Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Natural Sciences
Department of Physics

Rasmus Vikhamar-Sandberg

The Hubble tension and electromagnetic waves in curved spacetime

Master's thesis in Master of Science in Physics
Supervisor: Jon Andreas Støvneng
Co-supervisor: Iver H. Brevik
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Abstract

Measurement of today's value of the Hubble parameter H_0 by using type 1a supernovae as standard candles give $H_0 = 73.7 \pm 1.4\,\text{km/s/Mpc}$, while measurement using the CMB as a standard ruler give $H_0 = 67.4 \pm 0.5\,\text{km/s/Mpc}$. This is a problem. The measurements from type 1a supernovae use the luminosity light intensity relation (LLR) (1.1), whose derivation is based on premises that do not directly seem to relate to electromagnetism. The aim of this master thesis is to see if Maxwell's equations in curved spacetime imply a different relation between luminosity, observed light intensity, redshift and comoving distance of a light source than the LLR. The advanced and retarded Green's function that solve the Lorentz gauge conditional Maxwell's equations (2.13) for the four-potential A are constructed. But a problem arises. It is not clear if four-potentials produced by the two Green's functions actually fulfill the Lorentz gauge condition. Two sufficient conditions (6.5) for fulfillment of the Lorentz gauge condition are given. But if they hold is not checked. The result is therefore inconclusive. Assuming the Lorentz gauge condition is fulfilled, the conclusion is that Maxwell's equations in curved spacetime imply a relation between luminosity, light intensity, redshift and comoving distance identical to the LLR.

Figure 1: 7. December 185 AD saw Chinese astronomers a type 1a supernova on the sky. The four space telescopes Chandra, XMM-Newton, Spitzer, and WISE took infrared and X-ray photographies of its remnant to compile it into the image shown above.

Sammendrag

Måling av dagens verdi av Hubble-parameteren H_0 med type 1a supernovaer som standard candles gir $H_0 = 73.7 \pm 1.4\,\mathrm{km/s/Mpc}$, mens måling ved hjelp av CMB som standard ruler gir $H_0 = 67.4 \pm 0.5\,\mathrm{km/s/Mpc}$. Dette er et problem. Målinger fra type 1a supernovaer bruker en luminositetlysintensitetsrelasjon (LLR) (1.1) som er basert på premisser som ikke direkte ser ut til å ha noe med elektromagnetisme å gjøre. Målet med denne masteroppgava er å se om Maxwells likninger i krum romtid gir en annen sammenheng mellom luminositet, observert lysintensitet, rødforskyving og comoving avstand til en lyskilde enn LLR. Den avanserte og retarderte greensfunksjonen som løser lorentzgaugebetinga Maxwells likninger (2.13) for firepotensialet A. Men det oppstår et problem. Det er ikke klart om fire-potensialer produsert av de to greensfunksjonene faktisk oppfyller lorentzgaugebetingelsen. To tilstrekkelige krav (6.5) for tilfredsstilling av lorentzgaugebetingelsen gis. Men om de stemmer sjekkes ikke. Resultatet er derfor ikke entydig. Forutsatt at lorentzgaugebetingelsen er oppfylt er konklusjonen at Maxwells likninger i krum romtid gir en relasjon mellom luminositet, lysintensitet, rødforskyving og comoving avstand identisk lik LLR.

Preface

I chose to write a master thesis in cosmology not because I am particularly interested in cosmology, but because it involves general relativity and other kinds of physics that I like. I was first considering writing a thesis about viscous cosmology in which my supervisor Iver H. Brevik specializes. Brevik mentioned that there is a possibility that including viscosity in cosmology can resolve the Hubble tension. But I remembered one formula from a cosmology course I took in Switzerland that have a derivation I was unsatisfied with. "Maybe that formula is wrong and therefore causes the Hubble tension" I thought. I decided to use Maxwell's equations in curved spacetime to derive my own version of it. That turned out to be hard, until I stumbled upon a book called "The wave equation on a curved space-time", and everything got much easier.

This entire process has teached me more than what I have learned during the 3 years bachelor program at NTNU. That doesn't mean I have not learned much during the bachelor, but the bachelor program has given me foundation to learn new things faster. I have learned that fiber bundles and vector bundles are genius mathematical constructions to describe physics in coordinate independent ways. I have learned that distribution theory on pseudo-riemannian manifolds is the right language to use when working with partial differential equations. More specific to the master thesis have I learned that gravity can theoretically scatter light in addition to bending it. I have also learned that non-equilibrium thermodynamics is an incomplete theory that needs more research. I tried to apply it when I considered writing about viscous cosmology.

I will thank my mother Gunnhild Vikhamar and my brother Oskar Vikhamar-Sandberg for valuable feedback on the thesis. I will also thank my friends Gabriele Piattoli and Jostein Brunvær Steffensen for moral support and some help in the end. Finally, I will thank my supervisor Iver H. Brevik for his patiency and openness to let me explore unorthodox methods.

Acronyms

IVP initial value problem (22, 23, 50)

LLR luminosity light intensity relation (i, ii, vii, 2, 3, 53, 70–72)

ODE ordinary differential equation (34)

PDE partial differential equation (iii, 9)

wrt. with respect to (v, 11, 17, 19, 21, 42, 43, 50, 58, 77, 82, 83)

Notation and conventions

Throughout this thesis, let $(\mathcal{M}, g)$ be a $(n+1)$-dimensional orientable and time oriented lorentzian manifold of signature $(-+++)$, also called spacetime. It represents the universe.

Notation

$\forall$ reads "for all". $\exists$ reads "there is".

$\langle , \rangle$ is used for application of the metric tensor g on tangent vectors or cotangent vectors or for cotangent vectors applied on tangent vectors. A short hand notation $\langle X \rangle^2 \equiv \langle X, X \rangle$ is used.

Einstein's summation convention is used in component notation of tensors and other arrays, if not stated otherwise. Greek indices are spacetime indices and take values $\{0, 1, 2, 3\}$ while Latin indices are space indices and take values $\{1, 2, 3\}$.

Abstract index notation is used for tensors and tensor fields as an abstract coordinate independent notation for contraction between tensors. Abstract index notation should not be confused with index notation for components of tensors even though it is the exact same notation. Repeating indices, where one is lower and the other is upper, are contracted. For example, $A^{\mu\nu} B_{\alpha}{}^{\beta}{}_{\nu}$ is a tensor contraction over the index ν. Big Greek letters are used for multispacetime indices. For example $A_M \equiv A_{\mu_1, \dots, \mu_r}$, where $M \equiv (\mu_1, \dots, \mu_r)$.

A vertical line $|$ is used in a self-made abstract index notation for bitensor fields and distributions. A bitensor field is written as for example, $A^{\mu}{}_{\nu}|_{\alpha\beta}{}^{\gamma}(p, q)$. The spacetime indices to the left of $|$ belong to the left spacetime argument p and the spacetime indices to the right belong to argument q.

$\omega \in \Omega^{n+1}(\mathcal{M})$ denotes the volume form induced by the metric g. In any ordered basis $(e_0, \dots, e_n)$ in $T_p\mathcal{M}$ at a point $p \in \mathcal{M}$ $\quad \omega\big|_p = \pm\sqrt{-\det g} e^0 \wedge \cdots \wedge e^n$, where $\det g$ is the determinant of the matrix $g_{\mu\nu} \equiv \langle e_\mu, e_\nu \rangle$ and $(e^0, \dots, e^n)$ is the dual basis of $(e_0, \dots, e_n)$. The sign in front of $\sqrt{-\det g}$ depends on choice of orientation of $\mathcal{M}$ and the orientation of $(e_0, \dots, e_3)$ which doesn't matter as far as one is consistent.

d in front of a differential form denotes exterior derivative. For example $d\alpha \in \Omega^{k+1}(\mathcal{M})$ is the exterior derivative of $\alpha \in \Omega^k(\mathcal{M})$.

