $(\mathbb{R}^n, *_G)$ denote a rational stratified Lie group with step less than or equal two. Consider two sets of non-trivial parameters $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$, and $0 \leq \alpha_1, \alpha_2 \leq 1$. We have equality*

$$M^{s_1,\alpha_1}_{p_1,q_1}(G) = M^{s_2,\alpha_2}_{p_2,q_2}(\mathbb{R}^n)$$

with equivalent norms if and only if both

$$(p_1, q_1, s_1, \alpha_1) = (p_2, q_2, s_2, \alpha_2) \quad \textit{and} \quad (\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +).$$

Given two stratified Lie groups G and H with $\dim(G) = \dim(H)$, the spaces $M^{s_1,\alpha_1}_{p_1,q_1}(G)$ and $M^{s_2,\alpha_2}_{p_2,q_2}(H)$ will both consist of distributions on $\mathbb{R}^n$. Hence it makes sense to ask whether the inclusion $M^{s_1,\alpha_1}_{p_1,q_1}(G) \hookrightarrow M^{s_2,\alpha_2}_{p_2,q_2}(H)$ is bounded for certain parameters. However, when $\dim(G) \neq \dim(H)$ this approach is not possible as the distributions in each space are not comparable.

As a substitute, we would like to understand when there exist embeddings

$$F\colon M^{s_1,\alpha_1}_{p_1,q_1}(G) \longrightarrow M^{s_2,\alpha_2}_{p_2,q_2}(H)$$

that preserve the underlying coverings $Q^\alpha(G)$ and $Q^\alpha(H)$ in a suitable sense. These embeddings have recently been invented in [19] under the name *geometric embeddings*. We will give the precise definitions in Section B.6. The existence of geometric embeddings is at the moment to challenging to answer in its full generality. In Theorem B.6.4 we give a partial answer to when the Euclidean modulation spaces $M^{s,0}_{p,q}(\mathbb{R}^k)$ can embed geometrically into the generalized modulation spaces $M^{s,0}_{p,q}(G)$.

Theorem. *Let G be a rational stratified Lie group with step less than or equal two and with rank k. There exists a geometric embedding*

$$F\colon M^{s,0}_{p,q}(\mathbb{R}^{k'}) \to M^{s,0}_{p,q}(G)$$

for every $k' \leq k$, $1 \leq p, q < \infty$ and $s \in \mathbb{R}$. This is in general optimal as there are no geometric embeddings from $M^{s,0}_{p,q}(\mathbb{R}^{k'})$ to $M^{s,0}_{p,q}(\mathbb{R}^l)$ for $l < k'$.

The structure of the paper is as follows: In Section B.2 we introduce stratified Lie groups, admissible coverings, and related notions. We will also define the traditional α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ in Subsection B.2.3 to make the exposition more self-contained. The coverings $Q^\alpha(G)$ associated to the group G are defined in Section B.3 and we develop some of their basic properties. In Subsection B.3.3 we discuss when the elements in the covering $Q^\alpha(G)$ are images of a few reference sets under well-behaved affine transformations. As one might expect, this depends on how "polynomial" the group multiplication on $(\mathbb{R}^n, *_G)$ is.

In Section B.4 we define the spaces $M^{s,\alpha}_{p,q}(G)$ and investigate their duality relations. We moreover show that the rapidly decaying smooth functions $\mathcal{S}(\mathbb{R}^n)$ are contained in $M^{s,\alpha}_{p,q}(G)$. It is in Section B.5 that we answer the fifth question regarding uniqueness of the spaces $M^{s,\alpha}_{p,q}(G)$ and develop a few auxiliary results. We will study geometric embeddings in Section B.6. Finally, in Section B.7 we look back on the five questions posted in the introduction and outline some open problems and possible future directions.

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B.2 Preliminaries

Our notational conventions are fairly standard: We use the convention that $\mathbb{N}$ does not contain zero and we will write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ will be denoted by $|A|$, while the number of elements in a finite or countably infinite set B will be denoted by $\#B$. The *Fourier transform* on $\mathbb{R}^n$ will be denoted by $\mathcal{F}$ and we use the normalization convention

$$\mathcal{F}(f)(\omega) := \int_{\mathbb{R}^n} f(x) \cdot e^{-2\pi i x \cdot \omega}\, dx.$$

We will denote by $\mathcal{S}(\mathbb{R}^n)$ the space of smooth functions on $\mathbb{R}^n$ with rapid decay. Its topological dual space $\mathcal{S}'(\mathbb{R}^n)$ will be referred to as the *tempered distributions*. Denote by $L^p := L^p(\mathbb{R}^n)$ the p'th integrable Lebesgue measurable functions for $1 \leq p \leq \infty$ with the usual modification for $p = \infty$. The space $l^q(I)$ where I is a countable index set will denote the q'th summable sequences indexed by I where $1 \leq q < \infty$. Similarly, the space $l^\infty(I)$ denotes all bounded sequences on the index set I. When the index set I is clear from the context we will often simply write $l^q := l^q(I)$ for $1 \leq q \leq \infty$.

We will use the notation $\|\cdot\|_E$ for the usual Euclidean norm on $\mathbb{R}^n$ and reserve the notation $\|\cdot\|$ for the homogeneous quasi-norms on stratified Lie groups introduced in Subsection B.2.1. Finally, the notation $a_i \asymp b_i$ between two quantities a_i and b_i that (possibly) depends on an index $i \in I$ indicates that there exists an absolute constant $C > 0$ such that

$$\frac{1}{C} \cdot a_i \leq b_i \leq C \cdot a_i, \qquad i \in I.$$

B.2.1 Stratified Lie Groups

In this subsection we briefly outline the essence of stratified Lie groups and the basic constructions on them we will need in subsequent sections. As our intended audience include people with a background in harmonic analysis and time-frequency analysis, we have tried to keep the prerequisites at a minimum. Any statement that is not justified in this subsection can be found in [65, Chapter 1.6 and Chapter 3.1].

Definition B.2.1. Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then G is called *stratified* if there exists a *stratification*

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \quad [V_1, V_j] = \begin{cases} V_{j+1}, & \text{if } j = 1, \ldots, s-1 \\ \{0\}, & \text{if } j = s \end{cases}. \tag{B.2.1}$$

The number s is called the *step* of G while the number $k := \dim(V_1)$ is called the *rank* of G. Both numbers are invariant under different choices of stratifications. Elements in V_i are said to be of *degree* i for $i = 1, \ldots, s$ and we use the notation $\deg(X) = i$ for $X \in V_i$. It is clear that any stratified Lie group G is *nilpotent*, that is, the *adjoint map* $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_X(Y) := [X, Y]$ is a nilpotent linear map for all $X \in \mathfrak{g}$.

For stratified Lie groups the exponential map $\exp_G : \mathfrak{g} \to G$ is a global diffeomorphism and we denote its inverse by $\log_G : G \to \mathfrak{g}$. The *Baker-Campbell-Hausdorff formula* (BCH) gives the expression

$$\log_G\left(\exp_G(X) *_G \exp_G(Y)\right) = X + Y + \frac{1}{2}[X, Y] - \frac{1}{12}[Y, [X, Y]] + \cdots, \tag{B.2.2}$$

where there are only finitely many terms due to the nilpotency and they all involve iterated brackets between X and Y.

An important feature of stratified Lie groups is that they admit dilations: Define the maps $D_r : \mathfrak{g} \to \mathfrak{g}$ for $r > 0$ by

$$D_r(X) := r^{\deg(X)} X, \quad X \in \mathfrak{g}.$$

It is straightforward to see that the maps D_r are all Lie algebra isomorphisms. Since G is the connected and simply connected Lie group of $\mathfrak{g}$, there exist unique Lie group automorphisms $D_r^G\colon G \to G$ lifting the maps D_r for all $r > 0$. We call the maps $D_r^G\colon G \to G$ for $r > 0$ *dilations* on the Lie group G and they are explicitly given by

$$D_r^G(g) \coloneqq \exp_G \circ D_r \circ \log_G(g), \quad g \in G.$$

Any stratified Lie group G is *unimodular*, that is, the right and left Haar measures coincide. Let μ denote a choice of Haar measure on G. Then

$$\mu(A) \coloneqq \lambda(\log_G(A)), \tag{B.2.3}$$

where λ is a corresponding choice of Lebesgue measure on the vector space $\mathfrak{g}$ and $A \subset G$ is a Borel measurable set. Hence $\mu(D_r^G(A)) = r^Q \mu(A)$, where

$$Q \coloneqq \sum_{j=1}^{s} j \cdot \dim(V_j).$$

The number Q satisfies $\dim(G) \le Q$ and is the *homogeneous dimension* of G.

Recall that a *lattice* N in a Lie group G is a discrete subgroup such that there exists a G-invariant Borel measure $\mu_{G/N}$ on the quotient G/N with

$$\mu_{G/N}(G/N) < \infty.$$

Lattices in stratified Lie groups enjoy two properties that are not shared by lattices in general Lie groups (or in general locally compact groups):

- Any lattice N in a stratified Lie group G is *uniform*, that is, the quotient space G/N is compact. In fact, the compactness of G/N for a discrete subgroup N is equivalent to the existence of a G-invariant Borel measure $\mu_{G/N}$ on the quotient G/N with $\mu_{G/N}(G/N) < \infty$ [139, Theorem 2.1].

- Any lattice in a stratified Lie group is a finitely generated nilpotent group [139, Theorem 2.10].

Moreover, a stratified Lie group G admits a lattice if and only if there exists a basis $X_1, \ldots, X_n$ for its Lie algebra $\mathfrak{g}$ such that the *structure constants* c_{ij}^k defined by the relation

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k, \quad i, j = 1, \ldots, n,$$

are all rational numbers [139, Theorem 2.12]. Such stratified Lie groups are called *realizable over the rationals* or simply *rational*. The classification of nilpotent Lie algebras in [79] shows that every stratified Lie group of dimension less than seven is rational. We will mostly be interested in stratified Lie groups G that are rational and many results (such as Proposition B.3.5, Theorem B.5.6, and Theorem B.6.4) require this.

We can identify G as a manifold with $\mathbb{R}^n$ for $n = \dim(G)$ through the exponential map. The group operation $*_G$ on $\mathbb{R}^n$ such that G is isomorphic to $(\mathbb{R}^n, *_G)$ as a Lie group is polynomial by the BCH formula (B.2.2). Then relation (B.2.3) shows that the Haar measure μ on G transported to $\mathbb{R}^n$ through the exponential map is simply the Lebesgue measure λ on $\mathbb{R}^n$. However, lattices N in G are not in general identified with the standard lattices in $\mathbb{R}^n$, that is, the subgroups $\Gamma \subset \mathbb{R}^n$ on the form $\Gamma = A\mathbb{Z}^n$, where $A \in GL(n, \mathbb{R})$. Our motivation for identifying stratified Lie groups with $\mathbb{R}^n$ comes from the need to use the Euclidean Fourier transform when defining the generalized α-modulation spaces in Section B.4.

Let us now describe an alternative to the usual Euclidean norm $\|\cdot\|_E$ on $\mathbb{R}^n$ that is adapted to the stratified Lie group G: We say that a function $f\colon G \to \mathbb{C}$ is *l-homogeneous* for $l \in \mathbb{N}_0$ if

$$f\left(D_r^G(g)\right) = r^l f(g),$$

for every $r > 0$ and all $g \in G$. The function $f\colon G \to \mathbb{C}$ is called *symmetric* if $f(g) = f\left(g^{-1}\right)$ for every $g \in G$.

Definition B.2.2. A *homogeneous quasi-norm* on G is a 1-homogeneous continuous function $\|\cdot\|$ that is symmetric and has the property that $\|g\| = 0$ only holds when g is the identity element of G.

We will use the standard notation

$$B^{\|\cdot\|}(g, R) := \left\{h \in G : \|g^{-1} *_G h\| < R\right\}, \qquad g \in G, \quad R > 0.$$

The following proposition is proved in [65, Proposition 3.1.35] and shows that the choice of homogeneous quasi-norm is in many instances irrelevant.

Lemma B.2.3. *Let G be a stratified Lie group. Then G admits a homogeneous quasi-norm that is smooth away from the identity element. Moreover, any two homogeneous quasi-norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on G are equivalent in the sense that there exists $C > 0$ such that*

$$\frac{1}{C}\|g\|_1 \leq \|g\|_2 \leq C\|g\|_1,$$

for every $g \in G$.

The terminology "quasi-norm" is justified by [65, Proposition 3.1.38], showing that homogeneous quasi-norms satisfy

$$\|g *_G h\| \leq C\left(\|g\| + \|h\|\right), \quad g, h \in G, \tag{B.2.4}$$

where $C \geq 1$ is a constant that does not depend on the elements $g, h \in G$. In fact, it is always possible by [65, Proposition 3.1.39] to find a *homogeneous norm*, that is, a homogeneous quasi-norm $\|\cdot\|$ that additionally satisfies

$$\|g *_G h\| \leq \|g\| + \|h\|, \quad g, h \in G.$$

When considering stratified Lie groups in the rest of this paper, we implicitly assume the following standing assumption: We always chose the realization of G as $(\mathbb{R}^n, *_G)$ where $n = \dim(G)$ through the exponential map. The triple $(\mathbb{R}^n, *_G, \|\cdot\|)$ will for the rest of the paper denote the realization of G where $\|\cdot\|$ is a chosen homogeneous quasi-norm on $(\mathbb{R}^n, *_G)$.

Example B.2.4. Consider the *Heisenberg Lie algebra*

$$\mathfrak{h}_n := \operatorname{span}_{\mathbb{R}}\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$$

with non-trivial bracket relations

$$[X_i, Y_i] = Z, \quad i = 1, \dots, n.$$

The connected and simply connected Lie group $\mathbb{H}_n$ corresponding to $\mathfrak{h}_n$ is called the *Heisenberg group*. It follows from the BCH formula (B.2.2) that

$$\log_{\mathbb{H}_n}\left(\exp_{\mathbb{H}_n}(X) *_{\mathbb{H}_n} \exp_{\mathbb{H}_n}(Y)\right) = X + Y + \frac{1}{2}[X, Y], \quad X, Y \in \mathfrak{h}_n.$$

Through the exponential map, the Heisenberg group $\mathbb{H}_n$ is isomorphic as a Lie group to $(\mathbb{R}^{2n+1}, *_{\mathbb{H}_n})$ where

$$(x, \omega, t) *_{\mathbb{H}_n} (x', \omega', t) := \left(x + x',\ \omega + \omega',\ t + t' + \frac{1}{2}(x'\omega - x\omega')\right),$$

for $x, x', \omega, \omega' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}$. After this identification, the dilations $D_r^{\mathbb{H}_n}$ for $r > 0$ are given by

$$D_r^{\mathbb{H}_n}(x, \omega, t) = (rx, r\omega, r^2 t), \quad (x, \omega, t) \in \mathbb{R}^{2n+1}.$$

The homogeneous dimension of $\mathbb{H}_n$ is $Q = 2n + 2$ and a concrete example of a lattice in $(\mathbb{R}^{2n+1}, *_{\mathbb{H}_n})$ is

$$N := \left\{(x, \omega, t) \in \mathbb{R}^{2n+1} : x, \omega \in 2\mathbb{Z}^n,\ t \in \mathbb{Z}\right\}.$$

Moreover, the *homogeneous Cygan-Koranyi norm*

$$(x, \omega, t) \longmapsto \left((|x|^2 + |\omega|^2)^2 + 16t^2\right)^{\frac{1}{4}} \tag{B.2.5}$$

is an example of a homogeneous quasi-norm on the Heisenberg group.

B.2.2 Admissible Coverings

We give a brief review of admissible coverings and some related notions that we need in subsequent sections. Admissible coverings was originally formulated in [59] as special coverings on an arbitrary set. However, we will restrict ourselves to admissible coverings on $\mathbb{R}^n$ since every stratified Lie group G has a realization as $(\mathbb{R}^n, *_G)$ as explained in the previous section.

Definition B.2.5. A covering $\mathcal{Q} := (Q_i)_{i \in I}$ consisting of non-empty sets on $\mathbb{R}^n$ is called *admissible* if we have the uniform bound

$$\sup_{i \in I} \# \left\{ j \in I : Q_i \cap Q_j \neq \emptyset \right\} \leq N_{\mathcal{Q}}, \tag{B.2.6}$$

for some $N_{\mathcal{Q}} \in \mathbb{N}$. The admissible covering $\mathcal{Q}$ will be called a *concatenation* if we additionally have the equality

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} Q_i^{k*}, \tag{B.2.7}$$

for some (and hence all) $i \in I$, where we use the notation

$$Q_i^* := \left\{ Q_j \in \mathcal{Q} : Q_i \cap Q_j \neq \emptyset \right\}, \qquad Q_i^{k*} := \left(Q_i^{(k-1)*} \right)^*,$$

for $k \geq 2$ and $i \in I$.

Given an admissible covering $\mathcal{Q} := (Q_i)_{i \in I}$ we call the elements in Q_i^* the *neighbours* of the set $Q_i \in \mathcal{Q}$. Moreover, the smallest possible constant $N_{\mathcal{Q}}$ in (B.2.6) is called the *admissibility constant* of the admissible covering $\mathcal{Q}$. The admissibility condition (B.2.6) is needed to obtain non-trivial classes of functions that have a prescribed frequency decay with respect to the covering $\mathcal{Q}$. On the other hand, the concatenation property (B.2.7) will be necessary when we examine coverings from a metric space viewpoint in Section B.5 and Section B.6.

We will in Section B.5 need the notion of weight functions that are well-behaved with respect to an admissible covering $\mathcal{Q} := (Q_i)_{i \in I}$ on $\mathbb{R}^n$. To be precise, we will say that a function $\omega \colon I \to (0, \infty)$ is *$\mathcal{Q}$-moderate* if we have the uniform bound

$$\sup_{\{j : Q_i \cap Q_j \neq \emptyset\}} \frac{\omega(i)}{\omega(j)} \leq C_\omega,$$

where the constant C_ω does not depend on the index $i \in I$.

Given two admissible coverings $\mathcal{Q} := (Q_i)_{i \in I}$ and $\mathcal{P} := (P_j)_{j \in J}$ on $\mathbb{R}^n$, there are two common ways of comparing them:

- We say that $\mathcal{Q}$ is *almost subordinate* to $\mathcal{P}$ if there exists a $k \in \mathbb{N}$ such that for every $i \in I$ there is a $j \in J$ with $Q_i \subset P_j^{k*}$. We use the notation $\mathcal{Q} \leq \mathcal{P}$ and say that the coverings $\mathcal{Q}$ and $\mathcal{P}$ are *equivalent* if both $\mathcal{Q} \leq \mathcal{P}$ and $\mathcal{P} \leq \mathcal{Q}$ are satisfied.

- We say that $\mathcal{Q}$ is *weakly subordinate* to $\mathcal{P}$ if we have the bound

$$\sup_{i \in I} \# \left\{ j \in J : P_j \cap Q_i \neq \emptyset \right\} < \infty.$$

  If $\mathcal{Q}$ is weakly subordinate to $\mathcal{P}$ and vice versa, we call the coverings *weakly equivalent*.

It follows from [59, Proposition 3.5] that almost subordination implies weak subordination, although the converse is not true in general. It is generally difficult to show that one covering $\mathcal{Q}$ is almost subordinate to another covering $\mathcal{P}$. However, it is often easier to show that $\mathcal{Q}$ is weakly subordinate to $\mathcal{P}$. Whenever the coverings consist of open and path-connected sets, then it follows from [59, Proposition 3.6] that the two notions coincide.

An arbitrary admissible covering $\mathcal{Q} := (Q_i)_{i \in I}$ on $\mathbb{R}^n$ can have (at least) two problematic features: Firstly, the index set I might not be countable. Secondly, sets $Q_i \in \mathcal{Q}$ are allowed repeat in the collection $\mathcal{Q} := (Q_i)_{i \in I}$, only with different indices. The covering $\mathcal{Q}$ on $\mathbb{R}$ whose index set is $I := \mathbb{R} \times \{0, 1\}$ and is given by $Q_{(r,0)} = Q_{(r,1)} := \{r\}$ for $r \in \mathbb{R}$ is a simple admissible covering that embodies both problems simultaneously. Moreover, the covering $\mathcal{Q}$ is clearly not a concatenation as it is a partition. The following lemma shows that these problems disappear once we require the elements in the covering to be open sets.

Lemma B.2.6. *Let $\mathcal{Q} := (Q_i)_{i \in I}$ be an admissible covering on $\mathbb{R}^n$ consisting of open sets. Then I has to be countable and the covering $\mathcal{Q}$ is automatically a concatenation. Moreover, we can remove repeated elements in $\mathcal{Q} := (Q_i)_{i \in I}$ and obtain an equivalent covering.*

Proof. Since $\mathcal{Q}$ is an open covering on $\mathbb{R}^n$ we can find a countable subcovering $\mathcal{Q}' := (Q_j)_{j \in J}$ of $\mathcal{Q}$ with $J \subset I$. Consider the sets

$$A_j := \left\{ i \in I : Q_i \cap Q_j \neq \emptyset \right\}, \quad j \in J.$$

Then for every $i \in I$ we can find a set A_j with $j \in J$ such that $i \in A_j$ since $\mathcal{Q}'$ is a covering on $\mathbb{R}^n$. The set $\cup_{j \in J} A_j$ is countable and we obtain that I has to be countable as well.

The concatenation property (B.2.7) is equivalent to the following statement: Given $x, y \in \mathbb{R}^n$ we can find a sequence $Q_{i_1}, \ldots, Q_{i_k} \in \mathcal{Q}$ of elements in $\mathcal{Q}$ with

$x \in Q_{i_1}$ and $y \in Q_{i_k}$ such that $Q_{i_l} \cap Q_{i_{l+1}} \neq \emptyset$ for every $1 \leq l \leq k-1$. Such a sequence is called a *chain* from x to y in [59]. To see that this is always possible to find, consider the straight line

$$\gamma_{x,y} \colon [0,1] \to \mathbb{R}^n, \qquad \gamma_{x,y}(t) = ty + (1-t)x, \tag{B.2.8}$$

connecting x and y. Since the image of $\gamma_{x,y}$ is compact and the elements in $\mathcal{Q}$ are open, we can find a finite set of elements $(Q_j)_{j\in J}$ in $\mathcal{Q}$ such that

$$\mathrm{Im}(\gamma_{x,y}) \subset \bigcup_{j\in J} Q_j.$$

A standard topological argument using the openness of the elements $(Q_j)_{j\in J}$ shows that we can reorder $(Q_j)_{j\in J}$ to obtain a chain from x to y. The final statement is obvious from the definition of equivalent coverings. □

Remark. We would like to emphasize that the proof of Lemma B.2.6 goes through if, instead of $\mathbb{R}^n$, we consider a path-connected topological space X where any open covering on X has a countable subcover. The only modification is that we would need to pick an abstract continuous path from x to y guaranteed by the path-connectedness of X rather than the straight line given in (B.2.8). These conditions hold for all connected manifolds and hence include most settings considered in the literature.

B.2.3 α-Modulation Spaces

We now give the definitions of the Euclidean α-coverings and α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$. This will serve as a motivation for the generalization to stratified Lie groups described in the next sections.

Definition B.2.7. An admissible covering $\mathcal{Q}^\alpha := (Q_i^\alpha)_{i\in I}$ on $\mathbb{R}^n$ consisting of open and connected sets is called an *α-covering* for $0 \leq \alpha \leq 1$ if

- The sets $Q_i^\alpha \in \mathcal{Q}^\alpha$ satisfy $|Q_i^\alpha| \asymp (1+\|\xi_i\|_E^2)^{\frac{\alpha n}{2}}$ for $i \in I$ and for all $\xi_i \in Q_i^\alpha$.
- For each $i \in I$ we denote by $r\left(Q_i^\alpha\right)$ and $R\left(Q_i^\alpha\right)$ the numbers

$$r\left(Q_i^\alpha\right) := \sup\left\{r \in \mathbb{R} : B(c_r, r) \subset Q_i^\alpha \text{ for some } c_r \in \mathbb{R}\right\},$$
$$R\left(Q_i^\alpha\right) := \inf\left\{R \in \mathbb{R} : Q_i^\alpha \subset B(C_r, R) \text{ for some } C_r \in \mathbb{R}\right\}.$$

  There should exist a constant $K \geq 1$ such that

$$\sup_{i\in I} \frac{R\left(Q_i^\alpha\right)}{r\left(Q_i^\alpha\right)} \leq K. \tag{B.2.9}$$

There is much variation in the literature about the definition of α-coverings: In [28] the authors do not require (B.2.9) to hold. In [148, Chapter 9], the author considers a concrete covering that satisfies Definition B.2.7. Our definition is the same as in [27] and is motivated by the following remark.

Remark. It follows from [27, Lemma B.2] that any two α-coverings on $\mathbb{R}^n$ as we have defined them are weakly equivalent. Thus they are in fact equivalent since they consist of open and connected sets. To see that connectedness is a necessary condition, we can take $\mathcal{Q} := (Q_n)_{n\in\mathbb{Z}}$ to be the covering $Q_n := (n-1, n+1)$ and $\mathcal{P} := (U_k)_{k\in\mathbb{N}_0}$ to be the covering

$$U_0 := (-2, 2), \quad U_k := (-k-2, -k) \cup (k, k+2), \quad k \in \mathbb{N}.$$

Both coverings are 0-coverings on $\mathbb{R}$. However, they are clearly not equivalent since $\mathcal{P}$ is not almost subordinate to $\mathcal{Q}$.

Let $\mathcal{Q} := (Q_i)_{i\in I}$ be an admissible covering on $\mathbb{R}^n$. A *(smooth) bounded admissible partition of unity* subordinate to $\mathcal{Q}$ ($\mathcal{Q}$-BAPU) is a family of non-negative smooth functions $\Phi := (\psi_i)_{i\in I}$ on $\mathbb{R}^n$ such that

$$\operatorname{supp}(\psi_i) \subset Q_i, \qquad \sum_{i\in I} \psi_i \equiv 1, \qquad \sup_{i\in I} \left\|\mathcal{F}^{-1}\psi_i\right\|_{L^1} < \infty. \tag{B.2.10}$$

Definition B.2.8. Let $\mathcal{Q}^\alpha := (Q_i^\alpha)_{i\in I}$ be an α-covering on $\mathbb{R}^n$ and let $\Phi := (\psi_i)_{i\in I}$ be a $\mathcal{Q}^\alpha$-BAPU. Given the parameters $1 \le p, q \le \infty$, $s \in \mathbb{R}$, and $0 \le \alpha \le 1$ we define the *α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$* to be tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{M^{s,\alpha}_{p,q}} := \left(\sum_{i\in I} \left(1 + \|\xi_i\|_E^2\right)^{\frac{qs}{2}} \left\|\mathcal{F}^{-1}\left(\psi_i \cdot \mathcal{F}(f)\right)\right\|_{L^p}^q\right)^{\frac{1}{q}} < \infty,$$

where $\xi_i \in Q_i^\alpha$ for every $i \in I$. If $q = \infty$ we use the obvious modification from summation to supremum.

The α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ were first introduced in [80]. Since two α-coverings on $\mathbb{R}^n$ are equivalent, we obtain from [59, Theorem 3.7] that the resulting α-modulation spaces have equivalent norms. Moreover, one also obtain equivalent norms by choosing another bounded admissible partition of unity by [59, Theorem 2.3 B)]. The α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ are Banach spaces for all the parameter values $1 \le p, q \le \infty$, $s \in \mathbb{R}$, and $0 \le \alpha \le 1$.

Example B.2.9. If $\alpha = 0$ then an option for an α-covering on $\mathbb{R}^n$ is the *uniform covering*

$$\mathcal{U}(\mathbb{R}^n) := (Q_{m_1,\ldots,m_n})_{m_1,\ldots,m_n\in\mathbb{Z}}, \qquad Q_{m_1,\ldots,m_n} := (-1, 1)^n + (m_1, \ldots, m_n).$$

The resulting spaces $M^s_{p,q}(\mathbb{R}^n) := M^{s,0}_{p,q}(\mathbb{R}^n)$ are precisely the *modulation spaces with polynomial weights*. They are typically denoted by $M^{p,q}_{v_s}(\mathbb{R}^n)$ or simply $M^{p,q}_s(\mathbb{R}^n)$ in the literature and we refer the reader to [81, Chapter 11] for more information on them.

Example B.2.10. If $\alpha = 1$ and $n \geq 2$ we can use the *dyadic covering* given by $\mathcal{B}(\mathbb{R}^n) := (D_m)_{m=0}^{\infty}$ where $D_0 := B(0,2)$ and

$$D_m := \left\{x \in \mathbb{R}^n : 2^{m-1} < \|x\|_E < 2^{m+1}\right\}, \quad m \in \mathbb{N}. \tag{B.2.11}$$

The resulting spaces $\mathcal{B}^s_{p,q}(\mathbb{R}^n) := M^{s,1}_{p,q}(\mathbb{R}^n)$ are the *(inhomogeneous) Besov spaces*. For $n = 1$ the covering given in (B.2.11) is not connected and we would need to split each of the sets D_m for $m \geq 1$ into its two connected components and consider them individually to obtain a 1-covering.

To summarize, the α-modulation spaces are a one-parameter class of Banach spaces connecting the modulation spaces used in time-frequency analysis and the Besov spaces used in harmonic analysis.

B.3 Generalized α-Coverings

We will in this section define generalized α-coverings for $0 \leq \alpha \leq 1$ on $\mathbb{R}^n$ that reflect the stratified Lie group structure $(\mathbb{R}^n, *_G)$ and extend the α-coverings defined in Subsection B.2.3. We will emphasize the role of the homogeneous quasi-norms and lattices in the stratified Lie group $(\mathbb{R}^n, *_G)$.

B.3.1 Definition and Equivalence

From now on, we identify a stratified Lie group G with $(\mathbb{R}^n, *_G)$ through the exponential map and fix a homogeneous quasi-norm $\|\cdot\|$ on $(\mathbb{R}^n, *_G)$. By doing this, we have to keep track of that $\mathbb{R}^n$ is equipped with a both a group structure $*_G$ and a Lie algebra structure $\mathbb{R}^n \simeq \mathfrak{g} = V_1 \oplus \cdots \oplus V_s$, where s is the step of G. When writing elements $x \in (\mathbb{R}^n, *_G)$ in coordinates $x = (x_1, \ldots, x_n)$ we implicitly assume that we have chosen a basis $v_1, \ldots, v_n$ for $\mathbb{R}^n$ that is *adapted to the stratification*. This means that $v_1, \ldots, v_{\dim(V_1)}$ is a basis for V_1, $v_{\dim(V_1)+1}, \ldots, v_{\dim(V_2)}$ is a basis for V_2, and so on.

Definition B.3.1. Let $(\mathbb{R}^n, *_G, \|\cdot\|)$ be a stratified Lie group with homogeneous dimension Q where $\|\cdot\|$ is a chosen homogeneous quasi-norm. For a fixed $0 \leq \alpha \leq 1$ we call an admissible covering $\mathcal{P}^\alpha := (P^\alpha_i)_{i \in I}$ on $\mathbb{R}^n$ consisting of open and connected sets a *generalized α-covering* for the stratified Lie group $(\mathbb{R}^n, *_G)$ if it satisfies the following two properties:

- The sets P_i^α satisfy the estimates

$$|P_i^\alpha| \asymp \left(1 + \|\xi_i\|^2\right)^{\frac{\alpha Q}{2}}, \tag{B.3.1}$$

for each $i \in I$ and for all $\xi_i \in P_i^\alpha$.

- For each $i \in I$ we denote by $r^{\|\cdot\|}\left(P_i^\alpha\right)$ and $R^{\|\cdot\|}\left(P_i^\alpha\right)$ the numbers

$$r^{\|\cdot\|}\left(P_i^\alpha\right) := \sup\left\{r \in \mathbb{R} : B^{\|\cdot\|}(c_r, r) \subset P_i^\alpha \text{ for some } c_r \in \mathbb{R}\right\},$$
$$R^{\|\cdot\|}\left(P_i^\alpha\right) := \inf\left\{R \in \mathbb{R} : P_i^\alpha \subset B^{\|\cdot\|}(C_r, R) \text{ for some } C_r \in \mathbb{R}\right\}.$$

There should exist a constant $K \geq 1$ such that

$$\sup_{i \in I} \frac{R^{\|\cdot\|}\left(P_i^\alpha\right)}{r^{\|\cdot\|}\left(P_i^\alpha\right)} \leq K. \tag{B.3.2}$$

Notice that the numbers $r^{\|\cdot\|}\left(P_i^\alpha\right)$ and $R^{\|\cdot\|}\left(P_i^\alpha\right)$ are strictly positive since we assume that the sets P_i^α are open. Condition (B.3.2) is necessary to obtain that two generalized α-coverings are equivalent as will be shown in Proposition B.3.3. Notice that Lemma B.2.6 implies that the index set I has to be countable and that the covering $\mathcal{P}^\alpha$ is automatically a concatenation.

Remark. When $G = (\mathbb{R}^n, +)$ the homogeneous dimension satisfies $Q = n$ and we regain the definition of the α-coverings given in Definition B.2.7. The reason for realizing generalized α-coverings corresponding to stratified Lie groups on Euclidean space is to involve the Euclidean Fourier transform when we define generalized α-modulation spaces in Section B.4. Moreover, the heuristic reason we use the homogeneous dimension Q in (B.3.1) instead of the dimension n is that we would like to obtain “Besov type spaces” for $\alpha = 1$ that incorporate the intrinsic dilations D_r^G of the stratified Lie group $(\mathbb{R}^n, *_G)$. We will see later in Lemma B.3.7 that this intuition gives a concrete 1-covering that is similar to the dyadic covering given in Example B.2.10.

Whenever $\alpha = 0$ then condition (B.3.1) simply says that the Lebesgue measure of the sets P_i^0 is constant. One could wonder whether the uniform covering $\mathcal{U}(\mathbb{R}^n)$ in Example B.2.9 satisfies (B.3.2) and thus is a generalized α-covering for any rational stratified Lie group $(\mathbb{R}^n, *_G)$ other than $(\mathbb{R}^n, +)$. We will see in Proposition B.5.4 that this is not the case by using arguments from metric geometry. The reader might get some motivation for this approach by trying to prove this statement directly without additional tools.

Example B.3.2. For the Heisenberg group $\mathbb{H}_3$ with the homogeneous Cygan-Koranyi norm (B.2.5) we have that generalized α-coverings $\mathcal{P}^\alpha := (P_i^\alpha)_{i\in I}$ for $0 \le \alpha \le 1$ satisfy

$$|P_i^\alpha| \asymp \left(1 + \left((x^2+\omega^2)^2 + 16t^2\right)^{\frac{1}{2}}\right)^{2\alpha} \asymp \left(1 + x^4 + \omega^4 + 2x^2\omega^2 + 16t^2\right)^{\alpha},$$

for $(x, \omega, t) \in P_i^\alpha$.

We give explicit examples of generalized α-coverings in Subsection B.3.2. Before that, we turn to the question about equivalence. The following proposition implies that the specific choice of homogeneous quasi-norm and generalized α-covering does not matter when we define the generalized α-modulation spaces $M^{s,\alpha}_{p,q}(G)$ in Section B.4. The proof of the second statement in Proposition B.3.3 is inspired by the proof of the corresponding statement for Euclidean α-coverings given in [27, Appendix B].

Proposition B.3.3. A covering $\mathcal{P}^\alpha$ on $\mathbb{R}^n$ is a generalized α-covering for the stratified Lie group $(\mathbb{R}^n, *_G)$ independently of the choice of the homogeneous quasi-norm. Moreover, any two generalized α-coverings $\mathcal{Q}^\alpha$ and $\mathcal{P}^\alpha$ for $(\mathbb{R}^n, *_G)$ are equivalent.

Proof. Assume that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two homogeneous quasi-norms on $(\mathbb{R}^n, *_G)$ and that $\mathcal{P} := (P_i^\alpha)_{i\in I}$ is a covering that satisfies (B.3.1) with respect to $\|\cdot\|_1$. It follows from Lemma B.2.3 that

$$|P_i^\alpha| \asymp (1 + \|\xi_i\|_1^2)^{\frac{\alpha Q}{2}} \asymp (1 + \|\xi_i\|_2^2)^{\frac{\alpha Q}{2}},$$

for all $\xi_i \in P_i^\alpha$. Similarly, we have

$$r^{\|\cdot\|_1}(P_i^\alpha) \asymp r^{\|\cdot\|_2}(P_i^\alpha), \qquad R^{\|\cdot\|_1}(P_i^\alpha) \asymp R^{\|\cdot\|_2}(P_i^\alpha)$$

independently of $i \in I$. Hence condition (B.3.2) is satisfied for $\|\cdot\|_2$ when it is satisfied for $\|\cdot\|_1$ and the first statement follows.

For the last statement, it suffices by [59, Proposition 3.6] to show that $\mathcal{Q}^\alpha$ and $\mathcal{P}^\alpha$ are weakly equivalent since they both consist of connected and open sets. Let us fix a homogeneous quasi-norm $\|\cdot\|$ on $(\mathbb{R}^n, *_G)$ and denote by $\mu := |B^{\|\cdot\|}(0,1)|$. We first claim that

$$|P_i^\alpha| \asymp \left(r^{\|\cdot\|}(P_i^\alpha)\right)^Q \asymp \left(R^{\|\cdot\|}(P_i^\alpha)\right)^Q, \tag{B.3.3}$$

where Q denotes the homogeneous dimension of $(\mathbb{R}^n, *_G)$. Since the usual Lebesgue measure is the Haar measure on $(\mathbb{R}^n, *_G)$ we have

$$|B^{\|\cdot\|}(x,R)| = |x *_G B^{\|\cdot\|}(0,R)| = |B^{\|\cdot\|}(0,R)| = |D_R^G B^{\|\cdot\|}(0,1)| = R^Q\mu,$$

where $x \in \mathbb{R}^n$ is arbitrary and $R > 0$. Hence

$$\mu \cdot \left(r^{\|\cdot\|}(P_i^\alpha)\right)^Q \leq |P_i^\alpha| \leq \mu \cdot \left(R^{\|\cdot\|}(P_i^\alpha)\right)^Q,$$

for every $i \in I$. This implies (B.3.3) since

$$\mu \leq \frac{|P_i^\alpha|}{\left(r^{\|\cdot\|}(P_i^\alpha)\right)^Q} = \frac{\left(R^{\|\cdot\|}(P_i^\alpha)\right)^Q}{\left(r^{\|\cdot\|}(P_i^\alpha)\right)^Q} \cdot \frac{|P_i^\alpha|}{\left(R^{\|\cdot\|}(P_i^\alpha)\right)^Q} \leq K^Q \mu,$$

where $K \geq 1$ denotes the uniform bound in (B.3.2).

Assume that $Q_j^\alpha \cap P_i^\alpha \neq \emptyset$ for some $i \in I$ and some $j \in J$. Then (B.3.1) together with (B.3.3) give the estimate

$$R^{\|\cdot\|}(Q_j^\alpha) \asymp R^{\|\cdot\|}(P_i^\alpha) \asymp r^{\|\cdot\|}(P_i^\alpha).$$

Hence there exists a uniform constant $\kappa \geq 1$ such that

$$Q_j^\alpha \subset B^{\|\cdot\|}\left(c_i, \kappa r^{\|\cdot\|}(P_i^\alpha)\right), \quad c_i \in P_i^\alpha. \tag{B.3.4}$$

For every $i \in I$ we consider the constants

$$A_\alpha(i) := \#\left\{Q_j^\alpha \in \mathcal{Q}^\alpha : Q_j^\alpha \cap P_i^\alpha \neq \emptyset\right\}.$$

Assume that there exist a sequence $i_k \in I$ with $k \in \mathbb{N}$ such that $A_\alpha(i_k) \to \infty$. Then if $Q_{j_k}^\alpha \cap P_{i_k}^\alpha \neq \emptyset$, we have

$$\frac{|Q_{j_k}^\alpha|}{\left|B^{\|\cdot\|}\left(c_{i_k}, \kappa r^{\|\cdot\|}(P_{i_k}^\alpha)\right)\right|} \asymp \left(\frac{1}{\kappa\mu}\right)^Q. \tag{B.3.5}$$

Notice that the right-hand side of (B.3.5) does not depend on $k \in \mathbb{N}$. Thus (B.3.4) and (B.3.5) give a contradiction since $\mathcal{Q}^\alpha$ is assumed to be admissible. $\square$

Notice that Proposition B.3.3 still leaves open the possibility that a generalized α_1-covering $\mathcal{P}^{\alpha_1}$ and a generalized α_2-covering $\mathcal{Q}^{\alpha_2}$ for a stratified Lie group $(\mathbb{R}^n, *_G)$ might be equivalent whenever $\alpha_1 \neq \alpha_2$. We will prove in Theorem B.5.1 that is not possible.

B.3.2 Concrete Examples

We now turn to giving concrete examples of generalized α-coverings. It will be clear that we need to require that $(\mathbb{R}^n, *_G)$ is rational in the intermediate case $0 < \alpha < 1$. For $\alpha = 0$, the existence of a lattice $N \subset (\mathbb{R}^n, *_G)$ is convenient but not nessesary. For $\alpha = 1$ the existence of a lattice is irrelevant. The main difference from the Euclidean case is that we do not have the luxury of picking the "canonical" lattice $\mathbb{Z}^n$.

The Uniform Case: $\alpha = 0$

We would like to find a concrete generalized 0-covering for $(\mathbb{R}^n, *_G)$ that, similarly to the Euclidean case, reflects the group operation $*_G$. Such a covering was constructed for any locally compact group in [56] and we briefly review this in the setting of stratified Lie groups realized on Euclidean space.

For every stratified Lie group $(\mathbb{R}^n, *_G, \|\cdot\|)$ there exists a covering $\mathcal{U}$ on $\mathbb{R}^n$ constructed in the following manner: Fix the set $B^{\|\cdot\|}(0,1)$ and consider the collection

$$\left\{x *_G B^{\|\cdot\|}(0,1)\right\}_{x\in\mathbb{R}^n} = \left\{B^{\|\cdot\|}(x,1)\right\}_{x\in\mathbb{R}^n}.$$

Lemma B.3.4. *There exists a family of elements $\{x_i\}_{i\in I}$ with $x_i \in \mathbb{R}^n$ for every $i \in I$ such that*

$$\mathcal{U}(G) \coloneqq \left\{B^{\|\cdot\|}(x_i,1)\right\}_{i\in I}$$

is a generalized 0*-covering for the stratified Lie group* $(\mathbb{R}^n, *_G)$.

Proof. It follows from [56] that there exists a family of elements $\{x_i\}_{i\in I}$ with $x_i \in \mathbb{R}^n$ for every $i \in I$ such that $\mathcal{U}(G)$ is an admissible covering. To show that any ball

$$B^{\|\cdot\|}(x,R), \quad x \in \mathbb{R}^n,\ R > 0$$

is path-connected, it suffices to consider the unit ball $B_0 \coloneqq B^{\|\cdot\|}(0,1)$ by applying a left-translation and a scaling D_R^G. The path $t \mapsto (tx_1, \ldots, t^{v_j}x_j, \ldots, t^{v_n}x_n)$ for $t \in [0,1]$, $v_j \coloneqq \deg(x_j)$, and $x \coloneqq (x_1, \ldots, x_n) \in B_0$ connects the origin to x and lies within B_0 since

$$\|(tx_1, \ldots, t^{v_n}x_n)\| = \|D_t^G(x_1, \ldots, x_n)\| = |t|\|x\| < 1.$$

The balls $B^{\|\cdot\|}(x,R)$ are also open due to the continuity of the homogeneous quasi-norm $\|\cdot\|$.

We are left with checking the two conditions in the definition of a generalized α-covering: The first condition (B.3.1) follows readily since

$$\left|B^{\|\cdot\|}(x_i,R)\right| = R^Q\left|B^{\|\cdot\|}(0,1)\right| \asymp \left(1 + \|\xi_i\|^2\right)^{\frac{0\cdot Q}{2}},$$

where $\xi_i \in B^{\|\cdot\|}(x_i,R)$ and Q is the homogeneous dimension of $(\mathbb{R}^n, *_G)$. The second condition (B.3.2) is clearly satisfied with $K = 1$ since the covering consists of balls with respect to the homogeneous quasi-norm $\|\cdot\|$. □

Proposition B.3.3 implies that the choice of the family $\{x_i\}_{i\in I}$ is largely irrelevant as different families will produce equivalent coverings. We refer to $\mathcal{U}(G)$ as the *uniform covering* of the stratified Lie group $(\mathbb{R}^n, *_G)$. In the case where the stratified Lie group $(\mathbb{R}^n, *_G)$ is rational, we can be even more concrete: Fix a lattice $N \subset (\mathbb{R}^n, *_G)$ and fix $R > 0$ such that the collection

$$\mathcal{U}(G;N) := \left\{ B^{\|\cdot\|}(n,R) \right\}_{n \in N \setminus \{0\}}$$

is an admissible covering. This is possible since N is both uniform and discrete. We can again apply Proposition B.3.3 to see that $\mathcal{U}(G;N)$ is equivalent to the uniform covering $\mathcal{U}(G)$ and we consider $\mathcal{U}(G;N)$ as a concrete realization of $\mathcal{U}(G)$. When $G = (\mathbb{R}^n, +)$ and $N = \mathbb{Z}^n$ then the covering $\mathcal{U}(G;N)$ is precisely the covering introduced in Example B.2.9. Hence the uniform covering $\mathcal{U}(G)$ is a 0-covering that incorporates information about the group structure $*_G$.

The Intermediate Case: $0 < \alpha < 1$

We turn to the intermediate range $0 < \alpha < 1$ and give a concrete covering motivated by the most commonly used α-covering in the Euclidean setting, see [135] for its origin. This covering will require the existence of a lattice $N \subset (\mathbb{R}^n, *_G)$ and extends the covering $\mathcal{U}(G;N)$ introduced above.

Proposition B.3.5. Let $(\mathbb{R}^n, *_G, \|\cdot\|)$ be a rational stratified Lie group with a lattice $N \subset (\mathbb{R}^n, *_G)$. We will use the notation

$$\delta_\beta(\xi) := \|\xi\|^\beta \xi, \quad \xi \in \mathbb{R}^n, \qquad \beta := \frac{\alpha}{1-\alpha},$$

where we have fixed $0 \leq \alpha < 1$. There exists $r_1 > 0$ such that the collection

$$Q_r^\alpha(G;N) := \left\{ B^{\|\cdot\|}\left(\delta_\beta(k), r\|k\|^\beta\right) \right\}_{k \in N \setminus \{0\}}$$

is a generalized α-covering for any $r > r_1$. For $\alpha = 0$ the covering $Q_r^0(G;N)$ is simply $\mathcal{U}(G;N)$ introduced previously.

Proof. Since the statement about $\alpha = 0$ is clear and already justified previously, we will henceforth assume that $0 < \alpha < 1$. The topology induced on $\mathbb{R}^n$ by the balls with respect to the homogeneous quasi-norm $\|\cdot\|$ is equivalent to the usual Euclidean topology by [65, Proposition 3.1.37]. Hence since N is uniform we have that there exists $r_1 > 0$ such that the covering $Q_r^\alpha(G;N)$ is a covering for all $r > r_1$. The argument that $Q_r^\alpha(G;N)$ is admissible is the same as in the Euclidean case and is given in [27, Lemma 2.5 and Theorem 2.6]. We can duplicate the proof of Lemma B.3.4 to deduce all the properties needed for $Q_r^\alpha(G;N)$ to be a generalized α-covering except for the proof of condition (B.3.1). To show this, we need to make a few estimates:

If we let Q denote the homogeneous dimension of $(\mathbb{R}^n, *_G)$, then we have

$$\left|B^{\|\cdot\|}(\delta_\beta(k), r\|k\|^\beta)\right| = \left|D^G_{r\|k\|^\beta} B^{\|\cdot\|}(0,1)\right| \asymp \|k\|^{\frac{\alpha Q}{1-\alpha}}, \tag{B.3.6}$$

by the left-invariance and dilation properties of the Haar measure. By picking the center point $\delta_\beta(k)$ in the ball $B^{\|\cdot\|}(\delta_\beta(k), r\|k\|^\beta)$ we see that

$$\left(1 + \|\delta_\beta(k)\|^2\right)^{\frac{\alpha Q}{2}} = \left(1 + \|k\|^{2\beta+2}\right)^{\frac{\alpha Q}{2}} = \left(1 + \|k\|^{\frac{2}{1-\alpha}}\right)^{\frac{\alpha Q}{2}}. \tag{B.3.7}$$

Since we have excluded zero from the lattice N the estimate $\|k\|^{\frac{2}{1-\alpha}} \asymp 1 + \|k\|^{\frac{2}{1-\alpha}}$ is valid. Comparing this observation with (B.3.6) and (B.3.7) shows that the covering $Q_r^\alpha(G; N)$ is a generalized α-covering since

$$\left|B^{\|\cdot\|}(\delta_\beta(k), r\|k\|^\beta)\right|^{\frac{2}{\alpha Q}} \asymp \|k\|^{\frac{2}{1-\alpha}} \asymp 1 + \|k\|^{\frac{2}{1-\alpha}} = 1 + \|\delta_\beta(k)\|^2.$$

In the definition of a generalized α-covering, we need the above estimate for every $\xi_k \in B^{\|\cdot\|}(\delta_\beta(k), r\|k\|^\beta)$ and not only the center point $\delta_\beta(k)$. This follows from a straightforward computation using that

$$\|\delta_\beta(k)\| = \|\delta_\beta(k) - \xi_k + \xi_k\| \leq C(r\|k\|^\beta + \|\xi_k\|),$$

where $C > 0$ is the constant appearing in (B.2.4). □

The Dyadic Case: $\alpha = 1$

The covering given in Proposition B.3.5 is clearly not well-defined for $\alpha = 1$. We will give a concrete example of a generalized 1-covering that models the classical dyadic intervals underlying the Besov spaces given in Example B.2.10.

Definition B.3.6. Let $(\mathbb{R}^n, *_G, \|\cdot\|)$ be a stratified Lie group. The covering

$$\mathcal{B}(G) := \{D_m(G)\}_{m\in\mathbb{N}_0}$$

given by

$$D_0(G) := B^{\|\cdot\|}(0,2), \quad D_m(G) := B^{\|\cdot\|}(0,2^{m+1}) \setminus \overline{B^{\|\cdot\|}(0,2^{m-1})}, \quad m \in \mathbb{N},$$

is called the *Besov covering* with respect to the homogeneous quasi-norm $\|\cdot\|$ on the stratified Lie group $(\mathbb{R}^n, *_G)$.

The fact that the homogeneous quasi-norm $\|\cdot\|$ is not part of the notation $\mathcal{B}(G)$ will be justified in Lemma B.3.7. The Besov covering is an admissible covering consisting of open and connected sets. Hence it is a concatenation by Lemma B.2.6. The most important property of the covering $\mathcal{B}(G)$ is the scaling invariance

$$D^G_{2^k} D_m(G) = D_{m+k}(G), \quad m \geq 1,\ k \geq 0.$$

For an arbitrary homogeneous quasi-norm $\|\cdot\|$, one can not assure that $(2^m, 0, \ldots, 0) \in D_m(G)$ for $m \geq 1$. Although this is not a serious obstacle, we can fix this by using the homogeneous quasi-norm

$$\|(x_1, \ldots, x_n)\|_2 := \left(\sum_{j=1}^{n} |x_j|^{\frac{2}{v_j}} \right)^{\frac{1}{2}}, \tag{B.3.8}$$

where v_j denotes the degree of x_j. We emphasize that we denote the usual Euclidean norm by $\|\cdot\|_E$ to distinguish it from the homogeneous quasi-norm $\|\cdot\|_2$ in (B.3.8). If $(\mathbb{R}^n, *_G)$ has rank k then

$$\|(x_1, \ldots, x_k, 0, \ldots, 0)\|_2 = \|(x_1, \ldots, x_k, 0, \ldots, 0)\|_E.$$

With the homogeneous quasi-norm (B.3.8) it is clear that

$$(2^m, 0, \ldots, 0) \in D_m(G), \quad m \in \mathbb{N}_0. \tag{B.3.9}$$

Moreover, the group structure between the elements in (B.3.9) is the same as the Euclidean addition since they are in the first layer V_1. The following lemma shows that fixing the homogeneous quasi-norm $\|\cdot\|_2$ is justified and that the Besov covering is a concrete realization of a generalized 1-covering.

Lemma B.3.7. *Assume that $n > 1$. The Besov covering $\mathcal{B}(G)$ is a 1-covering for the stratified Lie group $(\mathbb{R}^n, *_G)$ independently of the choice of homogeneous quasi-norm.*

Proof. We first work with the homogeneous quasi-norm $\|\cdot\|_2$ given in (B.3.8). Let us begin by checking that the ratio property (B.3.2) is satisfied: For $m = 0$ we obviously have $r^{\|\cdot\|_2}(D_0(G)) = R^{\|\cdot\|_2}(D_0(G))$ and hence the ratio is one. For $m \geq 1$ we claim that we have the estimates

$$R^{\|\cdot\|_2}(D_m(G)) \leq 2^{m+1}, \qquad r^{\|\cdot\|_2}(D_m(G)) \geq 2^{m-1}.$$

The first is clear from the definition of $R^{\|\cdot\|_2}(D_m(G))$ while the second follows from considering a ball centered at the point $c_m = (2^m, \ldots, 0)$. In conclusion, this gives

$$\sup_{m \in \mathbb{N}} \frac{R^{\|\cdot\|_2}(D_m(G))}{r^{\|\cdot\|_2}(D_m(G))} \leq \sup_{m \in \mathbb{N}} \frac{2^{m+1}}{2^{m-1}} = 4.$$

To see that the size condition (B.3.1) is satisfied, we denote by $\mu \coloneqq |B^{\|\cdot\|_2}(0,1)|$ and estimate for $m \geq 1$ that

$$\begin{aligned}|D_m(G)| &= \left|B^{\|\cdot\|_2}(0,2^{m+1})\right| - \left|B^{\|\cdot\|_2}(0,2^{m-1})\right| \\ &= \left|D^G_{2^{m+1}} B^{\|\cdot\|_2}(0,1)\right| - \left|D^G_{2^{m-1}} B^{\|\cdot\|_2}(0,1)\right| \\ &= \left(\mu \frac{4^Q - 1}{2^Q}\right) 2^{Qm}.\end{aligned}$$

On the other hand, for any $\xi_m \in D_m(G)$ we have $\|\xi_m\| \asymp 2^m$ and hence

$$(1 + \|\xi_m\|_2^2)^{\frac{Q}{2}} \asymp 2^{Qm}.$$

Combining these estimates shows that the Besov covering $\mathcal{B}(G)$ with the homogeneous quasi-norm $\|\cdot\|_2$ is a 1-covering. Then we can apply Proposition B.3.3 and obtain that the choice of homogeneous quasi-norm defining the Besov covering $\mathcal{B}(G)$ is irrelevant as they all produce equivalent coverings. Hence we can safely use the homogeneous quasi-norm $\|\cdot\|_2$ given in (B.3.8) without loss of generality. When $n = 1$ we have that $(\mathbb{R}, *_G) \simeq (\mathbb{R}, +)$. In that case, we refer the reader to Example B.2.10 for a trivial modification of the result. □

Let $(\mathbb{R}^n, *_G)$ be a rational stratified Lie group with a lattice N. We will use the notation

$$Q^\alpha(G) \coloneqq Q^\alpha(G;N) = \begin{cases} \mathcal{U}(G;N), & \text{if } \alpha = 0 \\ Q_r^\alpha(G;N), & \text{if } 0 < \alpha < 1 \ , \\ \mathcal{B}(G), & \text{if } \alpha = 1 \end{cases} \tag{B.3.10}$$

where the number $r > 0$ is chosen large enough so that $Q_r^\alpha(G;N)$ is a concatenation. The specific value of $r > 0$ needed will be suppressed as it is of no relevance in our augments.

Remark. We have showed that the concatenation $Q^\alpha(G)$ depends (up to equivalence of coverings) only on the parameter $0 \leq \alpha \leq 1$ and the stratified Lie group $(\mathbb{R}^n, *_G)$. We can say even more by introducing the following terminology: The *growth vector* of a stratified Lie group $(\mathbb{R}^n, *_G)$ is the multi-index

$$\mathfrak{G}(G) \coloneqq (n_1, \ldots, n_s), \qquad n_i \coloneqq \dim(V_i), \quad i = 1, \ldots, s,$$

where V_i are as in (B.2.1). If $(\mathbb{R}^n, *_G)$ and $(\mathbb{R}^n, *_H)$ are two stratified Lie groups with $\mathfrak{G}(G) = \mathfrak{G}(H)$ then the homogeneous quasi-norm $\|\cdot\|_2$ given in (B.3.8) are equal for both $(\mathbb{R}^n, *_G)$ and $(\mathbb{R}^n, *_H)$. Moreover, they clearly have the same homogeneous dimension as well. Hence by using the homogeneous quasi-norm

$\|\cdot\|_2$ we see from Definition B.3.1 that a generalized α-covering for $(\mathbb{R}^n, *_G)$ is also a generalized α-covering for $(\mathbb{R}^n, *_H)$ and vice versa. From this we can conclude from Proposition B.3.3 that any generalized α-covering $\mathcal{P}^\alpha$ can be described by two parameters: The continuous parameter $0 \le \alpha \le 1$ and the discrete parameter $\mathfrak{G}(G) \in \mathbb{N}_0^s$ where s is the step of $(\mathbb{R}^n, *_G)$.

B.3.3 Almost Structured Coverings and BAPU's

Many coverings that arise in practice have the property that its elements are essentially given by well-behaved affine transformations of a few reference sets. This notion was studied in [29] and the following definition is a slight generalization appearing in [148].

Definition B.3.8. Let $Q := (Q_i)_{i\in I}$ be an admissible covering on $\mathbb{R}^n$. We call Q an *almost structured covering* if there exists a finite collection $(\mathcal{P}_s)_{s\in J}$ of bounded, open subsets of $\mathbb{R}^n$ called *reference sets* with the following properties:

- There is an invertible affine transformation $A_i := T_i + b_i$ for every $i \in I$ with $T_i \in GL(n, \mathbb{R})$ and $b_i \in \mathbb{R}^n$ such that
$$Q_i = A_i(P_s) = T_i(P_s) + b_i, \tag{B.3.11}$$
for some $s \in J$ depending on $i \in I$.
- If $Q_i \cap Q_j \neq \emptyset$ for some $i, j \in I$ then we have the uniform compatibility condition
$$\|T_i^{-1}T_j\| \le C_Q < \infty, \tag{B.3.12}$$
where C_Q does not depend on $i, j \in I$.
- There should exist a finite collection $(P_s')_{s\in J}$ of open sets with $\overline{P_s'} \subset P_s$ for every $s \in J$ such that $(A_i(P_s'))_{i\in I, s\in J}$ cover $\mathbb{R}^n$.

If the index set $J = \{s\}$ is a singleton, then the covering Q is called a *structured covering*.

Remarks.

- The elements in an almost structured covering $Q := (Q_i)_{i\in I}$ are automatically open by (B.3.11). Hence the index set I is always countable and Q is a concatenation by Lemma B.2.6. The reason one needs to consider almost structured coverings rather than structured coverings can be seen from the dyadic covering $\mathcal{B}(\mathbb{R}^2)$ given in Example B.2.10.

- We would also like to point out that structured or almost structured coverings are not preserved under equivalence of coverings: It is straightforward to construct an equivalent covering to, say, the uniform covering $\mathcal{U}(\mathbb{R}^2)$ that is not even almost structured. Hence questions such as "are all generalized α-coverings corresponding to a stratified Lie group $(\mathbb{R}^n, *_G)$ almost structured?" are not well-defined. To ask meaningful questions, we will have to consider the specific representative coverings given in (B.3.10).

The following originates from [29] and shows that the standard realization of the Euclidean α-coverings considered in the literature are indeed almost structured coverings.

Lemma B.3.9. *The coverings $\mathcal{Q}^\alpha(\mathbb{R}^n; \mathbb{Z}^n)$ are structured for $0 \le \alpha < 1$, while the coverings $\mathcal{B}(\mathbb{R}^n)$ for $n \ge 2$ are only almost structured.*

Hence one might expect that the coverings $\mathcal{Q}^\alpha(G)$ are all at least almost structured for any rational stratified Lie group $(\mathbb{R}^n, *_G)$. This is supported by the fact that a small modification of [64, Proposition 6.2] shows that the uniform covering $\mathcal{U}(\mathbb{H}_n; N)$ is a structured covering where $\mathbb{H}_n$ is the Heisenberg group and N is the lattice $N := (2\mathbb{Z})^{2n} \times \mathbb{Z}$. However, the following proposition shows that this is not true in general and depends on the step of the stratified Lie group in question.

Proposition B.3.10. Let $(\mathbb{R}^n, *_G)$ be a stratified Lie group where $n > 1$.