$\lrcorner$ is used for inner product or contraction between a tangent vector $X \in T_p\mathcal{M}$ and a k-form $\alpha \in \bigwedge^k T_p\mathcal{M}$ written like so: $X \lrcorner \alpha$.

R denotes the Riemann or Ricci tensor depending on number of indices.

∇ denotes the covariant derivative from the Levi-Civita connection.

$\Box \equiv \nabla^\mu \nabla_\mu$ is the notation for the d'Alembert operator.

$\Gamma(E)$, where $E \longrightarrow \mathcal{M}$ is a fiber bundle, denotes smooth sections of $E \longrightarrow \mathcal{M}$.

$\mathrm{Reg}(f)$ denotes the regular values of a smooth function $f \in C^\infty(\mathcal{M})$.

$\mathcal{T}_{(r,s)}(\mathcal{M}) \equiv \Gamma\left(T\mathcal{M}^{\otimes r} \otimes T^*\mathcal{M}^{\otimes s}\right)$ is short hand notation for smooth type (r,s) tensor fields.

$\mathcal{D}_{(r,s)}(\mathcal{M}) \equiv \Gamma_c\left(T\mathcal{M}^{\otimes s} \otimes T^*\mathcal{M}^{\otimes r}\right)$ denotes the space of type (r,s) test functions on spacetime $\mathcal{M}$. They are smooth type (s,r) tensor fields with compact support.

$\mathcal{D}'_{(r,s)}(\mathcal{M})$ denotes the space of type (r,s) tensor distributions on spacetime $\mathcal{M}$. It is defined as the algebraic dual of $\mathcal{D}_{(r,s)}(\mathcal{M})$.

If $u \in \mathcal{D}'_{(r,s)}(\mathcal{M})$, application of u on a test function $\phi \in \mathcal{D}_{(r,s)}(\mathcal{M})$ is written as $\langle u, \phi\rangle$, $\langle u(p), \phi(p)\rangle$, $\left\langle u^M{}_N, \phi^N{}_M\right\rangle$ or $\left\langle u^M{}_N(p), \phi^N{}_M(p)\right\rangle$. The point p is not a evaluation point. It is just an abstract symbol showing what is contracted against what. The indices M and N are also just abstract indices demonstrating contraction.

$\boxtimes$ is used for outer tensor product of two vector bundles $E \longrightarrow \mathcal{M}$ and $F \longrightarrow \mathcal{N}$ written $E \boxtimes F \longrightarrow \mathcal{M} \times \mathcal{N}$.

$[v]$, where v is a vector in a vector space, denotes the column representation of v wrt. some specified basis. Additionally, a sub- or superscript can be given to specify the basis. For example $[v]_{\mathcal{B}}$, where $\mathcal{B}$ is the specified basis.

$\hookrightarrow$ is used for inclusions and injections.

A° denotes the interior and $\overline{A}$ denotes the closure of a subset A of a topological space.

$A^B \equiv \prod_{b \in B} A$, when A and B are sets.

τ_X, where X is a topological space, denotes the topology of X.

Conventions

Speed of light $c = 1$ by letting length and time have same dimension.

The vacuum permeability $\mu_0 = 1$ by in addition letting voltage and electric current have same dimension.

When "submanifold" is mentioned, embedded submanifold is meant, unless stated otherwise.

Contents

Chapter 1

Introduction

There is a crisis in cosmology. We know our universe is expanding. We also know its expansion is accelerating [3]. However, when we try to measure how fast the universe expands with different methods, we get different results. And it doesn't stop there. The measurements come equipped with error bars, and the error bars do not even overlap! This is called "The crisis in cosmology". This is important because it tells us that something within field of cosmology is wrong.

Cosmology is limited by what we can measure. We can measure the electromagnetic radiation seen on the night sky. More specifically, we can measure quantities such as angles between light sources, light intensity, frequency of light, polarization of light and period of pulsating sources. From this there are three main quantities we use in cosmology: light intensity I, angle θ, redshift z and pulsation period T. What we can't measure are cosmic time and distances. Objects are simply too far away.

In order to calculate time and distances, astrophysicists have come up with two main methods, "the cosmic distance ladder" and "the reverse cosmic distance ladder". In short, the cosmic distance ladder uses several tools suitable for different length scales to calculate distances based on the observables (I, θ, z, T). The length scales are ordered from shortest to longest. The first tool is based on simple geometry. The second tool needs to be calibrated by the first tool, the third by the second and so on. The last tool measures the light from type 1a supernovae, which is a type of standard candle. The reverse distance ladder, however, starts with far distances and works towards us. One starts by looking at the angular power spectrum of the CMB and compares it to the distribution of galaxies we can see today. A big difference between the reverse distance ladder and the distance ladder is that the reverse distance ladder needs a cosmological model to evolve the universe's expansion history from far back in time till now. It is "model-dependent".

With each of these two methods to compute lengths it is possible to compute today's expansion rate H_0, called the Hubble parameter. The crisis in cosmology is essentially that cosmic distance ladder gives a value of $H_0 = 73.7 \pm 1.4\,\mathrm{km/s/Mpc}$ [4], while the reverse cosmic distance ladder gives $H_0 = 67.4 \pm 0.5\,\mathrm{km/s/Mpc}$ [2]. This means something must be wrong. There are two main possibilities. Either the measurements are wrong or the theory is wrong. If the measurements are wrong, it can either be that there are bigger errors than estimated or that we measure something else than we think. If the theory is wrong, there are many possibilities. Maybe the standard candles and standard rulers in the cosmic distance ladder aren't as standard as we think. Maybe the ΛCDM model is not good enough for modelling the expansion history of the universe. Maybe the cosmological principle is a bad approximation.

One of the distance measuring tools in the cosmic distance ladder are standard candles. A standard candle is a light source whose luminosity L is known. One can compute the distance to a standard candle by measuring the light intensity I and redshift z from it, since there is a relation between L, I, z and the comoving distance χ.

$$I = \frac{L}{4\pi(1+z)^2 a_0^2 \chi^2} \tag{1.1}$$

This equation is referred to as the luminosity light intensity relation (LLR) in this master thesis. A derivation of this relation goes as follows:

Let there be a comoving light source. It has constant comoving distance χ to Earth, since Earth is also assumed to be roughly comoving. The light source emits light at cosmic time t_1 that hits Earth at cosmic time t_0. A photon has a four-momentum $P(\lambda) \in T_{\gamma(\lambda)}M$ which is parallel transported along its trajectory γ which is a lightlike geodesic from the source to the the Earth. In normalized comoving coordinate basis $P(\lambda)$ is $(\hbar k^0(\lambda), \hbar \mathbf{k}(\lambda))$ and $k^0 = |\mathbf{k}|$. k^0 is the photon's frequency and $\mathbf{k}$ is its wave vector. Parallel transport in the flat FRW metric implies that $\frac{d}{d\lambda} a\left(t\left(\gamma(\lambda)\right)\right) k^0(\lambda) = 0$, so $a(t_1)k_1^0 = a(t_0)k_0^0$, where k_1^0 and k_0^0 are the emitted and observed frequency. Let $n(t,k)$ be the number of photons with four-momentum k ever emitted by the source until cosmic time t so that the total number of photons ever emitted is $n(t) = \int_{\mathbb{R}^3} \dot{n}(t,k) \frac{d^3k}{|\vec{k}|}$. Then $L = \int_{\mathbb{R}^3} \hbar k^0 \dot{n}(t_1, k) \frac{d^3k}{|\vec{k}|}$. The function $t_1(t_0)$ is the solution to $\chi = \int_{t_1}^{t_0} \frac{dt}{a(t)}$. Since χ is constant, differentiating by t_0 gives $\frac{dt_1}{dt_0} = \frac{a_1}{a_0}$. Set

$$S_{t_0} \equiv \left\{ (t_0, \boldsymbol{\chi}) \in M \;\middle|\; |\boldsymbol{\chi}| = \int_{t_1}^{t_0} \frac{dt}{a(t)} \right\}$$

to be the comoving sphere centered at the light source and tangenting the Earth. From the metric one can compute its area to be $4\pi a_0^2 \chi^2$. If we assume the light source has spherical symmetry in its rest frame, then

$$I(t_0) = \frac{\frac{d}{dt_0}\int_{\mathbb{R}^3} \hbar \frac{a_1}{a_0} k^0 n(t_1(t_0), k)\frac{d^3k}{|\vec{k}|}}{4\pi a_0^2 \chi^2}$$
$$= \frac{L + \left(H_1 - \frac{a_0}{a_1} H_0\right) E}{4\pi \frac{a_0^2}{a_1^2} a_0^2 \chi^2},$$

where $E = \int_{\mathbb{R}^3} \hbar k^0 n(t_1, k)\frac{d^3k}{|\vec{k}|}$ is the total amount of energy ever emitted by the light source until time t_1. Since the lifetime of a light source is much less than H_1^{-1} and H_0^{-1}, $I \approx \frac{L}{4\pi \frac{a_0^2}{a_1^2} a_0^2 \chi^2} = \frac{L}{4\pi (1+z)^2 a_0^2 \chi^2}$.

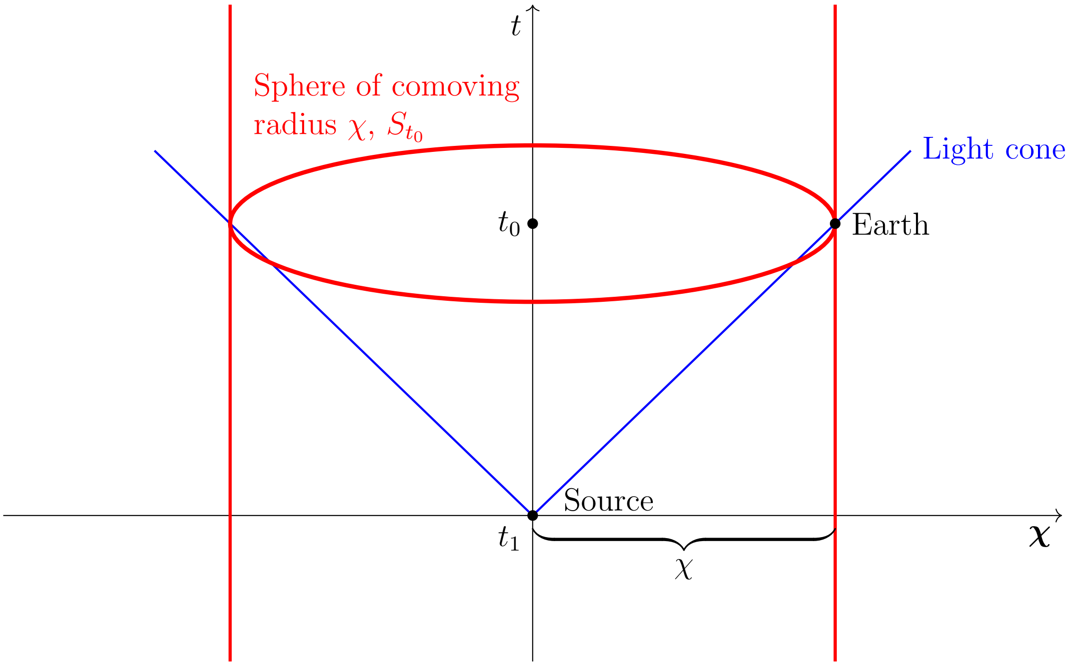

Figure 1.1: Visualization of light radiating from a source and hitting Earth at a comoving spere around the source.

This derivation is based on Sean Carroll's notes [1] p. 230.

The aim of this thesis is to test the following hypothesis:

Maxwell's equations in curved spacetime imply a different relationship among I, z, χ and L than the luminosity light intensity relation (1.1).

If this hypothesis is true, then this different formula for standard candles produces a different value for H_0, and maybe match the value produced by the reverse cosmic distance ladder.

In the following chapter, classical electromagnetism is briefly presented to set conventions. That is because several well-known equations in electromagnetism depend on choice of metric signature. Then a presentation of distribution theory on pseudo-riemannian manifolds follows in chapter 3. Chapter 4 applies that theory to construct an advanced and retarded Green's function to solve Maxwell's equations. The retarded Green's function is used in chapter 5 to derive the main result of the thesis. In the result/discussion chapter some possible shortcomings of this study are discussed. Suggestions for further study are raised in the conclusion chapter.

Chapter 2

Maxwell's equations in curved spacetime

2.1 The electromagnetic field

The electromagnetic field (also called electromagnetic tensor) is a two-form $F \in \Gamma\left(T^*\mathcal{M} \wedge T^*\mathcal{M}\right)$. It is defined experimentally by the Lorentz force equation

$$f^\mu = F^\mu{}_\nu j^\nu, \tag{2.1}$$

where $f \in T\mathcal{M}$ is the four-force density acting at a point on a piece charge , and $j \in T\mathcal{M}$ is the charge's four-current density (often just called four-current) at that point. To determine F experimentally, one brings in a small test charge to the system in question and measures the force acting on it. Let $p \in \mathcal{M}$ be the position of an observer with four-velocity $u \in T_p\mathcal{M}$ normalized by $\langle u \rangle^2 = -1$. Then charge density, current density, electric field and magnetic field relative to the observer u is

$$\rho = -u^\mu j_\mu \tag{2.2}$$
$$\mathbf{j} = \left(g^{\mu\nu} + u^\mu u^\nu\right) j_\mu e_\nu \in T_p\mathcal{M} \tag{2.3}$$
$$\mathbf{E} = u_\nu F^{\nu\mu} e_\mu \in T_p\mathcal{M} \tag{2.4}$$
$$\mathbf{B} = \frac{1}{2}\omega^\mu{}_{\nu\alpha\beta} u^\nu F^{\alpha\beta} e_\mu \in T_p\mathcal{M}, \tag{2.5}$$

where $(e_0, \ldots, e_3)$ is any ordered basis in $T_p\mathcal{M}$.

2.2 Maxwell's equations and the four-potential A

Maxwell's equations read

Gauss-Faraday law	$dF = 0$	(2.6)
Gauss-Ampére law	$\nabla_\mu F^{\nu\mu} = j^\nu,$	(2.7)

where $j \in \Gamma(T\mathcal{M})$ is the four-current density. Because $[\nabla_\nu, \nabla_\mu]F^{\mu\nu} = R^\mu{}_{\alpha\nu\mu}F^{\alpha\nu} + R^\mu{}_{\beta\nu\mu}F^{\mu\beta} = -R_{\alpha\nu}F^{\alpha\nu} + R_{\beta\mu}F^{\mu\beta} = 0 + 0 = 0$,

$$\begin{aligned}
\nabla_\nu j^\nu =& \nabla_\nu \nabla_\mu F^{\nu\mu} \\
=& \frac{1}{2}\nabla_\nu \nabla_\mu F^{\nu\mu} - \frac{1}{2}\nabla_\nu \nabla_\mu F^{\mu\nu} \\
=& \frac{1}{2}\left[\nabla_\nu, \nabla_\mu\right] F^{\nu\mu} \\
=& 0.
\end{aligned}$$

We get that the Gauss-Ampére law implies conservation of charge.