- The Besov covering $\mathcal{B}(G)$ is an almost structured covering that is never structured unless the group $(\mathbb{R}^n, *_G)$ is isomorphic to $(\mathbb{R}, +)$.
- Assume that $(\mathbb{R}^n, *_G)$ is rational and let N be a lattice. The coverings $\mathcal{Q}^\alpha(G; N)$ are structured for $0 \le \alpha < 1$ whenever the step of $(\mathbb{R}^n, *_G)$ is less than or equal two. However, the coverings $\mathcal{Q}^\alpha(G; N)$ for $0 \le \alpha < 1$ are not necessarily almost structured whenever the step of $(\mathbb{R}^n, *_G)$ is higher than two.

Proof. For the Besov covering $\mathcal{B}(G) = (D_m(G))_{m=0}^\infty$ given in Definition B.3.6 we consider $D_0(G)$ and $D_1(G)$ as the reference sets. Define the matrices

$$A_m = T_m := \begin{pmatrix} 2^{(m-1)\cdot v_1} & & \\ & \ddots & \\ & & 2^{(m-1)\cdot v_n} \end{pmatrix}, \qquad v_j := \deg(x_j), \quad m \ge 1.$$

By setting A_0 to be the identity matrix we then have that $A_0(D_0(G)) = D_0(G)$ and $A_m(D_1(G)) = D_m(G)$ since

$$A_m D_1(G) = D^G_{2^{m-1}} D_1(G) = D_m(G).$$

Notice that two elements $D_m(G)$ and $D_l(G)$ can only intersect in the case when $m \in \{l-1, l, l+1\}$. In any case, a straightforward computation gives that

$$\|T_l^{-1} T_m\| \leq 2^{\nu_n},$$

and the estimate (B.3.12) is satisfied. Finally, it is clear that the last requirement in Definition B.3.8 is satisfied by shrinking $D_0(G)$ and $D_1(G)$ slightly.

The Besov covering $\mathcal{B}(G)$ is not a structured covering when $n > 1$ since $D_0(G)$ is convex while the sets $D_m(G)$ for $m \geq 1$ are not. When $n = 1$ the stratification (B.2.1) has only one layer and hence $(\mathbb{R}, *_G) \simeq (\mathbb{R}, +)$. It is clear from the construction given in Example B.2.11 that the modification of the covering $\mathcal{B}(\mathbb{R})$ given by dividing each of the sets $D_m(\mathbb{R})$ for $m \in \mathbb{N}$ into its connected components is structured.

Let us now turn to the second statement. If $(\mathbb{R}^n, *_G)$ has step one, then we are in the Euclidean setting and the result follows from Lemma B.3.9. Assume that $(\mathbb{R}^n, *_G, \|\cdot\|)$ has step two and write $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$ with $l = n - k$ according to the decomposition given in (B.2.1). For $(a, b), (c, d) \in \mathbb{R}^k \oplus \mathbb{R}^l$ we can use the BCH-formula (B.2.2) to write their product as

$$(a, b) *_G (c, d) = \left(a + c, b + d + \frac{1}{2} P(a, c)\right),$$

where $P(a, c)$ is a linear polynomial in the components of a and c. This can be written as the block-matrix equation

$$(a, b) *_G (c, d) = \begin{pmatrix} I_{k\times k} & 0_{l\times k} \\ \rho(c) & I_{l\times l} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix},$$

where $\rho(c) \in M_{l\times k}(\mathbb{R})$, each of the entries in $\rho(c)$ depend linearly on the components of c, and $\rho(c) \cdot a = P(a, c)$. Consider now the covering $Q^\alpha(G; N)$ for $0 \leq \alpha < 1$ and write each element as

$$B^{\|\cdot\|}\left(\delta_\beta(k), r\|k\|^\beta\right) = \|k\|^{\frac{\alpha}{1-\alpha}} k *_G \left(D^G_{r\|k\|^{\frac{\alpha}{1-\alpha}}} B^{\|\cdot\|}(0, 1)\right), \quad \beta := \frac{\alpha}{1-\alpha},$$

for every $k \in N \setminus \{0\}$ where $\delta_\beta(k) := \|k\|^\beta k$. We set the reference set to be $B^{\|\cdot\|}(0, 1)$ and leave it to the reader to show that the affine transformations

$$A_k(x) := \|k\|^{\frac{\alpha}{1-\alpha}} k *_G D^G_{r\|k\|^{\frac{\alpha}{1-\alpha}}}(x), \qquad x \in \mathbb{R}^n, \quad k \in N \setminus \{0\},$$

make $Q^\alpha(G; N)$ into a structured covering.

Since $Q^\alpha(G; N)$ is almost structured whenever $\mathcal{U}(G; N)$ is almost structured, it suffices find a stratified Lie group $(\mathbb{R}^n, *_G)$ such that $\mathcal{U}(G; N)$ is not almost structured. Consider the stratified Lie group G whose Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g} := \operatorname{span}\{X_1, X_2, X_3, X_4\},$$

with bracket relations $[X_1, X_2] = X_3$ and $[X_1, X_3] = X_4$. This is a stratification where $V_1 = \operatorname{span}\{X_1, X_2\}$, $V_2 = \operatorname{span}\{X_3\}$ and $V_3 = \operatorname{span}\{X_4\}$. By using the BCH-formula (B.2.2) we can identify G with $(\mathbb{R}^4, *_G)$, where the multiplication $(z_1, z_2, z_3, z_4) = (x_1, x_2, x_3, x_4) *_G (y_1, y_2, y_3, y_4)$ has the form

$$\begin{aligned}
z_1 &= x_1 + y_1, \\
z_2 &= x_2 + y_2, \\
z_3 &= x_3 + y_3 + \frac{1}{2}\left(x_1 y_2 - x_2 y_1\right), \\
z_4 &= x_4 + y_4 + \frac{1}{2}\left(x_1 y_3 - x_3 y_1\right) + \frac{x_1}{12}\left(x_1 y_2 - x_2 y_1\right).
\end{aligned}$$

Consider the lattice $N := 12\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ in $(\mathbb{R}^4, *_G)$. Assume first that $Q(G; N)$ is a structured covering and let B be the reference set. Then for elements $k, k' \in N \setminus \{0\}$ we can find affine transformations A_k and $A_{k'}$ such that

$$A_k(B) = T_k(B) + b_k = B^{\|\cdot\|}(k, R), \quad A_{k'}(B) = T_{k'}(B) + b_{k'} = B^{\|\cdot\|}(k', R),$$

where $R > 0$ is a fixed number so that $Q(G; N)$ is an admissible covering. Then

$$T_{k'} T_k^{-1} B^{\|\cdot\|}(k, R) + (b_{k'} - T_{k'} T_k^{-1} b_k) = B^{\|\cdot\|}(k', R).$$

Hence if $k = (12, 2, 1, 1)$ and $k' = (12n, 2n, n, n)$ for $n \in \mathbb{N}$ we can increase n and obtain a contradiction due to the quadratic term in x_1 in the z_4-entry of the group product. This argument can easily be extended to show that $Q(G; N)$ is not an almost structured covering since infinitely many of the numbers $k' = (12n, 2n, n, n)$ for $n \in \mathbb{N}$ have to correspond to one of the (finite number of) reference sets. □

The fact that the coverings $Q^\alpha(G; N)$ are almost structured whenever the step of $(\mathbb{R}^n, *_G)$ is less than or equal two is closely related to the existence of $Q^\alpha(G; N)$-BAPU's. The following proposition follows from [149, Theorem 2.8] which is a slight generalization of the general existence result [29, Proposition 1].

Proposition B.3.11. Let $(\mathbb{R}^n, *_G)$ be a rational stratified Lie group with step less than or equal two and fix a lattice N. Then there exists a $Q^\alpha(G; N)$-BAPU for all $0 \leq \alpha \leq 1$.

Example B.3.12. Consider a rational stratified Lie group $(\mathbb{R}^n, *_G, \|\cdot\|)$ of step less than or equal two with a lattice N.

- For $0 \leq \alpha < 1$, then an explicit $Q^\alpha(G; N)$-BAPU can be constructed by adapting the argument in [27, Proposition A.1] as follows: Fix $r > r_1$ where

r_1 is the number appearing in Proposition B.3.5. Consider a positive and smooth function $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\operatorname{supp}(\Phi) \subset B^{\|\cdot\|}(0,r), \qquad \inf_{\xi \in B^{\|\cdot\|}(0,r_1)} \Phi(\xi) > 0. \tag{B.3.13}$$

To find the $Q_r^\alpha(G;N)$-BAPU we need, we simply scale the argument of Φ correctly: Define

$$g_k(\xi) := \Phi\left(D^G_{\|c_k\|^{-\alpha}}\left(c_k^{-1} *_G \xi\right)\right), \qquad c_k := \|k\|^{\frac{\alpha}{1-\alpha}} k, \quad k \in N \setminus \{0\}.$$

Then g_k is smooth and unwinding its definition shows that

$$\operatorname{supp}(g_k) \subset B^{\|\cdot\|}\left(\delta_\beta(k), r\|k\|^\beta\right), \quad k \in N \setminus \{0\}.$$

Moreover, the infimum bound in (B.3.13) ensures that for every $\xi \in \mathbb{R}^n$ there is a g_k such that $g_k(\xi) > 0$.

Define

$$\psi_k(\xi) := \frac{g_k(\xi)}{\sum_{l \in N \setminus \{0\}} g_l(\xi)}.$$

The L^1-bound in (B.2.10) is satisfied by an adaption of [28, Proposition 2.4]. For this to work it is essential that the step of $(\mathbb{R}^n, *_G)$ is less than or equal two so that the group multiplication $*_G$ can be represented by linear maps. Hence we obtain a $Q_r^\alpha(G;N)$-BAPU. The functions ψ_k have compact support since the balls induced by the homogeneous quasi-norm $\|\cdot\|$ are bounded (not uniformly) with respect to the Euclidean metric.

- For $\alpha = 1$ we can proceed as follows: Pick a positive and smooth function Φ_0 with $\operatorname{supp}(\Phi_0) \subset D_0(G)$ and $\Phi_0(x) = 1$ for every $x \in \mathbb{R}^n$ with $\|x\| \leq \frac{3}{2}$. Moreover, pick a positive and smooth function Φ_1 with $\operatorname{supp}(\Phi_1) \subset D_1(G)$ and with $\Phi_1(x) = 1$ for every $x \in \mathbb{R}^n$ with $\frac{3}{2} \leq \|x\| \leq \frac{7}{2}$. The collection $(\Phi_m)_{m=0}^\infty$ given by

$$\Phi_m(x) := \Phi_1\left(D^G_{2^{1-m}}(x)\right), \quad m \geq 2,\ x \in \mathbb{R}^n, \tag{B.3.14}$$

consists of smooth functions with $\operatorname{supp}(\Phi_m) \subset D_m(G)$ that are never vanishing simultaneously. Hence we define for $x \in \mathbb{R}^n$ and $m \in \mathbb{N}_0$ the normalized collection

$$\psi_m(x) := \frac{\Phi_m(x)}{\sum_{k=0}^\infty \Phi_k(x)} = \frac{\Phi_m(x)}{\Phi_{m-1}(x) + \Phi_m(x) + \Phi_{m+1}(x)},$$

where we set $\Phi_{-1} \equiv 0$ to make the last equality work for $m = 0$. The L^1-bound in (B.2.10) follows readily from the relation (B.3.14) and thus $(\psi_m)_{m=0}^{\infty}$ is a $\mathcal{B}(G)$-BAPU. Since the support of ψ_m is closed and contained in $B^{\|\cdot\|}(0, 2^{m+1})$, it is clear that ψ_m have compact support for every $m \geq 0$.

Notice that the existence of a lattice in the above example is not nessesary for the case $\alpha = 1$. In fact, the $\mathcal{B}(G)$-BAPU construction is valid for any stratified Lie group regardless of its step. For the rest of the paper, we will refer to a rational stratified Lie group with step less than or equal two as an *admissible Lie group* for simplicity.

B.4 Generalized α-Modulation Spaces

In this section, we define the generalized α-modulation spaces $M^{s,\alpha}_{p,q}(G)$ associated to a stratified Lie group $(\mathbb{R}^n, *_G)$. The spaces $M^{s,\alpha}_{p,q}(G)$ are built on the generalized α-coverings examined in the previous section. In many regards, the spaces $M^{s,\alpha}_{p,q}(G)$ behave similarly to their Euclidean counterparts $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$. However, we will show in later sections that they depend heavily on the stratified Lie group $(\mathbb{R}^n, *_G)$ in question. Firstly, let us define the correct reservoir defining the functions/distributions of interest.

Definition B.4.1. Consider the space $Z(\mathbb{R}^n) := \mathcal{F}\left(C_c^{\infty}(\mathbb{R}^n)\right)$ consisting of Fourier transforms of all smooth functions with compact support on $\mathbb{R}^n$. We equip the space $Z(\mathbb{R}^n)$ with the unique topology ensuring that the Fourier transform is a homeomorphism from $C_c^{\infty}(\mathbb{R}^n)$ to $Z(\mathbb{R}^n)$. Define the *Fourier type reservoir* as the dual space $Z'(\mathbb{R}^n)$ equipped with the weak* topology.

The Fourier transform extends by duality to a homeomorphism

$$\mathcal{F}\colon Z'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n) := \left(C_c^{\infty}(\mathbb{R}^n)\right)'.$$

We refer the reader to [148, Chapter 3] where the danger of using the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ as a reservoir instead of the more exotic space $Z'(\mathbb{R}^n)$ is discussed. This might seem contradictory as we defined the Euclidean α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ in Definition B.2.8 as subspaces of the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. However, it follows from [148, Theorem 8.3] that the Euclidean α-modulation spaces would embed into $\mathcal{S}'(\mathbb{R}^n)$ if we had defined them using the Fourier type reservoir $Z'(\mathbb{R}^n)$. Hence one might as well define the Euclidean α-modulation spaces as subspaces of tempered distributions without loss of generality.

Definition B.4.2. Let $(\mathbb{R}^n, *_G, \|\cdot\|)$ be a stratified Lie group with a homogeneous quasi-norm $\|\cdot\|$. Consider a generalized α-covering $\mathcal{P}^{\alpha} := (P_i^{\alpha})_{i\in I}$ on $\mathbb{R}^n$ where

$0 \leq \alpha \leq 1$ and assume that $\Phi := (\psi_i)_{i \in I}$ is a $\mathcal{P}^\alpha$-BAPU. The *generalized α-modulation space* $M^{s,\alpha}_{p,q}(G)$ for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ consists of all Fourier type distributions $f \in Z'(\mathbb{R}^n)$ such that

$$\|f\|_{M^{s,\alpha}_{p,q}(G)} := \left\| \left((1 + \|\xi_i\|^2)^{\frac{s}{2}} \left\| \mathcal{F}^{-1} \left(\psi_i \cdot \mathcal{F}(f) \right) \right\|_{L^p} \right)_{i \in I} \right\|_{l^q(I)} < \infty, \qquad \text{(B.4.1)}$$

where $\xi_i \in P^\alpha_i$ for every $i \in I$. The number s will be referred to as the *smoothness parameter* of the space $M^{s,\alpha}_{p,q}(G)$, while p and q are called the *integrability parameters*.

If the stratified Lie group $(\mathbb{R}^n, *_G)$ is isomorphic to the Euclidean space $(\mathbb{R}^n, +)$ with its usual addition, then Definition B.4.2 reduces to the usual α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$. Notice that

$$M^{s+\epsilon,\alpha}_{p,q}(G) \subset M^{s,\alpha}_{p,q}(G), \qquad M^{s,\alpha}_{p,q_1}(G) \subset M^{s,\alpha}_{p,q_2}(G),$$

for all $\epsilon > 0$ and whenever $q_1 \leq q_2$ due to the monotonicity of the l^q-norms. It follows from Proposition B.3.3 that any two generalized α-coverings for the same stratified Lie group are equivalent. Hence [59, Theorem 3.7] implies that $M^{s,\alpha}_{p,q}(G)$ does not depend on the specific generalized α-covering chosen. Moreover, it follows from [59, Theorem 2.3 B)] that different choices of $\mathcal{P}^\alpha$-BAPU's in Definition B.4.2 yield equivalent norms.

Remarks.

- As we have discussed in Subsection B.3.3, we can only guarantee the existence of the BAPU's needed in Definition B.4.2 in certain settings. This setting include all admissible Lie groups, which is the most interesting class when it comes to applications. However, will state some results for generalized α-modulation spaces on an arbitrary stratified Lie group with the convention that this might be vacuous when we do not know the existence of suitable BAPU's. In that way, some of the results we prove can still be used for a general stratified Lie group once the existence of a suitable BAPU has been established.

- Let us briefly comment on why the expression (B.4.1) is well-defined: Since $f \in Z'(\mathbb{R}^n)$ we have that $\mathcal{F}(f) \in \mathcal{D}'(\mathbb{R}^n)$. Then the product $\psi_i \cdot \mathcal{F}(f)$ is a compactly supported distribution. Hence we can consider $\psi_i \cdot \mathcal{F}(f)$ as a tempered distribution and thus

  $$\mathcal{F}^{-1} \left(\psi_i \cdot \mathcal{F}(f) \right) \in \mathcal{S}'(\mathbb{R}^n).$$

  Moreover, the distribution $\mathcal{F}^{-1} \left(\psi_i \cdot \mathcal{F}(f) \right)$ acts on rapidly decaying functions by integrating them against an entire function with polynomially bounded derivatives [142, Theorem 7.23]. Hence the expression in (B.4.1) is well-defined, although often infinite.

The generalized α-modulation spaces $M^{s,\alpha}_{p,q}(G)$ are complete for all values of the parameters $1 \leq p, q \leq \infty$, $s \in \mathbb{R}^n$, and $0 \leq \alpha \leq 1$ by [148, Theorem 3.21]. Motivated by the Euclidean setting, we will also refer to

$$M^s_{p,q}(G) \coloneqq M^{s,0}_{p,q}(G)$$

as the *modulation space* corresponding to the stratified Lie group $(\mathbb{R}^n, *_G)$. The modulation space $M^s_{p,q}(\mathbb{H}_n)$ corresponding to the Heisenberg group $\mathbb{H}_n$ has been investigated in [64].

Similarly, we will also refer to

$$\mathcal{B}^s_{p,q}(G) \coloneqq M^{s,1}_{p,q}(G)$$

as the *Besov space* corresponding to the stratified Lie group $(\mathbb{R}^n, *_G)$. One can view the Besov spaces $\mathcal{B}^s_{p,q}(G)$ as generalizations of the traditional Besov spaces $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ where the dilations are not uniform in different directions. The spaces

$$M_{p,q}(G) \coloneqq M^{0,0}_{p,q}(G), \qquad \mathcal{B}_{p,q}(G) \coloneqq \mathcal{B}^{0,1}_{p,q}(G)$$

will be called the *standard modulation spaces* and *standard Besov spaces* of $(\mathbb{R}^n, *_G)$, respectively. We begin by giving concrete realizations of the generalized α-modulation spaces.

Corollary B.4.3. Let $(\mathbb{R}^n, *_G, \|\cdot\|)$ be a rational stratified Lie group with a given lattice N. Fix the parameters $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$.

- An equivalent norm on the modulation space $M^s_{p,q}(G)$ is given by the expression

$$\left(\sum_{k \in N \setminus \{0\}} \left(1 + \|k\|^2\right)^{\frac{qs}{2}} \left\| \mathcal{F}^{-1}\left(\psi_k \cdot \mathcal{F}(f)\right) \right\|^q_{L^p} \right)^{\frac{1}{q}}, \tag{B.4.2}$$

  where $\{\psi_k\}_{k \in N \setminus \{0\}}$ is a $\mathcal{U}(G; N)$-BAPU and the covering $\mathcal{U}(G; N)$ is described in Subsection B.3.2.

- An equivalent norm on $M^{s,\alpha}_{p,q}(G)$ for $0 < \alpha < 1$ is given by the expression

$$\left(\sum_{k \in N \setminus \{0\}} \left(1 + \|k\|^{\frac{2}{(1-\alpha)}}\right)^{\frac{qs}{2}} \left\| \mathcal{F}^{-1}\left(\psi_k \cdot \mathcal{F}(f)\right) \right\|^q_{L^p} \right)^{\frac{1}{q}}, \tag{B.4.3}$$

  where $\{\psi_k\}_{k \in N \setminus \{0\}}$ is a $\mathcal{Q}^\alpha_r(G; N)$-BAPU and the covering $\mathcal{Q}^\alpha_r(G; N)$ is given in Subsection B.3.2.

- An equivalent norm on the Besov space $\mathcal{B}^s_{p,q}(G)$ is given by the expression

$$\left(\sum_{m=0}^{\infty} 2^{mqs} \left\|\mathcal{F}^{-1}\left(\psi_m \cdot \mathcal{F}(f)\right)\right\|_{L^p}^q\right)^{\frac{1}{q}}, \tag{B.4.4}$$

  where $\{\psi_m\}_{m=0}^{\infty}$ is a $\mathcal{B}(G)$-BAPU and the covering $\mathcal{B}(G)$ is described in Subsection B.3.2.

If $(\mathbb{R}^n, *_G, \|\cdot\|)$ is an admissible Lie group, then we can pick the explicit BAPU's given in Example B.3.12.

Proof. Consider the intermediate case $0 < \alpha < 1$: Picking the center point $\delta_\beta(k)$ in each of the balls in the covering $Q_r^\alpha(G; N)$ gives

$$\left(1 + \|\delta_\beta(k)\|^2\right)^{\frac{1}{2}} = \left(1 + \|k\|^{2(\beta+1)}\right)^{\frac{1}{2}} = \left(1 + \|k\|^{\frac{2}{1-\alpha}}\right)^{\frac{1}{2}}, \quad k \in N \setminus \{0\}.$$

Hence (B.4.3) follows and we obtain (B.4.2) since $\mathcal{U}(G; N) = Q_r^0(G; N)$ as explained in Proposition B.3.5. For the Besov covering $\mathcal{B}(G)$ the choice of homogeneous quasi-norm in (B.4.1) is irrelevant due to Lemma B.3.7. Hence we can freely choose the homogeneous quasi-norm $\|\cdot\|_2$ given in (B.3.8) and use consequence (B.3.9). Thus we obtain

$$\left(1 + \|\left(2^m, 0, \ldots, 0\right)\|_2^2\right)^{\frac{1}{2}} = (1 + 4^m)^{\frac{1}{2}} \asymp 2^m, \quad m \in \mathbb{N}_0,$$

and the statement follows. $\square$

Remark. We would like to emphasize that the expression (B.4.4) does not depend on the lattice N and is hence valid whenever $(\mathbb{R}^n, *_G)$ is not rational. The expressions (B.4.4) also shows the similarities with the classical Besov spaces $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ in the literature.

Let us introduce some notation that simplifies the expressions in Corollary B.4.3 for admissible Lie groups when $0 \leq \alpha < 1$: Consider the generalized α-covering $Q^\alpha(G; N)$ and a smooth $Q^\alpha(G; N)$-BAPU $\{\psi_k\}_{k \in N \setminus \{0\}}$ with compact support. Let $\mathcal{P}(\mathbb{R}^n; N)$ denote sequences $\{f_k\}_{k \in N \setminus \{0\}}$ where each $f_k \in \mathcal{S}'(\mathbb{R}^n)$ acts on rapidly decaying functions by integrating against a polynomially bounded function. Define the *Fourier multiplier operator*

$$\square_k^{\alpha, G} := \mathcal{F}^{-1} \psi_k \mathcal{F},$$

and the space

$$
\begin{aligned}
&l^{s,\alpha}_{p,q}(G;N) \\
&\quad := \left\{ \{f_k\}_{k\in N\setminus\{0\}} \in \mathcal{P}(\mathbb{R}^n;N) : \left\| \left(1+\|k\|^{\frac{2}{1-\alpha}}\right)^{\frac{s}{2}} \|f_k\|_{L^p} \right\|_{l^q(N\setminus\{0\})} < \infty \right\}.
\end{aligned}
$$

The generalized α-modulation space $M^{s,\alpha}_{p,q}(G)$ for an admissible Lie group $(\mathbb{R}^n, *_G)$ and parameters $0 \le \alpha < 1$, $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ can be written as

$$
M^{s,\alpha}_{p,q}(G) = \left\{ f \in Z'(\mathbb{R}^n) : \|\square^{\alpha,G}_k f\|_{l^{s,\alpha}_{p,q}(G;N)} < \infty \right\}. \tag{B.4.5}
$$

We emphasize that this is only valid since the image of the Fourier type distributions $Z'(\mathbb{R}^n)$ under the Fourier multiplier operator $\square^{\alpha,G}_k$ for $k \in N \setminus \{0\}$ and $0 \le \alpha < 1$ is contained in the tempered distributions.

The representation (B.4.5) is useful because it allows us to reduce certain questions about the generalized α-modulation spaces $M^{s,\alpha}_{p,q}(G)$ to the sequence-type spaces $l^{s,\alpha}_{p,q}(G;N)$. As an application, we now prove a duality relation between the spaces $M^{s,\alpha}_{p,q}(G)$ for $1 \le p, q < \infty$, $s \in \mathbb{R}$, and $0 \le \alpha < 1$. This is more or less a straightforward adaption of the Euclidean case given in [93, Theorem 2.1] with some minor modifications resulting from using an arbitrary lattice N instead of the concrete lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Proposition B.4.4. Let $(\mathbb{R}^n, *_G)$ be an admissible Lie group and fix parameters $1 \le p, q < \infty$, $s \in \mathbb{R}$, and $0 \le \alpha < 1$. The dual space of $M^{s,\alpha}_{p,q}(G)$ can be identified with $M^{-s,\alpha}_{p',q'}(G)$, where p' and q' are the conjugate variables of p and q, respectively.

Proof. Fix a lattice $N \subset (\mathbb{R}^n, *_G)$. Any $f \in M^{-s,\alpha}_{p',q'}(G)$ acts on $g \in M^{s,\alpha}_{p,q}(G)$ by

$$
\langle f, g\rangle := \sum_{k\in N\setminus\{0\}} \int_{\mathbb{R}^n} \square^{\alpha,G}_k f \cdot \square^{\alpha,G}_k g \, dx.
$$

A straightforward computation using Hölder's inequality twice shows that

$$
\begin{aligned}
|\langle f, g\rangle| &= \left| \sum_{k\in N\setminus\{0\}} \int_{\mathbb{R}^n} \left(1+\|k\|^{\frac{2}{1-\alpha}}\right)^{\frac{-s}{2}} \square^{\alpha,G}_k f \cdot \left(1+\|k\|^{\frac{2}{1-\alpha}}\right)^{\frac{s}{2}} \square^{\alpha,G}_k g \, dx \right| \\
&\le \|f\|_{M^{-s,\alpha}_{p',q'}(G)} \|g\|_{M^{s,\alpha}_{p,q}(G)},
\end{aligned}
$$

where we have used the explicit expressions in (B.4.3) and (B.4.2). Hence the action of f on $M^{s,\alpha}_{p,q}(G)$ is bounded and we have $M^{-s,\alpha}_{p',q'}(G) \subset (M^{s,\alpha}_{p,q}(G))^*$.

Conversely, an element $h \in (M^{s,\alpha}_{p,q}(G))^*$ induce an element $\tilde{h} \in \left(l^{s,\alpha}_{p,q}(G;N)\right)^*$ since we can identify $M^{s,\alpha}_{p,q}(G)$ with the image $\{\Box^{\alpha,G}_k f\}_{k\in N\setminus\{0\}} \in l^{s,\alpha}_{p,q}(G;N)$. A standard argument similar to the one given in [151, Proposition 3.3] shows that we have the duality relation

$$\left(l^{s,\alpha}_{p,q}(G;N)\right)^* \simeq l^{-s,\alpha}_{p',q'}(G;N), \qquad \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1.$$

Hence we can find $\{h_k\} \in l^{-s,\alpha}_{p',q'}(G;N)$ such that

$$\langle \tilde{h}, \{f_k\}\rangle = \sum_{k\in N\setminus\{0\}} \int_{\mathbb{R}^n} \overline{h_k(x)} f_k(x)\,dx, \quad \{f_k\} \in l^{s,\alpha}_{p,q}(G;N).$$

Thus for $g \in M^{s,\alpha}_{p,q}(G)$ we can use Plancherel to obtain

$$\langle h, g\rangle = \langle \tilde{h}, \Box^{\alpha,G}_k g\rangle = \int_{\mathbb{R}^n} \sum_{k\in N\setminus\{0\}} \overline{\Box^{\alpha,G}_k h_k(x)} g(x)\,dx,$$

and we can conclude that

$$h = \sum_{k\in N\setminus\{0\}} \Box^{\alpha,G}_k h_k.$$

A straightforward generalization of [93, Lemma 2.1] shows that

$$\|h\|_{M^{-s,\alpha}_{p',q'}(G)} \asymp \|\{h_k\}_{k\in N\setminus\{0\}}\|_{l^{-s,\alpha}_{p',q'}(G;N)} = \|h\|_{(M^{s,\alpha}_{p,q}(G))^*},$$

implying that $(M^{s,\alpha}_{p,q}(G))^* \subset M^{-s,\alpha}_{p',q'}(G)$. $\Box$

The duality relations for the Besov spaces are the obvious extensions of the Euclidean Besov spaces, namely

$$\left(\mathcal{B}^s_{p,q}(G)\right)^* \simeq B^{-s}_{p',q'}(G),$$

where $1 \le p, q < \infty$, $s \in \mathbb{R}$, and p', q' are the conjugate variables of p and q. We omit the proof since it is straightforward.

Corollary B.4.5. Let $(\mathbb{R}^n, *_G)$ be an admissible Lie group. The modulation spaces $M^s_{p,q}(G)$ and Besov spaces $\mathcal{B}^s_{p,q}(G)$ for $1 < p, q < \infty$ and $s \in \mathbb{R}$ are two families of reflexive Banach spaces that are closed under duality.

There is an abundance of properties one can prove when defining new function spaces. We will focus on two properties illustrating that the generalized α-modulation spaces $M^{s,\alpha}_{p,q}(G)$ are not degenerate:

(i) When $n = \dim(G)$ the spaces $M^{s,\alpha}_{p,q}(G)$ are large enough to contain the rapidly decaying smooth functions $\mathcal{S}(\mathbb{R}^n)$ as subspaces.

(ii) The spaces $M^{s_1,\alpha_1}_{p_1,q_1}(G)$ are really new spaces in the sense that they do not coincide with the traditional α-modulation spaces $M^{s_2,\alpha_2}_{p_2,q_2}(\mathbb{R}^n)$ for most of the parameters $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$, and $0 \leq \alpha_1, \alpha_2 \leq 1$.

Property (i) relies on basic properties of lattices in stratified Lie groups and is proved below. On the other hand, Property (ii) is more challenging and require several preliminary results. The main aim of Section B.5 is to prove Property (ii).

Proposition B.4.6. Let $(\mathbb{R}^n, *_G)$ be a rational stratified Lie group with $\dim(G) = n$. Then the rapidly decaying smooth functions $\mathcal{S}(\mathbb{R}^n)$ is contained in $M^{s,\alpha}_{p,q}(G)$ for all $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$.

Proof. The embeddings $M^{s,\alpha}_{p,q_1}(G) \subset M^{s,\alpha}_{p,q_2}(G)$ for $q_1 \leq q_2$ implies that it suffices to show the inclusion $\mathcal{S}(\mathbb{R}^n) \subset M^{s,\alpha}_{p,1}(G)$. We consider first the case $0 \leq \alpha < 1$ and use the norms (B.4.2) and (B.4.3). Since the Fourier transform of a Schwartz function is again a Schwartz function, we have that

$$\epsilon_k := \left\|\mathcal{F}^{-1}\left(\psi_k \cdot \mathcal{F}(f)\right)\right\|_{L^p} < \infty, \quad k \in N \setminus \{0\},$$

for each $f \in \mathcal{S}(\mathbb{R}^n)$ since $\mathcal{S}(\mathbb{R}^n) \subset L^p$ for every $1 \leq p \leq \infty$.

Fix a homogeneous norm $\|\cdot\|$ on $(\mathbb{R}^n, *_G)$ as we have remarked previously always exists. Define the metric

$$d_G(x, y) := \|x^{-1} *_G y\|, \quad x, y \in \mathbb{R}^n.$$

We claim that the number of points in N that are of distance less than $R > 0$ away from the origin with respect to the metric d_G grows with polynomial rate. It is clear that d_G is a left-invariant metric on $(\mathbb{R}^n, *_G)$. It restricts to a proper, left-invariant metric $d_G|_N$ on N. Since N is a finitely generated nilpotent group we know that $d_G|_N$ has polynomial growth by Gromov's celebrated polynomial growth theorem [87] and the claim follows.

It is straightforward to see that the numbers ϵ_k decay exponentially as the size of $k \in N \setminus \{0\}$ grows with respect to the metric $d_G|_N$. The weight

$$\left(1 + \|k\|^{\frac{2}{1-\alpha}}\right)^{\frac{s}{2}}$$

only contributes polynomially since both α and s are fixed. Hence it follows that

$$\mathcal{S}(\mathbb{R}^n) \subset M^{s,\alpha}_{p,1}(G) \subset M^{s,\alpha}_{p,q}(G),$$

since $l^1(N \setminus \{0\})$ contains all rapidly decreasing sequences. The case $\alpha = 1$ is an elementary adaption of the classical proof of the inclusion $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{B}^s_{p,q}(\mathbb{R}^n)$. $\square$

B.5 Uniqueness of the Generalized α-Modulation Spaces

B.5.1 Preliminary Results

We now turn to the question regarding the uniqueness of the spaces $M^{s,\alpha}_{p,q}(G)$ for an admissible Lie group $(\mathbb{R}^n, *_G)$. To be able to answer this, we need a stronger statement about the generalized α-coverings than what was proved in Proposition B.3.3. In the Euclidean case, parts of Theorem B.5.1 have been proven with different methods in [148, Lemma 9.5 and Lemma 9.12]. We remark that the proof of Theorem B.5.1 does build on Proposition B.3.3 and most of Section B.3 in a non-trivial way.

Theorem B.5.1. *Let $(\mathbb{R}^n, *_G)$ be a rational stratified Lie group and consider a generalized α_1-covering $\mathcal{P}^{\alpha_1}$ and a generalized α_2-covering $\mathcal{P}^{\alpha_2}$ for parameters $0 \leq \alpha_1, \alpha_2 \leq 1$. Then the following are equivalent:*

- *The covering $\mathcal{P}^{\alpha_1}$ is weakly subordinate to the covering $\mathcal{P}^{\alpha_2}$,*
- *The covering $\mathcal{P}^{\alpha_1}$ is almost subordinate to the covering $\mathcal{P}^{\alpha_2}$,*
- *The parameters satisfy $\alpha_1 \leq \alpha_2$.*

In particular, the coverings $\mathcal{P}^{\alpha_1}$ and $\mathcal{P}^{\alpha_2}$ are equivalent if and only if $\alpha_1 = \alpha_2$.

Proof. Since all the coverings in question consist of open and connected sets, it suffices to show the second equivalence as pointed out previously. Moreover, since we have proved in Proposition B.3.3 that any two generalized α-coverings are equivalent, it suffices to consider the explicit coverings $Q^{\alpha_1}(G)$ and $Q^{\alpha_2}(G)$ in (B.3.10). Let us fix a lattice N in $(\mathbb{R}^n, *_G)$ and a homogeneous quasi-norm $\|\cdot\|$.

We begin by considering the Besov case $\alpha_2 = 1$. We note that the size of the sets $D_m(G)$ grows exponentially with respect to $m \in \mathbb{N}_0$. However, the size of the elements in $Q^{\alpha_1} := (Q^{\alpha_1}_n(G))_{n \in N \setminus \{0\}}$ for $0 \leq \alpha_1 < 1$ grows polynomially when we order the index set $N \setminus \{0\}$ in a way such that $m \leq n$ whenever $\|m\| \leq \|n\|$. Hence the number

$$\#\left\{n \in N \setminus \{0\} : D_m(G) \cap Q^{\alpha_1}_n(G) \neq \emptyset\right\}$$

will grow unbounded as m increases. Hence $\mathcal{B}(G)$ is not weakly subordinate to any $Q^{\alpha_1}(G)$ for $0 \leq \alpha_1 < 1$.

Next, we need to show that $Q^{\alpha_1}(G)$ is weakly subordinate to the Besov covering $\mathcal{B}(G)$ whenever $0 \leq \alpha_1 < 1$. Pick the center point

$$\delta_\beta(n) := n\|n\|^{\frac{\alpha_1}{1-\alpha_1}} \in Q^{\alpha_1}_n(G) := B^{\|\cdot\|}(\delta_\beta(n), r\|n\|^\beta), \quad \beta := \frac{\alpha_1}{1-\alpha_1}.$$

Then if $2^{m-1} \leq \|\delta_\beta(n)\| \leq 2^{m+1}$ for some $m \in \mathbb{N}$ we have that

$$2^{(m-1)(1-\alpha_1)} \leq \|n\| \leq 2^{(m+1)(1-\alpha_1)}.$$

Hence for all $n \in N \setminus \{0\}$ where $\|n\|$ is sufficiently large it follows that

$$\#\left\{m \in \mathbb{N}_0 : Q_n^{\alpha_1}(G) \cap D_m(G) \neq \emptyset\right\} \in \{1, 2\}.$$

Since there are only a finite number of elements $n \in N \setminus \{0\}$ with norm less than a fixed tolerance, we have that $Q^{\alpha_1}(G)$ is weakly subordinate to the Besov covering $\mathcal{B}(G)$. Showing that $Q^{\alpha_1}(G)$ is weakly subordinate to $Q^{\alpha_2}(G)$ whenever $0 \leq \alpha_1 \leq \alpha_2 < 1$ is straightforward since the function

$$x \longmapsto \frac{x}{1-x}, \qquad x \in [0, 1), \tag{B.5.1}$$

is increasing. It only remains that $Q^{\alpha_2}(G)$ is not weakly subordinate to $Q^{\alpha_1}(G)$ when $0 \leq \alpha_1 < \alpha_2 < 1$.

We use the notation

$$A_{\alpha_1}^{\alpha_2}(n) := \#\left\{l \in N \setminus \{0\} : Q_n^{\alpha_2}(G) \cap Q_l^{\alpha_1}(G) \neq \emptyset\right\},$$

and will give an iterated argument to show that there is no uniform bound on $A_{\alpha_1}^{\alpha_2}(n)$ for all $n \in N \setminus \{0\}$. The size of the elements in $Q^{\alpha_1}(G)$ is given by

$$|Q_n^{\alpha_1}(G)| = \left|B^{\|\cdot\|}\left(\delta_\beta(n), r\|n\|^\beta\right)\right| = \left(r\|n\|^{\frac{\alpha_1}{1-\alpha_1}}\right)^Q \cdot \mu, \qquad \mu := \left|B^{\|\cdot\|}(0,1)\right|,$$

where Q is the homogeneous dimension of $(\mathbb{R}^n, *_G)$. Since the function given in (B.5.1) is increasing there exists for every $\epsilon > 0$ a threshold $R > 0$ such that for $\|n\| \geq R$ we have

$$\frac{|Q_n^{\alpha_1}(G)|}{|Q_n^{\alpha_2}(G)|} = \|n\|^{Q\left(\frac{\alpha_1}{1-\alpha_1} - \frac{\alpha_2}{1-\alpha_2}\right)} < \epsilon.$$

The fact that N is a uniform lattice gives that there is a number $C_N > 0$ such that $\|m^{-1}n\| \leq C_N$ for every $m \in N \setminus \{0\}$ that is a neighbor of n with respect to the covering $Q^{\alpha_2}(G)$. Hence

$$|Q_n^{\alpha_2}(G)| \asymp |Q_m^{\alpha_2}(G)|,$$

for all such m. Therefore we can, by increasing the threshold R, find a sequence $n_k \in N \setminus \{0\}$ such that $A_{\alpha_1}^{\alpha_2}(n_k) \to \infty$. This implies that Q^{α_2} is not weakly subordinate to $Q^{\alpha_1}(G)$ whenever $0 \leq \alpha_1 < \alpha_2 < 1$. $\square$

Before we turn to the uniqueness result we need to investigate the generalized α-coverings in the two extreme cases $\alpha \in \{0, 1\}$ more thoroughly. To do this, we will briefly review a procedure originating in [59] and more recently investigated in [115] and [19] that associates a metric space to any concatenation. Although investigating the extreme cases $\alpha \in \{0, 1\}$ could be done without this extra machinery, we will need this approach in Section B.6 anyway and hence save our sanity for later.

Associated to any concatenation $\mathcal{Q}$ on $\mathbb{R}^n$ is a metric $d_\mathcal{Q}$ on $\mathbb{R}^n$ that reflects the global properties of $\mathcal{Q}$. For two distinct points $x, y \in \mathbb{R}^n$ we define the distance $d_\mathcal{Q}(x, y)$ to be the minimal number k such that there is a sequence $Q_{i_1}, \dots, Q_{i_k}$ connecting x and y. To be more formal, we require that $x \in Q_{i_1}$, $y \in Q_{i_k}$ and that $Q_{i_j} \cap Q_{i_{j+1}} \neq \emptyset$ for every $j = 1, \dots, k-1$. Moreover, no such sequence of length $k-1$ should exist. We extend the definition by $d_\mathcal{Q}(x, x) = 0$ for all $x \in \mathbb{R}^n$ and refer to $(\mathbb{R}^n, d_\mathcal{Q})$ as the *associated metric space* to the concatenation $\mathcal{Q}$. Comparing two metric spaces $(\mathbb{R}^n, d_\mathcal{Q})$ and $(\mathbb{R}^m, d_\mathcal{P})$ corresponding to different coverings $\mathcal{Q}$ and $\mathcal{P}$ is done by employing the notion of quasi-isometric embeddings.

Definition B.5.2. A *quasi-isometric embedding* between $(\mathbb{R}^n, d_\mathcal{Q})$ and $(\mathbb{R}^m, d_\mathcal{P})$ is a map $f\colon (\mathbb{R}^n, d_\mathcal{Q}) \to (\mathbb{R}^m, d_\mathcal{P})$ with fixed parameters $L, C > 0$ such that

$$\frac{1}{L} d_\mathcal{Q}(x, y) - C \leq d_\mathcal{P}(f(x), f(y)) \leq L d_\mathcal{Q}(x, y) + C,$$

for all $x, y \in \mathbb{R}^n$. We say that f is a *quasi-isometry* if additionally $f(\mathbb{R}^n)$ is a *net* in $\mathbb{R}^m$, that is, we have the uniform bound

$$\sup_{y \in \mathbb{R}^m} \inf_{x \in f(\mathbb{R}^n)} d_\mathcal{P}(x, y) < \infty.$$

Proposition B.5.3. Let $\mathcal{Q}$ and $\mathcal{P}$ be two concatenations on $\mathbb{R}^n$. Then $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ if and only if the identity map $Id\colon (\mathbb{R}^n, d_\mathcal{Q}) \to (\mathbb{R}^n, d_\mathcal{P})$ is Lipschitz continuous. Moreover, the concatenations $\mathcal{Q}$ and $\mathcal{P}$ are equivalent if and only if the identity map $Id\colon (\mathbb{R}^n, d_\mathcal{Q}) \to (\mathbb{R}^n, d_\mathcal{P})$ is a quasi-isometry.

The proposition above originates from [59, Proposition 3.8] and was phrased in the language of quasi-isometries first for open sets of a Euclidean space in [115] and for more general coverings in [19]. It shows that the metric space approach extends the notion of almost subordination to coverings defined on different Euclidean spaces. We now use the metric space viewpoint of coverings to examine the boundary cases $\alpha \in \{0, 1\}$.

Proposition B.5.4. Consider the rational stratified Lie group $(\mathbb{R}^n, *_G)$. The Besov covering $\mathcal{B}(G)$ is equivalent to the Euclidean Besov covering $\mathcal{B}(\mathbb{R}^n)$ only when

$(\mathbb{R}^n, *_G)$ is isomorphic to $(\mathbb{R}^n, +)$. Similarly, the uniform covering $\mathcal{U}(G)$ is only equivalent to the Euclidean uniform covering $\mathcal{U}(\mathbb{R}^n)$ when $(\mathbb{R}^n, *_G)$ is isomorphic to $(\mathbb{R}^n, +)$.

Proof. We can by Lemma B.3.7 choose to work with the homogeneous quasi-norm $\|\cdot\|_2$ given in (B.3.8). It is straightforward to check that the points

$$p(m) \coloneqq (2^m, 0, \dots, 0), \qquad q(m) \coloneqq (0, \dots, 2^{sm+1})$$

are both in $D_m(G)$ for all $m \in \mathbb{N}$, where s is the step of $(\mathbb{R}^n, *_G)$. Hence $d_{\mathcal{B}(G)}(p(m), q(m)) = 1$. However, the distance $d_{\mathcal{B}(\mathbb{R}^n)}(p(m), q(m))$ tends to infinity as m increases as long as $s > 1$. Thus the identity map

$$Id \colon (\mathbb{R}^n, d_{\mathcal{B}(G)}) \to (\mathbb{R}^n, d_{\mathcal{B}(\mathbb{R}^n)})$$

is not a quasi-isometry and we can apply Proposition B.5.3 to obtain that the concatenations $\mathcal{B}(G)$ and $\mathcal{B}(\mathbb{R}^n)$ are not equivalent. When $s = 1$ there is only one layer in the stratification (B.2.1) and it is clear that $(\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +)$ in that case.

The second statement follows from the more general statement [19, Theorem 3.6] implying that the uniform coverings $\mathcal{U}(G)$ and $\mathcal{U}(H)$ of two rational stratified Lie groups $(\mathbb{R}^n, *_G)$ and $(\mathbb{R}^n, *_H)$ can only be equivalent if the groups have the same homogeneous dimension. The homogeneous dimension Q of $(\mathbb{R}^n, *_G)$ satisfies $Q = n$ only when the stratification (B.2.1) has only one layer. Hence we conclude that $(\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +)$ and the result follows. □

Remark. The Besov covering $\mathcal{B}(G)$ of a stratified Lie group $(\mathbb{R}^n, *_G)$ fits in a larger class of coverings investigated in [36] known as inhomogeneous covering induced by an expansive matrix. An *expansive matrix* A is a matrix such that all its eigenvalues have norm strictly greater than one. Consider a collection $C \coloneqq (C_j)_{j \in \mathbb{N}_0}$ such that C_0 and C_1 are the closures of two bounded and open sets and $C_j = A^{j-1}(C_1)$ for $j \geq 1$. If

$$\bigcup_{j=0}^{\infty} C_j = \mathbb{R}^n,$$

then the collection C is called an *inhomogeneous covering* induced by the expansive matrix A. For our Besov coverings $\mathcal{B}(G)$, we have $C_0 = \overline{D_0(G)}$, $C_1 = \overline{D_1(G)}$, and

$$A = \begin{pmatrix} 2^{\nu_1} & & \\ & \ddots & \\ & & 2^{\nu_n} \end{pmatrix}, \qquad \nu_j \coloneqq \deg(x_j).$$

The attentive reader will notice that there is a small discrepancy as the sets in the covering $\mathcal{B}(G)$ are open, while the one described above consists of the closures of these elements. This is of minor importance as the two versions are clearly equivalent. Written in this framework, we could use [36, Lemma 6.1 (b)] to derive the first statement in Proposition B.5.4.

We need a final lemma regarding the weights appearing in Corollary B.4.3 before answering the uniqueness of the generalized α-modulation spaces $M^{s,\alpha}_{p,q}(G)$.

Lemma B.5.5. *Let* $(\mathbb{R}^n, *_G)$ *be an admissible Lie group with a lattice* N *and denote by* $Q^\alpha(G)$ *the explicit generalized* α*-coverings given in* (B.3.10)*. The weights*

$$N \setminus \{0\} \ni k \longmapsto \left(1 + \|k\|^{\frac{2}{1-\alpha}}\right)^{\frac{s}{2}}, \quad 0 \le \alpha < 1,$$

are $Q^\alpha(G)$*-moderate and the weight*

$$\mathbb{N}_0 \ni m \longmapsto 2^{ms}$$

is $\mathcal{B}(G)$*-moderate.*

Proof. For the Besov case, recall that two elements $D_n(G)$ and $D_m(G)$ only intersect whenever $m \in \{n-1, n, n+1\}$. Hence the weight is $\mathcal{B}(G)$-moderate since

$$2^{(n+1)s}/2^{ns} = 2^{ns}/2^{(n-1)s} = 2^s.$$

Let us consider the case $\alpha = 0$; we omit the more cumbersome case $0 < \alpha < 1$ as it relies on the same idea along with computations that can be found in the proof of [148, Lemma 9.2]. Assume that $B^{\|\cdot\|}(k, R) \cap B^{\|\cdot\|}(l, R) \neq \emptyset$ for $k, l \in N \setminus \{0\}$ where $R > 0$ is large enough so that $\mathcal{U}(G; N)$ is a covering. Then the triangle inequality implies that $\|l^{-1} *_G k\| \le 2CR$ where $C \ge 1$ is the quasi-norm constant in (B.2.4). Hence we obtain

$$\begin{aligned}
\frac{\omega(k)}{\omega(l)} &= \left(\frac{1 + \|l *_G l^{-1} *_G k\|^2}{1 + \|l\|^2}\right)^{\frac{s}{2}} \\
&\le \left(\frac{1 + (\|l\| + 2R)^2}{1 + \|l\|^2}\right)^{\frac{s}{2}} \\
&= \left(1 + 4R^2 \frac{1 + R^{-1}\|l\|}{1 + \|l\|^2}\right)^{\frac{s}{2}} \\
&\le C_\omega,
\end{aligned}$$

where the constant C_ω does not depend on the choice of points $l, k \in N \setminus \{0\}$. $\square$

B.5.2 Main Result

We now have all the tools needed to answer the question regarding uniqueness of the spaces $M^{s,\alpha}_{p,q}(G)$. For the case $G = \mathbb{H}_n$ and $\alpha = 0$, this question has been settled in [64, Theorem 7.6]. The authors showed that $M^{s_1}_{p_1,q_1}(\mathbb{H}_n) \neq M^{s_2}_{p_2,q_2}(\mathbb{R}^{2n+1})$ unless $(p_1, q_1, s_1) = (p_2, q_2, s_2) = (2, 2, 0)$, in which case

$$M^0_{2,2}(\mathbb{H}_n) = M^0_{2,2}(\mathbb{R}^{2n+1}) = L^2(\mathbb{R}^{2n+1}).$$

We say that the parameters $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$ are *non-trivial* if $(p, q, s) \neq (2, 2, 0)$. We are now ready to state the uniqueness result.

Theorem B.5.6. *Consider an admissible Lie group* $(\mathbb{R}^n, *_G)$ *and two sets of non-trivial parameters* $(p_1, q_1, s_1, \alpha_1)$ *and* $(p_2, q_2, s_2, \alpha_2)$*. We have equality*

$$M^{s_1,\alpha_1}_{p_1,q_1}(G) = M^{s_2,\alpha_2}_{p_2,q_2}(\mathbb{R}^n)$$

with equivalent norms if and only if both

$$(p_1, q_1, s_1, \alpha_1) = (p_2, q_2, s_2, \alpha_2) \quad \textit{and} \quad (\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +).$$

Proof. Assume first that the data coincide, that is, $(p_1, q_1, s_1, \alpha_1) = (p_2, q_2, s_2, \alpha_2)$ and $(\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +)$. We can apply Proposition B.3.3 to obtain that $\mathcal{Q}^{\alpha_1}(G)$ is equivalent to $\mathcal{Q}^{\alpha_2}(\mathbb{R}^n)$. Then the first implication follows from [59, Theorem 3.7] stating that two decomposition spaces are equal with equivalent norms whenever we have equivalent underlying coverings and equal parameters. The difficult part is the converse, and the rest of the proof is devoted to this direction.

Assume that we have equality $M^{s_1,\alpha_1}_{p_1,q_1}(G) = M^{s_2,\alpha_2}_{p_2,q_2}(\mathbb{R}^n)$ with equivalent norms. We start by applying the very general result [148, Theorem 6.9] implying that $(p_1, q_1) = (p_2, q_2)$ and that the coverings $\mathcal{Q}^{\alpha_1}(G)$ and $\mathcal{Q}^{\alpha_2}(\mathbb{R}^n)$ are weakly equivalent. We needed Lemma B.5.5 to invoke this result. Since both coverings consist of open and path-connected sets, it follows that $\mathcal{Q}^{\alpha_1}(G)$ and $\mathcal{Q}^{\alpha_2}(\mathbb{R}^n)$ are equivalent. Our strategy to show $s_1 = s_2$ is to use [148, Theorem 6.9 (4b)] showing that the weights $\omega_{\alpha_1,G}$ and $\omega_{\alpha_2,\mathbb{R}^n}$ corresponding to the coverings $\mathcal{Q}^{\alpha_1}(G) = (Q^{\alpha_1}_i(G))_{i\in I}$ and $\mathcal{Q}^{\alpha_2}(\mathbb{R}^n) = (Q^{\alpha_2}_j(\mathbb{R}^n))_{j\in J}$ are equivalent whenever $M^{s_1,\alpha_1}_{p_1,q_1}(G) = M^{s_2,\alpha_2}_{p_2,q_2}(\mathbb{R}^n)$. Equivalence in this setting means that there exists a constant $C > 0$ such that

$$\frac{1}{C}\omega_{\alpha_1,G}(i) \leq \omega_{\alpha_2,\mathbb{R}^n}(j) \leq C\omega_{\alpha_1,G}(i), \tag{B.5.2}$$

for all indices i and j such that $Q^{\alpha_1}_i(G) \cap Q^{\alpha_2}_j(\mathbb{R}^n) \neq \emptyset$.

Let us begin with the Besov case: Assume that either $\alpha_1 = 1$ or $\alpha_2 = 1$. The first part of the proof of Theorem B.5.1 regarding exponential versus polynomial growth goes through to show that it is nessesary that both $\alpha_1 = \alpha_2 = 1$. We can now apply Proposition B.5.4 to obtain that $(\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +)$. It remains to show that the smoothness parameters s_1 and s_2 are equal. However, this is obvious using (B.5.2) and Lemma B.5.5. Hence all the parameters coincide and $(\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +)$. This finishes the Besov case and we can now assume that $0 \leq \alpha_1, \alpha_2 < 1$.

Fix a lattice N in $(\mathbb{R}^n, *_G)$ and assume that either $\alpha_1 = 0$ or $\alpha_2 = 0$. Since the elements in $\mathcal{U}(G; N)$ and $\mathcal{U}(\mathbb{R}^n; \mathbb{Z}^n)$ have constant size, it it is nessesary then that both $\alpha_1 = \alpha_2 = 0$ since the coverings are equivalent. It follows from Proposition B.5.4 that this forces $(\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +)$. Since the choice of lattice is irrelevant, we can choose the lattice $\mathbb{Z}^n$ for both coverings. We then get from (B.5.2) that there exists $C > 0$ such that

$$\frac{1}{C}(1 + \|k\|_E^2)^{\frac{s_1}{2}} \leq (1 + \|k\|_E^2)^{\frac{s_2}{2}} \leq C(1 + \|k\|_E^2)^{\frac{s_1}{2}},$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$. The equality $s_1 = s_2$ follows from considering the points $k = (m, 0, \ldots, 0) \in \mathbb{Z}^n \setminus \{0\}$ for $m \in \mathbb{N}$.

For the intermediate case $0 < \alpha_1, \alpha_2 < 1$ we will first show that $\alpha_1 = \alpha_2$ with a restriction argument. Let k denote the rank of $(\mathbb{R}^n, *_G)$ and let V_1 be the first layer in the stratification (B.2.1). Consider the restricted covering on $\mathbb{R}^k$ given by

$$\mathcal{Q}^{\alpha_1}(G \,|\, \mathbb{R}^k) := \left(Q_l^{\alpha_1}(G) \cap \left(\mathbb{R}^k \times \{0\}^{n-k}\right)\right)_{l \in N \setminus \{0\}}.$$

To be a bit pedantic, we have defined coverings as consisting of non-empty subsets so we would actually need to remove all the empty sets and renumber the index set $N \setminus \{0\}$ accordingly. However, this will play no role so we omit this insignificant detail. It is straightforward to see that $\mathcal{Q}^{\alpha_1}(G \,|\, \mathbb{R}^k)$ is an admissible covering. Each element in $\mathcal{Q}^{\alpha_1}(G \,|\, \mathbb{R}^k)$ is open and connected due to the subspace topology on $\mathbb{R}^k$. It is clear when using the homogeneous quasi-norm $\|\cdot\|_2$ given in (B.3.8) that $\mathcal{Q}^{\alpha_1}(G \,|\, \mathbb{R}^k)$ is an α_1-covering on $\mathbb{R}^k$. It now follows from Proposition B.3.3 that $\mathcal{Q}^{\alpha_1}(G \,|\, \mathbb{R}^k)$ is equivalent to $\mathcal{Q}^{\alpha_1}(\mathbb{R}^k)$. Since $\mathcal{Q}^{\alpha_1}(G)$ is equivalent to $\mathcal{Q}^{\alpha_2}(\mathbb{R}^n)$ we obtain by restricting that $\mathcal{Q}^{\alpha_1}(\mathbb{R}^k)$ is equivalent to $\mathcal{Q}^{\alpha_2}(\mathbb{R}^k)$. We can now apply Theorem B.5.1 to obtain that $\alpha := \alpha_1 = \alpha_2$.

The next step is to show that the homogeneous dimension Q of $(\mathbb{R}^n, *_G)$ is actually equal to n. Fix a lattice on the form $N := \gamma\mathbb{Z}^k \times N'$ for $\gamma > 0$ and use the notation $\vec{l} = (\gamma l, 0, \ldots, 0)$ for $l \in \mathbb{N}$. Then by using the homogeneous quasi-norm $\|\cdot\|_2$ we have $Q_{\vec{l}}^{\alpha}(G) \cap Q_{\vec{l}}^{\alpha}(\mathbb{R}^n) \neq \emptyset$ and the estimates

$$|Q_{\vec{l}}^{\alpha}(G)| \asymp l^{\frac{Q\alpha}{1-\alpha}}, \qquad |Q_{\vec{l}}^{\alpha}(\mathbb{R}^n)| \asymp l^{\frac{n\alpha}{1-\alpha}}, \qquad \frac{|Q_{\vec{l}}^{\alpha}(G)|}{|Q_{\vec{l}}^{\alpha}(\mathbb{R}^n)|} \asymp l^{(Q-n)\frac{\alpha}{1-\alpha}}. \quad \text{(B.5.3)}$$

Notice that for large l the last expression in (B.5.3) tends to zero whenever $Q > n$. Recall that the neighbors of $Q_l^\alpha(\mathbb{R}^k)$ are of roughly the same size as $Q_l^\alpha(\mathbb{R}^k)$ because of (B.3.1). This is a contradiction to the equivalence of the coverings $\mathcal{Q}^\alpha(G)$ and $\mathcal{Q}^\alpha(\mathbb{R}^n)$. Thus we conclude that $Q = n$ and this implies as previously mentioned that $(\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +)$.

We can now use the standard lattice $\mathbb{Z}^n$ and the standard Euclidean norm $\|\cdot\|_E$ for both coverings. From (B.5.2) there exists a $C > 0$ such that

$$\frac{1}{C}\left(1 + \|k\|_E^{\frac{2}{1-\alpha}}\right)^{\frac{s_1}{2}} \leq \left(1 + \|k\|_E^{\frac{2}{1-\alpha}}\right)^{\frac{s_2}{2}} \leq C\left(1 + \|k\|_E^{\frac{2}{1-\alpha}}\right)^{\frac{s_1}{2}},$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$. By again considering $k = (m, 0, \dots, 0) \in \mathbb{Z}^n \setminus \{0\}$ for $m \in \mathbb{N}$ we see that $s_1 = s_2$. Thus all the parameters coincide and $(\mathbb{R}^n, *_G) \simeq (\mathbb{R}^n, +)$. □

Remarks.

- When the parameters are trivial we have the equality

  $$M_{2,2}^{0,\alpha_1}(G) = M_{2,2}^{0,\alpha_2}(\mathbb{R}^n) = L^2(\mathbb{R}^n),$$

  where $n := \dim(G)$ and $0 \leq \alpha_1, \alpha_2 \leq 1$.

- Usually, we can treat the uniform covering $\mathcal{U}(G; N)$ as a special case of the covering $\mathcal{Q}^\alpha(G; N)$ corresponding to $\alpha = 0$. However, a careful inspection of the proof of Theorem B.5.6 shows that the approach in (B.5.3) breaks down for $\alpha = 0$. This is why we treated the uniform case separately with techniques from metric space geometry by invoking Proposition B.5.4. The reader should be aware that we needed to use both metric geometry arguments and the non-trivial results [148, Theorem 6.9] and [59, Theorem 3.7] to prove Theorem B.5.6.

Corollary B.5.7. Consider a generalized α_1-covering $\mathcal{P}^{\alpha_1}(G)$ and a generalized α_2-covering $\mathcal{P}^{\alpha_1}(H)$ corresponding to admissible Lie groups $(\mathbb{R}^n, *_G)$ and $(\mathbb{R}^n, *_H)$, respectively. Then $\mathcal{P}^{\alpha_1}(G)$ can only be equivalent to $\mathcal{P}^{\alpha_1}(H)$ whenever $\alpha_1 = \alpha_2$.

Proof. It suffices to consider the explicit coverings $\mathcal{Q}^{\alpha_1}(G)$ and $\mathcal{Q}^{\alpha_2}(H)$ given in (B.3.10) due to Proposition B.3.3. We have remarked in the proof of Theorem B.5.6 that $\alpha_1 = 0$ implies that $\alpha_2 = 0$ and that $\alpha_1 = 1$ implies that $\alpha_2 = 1$. Hence $0 < \alpha_1, \alpha_2 < 1$ and we can use the restriction trick in the proof of Theorem B.5.6 to reduce both coverings to $\mathbb{R}^k$, where

$$k := \min\{\operatorname{rank}(\mathbb{R}^n, *_G), \operatorname{rank}(\mathbb{R}^n, *_H)\}.$$

Thus $\alpha_1 = \alpha_2$ follows from Theorem B.5.1. □

B.6 Geometric Embeddings Between Modulation Spaces

Consider two admissible Lie groups $(\mathbb{R}^n, *_G)$ and $(\mathbb{R}^m, *_H)$ together with the spaces $M^{s_1,\alpha_1}_{p_1,q_1}(G)$ and $M^{s_2,\alpha_2}_{p_2,q_2}(H)$. We would like to understand when $M^{s_1,\alpha_1}_{p_1,q_1}(G)$ embeds into the space $M^{s_2,\alpha_2}_{p_2,q_2}(H)$ in a way that preserves the global features of the underlying coverings $Q^{\alpha_1}(G)$ and $Q^{\alpha_2}(H)$.

When $n = m$, we can simply consider whether the inclusion

$$M^{s_1,\alpha_1}_{p_1,q_1}(G) \subset M^{s_2,\alpha_2}_{p_2,q_2}(H)$$

is bounded. However, when $n \neq m$ we need to be able to compare the coverings $Q^{\alpha_1}(G)$ and $Q^{\alpha_2}(H)$ even though they are not on the same space. Hence the commonly used notions of subordinate and weakly subordinate coverings introduced in Subsection B.2.2 are no longer applicable. However, we see from Proposition B.5.3 that we should ask that the embedding

$$F\colon M^{s_1,\alpha_1}_{p_1,q_1}(G) \to M^{s_2,\alpha_2}_{p_2,q_2}(H)$$

in some way induces a quasi-isometric embedding

$$F_*\colon \left(\mathbb{R}^n, d_{Q^{\alpha_1}(G)}\right) \longrightarrow \left(\mathbb{R}^m, d_{Q^{\alpha_2}(H)}\right).$$

The correct formalization for this was investigated for a very general class of spaces known as *decomposition spaces* in [19]. We will briefly review the technical details adapted to our setting.

Definition B.6.1. Consider the generalized α-modulation space $M^{s,\alpha}_{p,q}(G)$ for some $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$ corresponding to an admissible Lie group $(\mathbb{R}^n, *_G)$. Fix a lattice $N \subset (\mathbb{R}^n, *_G)$. The *essential support* of an element $f \in M^{s,\alpha}_{p,q}(G)$ with respect to the generalized α-covering $Q^\alpha(G; N) = (Q^\alpha_i)_{i\in I}$ is defined to be

$$C[f] := \bigcup_{i\in I} \left\{Q^\alpha_i : \|\mathcal{F}^{-1}\left(\psi_i \cdot \mathcal{F}(f)\right)\|_{L^p} \neq 0\right\},$$

where $(\psi_i)_{i\in I}$ is any choice of $Q^\alpha(G; N)$-BAPU.

To clarify, the index set I in Definition B.6.1 is equal to $N \setminus \{0\}$ when $0 \leq \alpha < 1$ and equal to $\mathbb{N}_0$ when $\alpha = 1$. Although the essential support of $f \in M^{s,\alpha}_{p,q}(G)$ does depend on the choice of lattice N and the $Q^\alpha(G)$-BAPU, it will be clear that specific choices are irrelevant. The reason we need to utilize this general notion of support is that not every element in $M^{s,\alpha}_{p,q}(G)$ can be realized as a function on $\mathbb{R}^n$; this is already the case for the Euclidean modulation spaces $M_{p,q}(\mathbb{R}^n)$ for most values of $1 \leq p, q \leq \infty$.

For every $i \in I$ and $k \in \mathbb{N}_0$ we can find an element $g_{i,k} \in M^{s,\alpha}_{p,q}(G)$ such that the essential support of $g_{i,k}$ is contained in $\left(Q_i^\alpha\right)^{k*}$. We can even choose the elements $g_{i,k}$ to be smooth functions with compact support since $\mathcal{S}(\mathbb{R}^n)$ is contained in $M^{s,\alpha}_{p,q}(G)$ by Proposition B.4.6 and there exist smooth $Q^\alpha(G)$-BAPU's with compact support for all $0 \leq \alpha \leq 1$. Hence the following definition is well-defined.

Definition B.6.2. Consider the spaces $M^{s_1,\alpha_1}_{p_1,q_1}(G)$ and $M^{s_2,\alpha_2}_{p_2,q_2}(H)$ associated to the admissible Lie groups $(\mathbb{R}^n, *_G)$ and $(\mathbb{R}^n, *_H)$, respectively. We say that a map

$$F\colon M^{s_1,\alpha_1}_{p_1,q_1}(G) \to M^{s_2,\alpha_2}_{p_2,q_2}(H)$$

is a *geometric embedding* if F is an injective bounded map between normed spaces with the following additional requirement: There should exist constants $L, C > 0$ such that for any $k \in \mathbb{N}_0$ and any $f, g \in M^{s_1,\alpha_1}_{p_1,q_1}(G)$ with $C[f] \subset (Q_i^{\alpha_1})^{k*}$ and $C[g] \subset (Q_j^{\alpha_1})^{k*}$, we have

$$\frac{1}{L} d_{Q^{\alpha_1}(G)}(x, y) - C \leq d_{Q^{\alpha_2}(H)}(z, w) \leq L d_{Q^{\alpha_1}(G)}(x, y) + C,$$

where $x \in (Q_i^{\alpha_1})^{k*}$, $y \in (Q_j^{\alpha_1})^{k*}$, $z \in C[F(f)]$ and $w \in C[F(g)]$ are arbitrary. The spaces $M^{s_1,\alpha_1}_{p_1,q_1}(G)$ and $M^{s_2,\alpha_2}_{p_2,q_2}(H)$ are said to be *geometrically isomorphic* if there exists an invertible geometric embedding from $M^{s_1,\alpha_1}_{p_1,q_1}(G)$ to $M^{s_2,\alpha_2}_{p_2,q_2}(H)$ whose inverse is also a geometric embedding.

Remark. Notice that we have left out the choice of the lattice in Definition B.6.2 although it implicitly appears in the essential supports and in the distances. The fact that any two lattices in a stratified Lie group are quasi-isometric as metric spaces [121, Corollary 5.5.9] implies that the choice of lattices are irrelevant when discussing the existence or non-existence of geometric embeddings. It is also straightforward to see that specific choices of BAPU's does not change the existence question. Hence we can treat existence of geometric embeddings as a canonical property of generalized α-modulation spaces.

It is straightforward to see that a composition of geometric embeddings is again a geometric embedding. The most important property of a geometric embedding

$$F\colon M^{s_1,\alpha_1}_{p_1,q_1}(G) \to M^{s_2,\alpha_2}_{p_2,q_2}(H)$$

is that it induces a quasi-isometric embedding between the two metric spaces $\left(\mathbb{R}^n, d_{Q^{\alpha_1}(G)}\right)$ and $\left(\mathbb{R}^m, d_{Q^{\alpha_2}(H)}\right)$ [19, Proposition 4.6]. In our case, this can be described as follows: For $x \in \mathbb{R}^n$ we pick $i \in I$ such that $x \in Q_i^{\alpha_1}$ and choose a non-zero function $g_i \in \mathcal{S}(\mathbb{R}^n)$ with $C[g_i] \subset Q_i^{\alpha_1}$. There exists an element $y \in C[F(g)]$ since F is assumed to be injective. If we define

$$F_*\colon \left(\mathbb{R}^n, d_{Q^{\alpha_1}(G)}\right) \longrightarrow \left(\mathbb{R}^m, d_{Q^{\alpha_2}(H)}\right), \qquad F_*(x) = y,$$

then F_* is easily seen to be a quasi-isometric embedding.

The way to think about geometric embeddings is that they are Banach spaces embeddings that do not "scramble" the frequency information to much. It might change the frequency information slightly in some bounded region, but we have global control over the displacements. We will focus on the geometric embeddings of the generalized modulation spaces $M^s_{p,q}(G)$ with underlying coverings $\mathcal{U}(G)$. The following result was proved in [19, Theorem 5.2] and settles the question for Euclidean modulation spaces.

Proposition B.6.3. For $1 \leq p, q < \infty$ there is a tower of compatible geometric embeddings

$$M_{p,q}(\mathbb{R}) \xrightarrow{\Gamma_1^2} M_{p,q}(\mathbb{R}^2) \xrightarrow{\Gamma_2^3} \cdots \xrightarrow{\Gamma_{n-1}^n} M_{p,q}(\mathbb{R}^n) \xrightarrow{\Gamma_n^{n+1}} \cdots,$$

where there are no geometric embeddings in the other direction.

While Proposition B.6.3 was proved by using the short-time Fourier transform, this is not available to us and we need to use the stratified structure of our group. The following result can be seen as partly generalizing Proposition B.6.3 to our setting.

Theorem B.6.4. *Let $(\mathbb{R}^n, *_G)$ be an admissible Lie group with rank k. There exists a geometric embedding*

$$F\colon M^s_{p,q}(\mathbb{R}^{k'}) \to M^s_{p,q}(G)$$

for every $k' \leq k$, $1 \leq p, q < \infty$, and $s \in \mathbb{R}$. This is optimal in the sense that there does not necessarily exists a geometric embedding from $M^s_{p,q}(\mathbb{R}^l)$ to $M^s_{p,q}(G)$ whenever $l > k$.

Proof. It suffices to prove the embedding statement only for $k = k'$. Once this has been shown, the general statement follows from Proposition B.6.3 and the fact that the composition of two geometric embeddings is a geometric embedding. Let us first set the stage by deciding the correct lattice, homogeneous quasi-norm and BAPU. Through the exponential map, we can always find a lattice N such that $N = l\mathbb{Z}^k \times N'$, where l is some integer. In the Heisenberg case $\mathbb{H}_3$, we can take $l = 2$. In general however, we only know the existence of a $l \in \mathbb{N}$. We will work with the specific homogeneous quasi-norm $\|\cdot\|_2$ given in (B.3.8) and utilize that

$\|\cdot\|_2$ agree with the usual Euclidean norm on the subspace $\mathbb{R}^k \times \{0\}^{n-k}$. Fix a $\mathcal{U}(G;N)$-BAPU $(\psi_m)_{m\in N\setminus\{0\}}$ such that

$$\psi_m(x) = \phi_{\overline{m}}(x_1,\dots,x_k)\cdot\psi'_{m'}(x_{k+1},\dots,x_n)$$

where $\overline{m}$ denotes the projection onto $l\mathbb{Z}^k$ and $(\phi_{\overline{m}})_{\overline{m}\in l\mathbb{Z}^k}$ is a $\mathcal{U}(\mathbb{R}^k)$-BAPU. For the existence of such a $\mathcal{U}(G;N)$-BAPU, we refer the reader to Example B.3.12.

Define the map $F\colon \mathcal{S}(\mathbb{R}^k)\subset M^s_{p,q}(\mathbb{R}^k)\to M^s_{p,q}(G)$ given by

$$f \longmapsto F(f)(x) = \mathcal{F}_n^{-1}\left(\mathcal{F}_k(f)(x_1,\dots,x_k)\cdot\xi(x_{k+1})\cdots\xi(x_n)\right),$$

where $\xi\in C_c^\infty(\mathbb{R})$ is a positive bump function supported in $(-1/2,1/2)$ and $\mathcal{F}_k$ and $\mathcal{F}_n$ denote the Fourier transforms in k and n variables, respectively. It is clear from our choice of homogeneous quasi-norm that the induced map of metric spaces

$$F_*\colon (\mathbb{R}^k, d_{\mathcal{U}(\mathbb{R}^k)}) \to \left(\mathbb{R}^n, d_{\mathcal{U}(G)}\right)$$

can be taken to be the inclusion into the first k coordinates. This is clearly a quasi-isometric embedding since the first k-coordinates in $(\mathbb{R}^n, *_G)$ is an abelian subgroup isomorphic to $(\mathbb{R}^k,+)$. We will show boundedness of F on the Schwartz space $\mathcal{S}(\mathbb{R}^k)$ and then use that $\mathcal{S}(\mathbb{R}^k)$ is dense in $M^s_{p,q}(\mathbb{R}^k)$ for all $1\le p,q<\infty$ and $s\in\mathbb{R}$ [81, Proposition 11.3.4] to obtain boundedness on all of $M^s_{p,q}(\mathbb{R}^k)$.

We first compute that

$$\begin{aligned}\mathcal{F}_n^{-1}\left(\psi_m\cdot\mathcal{F}_n\left(F(f)\right)\right) &= \mathcal{F}_n^{-1}\left(\phi_{\overline{m}}\otimes\psi'_{m'}\cdot\mathcal{F}_k\left(f\otimes\xi\otimes\cdots\otimes\xi\right)\right)\\ &= \mathcal{F}_k^{-1}\left(\phi_{\overline{m}}\cdot\mathcal{F}_k\left(f\right)\right)\cdot\mathcal{F}_{n-k}^{-1}\left(\psi'_{m'}\cdot\xi\otimes\cdots\otimes\xi\right)\end{aligned}$$

for every $f\in\mathcal{S}(\mathbb{R}^k)$ and $m\in N\setminus\{0\}$. Due to the support condition on ξ, the function $\psi'_m\cdot\xi\otimes\cdots\otimes\xi$ is only non-zero whenever $m'=(0,\dots,0)$. Thus we obtain

$$\begin{aligned}\|F(f)\|^q_{M^s_{p,q}(G)} &= \sum_{m\in N\setminus\{0\}}\left(1+\|m\|_2^2\right)^{\frac{qs}{2}}\left\|\mathcal{F}_n^{-1}\left(\psi_m\cdot\mathcal{F}_n\left(F(f)\right)\right)\right\|^q_{L_p}\\ &\le C\sum_{\overline{m}\in l\mathbb{Z}^k}\left(1+\|\overline{m}\|_E^2\right)^{\frac{qs}{2}}\left\|\mathcal{F}_k^{-1}\left(\phi_{\overline{m}}\cdot\mathcal{F}_k\left(f\right)\right)\right\|^q_{L_p}\\ &= C\|f\|_{M^s_{p,q}(\mathbb{R}^k)},\end{aligned}$$

where $\|\cdot\|_E$ denotes the Euclidean norm in the coordinates $(x_1,\dots,x_k)$. Hence F is a geometric embedding. The optimally statement follows immediately from Proposition B.6.3. □

The following consequence of Theorem B.6.4 is both aesthetically pleasing and reveals the universality of the Feichtinger algebra on the real line.

Corollary B.6.5. The Feichtinger algebra $\mathcal{S}_0(\mathbb{R}) := M^{0,0}_{1,1}(\mathbb{R})$ embeds geometrically into $M_{p,q}(G)$ for all $1 \leq p, q < \infty$ and any admissible Lie group $(\mathbb{R}^n, *_G)$.

Proof. It follows from [81, Theorem 12.2.2] that the inclusion $\mathcal{S}_0(\mathbb{R}) \hookrightarrow M_{p,q}(\mathbb{R})$ is bounded for every $1 \leq p, q < \infty$. This induces the identity map on the metric spaces level and is hence a geometric embedding. Since every admissible Lie group $(\mathbb{R}^n, *_G)$ have rank greater or equal to one, the result follows from Theorem B.6.4. $\square$

In [19, Theorem 3.6] the authors proved that, given two rational stratified Lie groups $(\mathbb{R}^n, *_G)$ and $(\mathbb{R}^n, *_H)$, the metric spaces $(G, d_{\mathcal{U}(G)})$ and $(H, d_{\mathcal{U}(H)})$ are not quasi-isometric unless the growth vectors $\mathfrak{G}(G)$ and $\mathfrak{G}(H)$ are equal. This shows that two generalized modulation spaces $M^{s_1}_{p_1,q_1}(G)$ and $M^{s_2}_{p_2,q_2}(H)$ can only be geometrically isomorphic whenever the growth vectors $\mathfrak{G}(G)$ and $\mathfrak{G}(H)$ are the same. In particular, not only are the Heisenberg modulation spaces $M^{s_1}_{p_1,q_1}(\mathbb{H}_n)$ distinct from $M^{s_2}_{p_2,q_2}(\mathbb{R}^{2n+1})$ for any non-trivial values of the parameters by Theorem B.5.6, they are also not geometrically isomorphic.

B.7 Looking Back and Ahead

Let us return and comment on the five questions raised in the introduction in the setting of admissible Lie groups:

1) We have seen that the generalized α-modulation spaces $M^{s,\alpha}_{p,q}(G)$ have natural coverings associated to them. Moreover, we can choose the explicit coverings $Q^{\alpha}(G)$ given in (B.3.10) for most purposes. How the coverings $\mathcal{U}(G; N)$ underlying the modulation spaces $M^{s}_{p,q}(G)$ is related to the polynomial growth of the lattice N is further discussed in [19, Chapter 3].

2) We have seen that the elements in $M^{s,\alpha}_{p,q}(G)$ are rather exotic distributions on $\mathbb{R}^n$. However, the containment of the Schwartz functions

$$\mathcal{S}(\mathbb{R}^n) \subset M^{s,\alpha}_{p,q}(G)$$

as well as the explicit coverings $Q^{\alpha}(G)$ in (B.3.10) make the spaces $M^{s,\alpha}_{p,q}(G)$ more concrete. The coverings $Q^{\alpha}(G)$ are especially explicit in low dimensions since lattices can be explicitly found and one can use the explicit homogeneous quasi-norm $\|\cdot\|_2$ given in (B.3.8).

3) As we have mentioned, the modulation spaces on the Heisenberg group $M^s_{p,q}(\mathbb{H}_n)$ have been recently studied in [64]. This space have another description through representations of a particular stratified Lie group known as the *Dynin-Folland group*. We refer the reader to [64] for more information on this construction. Also in the Heisenberg case, the Besov spaces $\mathcal{B}^s_{p,q}(\mathbb{H}_3)$ are new and very concrete spaces where the non-Euclidean dilations on $\mathbb{R}^3$ can be visualized. Moreover, the Besov coverings $\mathcal{B}(G)$ fits within a previously examined framework as explained in the remark preceding Theorem B.5.6.

4) In contrast with the Euclidean setting, the coverings $Q^\alpha(G)$ are not always almost structured coverings when $(\mathbb{R}^n, *_G)$ is an arbitrary rational stratified Lie group as we showed in Proposition B.3.10. Moreover, the methods we have used depend more on geometric considerations (such as growth type) than the more prevalent analytic approach used in the Euclidean setting.

5) The uniqueness of the generalized α-modulation space $M^{s,\alpha}_{p,q}(G)$ was completely settled in Theorem B.5.6. We showed that the spaces $M^{s,\alpha}_{p,q}(G)$ do form new spaces when the parameters p, q, s, α are non-trivial.

We hope that we have convinced the reader that the spaces $M^{s,\alpha}_{p,q}(G)$ are worthy of further study. We have avoided the quasi-Banach regime where the integrability parameters p, q are also allowed to take the values $0 < p, q < 1$ to make the exposition less technical. We refer the reader to [148, Chapter 9] where $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ is investigated in the quasi-Banach setting.

The most obvious further work on the topic of generalized α-modulation spaces is to prove the existence of BAPU's for the coverings $Q^\alpha(G)$ given in (B.3.10) for an arbitrary rational stratified Lie group $(\mathbb{R}^n, *_G)$. This would remove the slightly artificial restriction of having step less than or equal two. Secondly, in light of the impressive results in [91], it would be interesting to investigate the validity of complex interpolation results for the generalized α-modulation spaces $M^{s,\alpha}_{p,q}(G)$. Let us also comment on a few other directions that have not yet been explored.

One of the main advantages of the traditional modulation spaces $M^s_{p,q}(\mathbb{R}^n)$ is that they admit a *coorbit description* in the following sense: For $f, g \in L^2(\mathbb{R}^n)$ with $g \neq 0$ we define the *short-time Fourier transform* (STFT) of f with respect to the window g to be

$$V_g f(x,\omega) := \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\, dt, \quad (x,\omega) \in \mathbb{R}^{2n}.$$

One can extend the domain of the STFT to $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ by duality. For an element $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ we have an alternative description of the space $M^s_{p,q}(\mathbb{R}^n)$

for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ as follows: A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the space $M^s_{p,q}(\mathbb{R}^n)$ if and only if

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |V_g f(x,\omega)|^p \, (1+|x|+|\omega|)^{ps} \, dx\right)^{\frac{q}{p}} \, d\omega\right)^{\frac{1}{q}} < \infty.$$

We refer the reader to [81, Chapter 11] for an approach to modulation spaces using the coorbit description.

The name coorbit description comes from the fact that the STFT is a manifestation of the unitary representation theory of the Heisenberg group [81, Chapter 9]. This falls within a larger framework developed in [61, 62] known as *coorbit theory*. Many properties of the modulation spaces $M^s_{p,q}(\mathbb{R}^n)$ are more easily understood through the coorbit description. It would be advantageous to find a coorbit description for the modulation spaces $M^s_{p,q}(G)$ where $(\mathbb{R}^n, *_G)$ is any rational stratified Lie group.

We would like to emphasize that the theory we have built for the boundary cases $M^s_{p,q}(G)$ and $\mathcal{B}^s_{p,q}(G)$ is interesting in itself. One can consider the *Sobolev spaces* $W^{s,p}(G) := B^s_{p,p}(G)$ associated to any stratified Lie group $(\mathbb{R}^n, *_G)$. We have from Theorem B.5.6 that the spaces $W^{s,p}(G)$ do not coincide with any of the Euclidean Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ unless $(s,p) = (0,2)$, in which case

$$W^{0,2}(G) = W^{0,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n).$$

In particular, the spaces $H^k(G) := W^{k,2}(G)$ for $k \in \mathbb{N}$ are alternatives to the spaces $H^k(\mathbb{R}^n)$ that permeates PDE theory and nearby disciplines. There are many notions of Sobolev spaces on stratified Lie groups in the literature, and it would be interesting to see how our approach fit in.

The modulation spaces $M^s_{p,q}(G)$ have not been considered previously in the literature except on the groups $\mathbb{R}^n$ and $\mathbb{H}_n$. In the Euclidean case, the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}^n) := M^{0,0}_{1,1}(\mathbb{R}^n)$ has several interesting properties: Every element in $\mathcal{S}_0(\mathbb{R}^n)$ is a continuous function and $\mathcal{S}_0(\mathbb{R}^n)$ is an algebra under both pointwise multiplication and convolution. Similar questions could be asked for the *nilpotent Feichtinger algebra*

$$\mathcal{S}_0(G) := M^{0,0}_{1,1}(G),$$

where $(\mathbb{R}^n, *_G)$ is a rational stratified Lie group. Moreover, it would be interesting to see whether the space $\mathcal{S}_0(G)$ satisfies a minimality characterization similarly to the Feichtinger algebra $\mathcal{S}_0(\mathbb{R}^n)$, see [81, Theorem 12.1.8].

Finally, one can ask if $\mathcal{S}_0(G)$ gives rise to a *Banach Gelfand triple* [63]

$$\mathcal{S}_0(G) \hookrightarrow L^2(\mathbb{R}^n) \hookrightarrow (\mathcal{S}_0(G))' \simeq M^0_{\infty,\infty}(G),$$

for any admissible Lie group $(\mathbb{R}^n, *_G)$. These and many more questions could be illuminating even in a special case such as the *free nilpotent Lie group* $\mathbf{F}_{k,s}$ with rank k and step s whose Lie algebra is defined in [48, Example 1.5]. This would generalize most of the known results as $\mathbf{F}_{n,1} = \mathbb{R}^n$ and $\mathbf{F}_{2,2} = \mathbb{H}_3$. We encourage the reader to explore these open questions and build on the work presented.

Paper C

Interpolation in Wavelet Spaces and the HRT-Conjecture

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Paper C

Interpolation in Wavelet Spaces and the HRT-Conjecture

Abstract

We investigate the wavelet spaces $\mathcal{W}_g(\mathcal{H}_\pi) \subset L^2(G)$ arising from square integrable representations $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ of a locally compact group G. We show that the wavelet spaces are rigid in the sense that non-trivial intersection between them imposes strong restrictions. Moreover, we use this to derive consequences for wavelet transforms related to convexity and functions of positive type.

Motivated by the reproducing kernel Hilbert space structure of wavelet spaces we examine an interpolation problem. In the setting of time-frequency analysis, this problem turns out to be equivalent to the HRT-Conjecture. Finally, we consider the problem of whether all the wavelet spaces $\mathcal{W}_g(\mathcal{H}_\pi)$ of a locally compact group G collectively exhaust the ambient space $L^2(G)$. We show that the answer is affirmative for compact groups, while negative for the reduced Heisenberg group.

C.1 Introduction

In recent years there have been several fruitful connections between time-frequency analysis and abstract notions in both representation theory [51, 84, 85] and non-commutative geometry [8, 107, 122, 123]. This is mutually beneficial: The abstract machinery can illuminate many results in time-frequency analysis. On the other hand, the concrete setting of time-frequency analysis provides a useful playground for testing general conjectures. Building on this viewpoint, we consider a generalization of the *Gabor spaces*

$$V_g(L^2(\mathbb{R}^n)) \subset L^2(\mathbb{R}^{2n}),$$

where $V_g f$ is the short-time Fourier transform (STFT) of $f \in L^2(\mathbb{R}^n)$ with respect to a non-zero window function $g \in L^2(\mathbb{R}^n)$.

The Gabor spaces have appeared explicitly in the time-frequency literature several times, e.g. [3, 105], as well as being implicitly present in much of the literature concerning the STFT. We refer the reader to [81, Proposition 3.4.1] where the connection between a certain Gabor space and the Bargmann-Fock space in complex analysis is described. Despite their importance, it is only recently that some of the basic properties of Gabor spaces have been examined in [124]. Our goal is to derive results that are of interest both in the general setting and in the case of Gabor spaces.

Let us briefly describe the general setup of the paper. Consider a square integrable representation $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ of a locally compact group G on a Hilbert space $\mathcal{H}_\pi$. We investigate the *wavelet spaces*

$$\mathcal{W}_g(\mathcal{H}_\pi) \subset L^2(G), \qquad \mathcal{W}_g f(x) := \langle f, \pi(x)g \rangle,$$

where $g \in \mathcal{H}_\pi$ is an admissible vector and $x \in G$. The Gabor space $V_g(L^2(\mathbb{R}^n))$ is up to a phase-factor the wavelet space corresponding to the Schrödinger representation of the reduced Heisenberg group $\mathbb{H}_r^n$. Wavelet spaces have appeared in the theory of coorbit spaces [60, 61, 62] and have been independently studied in e.g. [77, 89, 156]. The following result illustrates the rigidity of wavelet spaces.

Theorem C.1.1. *Let $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\rho \colon G \to \mathcal{U}(\mathcal{H}_\rho)$ be two square integrable representations with admissible vectors $g \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_\rho$. Assume that the corresponding wavelet spaces intersect non-trivially, that is,*

$$\mathcal{W}_g(\mathcal{H}_\pi) \cap \mathcal{W}_h(\mathcal{H}_\rho) \neq \{0\}.$$

Then $\mathcal{W}_g(\mathcal{H}_\pi) = \mathcal{W}_h(\mathcal{H}_\rho)$ and there exists a unitary intertwining operator $T \colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ satisfying $T(g) = h$.

A special case of Theorem C.1.1 reduces to the result in [77, Theorem 4.2]. There are also two other noteworthy consequences of Theorem C.1.1 related to functions of positive type and convexity.

Corollary C.1.2. Let $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\rho \colon G \to \mathcal{U}(\mathcal{H}_\rho)$ be square integrable representations with admissible vectors $g \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_\rho$, respectively. Then $\mathcal{W}_g g - \mathcal{W}_h h$ is never a non-zero function of positive type.

Corollary C.1.3. Let $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation of a unimodular group G with admissible vectors $g, g_1, g_2 \in \mathcal{H}_\pi$. Assume we can write $\mathcal{W}_g g$ as a convex combination

$$\mathcal{W}_g g = t \cdot \mathcal{W}_{g_1} g_1 + (1-t) \cdot \mathcal{W}_{g_2} g_2,$$

for some $t \in [0,1]$. Then we either have $g = cg_1$ or $g = cg_2$ for some $c \in \mathbb{T}$.

It is well known that any wavelet space carries the structure of a reproducing kernel Hilbert space. This allows us to consider an interpolation problem for the wavelet spaces as follows: Consider distinct points $\{x_1, \ldots, x_m\} \subset G$ and possibly non-distinct scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. We investigate whether there exists a function $F \in \mathcal{W}_g(\mathcal{H}_\pi)$ that *interpolates* these points, that is, $F(x_i) = \lambda_i$ for all $i = 1, \ldots, m$. When this problem is always solvable the wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$ is called *fully interpolating*. This is a notion that has been extensively investigated in the reproducing kernel Hilbert space literature, see [137, Chapter 3]. However, in the case of the wavelet spaces the interpolation problem is to our knowledge only briefly mentioned in [89].

We show in Proposition C.5.3 that no wavelet space corresponding to a compact or abelian group can be fully interpolating. In the Gabor case, the interpolation problem turns out to be equivalent to the HRT-Conjecture regarding independence of time-frequency shifts. We will review the HRT-Conjecture in Section C.6 and show how it relates to the interpolation problem in Proposition C.6.1. The partial results obtained for the HRT-Conjecture in the literature gives concrete examples of wavelet spaces that are fully interpolating. On the other hand, the interpolation problem gives an alternative view of the HRT-Conjecture that allows the tools from reproducing kernel Hilbert space theory to be applied.

A theme throughout the paper is to utilize the theory of reproducing kernel Hilbert spaces to deduce properties of wavelet spaces. As an illustration of this, we will give a short proof of the following folklore result showing that tensor products are naturally incorporated in our setting.

Proposition C.1.4. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\rho\colon H \to \mathcal{U}(\mathcal{H}_\rho)$ be two square integrable representations with admissible vectors $g \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_\rho$. There is an isomorphism of reproducing kernel Hilbert spaces

$$\mathcal{W}_{g\otimes h}(\mathcal{H}_\pi \hat{\otimes} \mathcal{H}_\rho) \simeq \mathcal{W}_g(\mathcal{H}_\pi) \hat{\otimes} \mathcal{W}_h(\mathcal{H}_\rho).$$

Finally, we would like to mention a problem where we are only able to obtain partial results. For a square integrable representation $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ we let $\mathcal{A}_\pi$ denote the equivalence classes of admissible vectors in $\mathcal{H}_\pi$ modulo rotations by elements of $\mathbb{T}$. We let $\widehat{G}_s$ denote the equivalence classes of square integrable representations of G and consider the possibly non-direct sum of vector spaces

$$\bigoplus_{\pi \in \widehat{G}_s} \operatorname*{span}_{g \in \mathcal{A}_\pi} \left\{\mathcal{W}_g f : f \in \mathcal{H}_\pi\right\} \subset L^2(G).$$

Is this sum dense in $L^2(G)$ when $\widehat{G}_s \neq \emptyset$? Phrased conceptually, we question whether the wavelet spaces are collectively large enough to approximate any square integrable function. We say that a locally compact group G is *wavelet complete* when

$$\overline{\bigoplus_{\pi \in \widehat{G}_s} \operatorname{span}_{g \in \mathcal{A}_\pi} \left\{ \mathcal{W}_g f \,:\, f \in \mathcal{H}_\pi \right\}} = L^2(G).$$

For compact groups the affirmative answer follows directly from Peter-Weyl theory. Since commutative locally compact groups G only have $\widehat{G}_s \neq \emptyset$ whenever they are compact, the conjecture is primarily interesting for non-abelian groups. The following result shows that wavelet completeness is a non-trivial notion.

Proposition C.1.5. The reduced Heisenberg groups $\mathbb{H}_r^n$ are not wavelet complete.

The structure of the paper is as follows: In Section C.2 we review the nessesary material regarding square integrable representations and reproducing kernel Hilbert spaces. The examination of wavelet spaces starts in Section C.3 where we discuss basic properties. In Section C.4 we show the disjointedness of the wavelet spaces and the resulting convexity consequence by utilizing abstract notions from the theory of functions of positive type.

The interpolation problem for the wavelet spaces will be taken up in Section C.5. We present the connection between the interpolation problem and the HRT-Conjecture in Section C.6. Finally, we examine wavelet completeness in Section C.7. The author would like to thank Are Austad, Stine M. Berge, Franz Luef, Eirik Skrettingland, Keith Taylor, Jordy Timo van Velthoven, and the anonymous referee for valuable input.

C.2 Preliminaries

We will begin by reviewing the two settings of interest, namely square integrable representations of locally compact groups and reproducing kernel Hilbert spaces. This is done to fix notation and terminology, as well as to make the rest of the paper accessible to a broader audience. Background information for both topics can be found respectively in the books [43, 44, 47, 67] and [23, 137].

C.2.1 Square Integrable Representations

Let G be a locally compact group, that is, a Hausdorff topological space that is also a group such that the multiplication map $(x, y) \mapsto xy$ and inversion map $x \mapsto x^{-1}$ are both continuous. The most important result when it comes to locally compact groups is the existence of a unique left-invariant Radon measure μ_L on G called the (left) *Haar measure* on G. Whenever there is any measure-theoretic construction on G mentioned, it will always be with respect to the left Haar measure. In particular, the integrability spaces $L^p(G)$ for $1 \leq p \leq \infty$ consist of measurable functions $f\colon G \to \mathbb{C}$ such that

$$\|f\|_{L^p(G)} := \left(\int_G |f(x)|^p \, d\mu_L(x)\right)^{\frac{1}{p}} < \infty.$$

Moreover, given $f, g \in L^1(G)$ the *convolution* between f and g is given by

$$(f *_G g)(x) := \int_G f(y)g(y^{-1}x) \, d\mu_L(y), \qquad x \in G.$$

We mention that the convolution product on $L^1(G)$ is commutative if and only if the group G is abelian.

Analogously to the left Haar measure, there exists a right Haar measure μ_R on G that is right-invariant. How much the two measures μ_L and μ_R deviate is captured in the *modular function* Δ on G. Its precise definition [67, Section 2.4] need not concern us. However, it is worth knowing that $\mu_L = \mu_R$ precisely when Δ is identically one. In this case, we write $\mu := \mu_L = \mu_R$ and say that G is *unimodular*. Unimodular groups are abundant as they include abelian groups, compact groups, and discrete groups.

Definition C.2.1. Let $\mathcal{U}(\mathcal{H}_\pi)$ denote the unitary operators on the Hilbert space $\mathcal{H}_\pi$. A group homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ of a locally compact group G is said to be a *unitary representation* if the function

$$\mathcal{W}_g f(x) := \langle f, \pi(x)g\rangle_{\mathcal{H}_\pi}$$

is continuous on G for any fixed $f, g \in \mathcal{H}_\pi$. We refer to $\mathcal{W}_g f$ as the *wavelet transform* of f with respect to g.

The terminology for the wavelet transform is motivated by the classical continuous wavelet transform in wavelet analysis, see e.g. [44]. It is clear that $\mathcal{W}_g f$ is a bounded function on G since

$$|\mathcal{W}_g f(x)| \leq \|f\|_{\mathcal{H}_\pi}\|\pi(x)g\|_{\mathcal{H}_\pi} = \|f\|_{\mathcal{H}_\pi}\|g\|_{\mathcal{H}_\pi}, \quad x \in G,\ f, g \in \mathcal{H}_\pi.$$

We will often fix $g \in \mathcal{H}_\pi$ and consider the map $\mathcal{W}_g\colon \mathcal{H}_\pi \to C_b(G)$ given by $\mathcal{W}_g(f) := \mathcal{W}_g f$, where $C_b(G)$ denotes the bounded continuous functions on G.

The spaces of primary interest for us will be $\mathcal{W}_g(\mathcal{H}_\pi)$ as g varies. However, as it stands now the conditions are to loose to deduce nice properties of the spaces $\mathcal{W}_g(\mathcal{H}_\pi)$. Firstly, we will require that the representation π is *irreducible*, that is, there does not exist any non-trivial closed subspaces $\mathcal{M} \subset \mathcal{H}_\pi$ such that $\pi(x)\eta \in \mathcal{M}$ for every $x \in G$ and $\eta \in \mathcal{M}$. The main tool when working with irreducible representations is Schur's lemma [67, Chapter 3]:

Lemma C.2.2. *Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a unitary representation of a locally compact group G. Then π is irreducible if and only if every bounded linear operator $T\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ satisfying $T \circ \pi(x) = \pi(x) \circ T$ for all $x \in G$ is in fact a constant multiple of the identity transform $Id_{\mathcal{H}_\pi}$.*

Bounded linear operators $T\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ satisfying $T \circ \pi(x) = \pi(x) \circ T$ for all $x \in G$ are called *intertwining operators*. The second requirement we need on π is one of integrability.

Definition C.2.3. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be an irreducible unitary representation of a locally compact group G. We say that a non-zero vector $g \in \mathcal{H}_\pi$ is *square integrable* if $\mathcal{W}_g g \in L^2(G)$. Similarly, we say that π is *square integrable* if there exists a square integrable vector in $\mathcal{H}_\pi$.

If $g \in \mathcal{H}_\pi$ is square integrable, then it actually follows that $\mathcal{W}_g f \in L^2(G)$ for all $f \in \mathcal{H}_\pi$. Moreover, the irreducibility of π implies with little effort that the map $\mathcal{W}_g\colon \mathcal{H}_\pi \to C_b(G)$ is one-to-one. An improvement of these remarks is the following result of M. Duflo and C. C. Moore [50] showing that the map $f \mapsto \mathcal{W}_g f$ is essentially an isometry.

Proposition C.2.4. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation. There exists a unique positive, densely defined operator

$$C_\pi\colon \mathrm{dom}(C_\pi) \subset \mathcal{H}_\pi \to \mathcal{H}_\pi$$

with a densely defined inverse such that

- A non-zero element $g \in \mathcal{H}_\pi$ is square integrable if and only if $g \in \mathrm{dom}(C_\pi)$.
- For $g_1, g_2 \in \mathrm{dom}(C_\pi)$ and $f_1, f_2 \in \mathcal{H}_\pi$ we have the orthogonality relation

$$\langle \mathcal{W}_{g_1} f_1, \mathcal{W}_{g_2} f_2 \rangle_{L^2(G)} = \langle f_1, f_2 \rangle_{\mathcal{H}_\pi} \overline{\langle C_\pi g_1, C_\pi g_2 \rangle}_{\mathcal{H}_\pi}. \tag{C.2.1}$$

- The operator C_π is injective and satisfies the invariance relation

$$\pi(x) C_\pi = \sqrt{\Delta(x)} C_\pi \pi(x),$$

  for all $x \in G$ where Δ denotes the modular function on G.

The operator C_π is called the *Duflo-Moore operator*.

We can always normalize a square integrable vector $g \in \mathcal{H}_\pi$ such that

$$\|C_\pi g\|_{\mathcal{H}_\pi} = 1.$$

A square integrable vector $g \in \mathcal{H}_\pi$ satisfying $\|C_\pi g\|_{\mathcal{H}_\pi} = 1$ is said to be *admissible*. This condition is mainly one of convenience, and we will primarily work with admissible vectors. When G is a unimodular group, then any square integrable representation π of G satisfies $\mathrm{dom}(C_\pi) = \mathcal{H}_\pi$ and $C_\pi = c_\pi \cdot Id_{\mathcal{H}_\pi}$ for some $c_\pi > 0$. In this case, any non-zero vector $g \in \mathcal{H}_\pi$ is square integrable and admissibility simply reads $\|g\|_{\mathcal{H}_\pi} = c_\pi^{-1}$.

C.2.2 Reproducing Kernel Hilbert Spaces

A Hilbert space $\mathcal{H}$ consisting of functions $f\colon X \to \mathbb{C}$ on a set X does not need to relate pointwise notions with the abstract Hilbert space structure. For instance, convergence of a sequence $f_n \to f$ in the norm on $\mathcal{H}$ does not need to imply pointwise convergence $f_n(x) \to f(x)$ for every $x \in X$. However, by imposing that the natural evaluation functionals $E_x(f) := f(x)$ for $f \in \mathcal{H}$ and fixed $x \in X$ are bounded one obtains a strong relation between pointwise notions and the Hilbert space structure.

Definition C.2.5. A *reproducing kernel Hilbert space* is a Hilbert space $\mathcal{H}$ consisting of functions $f\colon X \to \mathbb{C}$ on a set X such that, for each $x \in X$, the evaluation functionals

$$E_x(f) := f(x), \qquad f \in \mathcal{H},$$

are well-defined and bounded. If the collection $\{E_x\}_{x\in X}$ is uniformly bounded in norm we refer to $\mathcal{H}$ as *uniform*.

Examples of well known reproducing kernel Hilbert spaces are the Paley-Wiener spaces $PW_{[-A,A]}$ for $A > 0$ and the Hardy space $H^2(\mathbb{D})$. We refer the reader to [137] for a detailed discussion of these examples, while [23] gives examples of reproducing kernel Hilbert spaces related to stochastic processes.

There exists for each $x \in X$ a function $k_x \in \mathcal{H}$ such that $E_x(f) = \langle f, k_x\rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$. We refer to k_x as the *point kernel* corresponding to $x \in X$. The function $K\colon X \times X \to \mathbb{C}$ given by

$$K(x, y) := \langle k_y, k_x\rangle_{\mathcal{H}} = k_y(x)$$

is called the *reproducing kernel* of $\mathcal{H}$. If $f_n \to f$ in the norm on $\mathcal{H}$, then

$$|f_n(x) - f(x)| = |\langle f_n - f, k_x\rangle| \le \|f_n - f\|_{\mathcal{H}}\|E_x\|_{\mathcal{H}^*} \to 0. \tag{C.2.2}$$

There are two general properties of reproducing kernel Hilbert spaces we will need in the sequel:

- [137, Proposition 2.13] The reproducing kernel K of a reproducing kernel Hilbert space is a *kernel function*: Given $\Omega := \{x_1, \ldots, x_m\} \subset X$ the matrix

$$K_\Omega := \{K(x_i, x_j)\}_{i,j=1}^m \tag{C.2.3}$$

  is positive semi-definite, that is, the eigenvalues of K_Ω are all non-negative.

- [137, Proposition 2.3 and Theorem 2.4] The reproducing kernel uniquely determines the resulting reproducing kernel Hilbert space: If $\mathcal{H}_1$ and $\mathcal{H}_2$ are both reproducing kernel Hilbert spaces on a set X with the same reproducing kernel K, then $\mathcal{H}_1 = \mathcal{H}_2$ and $\|\cdot\|_{\mathcal{H}_1} = \|\cdot\|_{\mathcal{H}_2}$. Conversely, if two reproducing kernel Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide with equal norms, then the reproducing kernels for the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are equal.

Remark. The reader should be aware that there is little consensus in the literature regarding the terminology *positive definite*: Some authors, e.g. [137], use the term positive definite for the case $K_\Omega \geq 0$, while the majority will use the term positive definite to indicate that $K_\Omega > 0$. Hence we adopt the terminology *positive semi-definite* for $K_\Omega \geq 0$ and *strictly positive definite* for $K_\Omega > 0$ to minimize the possibility for any confusion.

It is important to note that the matrices K_Ω in (C.2.3) do not need to be invertible. If all the matrices K_Ω are strictly positive definite, then we refer to the reproducing kernel Hilbert space $\mathcal{H}$ as *fully interpolating*. The reason for this terminology will be clear in Section C.5.

C.3 Basic Properties of Wavelet Spaces

In this section we will define wavelet spaces and give their basic properties. This will connect the two topics reviewed in Section C.2 as the wavelet spaces have a natural reproducing kernel Hilbert space structure.

Definition C.3.1. Let $\pi \colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation of a locally compact group G and fix an admissible vector $g \in \mathcal{H}_\pi$. The space

$$\mathcal{W}_g(\mathcal{H}_\pi) \subset L^2(G)$$

is called the *(generalized) wavelet space* corresponding to the representation π and the admissible vector g.

The terminology is again motivated by the continuous wavelet transform in classical wavelet analysis. Notice that the wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$ is a Hilbert space since it is a closed subspace of $L^2(G)$. Moreover, the norm $\mathcal{W}_g(\mathcal{H}_\pi)$ inherits from $L^2(G)$ can be written by using (C.2.1) as

$$\|\mathcal{W}_g f\|_{L^2(G)} = \|f\|_{\mathcal{H}_\pi}, \qquad f \in \mathcal{H}_\pi.$$

An important property of the wavelet transform is that $\mathcal{W}_g$ is a unitary intertwining operator between π and the left-regular representation on the space $\mathcal{W}_g(\mathcal{H}_\pi)$: Let L_x denote the left translation on functions $F \in \mathcal{W}_g(\mathcal{H}_\pi)$ by $x \in G$, that is, $L_x F(y) := F(x^{-1}y)$ for $y \in G$. Then

$$\mathcal{W}_g(\pi(y)f)(x) = \langle \pi(y)f, \pi(x)g\rangle = \mathcal{W}_g(f)(y^{-1}x) = L_y \mathcal{W}_g(f)(x),$$

for $x, y \in G$ and $f \in \mathcal{H}_\pi$. This shows that the wavelet spaces are left-invariant subspaces of $L^2(G)$.

Example C.3.2. Consider the *reduced Heisenberg group* $\mathbb{H}_r^n := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$ with the product

$$\left(x, \omega, e^{2\pi i\tau}\right) \cdot \left(x', \omega', e^{2\pi i\tau'}\right) := \left(x + x', \omega + \omega', e^{2\pi i(\tau+\tau')} e^{\pi i(x'\cdot\omega - x\cdot\omega')}\right),$$

for $x, x', \omega, \omega' \in \mathbb{R}^n$ and $\tau, \tau' \in \mathbb{R}$. The group $\mathbb{H}_r^n$ is non-abelian and unimodular with Haar measure equal to the usual product measure on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$. The *Schrödinger representation* $\rho_r : \mathbb{H}_r^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ is the irreducible unitary representation given by

$$\rho_r\left(x, \omega, e^{2\pi i\tau}\right) := e^{2\pi i\tau} e^{\pi i x\cdot\omega} T_x M_\omega, \qquad \left(x, \omega, e^{2\pi i\tau}\right) \in \mathbb{H}_r^n, \tag{C.3.1}$$

where T_x and M_ω are the *time-shift* and *frequency-shift* operators on $L^2(\mathbb{R}^n)$ given by

$$T_x f(y) := f(y - x), \qquad M_\omega f(y) := e^{2\pi i y\cdot\omega} f(y), \quad x, \omega \in \mathbb{R}^n.$$

A straightforward computation shows that the n-dimensional Gaussian function $g_n(x) := e^{-\frac{\pi}{2}x^2}$ for $x \in \mathbb{R}^n$ is square integrable for the Schrödinger representation. Hence the Duflo-Moore operator satisfies $C_\pi = c_\pi \cdot Id_{L^2(\mathbb{R}^n)}$ for some $c_\pi > 0$ since $\mathbb{H}_r^n$ is unimodular. In fact, we have $c_\pi = 1$ due to [81, Theorem 3.2.1]. Thus any normalized function in $L^2(\mathbb{R}^n)$ is admissible.

It is common in time-frequency analysis to consider the *short-time Fourier transform (STFT)*

$$V_g f(x, \omega) := \int_{\mathbb{R}^n} f(t)\overline{g(t - x)} e^{-2\pi i t\cdot\omega}\, dt,$$

for $(x, \omega) \in \mathbb{R}^{2n}$ and $f, g \in L^2(\mathbb{R}^n)$. The STFT is related to the wavelet transform of the reduced Heisenberg group by the formula

$$\mathcal{W}_g f\left(x, \omega, e^{2\pi i\tau}\right) = e^{-2\pi i\tau} e^{\pi i x\cdot\omega} V_g f(x, \omega), \qquad \left(x, \omega, e^{2\pi i\tau}\right) \in \mathbb{H}_r^n. \quad \text{(C.3.2)}$$

The phase-factor $e^{-2\pi i\tau} e^{\pi i x\cdot\omega}$ in (C.3.2) is often irrelevant. Hence we will for the most part consider the STFT and the *Gabor spaces*

$$V_g(L^2(\mathbb{R}^n)) \subset L^2(\mathbb{R}^{2n}),$$

for $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{L^2(\mathbb{R}^n)} = 1$.

C.3.1 Wavelet Spaces as Reproducing Kernel Hilbert Spaces

The fact that the wavelet spaces have a reproducing kernel Hilbert space structure originally appeared in the influential paper [89]. Since then, it has been used in both special cases [3] and in the general setting [141]. We provide the statement and brief proof for completeness as our assumptions are slightly different than in [89] and include minor additions.

Proposition C.3.3. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation with admissible vector $g \in \mathcal{H}_\pi$. The wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$ is a uniform reproducing kernel Hilbert space. The point kernel k_x corresponding to $x \in G$ is the function $k_x = \mathcal{W}_g(\pi(x)g)$, while the reproducing kernel $K\colon G \times G \to \mathbb{C}$ is given by

$$K(x, y) = \langle \pi(y)g, \pi(x)g\rangle = \mathcal{W}_g(\pi(y)g)(x), \qquad x, y \in G.$$

If $f_n \to f$ in the norm on $\mathcal{H}_\pi$, then

$$\mathcal{W}_g f_n(x) \to \mathcal{W}_g f(x) \tag{C.3.3}$$

uniformly for all $x \in G$. Moreover, if $h \in \mathcal{H}_\pi$ is another admissible vector then $\Psi_{g,h}\colon \mathcal{W}_g(\mathcal{H}_\pi) \to \mathcal{W}_h(\mathcal{H}_\pi)$ given by

$$\Psi_{g,h}\left(\mathcal{W}_g f\right) := \mathcal{W}_h f, \qquad f \in \mathcal{H}_\pi, \tag{C.3.4}$$

is an isomorphism of Hilbert spaces.

Proof. For $F \in \mathcal{W}_g(\mathcal{H}_\pi)$ we have that $F(x) = \mathcal{W}_g\left(\mathcal{W}_g^* F\right)(x)$ since $\mathcal{W}_g$ is an isometry. Hence

$$F(x) = \mathcal{W}_g\left(\mathcal{W}_g^* F\right)(x) = \left\langle \mathcal{W}_g^* F, \pi(x)g\right\rangle = \left\langle F, \mathcal{W}_g\left(\pi(x)g\right)\right\rangle.$$

Since $k_x := \mathcal{W}_g\,(\pi(x)g) \in \mathcal{W}_g(\mathcal{H}_\pi)$ the wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$ is a reproducing kernel Hilbert space. The reproducing kernel K can be written by using the orthogonality relations (C.2.1) as

$$K(x,y) = \langle k_y, k_x\rangle = \left\langle \mathcal{W}_g\,(\pi(y)g)\,, \mathcal{W}_g\,(\pi(x)g)\right\rangle = \langle \pi(y)g, \pi(x)g\rangle.$$

If E_x is the evaluation functional at the point $x \in G$ then

$$\|E_x\| = \|k_x\| = \|\mathcal{W}_g\,(\pi(x)g)\,\| = \|\pi(x)g\| = \|g\|.$$

Thus $\mathcal{W}_g(\mathcal{H}_\pi)$ is uniform since the admissible vector $g \in \mathcal{H}_\pi$ is fixed. The computation (C.2.2) shows that the convergence in (C.3.3) is uniform. The map $\Psi_{g,h}$ is an isometry since

$$\|\mathcal{W}_h f\|_{\mathcal{W}_h(\mathcal{H}_\pi)} = \|f\|_{\mathcal{H}_\pi} = \|\mathcal{W}_g f\|_{\mathcal{W}_g(\mathcal{H}_\pi)},$$

for all $f \in \mathcal{H}_\pi$. Finally, $\Psi_{g,h}$ is surjective as every element in $\mathcal{W}_h(\mathcal{H}_\pi)$ is of the form $\mathcal{W}_h f$ for some $f \in \mathcal{H}_\pi$. □

Remark. The fact that the map $\Psi_{g,h}$ in (C.3.4) is an isomorphism shows that the wavelet spaces corresponding to different admissible vectors can not be too different, e.g. their dimensions coincide. However, the wavelet spaces are still different as reproducing kernel Hilbert spaces since the map $\Psi_{g,h}$ does not in general preserve the reproducing kernels.

The wavelet transform $\mathcal{W}_g\colon \mathcal{H}_\pi \to L^2(G)$ is an isometry when $g \in \mathcal{H}_\pi$ is an admissible vector. Hence the projection from $L^2(G)$ to $\mathcal{W}_g(\mathcal{H}_\pi)$ is precisely given by $\mathcal{W}_g \circ \mathcal{W}_g^*$. A classical result in coorbit theory [61] known as the *reproducing formula* describes this projection in terms of convolutions: The orthogonal projection from $L^2(G)$ to $\mathcal{W}_g(\mathcal{H}_\pi)$ is explicitly given by

$$\mathcal{W}_g \circ \mathcal{W}_g^*(F) = F *_G k_e, \qquad F \in L^2(G),$$

where $k_e(x) := \mathcal{W}_g g(x)$ is the point kernel corresponding to the identity element $e \in G$. The following basic result shows that the wavelet spaces automatically exhibit integrability properties that are not shared by general subspaces of $L^2(G)$.

Proposition C.3.4. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation and fix an admissible vector $g \in \mathcal{H}_\pi$. The wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$ is continuously embedded into $L^p(G)$ for all $p \in [2,\infty]$. However, the wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$ is not in general contained in $L^1(G)$.

Proof. Notice that $\mathcal{W}_g(\mathcal{H}_\pi)$ is continuously embedded in both $L^2(G)$ and $L^\infty(G)$: The first claim is obvious, while the second follows from the computation

$$\|F\|_{L^\infty(G)} = \sup_{x\in G} |\langle k_x, F\rangle| \le \sup_{x\in G} \|k_x\|_{\mathcal{W}_g(\mathcal{H}_\pi)}\|F\|_{\mathcal{W}_g(\mathcal{H}_\pi)} = \|g\|_{\mathcal{H}_\pi}\|F\|_{\mathcal{W}_g(\mathcal{H}_\pi)},$$

for $F \in \mathcal{W}_g(\mathcal{H}_\pi)$. This observation implies that $\mathcal{W}_g(\mathcal{H}_\pi)$ is continuously embedded into the intermediate spaces $L^p(G)$ for $p \in (2, \infty)$ as well since

$$\begin{aligned}\|F\|_{L^p(G)} &= \left(\int_G |F(x)|^{p-2}|F(x)|^2 \, d\mu_L(x)\right)^{\frac{1}{p}} \\ &\leq \|F\|_{L^\infty(G)}^{\frac{p-2}{p}} \|F\|_{\mathcal{W}_g(\mathcal{H}_\pi)}^{\frac{2}{p}} \\ &\leq \|g\|_{\mathcal{H}_\pi}^{\frac{p-2}{p}} \|F\|_{\mathcal{W}_g(\mathcal{H}_\pi)}.\end{aligned}$$

Counterexamples to the last statement can be found in the time-frequency setting since the STFT satisfies $V_g g \in L^1(\mathbb{R}^{2n})$ only when g is a continuous function on $\mathbb{R}^n$ by [81, Proposition 12.1.4]. $\square$

Throughout the paper, we aim to emphasize how the reproducing kernel Hilbert space structure of the wavelet spaces is paramount. As a first example, we have the following existence result.

Proposition C.3.5. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation of a second countable locally compact group G and fix an admissible vector $g \in \mathcal{H}_\pi$. There exists a countable set $\Lambda \subset G$ such that the discrete set of vectors $\{\pi(\lambda)g\}_{\lambda\in\Lambda}$ is complete in $\mathcal{H}_\pi$.

Proof. The second countability of G is by [150, Theorem 2] equivalent to the requirement that $L^2(G)$ is separable. Whence the subspace $\mathcal{W}_g(\mathcal{H}_\pi) \subset L^2(G)$ is also separable. By [23, Lemma 11] there exists a countable set $\Lambda \subset G$ such that the collection of point kernels $k_\lambda = \mathcal{W}_g(\pi(\lambda)g)$ for $\lambda \in \Lambda$ is dense in $\mathcal{W}_g(\mathcal{H}_\pi)$. Hence for $f \in \mathcal{H}_\pi$ the criterion

$$\langle \mathcal{W}_g f, \mathcal{W}_g(\pi(\lambda)g)\rangle = 0$$

for all $\lambda \in \Lambda$ forces $\mathcal{W}_g f \equiv 0$. The orthogonality relations (C.2.1) and the injectivity of the Duflo-Moore operator implies that $\langle f, \pi(\lambda)g\rangle = 0$ for all $\lambda \in \Lambda$ only when $f = 0$. $\square$

Remark. The second countability condition in Proposition C.3.5 is only a sufficient requirement. In the proof of Proposition C.3.5 we need that the wavelet spaces $\mathcal{W}_g(\mathcal{H}_\pi)$ are separable. This can happen when the ambient space $L^2(G)$ is not separable. In particular, the conclusion of Proposition C.3.5 holds for all square integrable representations corresponding to compact groups since the wavelet spaces are then finite-dimensional by [67, Theorem 5.2].

C.3.2 Tensor Product of Wavelet Spaces

Our setting involves both square integrable representations of locally compact groups as well as reproducing kernel Hilbert spaces. Both of these categories have a natural notion of a tensor product. We will use reproducing kernel Hilbert space arguments to show that these operations are compatible. Let us first briefly recall the different notions or tensor products involved.

Consider two reproducing kernel Hilbert spaces $\mathcal{H}_i$ of functions on sets X_i with reproducing kernels $K_i\colon X_i \times X_i \to \mathbb{C}$ for $i = 1, 2$. We can form the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of Hilbert spaces in the usual way by requiring that

$$\langle f_1 \otimes f_2, g_1 \otimes g_2\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} := \langle f_1, g_1\rangle_{\mathcal{H}_1}\langle f_2, g_2\rangle_{\mathcal{H}_2},$$

where $f_1, g_1 \in \mathcal{H}_1$ and $f_2, g_2 \in \mathcal{H}_2$. This extends to an inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ that is not in general complete. The completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with this inner product is denoted by $\mathcal{H}_1\hat{\otimes}\mathcal{H}_2$ and called the *tensor product* of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Not surprisingly, the tensor product $\mathcal{H}_1\hat{\otimes}\mathcal{H}_2$ can be identified with a reproducing kernel Hilbert space on the set $X_1 \times X_2$ as follows: Any element $u = \sum_{i=1}^n f_i \otimes g_i \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be identified with the function on $X_1 \times X_2$ given by $\tilde{u}(x, y) := \sum_{i=1}^n f_i(x)g_i(y)$. This association extends to the completion $\mathcal{H}_1\hat{\otimes}\mathcal{H}_2$ and gives a well-defined linear isometry between $\mathcal{H}_1\hat{\otimes}\mathcal{H}_2$ and the reproducing kernel Hilbert space on $X_1 \times X_2$ with reproducing kernel

$$K((x_1, y_1), (x_2, y_2)) := K_1(x_1, x_2)K_2(y_1, y_2), \quad x_1, x_2 \in X_1,\ y_1, y_2 \in X_2.$$

In the setting of unitary representations of locally compact groups we also have a notion of a tensor product. Consider two unitary representations $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\rho\colon H \to \mathcal{U}(\mathcal{H}_\rho)$ where G and H are locally compact groups. We can consider the *tensor product representation* $\pi \otimes \rho$ given on elementary tensors $f_1 \otimes f_2$ by

$$(\pi \otimes \rho)(x, y)(f_1 \otimes f_2) := \pi(x)f_1 \otimes \rho(y)f_2,$$

for $x \in G$, $y \in H$, $f_1 \in \mathcal{H}_\pi$, and $f_2 \in \mathcal{H}_\rho$. This extends to arbitrary elements in $\mathcal{H}_\pi\hat{\otimes}\mathcal{H}_\rho$ and hence defines a unitary representation $\pi\otimes\rho\colon G\times H \to \mathcal{U}(\mathcal{H}_\pi\hat{\otimes}\mathcal{H}_\rho)$. The following result shows that the two tensor product constructions we have described are compatible in a natural way.