$$\nabla_\mu j^\mu = 0. \tag{2.8}$$

Lemma 1 (Poincaré lemma)
Let $\mathcal{N}$ be a contractible manifold and $\alpha \in \Omega^k(\mathcal{N})$ and $d\alpha = 0$. Them $\exists \beta \in \Omega^{k-1}(\mathcal{N})$ so that

$$\alpha = d\beta.$$

This means that on a contractible open subset $U \subseteq \mathcal{M}$ of spacetime there is a covector field $A \in \Gamma(T^*U)$ such that

$$F = dA. \tag{2.9}$$

A is called a four-potential of F. Note that A is not unique, since $A' \equiv A + df$ is also a four-potential of F for all smooth functions $f \in C^\infty(\mathcal{M})$. Choice of A is called choice of gauge. To relate the four-potentials to more familiar electric and magnetic potential V and $\mathbf{A}$ relative to an observer $u \in T_p\mathcal{M}$ we have

$$V = -u^\mu A_\mu \tag{2.10}$$

$$\mathbf{A} = (g^{\mu\nu} + u^\mu u^\nu) A_\nu e_\nu \in T_p\mathcal{M}, \tag{2.11}$$

so in an orthonormal basis (u, e_1, e_2, e_3),

$$A^\mu = (V, \mathbf{A}). \tag{2.12}$$

Write the Gauss-Ampére equation in terms four-potential.

$$\begin{aligned} j^\nu &= -\nabla_\mu F^{\mu\nu} \\ &= -\nabla_\mu \left(\nabla^\mu A^\nu - \nabla^\nu A^\mu\right) \\ &= -\Box A^\nu + \left([\nabla_\mu, \nabla^\nu] + \nabla^\nu \nabla_\mu\right) A^\mu \\ &= -\Box A^\nu + \nabla^\nu \nabla_\mu A^\mu + R^\mu{}_{\alpha\mu}{}^\nu A^\alpha \\ &= -\Box A^\nu + \nabla^\nu \nabla_\mu A^\mu + R^\nu{}_\mu A^\mu. \end{aligned}$$

If we impose the Lorentz gauge condition

$$\nabla_\mu A^\mu = 0,$$

we get

Lorentz gauge conditional Gauss-Ampére law for A	$-\Box A^\nu + R^\nu{}_\mu A^\mu = j^\nu$	(2.13)
Lorentz gauge condition	$\nabla_\mu A^\mu = 0.$	(2.14)

To justify the existence of a Lorentz gauged four-potential of an electromagnetic field F, we know there exists a four-potential A of F from the Poincaré lemma. Pick an $f \in C^\infty(\mathcal{M})$ so that $\Box f = -\nabla_\mu A^\mu$. Then $A' \equiv A + df$ satisfy the Lorentz gauge condition. The existence of f can at least be guaranteed locally (Theorem 4.5.1 in [6]).

2.3 The electromagnetic energy momentum tensor

The Lorentz force equation (2.1) together with Maxwell's equations gives

$$\begin{aligned}
f^\nu &= F^\nu{}_\alpha j^\alpha && \text{Lorentz force} \\
&= -F^\nu{}_\alpha \nabla_\mu F^{\mu\alpha} && \text{Gauss-Ampére} \\
&= -\nabla_\mu \left(F^\nu{}_\alpha F^{\mu\alpha}\right) + \nabla_\mu F^\nu{}_\alpha F^{\mu\alpha} && \\
&= -\nabla_\mu \left(F^\nu{}_\alpha F^{\mu\alpha}\right) + \frac{1}{2}\left(\nabla_\mu F^\nu{}_\alpha - \nabla_\alpha F^\nu{}_\mu\right) F^{\mu\alpha} && F^{\mu\alpha} = -F^{\alpha\mu} \\
&= -\nabla_\mu \left(F^\nu{}_\alpha F^{\mu\alpha}\right) + \frac{1}{2}\left(\nabla_\mu F^\nu{}_\alpha + \nabla_\alpha F_\mu{}^\nu\right) F^{\mu\alpha} && \\
&= -\nabla_\mu \left(F^\nu{}_\alpha F^{\mu\alpha}\right) - \frac{1}{2}\nabla^\nu F_{\alpha\mu} F^{\mu\alpha} && \text{Gauss-Faraday} \\
&= -\nabla_\mu \left(F^\nu{}_\alpha F^{\mu\alpha}\right) + \frac{1}{4}\nabla^\nu \left(F_{\mu\alpha} F^{\mu\alpha}\right) && \\
&= -\nabla_\mu \left(F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}\right).
\end{aligned}$$

The energy momentum tensor $T \in \Gamma\left(T\mathcal{M} \otimes T\mathcal{M}\right)$ of the electromagnetic field is therefore defined as

$$T^{\mu\nu} \equiv F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \text{ and} \tag{2.15}$$

$$\nabla_\mu T^{\mu\nu} = -f^\nu. \tag{2.16}$$

Chapter 3

Distribution theory on pseudo-riemannian manifolds

3.1 Tensor distributions on pseudo-riemannian manifolds

Functions that are solutions to partial differential equations must have a certain degree of smoothness. However, this is sometimes too restricitve when trying to solve them. One way to work around that is to extend the function space to include some new objects that are not smooth functions. But in some sense they are possible to differentiate and do other operations on which one can do with smooth functions. Distributions are such objects. The idea of distributions is to utilize the concept of duality from linear algebra. Instead of describing a smooth tensor field $u \in \mathcal{T}_{(r,s)}(\mathcal{M})$ by evaluating it at points $p \in \mathcal{M}$, one integrates it against test functions $\phi \in \Gamma_c\left(T\mathcal{M}^{\otimes s} \otimes T^*\mathcal{M}^{\otimes r}\right)$ like so:

$$\langle u, \phi \rangle \equiv \int_{\mathcal{M}} u(q)^M{}_N \phi(q)^N{}_M \omega(q), \tag{3.1}$$

where ω is the volume form induced by the metric, and $M_r \equiv (\mu_1, \dots \mu_r)$ and $N_s \equiv (\nu_1, \dots \nu_r)$ are multispacetime indices. Define the space of test functions as the $\mathbb{R}$-vector space

$$\mathcal{D}_{(r,s)}(\mathcal{M}) \equiv \Gamma_c\left(T\mathcal{M}^{\otimes s} \otimes T^*\mathcal{M}^{\otimes r}\right). \tag{3.2}$$

Distributions are then defined as linear forms acting on test functions. For example the Dirac delta distribution δ_p with support at $p \in \mathcal{M}$ is defined by evaluating the test functions at p.

$$\langle \delta_p, \phi \rangle \equiv \phi(p) \qquad \forall \phi \in C_c^\infty(\mathcal{M}). \tag{3.3}$$

A small note about notation: sometimes it is convenient to write $\langle u(q), \phi(q) \rangle$ instead of $\langle u, \phi \rangle$. Or even write $u(q) = v(q)$. It does not mean that $u(q)$ is a function evaluated at q!

3.2 Topology and convergence

All vector spaces considered in this thesis are real. But the lemmas that follow work in the complex case also.

Lemma 2 (Universal property of subspace topology)
Let X be a topologic space and $A \xhookrightarrow{\iota} X$ a subspace (subset equipped with subspace topology). Then $\forall$ topological spaces Z and $\forall$ functions $Z \xrightarrow{f} A$

$$f \text{ is continuous} \quad \iff \quad \iota \circ f \text{ is continuous.}$$

$$\begin{tikzcd} Z \arrow[d, "f"] \arrow[dr, "\iota\circ f"] & \\ A \arrow[r, hook, "\iota"] & X \end{tikzcd}$$

Lemma 3 (Universal property of product topology)
Let $\{X_i\}_{i\in I}$ be an indexed family of topological spaces. Equip $\prod_{i\in I} X_i$ with the product topology. Then $\forall$ topological spaces Z and $\forall$ functions $Z \xrightarrow{f} \prod_{i\in I} X_i$

$$f \text{ is continuous} \quad \iff \quad \pi_i \circ f \text{ is continuous } \forall i \in I.$$

$$\begin{tikzcd} Z \arrow[d, "f"] \arrow[dr, "\pi_i\circ f"] & \\ \prod_{i\in I} X_i \arrow[r, "\pi_i"] & X_i \end{tikzcd}$$

Here, $\prod_{i\in I} X_i \xrightarrow{\pi_i} X_i$ denote the canonical projections.