Proposition C.3.6. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\rho\colon H \to \mathcal{U}(\mathcal{H}_\rho)$ be two square integrable representations with admissible vectors $g \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_\rho$. There is an isomorphism of reproducing kernel Hilbert spaces

$$\mathcal{W}_{g\otimes h}(\mathcal{H}_\pi\hat{\otimes}\mathcal{H}_\rho) \simeq \mathcal{W}_g(\mathcal{H}_\pi)\hat{\otimes}\mathcal{W}_h(\mathcal{H}_\rho).$$

Proof. The tensor product representation $\pi \otimes \rho$ is irreducible by [67, Theorem 7.12]. Let us check that $\pi \otimes \rho$ is square integrable and that $g \otimes h \in \mathcal{H}_\pi \hat{\otimes} \mathcal{H}_\rho$ is admissible. For $x \in G$ and $y \in H$ we have

$$\begin{aligned}
\left\|\mathcal{W}_{g\otimes h} g \otimes h\right\|_{L^2(G\times H)}^2 &= \int_{G\times H} |\langle g \otimes h, (\pi \otimes \rho)(x, y)(g \otimes h)\rangle|^2 \, d\mu_L^{G\times H}(x, y) \\
&= \int_{G\times H} |\langle g, \pi(x) g\rangle \langle h, \rho(y) h\rangle|^2 \, d\mu_L^{G\times H}(x, y) \\
&= \int_G |\langle g, \pi(x) g\rangle|^2 \, d\mu_L^G(x) \int_H |\langle h, \rho(y) h\rangle|^2 \, d\mu_L^H(y) \\
&= \|\mathcal{W}_g g\|_{L^2(G)}^2 \|\mathcal{W}_h h\|_{L^2(H)}^2 < \infty.
\end{aligned}$$

Moreover, we see from the above computation that we have the pointwise equality

$$\mathcal{W}_{g\otimes h} g \otimes h(x, y) = \mathcal{W}_g g(x) \mathcal{W}_h h(y), \qquad x \in G, \ y \in H,$$

as functions on $G\times H$. Since the reproducing kernels for the space $\mathcal{W}_{g\otimes h}(\mathcal{H}_\pi \hat{\otimes} \mathcal{H}_\rho)$ and the space $\mathcal{W}_g(\mathcal{H}_\pi) \hat{\otimes} \mathcal{W}_h(\mathcal{H}_\rho)$ coincide, the result follows from the uniqueness of reproducing kernels given in Subsection C.2.2. $\square$

Example C.3.7. Consider the Gabor space $V_{g_n}(L^2(\mathbb{R}^n))$ where $g_n(x) = e^{-\frac{\pi}{2}x^2}$ is the n-dimensional Gaussian function. Then Theorem C.3.6 implies that

$$V_{g_n}\left(L^2(\mathbb{R}^n)\right) \hat{\otimes} V_{g_n}\left(L^2(\mathbb{R}^n)\right) \simeq V_{g_n \otimes g_n}\left(L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)\right) \simeq V_{g_{2n}}\left(L^2(\mathbb{R}^{2n})\right),$$

where g_{2n} is the $2n$-dimensional Gaussian function. Although this is folklore knowledge, we emphasize the the simplicity of its derivation from the theory of reproducing kernel Hilbert spaces. Many function spaces in complex analysis, e.g. the Hardy spaces and Bergman spaces, satisfy similar tensorization rules, see e.g. [137, Proposition 5.13 and Proposition 5.14]. This is maybe not so surprising given the connection between the Gabor space $V_{g_n}(L^2(\mathbb{R}^n))$ and complex analysis given in [81, Proposition 3.4.1].

C.4 Rigidity of Wavelet Spaces

In this section we will investigate how wavelet spaces associated with (potentially) different representations are related. The main result in Theorem C.4.2 have several noteworthy consequences. The first consequence in Corollary C.4.3 is a new proof of one of the main results in [77, Theorem 4.2]. The other consequences, Corollary C.4.5 and Corollary C.4.6, are new and illustrate the broad utility of Theorem C.4.2. Let us first consider an example of the general setting where things are greatly simplified.

Example C.4.1. Let G be a locally compact group that is abelian and consider a square integrable representation $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$. It follows from Schur's Lemma C.2.2 that $\mathcal{H}_\pi \simeq \mathbb{C}$ and $\mathcal{U}(\mathcal{H}_\pi) \simeq \mathbb{T}$. We make these identifications and view π as a map from G to $\mathbb{T}$. What requirements do the square integrability impose? For $z \in \mathbb{C} \setminus \{0\}$ we have that

$$\int_G |\langle z, \pi(x)z\rangle|^2 \, d\mu(x) = |z|^4 \mu(G).$$

Hence π is square integrable if and only if $\mu(G) < \infty$. This is the case precisely when G is compact.

Since G is unimodular, it follows from Proposition C.2.4 that the Duflo-Moore operator C_π is a positive constant multiple of the identity. That the constant is equal to one can be seen by direct verification, or by an application of Peter-Weyl theory [47, Example 12.2.7]. Hence a complex number $z \in \mathbb{C}$ is admissible if and only if $z \in \mathbb{T}$. The wavelet spaces $\mathcal{W}_z(\mathbb{C})$ for $z \in \mathbb{T}$ are one-dimensional subspaces of $L^2(G)$ that are spanned by the elements $\mathcal{W}_z z$. Moreover, all the wavelet spaces $\mathcal{W}_z(\mathbb{C})$ coincide since $\mathcal{W}_z z = \mathcal{W}_1 1$ for all $z \in \mathbb{T}$.

Notice that everything said in Example C.4.1 is independent of the representation in question: In the abelian case, all the wavelet spaces coincide even when we have two different representations $\pi\colon G \to \mathbb{T}$ and $\rho\colon G \to \mathbb{T}$. On the other hand, we always have that any two admissible vectors $z, w \in \mathbb{T}$ (regardless of the choice of representations) are related by $z = cw$ for some $c \in \mathbb{T}$. These elementary remarks motivate the following general result.

Theorem C.4.2. *Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\rho\colon G \to \mathcal{U}(\mathcal{H}_\rho)$ be two square integrable representations with admissible vectors $g \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_\rho$. Assume that the corresponding wavelet spaces intersect non-trivially, that is,*

$$\mathcal{W}_g(\mathcal{H}_\pi) \cap \mathcal{W}_h(\mathcal{H}_\rho) \neq \{0\}.$$

Then $\mathcal{W}_g(\mathcal{H}_\pi) = \mathcal{W}_h(\mathcal{H}_\rho)$ and there exists a unitary intertwining operator T from $\mathcal{H}_\pi$ to $\mathcal{H}_\rho$ satisfying $T(g) = h$.

Proof. Notice that the subspace $\mathcal{W}_h(\mathcal{H}_\rho) \cap \mathcal{W}_g(\mathcal{H}_\pi) \subset \mathcal{W}_g(\mathcal{H}_\pi)$ is invariant under translations. Since π is irreducible and $\mathcal{W}_g : \mathcal{H}_\pi \to \mathcal{W}_g(\mathcal{H}_\pi)$ is a unitary intertwiner we have that $\mathcal{W}_h(\mathcal{H}_\rho) = \mathcal{W}_g(\mathcal{H}_\pi)$. The norms on $\mathcal{W}_g(\mathcal{H}_\pi) = \mathcal{W}_h(\mathcal{H}_\rho)$ both coincide with the restriction of the $L^2(G)$-norm. Hence $\mathcal{W}_g(\mathcal{H}_\pi)$ and $\mathcal{W}_h(\mathcal{H}_\rho)$ are reproducing kernel Hilbert spaces that coincide with equal norms. By the uniqueness statements given in Subsection C.2.2 the two reproducing kernels coincide

$$\mathcal{W}_g(\pi(y)g)(x) = \mathcal{W}_h(\rho(y)h)(x), \qquad x, y \in G.$$

Since $\mathcal{W}_g(\pi(y)g)(x) = L_y\mathcal{W}_g g(x)$ and $\mathcal{W}_h(\rho(y)h)(x) = L_y\mathcal{W}_h h(x)$, all the information we need is contained in the equality

$$\mathcal{W}_g g(x) = \mathcal{W}_h h(x), \qquad x \in G. \tag{C.4.1}$$

To define the map $T\colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ we first require that $T(g) = h$. Moreover, for T to be an intertwining operator, we need that

$$T(\pi(x)g) = \rho(x)h, \qquad x \in G.$$

Since π is irreducible the set $\mathcal{M}_g := \operatorname{span}\{\pi(x)g\}_{x\in G}$ is dense in $\mathcal{H}_\pi$. To see that T extends to all of $\mathcal{H}_\pi$ we will show that it is an isometry on the subspace $\mathcal{M}_g$: For $x, y \in G$ we have

$$\begin{aligned}\langle T(\pi(x)g), T(\pi(y)g)\rangle_{\mathcal{H}_\pi} &= \langle \rho(x)h, \rho(y)h\rangle_{\mathcal{H}_\pi}\\ &= \langle h, \rho(x^{-1}y)h\rangle_{\mathcal{H}_\pi}\\ &= \mathcal{W}_h h(x^{-1}y).\end{aligned}$$

Hence we obtain from (C.4.1) that

$$\langle T(\pi(x)g), T(\pi(y)g)\rangle_{\mathcal{H}_\pi} = \mathcal{W}_g g(x^{-1}y) = \langle \pi(x)g, \pi(y)g\rangle_{\mathcal{H}_\pi}.$$

The map T is surjective since $\operatorname{span}\{\rho(x)h\}_{x\in G}$ is dense in $\mathcal{H}_\rho$ due to the irreducibility of ρ. Hence T is a unitary map. For $g \in \mathcal{H}_\pi$ we can write $g = \sum_{i=1}^\infty c_i\pi(x_i)g$ for constants $c_i \in \mathbb{C}$ and elements $x_i \in G$. Then for $x \in G$ it follows that

$$\begin{aligned}T(\pi(x)g) &= T\left(\pi(x)\sum_{i=1}^\infty c_i\pi(x_i)g\right)\\ &= \sum_{i=1}^\infty c_i T\left(\pi(xx_i)g\right)\\ &= \sum_{i=1}^\infty c_i\rho(xx_i)h = \rho(x)T(g).\end{aligned}$$ □

Notice that Theorem C.4.2 trivially implies that whenever π and ρ are not equivalent, then we necessarily have trivial intersection

$$\mathcal{W}_g(\mathcal{H}_\pi) \cap \mathcal{W}_h(\mathcal{H}_\rho) = \{0\},$$

for any admissible vectors $g \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_\rho$. The first application of Theorem C.4.2 is a new proof of the result [77, Theorem 4.2] which we state in Corollary C.4.3 below. This was originally proved by utilizing the orthogonality relations (C.2.1) for the wavelet transform. Recently, the result has been re-proven in the Gabor case in [124, Lemma 3.3] with the use of quantum harmonic analysis. For us, the result follows immediately from Theorem C.4.2 together with Lemma C.2.2.

Corollary C.4.3. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation with admissible vectors $g, h \in \mathcal{H}_\pi$. If $\mathcal{W}_h(\mathcal{H}_\pi) \cap \mathcal{W}_g(\mathcal{H}_\pi) \neq \{0\}$ then we have $\mathcal{W}_h(\mathcal{H}_\pi) = \mathcal{W}_g(\mathcal{H}_\pi)$ and $h = cg$ for some $c \in \mathbb{T}$.

Remark. The orthogonality relations (C.2.1) shows that the wavelet spaces $\mathcal{W}_g(\mathcal{H}_\pi)$ and $\mathcal{W}_h(\mathcal{H}_\pi)$ are orthogonal if and only if

$$\langle C_\pi g, C_\pi h\rangle = 0,$$

where C_π is the Duflo-Moore operator. When $\langle C_\pi g, C_\pi h\rangle \neq 0$ the wavelet spaces still intersect trivially by Corollary C.4.3 except in the case $h = cg$ with $c \in \mathbb{T}$.

Before moving on, we show how we can combine Corollary C.4.3 with abstract results regarding functions of positive type to deduce concrete results for the wavelet transform.

Definition C.4.4. A function $f : G \to \mathbb{C}$ on a locally compact group G is said to be a function of *(strictly) positive type* if for any finite subset $\Omega := \{x_1, \ldots, x_m\} \subset G$, the matrix

$$\left\{f\left(x_j^{-1}x_i\right)\right\}_{i,j=1}^m$$

is (strictly positive definite) positive semi-definite.

Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation with an admissible vector $g \in \mathcal{H}_\pi$. Then $\mathcal{W}_g g$ is a function of positive type due to Proposition C.3.3 and the equality

$$\mathcal{W}_g g(x_j^{-1}x_i) = L_{x_j}\mathcal{W}_g g(x_i) = \mathcal{W}_g(\pi(x_j)g)(x_i), \qquad x_i, x_j \in G. \tag{C.4.2}$$

Corollary C.4.5. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\rho\colon G \to \mathcal{U}(\mathcal{H}_\rho)$ be square integrable representations with admissible vectors $g \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_\rho$, respectively. Then $\mathcal{W}_g g - \mathcal{W}_h h$ is never a non-zero function of positive type.

Proof. Assume that $\mathcal{W}_g g - \mathcal{W}_h h$ is a function of positive type. Then Aronszajn's inclusion theorem [6, Theorem 1.7.1] in reproducing kernel Hilbert space theory implies that $\mathcal{W}_h(\mathcal{H}_\rho) \subset \mathcal{W}_g(\mathcal{H}_\pi)$. Hence Theorem C.4.2 shows that $h = T(g)$ for some unitary intertwining operator $T\colon \mathcal{H}_\pi \to \mathcal{H}_\rho$. For $x \in G$ we thus have

$$\begin{aligned}\mathcal{W}_g g(x) - \mathcal{W}_h h(x) &= \langle g, \pi(x)g\rangle - \langle T(g), \rho(x)T(g)\rangle\\ &= \langle g, \pi(x)g\rangle - \langle T(g), T(\pi(x)g)\rangle\\ &= \langle g, \pi(x)g\rangle - \langle g, \pi(x)g\rangle\\ &= 0.\end{aligned}$$

□

For a locally compact group G we let $\mathcal{P}_c$ denote the functions $f\colon G \to \mathbb{C}$ of positive type such that $f(e) = c \in \mathbb{C}$, where e is the identity element of G.

Corollary C.4.6. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation of a unimodular group G with admissible vectors $g, g_1, g_2 \in \mathcal{H}_\pi$. Assume we can write $\mathcal{W}_g g$ as a convex combination

$$\mathcal{W}_g g = t \cdot \mathcal{W}_{g_1} g_1 + (1-t) \cdot \mathcal{W}_{g_2} g_2,$$

for some $t \in [0,1]$. Then we either have $g = cg_1$ or $g = cg_2$ for some $c \in \mathbb{T}$.

Proof. Notice that $\mathcal{W}_g g \in \mathcal{P}_{c_\pi^{-1}}(G)$ where $C_\pi = c_\pi \cdot Id_{\mathcal{H}_\pi}$ since

$$\mathcal{W}_g g(e) = \|g\|_{\mathcal{H}_\pi} = c_\pi^{-1}.$$

It follows from [11, Theorem C.5.2] that the functions $\mathcal{W}_g g$ are extreme points in the bounded convex set $P_{c_\pi^{-1}}(G)$. This implies that $\mathcal{W}_g g = \mathcal{W}_{g_1} g_1$ or $\mathcal{W}_g g = \mathcal{W}_{g_2} g_2$. We can now apply Corollary C.4.3 to conclude that $g = cg_1$ or $g = cg_2$ for some $c \in \mathbb{T}$. □

Example C.4.7. Let us check that everything works out for the STFT. Assume for normalized vectors $g, g_1, g_2 \in L^2(\mathbb{R}^n)$ that

$$V_g g(x,\omega) = t \cdot V_{g_1} g_1(x,\omega) + (1-t) \cdot V_{g_2} g_2(x,\omega), \tag{C.4.3}$$

for some $t \in [0,1]$ and for all $(x,\omega) \in \mathbb{R}^{2n}$. Then by multiplying with $e^{-2\pi i\tau} e^{\pi i x\cdot\omega}$ on both sides we obtain

$$\mathcal{W}_g g\left(x,\omega,e^{2\pi i\tau}\right) = t \cdot \mathcal{W}_{g_1} g_1\left(x,\omega,e^{2\pi i\tau}\right) + (1-t) \cdot \mathcal{W}_{g_2} g_2\left(x,\omega,e^{2\pi i\tau}\right),$$

for $\left(x,\omega,e^{2\pi i\tau}\right) \in \mathbb{H}_r^n$. We can now apply Corollary C.4.6 to see that $g = cg_1$ or $g = cg_2$ for some $c \in \mathbb{T}$.

In this specialized setting we describe an alternative proof using quantum mechanical reasoning. Assume again that (C.4.3) holds for some $t \in [0,1]$. For $g \in L^2(\mathbb{R}^n)$ the *Wigner distribution* Wg in quantum mechanics can be defined through the STFT by the formula

$$Wg(x,\omega) := 2^n e^{4\pi i x\cdot\omega} V_{P(g)} g(2x, 2\omega), \qquad P(g)(x) := g(-x).$$

Hence (C.4.3) is equivalent to

$$Wg(x,\omega) = t \cdot Wg_1(x,\omega) + (1-t) \cdot Wg_2(x,\omega),$$

for all $(x,\omega) \in \mathbb{R}^{2n}$. One can now use the Weyl-quantization to go between functions on $\mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$. In this correspondence the Wigner distributions Wg for $g \in L^2(\mathbb{R}^n)$ correspond to the positive rank-one operators $g \otimes g$ given by

$$(g \otimes g)(f) \coloneqq \langle f, g \rangle \cdot g, \qquad f \in L^2(\mathbb{R}^n).$$

Hence we obtain

$$g \otimes g = t \cdot g_1 \otimes g_1 + (1 - t) \cdot g_2 \otimes g_2.$$

One can easily see by evaluation that this forces the same conclusion, namely that $g = cg_1$ or $g = cg_2$ for some $c \in \mathbb{T}$.

C.5 Interpolation in Wavelet Spaces

We have seen on multiple occasions that the reproducing kernel Hilbert space structure of the wavelet spaces is immensely useful. We now focus in on that structure by considering a non-trivial interpolation problem. In this section we will describe the interpolation problem and show that the answer is not always affirmative. As we will see in the Section C.6, the interpolation problem turns out to be equivalent to the HRT-Conjecture for the Gabor spaces.

Definition C.5.1. Let X be a set and consider the points $\Omega \coloneqq \{x_1, \ldots, x_m\} \subset X$ and possibly non-distinct scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. We say that a function $F\colon X \to \mathbb{C}$ *interpolates* these points whenever $F(x_i) = \lambda_i$ for all $i = 1, \ldots, m$. The function F is called an *interpolating function*.

The question in interpolation theory is whether we can find an interpolating function with additional requirements. Typically, we have a Hilbert space $\mathcal{H}$ of functions on X and ask whether we can choose $F \in \mathcal{H}$ as an interpolating function. When $\mathcal{H}$ is a reproducing Hilbert space, we can give an explicit criterion through the reproducing kernel. We state this result for the case we have investigated.

Proposition C.5.2. Let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation and fix an admissible vector $g \in \mathcal{H}_\pi$. Consider distinct points $\Omega \coloneqq \{x_1, \ldots, x_m\} \subset G$ and possibly non-distinct scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. There exists an interpolating function $F \in \mathcal{W}_g(\mathcal{H}_\pi)$ if and only if the vector $(\lambda_1, \ldots, \lambda_m)^T \in \mathbb{C}^m$ is in the image of the $m \times m$ matrix

$$K_\Omega \coloneqq \left\{K(x_i, x_j)\right\}_{i,j=1}^{m},$$

where K is the reproducing kernel for the wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$.

The proof of Proposition C.5.2 follows from Proposition C.3.3 together with [137, Theorem 3.4]. We remarked in Subsection C.2.2 that the matrices K_Ω are always positive semi-definite. The interpolation problem in Proposition C.5.2 have a unique solution for all $\Omega = \{x_1, \ldots, x_m\} \subset G$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ if and only if the matrices K_Ω are all strictly positive definite. This is the case if and only if the function $\mathcal{W}_g g$ is a function of strictly positive type. This is the motivation for the terminology *fully interpolating* given in Subsection C.2.2. Notice that for the point kernels $k_{x_1}, \ldots, k_{x_m}$ we can write

$$\sum_{i,j=1}^{m} \overline{\alpha_i}\alpha_j k_{x_j}(x_i) = \left\langle \sum_{j=1}^{m} \alpha_j k_{x_j}, \sum_{i=1}^{m} \alpha_i k_{x_i} \right\rangle = \left\| \sum_{i=1}^{m} \alpha_i k_{x_i} \right\|^2 \geq 0,$$

for $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$. Hence $\mathcal{W}_g(\mathcal{H}_\pi)$ is fully interpolating precisely when there are no non-trivial linear combinations between the point kernels $k_{x_1}, \ldots, k_{x_m}$ for any points $x_1, \ldots, x_m \in G$.

Remark. It is straightforward to check that Proposition C.5.2 is also valid for the Gabor spaces $V_g(L^2(\mathbb{R}^n))$. In that case, the point kernel corresponding to $(x, \omega) \in \mathbb{R}^{2n}$ is $k_{(x,\omega)} = V_g(M_\omega T_x g)$. Notice however that we get the extra phase-factor

$$V_g(M_\omega T_x g)(s,t) = e^{-2\pi i x\cdot(t-\omega)} V_g g(s-x, t-\omega), \quad (s,t) \in \mathbb{R}^{2n}, \qquad \text{(C.5.1)}$$

in contrast with (C.4.2).

When $G = \{e\}$ the only wavelet space associated with G is the one-dimensional space $L^2(G)$. This is fully interpolating for trivial reasons. We exclude this case in future examples and refer to a locally compact group G as *non-trivial* when G has more than one element. The next result shows that a large class of wavelet spaces are not fully interpolating.

Proposition C.5.3. Let G be a non-trivial locally compact group. If G is either abelian or compact then no wavelet space associated to G is fully interpolating.

Proof. An abelian locally compact group G possesses square integrable representations if and only if the group is compact. In this case, the representation theory of compact groups shows that any irreducible unitary representation of G is finite-dimensional [67, Theorem 5.2]. Any irreducible unitary representation of G is also automatically square integrable due to the compactness of G.

If G is an infinite group, then we can always pick $\Omega = \{x_1, \ldots, x_m\} \subset G$ to have larger cardinality than the dimension of the representation considered. Then there is no way that $k_{x_1}, \ldots, k_{x_m}$ can be linearly independent. If G is a finite group, then the same argument goes through unless G have an irreducible representation whose dimension is greater or equal to the order of the group G. This is not possible since the class equation in finite representation theory gives that

$$|G| = \sum_{[\pi]} \dim(\mathcal{H}_\pi)^2,$$

where the sum runs over all equivalence classes of irreducible representation π of G. Since we have excluded G from being the trivial group, the result follows. $\square$

Example C.5.4. For the n-dimensional Gaussian function $g_n(x) := e^{-\frac{\pi}{2}x^2}$ we will show that the Gabor space $V_{g_n}(L^2(\mathbb{R}^n))$ is fully interpolating. A straightforward computation reveals that

$$V_{g_n}g_n(x,\omega) = e^{-\pi i x\cdot\omega}e^{-\frac{\pi}{4}x^2}e^{-\pi\omega^2}, \qquad (x,\omega) \in \mathbb{R}^{2n}.$$

Assume by contradiction that there is a linear dependence between the point kernels $k_{(x_k,\omega_k)}$ corresponding to distinct points $(x_k,\omega_k) \in \mathbb{R}^{2n}$ for $k = 1,\ldots,m$. The linear dependence explicitly gives

$$\sum_{k=1}^{m} \alpha_k e^{2\pi i x_k\cdot\omega_k}e^{-2\pi i x_k\cdot\omega}e^{-\pi i(x-x_k)\cdot(\omega-\omega_k)}e^{-\frac{\pi}{4}(x-x_k)^2}e^{-\pi(\omega-\omega_k)^2} = 0,$$

where $\alpha_1,\ldots,\alpha_n \in \mathbb{C}$ are not all zero. By setting

$$\beta_k := \alpha_k e^{\pi i x_k\cdot\omega_k}e^{-\frac{\pi}{4}x_k^2}e^{-\pi\omega_k^2}$$

we obtain

$$e^{-\pi i x\cdot\omega}e^{-\frac{\pi}{4}x^2}e^{-\pi\omega^2}\sum_{k=1}^{m}\beta_k e^{-2\pi i x_k\cdot\omega}e^{\pi i(x\cdot\omega_k+\omega\cdot x_k)}e^{\frac{\pi}{2}x\cdot x_k}e^{2\pi\omega\cdot\omega_k} = 0.$$

We can divide by the non-zero function $e^{-\pi i x\cdot\omega}e^{-\frac{\pi}{4}x^2}e^{-\pi\omega^2}$ and set $\omega = 0$ to get the simplified equation

$$\sum_{k=1}^{m}\beta_k e^{x\cdot\left(\frac{\pi}{2}x_k+i\omega_k\right)} = 0. \tag{C.5.2}$$

Notice that the coefficients β_k satisfy $\beta_k = 0$ if and only if $\alpha_k = 0$. The equation (C.5.2) contradicts the independence of the exponential functions $x \mapsto e^{x\cdot\lambda_k}$, see e.g. [35, Lemma 13.1], since $\lambda_k = \frac{\pi}{2}x_k + i\omega_k$ are distinct complex numbers.

Example C.5.5. To illustrate that the Gabor space $V_{g_1}(L^2(\mathbb{R})) \subset L^2(\mathbb{R}^2)$ is fully interpolating we consider the points $x_1 = (0,0)$, $x_2 = (1,0)$, and $x_3 = (0,1)$ in $\mathbb{R}^2$ along with $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Then there exists $F \in V_{g_1}(L^2(\mathbb{R}))$ such that

$F(x_i) = \lambda_i$ for $i = 1, 2, 3$. Moreover, the function $F \in V_{g_1}(L^2(\mathbb{R}))$ with minimal norm that interpolates these points will be on the form

$$F(x, \omega) = \alpha_1 V_{g_1} g_1(x, \omega) + \alpha_2 V_{g_1} g_1(x - 1, \omega) + \alpha_2 V_{g_1} g_1(x, \omega - 1),$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ by [137, Theorem 3.4]. It follows by straightforward computations that $\alpha_1 \simeq 0.6218$, $\alpha_2 \simeq 0.7360$, and $\alpha_3 \simeq 0.9876$.

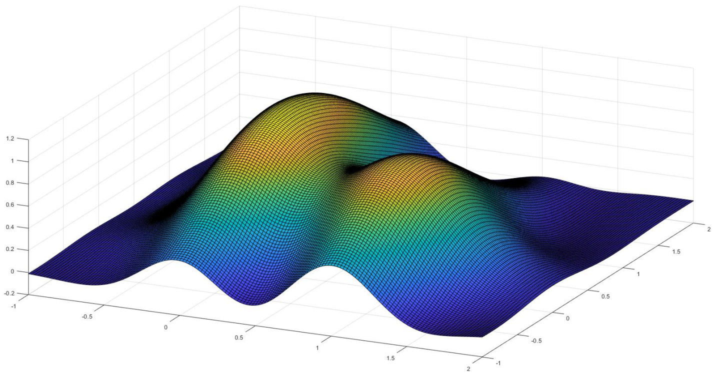

Figure C.1: The real part of the interpolating function $F(x, \omega)$ in Example C.5.5.

Remark. The function $V_{g_n} g_n$ on $\mathbb{R}^{2n}$ is not strictly positive definite even though the Gabor space $V_{g_n}(L^2(\mathbb{R}^n))$ is fully interpolating. This discrepancy is due to the extra phase-factor in (C.5.1). In fact, the function $V_g g(x, \omega)$ is not even positive definite: If this were the case, then the Fourier inverse $\mathcal{F}^{-1}(V_g g)$ would be a positive function on $\mathbb{R}^{2n}$ by Bochner's Theorem [137, Theorem 10.4]. However, the function $\mathcal{F}^{-1}(V_g g)(x, \omega) = e^{-\pi(x^2+\omega^2)} e^{2\pi i x \cdot \omega}$ is clearly not even real-valued.

C.6 Connection With the HRT-Conjecture

The question of whether the Gabor spaces are fully interpolating turns out to be equivalent to the infamous HRT-Conjecture. Recall that a subset $\mathcal{A} \subset \mathcal{H}$ of a vector space $\mathcal{H}$ is said to be *linearly independent* if every finite subset $F \subset \mathcal{A}$ is linearly independent in the classical sense. The following open conjecture reveals how little is understood about time-frequency shifts.

Conjecture (HRT)**.** Is the set

$$\{M_\omega T_x g\}_{(x,\omega) \in \mathbb{R}^{2n}}$$

linearly independent in $L^2(\mathbb{R}^n)$ for all non-zero $g \in L^2(\mathbb{R}^n)$?

The HRT-Conjecture was originally posed back in 1996 by C. Heil, J. Ramanathan, and P. Topiwala in the paper [97]. There have been many significant developments on the conjecture during the years, where techniques from von Neumann algebras [120], spectral theory [9], ergodic theory [96], and representation theory of the Heisenberg groups [42] have been used. We refer the reader to the introduction of the paper [134] for a reasonably extensive list of contributions to the HRT conjecture. Moreover, we recommend the survey papers [96, 98] on the HRT-Conjecture written by one of its founders. The following result shows that the HRT-Conjecture can be reformulated to a problem regarding reproducing kernel Hilbert spaces.

Proposition C.6.1. The HRT-Conjecture is equivalent to the statement that the Gabor spaces are fully interpolating.

Proof. Let us fix elements $(x_1, \omega_1), \ldots, (x_m, \omega_m) \in \mathbb{R}^{2n}$ and consider the collection

$$\{M_{\omega_k} T_{x_k} g\}_{k=1}^{m}. \tag{C.6.1}$$

We henceforth assume that $\|g\|_{L^2(\mathbb{R}^n)} = 1$ since normalizing $g \in L^2(\mathbb{R}^n)$ does not change whether the collection (C.6.1) is linearly independent.

Assume first that the collection (C.6.1) is linearly dependent, that is, there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ not all zero such that

$$\sum_{k=1}^{m} \alpha_k M_{\omega_k} T_{x_k} g = 0.$$

We can take the inner product with the function $M_\omega T_x g$ to obtain

$$\begin{aligned}
\sum_{k=1}^{m} \alpha_k \left\langle M_{\omega_k} T_{x_k} g, M_\omega T_x g \right\rangle &= \sum_{k=1}^{m} \alpha_k V_g(M_{\omega_k} T_{x_k} g)(x, \omega) \\
&= \sum_{k=1}^{m} \alpha_k k_{(x_k, \omega_k)}(x, \omega) \\
&= 0.
\end{aligned}$$

This gives a linear dependence between $k_{(x_1,\omega_1)}, \ldots, k_{(x_m,\omega_m)}$, showing that $V_g(L^2(\mathbb{R}^n))$ is not fully interpolating.

Conversely, assume that $V_g(L^2(\mathbb{R}^n))$ is not fully interpolating. Then there exists a linear dependence between the point kernels $k_{(x_1,\omega_1)}, \ldots, k_{(x_m,\omega_m)}$ for some points $(x_1, \omega_1), \ldots, (x_m, \omega_m) \in \mathbb{R}^{2n}$. Retracing the steps we took previously we conclude that

$$\sum_{k=1}^{m} \alpha_k \left\langle M_{\omega_k} T_{x_k} g, M_\omega T_x g \right\rangle = 0,$$

where $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ are not all zero. The proof of Proposition C.3.5 shows that the collection $\{M_\omega T_x g\}_{(x,\omega)\in\mathbb{R}^{2n}}$ is complete in $L^2(\mathbb{R}^n)$. This implies the linear dependence

$$\sum_{k=1}^{m} \alpha_k M_{\omega_k} T_{x_k} g = 0. \qquad \square$$

Proposition C.6.1 allows us to use the partial results available on the HRT-Conjecture in the literature to deduce that certain Gabor spaces are fully interpolating. In particular, it was known from the beginning [97, Proposition 4] that the HRT-Conjecture is true for the n-dimensional Gaussian function. In Example C.5.4 we proved, in light of Proposition C.6.1, the same thing by brute-force calculations with the short-time Fourier transform. We can use [97, Proposition 4] and Proposition C.6.1 to conclude that the Gabor spaces $V_g(L^2(\mathbb{R}^n))$ are fully interpolating whenever g is a Hermite function. It was shown in [97, Theorem 1] that the collection in (C.6.1) is linearly independent when $m \leq 3$. In light of Proposition C.6.1, this implies the following consequence for interpolation in Gabor spaces:

Corollary C.6.2. Consider three arbitrary points $(x_1, \omega_1), (x_2, \omega_2), (x_3, \omega_3) \in \mathbb{R}^{2n}$, three arbitrary values $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, and any normalized $g \in L^2(\mathbb{R}^n)$. One can always find $f \in L^2(\mathbb{R}^n)$ such that

$$V_g f(x_i, \omega_i) = \lambda_i, \qquad i = 1, 2, 3.$$

Remark. A careful read of the proof of Proposition C.6.1 reveals that the statement is true in the generalized setting. More precisely, let $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation with an admissible vector $g \in \mathcal{H}_\pi$. Then the collection $\{\pi(x)g\}_{x\in G}$ is linearly independent in $\mathcal{H}_\pi$ if and only if the wavelet space $\mathcal{W}_g(\mathcal{H}_\pi) \subset L^2(G)$ is fully interpolating. Hence the problem of whether the wavelet space $\mathcal{W}_g(\mathcal{H}_\pi)$ is fully interpolating is a convenient generalization of the HRT-Conjecture. In this reformulation, Proposition C.5.3 states that the generalized HRT-Conjecture is false for compact or abelian groups. Moreover, the generalized HRT-Conjecture is also false in the classical wavelet setting [98] as a result of the scaling relation in wavelet theory. Another generalization of the HRT-Conjecture is considered in [117].

Recently there has been an effort to prove the HRT-Conjecture for widely spaced index sets [116, 132]. In particular, it is showed in [116, Theorem 1] that the HRT-Conjecture holds for $g \in C_0(\mathbb{R}^n)$ and points $\Omega := \{(x_1, \omega_1), \dots, (x_m, \omega_m)\} \subset \mathbb{R}^{2n}$ that are widely spaced apart relative to the decay of g. Through our approach, we can deduce a similar result without the assumption that $g \in C_0(\mathbb{R}^n)$ since the STFT satisfies $V_g g \in C_0(\mathbb{R}^{2n})$ for all $g \in L^2(\mathbb{R}^n)$.

Corollary C.6.3. Let $g \in L^2(\mathbb{R}^n)$ be a non-zero function. There exists a constant $R > 0$ (depending only on g and $m \in \mathbb{N}$) such that for any collection of points $(x_1, \omega_1), \ldots, (x_m, \omega_m) \in \mathbb{R}^{2n}$ with

$$\min_{i \neq j} \sqrt{(x_j - x_i)^2 + (\omega_j - \omega_i)^2} \geq R, \qquad i, j = 1, \ldots, m, \tag{C.6.2}$$

the time-frequency shifts $\{M_{\omega_k} T_{x_k} g\}_{k=1}^m$ are linearly independent.

Proof. We assume that $\|g\|_{L^2(\mathbb{R}^n)} = 1$ as we can normalize g without altering the linear independence. The claim is equivalent, by Proposition C.6.1, to the fact that the matrix

$$\begin{aligned} \Omega_g &:= \left\{ \left\langle V_g(M_{\omega_j} T_{x_j} g), V_g(M_{\omega_i} T_{x_i} g) \right\rangle \right\}_{i,j=1}^m \\ &= \left\{ e^{-2\pi i x_j \cdot (\omega_i - \omega_j)} V_g g(x_i - x_j, \omega_i - \omega_j) \right\}_{i,j=1}^m \end{aligned}$$

is invertible. Notice that the diagonal terms of Ω_g are all 1's. Since $V_g g$ is continuous and vanishes at infinity, we can find $R > 0$ such that

$$\sum_{j=1}^m |V_g g(x_i - x_j, \omega_i - \omega_j)| \leq 1, \qquad i = 1, \ldots, m,$$

for all points $(x_1, \omega_1), \ldots, (x_m, \omega_m)$ satisfying the condition (C.6.2). This guarantees that the matrix Ω_g is diagonally dominant and hence invertible. □

Remark. We would like to bring up that Proposition C.6.1 is implicitly commented on in the paper [82] through frame theory terminology. More precisely, the author investigates the *Grammian matrix* corresponding to the time-frequency shifts $\{M_{\omega_k} T_{x_k} g\}_{k=1}^m$. The invertibility of the Grammian matrix is easily seen to be equivalent to the statement that the corresponding Gabor space $V_g(L^2(\mathbb{R}^n))$ is fully interpolating. We hope the connection with reproducing kernel Hilbert spaces adds a machinery that can help shed light on some aspects of the HRT-Conjecture.

C.7 Wavelet Completeness

In this final section we will look at how much of $L^2(G)$ the wavelet spaces $\mathcal{W}_g(\mathcal{H}_\pi)$ collectively fill up. Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be a square integrable representation and let $\mathcal{A}_\pi$ denote the equivalence classes of admissible vectors in $\mathcal{H}_\pi$ modulo rotations by elements of $\mathbb{T}$. From Example C.4.1 we see that the collection

$$\operatorname*{span}_{g \in \mathcal{A}_\pi} \left\{ \mathcal{W}_g f : f \in \mathcal{H}_\pi \right\} \subset L^2(G)$$

does not need to be dense in $L^2(G)$. To combat this we will start to vary the square integrable representation π as well. If $\widehat{G}_s$ denotes the equivalence classes of square integrable representations of G, then we consider

$$\bigoplus_{\pi\in\widehat{G}_s} \operatorname*{span}_{g\in\mathcal{A}_\pi}\left\{\mathcal{W}_g f : f\in\mathcal{H}_\pi\right\} \subset L^2(G). \tag{C.7.1}$$

It is straightforward to check that (C.7.1) is a well-defined direct sum, we refer to [69, Lemma 2.24] for details.

Example C.7.1. To make matters more concrete, let us first consider the group $G = \mathbb{T}$. Any unitary representation of $\mathbb{T}$ is equivalent through a unitary intertwining operator to one of the representations $\pi_n : \mathbb{T} \to \mathbb{T}$ for $n \in \mathbb{Z}$ given by

$$\pi_n\big(e^{i\theta}\big) := e^{in\theta}, \quad \theta \in \mathbb{R}.$$

For the representation π_n we see that

$$\mathcal{W}_1 1\big(e^{i\theta}\big) = \big\langle 1, \pi_n\big(e^{i\theta}\big)1\big\rangle = e^{-in\theta}.$$

This gives precisely the Fourier expansion of square integrable periodic functions since

$$\overline{\bigoplus_{\pi\in\widehat{G}_s} \operatorname*{span}_{g\in\mathcal{A}_\pi}\left\{\mathcal{W}_g f : f\in\mathcal{H}_\pi\right\}} = \overline{\bigoplus_{n\in\mathbb{Z}} \operatorname{span}\left\{e^{in\theta} : \theta\in\mathbb{R}\right\}} = L^2(\mathbb{T}).$$

Based on the observations above we formulate the following conjecture for a general locally compact group.

Conjecture (Wavelet Completeness)**.** Characterize the locally compact groups G that satisfy

$$\overline{\bigoplus_{\pi\in\widehat{G}_s} \operatorname*{span}_{g\in\mathcal{A}_\pi}\left\{\mathcal{W}_g f : f\in\mathcal{H}_\pi\right\}} = L^2(G). \tag{C.7.2}$$

We say that a locally compact group G is *wavelet complete* if (C.7.2) holds for G. For wavelet complete groups we can view the decomposition (C.7.2) conceptually as a generalized multiresolution analysis. An obvious condition that needs to be satisfied for G to be wavelet complete is $\widehat{G}_s \neq \emptyset$. Hence the integers $\mathbb{Z}$ and any other abelian non-compact group can not be wavelet complete. Any compact group is easily seen to be wavelet complete from Peter-Weyl theory [67, Theorem 5.11]. The following example illustrates that wavelet completeness is a non-trivial notion.

Proposition C.7.2. The reduced Heisenberg groups $\mathbb{H}_r^n$ are not wavelet complete.

Proof. A variant of the Stone-von Neumann Theorem [81, Corollary 9.3.5] implies that the only square integrable representations of $\mathbb{H}_r^n$ are the Schrödinger representation ρ_r given in (C.3.1) along with appropriate dilations

$$\rho_{r,m}\left(x, \omega, e^{2\pi i\tau}\right) := e^{2\pi i m\tau} e^{\pi i m x\cdot\omega} T_{mx} M_\omega,$$

for $m \in \mathbb{Z} \setminus \{0\}$. We will show that any $h \in L^2(\mathbb{H}_r^n)$ on the form

$$h\big(x, \omega, e^{2\pi i\tau}\big) = h(x, \omega)$$

is orthogonal to $\mathcal{W}_g f$ for all $f, g \in L^2(\mathbb{R}^n)$ and all the representations $\rho_{r,m}$. We compute that

$$\begin{aligned}
\langle h, \mathcal{W}_g f\rangle_{L^2(\mathbb{H}_r^n)} &= \int_{\mathbb{H}_r^n} h(x, \omega)\overline{\mathcal{W}_g f(x, \omega, e^{2\pi i\tau})}\, dx\, d\omega\, d\tau \\
&= \int_0^1 e^{2\pi i m\tau}\, d\tau \int_{\mathbb{R}^{2n}} h(x, \omega) e^{-\pi i m x\cdot\omega}\overline{V_g f(mx, \omega)}\, dx\, d\omega \\
&= 0,
\end{aligned}$$

since $m \in \mathbb{Z} \setminus \{0\}$. □

Paper D

The Affine Wigner Distribution

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Paper D

The Affine Wigner Distribution

Abstract

We examine the affine Wigner distribution from a quantization perspective with an emphasis on the underlying group structure. One of our main results expresses the scalogram as (affine) convolution of affine Wigner distributions. We strive to unite the literature on affine Wigner distributions and we provide the connection to the Mellin transform in a rigorous manner. Moreover, we present an affine ambiguity function and show how this can be used to illuminate properties of the affine Wigner distribution. In contrast with the usual Wigner distribution, we demonstrate that the affine Wigner distribution is never an analytic function.

Our approach naturally leads to several applications, one of which is an approximation problem for the affine Wigner distribution. We show that the deviation for a symbol to be an affine Wigner distribution can be expressed purely in terms of intrinsic operator-related properties of the symbol. Finally, we present a *positivity conjecture* regarding the non-negativity of the affine Wigner distribution.

D.1 Introduction

The most studied quadratic time-frequency representation is the *Wigner distribution* defined by

$$W_f(x,\omega) := \int_{\mathbb{R}^d} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega t}\,dt, \qquad (x,\omega)\in\mathbb{R}^{2d}. \tag{D.1.1}$$

Originally invented by Wigner in [154] almost a century ago, the Wigner distribution is essential in quantum mechanics as it gives the expectation values for Weyl quantization of symbols [52]. In recent decades, the Wigner distribution has found many applications in time-frequency analysis [81, Chapter 4] due to its connections with the short-time Fourier transform $V_g f$ defined precisely in (D.2.4). One of the more surprising connections is the convolution relation

$$|V_g f(x,\omega)|^2 = W_{P(g)} * W_f\,(x,\omega), \tag{D.1.2}$$

where P is the reflection operator $P(g)(x) := g(-x)$. The function

$$\mathrm{Spec}_g f := |V_g f(x,\omega)|^2$$

is called the *spectrogram* of f with window g. The spectrogram is an important tool for analyzing time-frequency content and has been used extensively in the engineering literature since its introduction.

Affine Wigner Distribution

Parallel to the theory of time-frequency analysis is the time-scale (or wavelet) paradigm. Although there have been many attempts at finding a suitable Wigner distribution in the time-scale setting, there is no general consensus in the literature. We will motivate the particular choice of a time-scale Wigner transform

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) := \int_{-\infty}^{\infty} \psi\left(\frac{aue^u}{e^u-1}\right)\overline{\phi\left(\frac{au}{e^u-1}\right)}e^{-2\pi i x u}\,du, \quad (x,a)\in \mathrm{Aff}. \tag{D.1.3}$$

The function $W_{\mathrm{Aff}}^{\psi} := W_{\mathrm{Aff}}^{\psi,\psi}$ is called the *affine Wigner distribution* due to its relation to the *affine group* $\mathrm{Aff} := \mathbb{R}\times\mathbb{R}_+$. It was derived through a quantization procedure in [73]. The authors showed that the affine Wigner distribution satisfies $W_{\mathrm{Aff}}^{\psi}\in L_r^2(\mathrm{Aff})$ for every $\psi\in L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, a^{-1}\,da)$, where $L_r^2(\mathrm{Aff})$ denotes all measurable functions on Aff that are square integrable with respect to the measure $a^{-1}\,da\,dx$.

The affine Wigner distribution W_{Aff}^{ψ} has appeared in the literature several times throughout the years; as a particular Bertrand distribution in [136], and as a tool for studying the quantum mechanics of the Morse potential in [130]. The basic properties of the affine Wigner distribution will be developed in a rigorous manner to fill gaps in the literature. In particular, for all sufficiently nice $\psi\in L^2(\mathbb{R}_+)$ we have the *marginal properties*

$$\int_{-\infty}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\,dx = |\psi(a)|^2 \quad\text{and}\quad \int_0^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\,\frac{da}{a} = |\mathcal{M}(\psi)(x)|^2.$$

The symbol $\mathcal{M}(\psi)(x)$ denotes the *Mellin transform* of $\psi\in L^2(\mathbb{R}_+)$ at the point $x\in\mathbb{R}$ given by

$$\mathcal{M}(\psi)(x) = \mathcal{M}_a(\psi)(x) := \int_0^{\infty}\psi(a)a^{-2\pi i x}\,\frac{da}{a}.$$

Scalogram Representation and the Affine Ambiguity Function

The first significant contribution is to develop a connection between the affine Wigner distribution and the *scalogram* defined by

$$\mathrm{Scal}_g f(x,a) := |\mathcal{W}_g f(x,a)|^2, \quad (x,a) \in \mathrm{Aff}, \tag{D.1.4}$$

where $\mathcal{W}_g f$ denotes the continuous wavelet transform of f with respect to g defined precisely in (D.2.8). By comparing with (D.1.2) in the time-frequency setting, one would expect a simple convolution relation to hold. However, as the group underlying the symmetries in the time-scale case is the non-unimodular affine group, we obtain the following result.

Theorem. *Let $f, g \in L^2(\mathbb{R})$ be such that their Fourier transforms $\widehat{f}$ and $\widehat{g}$ are supported in $\mathbb{R}_+$ and satisfy $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}_+)$. Then*

$$\mathrm{Scal}_g f(x,a) = \left(I\left(W_{\mathrm{Aff}}^{\widehat{g}}\right) *_{\mathrm{Aff}} \Delta W_{\mathrm{Aff}}^{\widehat{f}}\right)\left(\frac{x}{a}, \frac{1}{a}\right), \quad (x,a) \in \mathrm{Aff},$$

where Δ and I denote the modular function and the involution on the affine group, respectively.

We introduce the *affine ambiguity function* A_{Aff}^{ψ} for $\psi \in L^2(\mathbb{R}_+)$ given by

$$A_{\mathrm{Aff}}^{\psi}(x,a) := \int_0^\infty \psi\left(r\sqrt{a}\right) \overline{\psi\left(\frac{r}{\sqrt{a}}\right)} r^{-2\pi i x} \, \frac{dr}{r}, \quad (x,a) \in \mathrm{Aff}.$$

The affine ambiguity function is intimately related to the radar ambiguity function in time-frequency analysis [81, Chapter 4.2]. We will show that the affine Wigner distribution and the affine ambiguity function are related through the Mellin transform by

$$W_{\mathrm{Aff}}^{\psi}(x,a) = \mathcal{M}_y^{-1} \otimes \mathcal{M}_b \left[\left(\frac{\sqrt{b}\log(b)}{b-1}\right)^{2\pi i y} A_{\mathrm{Aff}}^{\psi}(y,b)\right](x,a). \tag{D.1.5}$$

The relation (D.1.5) is used to show that the affine Wigner distribution preserves Schwartz functions.

Analyticity and an Approximation Problem

It turns out that affine Wigner distributions are never analytic functions on the upper half-plane. However, the space $L_r^2(\mathrm{Aff})$ can be completely decomposed into "almost analytic" functions as the following result shows.

Proposition. We have the orthogonal decomposition

$$L_r^2(\mathrm{Aff}) = \bigoplus_{n=2}^{\infty} \mathcal{A}^n(\mathrm{Aff}) \oplus \mathcal{A}^{\perp,n}(\mathrm{Aff}), \tag{D.1.6}$$

where $\mathcal{A}^n(\mathrm{Aff})$ and $\mathcal{A}^{\perp,n}(\mathrm{Aff})$ denote the spaces of pure poly-analytic and pure anti-poly-analytic functions of order n, respectively.

As an application to the theory developed we consider the approximation problem of understanding, for a given $f \in L_r^2(\mathrm{Aff})$, the quantity

$$\inf_{\psi \in L^2(\mathbb{R}_+)} \left\| f - W_{\mathrm{Aff}}^{\psi} \right\|_{L_r^2(\mathrm{Aff})}. \tag{D.1.7}$$

Notice that (D.1.7) measures how far f is from being an affine Wigner distribution. The analogous problem in time-frequency analysis has been recently studied in [12]. For each symbol $f \in L_r^2(\mathrm{Aff})$ there is a Hilbert-Schmidt operator A_f on $L^2(\mathbb{R}_+)$ that is weakly defined by the relation

$$\left\langle A_f \psi, \phi \right\rangle_{L^2(\mathbb{R}_+)} = \left\langle f, W_{\mathrm{Aff}}^{\phi,\psi} \right\rangle_{L_r^2(\mathrm{Aff})}, \quad \psi, \phi \in L^2(\mathbb{R}_+). \tag{D.1.8}$$

The following result shows that the quantity (D.1.7) is linked to how much A_f deviates from being a rank-one operator.

Theorem. *Let $f \in L_r^2(\mathrm{Aff})$ be real-valued. Under a mild eigenvalue assumption on A_f we have*

$$\inf_{\psi \in L^2(\mathbb{R}_+)} \left\| f - W_{\mathrm{Aff}}^{\psi} \right\|_{L_r^2(\mathrm{Aff})} = \sqrt{\|A_f\|_{\mathcal{HS}}^2 - \|A_f\|_{op}^2},$$

where $\|\cdot\|_{\mathcal{HS}}$ and $\|\cdot\|_{op}$ are the Hilbert-Schmidt norm and operator norm, respectively. Moreover, the precise number of distinct minimizers can be deduced from the spectrum of A_f.

Motivation for the Affine Wigner Distribution

It is not immediately obvious why a Wigner distribution W_{Aff} in the affine setting should have the form given in (D.1.3). In [5] the authors define a Wigner distributions W_G on a general Lie group G. In the case of $G = \mathrm{Aff}$ we indeed have that W_G reduces to W_{Aff}. The general Wigner distribution W_G is the canonical choice for a Wigner distribution on G since it naturally related with Fourier transforms on the group. For the affine group, this relation [18, Section 5.1] takes the elegant form

$$A_f = \mathcal{F}_W^{-1} \mathcal{F}_{\mathrm{KO}}^{-1}(f), \qquad f \in L_r^2(\mathrm{Aff}),$$

where $\mathcal{F}_{\mathrm{W}}$ is the *affine Fourier-Wigner transform* and $\mathcal{F}_{\mathrm{KO}}$ is the *affine Fourier-Kirillov transform*. Since the affine Wigner distribution determines the affine Weyl quantization completely, this motivates further investigation into the affine Wigner distribution W_{Aff}.

Further Results

The affine Wigner distribution is developed further in the follow-up paper [18]. Let us mention two results in [18] that can help to additionally motivate the affine Wigner distribution:

Quantization of Coordinate Functions: In [18, Section 3.3] we extend the *affine Weyl quantization* $f \mapsto A_f$ to tempered distributions $f \in \mathcal{S}'(\mathrm{Aff})$. This offers the possibility of rigorously determining the quantizations A_{f_x} and A_{f_a} of the coordinate functions $f_x(x, a) := x$ and $f_a(x, a) := a$. We prove in [18, Theorem 3.11] the commutation relation

$$[A_{f_x}, A_{f_a}] = \frac{1}{2\pi i} A_{f_a}.$$

This is, up to re-normalization, precisely the infinitesimal structure of the affine group. Hence the affine Weyl quantization, and thus the affine Wigner distribution, is intimately linked with the Lie group structure of the affine group.

Cohen Class Operators: In [18, Section 6.3] we develop a theory affine Cohen class operators. This is motivated by the classical Cohen class operators on phase space [81, Section 4.5]. For a reasonable function f on Aff we define the associated *affine Cohen class function* as

$$Q_f(\psi, \phi) := W_{\mathrm{Aff}}^{\psi,\phi} *_{\mathrm{Aff}} \check{f}, \qquad \check{f}(x, a) := f((x, a)^{-1}).$$

This is a special case of an affine Cohen class function Q_S associated to an operator S, where one considers $S = A_f$. It turns out that any bilinear form $Q\colon L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \to L^\infty(\mathrm{Aff})$ is, under a mild continuity requirement, on the form $Q = Q_S$ for some bounded operator $S\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ by [18, Proposition 6.11]. As such, the affine Wigner distribution is essential in developing a well-behaved Cohen class theory on the affine group.

In addition to the two topics above, we show in [18, Proposition 6.2] that the affine Wigner distribution is also related to the localization operators of Daubechies and Paul given in [46]. Finally, in [18, Section 6.2] we relate the affine Wigner distribution to covariant integral quantizations developed by Gazeau and his collaborators in [4, 20, 21, 74, 75, 76].

Structure of the Paper

In Section D.2 we outline necessary definitions and briefly review the affine group as it will be central for many of the results we develop. In Section D.3 we derive basic properties of the affine Wigner distribution. We devote Section D.4 to uniting the literature and pointing out how the affine Wigner distribution can be derived by emphasizing symmetry. The convolution relation between the affine Wigner distribution and the scalogram will be proved in Section D.5.

In Section D.6 we define the affine ambiguity function and show how this allows us to extend the affine Weyl quantization (D.1.8) to the distributional setting. We prove the decomposition (D.1.6) of $L^2_r(\mathrm{Aff})$ in Section D.7. In addition to the approximation problem described above, we show in Section D.8 how basic questions regarding operators on $\mathbb{R}_+$ can be answered with our framework. Finally, we discuss the affine Grossmann-Royer operator and the affine positivity conjecture in Section D.9. The authors are grateful for helpful suggestions from Eirik Skrettingland and Luís Daniel Abreu.

D.2 Preliminaries

The notation $\mathscr{S}(\mathbb{R}^d)$ will be used for the Schwartz space of rapidly decaying smooth functions on $\mathbb{R}^d$. We write $\mathscr{S}(\mathbb{R}_+)$ for the smooth functions $\psi\colon \mathbb{R}_+ \to \mathbb{C}$ such that $\Psi(x) := \psi(e^x) \in \mathscr{S}(\mathbb{R})$. The corresponding dual spaces of tempered distributions are denoted by $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}_+)$, respectively. The Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ is given by

$$\mathcal{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\omega}\,dx, \quad \omega \in \mathbb{R}^d.$$

We will frequently use $L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, a^{-1}\,da)$ since $a^{-1}\,da$ is the Haar measure on $\mathbb{R}_+$.

D.2.1 The Classical Wigner Distribution

We begin by recalling basic definitions from time-frequency analysis and their connection with the Heisenberg group. The *cross-Wigner transform* $W(f,g)$ of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$W(f,g)(x,\omega) := \int_{\mathbb{R}^d} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega t}\,dt, \quad (x,\omega) \in \mathbb{R}^{2d}.$$

Notice that the Wigner distribution W_f given in (D.1.1) is precisely the diagonal term $W(f,f)$. The cross-Wigner transform satisfies the orthogonality property

$$\langle W(f_1,g_1), W(f_2,g_2)\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)}\overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R}^d)}. \qquad \text{(D.2.1)}$$

A key feature of the Wigner distribution is its connection with the Weyl calculus: For a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ the *Weyl (pseudo-differential) operator* L_σ corresponding to the symbol σ is the operator

$$L_\sigma f := \int_{\mathbb{R}^{2d}} e^{-\pi i \xi u} \hat{\sigma}(\xi, u) T_{-u} M_\xi f \, du \, d\xi. \tag{D.2.2}$$

The operators T_{-u} and M_ξ in (D.2.2) are respectively the *time-shift operator* and the *frequency-shift operator* defined by

$$T_x f(t) := f(t-x), \qquad M_\omega f(t) := e^{2\pi i \omega t} f(t), \quad x, \omega, t \in \mathbb{R}^d.$$

The association $\sigma \mapsto L_\sigma$ is called the *Weyl transform* and the operator L_σ maps $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ by [81, Lemma 14.3.1]. Moreover, the Weyl transform is a bijection between square integrable symbols $\sigma \in L^2(\mathbb{R}^{2d})$ and Hilbert-Schmidt operators $L_\sigma \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by a result of Poole [138, Proposition V.1].

The connection between the Weyl calculus and the cross-Wigner transform is the relation

$$\langle L_\sigma f, g \rangle_{L^2(\mathbb{R}^d)} = \langle \sigma, W(g, f) \rangle_{L^2(\mathbb{R}^{2d})},$$

for $\sigma \in L^2(\mathbb{R}^{2d})$ and $f, g \in L^2(\mathbb{R}^d)$. Since the Weyl transform is a quantization procedure, one can think of the inverse transformation $L_\sigma \mapsto \sigma$ as *dequantization*. In this terminology, the Wigner distribution W_f for $f \in L^2(\mathbb{R}^d)$ is the dequantization of the rank-one operator

$$L_{W_f} g := \langle g, f \rangle f, \qquad g \in L^2(\mathbb{R}^d). \tag{D.2.3}$$

The reader can consult [92, Chapter 13] and [78, Chapter 4] for more details about the Weyl transform from a quantum mechanical perspective.

Central to time-frequency analysis is the *short-time Fourier transform* $V_g f$ of $f, g \in L^2(\mathbb{R}^d)$ given by

$$V_g f(x, \omega) := \langle f, M_\omega T_x g \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} \, dt. \tag{D.2.4}$$

We have from [81, Lemma 4.3.1] that the cross-Wigner transform and the short-time Fourier transform is related by the formula

$$W(f, g)(x, \omega) = 2^d e^{4\pi i x \omega} V_{P(g)} f(2x, 2\omega),$$

where $P(g)(x) := g(-x)$. The short-time Fourier transform originates from the Schrödinger representation of the Heisenberg group, see [81, Chapter 9] for details.

D.2.2 The Affine Group

The two main operators in time-scale analysis are the time-shift operator T_x and the *dilation operator* D_a given by

$$D_a f(x) := \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right), \tag{D.2.5}$$

for $a > 0$ and $f \in L^2(\mathbb{R})$. One defines the *affine group* as $\mathrm{Aff} := (\mathbb{R} \times \mathbb{R}_+, \cdot_{\mathrm{Aff}})$, where the group operation is given by

$$(x, a) \cdot_{\mathrm{Aff}} (y, b) := (x + ay, ab), \quad (x, a), (y, b) \in \mathrm{Aff}.$$

The motivation for the group operation stems from calculation

$$(T_x D_a)(T_y D_b) = T_x T_{ay} D_a D_b = T_{x+ay} D_{ab}.$$

We can represent the affine group Aff and its Lie algebra $\mathfrak{aff}$ in the matrix form

$$\mathrm{Aff} = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} : a > 0,\, x \in \mathbb{R} \right\}, \quad \mathfrak{aff} = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} : u, v \in \mathbb{R} \right\}.$$

Essential for computations is the fact that the exponential map $\exp\colon \mathfrak{aff} \to \mathrm{Aff}$ given by

$$\exp \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^u & \frac{v(e^u - 1)}{u} \\ 0 & 1 \end{pmatrix}$$

is a global diffeomorpism. The left Haar measure on Aff is given by $a^{-2}\, da\, dx$, while the right Haar measure is $a^{-1}\, da\, dx$. We will use the notation $L^2_r(\mathrm{Aff})$ and $L^2_l(\mathrm{Aff})$ to indicate if we are using the right or left Haar measure, respectively. The left and right Haar measures on Aff can be written in the coordinates induced by the exponential map as

$$\frac{da\, dx}{a^2} = \frac{du\, dv}{\lambda(u)}, \qquad \frac{da\, dx}{a} = \frac{du\, dv}{\lambda(-u)},$$

where the function λ is given by

$$\lambda(u) := \frac{ue^u}{e^u - 1} = \frac{ue^{\frac{u}{2}}}{2\sinh(\frac{u}{2})}. \tag{D.2.6}$$

A natural way the affine group can act on $L^2(\mathbb{R})$ is by translations and dilations, namely as

$$f \longmapsto T_x D_a f, \qquad f \in L^2(\mathbb{R}). \tag{D.2.7}$$

This is a unitary representation, although it is not irreducible. The matrix coefficients of this representation are given by

$$\mathcal{W}_g f(x,a) := \langle f, T_x D_a g\rangle_{L^2(\mathbb{R})} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(y)\overline{g\left(\frac{y-x}{a}\right)}\, dy. \tag{D.2.8}$$

One typically refer to the map $(x,a) \mapsto \mathcal{W}_g f(x,a)$ as the *(continuous) wavelet transform* of f with respect to g. The continuous wavelet transform is analogous to the short-time Fourier transform and incorporates the possibility of observing f at different scales through g. Moreover, the magnifying aspect coming from the change of scales can characterize local regularity through decay properties of the wavelet transform, see [44, Theorem 2.9.2].

D.2.3 A Quantization Approach

We will briefly outline a procedure described in [73] to determine the affine Wigner distribution. The theory is based on Kirillov's theory of coadjoint orbits and we refer further explanations to the aforementioned paper.

The affine group Aff acts on its Lie algebra $\mathfrak{aff}$ through the *adjoint action*

$$\mathrm{Ad}_{(x,a)}(X) := \begin{pmatrix} u & av - xu \\ 0 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{aff},\ (x,a) \in \mathrm{Aff}. \tag{D.2.9}$$

A representation Φ of a Lie group G on a vector space V is always accompanied by a representation Φ^* of G on the dual space V^* defined by

$$\langle \Phi(g)^*\eta, v\rangle := \langle \eta, \Phi(g^{-1})v\rangle, \quad g \in G,\ v \in V,\ \eta \in V^*,$$

where the bracket denotes the natural pairing between V and V^*. In the case of the adjoint action in (D.2.9) we denote the accompanied representation on $\mathfrak{aff}^*$ by Ad^* and call it the *coadjoint representation* of the affine group. We can realize $\mathfrak{aff}^*$ as matrices on the form

$$\mathfrak{aff}^* \simeq \left\{ (x,y) := \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Any point on the form $(x,0) \in \mathfrak{aff}^*$ is a fixed point for the coadjoint representation. The upper and lower half-planes

$$\mathcal{H}_+ := \{(x,y) \in \mathfrak{aff}^* : y > 0\}, \qquad \mathcal{H}_- := \{(x,y) \in \mathfrak{aff}^* : y < 0\},$$

both constitute distinct orbits. For reasons of symmetry it suffices to understand the representation corresponding to $\mathcal{H}_+$. It is convenient to identify $\mathcal{H}_+ \simeq \mathrm{Aff}$ as sets and use the notation (x,a) for a general element in $\mathcal{H}_+$. From general coadjoint orbit theory [112, Chapter 1.2] it follows that Aff is equipped with a canonical symplectic structure. In fact, this symplectic structure is simply the right Haar measure $a^{-1}\, da\, dx$ on Aff.

The main idea of Kirillov's theory is to associate irreducible representations of the Lie group to orbits of the coadjoint representation in a one-to-one manner. A realization of the representation corresponding to $\mathcal{H}_+$ is given by acting on $\psi \in L^2(\mathbb{R}_+)$ by

$$U(x,a)\psi(r) := e^{2\pi i x r}\psi(ar) = \frac{1}{\sqrt{a}} M_x D_{\frac{1}{a}}\psi(r). \tag{D.2.10}$$

The representation U is (up to a normalization) the representation (D.2.7) on the Fourier side. Define the *Stratonovich -Weyl operator* on $L^2(\mathbb{R}_+)$ by the formula

$$\Omega(x,a)\psi(r) := a \int_{\mathbb{R}^2} e^{-2\pi i(xu+av)} U\left(\frac{ve^u}{\lambda(u)}, e^u\right)\psi(r)\, du\, dv,$$

where $\psi \in L^2(\mathbb{R}_+)$, $(x,a) \in \mathrm{Aff}$, and λ is the function defined in (D.2.6). The following result is given in [73, Corollary 4.3].

Proposition D.2.1. There is an isometric isomorphism between $L^2_r(\mathrm{Aff})$ and the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$. The isomorphism sends the function $f \in L^2_r(\mathrm{Aff})$ to the operator A_f on $L^2(\mathbb{R}_+)$ defined by

$$A_f\psi(r) := \int_{-\infty}^{\infty}\int_0^{\infty} f(x,a)\Omega(x,a)\psi(r)\, \frac{da\, dx}{a}.$$

The association $f \mapsto A_f$ is called *affine Weyl quantization*, while the direction $A_f \mapsto f$ is referred to as *affine dequantization*. Moreover, we call f the *(affine) symbol* of A_f. Recall that any Hilbert-Schmidt operator A on $L^2(\mathbb{R}_+)$ has an associated integral kernel $A_K \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ so that

$$A\psi(r) = \int_0^{\infty} A_K(r,s)\psi(s)\, \frac{ds}{s},$$

for all $\psi \in L^2(\mathbb{R}_+)$. If $A = A_f$, then one can recover $f \in L^2_r(\mathrm{Aff})$ from the formula

$$f(x,a) = \int_{-\infty}^{\infty} A_K\left(a\lambda(u), a\lambda(-u)\right) e^{-2\pi i x u}\, du.$$

Motivated by (D.2.3), the affine Wigner distribution should be defined as the affine dequantization of a rank-one operator. Hence we have the following definition.

Definition D.2.2. The *affine cross-Wigner transform* acts on $\psi, \phi \in L^2(\mathbb{R}_+)$ by

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) := \int_{-\infty}^{\infty} \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}e^{-2\pi i x u}\,du$$

for $(x,a) \in \mathrm{Aff}$. We refer to the diagonal $W_{\mathrm{Aff}}^{\psi} := W_{\mathrm{Aff}}^{\psi,\psi}$ as the *affine Wigner distribution* of ψ.

If $f \in L_r^2(\mathrm{Aff})$ is the symbol of the Hilbert-Schmidt operator A_f acting on $L^2(\mathbb{R}_+)$, then

$$\left\langle A_f\psi, \phi\right\rangle_{L^2(\mathbb{R}_+)} = \left\langle f, W_{\mathrm{Aff}}^{\phi,\psi}\right\rangle_{L_r^2(\mathrm{Aff})}, \quad \psi,\phi \in L^2(\mathbb{R}_+). \tag{D.2.11}$$

D.3 Basic Properties

We now derive some basic properties of the affine cross-Wigner transform. The affine cross-Wigner transform is related to the isometry

$$\Pi\colon L^2(\mathbb{R}^+ \times \mathbb{R}^+, (rs)^{-1}\,dr\,ds) \to L_r^2(\mathrm{Aff})$$

given by

$$\Pi(F)(u,a) := F(a\lambda(u), a\lambda(-u)).$$

Lemma D.3.1. *The affine cross-Wigner transform can be factorized as*

$$W_{\mathrm{Aff}}^{\psi,\phi} = \mathcal{F}_1\Pi(\psi \otimes \overline{\phi}), \quad \psi,\phi \in L^2(\mathbb{R}_+),$$

where $\mathcal{F}_1$ is the Fourier transform in the first component and $\psi\otimes\overline{\phi}(r,s) := \psi(r)\overline{\phi(s)}$ for $r, s \in \mathbb{R}_+$.

The factorization in Lemma D.3.1 is key for understanding essential properties of the affine cross-Wigner transform. We illustrate its use by extending the orthogonality property of the classical Wigner distribution in (D.2.1) to the affine setting.

Proposition D.3.2. The affine Wigner distribution satisfies the orthogonality relation

$$\int_{-\infty}^{\infty}\int_0^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\overline{W_{\mathrm{Aff}}^{\phi}(x,a)}\,\frac{da\,dx}{a} = |\langle\psi,\phi\rangle|^2, \tag{D.3.1}$$

for $\psi, \phi \in L^2(\mathbb{R}_+)$.

Proof. We use the factorization in Lemma D.3.1 and obtain

$$\begin{aligned}
\left\langle W_{\mathrm{Aff}}^{\psi}, W_{\mathrm{Aff}}^{\phi} \right\rangle_{L_r^2(\mathrm{Aff})} &= \left\langle \mathcal{F}_1 \Pi \left(\psi \otimes \overline{\psi}\right), \mathcal{F}_1 \Pi \left(\phi \otimes \overline{\phi}\right) \right\rangle_{L_r^2(\mathrm{Aff})} \\
&= \left\langle \Pi \left(\psi \otimes \overline{\psi}\right), \Pi \left(\phi \otimes \overline{\phi}\right) \right\rangle_{L_r^2(\mathrm{Aff})} \\
&= \left\langle \psi \otimes \overline{\psi}, \phi \otimes \overline{\phi} \right\rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}^+, (rs)^{-1}\, dr\, ds)} \\
&= |\langle \psi, \phi \rangle|^2. \qquad \square
\end{aligned}$$

We will refer to (D.3.1) as the *affine orthogonality relation* motivated by the analogous result for the classical Wigner distribution in (D.2.1). Through a different (but ultimately equivalent) approach to the affine Wigner distribution taken in [24] and [130], the affine orthogonality relation is already known. The usefulness of the affine orthogonality relation can be readily demonstrated with the following two corollaries.

Corollary D.3.3. Let $\{\psi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}_+)$. Then the collection $\{W_{\mathrm{Aff}}^{\psi_n, \psi_m}\}_{n,m \in \mathbb{N}}$ is an orthonormal basis for $L_r^2(\mathrm{Aff})$. In particular, we can expand any $f \in L_r^2(\mathrm{Aff})$ as

$$f = \sum_{n,m=0}^{\infty} \left\langle f, W_{\mathrm{Aff}}^{\mathcal{L}_n, \mathcal{L}_m} \right\rangle W_{\mathrm{Aff}}^{\mathcal{L}_n, \mathcal{L}_m},$$

where $\{\mathcal{L}_n\}_{n=0}^{\infty}$ is given by

$$\mathcal{L}_n(x) := \frac{e^{\frac{x}{2}}}{n! \sqrt{n+1}} \frac{d^n}{dx^n} \left(e^{-x} x^{n+1} \right). \tag{D.3.2}$$

Proof. The orthonormality of the functions $W_{\mathrm{Aff}}^{\psi_n, \psi_m}$ clearly follows from Proposition D.3.2. To see the completeness in $L_r^2(\mathrm{Aff})$ we assume that $f \in L_r^2(\mathrm{Aff})$ satisfies

$$\left\langle f, W_{\mathrm{Aff}}^{\psi_n, \psi_m} \right\rangle_{L_r^2(\mathrm{Aff})} = 0$$

for every $n, m \in \mathbb{N}$. Then equation (D.2.11) implies that $A_f = 0$ and hence $f \equiv 0$. $\square$

Corollary D.3.4. We have $W_{\mathrm{Aff}}^{\psi} = W_{\mathrm{Aff}}^{\phi}$ for $\psi, \phi \in L^2(\mathbb{R}_+)$ if and only if $\psi = c \cdot \phi$ with $|c| = 1$.

Proof. It is clear from the definition of W_{Aff} that $\psi = c \cdot \phi$ with $|c| = 1$ implies that $W_{\mathrm{Aff}}^{\psi} = W_{\mathrm{Aff}}^{\phi}$. Conversely, if we assume that $W_{\mathrm{Aff}}^{\psi} = W_{\mathrm{Aff}}^{\phi}$ then the affine orthogonality relation (D.3.1) shows that

$$|\langle \psi, \phi \rangle|^2_{L^2(\mathbb{R}_+)} = \|\psi\|^4_{L^2(\mathbb{R}_+)} = \|\phi\|^4_{L^2(\mathbb{R}_+)}.$$

Hence $\|\phi\|_{L^2(\mathbb{R}_+)} = \|\psi\|_{L^2(\mathbb{R}_+)}$ and $|\langle\psi, \phi\rangle|_{L^2(\mathbb{R}_+)} = \|\psi\|_{L^2(\mathbb{R}_+)}\|\phi\|_{L^2(\mathbb{R}_+)}$. This can only happen when $\psi = c \cdot \phi$ for some $|c| = 1$. □

The *marginal properties* [81, Lemma 4.3.6] for the classical Wigner distribution strengthen a quantum mechanical interpretation of the Wigner distribution. For the affine Wigner distribution, we need an analogue of the Fourier transform on the group $\mathbb{R}^+$. This is the *Mellin transform* given by

$$\mathcal{M}(\psi)(x) = \mathcal{M}_a(\psi)(x) := \int_0^\infty \psi(a)a^{-2\pi ix}\,\frac{da}{a},$$

for $x \in \mathbb{R}$ and $\psi \in L^2(\mathbb{R}_+)$. There is little consensus regarding the exponent of a in the literature and we recommend checking carefully which convention is used whenever the Mellin transform is encountered. The Mellin transform is a unitary map $\mathcal{M}\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ with inverse

$$\mathcal{M}^{-1}(f)(a) = \mathcal{M}_x^{-1}(f)(a) = \int_{-\infty}^\infty f(x)a^{2\pi ix}\,dx, \tag{D.3.3}$$

for $a \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R})$. Moreover, the Mellin transform of a dilated function satisfies

$$\mathcal{M}(D_r\psi)(x) = r^{-2\pi ix-\frac{1}{2}}\mathcal{M}(\psi)(x). \tag{D.3.4}$$

The following marginal properties have been stated in [143] where the proofs are referred to the unpublished Ph.D. thesis of R. G. Shenoy. We provide a new proof of this remarkable fact to fill in gaps in the original sources.

Proposition D.3.5. The affine Wigner distribution satisfies for $\psi \in \mathcal{S}(\mathbb{R}_+)$ the marginal properties

$$\int_{-\infty}^\infty W_{\mathrm{Aff}}^\psi(x,a)\,dx = |\psi(a)|^2,$$
$$\int_0^\infty W_{\mathrm{Aff}}^\psi(x,a)\,\frac{da}{a} = |\mathcal{M}(\psi)(x)|^2.$$

Proof. The first marginal property follows from Lemma D.3.1 and the realization that

$$\int_{-\infty}^\infty W_{\mathrm{Aff}}^\psi(x,a)\,dx = \mathcal{F}_1^{-1}\left(W_{\mathrm{Aff}}^\psi\right)(0,a).$$

The validity of the pointwise convergence in the Fourier inversion step is clear since $\psi \in \mathcal{S}(\mathbb{R}_+)$.

For the second marginal property, we utilize a change of variables in the definition of the affine Wigner distribution to get the alternative form

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) = \int_0^\infty u^{-2\pi ix}\psi\left(a\frac{u\log(u)}{u-1}\right)\overline{\phi\left(a\frac{\log(u)}{u-1}\right)}\,\frac{du}{u}.$$

The isometry property of the Mellin transform can then be used to obtain

$$\int_0^\infty W_{\text{Aff}}^{\psi}(x,a)\,\frac{da}{a} =$$
$$\int_0^\infty \int_{-\infty}^\infty u^{-2\pi i x}\mathcal{M}_a\left(\psi\left(a\frac{u\log(u)}{u-1}\right)\right)(\beta)\overline{\mathcal{M}_a\left(\psi\left(a\frac{\log(u)}{u-1}\right)\right)(\beta)}\,\frac{d\beta\,du}{u}.$$

By using the dilation relation (D.3.4) and the inverse Mellin transform (D.3.3) we end up with

$$\begin{aligned}
\int_0^\infty W_{\text{Aff}}^{\psi}(x,a)\,\frac{da}{a} &= \int_0^\infty \int_{-\infty}^\infty u^{-2\pi i x}u^{2\pi i\beta}|\mathcal{M}_a(\psi)(\beta)|^2\,\frac{d\beta\,du}{u}\\
&= \int_0^\infty u^{-2\pi i x}\mathcal{M}_\beta^{-1}(|\mathcal{M}_a(\psi)(\beta)|^2)(u)\,\frac{du}{u}\\
&= \mathcal{M}_u(\mathcal{M}_\beta^{-1}(|\mathcal{M}_a(\psi)(\beta)|^2)(u))(x)\\
&= |\mathcal{M}(\psi)(x)|^2.
\end{aligned}$$

Interchanging the order of integration and the pointwise convergence of the Mellin transform is easily justified under the assumption that $\psi \in \mathcal{S}(\mathbb{R}_+)$. □

Remark. It follows from Proposition D.3.5 that

$$\int_0^\infty \int_{-\infty}^\infty W_{\text{Aff}}^{\psi}(x,a)\,\frac{da\,dx}{a} = \int_0^\infty |\psi(a)|^2\,\frac{da}{a} = \|\psi\|^2_{L^2(\mathbb{R}_+)},$$

for all ψ in the dense subspace $\mathcal{S}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$. If $\|\psi\|_{L^2(\mathbb{R}_+)} = 1$ and W_{Aff}^{ψ} is everywhere non-negative, then the affine Wigner distribution would be a probability density function on the upper half-plane. We will elaborate on this in Section D.9.

If $\psi \in \mathcal{S}(\mathbb{R}_+)$ has compact support and $a \in \mathbb{R}_+$ is outside the support of ψ, then Proposition D.3.5 shows that

$$\int_{-\infty}^\infty W_{\text{Aff}}^{\psi}(x,a)\,dx = 0.$$

This extreme case can be improved with the following *finite support property*.

Proposition D.3.6. Assume $\psi \in L^2(\mathbb{R}_+)$ is continuous and supported in the interval $[r,s] \subset \mathbb{R}_+$. Then $W_{\text{Aff}}^{\psi}(x,a) = 0$ for all $x \in \mathbb{R}$ whenever $a \notin [r,s]$.

Proof. The functions $\psi(a\lambda(u))$ and $\psi(a\lambda(-u))$ are both non-zero if and only if

$$\lambda(u), \lambda(-u) \in L := \left[\frac{r}{a}, \frac{s}{a}\right].$$

If $a > s$ then $L \subset (0,1)$. Hence it suffices to show that $\lambda(u)$ and $\lambda(-u)$ can not take values in $(0,1)$ simultaneously. This follows since $\lambda(u)$ is an increasing function that only takes values in $(0,1)$ whenever $u < 0$. If $a < r$ then $L \subset (1,\infty)$. In this case, the result follows from the fact that $\lambda(u) > 1$ if and only if $u > 0$. □

D.4 Alternative Descriptions

Although the affine Wigner distribution was constructed rather recently, it has appeared in the literature several times in different disguises. We outline two instances of this and see how this enriches our understanding of the more subtle properties of the affine Wigner distribution.

Consider a function $\psi \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}_+)$ that is supported on $\mathbb{R}_+$ and let $f \in L^2(\mathbb{R})$ be such that $\hat{f} = \psi$. The affine Wigner distribution W_{Aff}^{ψ} is related to the Bertrand $P := (P_0, 1)$ distribution described in [136] by the formula

$$W_{\mathrm{Aff}}^{\psi}(x, a) = \frac{1}{a} P f \left(a, -\frac{x}{a}\right).$$

One refers to P as the *Bertrand P_0 distribution* and it is in both the *affine class* and the *hyperbolic class* described in [136]. From this we can gauge several invariance properties of the affine Wigner distribution:

- The fact that P is in the affine class gives the invariance properties

  $$W_{\mathrm{Aff}}^{M_\omega \psi}(x, a) = W_{\mathrm{Aff}}^{\psi}(x - a\omega, a)$$

  and

  $$W_{\mathrm{Aff}}^{D_r \psi}(x, a) = \frac{1}{r} W_{\mathrm{Aff}}^{\psi}\left(x, \frac{a}{r}\right). \tag{D.4.1}$$

  These invariance properties can be summarized as

  $$W_{\mathrm{Aff}}^{U(x,a)\psi}(y, b) = W_{\mathrm{Aff}}^{\psi}(y - bx, ab), \tag{D.4.2}$$

  where U is the action of the affine group on $L^2(\mathbb{R}_+)$ given in (D.2.10).

- The fact that P is in the hyperbolic class gives the invariance property

  $$W_{\mathrm{Aff}}^{\mathcal{H}(c, f_r)\psi}(x, a) = W_{\mathrm{Aff}}^{\psi}(x + c, a), \tag{D.4.3}$$

  where $\mathcal{H}(c, f_r)$ is the transformation

  $$\mathcal{H}(c, f_r)\psi(r) := e^{-2\pi i c \ln\left(\frac{r}{f_r}\right)} \psi(r), \quad r, f_r > 0,\ c \in \mathbb{R}.$$

  Notice that the *positive reference frequency* f_r only appears on the left-hand side of (D.4.3).

The affine Wigner distribution W_{Aff} can be derived in another way by emphasizing invariance properties as done in [24] and [130]. From this perspective, one starts with a general quadratic distribution and require invariance under a group extension of the affine group. This will produce the distribution

$$W^{\psi}(x,a) := \int_{-\infty}^{\infty} \psi(a\lambda(u))\overline{\psi(a\lambda(-u))}e^{-2\pi iux}\mu(u)\,du,$$

where $\mu(u)$ is a weight function that satisfies $\overline{\mu(u)} = \mu(-u)$. The requirement that W^{ψ} satisfies the affine orthogonality relation in (D.3.1) forces $\mu \equiv 1$ so that $W^{\psi} = W^{\psi}_{\mathrm{Aff}}$. Although one gets the orthogonality relation (D.3.1) for free with this approach, the connection with the affine Weyl quantization in (D.2.11) is then obscured. The affine Wigner distribution W_{Aff} is a special case of a family of distributions that are called *tomographic distributions* in [24].

Remark. There have been other attempts at defining a notion of affine Wigner distribution that do not coincide with our definition. As an example, we refer the reader to [76] and the recent successor paper [75] where an affine Wigner-like quasi-probability is defined through a semi-classical quantization approach. Although this is different from the approach in [73] that our work is based on, it has similarities in both motivation and properties.

D.5 Affine Convolution Representation of the Scalogram

Recall from the introduction that the classical Wigner distribution can represent the spectrogram through convolution

$$\mathrm{Spec}_g f(x,\omega) = W_{P(g)} * W_f(x,\omega) = W_{P(\hat{g})} * W_{\hat{f}}(\omega,-x), \tag{D.5.1}$$

where $P(g)(x) := g(-x)$. This relation was mentioned in [129, Eq 85] where the Wigner distribution went under the name *(instantaneous) spectrum-smoothing function*. It later appeared in [39, Eq 4.5], where it was used to show that the spectrogram is a Cohen class distribution. Finally, it was put on more rigorous foundations in [68, Proposition 1.99]. By attempting to use the classical Wigner distribution to represent the scalogram in (D.1.4) one obtains

$$\mathrm{Scal}_g f(x,a) = \int_{\mathbb{R}^2} W_f(\tau,\xi)W_g\left(\frac{\tau - x}{a}, a\xi\right)\,d\tau\,d\xi.$$

However, this only superficially looks like convolution as it does not incorporate one of the Haar measures on Aff.

We will use the affine Wigner distribution to get a proper convolution representation of the scalogram. Before stating the precise result, we recall some generalities from the theory of locally compact groups applied to the affine group:

The *affine convolution* between two functions f, g on the affine group is given whenever it is well-defined by

$$f *_{\mathrm{Aff}} g(x,a) := \int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,b) g\left((y,b)^{-1} \cdot_{\mathrm{Aff}} (x,a)\right) \frac{db\,dy}{b^2}.$$

A departure from the usual Euclidean convolution is that the affine convolution is not commutative.

The *modular function* Δ on any locally compact group measures the difference between the right and left Haar measure. We refer the reader to a precise definition in [67, Chapter 2.4] as we only need that the modular function on the affine group is

$$\Delta(x,a) = \frac{1}{a}, \qquad (x,a) \in \mathrm{Aff}.$$

Finally, the *(right) involution* of a function f on the affine group is given by

$$I(f)(x,a) := \Delta(x,a)\overline{f((x,a)^{-1})} = \frac{1}{a}\overline{f\left(-\frac{x}{a}, \frac{1}{a}\right)}, \qquad (x,a) \in \mathrm{Aff}.$$

The following convolution result should be compared with (D.5.1).

Theorem D.5.1. *Let $f, g \in L^2(\mathbb{R})$ be such that their Fourier transforms $\widehat{f}$ and $\widehat{g}$ are supported in $\mathbb{R}_+$ and satisfy $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}_+)$. Then*

$$\mathrm{Scal}_g f(x,a) = \left(I\left(W_{\mathrm{Aff}}^{\widehat{g}}\right) *_{\mathrm{Aff}} \Delta W_{\mathrm{Aff}}^{\widehat{f}}\right)\left(\frac{x}{a}, \frac{1}{a}\right), \quad (x,a) \in \mathrm{Aff}.$$

Proof. By using Parseval's identity and that the support of the Fourier transforms are in $\mathbb{R}_+$ we obtain

$$\mathrm{Scal}_g f(x,a) = \left|\langle f, T_x D_a g\rangle_{L^2(\mathbb{R})}\right|^2 = \left|\langle \widehat{f}, \sqrt{a} \cdot U(x,a)\widehat{g}\rangle_{L^2(\mathbb{R}_+)}\right|^2,$$

where $U(x,a)$ is given in (D.2.10). The affine orthogonality relation given in Proposition D.3.2 and the invariance property given in (D.4.2) together show that

$$\begin{aligned}
\mathrm{Scal}_g f(x,a) &= \int_{-\infty}^{\infty}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\widehat{f}}(y,b) \cdot a \cdot W_{\mathrm{Aff}}^{U(x,a)\widehat{g}}(y,b)\, \frac{db\,dy}{b} \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\widehat{f}}(y,b) \cdot a \cdot W_{\mathrm{Aff}}^{\widehat{g}}(y - bx, ab)\, \frac{db\,dy}{b} \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\widehat{f}}(y,b) \cdot ab \cdot W_{\mathrm{Aff}}^{\widehat{g}}\left((y,b) \cdot_{\mathrm{Aff}} (-x,a)\right) \frac{db\,dy}{b^2}.
\end{aligned}$$

We use the involution on the affine group to write

$$ab \cdot W_{\mathrm{Aff}}^{\widehat{g}}\left((y,b) \cdot_{\mathrm{Aff}} (-x,a)\right) = I\left(W_{\mathrm{Aff}}^{\widehat{g}}\right)\left((-x,a)^{-1} \cdot_{\mathrm{Aff}} (y,b)^{-1}\right).$$

Combining these observations shows that

$$\begin{aligned}\operatorname{Scal}_g f(x,a) &= \int_{-\infty}^{\infty}\int_0^{\infty} \frac{W_{\mathrm{Aff}}^{\widehat{f}}(y,b)}{b} \cdot I\left(W_{\mathrm{Aff}}^{\widehat{g}}\right)\left(\left(\frac{x}{a},\frac{1}{a}\right)\cdot_{\mathrm{Aff}}(y,b)^{-1}\right)\frac{db\,dy}{b} \\ &= \left(I\left(W_{\mathrm{Aff}}^{\widehat{g}}\right) *_{\mathrm{Aff}} \Delta W_{\mathrm{Aff}}^{\widehat{f}}\right)\left(\frac{x}{a},\frac{1}{a}\right). \qquad \square\end{aligned}$$

D.6 The Affine Ambiguity Function

The *cross-ambiguity function* in time-frequency analysis of $f, g \in L^2(\mathbb{R})$ is defined to be

$$A(f,g)(x,\omega) := \int_{-\infty}^{\infty} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}e^{-2\pi i t\omega}\,dt, \qquad (x,\omega)\in\mathbb{R}^2.$$

The *ambiguity function* $Af := A(f,f)$ of $f \in L^2(\mathbb{R})$ has been frequently used in radar applications [81, Chapter 4.2]. In the affine setting, we suggest the following analogue.

Definition D.6.1. The *affine cross-ambiguity function* of $\psi, \phi \in L^2(\mathbb{R}_+)$ is the function $A_{\mathrm{Aff}}^{\psi,\phi}$ on Aff defined by

$$A_{\mathrm{Aff}}^{\psi,\phi}(x,a) := \int_0^{\infty} \psi\left(r\sqrt{a}\right)\overline{\phi\left(\frac{r}{\sqrt{a}}\right)}r^{-2\pi i x}\,\frac{dr}{r}, \quad (x,a)\in \mathrm{Aff}.$$

Similarly as before, we call the function $A_{\mathrm{Aff}}^{\psi} := A_{\mathrm{Aff}}^{\psi,\psi}$ the *affine ambiguity function*.

In [143] the authors define a different notion of affine ambiguity function under the name *wide-band ambiguity function*. Notice that the definition of $A_{\mathrm{Aff}}^{\psi,\phi}$ incorporates the Haar measure on $\mathbb{R}_+$ in a natural way. Moreover, we will show that our definition possesses properties that justifies the terminology *affine ambiguity function*.

Lemma D.6.2. *For $\psi, \phi \in L^2(\mathbb{R}_+)$ we define the functions $\Psi(x) := \psi(e^x)$ and $\Phi(x) := \phi(e^x)$ for $x \in \mathbb{R}$. Then*

$$A_{\mathrm{Aff}}^{\psi,\phi}(\omega, e^x) = A(\Psi,\Phi)(x,\omega), \qquad (x,\omega)\in\mathbb{R}^2.$$

Moreover, the affine ambiguity function satisfies

$$|A_{\mathrm{Aff}}^{\psi}(x,a)| < A_{\mathrm{Aff}}^{\psi}(0,1) = \|\psi\|_{L^2(\mathbb{R}_+)}^2,$$

for every $(x,a) \neq (0,1)$.

The last statement in Lemma D.6.2 is a consequence of [81, Lemma 4.2.1]. We will use Lemma D.6.2 in the proof of the following result.

Proposition D.6.3. The affine cross-ambiguity function satisfies the orthogonality relation

$$\left\langle A^{\psi_1,\phi_1}_{\mathrm{Aff}}, A^{\psi_2,\phi_2}_{\mathrm{Aff}} \right\rangle_{L^2_r(\mathrm{Aff})} = \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}_+)} \overline{\langle \phi_1, \phi_2 \rangle}_{L^2(\mathbb{R}_+)},$$

for $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}_+)$.

Proof. Let $\Psi_i(x) := \psi_i(e^x)$ and $\Phi_i(x) := \phi_i(e^x)$ for $i = 1, 2$. Then Lemma D.6.2 gives that

$$\left\langle A^{\psi_1,\phi_1}_{\mathrm{Aff}}, A^{\psi_2,\phi_2}_{\mathrm{Aff}} \right\rangle_{L^2_r(\mathrm{Aff})} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\Psi_1, \Phi_1)(u, x) \overline{A(\Psi_2, \Phi_2)(u, x)}\, du\, dx.$$

From [81, Lemma 4.3.4] it follows that the ambiguity function is related to the classical cross-Wigner transform by

$$W(\Psi_i, \Phi_i) = \mathcal{F}\mathcal{U}A(\Psi_i, \Phi_i), \quad i = 1, 2,$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{U}$ is the rotation $\mathcal{U}F(x, \omega) := F(\omega, -x)$ for a function F on $\mathbb{R}^2$. Hence from (D.2.1) we obtain that

$$\left\langle A^{\psi_1,\phi_1}_{\mathrm{Aff}}, A^{\psi_2,\phi_2}_{\mathrm{Aff}} \right\rangle_{L^2_r(\mathrm{Aff})} = \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}_+)} \overline{\langle \phi_1, \phi_2 \rangle}_{L^2(\mathbb{R}_+)}. \qquad \square$$

Corollary D.6.4. Let $\psi \in L^2(\mathbb{R}_+)$ be normalized and let $U \subset \mathrm{Aff}$ be a Borel set. Assume there exists $\epsilon > 0$ such that

$$\iint_U |A^{\psi}_{\mathrm{Aff}}(x, a)|^2 \,\frac{da\, dx}{a} \geq 1 - \epsilon. \tag{D.6.1}$$

Then the right Haar measure $\mu_r(U)$ of U satisfies

$$\mu_r(U) \geq (1 - \epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2}{p-2}}, \qquad p > 2.$$

In particular, we have $\mu_r(U) \geq \max(2(1 - \epsilon)^2, 1 - \epsilon)$.

Proof. Notice that the assumption (D.6.1) is by Lemma D.6.2 equivalent to

$$\int_{U_1} \int_{\ln(U_2)} |A\Psi(u, x)|^2 \, du\, dx \geq 1 - \epsilon,$$

where $\Psi(x) := \psi(e^x)$. We can write $A\Psi(u, x) = e^{\pi i u x} V_\Psi \Psi(u, x)$, where V is the short-time Fourier transform given in (D.2.4). The assumption

$$\int_{U_1} \int_{\ln(U_2)} |V_\Psi \Psi(u, x)|^2 \, du\, dx \geq 1 - \epsilon$$

implies by Lieb's uncertainty principle [81, Theorem 3.3.3] that we have

$$\mu_r(U) = |U_1 \times \ln(U_2)| \geq (1-\epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2}{p-2}}, \qquad p > 2.$$

The final claim follows from considering $p = 4$ and $p \to \infty$. $\square$

We now relate the affine ambiguity function to the affine Wigner distribution. Define

$$\Theta(y, b) := \left(\frac{\sqrt{b}\log(b)}{b-1}\right)^{2\pi i y},$$

for $y \in \mathbb{R}$ and $b > 0$ with the convention that $\Theta(y, 1) = 1$ for all $y \in \mathbb{R}$. If we write $b = e^u$ for $u = \log(b)$, then

$$\frac{\sqrt{b}\log(b)}{b-1} = \sqrt{\lambda(u)\lambda(-u)},$$

where λ is the function given in (D.2.6). Hence we can think of $\Theta(y, b)$ as arising from a symmetrization of the function λ. We leave the verification of the following result to the reader as it is straightforward.

Lemma D.6.5. *For $\psi, \phi \in L^2(\mathbb{R}_+)$ we have the equality*

$$W_{\mathrm{Aff}}^{\psi,\phi}(x, a) = \mathcal{M}_y^{-1} \otimes \mathcal{M}_b \left[\Theta(y, b) \cdot A_{\mathrm{Aff}}^{\psi,\phi}(y, b)\right](x, a),$$

where $(x, a) \in \mathrm{Aff}$ and $\mathcal{M}$ is the Mellin transform.

It is of importance to extend the affine Weyl quantization to tempered distributions. To do this, we first need the following definition.

Definition D.6.6. Let $\mathcal{S}(\mathrm{Aff})$ denote the smooth functions $f\colon \mathrm{Aff} \to \mathbb{C}$ that satisfy

$$(x, \omega) \longmapsto f(x, e^{\omega}) \in \mathcal{S}(\mathbb{R}^2).$$

The space $\mathcal{S}(\mathrm{Aff})$ is called the *rapidly decaying smooth functions* on Aff. The dual space of $\mathcal{S}(\mathrm{Aff})$ will be denoted by $\mathcal{S}'(\mathrm{Aff})$ and called the *tempered distributions* on Aff.

The following result illustrates how we can use the Mellin transform and the affine ambiguity function to deduce properties of the affine Wigner distribution.

Proposition D.6.7. For $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$ the affine Wigner distribution satisfies $W_{\mathrm{Aff}}^{\psi,\phi} \in \mathcal{S}(\mathrm{Aff})$.

Proof. Let $\Psi(x) := \psi(e^x)$ and $\Phi(x) := \phi(e^x)$. By Lemma D.6.2 and Lemma D.6.5 we want to show that

$$(x,\omega) \longmapsto \mathcal{M}_y^{-1} \otimes \mathcal{M}_b \left[\Theta(y,b) \cdot A^{\Psi,\Phi}(\log(b), y)\right](x, e^{\omega}) \in \mathcal{S}(\mathbb{R}^2).$$

The cross-ambiguity function A is by [81, Theorem 11.2.5] a map

$$A \colon \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^2).$$

Hence $A(y,b) := A^{\Psi,\Phi}(\log(b), y) \in \mathcal{S}(\mathrm{Aff})$. Since $\Theta(y,b)$ is a smooth function with polynomially bounded derivatives, the same is true for $\Theta(y,b) \cdot A(y,b)$. The claim follows since the Mellin transform is related to the Fourier transform by the formula $\mathcal{M}(\psi)(x) = \mathcal{F}(\Psi)(x)$. □

Corollary D.6.8. The affine Weyl quantization A_f of $f \in \mathcal{S}'(\mathrm{Aff})$ is well-defined as an operator

$$A_f : \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}'(\mathbb{R}_+).$$

Example D.6.9. Consider the point measure $\delta_{\mathrm{Aff}}(x,a) \in \mathcal{S}'(\mathrm{Aff})$ defined by

$$\langle \delta_{\mathrm{Aff}}(x,a), f \rangle = \overline{f(x,a)},$$

for $f \in \mathcal{S}(\mathrm{Aff})$ and $(x,a) \in \mathrm{Aff}$. We compute for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$ that

$$\left\langle A_{\delta_{\mathrm{Aff}}(x,a)} \psi, \phi \right\rangle = \left\langle \delta_{\mathrm{Aff}}(x,a), W_{\mathrm{Aff}}^{\phi,\psi} \right\rangle = \overline{W_{\mathrm{Aff}}^{\phi,\psi}(x,a)} = W_{\mathrm{Aff}}^{\psi,\phi}(x,a).$$

Hence the operator $A_{\delta_{\mathrm{Aff}}(x,a)}$ is weakly defined through the values of the affine Wigner distribution.

D.7 An Almost Analytic Decomposition

Recall that analytic and anti-analytic functions f are characterized by the equations $\partial_{\bar{z}} f(z) = 0$ and $\partial_z f(z) = 0$, respectively. The fact that $W_{\mathrm{Aff}}^{\psi,\phi} \in L_r^2(\mathrm{Aff})$ when $\psi, \phi \in L^2(\mathbb{R}_+)$ allows us to exclude (anti-)analytic functions from being affine Wigner distributions.

Proposition D.7.1. There are no analytic or anti-analytic functions in $L_r^2(\mathrm{Aff})$.

Proof. The conclusion is easier to obtain by looking at the isomorphic spaces in the unit disc $\mathbb{D}$ by applying the standard linear fractional transformation. Under this transformation, the analytic functions in $L_r^2(\mathrm{Aff})$ are transformed to the analytic functions f in the unit disc satisfying the integrability condition

$$\int_{\mathbb{D}} \frac{|f(z)|^2}{1 - |z|^2} \, dz < \infty.$$

Any such analytic function will have to vanish as it approaches the boundary circle. Thus they are identically zero inside the unit disc as well by the unique continuation principle for analytic functions. The case of anti-analytic functions is similar. □

Remark. Proposition D.7.1 shows a big difference between the affine Wigner distribution and both the classical Wigner distribution and the wavelet transform; the classical Wigner distribution can produce Gaussians, while one can obtain plenty of analytic functions from the wavelet transform [44, Chapter 2.5].

Definition D.7.2. Consider a function $f\colon \mathrm{Aff} \to \mathbb{C}$.

- The function f is called *poly-analytic of order* if $n \in \mathbb{N}$ if $\partial_{\overline{z}}^n f = 0$. We write $f \in \mathcal{A}^n(\mathrm{Aff})$ to signify that f is poly-analytic of order n, but not poly-analytic of order $n-1$.
- The function f is called *anti-poly-analytic of order* if $n \in \mathbb{N}$ if $\partial_z^n f = 0$. We write $f \in \mathcal{A}^{\perp,n}(\mathrm{Aff})$ to signify that f is anti-poly-analytic of order n, but not anti-poly-analytic of order $n-1$.

The following result is inspired by [146] and shows that $L_r^2(\mathrm{Aff})$ decomposes completely into poly-analytic and anti-poly-analytic functions.

Proposition D.7.3. The space $L_r^2(\mathrm{Aff})$ has the orthogonal decomposition

$$L_r^2(\mathrm{Aff}) = \bigoplus_{n=2}^{\infty} \left(\mathcal{A}^n(\mathrm{Aff}) \oplus \mathcal{A}^{\perp,n}(\mathrm{Aff})\right).$$

Proof. Notice first that

$$L_r^2(\mathrm{Aff}) \simeq \Big(L^2(\mathbb{R}_+, dx) \otimes L^2(\mathbb{R}_+, a^{-1}\, da)\Big) \oplus \Big(L^2(\mathbb{R}_-, dx) \otimes L^2(\mathbb{R}_+, a^{-1}\, da)\Big).$$

Hence for $n \geq 2$ it suffices to show the decompositions

$$\mathcal{A}^n(\mathrm{Aff}) \simeq L^2(\mathbb{R}_+, dx) \otimes \mathrm{span}\,\{\mathcal{L}_{n-2}\}, \quad \mathcal{A}^{\perp,n}(\mathrm{Aff}) \simeq L^2(\mathbb{R}_-, dx) \otimes \mathrm{span}\,\{\mathcal{L}_{n-2}\},$$

where $\{\mathcal{L}_n\}_{n=0}^{\infty}$ is the orthogonal basis for $L^2(\mathbb{R}_+)$ defined in (D.3.2). We will only show the decomposition of $\mathcal{A}^n(\mathrm{Aff})$ since the decomposition of $\mathcal{A}^{\perp,n}(\mathrm{Aff})$ is similar.

Consider the map $\Phi\colon L_r^2(\mathrm{Aff}) \to L_r^2(\mathrm{Aff})$ given by

$$\Phi f(x, a) := \mathcal{F}_1(f)\left(x, \frac{a}{2|x|}\right),$$

where $\mathcal{F}_1$ is the Fourier transform in the first component. It is straightforward to check that Φ is a unitary map. The image of $\mathcal{A}^n(\mathrm{Aff})$ under Φ consists of all functions in $L^2_r(\mathrm{Aff})$ that satisfy

$$\Phi \circ \partial^n_{\overline{z}} \circ \Phi^{-1} f = \Phi(\partial_x + i\partial_a)^n \Phi^{-1} f = 0, \tag{D.7.1}$$

but do not satisfy (D.7.1) for $n-1$. A computation shows that functions $f \in L^2_r(\mathrm{Aff})$ satisfying (D.7.1) are precisely those that satisfy the homogeneous equation

$$|x|^n \left(\mathrm{sign}(x) + 2\partial_a\right)^n f(x,a) = 0, \tag{D.7.2}$$

but do not satisfy (D.7.2) for $n-1$. It is well known that the solution is precisely

$$f(x,a) = g(x)\mathcal{L}_{n-2}(a), \qquad g \in L^2(\mathbb{R}_+, dx).$$

Hence we obtain the decomposition for $\mathcal{A}^n(\mathrm{Aff})$ and the result follows. □

Remark. Notice that Proposition D.7.3 does not claim that $\mathcal{A}^n(\mathrm{Aff})$ and $\mathcal{A}^{\perp,n}(\mathrm{Aff})$ are orthogonal as $\mathcal{A}^n(\mathrm{Aff}) \cap \mathcal{A}^{\perp,n}(\mathrm{Aff}) \neq \{0\}$ for all $n \geq 2$. The poly-analytic functions have appeared prominently in the work of Abreu, see e.g. [1], in the context of wavelet analysis and sampling theory.

D.8 Applications

D.8.1 An Approximation Problem

Let us use the notation

$$\mathfrak{W}(\mathrm{Aff}) := \left\{ W^{\psi}_{\mathrm{Aff}} : \psi \in L^2(\mathbb{R}_+) \right\} \subset L^2_r(\mathrm{Aff}),$$

and call $\mathfrak{W}(\mathrm{Aff})$ the *affine Wigner space*. The affine orthogonality relation (D.3.1) implies that $\mathfrak{W}(\mathrm{Aff})$ is a closed subset of $L^2_r(\mathrm{Aff})$. Although we can create orthonormal bases for $L^2_r(\mathrm{Aff})$ by using the affine cross-Wigner transform as done in Corollary D.3.3, the space $\mathfrak{W}(\mathrm{Aff})$ is a proper subset of $L^2_r(\mathrm{Aff})$.

It is natural to ask how far a function $f \in L^2_r(\mathrm{Aff})$ is from being in $\mathfrak{W}(\mathrm{Aff})$. Hence we are interested in the following *affine Wigner approximation problem*

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L^2_r(\mathrm{Aff})}. \tag{D.8.1}$$

The analogous problem for the classical Wigner distribution has been recently investigated in [12]. Our quantization based approach will as a byproduct produce a new proof of the classical Wigner approximation problem in [12].

For $g = W_{\mathrm{Aff}}^{\psi}$ it follows from (D.2.11) and the affine orthogonality relation (D.3.1) that A_g is the rank-one operator

$$A_g \phi = \langle \phi, \psi \rangle \psi.$$

The converse is also clear, so there is a one-to-one correspondence between affine Wigner distributions and positive rank-one operators. Hence the distance (D.8.1) should somehow be related to how far A_f is from being a rank-one operator. In Corollary D.8.2 we will see that this heuristic is correct for a large class of functions $f \in L_r^2(\mathrm{Aff})$. We use the notation

$$\lambda_{\max}^{+}(A_f) := \max\left\{ \max_{\lambda \in \mathrm{Spec}(A_f)} \lambda, 0 \right\}.$$

Theorem D.8.1. *The affine Wigner approximation problem for a real-valued function $f \in L_r^2(\mathrm{Aff})$ has the explicit solution*

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L_r^2(\mathrm{Aff})} = \sqrt{\|f\|_{L_r^2(\mathrm{Aff})}^2 - \lambda_{\max}^{+}(A_f)^2}. \tag{D.8.2}$$

A minimizing function $h \in \mathfrak{W}(\mathrm{Aff})$ to the affine Wigner approximation problem always exists. Moreover, when $\lambda_{\max}^{+}(A_f) > 0$ the number of minimizers is equal to the multiplicity of $\lambda_{\max}^{+}(A_f)$.

Proof. Notice that A_f is self-adjoint since

$$\left\langle A_f \psi, \phi \right\rangle = \left\langle A_{\overline{f}} \psi, \phi \right\rangle = \left\langle \overline{f}, W_{\mathrm{Aff}}^{\phi,\psi} \right\rangle = \overline{\left\langle f, W_{\mathrm{Aff}}^{\psi,\phi} \right\rangle} = \left\langle \psi, A_f \phi \right\rangle,$$

for $\psi, \phi \in L^2(\mathbb{R}_+)$. Thus the spectral theory for compact, self-adjoint operators implies that the spectrum $\mathrm{Spec}(A_f) = \{\lambda_k\}_{k=0}^{\infty}$ of A_f is countable with $0 \in \mathrm{Spec}(A_f)$ as the only possible accumulation point. Moreover, there is by [67, Theorem 1.52] an orthonormal basis $\{\phi_k\}_{k=0}^{\infty}$ for $L^2(\mathbb{R}_+)$ such that ϕ_k is an eigenvector for A_f corresponding to the eigenvalue λ_k. The convention is that eigenvalues are repeated according to their multiplicity.

We claim that we can write

$$A_f = \sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k},$$

where the convergence is in the Hilbert-Schmidt norm. Notice that convergence of $\sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$ to A_f is guaranteed in the operator norm from the theory of compact operators [32, Theorem 3.5]. Hence it suffices to show that $\sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$ converges in the Hilbert-Schmidt norm; this will imply together with the norm inequality $\|\cdot\|_{op} \leq \|\cdot\|_{\mathcal{HS}}$ that $\sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$ must converge to A_f in the Hilbert-Schmidt norm. Due to completeness, it suffices to show that $\sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$ is a Cauchy sequence. For $n, m \in \mathbb{N}$ with $n < m$ we have

$$\left\| \sum_{k=n}^{m} \lambda_k \phi_k \otimes \overline{\phi_k} \right\|_{\mathcal{HS}}^2 = \sum_{k,k'=n}^{m} \lambda_k \overline{\lambda_{k'}} \left\langle \phi_k \otimes \overline{\phi_k}, \phi_{k'} \otimes \overline{\phi_{k'}} \right\rangle_{\mathcal{HS}} = \sum_{k=n}^{m} |\lambda_k|^2,$$

where $\| \cdot \|_{\mathcal{HS}}$ denotes the Hilbert-Schmidt norm. The claim follows since A_f is Hilbert-Schmidt.

We can now by Proposition D.2.1 write

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L_r^2(\mathrm{Aff})} = \inf_{\psi \in L^2(\mathbb{R}_+)} \left\| \sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k} - \psi \otimes \overline{\psi} \right\|_{\mathcal{HS}}. \tag{D.8.3}$$

Assume that $\lambda_j = \lambda_{\max}^{+}(A_f)$. Then (D.8.3) is clearly minimized when $\psi = \sqrt{\lambda_j}\phi_j$. By orthogonality, we can rewrite (D.8.3) and obtain

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L_r^2(\mathrm{Aff})} = \sqrt{\|A_f\|_{\mathcal{HS}}^2 - \lambda_{\max}^{+}(A_f)^2} = \sqrt{\|f\|_{L_r^2(\mathrm{Aff})}^2 - \lambda_{\max}^{+}(A_f)^2}.$$

We always have a minimizer as we can take $h = W_{\mathrm{Aff}}^{\psi}$. The statement about uniqueness of minimizers is clear from (D.8.3). $\square$

Remarks.

- From the spectral theory of compact, self-adjoint operators, it also follows that the eigenspaces corresponding to non-zero eigenvalues are finite-dimensional. Hence, for a given $f \in L_r^2(\mathrm{Aff})$, there is at most a finite number of minimizers $h_1, \ldots, h_k \in \mathfrak{W}(\mathrm{Aff})$ so that

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L_r^2(\mathrm{Aff})} = \|f - h_i\|_{L_r^2(\mathrm{Aff})}, \qquad i = 1, \ldots, k.$$

- The proof of Theorem D.8.1 goes through almost verbatim to show the analogous result for the classical Wigner distribution. The analogous formula to (D.8.2) for the classical Wigner distribution was shown in [12, Theorem 3] using a variational calculus approach. That the number of minimizers can be easily deduced from the spectrum of the quantized operator seems new even for the classical Wigner distribution.

- Assume that $f \in L_r^2(\mathrm{Aff})$ is such that A_f is a negative operator. Then $\lambda_{\max}^{+}(A_f) = 0$ and it is clear from (D.8.3) that the zero function is the unique minimizer.

Corollary D.8.2. Let $f \in L^2_r(\mathrm{Aff})$ be real-valued and assume that

$$\lambda^+_{\max}(A_f) = \max_{\lambda \in \mathrm{Spec}(A_f)} |\lambda|.$$

Then

$$\min_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L^2_r(\mathrm{Aff})} = \sqrt{\|A_f\|^2_{\mathcal{HS}} - \|A_f\|^2_{op}}. \tag{D.8.4}$$

Proof. Since A_f is self-adjoint it follows from [67, Proposition 1.24] that

$$\|A_f\|_{op} = \max_{\lambda \in \mathrm{Spec}(A_f)} |\lambda|.$$

□

Remark. Notice that under the assumptions in Corollary D.8.2, the heuristic we presented regarding rank-one operators holds true: If A_f is a rank-one operator, then the Hilbert-Schmidt norm and the operator norm coincide. Hence (D.8.4) is zero and thus f is in the affine Wigner space $\mathfrak{W}(\mathrm{Aff})$. Conversely, the equations

$$\|A_f\|^2_{op} = \max_{\lambda \in \mathrm{Spec}(A_f)} \lambda^2, \qquad \|A_f\|^2_{\mathcal{HS}} = \sum_{\lambda \in \mathrm{Spec}(A_f)} \lambda^2 \tag{D.8.5}$$

imply that (D.8.4) is zero precisely when A_f is a rank-one operator.

Example D.8.3. Let $f \in L^2_r(\mathrm{Aff})$ be such that A_f is a positive operator with rank $k > 0$. Then (D.8.5) implies that

$$\|A_f\|^2_{op} \geq \frac{\|A_f\|^2_{\mathcal{HS}}}{k}.$$

Hence we obtain from (D.8.4) that

$$\min_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L^2_r(\mathrm{Aff})} = \sqrt{\|A_f\|^2_{\mathcal{HS}} - \|A_f\|^2_{op}} \leq \sqrt{\frac{k-1}{k}} \|f\|_{L^2_r(\mathrm{Aff})}.$$

This has the following consequence: Let $f_1, f_2 \in L^2_r(\mathrm{Aff})$ both correspond to positive operators A_{f_1} and A_{f_2} with finite rank. If $\mathrm{rank}(A_{f_1}) \ll \mathrm{rank}(A_{f_2})$, then f_1 will be closer to the affine Wigner space than f_2, unless the energy of f_2 is significantly smaller than that of f_1.

D.8.2 Dilation Invariant Operators

An operator $A \colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is said to be *dilation invariant* if

$$A = D_r \circ A \circ D_r^*, \tag{D.8.6}$$

for all $r > 0$ where D_r is the dilation operator in (D.2.5). We use the affine Weyl quantization to show the following result.

Proposition D.8.4. There are no non-zero dilation invariant Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$.

Proof. Assume by contradiction that $A\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is a dilation invariant Hilbert-Schmidt operator and write $A = A_f$ for $f \in L^2_r(\mathrm{Aff})$. It follows from (D.4.1) and that

$$W_{\mathrm{Aff}}^{D_{\frac{1}{r}}\psi, D_{\frac{1}{r}}\phi}(x,a) = r \cdot W_{\mathrm{Aff}}^{\psi,\phi}(x, ra), \qquad \psi, \phi \in L^2(\mathbb{R}_+),$$

for $r > 0$ and $(x,a) \in \mathrm{Aff}$. Hence (D.2.11) implies that

$$\left\langle D_r A_f D_{\frac{1}{r}}\psi, \phi \right\rangle = \left\langle f, W_{\mathrm{Aff}}^{D_{\frac{1}{r}}\phi, D_{\frac{1}{r}}\psi} \right\rangle = \int_{-\infty}^{\infty} \int_0^{\infty} r f\left(x, \frac{a}{r}\right) W_{\mathrm{Aff}}^{\psi,\phi}(x,a)\,\frac{da\,dx}{a}.$$

On the other hand, since A_f is dilation invariant we also have

$$\left\langle D_r A_f D_{\frac{1}{r}}\psi, \phi \right\rangle = \left\langle A_f \psi, \phi \right\rangle = \int_{-\infty}^{\infty} \int_0^{\infty} f\,(x,a)\, W_{\mathrm{Aff}}^{\psi,\phi}(x,a)\,\frac{da\,dx}{a}.$$

This forces $f \in L^2_r(\mathrm{Aff})$ by Corollary D.3.3 to satisfy the homogeneity relation

$$f(x,a) = r f\left(x, \frac{a}{r}\right),$$

for all $r > 0$ and almost every $(x,a) \in \mathrm{Aff}$. However, this implies that

$$\begin{aligned}
\|f\|^2_{L^2_r(\mathrm{Aff})} &= \int_{-\infty}^{\infty} \int_0^{\infty} |f(x,a)|^2\,\frac{da\,dx}{a} \\
&= r^2 \int_{-\infty}^{\infty} \int_0^{\infty} \left| f\left(x, \frac{a}{r}\right)\right|^2 \frac{da\,dx}{a} \\
&= r^2 \|f\|^2_{L^2_r(\mathrm{Aff})}.
\end{aligned}$$

Hence f is not in $L^2_r(\mathrm{Aff})$ unless $f = 0$, in which case A_f is the zero operator. □

Remark. Notice that the proof of Proposition D.8.4 actually shows that there can be no non-zero Hilbert-Schmidt operator A that satisfies (D.8.6) even for a single $r \neq 1$.

Example D.8.5. Although we showed in Proposition D.8.4 that there are no non-zero dilation invariant Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$, there are non-zero projections in $L^2(\mathbb{R}_+)$ that are dilation invariant. As an example, consider the orthogonal projection $P\colon L^2(\mathbb{R}_+) \to \mathcal{M}_{(0,\infty)}$ where $\mathcal{M}_{(0,\infty)}$ is the space of all $\psi \in L^2(\mathbb{R}_+)$ such that the Mellin transform of ψ satisfies

$$\mathrm{supp}(\mathcal{M}(\psi)) \subset \mathbb{R}_+.$$

The projection P is dilation invariant due to (D.3.4).

D.8.3 Trace-Class Operators

Finally, we give an application to trace-class operators motivated by the elegant result [78, Proposition 162].

Proposition D.8.6. Let $T\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be a trace-class operator. Then we can write $T = A_f \circ A_g$ for $f, g \in L^2_r(\mathrm{Aff})$. Moreover, the trace of T can be calculated by the formula

$$\mathrm{Tr}(T) = \mathrm{Tr}(A_f \circ A_g) = \int_{-\infty}^{\infty} \int_0^{\infty} f(x,a)g(x,a)\,\frac{da\,dx}{a}.$$

Proof. Any trace-class operator T on $L^2(\mathbb{R}_+)$ can be written as a composition of two Hilbert-Schmidt operators $T = A \circ B$. The bijective correspondence in Proposition D.2.1 shows that $A = A_f$ and $B = A_g$ for $f, g \in L^2_r(\mathrm{Aff})$. Finally, we have

$$\begin{aligned}\mathrm{Tr}(T) &= \mathrm{Tr}(A_f \circ A_g)\\ &= \left\langle A_g, A_f^* \right\rangle_{\mathcal{HS}}\\ &= \left\langle g, \overline{f} \right\rangle_{L^2_r(\mathrm{Aff})}\\ &= \int_{-\infty}^{\infty} \int_0^{\infty} f(x,a)g(x,a)\,\frac{da\,dx}{a}.\end{aligned}$$ □

Remark. Notice that

$$\overline{\mathrm{Tr}\,(T)} = \mathrm{Tr}\left(A_g^* \circ A_f^*\right) = \mathrm{Tr}\left(A_{\overline{g}} \circ A_{\overline{f}}\right) = \int_{-\infty}^{\infty} \int_0^{\infty} \overline{f(x,a)g(x,a)}\,\frac{da\,dx}{a}.$$

In particular, the trace of T is real-valued whenever f and g are real-valued.

D.9 Further Research

D.9.1 The Affine Grossmann-Royer Operator

A standard tool for deriving properties of the classical Wigner distribution is the *Grossmann-Royer operator* $\widehat{R}(x,\omega)$ defined by the relation

$$W(f,g)(x,\omega) = \left\langle \widehat{R}(x,\omega)f, g \right\rangle_{L^2(\mathbb{R}^d)},$$

for $f, g \in L^2(\mathbb{R}^d)$ and $(x,\omega) \in \mathbb{R}^{2d}$. An essential property of the Grossmann-Royer operator $\widehat{R}(x,\omega)$ is that

$$\left\|\widehat{R}(x,\omega)f\right\|_{L^2(\mathbb{R}^d)} = 2^d \cdot \|f\|_{L^2(\mathbb{R}^d)},$$

for all $f \in L^2(\mathbb{R}^d)$ and $(x,\omega) \in \mathbb{R}^{2d}$. This is immensely useful; to see that the classical cross-Wigner transform is bounded one simply needs to apply Cauchy-Schwarz inequality to obtain

$$\sup_{(x,\omega)\in\mathbb{R}^{2d}} |W(f,g)(x,\omega)| \leq 2^d \cdot \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{D.9.1}$$

Analogously, we define the *affine Grossmann-Royer operator* $\widehat{R}_{\mathrm{Aff}}(x,a)$ by the relation

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) = \left\langle \widehat{R}_{\mathrm{Aff}}(x,a)\psi, \phi \right\rangle_{L^2(\mathbb{R}_+)},$$

for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$ and $(x,a) \in \mathrm{Aff}$. We restrict our attention to Schwartz functions for convenience since then $W_{\mathrm{Aff}}^{\psi,\phi} \in \mathcal{S}(\mathrm{Aff})$ and hence have well-defined point values. Notice that the affine Grossmann-Royer operator $\widehat{R}_{\mathrm{Aff}}(x,a)$ is precisely the affine Weyl quantization of the point mass $\delta_{\mathrm{Aff}}(x,a)$ given in Example D.6.9. The affine Grossmann-Royer operator have the explicit form

$$\widehat{R}_{\mathrm{Aff}}(x,a)\psi(r) = \frac{e^{2\pi i x \lambda^{-1}\left(\frac{r}{a}\right)}\lambda^{-1}\left(\frac{r}{a}\right)\left(1 - e^{\lambda^{-1}\left(\frac{r}{a}\right)}\right)}{1 + \lambda^{-1}\left(\frac{r}{a}\right) - e^{\lambda^{-1}\left(\frac{r}{a}\right)}} \cdot \psi\left(re^{-\lambda^{-1}\left(\frac{r}{a}\right)}\right),$$

for $\psi \in \mathcal{S}(\mathbb{R}_+)$, $r > 0$, and $(x,a) \in \mathrm{Aff}$ where λ is the function given in (D.2.6).