Lemma 4
Let V be a vector space, and equip its algebraic dual V' with product topology. Then V' is a Hausdorff topological vector space.

Proof is on p. 77.

Lemma 5
Let V, W be two vector spaces, V' and W' their algebraic duals and $V \xrightarrow{T} W$ linear. Equip V' and

W' with product topology. Then the adjoint map

$$W' \xrightarrow{T'} V'$$
$$\phi \mapsto \langle \phi, T\cdot \rangle$$

is continuous.

Proof is on p. 78.

Def 1
A type (r,s) tensor distribution is an element of the dual of $\mathcal{D}_{(r,s)}(\mathcal{M})$ denoted $\mathcal{D}'_{(r,s)}(\mathcal{M})$. $\mathcal{D}'_{(r,s)}(\mathcal{M})$ is given the product topology.

In literature, this topology is often referred to as weak* topology. Subspace topology of product topology and weak* topology are actually the same thing. Because of lemma 5, whenever we have a linear transform $\mathcal{D}_{(k,l)} \xrightarrow{T} \mathcal{D}_{(r,s)}(\mathcal{M})$, it induces a *continuous* linear transform $\mathcal{D}'_{(r,s)} \xrightarrow{T'} \mathcal{D}'_{(k,l)}(\mathcal{M})$. Also, distributions are usually required to be continuous wrt. a certain topology on the test functions. This is not required here since it has no use in this thesis.

The purpose of distributions is to generalize smooth fields. If $u \in \mathcal{T}_{(r,s)}(\mathcal{M})$ and $\langle u, \phi \rangle = 0 \; \forall \phi \in \mathcal{D}_{(r,s)}$, then $u = 0$. That means smooth tensor fields are uniquely determined by how they act on test functions, so there is an injection

$$\mathcal{T}_{(r,s)}(\mathcal{M}) \hookrightarrow \mathcal{D}'_{(r,s)}(\mathcal{M}). \tag{3.4}$$

This is essential for distributions being a generalization because if (3.4) was not injective, distributions that are smooth in a region could have non-unique values in that region. This is important when trying to multiply distributions (definition 3.13).

One advantage of topologizing the space of distributions is *regularization*. Regularization means to find a net of smooth functions $\{f_\lambda\}_{\lambda \in \Lambda}$ that converges to a distribution $u = \lim_{\lambda \in \Lambda} f_\lambda \in \mathcal{D}'_{(r,s)}(\mathcal{M})$. Smooth functions are easier to work with than general distributions. After applying continuous operations T to the regularized distribution f_λ one takes the limit. By continuity $Tu = T\left(\lim_{\lambda \in \Lambda} f_\lambda\right) = \lim_{\lambda \in \Lambda} T f_\lambda$. Another advantage is simply that one can define new distributions as limits of other distributions. That technique is utilized when constructing Green's functions in chapter 4.

3.3 Operations on distributions

If $u \in \mathcal{T}_{(r,s)}(\mathcal{M})$ is a smooth tensor field, one can multiply it with another smooth tensor field $f \in \mathcal{T}_{(k,l)}(\mathcal{M})$. When integrated against a test function one gets

$$\begin{aligned}\langle f \otimes u, \phi\rangle &= \int_{\mathcal{M}} f^{A}{}_{B} u^{M}{}_{N} \phi^{BN}{}_{AM} \omega \\ &= \left\langle u^{A}{}_{B}, f^{M}{}_{N} \phi^{BN}{}_{AM} \right\rangle .\end{aligned}$$

Two smooth tensor fields $u, v \in \mathcal{T}_{(r,s)}(\mathcal{M})$ can be added, and when integrated against test functions one gets

$$\begin{aligned}\langle u + v, \phi\rangle &= \int_{\mathcal{M}} \left(u^{M}{}_{N} + v^{M}{}_{N}\right) \phi^{N}{}_{M} \omega \\ &= \langle u, \phi\rangle + \langle v, \phi\rangle .\end{aligned}$$

This way, those operations can be extended to distributions.

$$\langle f u, \phi\rangle \equiv \left\langle u^{A_k}{}_{B_l}, f^{M}{}_{N} \phi^{B_l N}{}_{A_k M} \right\rangle \tag{3.5}$$

$$\langle u + v, \phi\rangle \equiv \langle u, \phi\rangle + \langle v, \phi\rangle . \tag{3.6}$$

By lemma 5 the map

$$\begin{aligned}\mathcal{D}'_{(r,s)}(\mathcal{M}) &\longrightarrow \mathcal{D}'_{(r+k,s+l)}(\mathcal{M}) \\ u &\mapsto f \otimes u\end{aligned}$$

is continuous $\forall f \in \mathcal{D}_{(k,l)}(\mathcal{M})$. By lemma 4 the maps

$$\begin{aligned}\mathbb{R} \times \mathcal{D}'_{(r,s)}(\mathcal{M}) &\xrightarrow{\cdot} \mathcal{D}'_{(r,s)}(\mathcal{M}) \\ (\lambda, u) &\mapsto \lambda \cdot u\end{aligned} \qquad \text{and} \qquad \begin{aligned}\mathcal{D}'_{(r,s)}(\mathcal{M}) \times \mathcal{D}'_{(r,s)}(\mathcal{M}) &\xrightarrow{+} \mathcal{D}'_{(r,s)}(\mathcal{M}) \\ (u, v) &\mapsto u + v\end{aligned}$$

are also continuous.

Another important operation is covariant differentiation of tensor fields. If $u \in \mathcal{T}_{(r,s)}(\mathcal{M})$, $\nabla u \in \Gamma\left(T\mathcal{M}^{\otimes r} \otimes T^{*}\mathcal{M}^{\otimes (s+1)}\right)$. From the divergence theorem and compactness of $\operatorname{supp} \phi$ one gets

$$\begin{aligned}\langle \nabla u, \phi\rangle &= \int_{\mathcal{M}} \nabla_{\nu} u^{M}{}_{N} \phi^{\nu N}{}_{M} \omega \\ &= \int_{\mathcal{M}} \Big(\nabla_{\nu} \underbrace{\left(u^{M}{}_{N} \phi^{\nu N}{}_{M}\right)}_{\text{vector field}} - u^{M}{}_{N} \nabla_{\nu} \phi^{\nu N}{}_{M}\Big) \omega \\ &= -\int_{\mathcal{M}} u^{M}{}_{N} \nabla_{\nu} \phi^{\nu N}{}_{M} \omega .\end{aligned}$$

Define therefore covariant differentiation of tensor distributions by

$$\langle \nabla u, \phi \rangle \equiv - \left\langle u^M{}_N, \nabla_\nu \phi^{\nu N}{}_M \right\rangle . \tag{3.7}$$

This definition generalizes covariant differentiation of smooth tensor fields. By lemma 5 the map

$$\begin{aligned} \mathcal{D}'_{(r,s)}(\mathcal{M}) &\xrightarrow{\nabla} \mathcal{D}'_{(r,s+1)}(\mathcal{M}) \\ u &\mapsto \nabla u \end{aligned}$$

is continuous. Direct computation by application on test functions shows that Leibniz's rule holds also for product of a tensor field $f \in \mathcal{T}_{(k,l)}(\mathcal{M})$ and a tensor distribution $u \in \mathcal{D}'_{(r,s)}(\mathcal{M})$.

$$\nabla (f \otimes u) = \nabla f \otimes u + f \otimes \nabla u. \tag{3.8}$$

3.4 Extending distributions to act on bigger set of functions

Def 2 (Restriction of distributions)
Let $u \in \mathcal{D}'_{(r,s)}(\mathcal{M})$ and $U \in \tau_{\mathcal{M}}$. The restriction $u\big|_U \in \mathcal{D}'_{(r,s)}(U)$ is defined by

$$\left\langle u\big|_U, \phi \right\rangle \equiv \left\langle u(p), \begin{cases} \phi(p), & p \in U \\ 0, & p \in \mathcal{M} \setminus \operatorname{supp} \phi \end{cases} \right\rangle \qquad \forall \phi \in \mathcal{D}_{(r,s)}(U). \tag{3.9}$$

This is well-defined since $\mathcal{M} \ni p \mapsto \begin{cases} \phi(p), & p \in U \\ 0, & p \in \mathcal{M} \setminus \operatorname{supp} \phi \end{cases}$ is smooth by the gluing lemma of smooth functions, and since $\operatorname{supp} \phi$ is compact, it is a test function on $\mathcal{M}$.

Def 3 (Support)
Let $u \in \mathcal{D}'_{(r,s)}(\mathcal{M})$. The support of u is

$$\operatorname{supp} u \equiv \mathcal{M} \setminus \left\{ p \in \mathcal{M} \;\middle|\; \exists U \in \tau_{\mathcal{M}} \text{ so that } p \in U \text{ and } u\big|_U = 0 \right\}. \tag{3.10}$$

Note that support is always closed.

When a tensor distribution $u \in \mathcal{D}'_{(r,s)}(\mathcal{M})$ acts on a test function $\phi \in \mathcal{D}_{(r,s)}(\mathcal{M})$, u only cares about the values of ϕ on an arbitrarily small neighbourhood of $\operatorname{supp} u \cap \operatorname{supp} \phi$. This suggests the following definition of extension of u.

Def 4 (Extension of distributions)
Let $u \in \mathcal{D}'_{(r,s)}(\mathcal{M})$. u can be extended to a function on the set

$$\mathcal{T}_{(r,s)}(\operatorname{supp} u, \mathcal{M}) \equiv \left\{\phi \in \mathcal{T}_{(r,s)}(U_\phi) \;\middle|\; \operatorname{supp} u \subseteq U_\phi \in \tau_{\mathcal{M}} \text{ and } \operatorname{supp} \phi \cap \operatorname{supp} u \text{ is compact}\right\}$$

by defining

$$\langle u, \phi \rangle \equiv \langle u, \tilde{\phi} \rangle \qquad \forall \phi \in \mathcal{T}_{(r,s)}(\operatorname{supp} u, \mathcal{M}), \tag{3.11}$$

where $\tilde{\phi} \in \mathcal{D}_{(r,s)}(\mathcal{M})$ is any test function so that $\tilde{\phi} = \phi$ on a neighbourhood V of $\operatorname{supp} u \cap \operatorname{supp} \phi$ *and* $\operatorname{supp} \tilde{\phi} \subseteq \operatorname{supp} \phi$.

Proof for that the extension is well-defined is found on p. 78. Note that $\mathcal{T}_{(r,s)}(\operatorname{supp} \phi, \mathcal{M})$ is not a vector space. That means linearity is lost when a distribution $u \in \mathcal{D}_{(r,s)}(\mathcal{M})$ is extended to to act on $\mathcal{T}_{(r,s)}(\operatorname{supp} u, \mathcal{M})$.

3.5 Singular support and multiplication of distributions

Def 5 (Singular support)
Let $u \in \mathcal{D}'_{(r,s)}(\mathcal{M})$. The singular support of u is

$$\operatorname{singsupp} u \equiv \mathcal{M} \setminus \left\{p \in \mathcal{M} \;\middle|\; \exists \text{ open } U \text{ around } p \text{ so that } u\big|_U \in \mathcal{T}_{(r,s)}(U)\right\}. \tag{3.12}$$

Note that like support, singular support is also closed, and $\operatorname{singsupp} u \subseteq \operatorname{supp} u$. If two scalar distributions $u, v \in \mathcal{D}'(\mathcal{M})$ have disjoint singular support, it is possible to multiply them in a way that extends multiplication of smooth functions. Pick a $\rho \in C^\infty(\mathcal{M})$ so that $\rho\big|_{\operatorname{singsupp} u} = 0$ and $\rho\big|_{\operatorname{singsupp} v} = 1$. Define the product as

$$\langle u \cdot v, \phi \rangle \equiv \langle v, \rho u \phi \rangle + \langle u, (1-\rho) v \phi \rangle. \tag{3.13}$$

The proof for that this is well-defined is on p. 79. The product is associative, commutative and extending the product of smooth functions. Also for multiplication of distributions $u, v \in \mathcal{D}'(\mathcal{M})$ with disjoint singular support, Leibniz's rules holds.

$$(u \cdot v)' = u' \cdot v + u \cdot v'. \tag{3.14}$$

3.6 Composition

An operation on distributions that is much used in physics is composing a scalar distribution $u \in \mathcal{D}'(\mathbb{R}^m)$ with a smooth $S \in C^\infty(\mathcal{M}, \mathbb{R}^m)$, where $\operatorname{supp} u \subseteq \operatorname{Reg}(S)$ is assumed. Let $s \in \operatorname{Reg}(S)$. The preimage theorem says that $S^{-1}(\{s\}) \subseteq \mathcal{M}$ is an orientable submanifold of codimension m.

Theorem 1 (Leray form)
Let $\mathcal{M} \xrightarrow{S} \mathbb{R}^m$ be smooth and $s \in \mathrm{Reg}(S)$. Then $\exists! \omega_{S,s} \in \Omega^{\dim \mathcal{M} - m}\left(S^{-1}(\{s\})\right)$ so that

$$dS_q^1 \wedge \ldots dS_q^m \wedge \omega_{S,s} = \omega(q) \qquad \forall q \in S^{-1}(\{s\}). \tag{3.15}$$

$\omega_{S,s}$ is called a Leray form.

Proof can be found by using induction on lemma 2.9.2 in [6].

Theorem 2 (Composition)
Let $u \in \mathcal{D}'(\mathbb{R})$ and $S \in C^\infty(\mathcal{M})$ with $\operatorname{supp} u \subseteq \mathrm{Reg}(S)$. $\forall \phi \in \mathcal{D}(\mathcal{M})$ define

$$\begin{aligned} \mathrm{Reg}\left(S\big|_{\operatorname{supp}\phi}\right) &\xrightarrow{\phi_S} \mathbb{R} \\ s &\mapsto \int_{S\big|^{-1}_{\operatorname{supp}\phi}(\{s\})} \phi\,\omega_{S,s}, \end{aligned} \tag{3.16}$$

where $S^{-1}(\{s\})$ is given the orientation so that $\omega_{S,s} > 0$. Then $\phi_S \in \mathcal{T}_{(0,0)}(\operatorname{supp} u, \mathcal{M})$ and the composition or pullback $u \circ S$, defined by

$$\langle u \circ S, \phi \rangle \equiv \langle u, \phi_S \rangle \qquad \forall \phi \in \mathcal{D}(\mathcal{M}), \tag{3.17}$$

is a scalar distribution on $\mathcal{M}$. Furthermore, the operation generalizes composition when u is smooth, and the map

$$\begin{aligned} \{u \in \mathcal{D}'(\mathbb{R}) \mid \operatorname{supp} u \subseteq \mathrm{Reg}(S)\} &\longrightarrow \mathcal{D}'(\mathcal{M}) \\ u &\mapsto u \circ S \end{aligned}$$

is continuous.

The next theorem is a version of the chain rule for distributions.

Theorem 3
Let $u \in \mathcal{D}'(\mathbb{R})$, $S \in C^\infty(\mathcal{M})$ and $\operatorname{supp} u \subseteq \mathrm{Reg}(S)$. Then

$$d(u \circ S) = (u' \circ S) \cdot dS. \tag{3.18}$$

This is theorem 2.9.2 in [6].