Trying to generalize the strategy in (D.9.1) runs into a problem: The affine Grossmann-Royer operator is not a bounded operator on $\mathcal{S}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ with respect to the norm $\|\cdot\|_{L^2(\mathbb{R}_+)}$. However, if $\psi \in \mathcal{S}(\mathbb{R}_+)$ is supported in the interval $\left[\frac{1}{k}, k\right]$ for some $k > 0$, then there is a constant $C_k > 0$ such that

$$\left\|\widehat{R}_{\mathrm{Aff}}(x,a)\psi\right\|_{L^2(\mathbb{R}_+)} \leq C_k \cdot \|\psi\|_{L^2(\mathbb{R}_+)}.$$

We call the optimal constant C_k in the inequality above the *k-support constant*. Hence if $\phi \in \mathcal{S}(\mathbb{R}_+)$ we have

$$\sup_{(x,a)\in\mathrm{Aff}} \left|W_{\mathrm{Aff}}^{\psi,\phi}(x,a)\right| \leq C_k \cdot \|\psi\|_{L^2(\mathbb{R}_+)}\|\phi\|_{L^2(\mathbb{R}_+)}.$$

A trivial adaption of [81, Lemma 4.3.7] gives the following *relative uncertainty principle*.

Proposition D.9.1. Let $\psi \in \mathcal{S}(\mathbb{R}_+)$ be supported in the interval $\left[\frac{1}{k}, k\right]$ for some $k > 0$ and let $U \subset \mathrm{Aff}$ be a Borel set. Assume there exists $\epsilon \geq 0$ such that

$$\int_U W_{\mathrm{Aff}}^{\psi}(x,a)\,\frac{da\,dx}{a} \geq (1-\epsilon)\|\psi\|_{L^2(\mathbb{R}_+)}^2.$$

Then the right Haar measure of U satisfies $\mu_r(U) \geq (1-\epsilon)C_k^{-1}$.

Motivated by Proposition D.9.1, it is of interest to investigate the k-support constant C_k both numerically and asymptotically. The affine Grossmann-Royer operator is investigated more thoroughly in the follow-up paper [18].

D.9.2 The Affine Positivity Conjecture

One of the major results about the classical Wigner distribution is regarding positivity; when is W_f a non-negative function on $\mathbb{R}^{2d}$? Normalized functions $f \in L^2(\mathbb{R}^d)$ such that W_f is non-negative would generate probability density functions on $\mathbb{R}^{2d}$ that represent the time-frequency distribution of f. However, a well known result of Hudson [81, Theorem 4.4.1] shows that this can only happen for suitably perturbed Gaussians.

Turning to the affine setting, we would like to determine the normalized functions $\psi \in L^2(\mathbb{R}_+)$ such that W_{Aff}^{ψ} is a non-negative function on the affine group. In [130] the authors showed that the affine Wigner distribution $W_{\mathrm{Aff}}^{\psi_s}$ is non-negative if ψ_s is the so called *Morse ground state*

$$\psi_s(r) := \frac{r^s e^{-\frac{r}{2}}}{\Gamma(2s)}, \qquad s \geq 0.$$

We will only consider ψ_s for $s > 0$ as $\psi_0 \notin L^2(\mathbb{R}_+)$. More generally, one can use the invariance properties (D.4.2) and (D.4.3) to show that the affine Wigner distribution W_{Aff}^{ψ} of

$$\psi(r) = Cr^{-i(x+ia)}e^{i(y+ib)r} \qquad C \in \mathbb{C},\ (x,a),(y,b) \in \mathrm{Aff}, \tag{D.9.2}$$

is non-negative. The functions (D.9.2) are called the *generalized Klauder wavelets* in [66, Equation (41)] that are in $L^2(\mathbb{R}_+)$.

It is of interest to determine the following *affine positivity conjecture*, which is a reformulation of an open question in [66]:

If W_{Aff}^{ψ} is non-negative for $\psi \in L^2(\mathbb{R}_+)$, then ψ is a generalized Klauder wavelet.

The Klauder wavelets have in [99] been shown to be the only functions that generate analytic spaces for the continuous wavelet transform. This gives a concrete connection between Klauder wavelets and Gaussians, since Gaussians are the only functions in the classical case that generate analytic spaces for the short-time Fourier transform by [7, Theorem 3.1].

Paper E

Affine Quantum Harmonic Analysis

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Paper E

Affine Quantum Harmonic Analysis

Abstract

We develop a quantum harmonic analysis framework for the affine group. This encapsulates several examples in the literature such as affine localization operators, covariant integral quantizations, and affine quadratic time-frequency representations. In the process, we develop a notion of admissibility for operators and extend well known results to the operator setting. A major theme of the paper is the interaction between operator convolutions, affine Weyl quantization, and admissibility.

E.1 Introduction

The affine group and the Heisenberg group play prominent roles in wavelet theory and Gabor analysis, respectively. As is well known, the representation theory of the Heisenberg group is intrinsically linked to quantization on phase space $\mathbb{R}^{2n}$. Similarly, the relation between quantization schemes on the affine group and its representation theory has received some attention and several schemes have been proposed, e.g. [17, 73, 76]. However, there are still many open questions awaiting a definite answer in the case of the affine group.

As has been shown by two of the authors in [125], the theory of *quantum harmonic analysis on phase space* introduced by Werner [153] provides a coherent framework for many aspects of quantization and Gabor analysis associated with the Heisenberg group. Based on this connection, advances in the understanding of time-frequency analysis have been made [124, 126, 127]. In this paper we aim to develop a variant of Werner's quantum harmonic analysis in [153] for time-scale analysis. This is based on unitary representations of the affine group in a similar way to the Schrödinger representation of the Heisenberg group being used in Werner's framework. We will refer to this theory on the affine group as *affine quantum harmonic analysis*.

Affine Operator Convolutions

In Werner's quantum harmonic analysis on phase space, a crucial component is extending convolutions to operators. Recall that the affine group Aff has the underlying set $\mathbb{R} \times \mathbb{R}_+$ and group operation modeling composition of affine transformations. A key feature of this group is that the left Haar measure $a^{-2}dx\,da$ and the right Haar measure $a^{-1}dx\,da$ are not equal, making the group *non-unimodular*. Both measures play a role in affine quantum harmonic analysis, making the theory more involved than the case of the Heisenberg group. In addition to the standard function (right-)convolution on the affine group

$$f *_{\mathrm{Aff}} g(x,a) := \int_{\mathrm{Aff}} f(y,b)g((x,a)\cdot(y,b)^{-1})\,\frac{dy\,db}{b},$$

we introduce the following *operator convolutions* for operators on the space $L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, r^{-1}\,dr)$ in Section E.3:

- Let $f \in L^1_r(\mathrm{Aff}) := L^1(\mathrm{Aff}, a^{-1}dx\,da)$ and let S be a trace-class operator on $L^2(\mathbb{R}_+)$. We define the *convolution* $f \star_{\mathrm{Aff}} S$ between f and S to be the operator on $L^2(\mathbb{R}_+)$ given by

$$f \star_{\mathrm{Aff}} S := \int_{\mathrm{Aff}} f(x,a)U(-x,a)^*SU(-x,a)\,\frac{dx\,da}{a},$$

  where U is the unitary representation of Aff on $L^2(\mathbb{R}_+)$ given by

$$U(x,a)\psi(r) := e^{2\pi i x r}\psi(ar).$$

- Let S be a trace-class operator and let T be a bounded operator on $L^2(\mathbb{R}_+)$. Then we define the *convolution* $S \star_{\mathrm{Aff}} T$ between S and T to be the function on Aff given by

$$S \star_{\mathrm{Aff}} T(x,a) := \mathrm{tr}(SU(-x,a)^*TU(-x,a)).$$

The three convolutions are compatible in the following sense: Let $f, g \in L^1_r(\mathrm{Aff})$ and denote by S a trace-class operator and by T a bounded operator, both on $L^2(\mathbb{R}_+)$. Then

$$(f \star_{\mathrm{Aff}} S) \star_{\mathrm{Aff}} T = f *_{\mathrm{Aff}} (S \star_{\mathrm{Aff}} T),$$
$$f \star_{\mathrm{Aff}} (g \star_{\mathrm{Aff}} S) = (f *_{\mathrm{Aff}} g) \star_{\mathrm{Aff}} S.$$

Interplay Between Affine Weyl Quantization and Convolutions

Integral to the theory in this paper is the affine Wigner distribution and the associated affine Weyl quantization. The *affine (cross-)Wigner distribution* $W_{\mathrm{Aff}}^{\psi,\phi}$ of elements $\phi, \psi \in L^2(\mathbb{R}_+)$ is the function on Aff given by

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) := \int_{-\infty}^{\infty} \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}e^{-2\pi i x u}\,du, \tag{E.1.1}$$

where $\lambda\colon \mathbb{R} \to \mathbb{R}$ is explicitly given by

$$\lambda(u) := \frac{1}{{}_1F_1(1,2;-u)} = \frac{ue^u}{e^u - 1},$$

where ${}_1F_1$ is *Kummer's confluent hypergeometric function*. The function λ will play a central role throughout the paper.

Although at first glance the definition (E.1.1) might look unnatural, it can be motivated through the representation theory of the affine group as illustrated in [5]. We will elaborate on this viewpoint in Section E.5. One defines the *affine Weyl quantization* of $f \in L_r^2(\mathrm{Aff}) := L^2(\mathrm{Aff}, a^{-1}dx\,da)$ as the operator A_f given by

$$\left\langle A_f\,\phi, \psi\right\rangle_{L^2(\mathbb{R}_+)} = \left\langle f, W_{\mathrm{Aff}}^{\psi,\phi}\right\rangle_{L_r^2(\mathrm{Aff})}, \qquad \text{for all } \phi, \psi \in L^2(\mathbb{R}_+).$$

We will explore the intimate relation between the convolutions and the affine Weyl quantization. The following theorem, being a combination of Proposition E.3.6 and Proposition E.3.7, highlights this relation.

Theorem A. Let $f, g \in L_r^2(\mathrm{Aff})$, where g is additionally in $L_r^1(\mathrm{Aff})$ and square integrable with respect to the left Haar measure. Then

$$\begin{aligned} g \star_{\mathrm{Aff}} A_f &= A_{g *_{\mathrm{Aff}} f}, \\ A_g \star_{\mathrm{Aff}} A_f &= f *_{\mathrm{Aff}} \check{g}, \end{aligned}$$

where $\check{g}(x,a) := g((x,a)^{-1})$.

We will exploit the previous theorem to define the affine Weyl quantization of tempered distributions in Subsection E.3.3. To do this rigorously, we will utilize a Schwartz space $\mathcal{S}(\mathrm{Aff})$ on the affine group introduced in [17]. An important example we prove in Theorem E.3.10 is the affine Weyl quantization of the coordinate functions:

Theorem B. Let $f_x(x,a) := x$ and $f_a(x,a) := a$ be the coordinate functions on Aff. The affine Weyl quantizations A_{f_x} and A_{f_a} satisfy the commutation relation

$$[A_{f_x}, A_{f_a}] = \frac{1}{2\pi i}A_{f_a}.$$

This is, up to re-normalization, precisely the infinitesimal structure of the affine group.

We define *affine parity operator* P_{Aff} as

$$P_{\mathrm{Aff}} := A_{\delta_{(0,1)}},$$

where $\delta_{(0,1)}$ denotes the Dirac distribution at the identity element $(0,1) \in \mathrm{Aff}$. The following result, which will be rigorously stated in Subsection E.3.5, builds on these definitions.

Theorem C. The affine Weyl quantization A_g of $g \in \mathcal{S}(\mathrm{Aff})$ can be written as

$$A_g = g \star_{\mathrm{Aff}} P_{\mathrm{Aff}}.$$

Moreover, for ϕ, ψ such that $\phi(e^x), \psi(e^x) \in \mathcal{S}(\mathbb{R})$, the affine Weyl symbol $W_{\mathrm{Aff}}^{\psi,\phi}$ of the rank-one operator $\psi \otimes \phi$ can be written as

$$W_{\mathrm{Aff}}^{\psi,\phi} = (\psi \otimes \phi) \star_{\mathrm{Aff}} P_{\mathrm{Aff}}.$$

Operator Admissibility

One of the key features of representations of non-unimodular groups is the concept of admissibility. Recall that the *Duflo-Moore operator* $\mathcal{D}^{-1}$ corresponding to the representation U is the densely defined positive operator on $L^2(\mathbb{R}_+)$ given by $\mathcal{D}^{-1}\psi(r) = r^{-1/2}\psi(r)$. We will often use that $\mathcal{D}^{-1}$ has a densely defined inverse given by $\mathcal{D}\psi(r) = r^{1/2}\psi(r)$. A function ψ is said to be an *admissible wavelet* if $\psi \in \mathrm{dom}(\mathcal{D}^{-1})$. It is well known [50] that admissible wavelets satisfy the orthogonality relation

$$\int_{\mathrm{Aff}} |\langle \phi, U(-x,a)^*\psi\rangle_{L^2(\mathbb{R}_+)}|^2 \, \frac{dx\, da}{a} = \|\phi\|_{L^2(\mathbb{R}_+)}^2 \|\mathcal{D}^{-1}\psi\|_{L^2(\mathbb{R}_+)}^2. \tag{E.1.2}$$

We extend the definition of admissibility to operators as follows:

Definition. Let S be a non-zero bounded operator on $L^2(\mathbb{R}_+)$ that maps $\mathrm{dom}(\mathcal{D})$ into $\mathrm{dom}(\mathcal{D}^{-1})$. We say that S is *admissible* if the composition $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ is bounded on $\mathrm{dom}(\mathcal{D}^{-1})$ and extends to a trace-class operator $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ on $L^2(\mathbb{R}_+)$.

Note that the rank-one operator $S := \psi \otimes \psi$ for $\psi \in L^2(\mathbb{R}_+)$ is admissible precisely when ψ is an admissible wavelet. In Subsection E.4.2 we show that a large class of admissible operators can be constructed from Laguerre bases. The following result, which we prove in Corollary E.4.5, extends (E.1.2) to the operator setting and is motivated by [153, Lemma 3.1].

Theorem D. Let S be an admissible operator on $L^2(\mathbb{R}_+)$. For any trace-class operator T on $L^2(\mathbb{R}_+)$, we have that $T \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$ with

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\, da}{a} = \mathrm{tr}(T)\,\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1}).$$

Determining whether an operator is admissible or not can be a daunting task. We managed in Corollary E.4.9 to find an elegant characterization in terms of operator convolutions of admissible operators that are additionally positive trace-class operators.

Theorem E. Let S be a non-zero, positive trace-class operator. Then S is admissible if and only if $S \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$.

The following result is derived in Subsection E.4.4 and uses the affine Weyl quantization to show that admissibility is an operator manifestation of the non-unimodularity of the affine group.

Theorem F.

- Let $f \in L^1_r(\mathrm{Aff})$ be such that A_f is a trace-class operator on $L^2(\mathbb{R}_+)$. Then
$$\mathrm{tr}(A_f) = \int_{\mathrm{Aff}} f(x,a)\,\frac{dx\,da}{a}.$$
- Let $g \in L^1_l(\mathrm{Aff}) := L^1(\mathrm{Aff}, a^{-2}dx\,da)$ be such that A_g is an admissible Hilbert-Schmidt operator. Then
$$\mathrm{tr}\left(\mathcal{D}^{-1}A_g\mathcal{D}^{-1}\right) = \int_{\mathrm{Aff}} g(x,a)\,\frac{dx\,da}{a^2}.$$

Relationship with Fourier Transforms

For completeness, we will also investigate how notions of Fourier transforms on the affine group fit into the theory, and use known results from abstract harmonic analysis to explore the relationship between affine Weyl quantization and affine Fourier transforms. Recall that the *integrated representation* $U(f)$ of $f \in L^1_l(\mathrm{Aff})$ is the operator on $L^2(\mathbb{R}_+)$ given by

$$U(f)\psi := \int_{\mathrm{Aff}} f(x,a)U(x,a)\psi\,\frac{dx\,da}{a^2}, \qquad \psi \in L^2(\mathbb{R}_+).$$

We define the following operator Fourier transform in the affine setting.

Definition. The *affine Fourier-Wigner transform* is the isometry $\mathcal{F}_W$ sending a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ to a function in $L^2_r(\mathrm{Aff})$ such that

$$\mathcal{F}_W^{-1}(f) := U(\check{f}) \circ \mathcal{D}, \qquad f \in \mathrm{Im}(\mathcal{F}_W) \cap L^1_r(\mathrm{Aff}).$$

The following result is proved in Proposition E.5.7 and provides a connection between the affine Fourier-Wigner transform and admissibility.

Theorem G. Let A be a trace-class operator on $L^2(\mathbb{R}_+)$. The following are equivalent:

1) $\mathcal{F}_W(A\mathcal{D}^{-1}) \in L^2_r(\mathrm{Aff})$.

2) $A\mathcal{D}^{-1}$ extends from $\mathrm{dom}(\mathcal{D}^{-1})$ to a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$.

3) A^*A is admissible.

Another Fourier transform of interest is the (modified) *Fourier-Kirillov transform* on the affine group $\mathcal{F}_{\mathrm{KO}}$ given by

$$(\mathcal{F}_{\mathrm{KO}}f)(x,a) := \sqrt{a}\int_{\mathbb{R}^2} f\left(\frac{v}{\lambda(-u)}, e^u\right) e^{-2\pi i(xu+av)}\,\frac{du\,dv}{\sqrt{\lambda(-u)}}, \qquad f \in \mathrm{Im}(\mathcal{F}_W).$$

As in quantum harmonic analysis on phase space, we have that the affine Weyl quantization is the composition of these Fourier transforms, see Proposition E.5.8. In the affine setting we have in general that

$$\mathcal{F}_W(f \star_{\mathrm{Aff}} S) \neq \mathcal{F}_{KO}(f)\mathcal{F}_W(S), \qquad \mathcal{F}_{KO}(S \star_{\mathrm{Aff}} T) \neq \mathcal{F}_W(S)\mathcal{F}_W(T).$$

This contrasts the analogous result in Werner's original quantum harmonic analysis, see (E.5.5). In spite of this, not all properties typically associated with the Fourier transform are lost: In Subsection E.5.2 we prove a quantum Bochner theorem in the affine setting.

Main Applications

In Section E.6 we show that affine quantum harmonic analysis provides a conceptual framework for the study of *covariant integral quantizations* and a version of the *Cohen class* for the affine group. In addition, we show in Subsection E.6.1 that if S is a rank-one operator, then the study of operators $f \star_{\mathrm{Aff}} S$ for functions f on Aff reduces to the study of time-scale localization operators [46].

We have seen that affine Weyl quantization is given by $f \mapsto f \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$ for $f \in \mathcal{S}(\mathrm{Aff})$. Inspired by this, we consider a whole class of quantization procedures: For any suitably nice operator S on $L^2(\mathbb{R}_+)$ we define a quantization procedure Γ_S for functions f on Aff by

$$\Gamma_S(f) := f \star_{\mathrm{Aff}} S.$$

This class of quantization procedures coincides with the *covariant integral quantizations* studied by Gazeau and his collaborators motivated by applications in physics, see e.g. [74, 75, 76]. Our results on affine quantum harmonic analysis are therefore also results on covariant integral quantizations. In particular, the abstract notion of admissibility of an operator S implies that Γ_S satisfies the simple property

$$\Gamma_S(1) = c \cdot I_{L^2(\mathbb{R}_+)},$$

where c is some constant, $I_{L^2(\mathbb{R}_+)}$ is the identity operator on $L^2(\mathbb{R}_+)$, and $1(x,a) := 1$ for all $(x,a) \in \mathrm{Aff}$.

As the name suggests, covariant integral quantizations Γ_S satisfy a *covariance* property, namely

$$U(-x,a)^*\Gamma_S(f)U(-x,a) = \Gamma_S(R_{(x,a)^{-1}}f),$$

where R denotes right translations of functions on Aff. In Theorem E.6.5 we point out that, by a result on positive operator valued measures [34, 113], this covariance assumption together with other mild assumptions completely characterize the covariant integral quantizations. We have also seen that the affine cross-Wigner distribution is given for sufficiently nice ψ, ϕ by $W_{\mathrm{Aff}}^{\psi,\phi} = (\psi \otimes \phi) \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$. Inspired by this and the description in [126] of the Cohen class of time-frequency distributions on $\mathbb{R}^{2n}$, we make the following definition.

Definition. A bilinear map $Q\colon L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \to L^\infty(\mathrm{Aff})$ belongs to the *affine Cohen class* if $Q = Q_S$ for some operator S on $L^2(\mathbb{R}_+)$, where

$$Q_S(\psi,\phi)(x,a) := (\psi \otimes \phi) \star_{\mathrm{Aff}} S(x,a) = \langle SU(-x,a)\psi, U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)}.$$

We will show how properties of S (such as admissibility) influence properties of Q_S, and obtain an abstract characterization of the affine Cohen class. Readers familiar with the Cohen class on $\mathbb{R}^{2n}$ [40] will know that it is defined in terms of convolutions with the Wigner function. In the affine setting, we have the analogous result

$$Q_{A_f}(\psi,\phi) = W_{\mathrm{Aff}}^{\psi,\phi} *_{\mathrm{Aff}} \check{f}.$$

As we explain in Proposition E.6.14, the *affine class of quadratic time-frequency representations* from [136] may be identified with a subclass of the affine Cohen class.

Structure of the Paper

In Section E.2 we recall necessary background material for completeness. In particular, Subsection E.2.2 should serve as a brief reference for quantum harmonic analysis on phase space. We define affine operator convolution in Subsection E.3.1 and show the relationship with the affine Weyl quantization in Subsection E.3.2.

The affine parity operator will be introduced in Subsection E.3.4, and its relationship to affine Weyl quantization will be explored in Subsection E.3.5.

We have dedicated the entirety of Section E.4 to operator admissibility. Section E.5 discusses affine Weyl quantization from the viewpoint of representation theory. In particular, in Subsection E.5.2 we derive a Bochner type theorem for our setting. In Subsection E.6.1 and Subsection E.6.2 we relate our work to time-scale localization operators and covariant integral quantizations, respectively. Finally, in Subsection E.6.3 we define the affine Cohen class and derive some basic properties.

E.2 Preliminaries

Notation: Given a Hilbert space $\mathcal{H}$ we let $\mathcal{L}(\mathcal{H})$ denote the bounded operators on $\mathcal{H}$. The notation $\mathcal{S}_p(\mathcal{H})$ for $1 \leq p < \infty$ will be used for the *Schatten-p class operators* on $\mathcal{H}$. We remark that $\mathcal{S}_1(\mathcal{H})$and $\mathcal{S}_2(\mathcal{H})$ are respectively the trace-class operators and the Hilbert-Schmidt operators on $\mathcal{H}$. The space $\mathcal{S}_\infty(\mathcal{H})$ is by definition $\mathcal{L}(\mathcal{H})$ for duality reasons. When the Hilbert space in question is $\mathcal{H} = L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, r^{-1}\,dr)$, we will simplify the notation to $\mathcal{S}_p := \mathcal{S}_p(L^2(\mathbb{R}_+))$ for readability. We will denote by $\mathscr{S}(\mathbb{R}^n)$ the space of Schwartz functions on $\mathbb{R}^n$. For a function f on a group G, the function $\check{f}$ is defined by $\check{f}(g) := f(g^{-1})$ for all $g \in G$.

E.2.1 Basic Constructions on the Affine Group

We begin by giving a brief introduction to the affine group and relevant constructions on it. The *(reduced) affine group* $(\mathrm{Aff}, \cdot_{\mathrm{Aff}})$ is the Lie group whose underlying set is the upper half plane $\mathrm{Aff} := \mathbb{R} \times \mathbb{R}_+ := \mathbb{R} \times (0, \infty)$, while the group operation is given by

$$(x, a) \cdot_{\mathrm{Aff}} (y, b) := (ay + x, ab), \qquad (x, a), (y, b) \in \mathrm{Aff}.$$

We will often neglect the subscript in the group operation to improve readability. Moreover, we use the notation $L_{(x,a)}$ and $R_{(x,a)}$ to denote respectively the *left-translation* and *right-translation* by $(x, a) \in \mathrm{Aff}$, acting on a function $f \colon \mathrm{Aff} \to \mathbb{C}$ by

$$\begin{aligned}\left(L_{(x,a)}f\right)(y, b) &:= f((x, a)^{-1} \cdot_{\mathrm{Aff}} (y, b)),\\ \left(R_{(x,a)}f\right)(y, b) &:= f((y, b) \cdot_{\mathrm{Aff}} (x, a)).\end{aligned}$$

Recall that the *translation operator* T_x and the *dilation operator* D_a are respectively given by

$$T_x f(y) := f(y - x), \qquad D_a f(y) := \frac{1}{\sqrt{a}} f\left(\frac{y}{a}\right), \qquad x, y \in \mathbb{R},\ a \in \mathbb{R}_+. \quad \text{(E.2.1)}$$

The following computation motivates the group operation on the affine group:

$$(T_x D_a)(T_y D_b) = T_x T_{ay} D_a D_b = T_{x+ay} D_{ab}.$$

We can represent the affine group Aff and its Lie algebra $\mathfrak{aff}$ in matrix form

$$\mathrm{Aff} = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} : a > 0,\, x \in \mathbb{R} \right\}, \quad \mathfrak{aff} = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} : u, v \in \mathbb{R} \right\}.$$

The Lie algebra structure of $\mathfrak{aff}$ is completely determined by

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

An important feature of the affine group is that it is non-unimodular; the left and right Haar measures are respectively given by

$$\mu_L(x,a) = \frac{dx\,da}{a^2}, \qquad \mu_R(x,a) = \frac{dx\,da}{a}.$$

As such, the modular function on the affine group is given by $\Delta(x,a) = a^{-1}$.

The affine group is *exponential*, meaning that the exponential map

$$\exp\colon \mathfrak{aff} \to \mathrm{Aff}$$

given by

$$\exp \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^u & \frac{v(e^u-1)}{u} \\ 0 & 1 \end{pmatrix}$$

is a global diffeomorphism. Hence we can write the left and right Haar measures in exponential coordinates by the formulas

$$\mu_L(x,a) = \frac{du\,dv}{\lambda(u)}, \qquad \mu_R(x,a) = \frac{du\,dv}{\lambda(-u)}, \qquad \lambda(u) \coloneqq \frac{ue^u}{e^u - 1}. \tag{E.2.2}$$

Elementary properties of the function λ can be found in [73, Section 3]. Throughout the paper, we will use the spaces

$$L^p_l(\mathrm{Aff}) \coloneqq L^p(\mathrm{Aff}, \mu_L), \qquad L^p_r(\mathrm{Aff}) \coloneqq L^p(\mathrm{Aff}, \mu_R),$$

for $1 \le p \le \infty$. Using that $(x,u) \mapsto (x, e^u)$ maps Aff to $\mathbb{R}^2$ we can define the Schwartz space on Aff.

Definition E.2.1. Let $\mathcal{S}(\mathrm{Aff})$ denote the smooth functions $f\colon \mathrm{Aff} \to \mathbb{C}$ such that

$$(x,\omega) \longmapsto f(x, e^\omega) \in \mathcal{S}(\mathbb{R}^2).$$

We refer to $\mathcal{S}(\mathrm{Aff})$ as the space of *rapidly decaying smooth functions* (or *Schwartz functions*) on the affine group.

There is a natural topology on $\mathcal{S}(\mathrm{Aff})$ induced by the semi-norms

$$\|f\|_{\alpha,\beta} := \sup_{x,\,\omega\in\mathbb{R}} |x|^{\alpha_1}|\omega|^{\alpha_2}\left|\partial_x^{\beta_1}\partial_\omega^{\beta_2} f(x,e^\omega)\right|, \tag{E.2.3}$$

for $\alpha := (\alpha_1,\alpha_2)$ and $\beta := (\beta_1,\beta_2)$ in $\mathbb{N}_0\times\mathbb{N}_0$. With these semi-norms, the space $\mathcal{S}(\mathrm{Aff})$ becomes a Fréchet space. The space of bounded, anti-linear functionals on $\mathcal{S}(\mathrm{Aff})$ is denoted by $\mathcal{S}'(\mathrm{Aff})$ and called the space of *tempered distributions* on Aff.

E.2.2 Quantum Harmonic Analysis on the Heisenberg Group

Before delving into quantum harmonic analysis on the affine group, it is advantageous to review the Heisenberg setting, originally introduced by Werner [153]. There are three primary constructions that appear: (a) A quantization scheme, (b) an integrated representation, and (c) a way to define convolution that incorporates operators. We give a brief overview of these three constructions and refer the reader to [81, 125, 153] for more details.

Weyl Quantization

The *cross-Wigner distribution* of $\phi,\psi\in L^2(\mathbb{R}^n)$ is given by

$$W(\phi,\psi)(x,\omega) := \int_{\mathbb{R}^n}\phi\left(x+\frac{t}{2}\right)\overline{\psi\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega t}\,dt, \quad (x,\omega)\in\mathbb{R}^{2n}.$$

When $\phi=\psi$ we refer to $W\phi := W(\phi,\phi)$ as the *Wigner distribution* of $\phi\in L^2(\mathbb{R}^n)$. The cross-Wigner distribution satisfies the orthogonality relation

$$\langle W(\phi_1,\psi_1),W(\phi_2,\psi_2)\rangle_{L^2(\mathbb{R}^{2n})} = \langle\phi_1,\phi_2\rangle_{L^2(\mathbb{R}^n)}\overline{\langle\psi_1,\psi_2\rangle}_{L^2(\mathbb{R}^n)},$$

for $\phi_1,\phi_2,\psi_1,\psi_2\in L^2(\mathbb{R}^n)$. Moreover, the Wigner distribution satisfies for an element $\phi\in\mathcal{S}(\mathbb{R}^n)$ the marginal properties

$$\int_{\mathbb{R}^n} W\phi(x,\omega)\,d\omega = |\phi(x)|^2, \qquad \int_{\mathbb{R}^n} W\phi(x,\omega)\,dx = |\hat{\phi}(x)|^2.$$

Our primary interest in the cross-Wigner distribution stems from the following connection: For each $f\in L^2(\mathbb{R}^{2n})$ we define the operator $L_f : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ by the formula

$$\langle L_f\phi,\psi\rangle_{L^2(\mathbb{R}^n)} = \langle f, W(\psi,\phi)\rangle_{L^2(\mathbb{R}^{2n})}, \qquad \phi,\psi\in L^2(\mathbb{R}^n).$$

Then L_f is the *Weyl quantization* of f, see [81, Chapter 14] for details. It is a non-trivial fact, see [138], that the Weyl quantization gives a well-defined isomorphism between $L^2(\mathbb{R}^{2n})$ and $\mathcal{S}_2(L^2(\mathbb{R}^n))$, the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$.

Integrated Schrödinger Representation

Recall that the Heisenberg group $\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and with the group multiplication

$$(x, \omega, t) \cdot (x', \omega', t') := \left(x + x', \omega + \omega', t + t' + \frac{1}{2} (x'\omega - x\omega') \right).$$

The Heisenberg group is omnipresent in modern mathematics and theoretical physics, see [100]. For a Hilbert space $\mathcal{H}$ we let $\mathcal{U}(\mathcal{H})$ denote the unitary operators on $\mathcal{H}$. The most important representation of the Heisenberg group is the *Schrödinger representation* $\rho\colon \mathbb{H}^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ given by

$$\rho(x, \omega, t)\phi(y) := e^{2\pi i t} e^{-\pi i x \omega} M_\omega T_x \phi(y),$$

where T_x is the n-dimensional analogue of the translation operator defined in (E.2.1) and M_ω is the *modulation operator* given by

$$M_\omega \phi(y) := e^{2\pi i \omega y} \phi(y), \qquad \phi \in L^2(\mathbb{R}^n).$$

The Schrödinger representation is both irreducible and unitary.

Let us use the abbreviated notation $z := (x, \omega) \in \mathbb{R}^{2n}$ and $\pi(z) := M_\omega T_x$. Ignoring the central variable t, we can consider the *integrated Schrödinger representation*

$$\rho\colon L^1(\mathbb{R}^{2n}) \to \mathcal{L}(L^2(\mathbb{R}^n))$$

given by

$$\rho(f) := \int_{\mathbb{R}^{2n}} f(z) e^{-\pi i x \omega} \pi(z)\, dz, \tag{E.2.4}$$

where $\mathcal{L}(L^2(\mathbb{R}^n))$ denotes the bounded linear operators on $L^2(\mathbb{R}^n)$. We remark that the integral in (E.2.4) is defined weakly. It turns out, see [68, Theorem 1.30], that the integrated representation ρ extends from $L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$ to a unitary map $\rho\colon L^2(\mathbb{R}^{2n}) \to \mathcal{S}_2(L^2(\mathbb{R}^n))$.

Operator Convolution

Given a function $f \in L^1(\mathbb{R}^{2n})$ and a trace-class operator $S \in \mathcal{S}_1(L^2(\mathbb{R}^n))$, their convolution is the trace-class operator on $L^2(\mathbb{R}^n)$ defined by

$$f \star S := \int_{\mathbb{R}^{2n}} f(z) \pi(z) S \pi(z)^*\, dz.$$

The convolution $f \star S$ satisfies the estimate $\|f \star S\|_{\mathcal{S}_1} \le \|f\|_{L^1} \|S\|_{\mathcal{S}_1}$.

One can also define the convolution between two operators: For two trace-class operators $S, T \in \mathcal{S}_1(L^2(\mathbb{R}^n))$ we define their convolution to be the function on $\mathbb{R}^{2n}$ given by

$$S \star T(z) := \operatorname{tr}(S\pi(z)PTP\pi(z)^*),$$

where $P\psi(t) := \psi(-t)$ is the *parity operator*. The convolution $S \star T$ satisfies the estimate $\|S \star T\|_{L^1} \leq \|S\|_{\mathcal{S}_1}\|T\|_{\mathcal{S}_1}$ and the important relation [153, Lemma 3.1]

$$\int_{\mathbb{R}^{2n}} S \star T(z)\, dz = \operatorname{tr}(S)\operatorname{tr}(T). \tag{E.2.5}$$

To see the connection with the Wigner distribution, we note that the cross-Wigner distribution of $\psi, \phi \in L^2(\mathbb{R}^n)$ can be written as

$$W(\psi, \phi) = \psi \otimes \phi \star P, \tag{E.2.6}$$

where $\psi \otimes \phi$ denotes the rank-one operator on $L^2(\mathbb{R}^n)$ given by

$$(\psi \otimes \phi)(\xi) := \langle \xi, \phi \rangle_{L^2(\mathbb{R}^n)}\psi \qquad \text{for } \xi \in L^2(\mathbb{R}^n).$$

Similarly, the Weyl quantization of $f \in L^1(\mathbb{R}^{2n})$ may be expressed in terms of operator convolutions:

$$L_f = f \star P. \tag{E.2.7}$$

Hence convolution with the parity operator P gives a convenient way to represent the Wigner distribution and the Weyl quantization.

Finally, there is a Fourier transform for operators: Given a trace-class operator $S \in \mathcal{S}_1(L^2(\mathbb{R}^n))$ we define the *Fourier-Wigner transform* $\mathcal{F}_W(S)$ of S to be the function on $\mathbb{R}^{2n}$ given by

$$\mathcal{F}_W(S)(z) := e^{i\pi x\omega}\operatorname{tr}(S\pi(z)^*), \qquad z \in \mathbb{R}^{2n}. \tag{E.2.8}$$

The Fourier-Wigner transform extends to a unitary map

$$\mathcal{F}_W : \mathcal{S}_2(L^2(\mathbb{R}^n)) \to L^2(\mathbb{R}^{2n}),$$

where it turns out to be the inverse of the integrated Schrödinger representation given in (E.2.4). By [68, Proposition 2.5] it is related to the Weyl transform by the elegant formula

$$f = \mathcal{F}_\sigma(\mathcal{F}_W(L_f)), \tag{E.2.9}$$

where $\mathcal{F}_\sigma$ denotes the *symplectic Fourier transform*.

E.2.3 Affine Weyl Quantization

We briefly describe affine Weyl quantization and how this gives rise to the affine Wigner distribution. There is a unitary representation π of the affine group Aff on $L^2(\mathbb{R}_+, r^{-1}\,dr)$ given by

$$U(x,a)\psi(r) := e^{2\pi i x r}\psi(ar) = \frac{1}{\sqrt{a}}M_x D_{\frac{1}{a}}\psi(r). \tag{E.2.10}$$

Since $r^{-1}\,dr$ is the Haar measure on $\mathbb{R}_+$ we will write $L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, r^{-1}\,dr)$. Later we also consider another measure on $\mathbb{R}_+$ and will be more explicit when the situation requires it.

To define the quantization scheme we will utilize the *Stratonovich-Weyl operator* on $L^2(\mathbb{R}_+)$ given by

$$\Omega(x,a)\psi(r) := a\int_{\mathbb{R}^2} e^{-2\pi i(xu+av)}U\left(\frac{ve^u}{\lambda(u)}, e^u\right)\psi(r)\,du\,dv. \tag{E.2.11}$$

The following result was shown in [73] and provides us with an affine analogue of Weyl quantization.

Proposition E.2.2 ([73])**.** There is a norm-preserving isomorphism between the space $L_r^2(\mathrm{Aff})$ and the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$. The isomorphism sends $f \in L_r^2(\mathrm{Aff})$ to the operator A_f on $L^2(\mathbb{R}_+)$ defined weakly by

$$A_f\psi(r) := \int_{-\infty}^{\infty}\int_0^{\infty} f(x,a)\Omega(x,a)\psi(r)\,\frac{da\,dx}{a}, \qquad \psi \in L^2(\mathbb{R}_+).$$

We will refer to the association $f \mapsto A_f$ as *affine Weyl quantization*, while f is called the *affine (Weyl) symbol* of A_f. To emphasize the correspondence between a Hilbert-Schmidt operator A and its affine symbol f we use the notation $f_A := f$. The affine Weyl symbol of an operator A is explicitly given by

$$f_A(x,a) = \int_{-\infty}^{\infty} A_K\left(a\lambda(u), a\lambda(-u)\right)e^{-2\pi i x u}\,du, \tag{E.2.12}$$

where $A_K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ is the integral kernel of A defined by

$$A\psi(r) = \int_0^{\infty} A_K(r,s)\psi(s)\,\frac{ds}{s}, \qquad \psi \in L^2(\mathbb{R}_+).$$

By taking the affine Weyl symbol of the rank-one operator $\psi \otimes \phi$ on $L^2(\mathbb{R}_+)$ given by

$$\psi \otimes \phi(\xi) := \langle \xi, \phi\rangle_{L^2(\mathbb{R}_+)}\psi$$

for $\psi, \phi, \xi \in L^2(\mathbb{R}_+)$, we obtain the following definition.

Definition E.2.3. For $\phi, \psi \in L^2(\mathbb{R}_+)$ we define the *affine (cross-)Wigner distribution* $W_{\mathrm{Aff}}^{\psi,\phi}$ to be the function on Aff given for $(x, a) \in \mathrm{Aff}$ by

$$\begin{aligned} W_{\mathrm{Aff}}^{\psi,\phi}(x,a) &:= \int_{-\infty}^{\infty} \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}e^{-2\pi i x u}\, du \\ &= \int_{-\infty}^{\infty} \psi\left(\frac{aue^u}{e^u - 1}\right)\overline{\phi\left(\frac{au}{e^u-1}\right)}e^{-2\pi i x u}\, du. \end{aligned}$$

When $\phi = \psi$ we refer to $W_{\mathrm{Aff}}^{\psi} := W_{\mathrm{Aff}}^{\psi,\psi}$ as the *affine Wigner distribution* of ψ. The weak interpretation of the integral defining A_f means that we have the relation

$$\left\langle A_f\phi, \psi\right\rangle_{L^2(\mathbb{R}_+)} = \left\langle f, W_{\mathrm{Aff}}^{\psi,\phi}\right\rangle_{L^2_r(\mathrm{Aff})}, \tag{E.2.13}$$

for $f \in L^2_r(\mathrm{Aff})$ and $\phi, \psi \in L^2(\mathbb{R}_+)$. The affine Wigner distribution satisfies the orthogonality relation

$$\int_{-\infty}^{\infty}\int_0^{\infty} W_{\mathrm{Aff}}^{\psi_1,\psi_2}(x,a)\overline{W_{\mathrm{Aff}}^{\phi_1,\phi_2}(x,a)}\,\frac{da\,dx}{a} = \langle\psi_1,\phi_1\rangle_{L^2(\mathbb{R}_+)}\overline{\langle\psi_2,\phi_2\rangle}_{L^2(\mathbb{R}_+)},$$

for $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}_+)$. Moreover, the affine Wigner distribution also satisfies the marginal property

$$\int_{-\infty}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\, dx = |\psi(a)|^2, \quad (x,a) \in \mathrm{Aff}, \tag{E.2.14}$$

for all rapidly decaying smooth functions ψ on $\mathbb{R}_+$. We remark that a *rapidly decaying smooth function* (also called a *Schwartz function*) $\psi\colon \mathbb{R}_+ \to \mathbb{C}$ is by definition a smooth function such that $x \mapsto \psi(e^x)$ is a rapidly decaying function on $\mathbb{R}$. The space of all rapidly decaying smooth functions on $\mathbb{R}_+$ will be denoted by $\mathcal{S}(\mathbb{R}_+)$. We will later also need the space $\mathcal{S}'(\mathbb{R}_+)$ of bounded, anti-linear functionals on $\mathcal{S}(\mathbb{R}_+)$ called the *tempered distributions* on $\mathbb{R}_+$. For more information regarding the affine Wigner distribution the reader is referred to [17].

E.3 Affine Operator Convolutions

In this part we introduce operator convolutions in the affine setting. We show that this notion is intimately related to affine Weyl quantization in Subsection E.3.2. In Subsection E.3.4 we will introduce the affine Grossmann-Royer operator, which will be essential in Subsection E.3.5 where we prove the main connection between the affine Weyl quantization and the operator convolutions in Theorem E.3.20.

E.3.1 Definitions and Basic Properties

We begin by defining operator convolutions in the affine setting and derive basic properties. Recall that the usual convolution on the affine group with respect to the right Haar measure is given by

$$f *_{\mathrm{Aff}} g(x,a) := \int_{\mathrm{Aff}} f(y,b) g((x,a)\cdot(y,b)^{-1}) \, \frac{dy\, db}{b}.$$

Remark. Other sources, e.g. [67], use the left Haar measure and define the convolution to be

$$f *_{\mathrm{Aff}_L} g((x,a)) := \check{f} *_{\mathrm{Aff}} \check{g}((x,a)^{-1}),$$

where $\check{f}(x,a) := f((x,a)^{-1})$. We will mainly work with the right Haar measure, and our definition ensures that

$$\|f *_{\mathrm{Aff}} g\|_{L^1_r(\mathrm{Aff})} \leq \|f\|_{L^1_r(\mathrm{Aff})} \|g\|_{L^1_r(\mathrm{Aff})}.$$

Additionally, we have that

$$R_{(x,a)}(f *_{\mathrm{Aff}} g) = (R_{(x,a)} f) *_{\mathrm{Aff}} g.$$

Definition E.3.1. Let $f \in L^1_r(\mathrm{Aff})$ and let S be a trace-class operator on $L^2(\mathbb{R}_+)$. We define the *convolution* $f \star_{\mathrm{Aff}} S$ between f and S to be the operator on $L^2(\mathbb{R}_+)$ given by

$$f \star_{\mathrm{Aff}} S := \int_{\mathrm{Aff}} f(x,a) U(-x,a)^* S U(-x,a) \, \frac{dx\, da}{a},$$

where U is the unitary representation given in (E.2.10). The integral is a convergent Bochner integral in the space of trace-class operators.

Remarks.

- As we will see later, using $U(-x,a)$ instead of $U(x,a)$ in Definition E.3.1 ensures that the convolution is compatible with the following covariance property of the affine Wigner distribution:

$$W_{\mathrm{Aff}}^{U(-x,a)\phi, U(-x,a)\psi}(y,b) = W_{\mathrm{Aff}}^{\phi,\psi}((y,b)\cdot(x,a)).$$

- The notation $\star$ has a different meaning in [73], where it is used to denote the so-called Moyal product of two functions defined on Aff.

Definition E.3.2. Let S be a trace-class operator and let T be a bounded operator on $L^2(\mathbb{R}_+)$. Then we define the *convolution* $S \star_{\mathrm{Aff}} T$ between S and T to be the function on Aff given by

$$S \star_{\mathrm{Aff}} T(x,a) := \mathrm{tr}(S U(-x,a)^* T U(-x,a)).$$

Remark. Recently, [37] defined another notion of convolution of trace-class operators. Unlike our definition, this convolution produces a new trace-class operator, with the aim of interpreting the trace-class operators as an analogue of the Fourier algebra.

It is straightforward to check that if f is a positive function and S, T are positive operators, then $f \star_{\mathrm{Aff}} S$ is a positive operator and $S \star_{\mathrm{Aff}} T$ is a positive function. Moreover, we have the elementary estimates

$$\|f \star_{\mathrm{Aff}} S\|_{\mathcal{S}_1} \leq \|f\|_{L^1_r(\mathrm{Aff})} \|S\|_{\mathcal{S}_1} \tag{E.3.1}$$

and

$$\|S \star_{\mathrm{Aff}} T\|_{L^\infty(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_1} \|T\|_{\mathcal{L}(L^2(\mathbb{R}_+))}. \tag{E.3.2}$$

The following result is proved by a simple computation.

Lemma E.3.3. *For $\psi, \phi \in L^2(\mathbb{R}_+)$ and $S \in \mathcal{L}(L^2(\mathbb{R}_+))$ we have*

$$(\psi \otimes \phi) \star_{\mathrm{Aff}} S(x, a) = \langle SU(-x, a)\psi, U(-x, a)\phi \rangle_{L^2(\mathbb{R}_+)}.$$

In particular, for η, $\xi \in L^2(\mathbb{R}_+)$ we have

$$(\psi \otimes \phi) \star_{\mathrm{Aff}} (\eta \otimes \xi)(x, a) = \langle \psi, U(-x, a)^* \xi \rangle_{L^2(\mathbb{R}_+)} \overline{\langle \phi, U(-x, a)^* \eta \rangle_{L^2(\mathbb{R}_+)}},$$

and

$$(\psi \otimes \psi) \star_{\mathrm{Aff}} (\xi \otimes \xi)(x, a) = |\langle \psi, U(-x, a)^* \xi \rangle_{L^2(\mathbb{R}_+)}|^2.$$

A natural question to ask is whether the three different notions of convolution we have introduced are compatible. The following proposition gives an affirmative answer to this question.

Proposition E.3.4. Let $f, g \in L^1_r(\mathrm{Aff})$, $S \in \mathcal{S}_1$, and let T be a bounded operator on $L^2(\mathbb{R}_+)$. Then we have the compatibility equations

$$(f \star_{\mathrm{Aff}} S) \star_{\mathrm{Aff}} T = f *_{\mathrm{Aff}} (S \star_{\mathrm{Aff}} T),$$
$$f \star_{\mathrm{Aff}} (g \star_{\mathrm{Aff}} S) = (f *_{\mathrm{Aff}} g) \star_{\mathrm{Aff}} S.$$

Proof. The first equality follows since the left-hand side can be written as

$$\begin{aligned}
&\int_{\mathrm{Aff}} f(y, b)\, \mathrm{tr}(SU(-y, b)U(-x, a)^* TU(-x, a)U(-y, b)^*)\, \frac{dy\, db}{b} \\
&= \int_{\mathrm{Aff}} f(y, b)\, \mathrm{tr}(U(-y, b)^* SU(-y, b)U(-x, a)^* TU(-x, a))\, \frac{dy\, db}{b} \\
&= \mathrm{tr}\left(\int_{\mathrm{Aff}} f(y, b) U(-y, b)^* SU(-y, b)\, \frac{dy\, db}{b} U(-x, a)^* TU(-x, a) \right) \\
&= ((f \star_{\mathrm{Aff}} S) \star_{\mathrm{Aff}} T)\, (x, a).
\end{aligned}$$

We are allowed to take the trace outside the integral since the second to last line is essentially the duality action of the bounded operator $U(-x,a)^*TU(-x,a)$ on a convergent Bochner integral in the space of trace-class operators.

For the second equality, change variables and write the left-hand side as

$$\begin{aligned}
&\int_{\mathrm{Aff}}\int_{\mathrm{Aff}} f(x,a)g((z,c)\cdot(x,a)^{-1})U(-z,c)^*SU(-z,c)\,\frac{dx\,da}{a}\,\frac{dz\,dc}{c}\\
&=\int_{\mathrm{Aff}}\int_{\mathrm{Aff}} f(x,a)g(y,b)U(-x,a)^*U(-y,b)^*SU(-y,b)U(-x,a)\,\frac{dy\,db}{b}\,\frac{dx\,da}{a}\\
&=\int_{\mathrm{Aff}} f(x,a)U(-x,a)^*\int_{\mathrm{Aff}} g(y,b)U(-y,b)^*SU(-y,b)\,\frac{dy\,db}{b}U(-x,a)\,\frac{dx\,da}{a}\\
&=f\star_{\mathrm{Aff}}(g\star_{\mathrm{Aff}}S).
\end{aligned}$$

Changing the order of integration above is allowed by Fubini's theorem for Bochner integrals [106, Proposition 1.2.7]. Fubini's theorem is applicable since

$$\int_{\mathrm{Aff}}\int_{\mathrm{Aff}} |f(x,a)|\cdot|g((z,c)\cdot(x,a)^{-1})|\cdot\|U(-z,c)^*SU(-z,c)\|_{\mathcal{S}_1}\,\frac{dx\,da}{a}\,\frac{dz\,dc}{c}$$

is bounded from above by

$$\|S\|_{\mathcal{S}_1}\int_{\mathrm{Aff}}|f(x,a)|\,\frac{dx\,da}{a}\int_{\mathrm{Aff}}|g(z,c)|\,\frac{dz\,dc}{c}<\infty. \qquad \square$$

E.3.2 Relationship With Affine Weyl Quantization

The goal of this subsection is to connect the affine Weyl quantization described in Subsection E.2.3 with the convolutions defined in Subsection E.3.1. We first establish a preliminary result describing how right multiplication on the affine group affects the affine Weyl quantization.

Lemma E.3.5. *Let $A_f \in \mathcal{S}_2$ with affine Weyl symbol $f \in L^2_r(\mathrm{Aff})$. For an element $(x,a)\in\mathrm{Aff}$, the affine Weyl symbol of $U(-x,a)^*A_fU(-x,a)$ is $R_{(x,a)^{-1}}f$.*

Proof. The result follows from (E.2.13) and the computation

$$\begin{aligned}
\langle U(-x,a)^*A_fU(-x,a)\psi,\phi\rangle_{L^2(\mathbb{R}_+)} &= \langle A_fU(-x,a)\psi,U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)}\\
&=\langle f, W_{\mathrm{Aff}}^{U(-x,a)\phi,U(-x,a)\psi}\rangle_{L^2_r(\mathrm{Aff})}\\
&=\langle f, R_{(x,a)}W_{\mathrm{Aff}}^{\phi,\psi}\rangle_{L^2_r(\mathrm{Aff})}\\
&=\langle R_{(x,a)^{-1}}f, W_{\mathrm{Aff}}^{\phi,\psi}\rangle_{L^2_r(\mathrm{Aff})}. \qquad \square
\end{aligned}$$

We are now ready to prove the first result showing the connection between convolution and affine Weyl quantization.

Proposition E.3.6. Assume that $A_f \in \mathcal{S}_2$ with affine Weyl symbol $f \in L^2_r(\mathrm{Aff})$, and let $g \in L^1_r(\mathrm{Aff})$. Then the affine Weyl symbol of $g \star_{\mathrm{Aff}} A_f$ is $g *_{\mathrm{Aff}} f$, that is,

$$g \star_{\mathrm{Aff}} A_f = A_{g *_{\mathrm{Aff}} f}.$$

Proof. The operator $g \star_{\mathrm{Aff}} A_f$ is defined as the $\mathcal{S}_2$-convergent Bochner integral

$$g \star_{\mathrm{Aff}} A_f = \int_{\mathrm{Aff}} g(x,a) U(-x,a)^* A_f U(-x,a) \, \frac{dx\, da}{a}.$$

By Proposition E.2.2, the map $\mathfrak{W} \colon \mathcal{S}_2 \to L^2_r(\mathrm{Aff})$ given by $\mathfrak{W}(A_f) := f$ is unitary. Since bounded operators commute with convergent Bochner integrals, we have using Lemma E.3.5 that

$$\begin{aligned}
\mathfrak{W}\left(g \star_{\mathrm{Aff}} A_f\right) &= \int_{\mathrm{Aff}} g(x,a) \mathfrak{W}\left(U(-x,a)^* A_f U(-x,a)\right) \, \frac{dx\, da}{a} \\
&= \int_{\mathrm{Aff}} g(x,a) R_{(x,a)^{-1}} \mathfrak{W}\left(A_f\right) \, \frac{dx\, da}{a} \\
&= g *_{\mathrm{Aff}} f.
\end{aligned}$$

□

We can also express the convolution of two operators in terms of their affine Weyl symbols.

Proposition E.3.7. Let $A_f, A_g \in \mathcal{S}_2$ with affine Weyl symbols $f, g \in L^2_r(\mathrm{Aff})$. If additionally $g \in L^2_l(\mathrm{Aff})$, then we have

$$A_f \star_{\mathrm{Aff}} A_g = f *_{\mathrm{Aff}} \check{g},$$

where $\check{g}(x,a) := g((x,a)^{-1})$ for $(x,a) \in \mathrm{Aff}$.

Proof. Using Proposition E.2.2 and Lemma E.3.5 we compute that

$$\begin{aligned}
(A_f \star_{\mathrm{Aff}} A_g)(x,a) &= \mathrm{tr}(A_f U(-x,a)^* A_g U(-x,a)) \\
&= \langle A_f, U(-x,a)^* A_g^* U(-x,a) \rangle_{\mathcal{S}_2} \\
&= \langle f, R_{(x,a)^{-1}} \overline{g} \rangle_{L^2_r(\mathrm{Aff})} \\
&= \int_{\mathrm{Aff}} f(y,b) g((y,b) \cdot (x,a)^{-1}) \, \frac{dy\, db}{b} \\
&= \int_{\mathrm{Aff}} f(y,b) \check{g}((x,a) \cdot (y,b)^{-1}) \, \frac{dy\, db}{b} \\
&= f *_{\mathrm{Aff}} \check{g}(x,a).
\end{aligned}$$

The result follows as $\check{g} \in L^2_r(\mathrm{Aff})$ if and only if $g \in L^2_l(\mathrm{Aff})$. □

E.3.3 Affine Weyl Quantization of Coordinate Functions

Of particular interest is the affine Weyl quantization of the coordinate functions $f_x(x,a) := x$ and $f_a(x,a) := a$ for $(x,a) \in \mathrm{Aff}$. Due to the fact that the coordinate functions are not in $L^2_r(\mathrm{Aff})$, we first need to interpret the quantizations A_{f_x} and A_{f_a} in a rigorous manner.

Lemma E.3.8. *For any $f \in \mathcal{S}'(\mathrm{Aff})$ we can define A_f as the map*

$$A_f : \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}'(\mathbb{R}_+)$$

defined by the relation

$$\langle A_f \psi, \phi \rangle_{\mathcal{S}',\mathcal{S}} = \left\langle f, W^{\phi,\psi}_{\mathrm{Aff}} \right\rangle_{\mathcal{S}',\mathcal{S}}, \qquad \psi, \phi \in \mathcal{S}(\mathbb{R}_+). \tag{E.3.3}$$

Additionally, the map $f \mapsto A_f$ is injective.

Proof. It was shown in [17, Corollary 6.6] that for elements $\phi, \psi \in \mathcal{S}(\mathbb{R}_+)$ then $W^{\phi,\psi}_{\mathrm{Aff}} \in \mathcal{S}(\mathrm{Aff})$. Hence the pairing on the right-hand side of (E.3.3) is well-defined.

For the injectivity it suffices to show that $A_f = 0$ implies that $f = 0$. Let us first reformulate this slightly: If $A_f = 0$, then we have that

$$\langle A_f \psi, \phi \rangle_{\mathcal{S}',\mathcal{S}} = \left\langle f, W^{\phi,\psi}_{\mathrm{Aff}} \right\rangle_{\mathcal{S}',\mathcal{S}} = 0$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$. We could conclude that $f = 0$ if we knew that any $g \in \mathcal{S}(\mathrm{Aff})$ could be approximated (in the Fréchet topology) by linear combinations of elements on the form $W^{\phi,\psi}_{\mathrm{Aff}}$ for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$. To see that this is the case, we translate the problem to the Heisenberg setting.

The *Mellin transform* $\mathcal{M}$ is given by

$$\mathcal{M}(\phi)(x) = \mathcal{M}_r(\phi)(x) := \int_0^\infty \phi(r) r^{-2\pi i x} \, \frac{dr}{r}.$$

Define the functions Ψ and Φ to be $\Psi(x) := \psi(e^x)$ and $\Phi(x) := \phi(e^x)$ for $\psi, \phi \in L^2(\mathbb{R}_+)$. A reformulation of [17, Lemma 6.4] shows that we have the relation

$$W^{\psi,\phi}_{\mathrm{Aff}}(x,a) = \mathcal{M}_y^{-1} \otimes \mathcal{M}_b \left[\left(\frac{\sqrt{b}\log(b)}{b-1} \right)^{2\pi i y} \mathcal{F}_\sigma W(\Psi, \Phi)\big(\log(b), y\big) \right](x,a),$$

where W is the cross-Wigner distribution. The correspondence preserves Schwartz functions, due to the term

$$\left(\frac{\sqrt{b}\log(b)}{b-1} \right)^{2\pi i y}$$

being smooth with polynomially bounded derivatives. This gives a bijective correspondence between $W_{\mathrm{Aff}}^{\psi,\phi} \in \mathcal{S}(\mathrm{Aff})$ and $W(\Psi,\Phi) \in \mathcal{S}(\mathbb{R}^2)$.

As such, the injectivity question is reduced to asking whether the linear span of elements on the form $W(f,g)$ for $f,g \in \mathcal{S}(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R}^2)$. One way to verify this well known fact is to note that the map $f \otimes g \mapsto W(f,g)$, where $f \otimes g(x,y) = f(x)g(y)$, extends to a topological isomorphism on $\mathcal{S}(\mathbb{R}^2)$, see for instance [81, Equation (14.21)] for the formula of this isomorphism. The density of elements on the form $W(f,g)$ for $f,g \in \mathcal{S}(\mathbb{R})$ therefore follows as the functions $h_m \otimes h_n$, where $\{h_n\}_{n=0}^{\infty}$ are the Hermite functions, span a dense subspace of $\mathcal{S}(\mathbb{R}^2)$ by [140, Theorem V.13]. □

Example E.3.9. Consider the constant function on the affine group given by $1(x,a) := 1$ for all $(x,a) \in \mathrm{Aff}$. Then the affine Weyl quantization A_1 is the identity operator on $L^2(\mathbb{R}_+)$ since for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$ we have

$$\begin{aligned}\langle A_1\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} &= \langle 1, W_{\mathrm{Aff}}^{\phi,\psi}\rangle_{\mathcal{S}',\mathcal{S}} \\ &= \int_{\mathrm{Aff}} \overline{W_{\mathrm{Aff}}^{\phi,\psi}(x,a)}\, \frac{da\, dx}{a} \\ &= \int_0^{\infty} \psi(a)\overline{\phi(a)}\, \frac{da}{a} \\ &= \langle \psi, \phi\rangle_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Notice that we used a straightforward generalization of the marginal property of the affine Wigner distribution given in (E.2.14), see the proof of [17, Proposition 3.4] for details.

To motivate the next result, consider the coordinate functions $\sigma_x(x,\omega) := x$ and $\sigma_\omega(x,\omega) := \omega$ for $(x,\omega) \in \mathbb{R}^{2n}$. The Weyl quantizations L_{σ_x} and L_{σ_ω} are the well known *position operator* and *momentum operator* in quantum mechanics. In particular, the commutator

$$\left[L_{\sigma_x}, L_{\sigma_\omega}\right] := L_{\sigma_x} \circ L_{\sigma_\omega} - L_{\sigma_\omega} \circ L_{\sigma_x}$$

is a constant times the identity by [92, Proposition 3.8]. This is precisely the relation for the Lie algebra of the Heisenberg group. In light of this, the following proposition shows that the affine Weyl quantization has the expected expression for the coordinate functions.