3.7 Bitensor distributions

Two pseudo-riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ have natural product $(\mathcal{M} \times \mathcal{N}, g \oplus h)$ which is also a pseudo-riemannian manifold. The metric $g \oplus h$ is defined by

$$\begin{aligned}(T_p\mathcal{M} \oplus T_q\mathcal{N}) \times (T_p\mathcal{M} \oplus T_q\mathcal{N}) &\xrightarrow{g\oplus h_{(p,q)}} \mathbb{R} \\ (X_1 \oplus Y_1, X_2 \oplus Y_2) &\mapsto g(X_1, X_2) + h(Y_1, Y_2) \qquad \forall (p,q) \in \mathcal{M} \times \mathcal{N}.\end{aligned}$$

Given two vector bundles $E \longrightarrow \mathcal{M}$ and $F \longrightarrow \mathcal{N}$ one can construct the external tensor product $E \boxtimes F \longrightarrow \mathcal{M} \times \mathcal{N}$. It is a new vector bundle whose fibers are tensor products of E and F's fibers. That is, $(E \boxtimes F)_{(p,q)} = E_p \otimes F_q \quad \forall (p,q) \in \mathcal{M} \times \mathcal{N}$.

Def 6

1. *A <u>scalar bitensor field</u> is an element of* $C^\infty(\mathcal{M} \times \mathcal{M})$.
2. *A <u>type $(r,s) \boxtimes (k,l)$ bitensor field</u> is an element of*

$$\mathcal{T}_{(r,s)\boxtimes(k,l)}(\mathcal{M}) \equiv \Gamma\left(\left(T\mathcal{M}^{\otimes r} \otimes T^*\mathcal{M}^{\otimes s}\right) \boxtimes \left(T\mathcal{M}^{\otimes k} \otimes T^*\mathcal{M}^{\otimes l}\right)\right).$$

It is handy with an abstract index notation for bitensor fields when doing tensor contractions. For $A \in \mathcal{T}_{(r,s)\boxtimes(k,l)}(\mathcal{M})$ write

$$A^M{}_N|^P{}_\Sigma(p,q).$$

Here, M, N, P and Σ are multispacetime indices of length r, s, k and l, and $(p,q) \in \mathcal{M} \times \mathcal{M}$. The notation can also serve as component notation. Let's generalize smooth bitensor fields to distributions. Define first type $(r,s) \boxtimes (k,l)$ bitensor test functions as

$$\mathcal{D}_{(r,s)\boxtimes(k,l)}(\mathcal{M} \times \mathcal{M}) \equiv \Gamma_c\left(\left(T\mathcal{M}^{\otimes s} \otimes T^*\mathcal{M}^{\otimes r}\right) \boxtimes \left(T\mathcal{M}^{\otimes l} \otimes T^*\mathcal{M}^{\otimes k}\right)\right). \tag{3.19}$$

Def 7
A <u>type $(r,s) \boxtimes (k,l)$ bitensor distribution</u> is an element of the dual of $\mathcal{D}_{(r,s)\boxtimes(k,l)}(\mathcal{M} \times \mathcal{M})$*, denoted*

$$\mathcal{D}'_{(r,s)\boxtimes(k,l)}(\mathcal{M} \times \mathcal{M}).$$

$\mathcal{D}'_{(r,s)\boxtimes(k,l)}(\mathcal{M} \times \mathcal{M})$ *is equipped with the weak* topology.*

Likewise as for tensor distributions, there is an inclusion of smooth bitensor fields,

$$\mathcal{T}_{(r,s)\boxtimes(k,l)}(\mathcal{M}) \hookrightarrow \mathcal{D}'_{(r,s)\boxtimes(k,l)}(\mathcal{M}). \tag{3.20}$$

A bitensor distribution $u \in \mathcal{D}'_{(r_1,s_1)\boxtimes(k_1,l_1)}(\mathcal{M})$ can be multiplied by a bitensor field $f \in \mathcal{T}_{(r_2,s_2)\boxtimes(k_2,l_2)}(\mathcal{M})$ like so:

$$\langle f \otimes u, \phi \rangle \equiv \big\langle u^A{}_B|^M{}_N, \quad f^P{}_\Sigma|^H{}_\Theta \cdot \phi^{\Sigma B}{}_{PA}|^{\Theta N}{}_{HM} \big\rangle \\ \forall \phi \in \mathcal{D}_{(r_1+r_2,s_1+s_2)\boxtimes(k_1+k_2,l_1+l_2)}(\mathcal{M}). \tag{3.21}$$

This generalizes standard fiberwise tensor product of smooth bitensor fields. Bitensor distributions can be differentiated covariantly the same way as tensor distributions, but now there are two derivatives.

$$\mathcal{D}'_{(r,s)\boxtimes(k,l)}(\mathcal{M}) \xrightarrow{\overset{\text{left}}{\nabla}} \mathcal{D}'_{(r,s+1)\boxtimes(k,l)}(\mathcal{M})$$

and

$$\mathcal{D}'_{(r,s)\boxtimes(k,l)}(\mathcal{M}) \xrightarrow{\overset{\text{right}}{\nabla}} \mathcal{D}'_{(r,s)\boxtimes(k,l+1)}(\mathcal{M}).$$

They are defined by

$$\left\langle \overset{\text{left}}{\nabla} u, \phi \right\rangle \equiv - \left\langle u^M{}_N|^B{}_A, \quad \overset{\text{left}}{\nabla}_\nu \phi^{\nu N}{}_M|^A{}_B \right\rangle \tag{3.22}$$

and

$$\left\langle \overset{\text{right}}{\nabla} u, \phi \right\rangle \equiv - \left\langle u^M{}_N|^B{}_A, \quad \overset{\text{right}}{\nabla}_\alpha \phi^N{}_M|^{\alpha A}{}_B \right\rangle. \tag{3.23}$$

Let me suggest a more compact abstract index notation of left and right covariant differentiation of bitensor distributions. Write $\overset{\text{left}}{\nabla} u \in \mathcal{D}'_{(r,s+1)\boxtimes(k,l)}(\mathcal{M})$ and $\overset{\text{right}}{\nabla} u \in \mathcal{D}'_{(r,s+1)\boxtimes(k,l)}(\mathcal{M})$ as

$$u^M{}_{N;\nu}|^A{}_B \qquad \text{and} \qquad u^M{}_N|^A{}_{B;\beta}. \tag{3.24}$$

Again, by lemma 5 multiplication by smooth bitensor field and left and right covariant differentiation are continuous operations $\mathcal{D}'_{(r,s)\boxtimes(k,l)}(\mathcal{M}) \longrightarrow \mathcal{D}'_{(a,b)\boxtimes(c,d)}(\mathcal{M})$ wrt. the weak* topology. Leibniz's rule still applies, so $\forall u \in \mathcal{D}_{(r_1,s_1)\boxtimes(k_1,l_1)}(\mathcal{M})$ and $\forall v \in \mathcal{D}_{(r_2,s_2)\boxtimes(k_2,l_2)}(\mathcal{M})$

$$\overset{\text{left}}{\nabla}(u \otimes v) = \overset{\text{left}}{\nabla} u \otimes v + u \otimes \overset{\text{left}}{\nabla} v \qquad \text{and} \qquad \overset{\text{right}}{\nabla}(u \otimes v) = \overset{\text{right}}{\nabla} u \otimes v + u \otimes \overset{\text{right}}{\nabla} v. \tag{3.25}$$

The same goes for the chain rule. $\forall S \in C^\infty(\mathcal{M} \times \mathcal{M})$ and $\forall u \in \mathcal{D}'(\mathbb{R})$ where $\operatorname{supp} u \subseteq \operatorname{Reg}(S)$

$$\overset{\text{left}}{d}(u \circ S) = (u' \circ S) \cdot \overset{\text{left}}{d} S \qquad \text{and} \qquad \overset{\text{right}}{d}(u \circ S) = (u' \circ S) \cdot \overset{\text{right}}{d} S. \tag{3.26}$$

Chapter 4

Green's functions to solve Maxwell's equations

In this chapter, two Green's functions are developed to solve the Lorentz gauge conditional Gauss-Ampére equation (2.13) for the four-potential A in four-dimensional spacetime. Define the wave operator

$$\Gamma(T\mathcal{M}) \xrightarrow{\mathcal{L} \equiv -\Box + R} \Gamma(T\mathcal{M}) \tag{4.1}$$

$$A^\mu \mapsto -\Box A^\mu + R^\mu{}_\nu A^\nu. \tag{4.2}$$

Then (2.13) can be written as

$$\boxed{\mathcal{L}A = j.} \tag{4.3}$$

Note that the operator $\mathcal{L}$ is self-adjoint. If u is a smooth vector field and ϕ a vector test function,

$$\langle \mathcal{L}u, \phi \rangle = \langle u, \mathcal{L}\phi \rangle. \tag{4.4}$$

This is useful later. To develope the two Green's functions, three bitensor fields need to be defined.