Theorem E.3.10. *Let f_x and f_a be the coordinate functions on the affine group. The affine Weyl quantizations A_{f_x} and A_{f_a} are well-defined as maps from $\mathcal{S}(\mathbb{R}_+)$ to $\mathcal{S}'(\mathbb{R}_+)$ and are explicitly given by*

$$A_{f_x}\psi(r) = \frac{1}{2\pi i} r\psi'(r), \qquad A_{f_a}\psi(r) = r\psi(r), \qquad \psi \in \mathcal{S}(\mathbb{R}_+).$$

In particular, we have the commutation relation

$$[A_{f_x}, A_{f_a}] = \frac{1}{2\pi i} A_{f_a}.$$

This is, up to re-normalization, precisely the infinitesimal structure of the affine group.

Proof. Let us begin by computing A_{f_x}. We can change the order of integrating by Fubini's theorem and obtain for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$ that

$$\begin{aligned}\langle A_{f_x}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} &= \left\langle f_x, W_{\mathrm{Aff}}^{\phi,\psi} \right\rangle_{\mathcal{S}',\mathcal{S}} \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} x \overline{\int_{-\infty}^{\infty} \phi(a\lambda(u))\overline{\psi(a\lambda(-u))} e^{-2\pi i x u}\, du}\, \frac{da\, dx}{a} \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} \left(\int_{-\infty}^{\infty} x e^{2\pi i x u}\, dx\right) \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}\, \frac{da\, du}{a}.\end{aligned}$$

Notice that the inner integral is equal to

$$\int_{-\infty}^{\infty} x e^{2\pi i x u}\, dx = \frac{1}{2\pi i}\delta_0'(u),$$

where

$$\int_{-\infty}^{\infty} \delta_0'(u)\psi(u)\, du = \psi'(0).$$

Hence we have the relation

$$\langle A_{f_x}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{2\pi i}\int_0^{\infty} \frac{\partial}{\partial u}\left(\psi(a\lambda(u))\overline{\phi(a\lambda(-u))}\right)\Big|_{u=0} \frac{da}{a}.$$

By using the formulas $\lambda(0) = 1$ and $\lambda'(0) = 1/2$ we can simplify and obtain

$$\langle A_{f_x}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{4\pi i}\int_0^{\infty} a \cdot \left(\psi'(a)\overline{\phi(a)} - \psi(a)\overline{\phi'(a)}\right) \frac{da}{a}.$$

Using integration by parts we obtain the claim since

$$\langle A_{f_x}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} = \int_0^{\infty} \left[\frac{1}{2\pi i} a\psi'(a)\right] \overline{\phi(a)}\, \frac{da}{a}.$$

For A_{f_a} we have by similar calculations as above that

$$\begin{aligned}\langle A_{f_a}\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} &= \int_{-\infty}^{\infty}\int_0^{\infty} \left(\int_{-\infty}^{\infty} 1 \cdot e^{2\pi i x u}\, dx\right) a \cdot \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}\, \frac{da\, du}{a} \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} \delta_0(u)\left(a \cdot \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}\right) \frac{da\, du}{a} \\ &= \int_0^{\infty} a\psi(a)\overline{\phi(a)}\, \frac{da}{a}.\end{aligned}$$

The commutation relation follows from straightforward computation. □

E.3.4 The Affine Grossmann-Royer Operator

In this subsection we introduce the affine Grossmann-Royer operator with the aim of obtaining an affine parity operator analogous to the (Heisenberg) parity operator P in Subsection E.2.2. The main reason for this is to obtain affine version of the formulas (E.2.6) and (E.2.7) so that we can describe the affine Weyl quantization through convolution. Recall that the *(Heisenberg) Grossmann-Royer operator* $R(x, \omega)$ for $(x, \omega) \in \mathbb{R}^{2n}$ is defined by the relation

$$W(f, g)(x, \omega) = \langle R(x, \omega) f, g \rangle_{L^2(\mathbb{R}^n)}, \qquad f, g \in L^2(\mathbb{R}^n).$$

Analogously, we have the following definition.

Definition E.3.11. We define the *affine Grossmann-Royer operator $R_{\mathrm{Aff}}(x, a)$* for $(x, a) \in \mathrm{Aff}$ by the relation

$$W_{\mathrm{Aff}}^{\psi,\phi}(x, a) = \langle R_{\mathrm{Aff}}(x, a)\psi, \phi \rangle_{\mathcal{S}', \mathcal{S}}, \qquad \psi, \phi \in \mathcal{S}(\mathbb{R}_+).$$

We restrict our attention to Schwartz functions for convenience since then $W_{\mathrm{Aff}}^{\psi,\phi} \in \mathcal{S}(\mathrm{Aff})$ by [17, Corollary 6.6], and hence have well-defined point values. The Grossmann-Royer operator $R_{\mathrm{Aff}}(x, a)$ is precisely the affine Weyl quantization of the point mass $\delta_{\mathrm{Aff}}(x, a) \in \mathcal{S}'(\mathrm{Aff})$ for $(x, a) \in \mathrm{Aff}$ defined by

$$\langle \delta_{\mathrm{Aff}}(x, a), f \rangle_{\mathcal{S}', \mathcal{S}} := \overline{f(x, a)}, \qquad f \in \mathcal{S}(\mathrm{Aff}).$$

Since this is also true for the Stratonovich-Weyl operator $\Omega(x, a)$ given in (E.2.11), it follows that $R_{\mathrm{Aff}}(x, a) = \Omega(x, a)$ for all $(x, a) \in \mathrm{Aff}$. From [73, Page 12] it follows that we have the *affine covariance relation*

$$U(-x, a)^* R_{\mathrm{Aff}}(0, 1) U(-x, a) = R_{\mathrm{Aff}}(x, a).$$

The following result, which is a straightforward computation, shows that $R_{\mathrm{Aff}}(x, a)$ is an unbounded and densely defined operator on $L^2(\mathbb{R}_+)$.

Lemma E.3.12. *Fix $\psi \in \mathcal{S}(\mathbb{R}_+)$ and $(x, a) \in \mathrm{Aff}$. The affine Grossmann-Royer operator $R_{\mathrm{Aff}}(x, a)$ has the explicit form*

$$R_{\mathrm{Aff}}(x, a)\psi(r) = \frac{e^{2\pi i x \lambda^{-1}\left(\frac{r}{a}\right)} \lambda^{-1}\left(\frac{r}{a}\right)\left(1 - e^{\lambda^{-1}\left(\frac{r}{a}\right)}\right)}{1 + \lambda^{-1}\left(\frac{r}{a}\right) - e^{\lambda^{-1}\left(\frac{r}{a}\right)}} \cdot \psi\left(r e^{-\lambda^{-1}\left(\frac{r}{a}\right)}\right),$$

where λ is the function given in (E.2.2).

We will be particularly interested in the *affine parity operator* P_{Aff} given by the affine Grossmann-Royer operator at the identity element, that is,

$$P_{\text{Aff}}(\psi)(r) := R_{\text{Aff}}(0,1)\psi(r) = \frac{\lambda^{-1}(r)(1 - e^{\lambda^{-1}(r)})}{1 + \lambda^{-1}(r) - e^{\lambda^{-1}(r)}}\psi\left(re^{-\lambda^{-1}(r)}\right),$$

for $\psi \in \mathcal{S}(\mathbb{R}_+)$. The affine parity operator P_{Aff} is symmetric as an unbounded operator on $L^2(\mathbb{R}_+)$. Moreover, we see from the relation

$$e^{\lambda^{-1}(r)} - 1 = \frac{\lambda^{-1}(r)e^{\lambda^{-1}(r)}}{r}$$

that we have the alternative formula

$$P_{\text{Aff}}(\psi)(r) = \frac{\lambda^{-1}(r)}{1 - re^{-\lambda^{-1}(r)}}\psi\left(re^{-\lambda^{-1}(r)}\right). \tag{E.3.4}$$

An important commutation relation for the (Heisenberg) Grossmann-Royer operator $R(x,\omega)$ for $(x,\omega) \in \mathbb{R}^{2n}$ is given by

$$P \circ R(x,\omega) = R(-x,-\omega) \circ P. \tag{E.3.5}$$

The following proposition shows that the analogue of (E.3.5) breaks down in the affine setting due to Aff being non-unimodular. As the proof is a straightforward computation, we leave the details to the reader.

Proposition E.3.13. The commutation relation

$$P_{\text{Aff}} \circ R_{\text{Aff}}(x,a) = R_{\text{Aff}}\left((x,a)^{-1}\right) \circ P_{\text{Aff}}$$

holds precisely for those $(x,a) \in \text{Aff}$ such that $\Delta(x,a) = \frac{1}{a} = 1$.

We will now show that both the function λ in (E.2.2) and the affine parity operator P_{Aff} are related to the Lambert W function. Recall that the *(real) Lambert W function* is the multivalued function defined to be the inverse relation of the function $f(x) = xe^x$ for $x \in \mathbb{R}$. The function $f(x)$ for $x < 0$ is not injective. There exist for each $y \in (-1/e, 0)$ precisely two values $x_1, x_2 \in (-\infty, 0)$ such that

$$x_1 e^{x_1} = x_2 e^{x_2} = y.$$

As the solutions appear in pairs, we can define σ to be the function that permutes these solutions, that is, $\sigma(x_1) = x_2$ and $\sigma(x_2) = x_1$. For $y = -1/e$ there is only one solution to the equation $xe^x = y$, namely $x = -1$. Hence we define $\sigma(-1) = -1$. We can represent the function σ as

$$\sigma(x) = \begin{cases} W_0(xe^x), & x < -1 \\ -1, & x = -1 \\ W_{-1}(xe^x), & -1 < x < 0 \end{cases},$$

where W_0, W_{-1} are the two branches of the Lambert W function satisfying

$$W_0(xe^x) = x, \qquad \text{for } x \geq -1$$

and

$$W_{-1}(xe^x) = x, \qquad \text{for } x \leq -1.$$

Lemma E.3.14. *The inverse of λ is given by*

$$\lambda^{-1}(r) = \log\left(\frac{-r}{\sigma(-r)}\right) = \sigma(-r) + r, \qquad r > 0.$$

Proof. To find the inverse of λ we solve the equation

$$r = \lambda(u) = \frac{ue^u}{e^u - 1} = \frac{-u}{e^{-u} - 1}.$$

A simple computation shows that $-r = -u - re^{-u}$. Making the substitution $v = e^{-u}$ together with straightforward manipulations shows that

$$-re^{-r} = -rve^{-rv}. \tag{E.3.6}$$

The trivial solution to (E.3.6) is given by solving the equation $-r = -rv$. Checking with the original equation, this can not give the inverse of λ. We get the first equality from the definition of σ together with recalling that $u = -\log(v)$. The final equality follows from

$$\log\left(\frac{-r}{\sigma(-r)}\right) = \log\left(\frac{-r}{\sigma(-r)} \frac{\sigma(-r)e^{\sigma(-r)}}{-re^{-r}}\right) = \sigma(-r) + r. \qquad \square$$

Remark. A minor variation of σ appeared in [73, Section 3] where it was defined by the relation in Lemma E.3.14. The advantage of understanding the connection to the Lambert W function is that properties such as $\sigma(\sigma(x)) = x$ for every $x < 0$ become trivial in this description.

Corollary E.3.15. The affine parity operator P_{Aff} can be written as

$$P_{\mathrm{Aff}}(\psi)(r) = \frac{\sigma(-r) + r}{\sigma(-r) + 1}\psi(-\sigma(-r)), \qquad \psi \in \mathcal{S}(\mathbb{R}_+).$$

In particular, we have $P_{\mathrm{Aff}}(\psi)(1) = 2\psi(1)$.

Proof. The formula for $P_{\mathrm{Aff}}(\psi)$ is obtained from Lemma E.3.14 together with (E.3.4). To find the value $P_{\mathrm{Aff}}(\psi)(1)$, we use (E.3.4) and the fact that

$$\psi\left(re^{-\lambda^{-1}(r)}\right)\Big|_{r=1} = \psi(1).$$

Hence the claim follows from L'Hopital's rule since

$$\lim_{r\to 1}\frac{\lambda^{-1}(r)}{\lambda^{-1}(r)+1-r} = \frac{(\lambda^{-1})'(1)}{(\lambda^{-1})'(1)-1} = 2. \qquad \square$$

E.3.5 Operator Convolution for Tempered Distributions

This subsection is all about expressing the affine Weyl quantization of a function $f \in \mathcal{S}(\mathrm{Aff})$ by using affine convolution. To be able to do this, we will first define what it means for A_f to be a Schwartz operator.

Definition E.3.16. We say that a Hilbert-Schmidt operator $A\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is a *Schwartz operator* if the integral kernel A_K of A satisfies $A_K \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+)$, that is, if

$$(x,\omega) \longmapsto A_K(e^x, e^\omega) \in \mathcal{S}(\mathbb{R}^2).$$

Proposition E.3.17. A Hilbert-Schmidt operator $A \in \mathcal{S}_2$ is a Schwartz operator if and only if $A = A_f$ for some $f \in \mathcal{S}(\mathrm{Aff})$.

Proof. Assume that A is a Schwartz operator. In [73, Equation (4.8)] it is shown that the integral kernel A_K of A is related to the affine Weyl symbol f_A of A by the formula

$$A_K(r,s) = \int_{-\infty}^{\infty} f_A\left(x, \frac{r-s}{\log(r/s)}\right) e^{2\pi i x \log(r/s)}\, dx.$$

Since the inverse-Fourier transform preserves Schwartz functions, together with the definition of $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+)$, we have that

$$(r,s) \longmapsto f_A\left(\log(r/s), \frac{r-s}{\log(r/s)}\right) \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+).$$

By performing the change of variable $x = \log(r/s)$ and $s = e^\omega$ for $\omega \in \mathbb{R}$ we obtain

$$(x,\omega) \longmapsto f_A\left(x, e^\omega \frac{e^x - 1}{x}\right) \in \mathcal{S}(\mathbb{R}^2).$$

Finally, by letting $u = \log((e^x - 1)/x) + \omega$ we see that

$$(x,u) \longmapsto f_A(x, e^u) \in \mathcal{S}(\mathbb{R}^2),$$

due to the fact that $x \mapsto \log((e^x - 1)/x)$ has polynomial growth.

Conversely, assume that $A = A_f$ for $f \in \mathcal{S}(\mathrm{Aff})$. The integral kernel A_K is then given by

$$A_K(r,s) = \mathcal{F}_1^{-1}(f)\left(\log(r/s), \frac{r-s}{\log(r/s)}\right).$$

By using that the inverse-Fourier transform $\mathcal{F}_1^{-1}$ in the first component preserves $\mathcal{S}(\mathrm{Aff})$ together with similar substitutions as previously, we have that the integral kernel satisfies $A_K \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+)$. □

We will use the notation $\mathcal{S}(L^2(\mathbb{R}_+))$ for all Schwartz operators on $L^2(\mathbb{R}_+)$. There is a natural topology on $\mathcal{S}(L^2(\mathbb{R}_+))$ induced by the semi-norms

$$\|A_f\|_{\alpha,\beta} := \|f\|_{\alpha,\beta},$$

where $\|\cdot\|_{\alpha,\beta}$ are the semi-norms on $\mathcal{S}(\mathrm{Aff})$ given in (E.2.3).

Proposition E.3.18. The affine convolution gives a well-defined map

$$\mathcal{S}(\mathrm{Aff}) \star_{\mathrm{Aff}} \mathcal{S}(L^2(\mathbb{R}_+)) \to \mathcal{S}(L^2(\mathbb{R}_+)).$$

Moreover, for fixed $A \in \mathcal{S}(L^2(\mathbb{R}_+))$ the map

$$\mathcal{S}(\mathrm{Aff}) \ni f \longmapsto f \star_{\mathrm{Aff}} A \in \mathcal{S}(L^2(\mathbb{R}_+))$$

is continuous.

Proof. Let $f \in \mathcal{S}(\mathrm{Aff})$ and $A \in \mathcal{S}(L^2(\mathbb{R}_+))$. Then $A = A_g$ for some $g \in \mathcal{S}(\mathrm{Aff})$ and we have by Proposition E.3.6 that

$$f \star_{\mathrm{Aff}} A = f \star_{\mathrm{Aff}} A_g = A_{f *_{\mathrm{Aff}} g}. \tag{E.3.7}$$

Hence the first statement reduces to showing that the usual affine group convolution is a well-defined map

$$\mathcal{S}(\mathrm{Aff}) *_{\mathrm{Aff}} \mathcal{S}(\mathrm{Aff}) \to \mathcal{S}(\mathrm{Aff}).$$

After a change of variables, the question becomes whether the map

$$(x,u) \longmapsto (f *_{\mathrm{Aff}} g)(x, e^u) = \int_{\mathbb{R}^2} f(y, e^z) g(x - ye^{u-z}, e^{u-z})\, dy\, dz \tag{E.3.8}$$

is an element in $\mathcal{S}(\mathbb{R}^2)$. It is straightforward to check that (E.3.8) is a smooth function. Moreover, since f and g are both in $\mathcal{S}(\mathrm{Aff})$, it suffices to show that

(E.3.8) decays faster than any polynomial towards infinity; we can then iterate the argument to obtain the required decay statements for the derivatives.

We claim that

$$\sup_{x,u} |x|^k |u|^l |g(x - ye^{u-z}, e^{u-z})| \leq A^g_{k,l} (1 + |y|)^k (1 + |z|)^l, \tag{E.3.9}$$

where $A^g_{k,l}$ is a constant that depends only on the indices $k, l \in \mathbb{N}_0$ and $g \in \mathcal{S}(\mathrm{Aff})$. To show this, we need to individually consider three cases:

- Assume that we only take the supremum over x and u satisfying $2|z| \geq |u|$ and $2|y| \geq |x|$. Then clearly (E.3.9) is satisfied with $A^g_{k,l} = 2^{k+l} \max |g|$.
- Assume that we only take the supremum over u satisfying $2|z| \leq |u|$ and let $x \in \mathbb{R}$ be arbitrary. Then e^{u-z} is outside the interval $[e^{-|u|/2}, e^{|u|/2}]$. Since $g \in \mathcal{S}(\mathrm{Aff})$ the left-hand side of (E.3.9) will eventually decrease when increasing u. When $y \leq 0$ the left hand-side of (E.3.9) will also obviously eventually decrease by increasing x. When $y > 0$ then any increase of x would necessitate an increase of u on the scale of $u \sim \ln(x)$ to compensate so that the first coordinate in g does not blow up. However, this again forces the second coordinate to grow on the scale of x and we would again, due to $g \in \mathcal{S}(\mathrm{Aff})$, have that the left hand-side of (E.3.9) would eventually decrease.
- Finally, we can take the supremum over x and u satisfying $2|z| \geq |u|$ and $2|y| \leq |x|$. As this case uses similar arguments as above, we leave the straightforward verification to the reader.

Using (E.3.9) we have that

$$\begin{aligned}\sup_{x,u} &|x^k u^l (f *_{\mathrm{Aff}} g)(x, e^u)| \\ &\leq A^g_{k,l} \int_{\mathbb{R}^2} |f(y, e^z)| (1 + |y|)^k (1 + |z|)^l \, dy\, dz < \infty,\end{aligned} \tag{E.3.10}$$

where the last inequality follows from that $f \in \mathcal{S}(\mathrm{Aff})$. Finally, the continuity of the map $f \mapsto f \star_{\mathrm{Aff}} A$ follows from (E.3.7) and (E.3.10). □

Remark. Notice that the proof of Proposition E.3.18 shows that affine convolution between $f, g \in \mathcal{S}(\mathrm{Aff})$ satisfies $f *_{\mathrm{Aff}} g \in \mathcal{S}(\mathrm{Aff})$. This fact, together with Proposition E.3.17, strengthens the claim that $\mathcal{S}(\mathrm{Aff})$ is the correct definition for Schwartz functions on the group Aff.

The main result in this section is Theorem E.3.20 presented below. To state the result rigorously, we first need to make sense of the convolution between Schwartz functions $g \in \mathcal{S}(\mathrm{Aff})$ and the affine parity operator P_{Aff}. As motivation for our definition we will use the following computation: Let $S, T \in \mathcal{S}_2$ with affine Weyl symbols $f_S, f_T \in L^2_r(\mathrm{Aff})$. Fix $g \in \mathcal{S}(\mathrm{Aff})$ and consider the affine Weyl symbol $f_{g\star_{\mathrm{Aff}}S}$ corresponding to the convolution $g \star_{\mathrm{Aff}} S$. Then

$$\begin{aligned}
\langle f_{g\star_{\mathrm{Aff}}S}, f_T\rangle_{L^2_r(\mathrm{Aff})} &= \langle g \star_{\mathrm{Aff}} S, T\rangle_{\mathcal{S}_2} \\
&= \left\langle S, \int_{\mathrm{Aff}} \overline{g(x,a)} U(-x,a) T U(-x,a)^* \,\frac{dx\,da}{a}\right\rangle_{\mathcal{S}_2} \\
&= \left\langle f_S, \int_{\mathrm{Aff}} \overline{g(x,a)} R_{(x,a)} f_T \,\frac{dx\,da}{a}\right\rangle_{L^2_r(\mathrm{Aff})}.
\end{aligned}$$

With this motivation in mind we get the following definition.

Definition E.3.19. Let $S \colon \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}'(\mathbb{R}_+)$ be the operator with affine Weyl symbol $f_S \in \mathcal{S}'(\mathrm{Aff})$ and let $g \in \mathcal{S}(\mathrm{Aff})$. Then $g \star_{\mathrm{Aff}} S$ is defined by its Weyl symbol $f_{g\star_{\mathrm{Aff}}S} \in \mathcal{S}'(\mathrm{Aff})$ satisfying

$$\langle f_{g\star_{\mathrm{Aff}}S}, h\rangle_{\mathcal{S}',\mathcal{S}} := \left\langle f_S, \int_{\mathrm{Aff}} \overline{g(x,a)} R_{(x,a)} h \,\frac{dx\,da}{a}\right\rangle_{\mathcal{S}',\mathcal{S}},$$

for all $h \in \mathcal{S}(\mathrm{Aff})$.

Recall that the injectivity in Lemma E.3.8 ensures that the operator S in Definition E.3.19 is well-defined. The argument to show $f_{g\star_{\mathrm{Aff}}S} \in \mathcal{S}'(\mathrm{Aff})$ is similar to the one presented in Proposition E.3.18. Hence $g \star_{\mathrm{Aff}} S$ is well-defined.

Remark. We could similarly have defined $S \star_{\mathrm{Aff}} A_f$ for $S \in \mathcal{S}(L^2(\mathbb{R}_+))$ and $f \in \mathcal{S}'(\mathrm{Aff})$ by using Proposition E.3.7. For brevity, we restrict ourselves in the next theorem to the case where $S := \phi \otimes \psi$ for $\psi, \phi \in \mathcal{S}(\mathrm{Aff})$. In this case, we can extend Lemma E.3.3 and define

$$(\phi \otimes \psi) \star_{\mathrm{Aff}} A_f := \langle A_f U(-x,a)\psi, U(-x,a)\phi\rangle_{\mathcal{S}',\mathcal{S}}.$$

We can now finally state the main theorem in this section.

Theorem E.3.20. *The affine Weyl quantization A_g of $g \in \mathcal{S}(\mathrm{Aff})$ can be written as*

$$A_g = g \star_{\mathrm{Aff}} P_{\mathrm{Aff}},$$

where P_{Aff} is the affine parity operator. Moreover, for $\psi, \phi \in \mathcal{S}(\mathbb{R}_+)$ we have that the affine Weyl symbol $W^{\psi,\phi}_{\mathrm{Aff}}$ of the rank-one operator $\psi \otimes \phi$ can be written as

$$W^{\psi,\phi}_{\mathrm{Aff}} = (\psi \otimes \phi) \star_{\mathrm{Aff}} P_{\mathrm{Aff}}.$$

Proof. Recall that the affine parity operator P_{Aff} is the affine Weyl quantization of the point measure $\delta_{(0,1)} \in \mathcal{S}'(\mathrm{Aff})$. As such, the convolution $g \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$ is well-defined with the interpretation given in Definition E.3.19. The affine Weyl symbol $f_{g\star_{\mathrm{Aff}}P_{\mathrm{Aff}}}$ of $g \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$ is acting on $h \in \mathcal{S}(\mathrm{Aff})$ by

$$\begin{aligned}
\langle f_{g\star_{\mathrm{Aff}}P_{\mathrm{Aff}}}, h\rangle_{\mathcal{S}',\mathcal{S}} &:= \left\langle \delta_{(0,1)}, \overline{\int_{\mathrm{Aff}} \overline{g(x,a)} R_{(x,a)} h \, \frac{dx\,da}{a}} \right\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \int_{\mathrm{Aff}} \overline{g(x,a)} h((0,1)\cdot(x,a)) \, \frac{dx\,da}{a} \\
&= \int_{\mathrm{Aff}} g(x,a)\overline{h(x,a)} \, \frac{dx\,da}{a} \\
&= \langle g, h\rangle_{L^2_r(\mathrm{Aff})}.
\end{aligned}$$

Since $\mathcal{S}(\mathrm{Aff}) \subset L^2_r(\mathrm{Aff})$ is dense, we can conclude that $f_{g\star_{\mathrm{Aff}}P_{\mathrm{Aff}}} = g$ and thus $A_g = g \star_{\mathrm{Aff}} P_{\mathrm{Aff}}$. For the second statement, we get that

$$\begin{aligned}
((\psi \otimes \phi) \star_{\mathrm{Aff}} P_{\mathrm{Aff}})(x,a) &= \langle P_{\mathrm{Aff}} U(-x,a)\psi, U(-x,a)\phi\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \langle R_{\mathrm{Aff}}(x,a)\psi, \phi\rangle_{\mathcal{S}',\mathcal{S}} \\
&= W^{\psi,\phi}_{\mathrm{Aff}}(x,a).
\end{aligned}$$

□

E.4 Operator Admissibility

For operator convolutions on the Heisenberg group, we have from (E.2.5) the important integral relation

$$\int_{\mathbb{R}^{2n}} S \star T(z)\, dz = \operatorname{tr}(S)\operatorname{tr}(T).$$

A similar formula for the integral of operator convolutions will not hold generally in the affine setting. We therefore search for a class of operators where such a relation does hold: The *admissible operators*. As a first step, we recall the notion of *admissible functions*.

Definition E.4.1. We say that $\psi \in L^2(\mathbb{R}_+)$ is *admissible* if

$$\int_0^\infty \frac{|\psi(r)|^2}{r} \frac{dr}{r} < \infty.$$

This definition of admissibility is motivated by the theorem of Duflo and Moore [50], see also [89]. The *Duflo-Moore operator* $\mathcal{D}^{-1}$ in our setting is given by

$$\mathcal{D}^{-1}\psi(r) := \frac{\psi(r)}{\sqrt{r}}.$$

It is clear that the Duflo-Moore operator $\mathcal{D}^{-1}$ is a densely defined, self-adjoint positive operator on $L^2(\mathbb{R}_+)$ with a densely defined inverse, namely

$$\mathcal{D}\psi(r) := \sqrt{r}\psi(r).$$

Hence a function $\psi \in L^2(\mathbb{R}_+)$ is admissible if and only if $\mathcal{D}^{-1}\psi \in L^2(\mathbb{R}_+)$. We will on several occasions use the commutation relations

$$\mathcal{D}U(x,a) = \sqrt{\frac{1}{a}}U(x,a)\mathcal{D}, \quad U(x,a)^*\mathcal{D}^{-1} = \sqrt{a}\mathcal{D}^{-1}U(x,a)^*, \tag{E.4.1}$$

for $(x,a) \in \mathrm{Aff}$. The following orthogonality relation is a trivial reformulation of the classic orthogonality relations for wavelets, see for instance [90].

Proposition E.4.2. Let $\phi, \psi, \xi, \eta \in L^2(\mathbb{R}_+)$ and assume that ψ and η are admissible. Then

$$\begin{aligned}\int_{\mathrm{Aff}} \langle \phi, U(-x,a)^*\psi\rangle_{L^2(\mathbb{R}_+)} \overline{\langle \xi, U(-x,a)^*\eta\rangle_{L^2(\mathbb{R}_+)}} \, \frac{dx\, da}{a} \\ = \langle \phi, \xi\rangle_{L^2(\mathbb{R}_+)} \langle \mathcal{D}^{-1}\eta, \mathcal{D}^{-1}\psi\rangle_{L^2(\mathbb{R}_+)}.\end{aligned}$$

In particular, we have

$$\begin{aligned}\int_{\mathrm{Aff}} \langle \phi, U(-x,a)^*\psi\rangle_{L^2(\mathbb{R}_+)} \overline{\langle \xi, U(-x,a)^*\psi\rangle_{L^2(\mathbb{R}_+)}} \, \frac{dx\, da}{a} \\ = \langle \phi, \xi\rangle_{L^2(\mathbb{R}_+)} \|\mathcal{D}^{-1}\psi\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Remark. By Proposition E.4.2, admissibility of $\psi \in L^2(\mathbb{R}_+)$ is equivalent to the condition

$$\int_{\mathrm{Aff}} |\langle \psi, U(-x,a)^*\psi\rangle_{L^2(\mathbb{R}_+)}|^2 \, \frac{dx\, da}{a} < \infty.$$

E.4.1 Admissibility for Operators

Our goal is now to extend the notion of admissibility to bounded operators on $L^2(\mathbb{R}_+)$, with the aim of obtaining a class of operators where a formula for the integral of operator convolutions similar to (E.2.5) holds. We will often use that any compact operator S on $L^2(\mathbb{R}_+)$ has a *singular value decomposition*

$$S = \sum_{n=1}^{N} s_n \xi_n \otimes \eta_n, \qquad N \in \mathbb{N} \cup \{\infty\}, \tag{E.4.2}$$

where $\{\xi_n\}_{n=1}^N$ and $\{\eta_n\}_{n=1}^N$ are orthonormal sets in $L^2(\mathbb{R}_+)$. The *singular values* $\{s_n\}_{n=1}^N$ with $s_n > 0$ will converge to zero when $N = \infty$. If S is a trace-class operator we have $\{s_n\}_{n=1}^N \in \ell^1(\mathbb{N})$ with $\|S\|_{\mathcal{S}_1} = \|s_n\|_{\ell^1}$. Since the admissible functions in $L^2(\mathbb{R}_+)$ form a dense subspace, we can always find an orthonormal basis consisting of admissible functions.

The next result concerns bounded operators $\mathcal{D}S\mathcal{D}$ for a trace-class operator S. To be precise, this means that we assume that S maps $\mathrm{dom}(\mathcal{D}^{-1})$ into $\mathrm{dom}(\mathcal{D})$, and that the operator $\mathcal{D}S\mathcal{D}$ defined on $\mathrm{dom}(\mathcal{D})$ extends to a bounded operator.

Theorem E.4.3. *Let $S \in \mathcal{S}_1$ satisfy that $\mathcal{D}S\mathcal{D} \in \mathcal{L}(L^2(\mathbb{R}_+))$. For any $T \in \mathcal{S}_1$ we have that $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} \in L^1_r(\mathrm{Aff})$ with*

$$\|T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_1} \|T\|_{\mathcal{S}_1},$$

and

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}(x,a) \, \frac{dx\, da}{a} = \mathrm{tr}(T)\, \mathrm{tr}(S). \tag{E.4.3}$$

Proof. We divide the proof into three steps.
Step 1: We first assume that $T = \psi \otimes \phi$ for $\psi, \phi \in \mathrm{dom}(\mathcal{D})$. Recall that S can be written in the form (E.4.2). From Lemma E.3.3 and (E.4.1) we find that

$$\begin{aligned} T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}(x,a) &= \langle S\mathcal{D}U(-x,a)\psi, \mathcal{D}U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)} \\ &= \frac{1}{a}\langle SU(-x,a)\mathcal{D}\psi, U(-x,a)\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)} \\ &= \sum_{n=1}^N s_n \frac{1}{a}\langle U(-x,a)\mathcal{D}\psi, \eta_n\rangle_{L^2(\mathbb{R}_+)} \langle \xi_n, U(-x,a)\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)}. \end{aligned}$$

Integrating with respect to the right Haar measure and using that $(x,a) \mapsto (x,a)^{-1}$ interchanges left and right Haar measure, we get

$$\begin{aligned} &\int_{\mathrm{Aff}} |\langle U(-x,a)\mathcal{D}\psi, \eta_n\rangle_{L^2(\mathbb{R}_+)} \langle \xi_n, U(-x,a)\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)}| \frac{1}{a} \, \frac{dx\, da}{a} \\ &= \int_{\mathrm{Aff}} |\langle U(-x,a)^*\mathcal{D}\psi, \eta_n\rangle_{L^2(\mathbb{R}_+)} \langle \xi_n, U(-x,a)^*\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)}| \, \frac{dx\, da}{a} \\ &\leq \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)}, \end{aligned}$$

where the last line uses Proposition E.4.2. It follows that the sum in the expression for $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}(x,a)$ converges absolutely in $L^1_r(\mathrm{Aff})$ with

$$\|T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \leq \left(\sum_{n=1}^N s_n\right) \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)} = \|S\|_{\mathcal{S}_1} \|T\|_{\mathcal{S}_1}.$$

Equation (E.4.3) follows in a similar way by integrating the sum expressing the function $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ and using Proposition E.4.2.

Step 2: We now assume that $T = \psi \otimes \phi$ for arbitrary $\psi, \phi \in L^2(\mathbb{R}_+)$. Pick sequences $\{\psi_n\}_{n=1}^{\infty}, \{\phi_n\}_{n=1}^{\infty}$ in $\mathrm{dom}(\mathcal{D})$ converging to ψ and ϕ, respectively, and let $T_n = \psi_n \otimes \phi_n$. It is straightforward to check that T_n converges to T in $\mathcal{S}_1$. By (E.3.2) this implies that $T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ converges uniformly to $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$. On the other hand, $T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ is a Cauchy sequence in $L^1_r(\mathrm{Aff})$: For $m, n \in \mathbb{N}$ we find by Step 1 that

$$\begin{aligned}
&\|T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} - T_m \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \\
&\leq \|(\psi_n - \psi_m) \otimes \phi_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} + \|\psi_m \otimes (\phi_n - \phi_m) \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \\
&\leq \|S\|_{\mathcal{S}_1} \|\psi_n - \psi_m\|_{L^2(\mathbb{R}_+)} \|\phi_n\|_{L^2(\mathbb{R}_+)} + \|S\|_{\mathcal{S}_1} \|\psi_m\|_{L^2(\mathbb{R}_+)} \|\phi_m - \phi_n\|_{L^2(\mathbb{R}_+)},
\end{aligned}$$

which clearly goes to zero as $m, n \to \infty$. This means that $T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ converges in $L^1_r(\mathrm{Aff})$, and the limit must be $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ as we already know that $T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ converges uniformly to this function. In particular, this implies

$$\begin{aligned}
\|T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} &= \lim_{n\to\infty} \|T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} \\
&\leq \lim_{n\to\infty} \|\psi_n\|_{L^2(\mathbb{R}_+)} \|\phi_n\|_{L^2(\mathbb{R}_+)} \|S\|_{\mathcal{S}_1} \\
&= \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)} \|S\|_{\mathcal{S}_1}.
\end{aligned}$$

Equation (E.4.3) also follows by taking the limit of $\int_{\mathrm{Aff}} T_n \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}(x,a)\, \frac{dx\,da}{a}$.
Step 3: We now assume that $T \in \mathcal{S}_1$. Consider the singular value decomposition of T given by

$$T = \sum_{m=1}^{M} t_m \psi_m \otimes \phi_m$$

for $M \in \mathbb{N} \cup \{\infty\}$. By (E.3.2) we have, with uniform convergence of the sum, that

$$T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} = \sum_{m=1}^{M} t_m \psi_m \otimes \phi_m \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}. \tag{E.4.4}$$

Notice that Step 2 implies that the convergence is also in $L^1_r(\mathrm{Aff})$, since

$$\begin{aligned}
\sum_{m=1}^{M} t_m \|\psi_m \otimes \phi_m \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{L^1_r(\mathrm{Aff})} &\leq \sum_{m=1}^{M} t_m \|\psi_m\|_{L^2(\mathbb{R}_+)} \|\phi_m\|_{L^2(\mathbb{R}_+)} \|S\|_{\mathcal{S}_1} \\
&= \|T\|_{\mathcal{S}_1} \|S\|_{\mathcal{S}_1}.
\end{aligned}$$

In particular, $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D} \in L^1_r(\mathrm{Aff})$. Finally, (E.4.3) follows by integrating (E.4.4) and using that the sum converges in $L^1_r(\mathrm{Aff})$ and Step 2. $\square$

The integral relation (E.4.3) is somewhat artificial in the sense that it introduces $\mathcal{D}$ in the integrand. We will typically be interested in the integral of $T \star_{\mathrm{Aff}} S$, not of $T \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$. This motivates the following definition.

Definition E.4.4. Let S be a non-zero bounded operator on $L^2(\mathbb{R}_+)$ that maps $\mathrm{dom}(\mathcal{D})$ into $\mathrm{dom}(\mathcal{D}^{-1})$. We say that S is *admissible* if the composition $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ is bounded on $\mathrm{dom}(\mathcal{D}^{-1})$ and extends to a trace-class operator $\mathcal{D}^{-1}S\mathcal{D}^{-1} \in \mathcal{S}_1$.

Assume now that S is admissible, and define $R := \mathcal{D}^{-1}S\mathcal{D}^{-1}$. Clearly R maps $\mathrm{dom}(\mathcal{D}^{-1})$ into $\mathrm{dom}(\mathcal{D})$ as we assume that S maps $\mathrm{dom}(\mathcal{D})$ into $\mathrm{dom}(\mathcal{D}^{-1})$. The following corollary is therefore immediate from Theorem E.4.3. We also note that it extends [113, Corollary 1] to non-positive, non-compact operators.

Corollary E.4.5. Let $S \in \mathcal{L}(L^2(\mathbb{R}_+))$ be an admissible operator. For any $T \in \mathcal{S}_1$ we have that $T \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$ with

$$\|T \star_{\mathrm{Aff}} S\|_{L^1_r(\mathrm{Aff})} \leq \|\mathcal{D}^{-1}S\mathcal{D}^{-1}\|_{\mathcal{S}_1}\|T\|_{\mathcal{S}_1},$$

and

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a)\,\frac{dx\,da}{a} = \mathrm{tr}(T)\,\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1}).$$

Example E.4.6. A rank-one operator $S := \eta \otimes \xi$ for non-zero η, ξ is an admissible operator if and only if $\eta, \xi \in L^2(\mathbb{R}_+)$ are admissible functions. Requiring that S maps $\mathrm{dom}(\mathcal{D})$ into $\mathrm{dom}(\mathcal{D}^{-1})$ clearly implies that $\eta \in \mathrm{dom}(\mathcal{D}^{-1})$, i.e. η is admissible. For $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ to be trace-class, the map

$$\psi \mapsto \|\mathcal{D}^{-1}S\mathcal{D}^{-1}\psi\|_{L^2(\mathbb{R}_+)} = |\langle \mathcal{D}^{-1}\psi, \xi\rangle_{L^2(\mathbb{R}_+)}| \cdot \|\mathcal{D}^{-1}\eta\|_{L^2(\mathbb{R}_+)},$$

for $\psi \in \mathrm{dom}(\mathcal{D}^{-1})$ must at least be bounded for $\|\psi\|_{L^2(\mathbb{R}_+)} \leq 1$. This is bounded if and only if

$$\psi \mapsto \langle \mathcal{D}^{-1}\psi, \xi\rangle_{L^2(\mathbb{R}_+)}$$

is bounded, which is precisely the condition that $\xi \in \mathrm{dom}\left((\mathcal{D}^{-1})^*\right) = \mathrm{dom}(\mathcal{D}^{-1})$. Hence our notion of admissibility for operators naturally extends the classical function admissibility. In the case of rank-one operators, it follows from Lemma E.3.3 and the computation

$$\mathrm{tr}(\mathcal{D}^{-1}(\eta \otimes \xi)\mathcal{D}^{-1}) = \langle \mathcal{D}^{-1}\eta, \mathcal{D}^{-1}\xi\rangle_{L^2(\mathbb{R}_+)}$$

that Corollary E.4.5 reduces to Proposition E.4.2.

When both S and T are admissible trace-class operators, their convolution $T \star_{\mathrm{Aff}} S$ behaves well with respect to both the left and right Haar measures.

Corollary E.4.7. Let S and T be admissible trace-class operators on $L^2(\mathbb{R}_+)$. Then the convolution $T \star_{\mathrm{Aff}} S$ satisfies $T \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff}) \cap L^1_l(\mathrm{Aff})$ and

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a} = \operatorname{tr}(T) \operatorname{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}),$$
$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a^2} = \operatorname{tr}(S) \operatorname{tr}(\mathcal{D}^{-1} T \mathcal{D}^{-1}).$$

Proof. The first equation and the claim that $T \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$ is Corollary E.4.5. The second equation and the claim that $T \star_{\mathrm{Aff}} S \in L^1_l(\mathrm{Aff})$ follows since

$$T \star_{\mathrm{Aff}} S(x,a) = S \star_{\mathrm{Aff}} T((x,a)^{-1}). \qquad \square$$

We now turn to the case where S is a positive compact operator. We first note that admissibility in this case becomes a statement about the eigenvectors and eigenvalues of S.

Proposition E.4.8. Let S be a non-zero positive compact operator with spectral decomposition

$$S = \sum_{n=1}^{N} s_n \xi_n \otimes \xi_n$$

for $N \in \mathbb{N} \cup \{\infty\}$. Then S is admissible if and only each ξ_n is admissible and

$$\sum_{n=1}^{N} s_n \|\mathcal{D}^{-1}\xi_n\|^2_{L^2(\mathbb{R}_+)} < \infty.$$

Proof. We first assume that S is admissible. By linearity and Lemma E.3.3 we get for $\xi \in L^2(\mathbb{R}_+)$ with $\|\xi\|_{L^2(\mathbb{R}_+)} = 1$ that

$$\xi \otimes \xi \star_{\mathrm{Aff}} S(x,a) = \sum_{n=1}^{N} s_n |\langle \xi, U(-x,a)^* \xi_n \rangle_{L^2(\mathbb{R}_+)}|^2. \tag{E.4.5}$$

Integrating (E.4.5) using the monotone convergence theorem and Proposition E.4.2, we obtain

$$\int_{\mathrm{Aff}} \xi \otimes \xi \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a} = \sum_{n=1}^{N} s_n \|\mathcal{D}^{-1}\xi_n\|^2_{L^2(\mathbb{R}_+)}.$$

The claim now follows from Corollary E.4.5.

For the converse, it is clear by the assumption that the operator

$$\sum_{n=1}^{N} s_n (\mathcal{D}^{-1}\xi_n) \otimes (\mathcal{D}^{-1}\xi_n) \tag{E.4.6}$$

is a trace-class operator. It only remains to show that S maps $\operatorname{dom}(\mathcal{D})$ into $\operatorname{dom}(\mathcal{D}^{-1})$ and that $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ is given by (E.4.6). This is easily shown when N is finite, so we do the proof for $N = \infty$.

The partial sums for $\psi \in L^2(\mathbb{R}_+)$ are denoted by

$$(S\psi)_M \coloneqq \sum_{n=1}^{M} s_n \langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)} \xi_n,$$

and converge in the sense that $(S\psi)_M \to S\psi$ as $M \to \infty$. Furthermore, it is clear that $(S\psi)_M$ is in the domain of $\mathcal{D}^{-1}$ for each M as each ξ_n is admissible. We also have that

$$\mathcal{D}^{-1}(S\psi)_M = \sum_{n=1}^{M} s_n \langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)} \mathcal{D}^{-1}\xi_n.$$

The sequence of partial sums $\mathcal{D}^{-1}(S\psi)_M$ also converges in $L^2(\mathbb{R}_+)$, since by using Hölder's inequality and Bessel's inequality we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} s_n |\langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)}| \|\mathcal{D}^{-1}\xi_n\|_{L^2(\mathbb{R}_+)} \\
&\leq \left(\sum_{n=1}^{\infty} |\langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)}|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} s_n^2 \|\mathcal{D}^{-1}\xi_n\|_{L^2(\mathbb{R}_+)}^2 \right)^{1/2} \\
&\lesssim \|\psi\|_{L^2(\mathbb{R}_+)} \left(\sum_{n=1}^{\infty} s_n \|\mathcal{D}^{-1}\xi_n\|_{L^2(\mathbb{R}_+)}^2 \right)^{1/2}.
\end{aligned}$$

Since $\mathcal{D}^{-1}$ is a closed operator, we get that $S\psi$ belongs to the domain of $\mathcal{D}^{-1}$ and

$$\mathcal{D}^{-1}S\psi = \sum_{n=1}^{\infty} s_n \langle \psi, \xi_n \rangle_{L^2(\mathbb{R}_+)} \mathcal{D}^{-1}\xi_n.$$

For any $\phi \in \operatorname{dom}(\mathcal{D}^{-1})$, we have that

$$\mathcal{D}^{-1}S\mathcal{D}^{-1}\phi = \sum_{n=1}^{\infty} s_n \langle \mathcal{D}^{-1}\phi, \xi_n \rangle_{L^2(\mathbb{R}_+)} \mathcal{D}^{-1}\xi_n = \sum_{n=1}^{\infty} s_n \langle \phi, \mathcal{D}^{-1}\xi_n \rangle_{L^2(\mathbb{R}_+)} \mathcal{D}^{-1}\xi_n,$$

so $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ agrees with (E.4.6) on this dense subspace. In fact, they agree on all of $L^2(\mathbb{R}_+)$ since

$$\|\mathcal{D}^{-1}S\mathcal{D}^{-1}\phi\|_{L^2(\mathbb{R}_+)} \leq \|\phi\|_{L^2(\mathbb{R}_+)} \sum_{n=1}^{\infty} s_n \|\mathcal{D}^{-1}\xi_n\|_{L^2(\mathbb{R}_+)}^2,$$

shows that $\mathcal{D}^{-1}S\mathcal{D}^{-1}$ extends to a bounded operator. □

As a consequence of Proposition E.4.8, we obtain a compact reformulation of admissibility for positive trace-class operators.

Corollary E.4.9. Let T be a non-zero positive trace-class operator on $L^2(\mathbb{R}_+)$, and let S be a non-zero positive compact operator. If

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a} < \infty,$$

then S is admissible with

$$\operatorname{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}) = \frac{1}{\operatorname{tr}(T)} \int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a}.$$

In particular, if S is a non-zero, positive trace-class operator, then S is admissible if and only if $S \star_{\mathrm{Aff}} S \in L^1_r(\mathrm{Aff})$.

Proof. Let

$$S = \sum_{n=1}^{N} s_n \xi_n \otimes \xi_n$$

be the spectral decomposition of S. An argument similar to the one given in the proof of Proposition E.4.8 shows that

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a) \, \frac{dx\,da}{a} = \operatorname{tr}(T) \sum_{n=1}^{N} s_n \|\mathcal{D}^{-1} \xi_n\|^2_{L^2(\mathbb{R}_+)}.$$

The claims now follow immediately from Proposition E.4.8. □

E.4.2 Admissible Operators from Laguerre Functions

Although we derived several basic properties of admissible operators in Subsection E.4.1, we have not given any way to construct such operators in practice. Our construction is based on the following observation: From Proposition E.4.8 we know that if

$$S = \sum_{n=1}^{\infty} s_n \varphi_n \otimes \varphi_n$$

is a non-zero positive compact operator with

$$\sum_{n=1}^{\infty} s_n \|\mathcal{D}^{-1} \varphi_n\|^2_{L^2(\mathbb{R}_+)} < \infty,$$

then S is admissible. So if we can find an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of admissible functions such that we can control the terms $\|\mathcal{D}^{-1}\varphi_n\|_{L^2(\mathbb{R}_+)}$, then we can construct admissible operators as infinite linear combinations of rank-one operators. It turns out that the Laguerre basis works extremely well in this regard.

Definition E.4.10. For fixed $\alpha \in \mathbb{R}_+$ we define the *Laguerre basis* $\left\{\mathcal{L}_n^{(\alpha)}\right\}_{n=0}^{\infty}$ for $L^2(\mathbb{R}_+)$ by

$$\mathcal{L}_n^{(\alpha)}(r) := \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}\, r^{\frac{\alpha+1}{2}} e^{-\frac{r}{2}} L_n^{(\alpha)}(r), \qquad n \in \mathbb{N}_0,\ r \in \mathbb{R}_+,$$

where Γ denotes the gamma function and $L_n^{(\alpha)}$ denotes the *generalized Laguerre polynomials* given by

$$L_n^{(\alpha)}(r) := \frac{r^{-\alpha} e^r}{n!} \frac{d^n}{dr^n}\left(e^{-r} r^{n+\alpha}\right) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{r^k}{k!}.$$

The classical orthogonality relation

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}, \tag{E.4.7}$$

for the generalized Laguerre polynomials ensures that the Laguerre bases are orthonormal bases for $L^2(\mathbb{R}_+)$ for any fixed $\alpha \in \mathbb{R}_+$. The following result shows that the Laguerre basis is especially compatible with the Duflo-Moore operator $\mathcal{D}^{-1}$.

Proposition E.4.11. For any $\alpha \in \mathbb{R}_+$ and $n \in \mathbb{N}_0$ we have

$$\left\|\mathcal{D}^{-1}\mathcal{L}_n^{(\alpha)}\right\|_{L^2(\mathbb{R}_+)}^2 = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-r} r^{\alpha-1} \left(L_n^{(\alpha)}(r)\right)^2 dr = \frac{1}{\alpha}. \tag{E.4.8}$$

Proof. The first equality in (E.4.8) follows from unwinding the definitions. For the second equality in (E.4.8), we will use the well known identity

$$L_n^{(\alpha)}(r) = \sum_{j=0}^{n} L_j^{(\alpha-1)}(r)$$

together with the orthogonality relation (E.4.7). This gives

$$\begin{aligned}
\int_0^\infty e^{-r} r^{\alpha-1} \left(L_n^{(\alpha)}(r)\right)^2 dr &= \sum_{i,j=0}^{n} \int_0^\infty e^{-r} r^{\alpha-1} L_i^{(\alpha-1)}(r) L_j^{(\alpha-1)}(r)\, dr \\
&= \sum_{i=0}^{n} \frac{\Gamma(i+\alpha)}{i!} \\
&= \frac{1}{\alpha}\frac{\Gamma(n+\alpha+1)}{n!},
\end{aligned}$$

where the last equality follows from a straightforward induction argument. □

The following consequence from Proposition E.4.8 shows that we can explicitly construct admissible operators by using the Laguerre basis.

Corollary E.4.12. Let $\{s_n\}_{n=0}^{\infty} \in \ell^1(\mathbb{N})$ be a sequence of non-negative numbers and let $\alpha \in \mathbb{R}_+$. Then

$$S := \sum_{n=0}^{\infty} s_n \mathcal{L}_n^{(\alpha)} \otimes \mathcal{L}_n^{(\alpha)}$$

is an admissible operator with

$$\mathrm{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}) = \frac{1}{\alpha} \sum_{n=0}^{\infty} s_n.$$

Remark. The corollary may be considered a reformulation with slightly different proof of the calculations in [76, Section 3.3], where a resolution of the identity operator is constructed from thermal states that are diagonal in the Laguerre basis. We will return to resolutions of the identity operator and the relation to admissibility in Subsection E.6.2.

E.4.3 Connection with Convolutions

We will now see how admissibility relates to the convolution of a function with an operator. The following result shows that we can use convolutions to generate new admissible operators from a given admissible operator.

Proposition E.4.13. Let $f \in L^1_l(\mathrm{Aff}) \cap L^1_r(\mathrm{Aff})$ be a non-zero positive function. If S is a positive, admissible trace-class operator on $L^2(\mathbb{R}_+)$, then so is $f \star_{\mathrm{Aff}} S$ with

$$\mathrm{tr}\left(\mathcal{D}^{-1}(f \star_{\mathrm{Aff}} S)\mathcal{D}^{-1}\right) = \int_{\mathrm{Aff}} f(x,a)\, \frac{dx\, da}{a^2}\, \mathrm{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}).$$

Proof. It is clear from (E.3.1) that $f \star_{\mathrm{Aff}} S$ is a trace-class operator, and positivity follows from the definition of the convolution $f \star_{\mathrm{Aff}} S$. Let T be a non-zero positive trace-class operator on $L^2(\mathbb{R}_+)$. It suffices by Corollary E.4.9 to show that

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} (f \star_{\mathrm{Aff}} S)(y,b)\, \frac{dy\, db}{b} = \mathrm{tr}(T) \int_{\mathrm{Aff}} f(x,a)\, \frac{dx\, da}{a^2}\, \mathrm{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}).$$

We have that

$$
\begin{aligned}
&T \star_{\mathrm{Aff}} (f \star_{\mathrm{Aff}} S)(y,b) \\
&= \operatorname{tr}\left(TU(-y,b)^* \int_{\mathrm{Aff}} f(x,a)U(-x,a)^* SU(-x,a) \,\frac{dx\,da}{a} U(-y,b) \right) \\
&= \int_{\mathrm{Aff}} f(x,a) \operatorname{tr}(TU((-x,a)\cdot(-y,b))^* SU((-x,a)\cdot(-y,b)) \,\frac{dx\,da}{a} \\
&= \int_{\mathrm{Aff}} f(x,a) T \star_{\mathrm{Aff}} S((x,a)\cdot(y,b)) \,\frac{dx\,da}{a}.
\end{aligned}
$$

We may then use Fubini's theorem, which applies by our assumptions on f and S, to show that

$$
\begin{aligned}
&\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} (f \star_{\mathrm{Aff}} S)(y,b) \,\frac{dy\,db}{b} \\
&= \int_{\mathrm{Aff}} f(x,a) \int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S((x,a)\cdot(y,b)) \,\frac{dy\,db}{b}\,\frac{dx\,da}{a} \\
&= \int_{\mathrm{Aff}} f(x,a) \,\frac{dx\,da}{a} \Delta(x,a) \int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(y,b) \,\frac{dy\,db}{b} \\
&= \int_{\mathrm{Aff}} f(x,a) \,\frac{dx\,da}{a^2} \operatorname{tr}(T) \operatorname{tr}\left(\mathcal{D}^{-1} S \mathcal{D}^{-1}\right),
\end{aligned}
$$

where we used the admissibility of S and Theorem E.4.5 in the last line. □

Remark. We can give a simple heuristic argument for Proposition E.4.13 by ignoring that $\mathcal{D}^{-1}$ is unbounded as follows: We have by using (E.4.1) that

$$
\begin{aligned}
\mathcal{D}^{-1}(f \star_{\mathrm{Aff}} S)\mathcal{D}^{-1} &= \int_{\mathrm{Aff}} f(x,a)\mathcal{D}^{-1}U(-x,a)^* SU(-x,a)\mathcal{D}^{-1} \,\frac{dx\,da}{a} \\
&= \int_{\mathrm{Aff}} f(x,a)U(-x,a)^* \mathcal{D}^{-1} S \mathcal{D}^{-1} U(-x,a) \,\frac{dx\,da}{a^2}.
\end{aligned}
$$

Since $\mathcal{D}^{-1} S \mathcal{D}^{-1}$ is a trace-class operator, the integral above is a convergent Bochner integral and we obtain the desired equality.

E.4.4 Admissibility as a Measure of Non-Unimodularity

In this subsection we will delve more into how the non-unimodularity of the affine group affects the affine Weyl quantization. As we will see, both the left and right Haar measures take on an active role in this picture.

Proposition E.4.14. Let S be an admissible Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ such that its affine Weyl symbol f_S satisfies $f_S \in L^1_l(\mathrm{Aff})$. Then

$$
\operatorname{tr}\left(\mathcal{D}^{-1} S \mathcal{D}^{-1}\right) = \int_{\mathrm{Aff}} f_S(x,a) \,\frac{dx\,da}{a^2}.
$$

Proof. Let $T := \varphi \otimes \varphi$ for some non-zero $\varphi \in \mathcal{S}(\mathbb{R}_+)$. Then the affine Weyl symbol of T is $f_T = W_{\mathrm{Aff}}^{\varphi} \in \mathcal{S}(\mathrm{Aff})$. We know by Corollary E.4.5 that

$$\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a)\,\frac{dx\,da}{a} = \mathrm{tr}(T)\,\mathrm{tr}\left(\mathcal{D}^{-1} S \mathcal{D}^{-1}\right).$$

On the other hand, Fubini's theorem together with Proposition E.3.7 allows us to calculate that

$$\begin{aligned}\int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} S(x,a)\,\frac{dx\,da}{a} &= \int_{\mathrm{Aff}} f_T *_{\mathrm{Aff}} \check{f_S}(x,a)\,\frac{dx\,da}{a} \\ &= \int_{\mathrm{Aff}} f_T(y,b) \int_{\mathrm{Aff}} f_S((y,b)(x,a)^{-1})\,\frac{dx\,da}{a}\,\frac{dy\,db}{b} \\ &= \int_{\mathrm{Aff}} f_T(y,b)\,\frac{dy\,db}{b} \int_{\mathrm{Aff}} f_S(x,a)\,\frac{dx\,da}{a^2}.\end{aligned}$$

The marginal properties of the affine Wigner distribution (E.2.14) show that

$$\int_{\mathrm{Aff}} f_T(y,b)\,\frac{dy\,db}{b} = \|\varphi\|_{L^2(\mathbb{R}_+)}^2 = \mathrm{tr}(T).$$

The claim now follows from combining the calculations above. □

Remark. Assuming that T is a trace-class operator we have that

$$\mathrm{tr}(T) = \int_{\mathrm{Aff}} f_T(x,a)\,\frac{dx\,da}{a},$$

which follows from a similar proof to the one in Proposition E.4.14. This gives the interesting heuristic interpretation that taking $\mathcal{D}^{-1} T \mathcal{D}^{-1}$ of an operator T coincides with multiplying f_T by $\frac{1}{a}$.

The following result shows that the affine Wigner distribution satisfies both left and right integrability when more is assumed of the input. This should be compared with the Heisenberg case where the Heisenberg group $\mathbb{H}^n$ is unimodular.

Theorem E.4.15. *Assume that $\phi, \psi, \mathcal{D}\phi, \mathcal{D}\psi \in L^2(\mathbb{R}_+)$. Then the affine Wigner distribution satisfies*

$$W_{\mathrm{Aff}}^{\phi,\psi} \in L_r^2(\mathrm{Aff}) \cap L_l^2(\mathrm{Aff}).$$

Proof. We already know that $W_{\mathrm{Aff}}^{\phi,\psi}$ is in $L_r^2(\mathrm{Aff})$ by the orthogonality relation for the affine Wigner distribution. Using the definition of the affine Wigner distribution and Plancherel's theorem, we have that

$$\begin{aligned}\|W_{\mathrm{Aff}}^{\phi,\psi}\|_{L_l^2(\mathrm{Aff})} &= \int_{\mathrm{Aff}} \left|\phi(a\lambda(x))|^2|\psi(a\lambda(-x))\right|^2\,\frac{dx\,da}{a^2} \\ &= \int_0^\infty \int_0^\infty |\phi(v)|^2|\psi(w)|^2 \frac{v-w}{\log(v/w)}\frac{dw\,dv}{vw},\end{aligned}$$

where we used the change of variables $v = a\lambda(x)$ and $w = a\lambda(-x)$ in the last line. By our assumptions on ϕ and ψ, it will suffice to show that for all $v, w \in \mathbb{R}_+$ we have the upper bound

$$\frac{v - w}{vw\log(v/w)} \leq 2 \cdot \max\left\{1, \frac{1}{v}, \frac{1}{w}, \frac{1}{vw}\right\}.$$

It will be enough by symmetry to consider $\Lambda = \{(v, w) \in \mathbb{R}_+ \times \mathbb{R}_+ : v > w\}$. We have the decomposition $\Lambda = C_1 \cup C_2 \cup C_3$, where

$$\begin{aligned}
C_1 &:= \Big\{(v, w) \in \Lambda : w \leq -2\sigma(-v/2)\Big\},\\
C_2 &:= \left\{(v, w) \in \Lambda : w \geq \frac{-1}{\sigma(-1/v)}\right\},\\
C_3 &:= \left\{(v, w) \in \Lambda : -2\sigma(-v/2) \leq w \leq \frac{-1}{\sigma(-1/v)}\right\},
\end{aligned}$$

where σ is the function appearing in Lemma E.3.14.

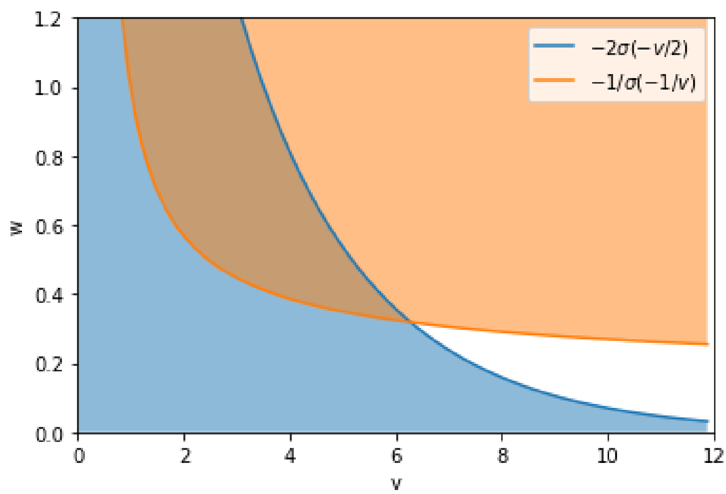

Figure E.1: A drawing marking the beginning and end of the different domains.

- The level surface $g(v, w) = (v - w)/\log(v/w) = C$ for $C > 0$ is given by the equation

$$w = -C\sigma\left(-\frac{v}{C}\right). \tag{E.4.9}$$

On C_1 we are below the level surface (E.4.9) with $C = 2$. Notice that $(1, 0.5) \in C_1$ with $g(1, 0.5) = \log(\sqrt{2}) < 2$. The continuity of g forces the inequality $g(v, w) \leq 2$ for all $(v, w) \in C_1$. Hence

$$\frac{v - w}{vw \log(v/w)} \leq \frac{2}{vw}.$$

- Notice that
$$\frac{v - w}{vw \log(v/w)} = \frac{\frac{1}{v} - \frac{1}{w}}{\log((1/v)/(1/w))}.$$
Hence the case of C_2 follows from the previous the argument for C_1 by considering the level surface of
$$g(1/v, 1/w) = 1.$$

- It is straightforward to verify that $v > 2$ and $w < 1$ when $(v, w) \in C_3$. Hence we obtain for any $(v, w) \in C_3$ that
$$\frac{v - w}{wv \log(v/w)} \leq \frac{v}{wv \log(2)} \leq 2/w. \qquad \square$$

Remarks.

- The connection from this result to admissibility is that the assumptions boil down to $S = \mathcal{D}\psi \otimes \mathcal{D}\phi$ being an admissible operator.

- Let A be a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ with integral kernel A_K. Then one can gauge from the proof of Theorem E.4.15 that the affine Weyl symbol f_A satisfies $f_A \in L_r^2(\mathrm{Aff}) \cap L_l^2(\mathrm{Aff})$ if and only if the integral kernel A_K satisfies
$$A_K \in L^2\left(\mathbb{R}_+ \times \mathbb{R}_+, \frac{s - t}{st \log(s/t)} dt\, ds\right) \cap L^2\left(\mathbb{R}_+ \times \mathbb{R}_+, \frac{1}{st} dt\, ds\right).$$

E.4.5 Extending the Setting

Except for Subsection E.3.5, we have so far considered convolutions between rather well-behaved functions and operators and obtained norm estimates for the norms of $L_r^1(\mathrm{Aff})$, $L^\infty(\mathrm{Aff})$, $\mathcal{S}_1$, and $\mathcal{L}(L^2(\mathbb{R}_+))$. We have seen that

$$\|f \star_{\mathrm{Aff}} S\|_{\mathcal{S}_1} \leq \|f\|_{L_r^1(\mathrm{Aff})} \|S\|_{\mathcal{S}_1},$$
$$\|T \star_{\mathrm{Aff}} S\|_{L^\infty(\mathrm{Aff})} \leq \|T\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \|S\|_{\mathcal{S}_1}.$$

The following result generalizes these inequalities to other Schatten-classes.

Proposition E.4.16. Let $1 \leq p \leq \infty$ and let q be its conjugate exponent given by $p^{-1} + q^{-1} = 1$. If $S \in \mathcal{S}_p, T \in \mathcal{S}_q$, and $f \in L^1_r(\mathrm{Aff})$, then the following hold:

- $f \star_{\mathrm{Aff}} S \in \mathcal{S}_p$ with $\|f \star_{\mathrm{Aff}} S\|_{\mathcal{S}_p} \leq \|f\|_{L^1_r(\mathrm{Aff})}\|S\|_{\mathcal{S}_p}$.
- $T \star_{\mathrm{Aff}} S \in L^\infty(\mathrm{Aff})$ with $\|T \star_{\mathrm{Aff}} S\|_{L^\infty(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_p}\|T\|_{\mathcal{S}_q}$.

Proof. For $p < \infty$, we can clearly interpret the definition of $f \star_{\mathrm{Aff}} S$ as a convergent Bochner integral in $\mathcal{S}_p$. The first inequality follows from [106, Proposition 1.2.2]. For $p = \infty$, we avoid the unpleasantness of Bochner integration in non-separable Banach spaces by interpreting $f \star_{\mathrm{Aff}} S$ weakly by

$$\langle f \star_{\mathrm{Aff}} S\psi, \phi\rangle_{L^2(\mathbb{R}_+)} = \int_{\mathrm{Aff}} f(x,a)\langle SU(-x,a)\psi, U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\, da}{a},$$

for $\psi, \phi \in L^2(\mathbb{R}_+)$. A standard argument shows that $f \star_{\mathrm{Aff}} S$ is a bounded operator with

$$\|f \star_{\mathrm{Aff}} S\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \leq \|f\|_{L^1_r(\mathrm{Aff})}\|S\|_{\mathcal{L}(L^2(\mathbb{R}_+))}.$$

The second inequality follows from [144, Theorem 2.8]. □

We have already seen in Subsection E.4.1 that we can say more about operator convolutions when one of the operators is admissible. As the next lemma shows, admissibility is also the correct condition to ensure that $f \star_{\mathrm{Aff}} S$ defines a bounded operator for all $f \in L^\infty(\mathrm{Aff})$.

Lemma E.4.17. *Let $S \in \mathcal{S}_1$ and $f \in L^\infty(\mathrm{Aff})$. Define the operator $f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ weakly for $\psi, \phi \in \mathrm{Dom}(\mathcal{D})$ by*

$$\begin{aligned}\langle f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\psi, \phi\rangle_{L^2(\mathbb{R}_+)} \\ = \int_{\mathrm{Aff}} f(x,a)\langle S\mathcal{D}U(-x,a)\psi, \mathcal{D}U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\, da}{a}.\end{aligned} \tag{E.4.10}$$

Then $f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ uniquely extends to a bounded linear operator on $L^2(\mathbb{R}_+)$ satisfying

$$\|f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \leq \|f\|_{L^\infty(\mathrm{Aff})}\|S\|_{\mathcal{S}_1}.$$

In particular, if R is an admissible operator, then $f \star_{\mathrm{Aff}} R \in \mathcal{L}(L^2(\mathbb{R}_+))$ with

$$\|f \star_{\mathrm{Aff}} R\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \leq \|f\|_{L^\infty(\mathrm{Aff})}\|\mathcal{D}^{-1}R\mathcal{D}^{-1}\|_{\mathcal{S}_1}.$$

Proof. By using (E.4.1) we get that

$$\begin{aligned}
&\langle f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\psi, \phi\rangle_{L^2(\mathbb{R}_+)} \\
&= \int_{\mathrm{Aff}} f(x,a)\langle SU(-x,a)\mathcal{D}\psi, U(-x,a)\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\,da}{a^2} \\
&= \int_{\mathrm{Aff}} \check{f}(x,a)\langle SU(-x,a)^*\mathcal{D}\psi, U(-x,a)^*\mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\,da}{a} \\
&= \int_{\mathrm{Aff}} \check{f}(x,a)(S \star_{\mathrm{Aff}} (\mathcal{D}\psi \otimes \mathcal{D}\phi))(x,a) \frac{dx\,da}{a}.
\end{aligned}$$

Clearly $\mathcal{D}\psi \otimes \mathcal{D}\phi$ is an admissible operator with

$$|\operatorname{tr}(\mathcal{D}^{-1}(\mathcal{D}\psi \otimes \mathcal{D}\phi)\mathcal{D}^{-1})| = |\langle \psi, \phi\rangle|_{L^2(\mathbb{R}_+)} \leq \|\psi\|_{L^2(\mathbb{R}_+)}\|\phi\|_{L^2(\mathbb{R}_+)}.$$

By Corollary E.4.5 we therefore get

$$\left|\langle f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}\psi, \phi\rangle_{L^2(\mathbb{R}_+)}\right| \leq \|f\|_{L^\infty(\mathrm{Aff})}\|S\|_{\mathcal{S}_1}\|\psi\|_{L^2(\mathbb{R}_+)}\|\phi\|_{L^2(\mathbb{R}_+)}.$$

The density of $\mathrm{dom}(\mathcal{D})$ implies that $f \star_{\mathrm{Aff}} \mathcal{D}S\mathcal{D}$ extends to a bounded operator on $L^2(\mathbb{R}_+)$. ∎

Armed with Lemma E.4.17 and Corollary E.4.5, we prove the following result describing L^p and $\mathcal{S}_p$ properties of convolutions with admissible operators. The proof is essentially an application of complex interpolation: We refer to the results [144, Theorem 2.10] and [22, Theorem 5.1.1] for the interpolation theory of $\mathcal{S}_p$ and $L^p_r(\mathrm{Aff})$.

Proposition E.4.18. Let $1 \leq p \leq \infty$ and let q be its conjugate exponent given by $p^{-1} + q^{-1} = 1$. If $R \in \mathcal{S}_p$, $g \in L^p_r(\mathrm{Aff})$, and S is an admissible trace-class operator, then:

- $g \star_{\mathrm{Aff}} S \in \mathcal{S}_p$ with $\|g \star_{\mathrm{Aff}} S\|_{\mathcal{S}_p} \leq \|S\|_{\mathcal{S}_1}^{1/p}\|\mathcal{D}^{-1}S\mathcal{D}^{-1}\|_{\mathcal{S}_1}^{1/q}\|g\|_{L^p_r(\mathrm{Aff})}$.
- $R \star_{\mathrm{Aff}} S \in L^p_r(\mathrm{Aff})$ with $\|R \star_{\mathrm{Aff}} S\|_{L^p_r(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_1}^{1/q}\|\mathcal{D}^{-1}S\mathcal{D}^{-1}\|_{\mathcal{S}_1}^{1/p}\|R\|_{\mathcal{S}_p}$.

Proof. For $g \in L^1_r(\mathrm{Aff}) \cap L^\infty(\mathrm{Aff})$, we have for $p = \infty$ that Lemma E.4.17 gives

$$\|g \star_{\mathrm{Aff}} S\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \leq \|\mathcal{D}^{-1}S\mathcal{D}^{-1}\|_{\mathcal{S}_1}\|g\|_{L^\infty(\mathrm{Aff})}.$$

Since we also have $\|g \star_{\mathrm{Aff}} S\|_{\mathcal{S}_1} \leq \|g\|_{L^1_r(\mathrm{Aff})}\|S\|_{\mathcal{S}_1}$, the first result follows by complex interpolation. For the second claim, if $R \in \mathcal{S}_1$ we know from Corollary E.4.5 that

$$\|R \star_{\mathrm{Aff}} S\|_{L^1_r(\mathrm{Aff})} \leq \|\mathcal{D}^{-1}S\mathcal{D}^{-1}\|_{\mathcal{S}_1}\|R\|_{\mathcal{S}_1}.$$

The result follows by complex interpolation since

$$\|R \star_{\mathrm{Aff}} S\|_{L^\infty(\mathrm{Aff})} \leq \|S\|_{\mathcal{S}_1}\|R\|_{\mathcal{L}(L^2(\mathbb{R}_+))}.$$ ∎

E.5 From the Viewpoint of Representation Theory

We will for completeness investigate how various notions of affine Fourier transforms fit into our framework. As we will see, known results from abstract wavelet analysis give connections between affine Weyl quantization, affine Fourier transforms, and admissibility for operators.

E.5.1 Affine Fourier Transforms

Definition E.5.1. For $f \in L^1_l(\mathrm{Aff})$ we define the *(left) integrated representation* $U(f)$ to be the operator given by

$$U(f)\psi := \int_{\mathrm{Aff}} f(x,a)U(x,a)\psi \, \frac{dx\,da}{a^2}, \qquad \psi \in L^2(\mathbb{R}_+).$$

The *inverse affine Fourier-Wigner transform* $\mathcal{F}_W^{-1}(f)$ of $f \in L^1_r(\mathrm{Aff})$ is given by

$$\mathcal{F}_W^{-1}(f) := U(\check{f}) \circ \mathcal{D}, \qquad \check{f}(x,a) := f((x,a)^{-1}).$$

The inverse affine Fourier-Wigner transform $\mathcal{F}_W^{-1}(f)$ of $f \in L^1_r(\mathrm{Aff})$ is explicitly given by

$$\mathcal{F}_W^{-1}(f)\psi(s) = \int_0^\infty \sqrt{r}\mathcal{F}_1(f)(r, s/r)\psi(r)\,\frac{dr}{r},$$

where $\mathcal{F}_1$ denotes the Fourier transform in the first coordinate and $\psi \in L^2(\mathbb{R}_+)$. Hence the integral kernel of $\mathcal{F}_W^{-1}(f)$ is given by

$$K_f(s,r) = \sqrt{r}(\mathcal{F}_1 f)(r, s/r), \qquad s, r \in \mathbb{R}_+. \tag{E.5.1}$$

It is straightforward to verify that we have the estimate

$$\|\mathcal{F}_W^{-1}(f)\|_{\mathcal{S}_2} \leq \|f\|_{L^2_r(\mathrm{Aff})},$$

for every $f \in L^1_r(\mathrm{Aff}) \cap L^2_r(\mathrm{Aff})$. Hence we can extend $\mathcal{F}_W^{-1}$ to be defined on $L^2_r(\mathrm{Aff})$ and we have that $\mathcal{F}_W^{-1}(f) \in \mathcal{S}_2$ for any $f \in L^2_r(\mathrm{Aff})$.

Proposition E.5.2. The inverse affine Fourier-Wigner transform is a unitary transformation $\mathcal{F}_W^{-1} \colon \mathcal{Q}_1 \to \mathcal{S}_2$, where

$$\mathcal{Q}_1 := \{f \in L^2_r(\mathrm{Aff}) \mid \operatorname{ess\,supp}(\mathcal{F}_1(f)) \subset \mathbb{R}_+ \times \mathbb{R}_+\}.$$

Proof. Any function $K \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ can be written uniquely on the form K_f in (E.5.1) for some $f \in \mathcal{Q}_1$. Moreover, we have

$$\|K_f\|_{L^2(\mathbb{R}_+\times\mathbb{R}_+)} = \sqrt{\int_0^\infty \int_0^\infty |\mathcal{F}_1 f(r, s/r)|^2 \, dr\, \frac{ds}{s}} = \|f\|_{L^2_r(\mathrm{Aff})}.$$

Since there is a norm-preserving correspondence between integral kernels in $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ and Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$, the claim follows. $\square$

It is straightforward to check that the inverse affine Fourier-Wigner transform $\mathcal{F}_W^{-1}$ satisfies for $f, g \in \mathcal{Q}_1$ the properties

- $\mathcal{F}_W^{-1}(f)^* = \mathcal{F}_W^{-1}(\Delta^{1/2} f^*), \qquad f^*(x,a) := \overline{f((x,a)^{-1})}$.
- $\mathcal{F}_W^{-1}(f *_{\mathrm{Aff}} g) = \mathcal{F}_W^{-1}(f) \circ \mathcal{D}^{-1} \circ \mathcal{F}_W^{-1}(g) = U(\check{f}) \circ \mathcal{F}_W^{-1}(g)$.
- $U(x,a) \circ \mathcal{F}_W^{-1}(f) = \mathcal{F}_W^{-1}(R_{(x,a)}(f))$.
- $\mathcal{F}_W^{-1}(f) \circ U(x,a) = \mathcal{F}_W^{-1}\left(\sqrt{a} L_{(x,a)^{-1}}(f)\right)$.

Definition E.5.3. The *affine Fourier-Wigner transform* $\mathcal{F}_W : \mathcal{S}_2 \to \mathcal{Q}_1$ is defined to be the inverse of $\mathcal{F}_W^{-1}|_{\mathcal{Q}_1}$.

Remarks.

- To avoid overly cluttered notation, we have used the symbol $\mathcal{F}_W$ for both the classical Fourier-Wigner transform in Subsection E.2.2, and the affine Fourier-Wigner transform. It should be clear from the context which operator we are referring to.
- Recall that the right multiplication R acts on elements in $L^2_r(\mathrm{Aff})$ by

  $$R_{(y,b)} f(x,a) := f((x,a)(y,b))$$

  for $(x,a), (y,b) \in \mathrm{Aff}$. For a closed subspace $\mathcal{H} \subset L^2_r(\mathrm{Aff})$ invariant under R, we write $R|_{\mathcal{H}} \cong U$ if there exists a unitary map $T\colon \mathcal{H} \to L^2(\mathbb{R}_+)$ satisfying

  $$T \circ R(x,a) f = U(x,a) \circ T f,$$

  for all $f \in \mathcal{H}$ and $(x,a) \in \mathrm{Aff}$. Define

  $$L^2_U(\mathrm{Aff}) := \overline{\mathrm{span}}\{\mathcal{H} \subset L^2_r(\mathrm{Aff}) : R|_{\mathcal{H}} \cong U\}.$$

  From [50, Lemma 3] we deduce that

  $$L^2_U(\mathrm{Aff}) = \mathcal{Q}_1,$$

  as both spaces are the image of the Hilbert-Schmidt operators under the Fourier-Wigner transform. Note that [50] uses left Haar measure, but translating to right Haar measure is an easy exercise using that $f \mapsto \check{f}$ is a unitary equivalence from the left regular representation on $L^2_l(\mathrm{Aff})$ to the right regular representation on $L^2_r(\mathrm{Aff})$.

Example E.5.4. Let $\phi, \psi \in L^2(\mathbb{R}_+)$ with $\psi \in \operatorname{dom}(\mathcal{D})$. If

$$f(x,a) = \langle \phi, U(x,a)^* \mathcal{D}\psi \rangle_{L^2(\mathbb{R}_+)},$$

one finds using Proposition E.4.2 that $f \in L^2_r(\mathrm{Aff})$ and

$$\langle \mathcal{F}_W^{-1}(f)\xi, \eta \rangle_{L^2(\mathbb{R}_+)} = \langle (\phi \otimes \psi)\xi, \eta \rangle_{L^2(\mathbb{R}_+)}$$

for $\eta \in L^2(\mathbb{R}_+)$ and $\xi \in \operatorname{dom}(\mathcal{D})$. This implies that $\mathcal{F}_W^{-1}(f) = \phi \otimes \psi$, in other words for $(x,a) \in \mathrm{Aff}$ that

$$\mathcal{F}_W(\phi \otimes \psi)(x,a) = \langle \phi, U(x,a)^* \mathcal{D}\psi \rangle_{L^2(\mathbb{R}_+)}.$$

For the Heisenberg group, the Fourier-Wigner transform has a very convenient expression for trace-class operators, see (E.2.8). The corresponding expression on the affine group is $\mathcal{F}_W(A)(x,a) = \operatorname{tr}(A\mathcal{D}U(x,a))$, and the next result shows that it holds as long as the objects in the formula are well-defined. The result is due to Führ in this generality [69, Theorem 4.15], and builds on an earlier result due to Duflo and Moore [50, Corollary 2].

Proposition E.5.5 (Führ, Duflo, and Moore)**.** Let $A \in \mathcal{S}_1$ be such that $A\mathcal{D}^{-1}$ extends to a Hilbert-Schmidt operator. Then

$$\mathcal{F}_W(A\mathcal{D}^{-1})(x,a) = \operatorname{tr}(AU(x,a)).$$

Proof. To see how the result follows from [69, Theorem 4.15], we need some terminology regarding direct integrals, see [69, Section 3.3]. Recall that the Plancherel theorem [69, Theorem 3.48] supplies a measurable field of Hilbert spaces indexed by the dual group $\{\mathcal{H}_\pi\}_{[\pi]\in\hat{G}}$. For the affine group $G = \mathrm{Aff}$, the Plancherel measure is the counting measure supported on the two representations $\pi_1(x,a) = U(x,a)$ on $L^2(\mathbb{R}_+)$ and $\pi_2(x,a) = U(x,a)$ on $L^2(\mathbb{R}_-) := L^2(\mathbb{R}_-, r^{-1}\,dr)$. So we can construct an element $\{A_{[\pi]}\}_{[\pi]\in\hat{G}}$ of the direct integral

$$\int_{\hat{G}}^{\oplus} HS(\mathcal{H}_\pi)\,d\hat{\mu}([\pi])$$

by choosing $A_{[\pi_1]} = A\mathcal{D}^{-1}$ and $A_{[\pi]} = 0$ for $[\pi] \neq [\pi_1]$. Inserting this measurable field of trace-class operators into [69, Theorem 4.15] then gives the conclusion. □

For $f, g \in L^2(\mathbb{R})$ we denote by $\mathrm{Scal}_g f$ the *scalogram* of f with respect to g given by $\mathrm{Scal}_g f(x,a) := |\mathcal{W}_g f(x,a)|^2$ where $\mathcal{W}_g f$ is the continuous wavelet transform

$$\mathcal{W}_g f(x,a) := \frac{1}{\sqrt{a}} \int_{-\inf}^{\inf} f f(t) \overline{g\left(\frac{t-x}{a}\right)}\,dt.$$

The following result, which follows from Lemma E.3.3 and Example E.5.4, gives a connection between the affine Fourier-Wigner transform, affine convolutions, and the scalogram.

Corollary E.5.6. Let $f, g \in L^2(\mathbb{R})$ such that $\psi := \hat{f}$ and $\phi := \hat{g}$ are supported in $\mathbb{R}_+$ and are in $L^2(\mathbb{R}_+)$. If ψ is admissible then

$$|\mathcal{F}_W(\phi \otimes \mathcal{D}^{-1}\psi)(x,a)|^2 = (\phi \otimes \phi) \star_{\mathrm{Aff}} (\psi \otimes \psi)(-x,a) = \frac{1}{a}\mathrm{Scal}_g f(x,a). \quad \text{(E.5.2)}$$

Remark. The condition that ψ is admissible in Corollary E.5.6 is only necessary for the first equality in (E.5.2). Recall that the affine Wigner distribution W^{ψ}_{Aff} is the affine Weyl symbol of the rank-one operator $\psi \otimes \psi$. If we use Proposition E.3.7 together with Corollary E.5.6, then we recover [17, Theorem 5.1].

Corollary E.5.6 shows that we have the simple relation

$$|\mathcal{F}_W(A\mathcal{D}^{-1})(x,a)|^2 = A \star_{\mathrm{Aff}} A(-x,a) \quad \text{(E.5.3)}$$

for positive rank-one operators A. By Corollary E.4.9, admissibility therefore means that $\mathcal{F}_W(A\mathcal{D}^{-1}) \in L^2_r(\mathrm{Aff})$ in this case. For more general operators, (E.5.3) will no longer hold. However, we still obtain a result relating admissibility to the Fourier-Wigner transform. Note that in the first statement in Proposition E.5.7 if $A \in \mathcal{S}_1$ we interpret $\mathcal{F}_W(A\mathcal{D}^{-1}) := \mathrm{tr}(AU(x,a))$ if we do not know that $A\mathcal{D}^{-1}$ extends to a Hilbert-Schmidt operator.

Proposition E.5.7. Let A be a trace-class operator on $L^2(\mathbb{R}_+)$. Then the following are equivalent:

1) $\mathcal{F}_W(A\mathcal{D}^{-1}) \in L^2_r(\mathrm{Aff})$.

2) $A\mathcal{D}^{-1}$ extends from $\mathrm{dom}(\mathcal{D}^{-1})$ to a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$.

3) A^*A is admissible.

Proof. The equivalence of 1) and 2) follows from [69, Theorem 4.15], by applying that theorem to the element $\{A_{[\pi]}\}_{[\pi]\in\hat{G}}$ of the direct integral (see proof of Proposition E.5.5)

$$\int_{\hat{G}}^{\oplus} HS(\mathcal{H}_\pi)\, d\hat{\mu}([\pi])$$

given by choosing $A_{[\pi_1]} = A$ and $A_{[\pi]} = 0$ for $[\pi] \neq [\pi_1]$.

The equivalence of 2) and 3) is clear apart from technicalities resulting from the unboundedness of $\mathcal{D}^{-1}$. If we assume 2), then [142, Theorem 13.2] gives that $(A\mathcal{D}^{-1})^* = \mathcal{D}^{-1}A^*$, where the equality includes equality of domains. As the domain of the left term is all of $L^2(\mathbb{R}_+)$ by assumption, this means that the range of A^* is contained in $\mathrm{dom}(\mathcal{D}^{-1})$. In particular, A^*A maps $\mathrm{dom}(\mathcal{D})$ into $\mathrm{dom}(\mathcal{D}^{-1})$, and as we also have $\mathcal{D}^{-1}A^*A\mathcal{D}^{-1} = (A\mathcal{D}^{-1})^*A\mathcal{D}^{-1}$ where $A\mathcal{D}^{-1}$ is Hilbert-Schmidt, A^*A satisfies all requirements for being admissible.

Conversely, if A^*A is admissible, then we have for $\psi \in \mathrm{dom}(\mathcal{D}^{-1})$

$$\begin{aligned}\|A\mathcal{D}^{-1}\psi\|^2_{L^2(\mathbb{R}_+)} &= \langle \mathcal{D}^{-1}A^*A\mathcal{D}^{-1}\psi, \psi\rangle_{L^2(\mathbb{R}_+)} \\ &\leq \|\mathcal{D}^{-1}A^*A\mathcal{D}^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}_+))}\|\psi\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

So $A\mathcal{D}^{-1}$ extends to a bounded operator, and as this operator satisfies that

$$(A\mathcal{D}^{-1})^*A\mathcal{D}^{-1} = \mathcal{D}^{-1}A^*A\mathcal{D}^{-1}$$

is trace-class, $A\mathcal{D}^{-1}$ is a Hilbert-Schmidt operator. □

Remark. Recall that we consider $\mathcal{F}_W$ a Fourier transform of operators. The inequality

$$\|\mathcal{F}_W(A\mathcal{D}^{-1})\|_{L^\infty(\mathrm{Aff})} \leq \|A\|_{\mathcal{S}_1}$$

and the equality

$$\|A\|_{\mathcal{S}_2} = \|\mathcal{F}_W(A)\|_{L^2_r(\mathrm{Aff})}$$

might therefore be interpreted as the endpoints $p = \infty$ and $p = 2$ of a Hausdorff-Young inequality, where the appearance of $\mathcal{D}^{-1}$ suggests that the definition of the Fourier-Wigner transform must depend on p. In fact, a Hausdorff-Young inequality of this kind—formulated in the other direction, i.e. for maps from functions on Aff to operators—was shown in [53, Theorem 1.41] for $1 \leq p \leq 2$.

There is a second Fourier transform related to the affine group that comes from representation theory. We define the *affine Fourier-Kirillov transform* as the map $\mathcal{F}_{\mathrm{KO}}\colon Q_1 \to L^2_r(\mathrm{Aff})$ given by

$$(\mathcal{F}_{\mathrm{KO}}f)(x,a) := \sqrt{a}\int_{\mathbb{R}^2} f\left(\frac{v}{\lambda(-u)}, e^u\right) e^{-2\pi i(xu+av)}\,\frac{du\,dv}{\sqrt{\lambda(-u)}}, \qquad (x,a) \in \mathrm{Aff}.$$

More information about the Fourier-Kirillov transform can be found in [112]. The following result, which is motivated by (E.2.9) and is a slight generalization of [5, Section VIII.6], shows that the affine Weyl quantization is intrinsically linked with the Fourier transforms on the affine group.

Proposition E.5.8. Let A_f be a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ with affine symbol $f \in L^2_r(\mathrm{Aff})$. Then

$$\mathcal{F}_W(A_f) = \mathcal{F}_{\mathrm{KO}}^{-1}(f).$$

Proof. Recall from (E.5.1) that the integral kernel of $\mathcal{F}_W^{-1}(g)$ for $g \in Q_1$ is given by

$$K_g(s,r) = \sqrt{r}(\mathcal{F}_1 g)(r, s/r), \qquad s,r \in \mathbb{R}_+.$$

Hence by using (E.2.12) and a change of variables, we see that the affine Weyl symbol of $\mathcal{F}_W^{-1}(g)$ is given at the point $(x,a) \in \mathrm{Aff}$ by

$$\begin{aligned}
&\int_{-\infty}^{\infty} \sqrt{a\lambda(-u)}\mathcal{F}_1(g)(a\lambda(-u), e^u)e^{-2\pi i x u}\, du \\
&= \int_{\mathbb{R}^2} \sqrt{a\lambda(-u)}g(v, e^u)e^{-2\pi i(xu + av\lambda(-u))}\, du\, dv \\
&= \sqrt{a}\int_{\mathbb{R}^2} g\left(\frac{v}{\lambda(-u)}, e^u\right) e^{-2\pi i(xu+av)}\, \frac{du\, dv}{\sqrt{\lambda(-u)}} \\
&= (\mathcal{F}_{\mathrm{KO}} g)(x,a).
\end{aligned}$$

□

Remarks.

- In [128] the authors define an alternative quantization scheme on general type 1 groups. Their quantization scheme together with the affine Weyl quantization is used in [128] to define a quantization scheme on the cotangent bundle $T^*\mathrm{Aff}$.

- Consider A_f for some $f \in L^2_r(\mathrm{Aff})$. Inserting $f = \mathcal{F}_{KO}\mathcal{F}_W(A_f)$ into Proposition E.4.14 allows us to obtain a formal expression for $\mathrm{tr}(\mathcal{D}^{-1}A_f\mathcal{D}^{-1})$ in terms of $\mathcal{F}_W(A_f)$: A formal calculation gives that for sufficiently nice operators A_f we have

$$\mathrm{tr}(\mathcal{D}^{-1}A_f\mathcal{D}^{-1}) = \int_0^\infty [\mathcal{F}_1\mathcal{F}_W(A_f)](a,1)\,\frac{da}{a^{3/2}}, \tag{E.5.4}$$

  where $\mathcal{F}_1$ is the Fourier transform in the first coordinate. This is similar to a condition in [76, Corollary 5.2], where finiteness of (E.5.4) is used as a necessary condition for $1 \star_{\mathrm{Aff}} A_f = I_{L^2(\mathbb{R}_+)}$ to hold, where $1(x,a) := 1$ for all $(x,a) \in \mathrm{Aff}$. We will see in Subsection E.6.2 that this is closely related to admissibility of A_f. Unfortunately, the formal calculation leading to (E.5.4) does not give clear conditions on A_f for the equality to hold.

E.5.2 Affine Quantum Bochner Theorem

On the Heisenberg group, the Fourier-Wigner transform behaves in many ways like the Fourier transform on functions. In particular, for the function $f \in L^1(\mathbb{R}^{2n})$ and the operators $S, T \in \mathcal{S}_1(\mathbb{R}^n)$ we get the *decoupling equations*

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S), \qquad \mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T), \tag{E.5.5}$$

where $\mathcal{F}_\sigma$ denotes the symplectic Fourier transform and $\mathcal{F}_W$ denotes the classical Fourier-Wigner transform introduced in Subsection E.2.2. Although the affine versions of the equations (E.5.5) do not hold, one can develop as a special case of the result [69, Theorem 4.12] a version of Bochner's theorem for the affine Fourier-Wigner transform. This is analogous to the *quantum Bochner theorem* [153, Proposition 3.2] for the Heisenberg group.

Bochner's classical theorem [67, Theorem 4.19] characterizes functions that are Fourier transforms of positive measures. The Bochner theorem for the affine Fourier-Wigner transform answers the following question: Which functions on Aff are of the form $\mathcal{F}_W(S)$, where S is a positive trace-class operator? As in Bochner's classical theorem, it turns out that the correct notion to consider is functions of positive type. Recall that a function $f\colon \mathrm{Aff} \to \mathbb{C}$ is a *function of positive type* if for any finite selection of points $\Omega := \{(x_1, a_1), \ldots, (x_n, a_n)\} \subset \mathrm{Aff}$ the matrix A_Ω with entries

$$(A_\Omega)_{i,j} := f((x_i, a_i)^{-1}(x_j, a_j))$$

is positive semi-definite. Before stating the general result we consider an illuminating special case.

Example E.5.9. Let $A := \phi \otimes \psi$ is a rank-one operator for $\phi, \psi \in L^2(\mathbb{R}_+)$. We will show that

$$\mathcal{F}_W(A\mathcal{D}^{-1})(x, a) = \langle U(x, a)\phi, \psi\rangle_{L^2(\mathbb{R}_+)} \tag{E.5.6}$$

is a function of positive type on Aff if and only if A is a positive operator. If A is positive, then a standard fact [67, Proposition 3.15] shows that (E.5.6) is a function of positive type. Conversely, we have from [67, Corollary 3.22] that

$$\mathcal{F}_W(\phi \otimes \psi\mathcal{D}^{-1})((x, a)^{-1}) = \overline{\mathcal{F}_W(\psi \otimes \phi\mathcal{D}^{-1})(x, a)} = \overline{\mathcal{F}_W(\phi \otimes \psi\mathcal{D}^{-1})(x, a)}.$$

Hence

$$\langle U(x, a)\phi, \psi\rangle_{L^2(\mathbb{R}_+)} = \langle U(x, a)\psi, \phi\rangle_{L^2(\mathbb{R}_+)}$$

and it follows from [77, Theorem 4.2] that $\phi = c \cdot \psi$ for some $c \in \mathbb{C}$. We can conclude from [67, Corollary 3.22] that $c \geq 0$ since

$$\mathcal{F}_W(c\psi \otimes \psi\mathcal{D}^{-1})(0, 1) = c \cdot \|\psi\|_{L^2(\mathbb{R}_+)} \geq 0.$$

We are now ready to state the main result regarding positivity. This result is actually a special case of the general result [69, Theorem 4.12].

Theorem E.5.10. *Let A be a trace-class operator on $L^2(\mathbb{R}_+)$. Then A is a positive operator if and only if the function*

$$\mathcal{F}_W(A\mathcal{D}^{-1})(x, a) = \mathrm{tr}(AU(x, a))$$

is of positive type on Aff.

Proof. We use the same notation as in the proof of Proposition E.5.5. For $G = \mathrm{Aff}$, the abstract result in [69] says that if

$$\{A_{[\pi]}\}_{[\pi]\in\hat{G}} \in \int_{\hat{G}}^{\oplus} HS(\mathcal{H}_\pi)\,d\hat{\mu}([\pi])$$

consists of trace-class operators, then $A_{[\pi]}$ is positive a.e. with respect to $\hat{\mu}$ if and only if the function $\int_{\hat{G}} \mathrm{tr}(A_{[\pi]}\pi(g)^*)\,d\hat{\mu}([\pi])$ is of positive type.

As in the proof of Proposition E.5.7, we pick $A_{[\pi_1]} = A$ and $A_{[\pi]} = 0$ for $[\pi] \neq [\pi_1]$. The resulting section consists of positive operators for a.e. $[\pi]$ if and only if A is positive. By the abstract result in [69], this happens if and only if

$$\int_{\hat{G}} \mathrm{tr}(A_{[\pi]}\pi(g)^*)\,d\hat{\mu}([\pi]) = \mathrm{tr}(AU(x,a)^*)$$

is a function of positive type. The definition of functions of positive type gives that this is equivalent to $\mathrm{tr}(AU(x,a))$ being of positive type. □

E.6 Examples

In this section, we show how the theory developed in this paper provides a common framework for various operators and functions studied by other authors. We also introduce an analogue of the Cohen class of time-frequency distributions for the affine group, and deduce its relation to the previously studied *affine quadratic time-frequency representations*.

E.6.1 Affine Localization Operators

There is no general consensus of a localization operator in the affine setting. We will use the following definition based on the convolution framework.

Definition E.6.1. Let $f \in L^1_r(\mathrm{Aff})$ and $\varphi \in L^2(\mathbb{R}_+)$. We say that

$$A := f \star_{\mathrm{Aff}} (\varphi \otimes \varphi)$$

is an *affine localization operator* on $L^2(\mathbb{R}_+)$.

Inequality (E.3.1) shows that an affine localization operator A is a trace-class operator on $L^2(\mathbb{R}_+)$ with

$$\|A\|_{\mathcal{S}_1} \leq \|f\|_{L^1_r(\mathrm{Aff})}\|\varphi\|^2_{L^2(\mathbb{R}_+)}.$$

Moreover, Proposition E.4.13 implies that A is admissible whenever φ is admissible and $f \in L^1_l(\mathrm{Aff}) \cap L^1_r(\mathrm{Aff})$.

We will now see that the affine localization operators are naturally unitarily equivalent to the more commonly defined localization operators on the Hardy space $H^2_+(\mathbb{R})$. Recall that the space $H^2_+(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ consisting of elements ψ whose Fourier transform $\mathcal{F}\psi$ is supported on $\mathbb{R}_+$. Note that the composition $\mathcal{D}\mathcal{F}$ is a unitary map from $H^2_+(\mathbb{R})$ to $L^2(\mathbb{R}_+)$. An *admissible wavelet* $\xi \in H^2_+(\mathbb{R})$ satisfies by definition that

$$c_\xi := \int_0^\infty \frac{|\mathcal{F}(\xi)(\omega)|^2}{\omega}\, d\omega < \infty.$$

Hence $\mathcal{D}\mathcal{F}\xi \in L^2(\mathbb{R}_+)$ is an admissible function in the sense of Definition E.4.1.

In [156, Theorem 18.13] the localization operator A^ξ_f on $H^2_+(\mathbb{R})$, given an admissible wavelet $\xi \in H^2_+(\mathbb{R})$ and $f \in L^1_l(\mathrm{Aff})$, is defined by

$$A^\xi_f \psi := c_\xi \int_{\mathrm{Aff}} f(x,a)\langle \xi, \pi(x,a)\xi\rangle_{H^2_+(\mathbb{R})} \pi(x,a)\xi \,\frac{dx\,da}{a^2}, \quad \xi \in H^2_+(\mathbb{R}),$$

where π acts on $H^2_+(\mathbb{R})$ by

$$\pi(x,a)\xi(t) := \frac{1}{\sqrt{a}}\xi\left(\frac{t-x}{a}\right), \qquad \xi \in H^2_+(\mathbb{R}). \tag{E.6.1}$$

The next proposition is straightforward and relates operators on the form A^ξ_f with affine localization operators.

Proposition E.6.2. Consider $f \in L^1_l(\mathrm{Aff})$ and an admissible wavelet $\xi \in H^2_+(\mathbb{R})$. Then

$$(\mathcal{D}\mathcal{F})A^\xi_f(\mathcal{D}\mathcal{F})^* = c_\xi \cdot \check{f} \star_{\mathrm{Aff}} (\mathcal{D}\mathcal{F}\xi \otimes \mathcal{D}\mathcal{F}\xi).$$

Remarks.

- From Proposition E.6.2 it follows that Proposition E.4.18 is a generalization of the result [156, Theorem 18.13].
- In [46], Daubechies and Paul define localization operators in the same way as in [156], except that they use $\pi(-x,a)$ instead of $\pi(x,a)$ in (E.6.1) and consider symbols f on the full affine group $\mathrm{Aff}_F := \mathbb{R} \times \mathbb{R}^*$. The eigenfunctions and eigenvalues of the resulting localization operators acting on $L^2(\mathbb{R})$ are studied in detail in [46] when the window is related to the first Laguerre function, and $f = \chi_{\Omega_C}$ where

  $$\Omega_C := \{(x,a) \in \mathrm{Aff} : |(x,a) - (0,C)|^2 \le (C^2 - 1)\}.$$

  The corresponding inverse problem, i.e. conditions on the eigenfunctions of the localization operator that imply that $\Omega = \Omega_C$, is studied in [2].

- Localization operators with windows related to Laguerre functions have also been extensively studied by Hutník, see for instance [102, 103, 104], with particular emphasis on symbols f depending only on either x or a. When $f(x, a) = f(a)$, it is shown that the resulting localization operator is unitarily equivalent to multiplication with some function γ_f. This correspondence allows properties of the localization operator to be deduced from properties of γ_f.

E.6.2 Covariant Integral Quantizations

Operators of the form $f \star_{\mathrm{Aff}} S$ form the basis of the study of covariant integral quantizations by Gazeau and his collaborators in [4, 20, 21, 74, 75, 76]. Apart from differing conventions that we clarify at the end of this section, *covariant integral quantizations* on Aff are maps Γ_S sending functions on Aff to operators given by

$$\Gamma_S(f) \coloneqq f \star_{\mathrm{Aff}} S,$$

for some fixed operator S.

By varying S we obtain several quantization maps Γ with properties depending on the properties of S. Examples of such quantization procedures with a different parametrization of Aff are studied in [21, 76]. Their approach is to define S either by $\mathcal{F}_W(S)$ or by its kernel as an integral operator, and deduce conditions on this function that ensures the condition

$$1 \star_{\mathrm{Aff}} S = I_{L^2(\mathbb{R}_+)}.$$

Example E.6.3. The affine Weyl quantization is an example of a covariant integral quantization Γ_S, where S is not a bounded operator. It corresponds to choosing $S = P_{\mathrm{Aff}}$ by Theorem E.3.20.

Remark. The example above leads to a natural question: Could there be other operators P such that $f \star_{\mathrm{Aff}} P$ behaves as an affine analogue of Weyl quantization? Since Weyl quantization on $\mathbb{R}^{2n}$ is given by convolving with the parity operator, a natural guess is

$$P\psi(r) = \psi(1/r), \qquad \psi \in L^2(\mathbb{R}_+).$$

The resulting quantization $\Gamma_P(f) = f \star_{\mathrm{Aff}} P$ has been studied by Gazeau and Murenzi in [76, Section 7]. It has the advantage that P is a bounded operator, but unfortunately by [76, Proposition 7.5] it does not satisfy the natural dequantization rule

$$f = \Gamma_P(f) \star_{\mathrm{Aff}} P.$$

We also mention that Gazeau and Bergeron have shown that this choice of P is merely a special case corresponding to $\nu = -1/2$ of a class P_ν of operators defining possible affine versions of the Weyl quantization [21, Section 4.5].

In quantization theory one typically wishes that the domain of Γ_S contains $L^\infty(\mathrm{Aff})$. This, by Lemma E.4.17, leads us to chose $S = \mathcal{D}T\mathcal{D}$ for some trace-class operator T. In particular, one requires that $\Gamma_S(1) = I_{L^2(\mathbb{R}_+)}$, which can be easily satisfied as the following proposition shows.

Proposition E.6.4. Let T be a trace-class operator on $L^2(\mathbb{R}_+)$. Then

$$1 \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D} = \mathrm{tr}(T) I_{L^2(\mathbb{R}_+)}.$$

Proof. Let $\psi, \phi \in \mathrm{dom}(D)$. We have by (E.4.10) that

$$\begin{aligned}\langle 1 \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D}\psi, \phi\rangle_{L^2(\mathbb{R}_+)} &= \int_{\mathrm{Aff}} \langle U(-x,a)^* \mathcal{D}T\mathcal{D}U(-x,a)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} \frac{dx\, da}{a} \\ &= \int_{\mathrm{Aff}} T \star_{\mathrm{Aff}} (\mathcal{D}\psi \otimes \mathcal{D}\phi) \frac{dx\, da}{a} \\ &= \mathrm{tr}(T)\langle \psi, \phi\rangle_{L^2(\mathbb{R}_+)},\end{aligned}$$

where the last equality uses Theorem E.4.3. □

Following the terminology used by Gazeau et al., we have a *resolution of the identity operator* of the form

$$I_{L^2(\mathbb{R}_+)} = \Gamma_{\mathcal{D}T\mathcal{D}}(1) = \int_{\mathrm{Aff}} U(-x,a)^* \mathcal{D}T\mathcal{D}U(-x,a) \frac{dx\, da}{a},$$

where $\mathrm{tr}(T) = 1$ and the integral has the usual weak interpretation.

Given a positive trace-class operator T with $\mathrm{tr}(T) = 1$, we know that

$$\Gamma_{\mathcal{D}T\mathcal{D}}(f) = f \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D}$$

defines a bounded map $\Gamma_{\mathcal{D}T\mathcal{D}} \colon L^\infty(\mathrm{Aff}) \to \mathcal{L}(L^2(\mathbb{R}_+))$ with $\Gamma_{\mathcal{D}T\mathcal{D}}(1) = I_{L^2(\mathbb{R}_+)}$. Moreover, $\Gamma_{\mathcal{D}T\mathcal{D}}$ maps positive functions to positive operators and by a variation of Lemma E.3.5 satisfies the covariance property

$$U(-x,a)^* \Gamma_{\mathcal{D}T\mathcal{D}}(f) U(-x,a) = \Gamma(R_{(x,a)^{-1}} f).$$

The following result, which is a modification of the remark given at the end of [113], shows a remarkable converse to these observations.

Theorem E.6.5. *Let* $\Gamma \colon L^\infty(\mathrm{Aff}) \to \mathcal{L}(L^2(\mathbb{R}_+))$ *be a linear map satisfying*

1. Γ *sends positive functions to positive operators,*

2. $\Gamma(1) = I_{L^2(\mathbb{R}_+)}$,

3. Γ *is continuous from the weak* topology on* $L^\infty(\mathrm{Aff})$ *(as the dual space of* $L^1_r(\mathrm{Aff})$*) to the weak* topology on* $\mathcal{L}(L^2(\mathbb{R}_+))$*,*

4. $U(-x,a)^*\Gamma(f)U(-x,a) = \Gamma(R_{(x,a)^{-1}}f)$.

Then there exists a unique positive trace-class operator T *with* $\mathrm{tr}(T) = 1$ *such that*

$$\Gamma(f) = f \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D}.$$

Proof. The map $\Gamma \mapsto \Gamma_l$ where $\Gamma_l(f) = \Gamma(\check{f})$ is a bijection from maps Γ satisfying the four assumptions to maps Γ_l satisfying

i) Γ_l sends positive functions to positive operators,

ii) $\Gamma_l(1) = I_{L^2(\mathbb{R}_+)}$,

iii) Γ_l is continuous from the weak* topology on $L^\infty(\mathrm{Aff})$ (as the dual space of $L^1_l(\mathrm{Aff})$) to the weak* topology on $\mathcal{L}(L^2(\mathbb{R}_+))$,

iv) $U(-x,a)^*\Gamma_l(f)U(-x,a) = \Gamma_l(L_{(x,a)^{-1}}f)$.

The remark in [113] applied to $G = \mathrm{Aff}$ and $U(-x,a)$ says that if a map Γ_l satisfies i)-iv) then it must be given for $\psi, \phi \in \mathrm{dom}(\mathcal{D})$ by

$$\langle \Gamma_l(f)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} = \int_{\mathrm{Aff}} f(x,a)\langle U(-x,a)TU(-x,a)^*\mathcal{D}\psi, \mathcal{D}\phi\rangle_{L^2(\mathbb{R}_+)} \,\frac{dx\,da}{a},$$

for some trace-class operator T as in the theorem. The relation (E.4.1) gives that

$$\begin{aligned}\langle \Gamma_l(f)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} &= \int_{\mathrm{Aff}} f(x,a)\langle U(-x,a)\mathcal{D}T\mathcal{D}U(-x,a)^*\psi, \phi\rangle_{L^2(\mathbb{R}_+)} \,\frac{dx\,da}{a^2}\\ &= \int_{\mathrm{Aff}} \check{f}(x,a)\langle U(-x,a)^*\mathcal{D}T\mathcal{D}U(-x,a)\psi, \phi\rangle_{L^2(\mathbb{R}_+)} \,\frac{dx\,da}{a}.\end{aligned}$$

Hence $\Gamma_l(f) = \check{f} \star_{\mathrm{Aff}} \mathcal{D}T\mathcal{D}$ and the result follows. □

Quantization using admissible trace-class operators

As we have mentioned, the properties of the quantization map $\Gamma(f) = f \star_{\mathrm{Aff}} S$ depend on the properties of S. From Lemma E.4.17 we know that if S is admissible, i.e. we can write $S = \mathcal{D}T\mathcal{D}$ for some trace-class operator T, then

$$\Gamma_S \colon L^\infty(\mathrm{Aff}) \to \mathcal{L}(L^2(\mathbb{R}_+))$$

is bounded. If we further assume that S is a trace-class operator, then Proposition E.4.18 shows that Γ_S is bounded from $L^p_r(\mathrm{Aff})$ to $\mathcal{S}_p$ for all $1 \leq p \leq \infty$. In this sense, the ideal class of covariant integral quantizations Γ_S are those given by *admissible trace-class operators.*

Example E.6.6. If $\varphi \in L^2(\mathbb{R}_+)$ is an admissible function, then $\varphi \otimes \varphi$ is an admissible operator. The resulting quantization $\Gamma_{\varphi\otimes\varphi}$ is then a special case of the quantization procedures introduced by Berezin [13]; Berezin calls f the *contravariant symbol* of $\Gamma_{\varphi\otimes\varphi}(f)$. In this sense, the quantization procedures Γ_S for admissible S generalize Berezin's contravariant symbols.

Relation to the Conventions of Gazeau and Murenzi

Gazeau and Murenzi [76] work with another parametrization of the affine group, namely $\Pi_+ := \mathbb{R}_+ \times \mathbb{R}$ where the group operation between $(q_1, p_1), (q_2, p_2) \in \Pi_+$ is given by

$$(q_1, p_1) \cdot (q_2, p_2) := (q_1 q_2, p_2/q_1 + p_1).$$

There is a unitary representation $U_G \colon \Pi_+ \to \mathcal{U}(L^2(\mathbb{R}_+, dr))$ given by

$$U_G(q,p)\psi(r) := \sqrt{\frac{1}{q}} e^{ipr}\psi(r/q) = \sqrt{\frac{1}{q}} U(p/2\pi, 1/q)\psi(r).$$

Given a function $\tilde{f}$ on Π_+ and an operator S on $L^2(\mathbb{R}_+, dr)$, Gazeau and Murenzi define

$$A^S_{\tilde{f}} := \frac{1}{C_S} \int_{-\infty}^{\infty} \int_0^{\infty} \tilde{f}(q,p) U_G(q,p) S U_G(q,p)^* \, dq \, dp,$$

where we assume that S satisfies

$$\int_{-\infty}^{\infty} \int_0^{\infty} U_G(q,p) S U_G(q,p)^* \, dq \, dp = C_S \cdot I_{L^2(\mathbb{R}_+, dr)}.$$

The next proposition is straightforward and shows that Gazeau and Murenzi's framework is easily related to our affine operator convolutions.

Proposition E.6.7. Let S be an operator on $L^2(\mathbb{R}_+, dr)$. Then $\mathcal{D}^{-1} S \mathcal{D}$ is an operator on $L^2(\mathbb{R}_+, r^{-1}dr)$ and

$$\mathcal{D} A^S_{\tilde{f}} \mathcal{D}^{-1} = \frac{2\pi}{C_S} f \star_{\mathrm{Aff}} (\mathcal{D} S \mathcal{D}^{-1}),$$

where $f(x,a) = \tilde{f}(a, \frac{2\pi x}{a})$ for $(x,a) \in \mathrm{Aff}$.

E.6.3 Affine Cohen Class Distributions

The cross-Wigner distribution $W(\psi, \phi)$ of $\psi, \phi \in L^2(\mathbb{R}^n)$ is known to have certain undesirable properties. A typical example is that one would like to interpret

$W(\psi,\phi)$ as a probability distribution, but $W(\psi,\phi)$ is seldom a non-negative function as shown by Hudson in [101]. To remedy this, Cohen introduced in [40] a new class of time-frequency distributions Q_f given by

$$Q_f(\psi,\phi) \coloneqq W(\psi,\phi) * f, \tag{E.6.2}$$

where f is a tempered distribution on $\mathbb{R}^{2n}$. In light of our setup, it is natural to investigate the affine analogue of the Cohen class.

Definition E.6.8. We say that a bilinear map

$$Q \colon L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \to L^\infty(\mathrm{Aff})$$

belongs to the *affine Cohen class* if $Q = Q_S$ for some $S \in \mathcal{L}(L^2(\mathbb{R}_+))$, where

$$Q_S(\psi,\phi)(x,a) \coloneqq (\psi \otimes \phi) \star_{\mathrm{Aff}} S(x,a) = \langle SU(-x,a)\psi, U(-x,a)\phi\rangle_{L^2(\mathbb{R}_+)}.$$

We will write $Q_S(\psi) \coloneqq Q_S(\psi,\psi)$.

By Proposition E.3.7 we get for $S = A_f$ that

$$Q_S(\psi,\phi) = W_{\mathrm{Aff}}^{\psi,\phi} *_{\mathrm{Aff}} \check{f},$$

which shows that our definition of the affine Cohen class is a natural analogue of (E.6.2). It is straightforward to verify that $Q_S(\psi,\phi)$ is a continuous function on Aff for all $\psi,\phi \in L^2(\mathbb{R}_+)$ and $S \in \mathcal{L}(L^2(\mathbb{R}_+))$. Since the affine Cohen class is defined in terms of the operator convolutions, we get some simple properties: The statements 1 and 2 in Proposition E.6.9 follow from Proposition E.4.18 and Corollary E.4.5. Statement 3 is a simple calculation and the last statement follows from a short polarization argument.

Proposition E.6.9. Let $S \in \mathcal{L}(L^2(\mathbb{R}_+))$. Then for $\psi,\phi \in L^2(\mathbb{R}_+)$ we have the following properties:

1. The function $Q_S(\psi,\phi)$ satisfies

$$\|Q_S(\psi,\phi)\|_{L^\infty(\mathrm{Aff})} \leq \|S\|_{\mathcal{L}(L^2(\mathbb{R}_+))}\|\psi\|_{L^2(\mathbb{R}_+)}\|\phi\|_{L^2(\mathbb{R}_+)}.$$

2. If S is admissible, then $Q_S(\psi,\phi) \in L^1_r(\mathrm{Aff})$ and

$$\int_{\mathrm{Aff}} Q_S(\psi,\phi)(x,a)\,\frac{dx\,da}{a} = \langle \psi,\phi\rangle_{L^2(\mathbb{R}_+)}\operatorname{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1}).$$

3. We have the covariance property

$$Q_S(U(-x,a)\psi, U(-x,a)\phi)(y,b) = Q_S(\psi,\phi)((y,b)\cdot(x,a)) \tag{E.6.3}$$

for all $(x,a),(y,b) \in \mathrm{Aff}$.

4. The function $Q_S(\psi, \psi)$ is (real-valued) positive for all $\psi \in L^2(\mathbb{R}_+)$ if and only if S is (self-adjoint) positive.

Example E.6.10.

- For $\psi, \phi \in L^2(\mathbb{R}_+)$ we have
$$Q_{\phi \otimes \phi}(\psi)(x, a) = |\langle \psi, U(-x, a)^* \phi \rangle_{L^2(\mathbb{R}_+)}|^2,$$
which by Corollary E.5.6 is simply a Fourier transform away from being a scalogram.

- If we relax the requirement that S is bounded in Definition E.6.8, then it follows from Theorem E.3.20 that
$$Q_{P_{\mathrm{Aff}}}(\psi) = W^{\psi}_{\mathrm{Aff}}$$
for $\psi \in \mathcal{S}(\mathbb{R}_+)$. Hence the affine Wigner distribution can be represented as a (generalized) affine Cohen class operator. If we define an alternative affine Weyl quantization using an operator P as in Subsection E.6.2, then it is clear that Q_P gives an alternative Wigner function. See [76, Section 7.2] for the case of $P\psi(r) = \psi(1/r)$.

The covariance property (E.6.3) and some rather weak continuity conditions completely characterize the affine Cohen class, as is shown in the following result.

Proposition E.6.11. Let $Q \colon L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \to L^\infty(\mathrm{Aff})$ be a bilinear map. Assume that for all $\psi, \phi \in L^2(\mathbb{R}_+)$ we know that $Q(\psi, \phi)$ is a continuous function on Aff that satisfies (E.6.3) and the estimate
$$|Q(\psi, \phi)(0, 1)| \lesssim \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)}.$$
Then there exists a unique $S \in \mathcal{L}(L^2(\mathbb{R}_+))$ such that $Q = Q_S$.

Proof. By assumption, the map $(\psi, \phi) \mapsto Q(\psi, \phi)(0, 1)$ is a bounded bilinear form. Hence there exists a unique bounded operator S such that
$$\langle S\psi, \phi \rangle_{L^2(\mathbb{R}_+)} = Q(\psi, \phi)(0, 1).$$
To see that $Q = Q_S$, note that we have
$$\begin{aligned} Q(\psi, \phi)(x, a) &= Q(U(-x, a)\psi, U(-x, a)\phi)(0, 1) \\ &= \langle SU(-x, a)\psi, U(-x, a)\phi \rangle_{L^2(\mathbb{R}_+)} \\ &= Q_S(\psi, \phi)(x, a). \end{aligned}$$
□

At this point we have seen that operators S define a quantization procedure $\Gamma_S(f) = f \star_{\mathrm{Aff}} S$ as in Subsection E.6.2, and an affine Cohen class distribution Q_S. The connection between these concepts is provided by the next proposition.

Proposition E.6.12. Let S be a positive, compact operator on $L^2(\mathbb{R}_+)$ and consider a positive function $f \in L^1_r(\mathrm{Aff})$. Then $f \star_{\mathrm{Aff}} S$ is a positive, compact operator. Denote by $\{\lambda_n\}_{n=1}^\infty$ its eigenvalues in non-increasing order with associated orthogonal eigenvectors $\{\phi_n\}_{n=1}^\infty$. Then

$$\lambda_n = \max_{\|\psi\|=1} \left\{ \int_{\mathrm{Aff}} f(x,a) Q_S(\psi,\psi)(x,a) \, \frac{dx\,da}{a} : \psi \perp \phi_k \text{ for } k = 1,\dots,n-1 \right\}.$$

Proof. The integral defining $f \star_{\mathrm{Aff}} S$ is a Bochner integral of compact operators converging in the operator norm, hence it defines a compact operator. It is straightforward to check that $f \star_{\mathrm{Aff}} S$ is also a positive operator. Furthermore, for $\psi \in L^2(\mathbb{R}_+)$ we have

$$\begin{aligned}\langle f \star_{\mathrm{Aff}} S\psi, \psi \rangle_{L^2(\mathbb{R}_+)} &= \int_{\mathrm{Aff}} f(x,a) \langle SU(-x,a)\psi, U(-x,a)\psi \rangle_{L^2(\mathbb{R}_+)} \, \frac{dx\,da}{a} \\ &= \int_{\mathrm{Aff}} f(x,a) Q_S(\psi,\psi)(x,a) \, \frac{dx\,da}{a}.\end{aligned}$$

The result thus follows from Courant's minimax theorem [119, Theorem 28.4]. □

Example E.6.13. Consider a localization operator $\chi_\Omega \star_{\mathrm{Aff}} \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}_+)$ and a compact subset $\Omega \subset \mathrm{Aff}$. The first eigenfunction ϕ_0 of this operator maximizes the quantity

$$\langle \chi_\Omega \star_{\mathrm{Aff}} (\varphi \otimes \varphi)\phi_0, \phi_0 \rangle_{L^2(\mathbb{R}_+)} = \int_\Omega |\langle \varphi_0, U(-x,a)^* \varphi \rangle_{L^2(\mathbb{R}_+)}|^2 \, \frac{dx\,da}{a}.$$

Hence in this sense, the eigenfunctions are the best localized functions in Ω, which explains the terminology of localization operators.

Relation to the Affine Quadratic Time-Frequency Representations

The signal processing literature contains a wealth of two-dimensional representations of signals. Among them we find the *affine class of quadratic time-frequency representations*, see [136]. A member of the affine class of quadratic time-frequency representations is a map sending functions ψ on $\mathbb{R}$ to a function $Q^A_\Phi(\psi)$ on $\mathbb{R}^2$ given by

$$Q^A_\Phi(\psi)(x,a) = \frac{1}{a} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(t/a, s/a) e^{2\pi i x(t-s)} \psi(t) \overline{\psi(s)} \, dt \, ds$$

for some kernel function Φ on $\mathbb{R}^2$.

There are clearly a few differences between our setup and the affine class of quadratic time-frequency representations. The domain of the affine class consists of functions on $\mathbb{R}$, whereas the affine Cohen class acts on functions on $\mathbb{R}_+$. For a function ψ on $\mathbb{R}_+$ we therefore define

$$\psi_0(t) = \begin{cases} \psi(t) & t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we recall that a function K_S defined on $\mathbb{R}_+ \times \mathbb{R}_+$ defines an integral operator S with respect to the measure $\frac{dt}{t}$ by

$$S\psi(s) = \int_0^\infty K_S(s,t)\psi(t)\, \frac{dt}{t}.$$

The following formal result is straightforward to verify.

Proposition E.6.14. Let S be an integral operator with kernel K_S and define

$$\Phi_S(s,t) := \begin{cases} \frac{K_S(t,s)}{\sqrt{st}} & \text{if } s,t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $x > 0$ and ψ defined on $\mathbb{R}_+$ we have

$$Q_S(\mathcal{D}\psi, \mathcal{D}\psi)(x,a) = Q^A_{\Phi_S}(\psi_0)(-x/a, a).$$

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