4.1 Synge's world function

Imagine two points p, q in spacetime $\mathcal{M}$. We want a coordinate invariant way to measure the distance between the points. If there is a unique geodesic between the points, one can take the *square geodesic* *distance* defined by $\langle X\rangle^2$ where $p = \exp_q(X)$. Note that $X \in T_q\mathcal{M}$ is unique from the uniqueness of the geodesic, and one would get the same if one chose the $Y \in T_p\mathcal{M}$ so that $q = \exp_p(Y)$. However, geodesics between points need neither to exist nor to be unique. Therefore, we must restrict the region we are looking at.

Def 8
An open set $\Omega \subseteq \mathcal{M}$ is called geodesically convex if $\forall p, q \in \Omega\ \exists!$ geodesic between p and q.

It can be proven that all points of $\mathcal{M}$ has a geodesically convex neighbourhood. Define the world function on a geodesically convex domain $\Omega \subseteq \mathcal{M}$ as follows

The world function
$$\begin{aligned} \Omega \times \Omega &\xrightarrow{\sigma} \mathbb{R} \\ (p, q) &\mapsto \frac{1}{2}\langle X\rangle^2, \qquad \text{where } p = \exp_q(X). \end{aligned} \tag{4.5}$$

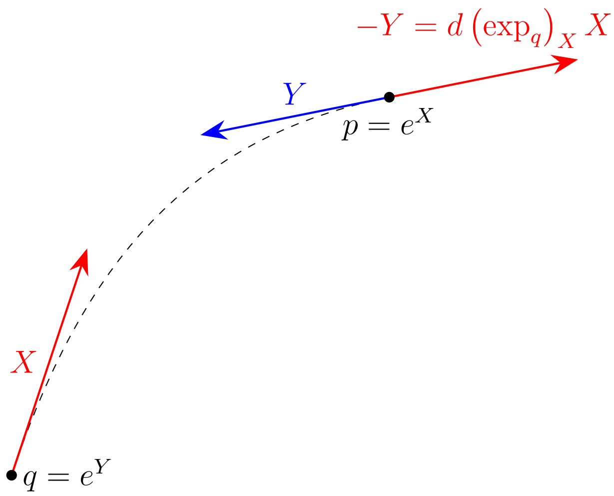

Figure 4.1: The world function can be computed from half of the square length of $X = \exp_q^{-1}(p)$ or from half of the square length of $Y = \exp_p^{-1}(q)$. It doesn't matter which one chooses, and the world function is hence symmetric.

The world function has some properties that are useful later in this chapter. It is smooth separately in each argument, because the exponential map acts diffeomorphically on tangent vectors. The inverse exponential map

$$\begin{aligned} \Omega \times \Omega &\xrightarrow{\exp^{-1}} T\Omega \\ (q, p) &\mapsto \exp_q^{-1}(p) \end{aligned} \tag{4.6}$$

is assumed continuous wrt. the product topology on $\Omega \times \Omega$. Consequently, the world function also becomes continuous wrt. the product topology. Now, keep $q \in \Omega$ constant, choose normal coordinates $(X^0, \ldots, X^3)$ around q and differentiate wrt. p.

$$\begin{aligned}
\overset{\text{left}}{\partial}_\mu \sigma(p,q) &= \nabla_\mu \left(\frac{1}{2} X_\nu X^\nu \right) \\
&= X_\mu \\
&= -\left(\exp_p^{-1}(q)\right)_\mu .
\end{aligned}$$

Then contract it with itself.

$$\begin{aligned}
\overset{\text{left}}{\partial}{}^\mu \sigma(p,q) \overset{\text{left}}{\partial}_\mu \sigma(p,q) &= X^\mu X_\mu \\
&= 2\sigma(p,q).
\end{aligned}$$

Now compute the d'Alembert of σ in the normal coordinates around q.

$$\begin{aligned}
\nabla_\mu \overset{\text{left}}{\partial}{}^\mu \sigma &= \nabla_\mu X^\mu \\
&= \partial_\mu X^\mu + \Gamma^{\alpha\mu}{}_\mu X^\alpha \\
&= \delta^\mu_\mu + \frac{1}{2} X^\alpha \frac{\partial_\alpha \det g}{\det g} \\
&= (n+1) + \frac{1}{2} \frac{\partial_{-Y} \det g}{\det g}.
\end{aligned}$$

We get four properties of σ that are used later.

i)	$\sigma(p,q) = \sigma(q,p)$	(4.7)
ii)	$\overset{\text{left}}{\partial}_\mu \sigma(p,q) = -Y_\mu$	(4.8)
iii)	$\left\langle \overset{\text{left}}{d}\sigma, \overset{\text{left}}{d}\sigma \right\rangle = 2\sigma$	(4.9)
iv)	$\overset{\text{left}}{\Box}\sigma = \dim \mathcal{M} + \frac{1}{2} \frac{\partial_{-Y} \det g(p)}{\det g(p)}, \qquad \text{where } Y \equiv \exp_p^{-1}(q).$	(4.10)

4.2 The van Vleck determinant

The van Vleck determinant is a function of same signature as the world function σ. That is, it is defined on geodesics joining two points $p, q \in \mathcal{M}$. It is defined as follows.

The van Vleck determinant $\quad \Omega \times \Omega \xrightarrow{\kappa} \mathbb{R}$ (4.11)

$$(p,q) \mapsto \frac{\left|\det \overset{\text{leftright}}{\partial \;\; \partial} \sigma(p,q)\right|^{1/2}}{\left|\det g(p) \cdot \det g(q)\right|^{1/4}},$$

where the differentiation is done in any coordinates around p and q. The same coordinates must be used for $g(p)$ and $g(q)$.

This is well-defined since if we let x and x' be two coordinate systems around p and let y and y' be two coordinate systems around q,

$$\begin{aligned}
\sigma_{;\mu}|_{;\nu} &= \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial y'^{\beta}}{\partial y^{\nu}} \sigma'_{;\alpha}|_{;\beta} \\
g_{\mu\nu}(p) &= \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta}(p) \\
g_{\mu\nu}(q) &= \frac{\partial y'^{\alpha}}{\partial y^{\mu}} \frac{\partial y'^{\beta}}{\partial y^{\nu}} g'_{\alpha\beta}(q)
\end{aligned}$$

and

$$\begin{aligned}
\frac{|\det \sigma_{;\mu}|_{;\nu}|^{1/2}}{|\det g(p)|^{1/4} \cdot |\det g(q)|^{1/4}} &= \frac{\left|\det \frac{\partial x'}{\partial x}\right|^{1/2} \cdot \left|\det \frac{\partial y'}{\partial y}\right|^{1/2} \cdot |\det \sigma_{;\mu}|_{;\nu}|^{1/2}}{\left|\det \frac{\partial x'}{\partial x}\right|^{1/2} \cdot |\det g(p)|^{1/4} \cdot \left|\det \frac{\partial y'}{\partial y}\right|^{1/2} \cdot |\det g(q)|^{1/4}} \\
&= \frac{\left|\det \sigma'_{;\mu}|_{;\nu}\right|^{1/2}}{|\det g'(p)|^{1/4} \cdot |\det g'(q)|^{1/4}}.
\end{aligned}$$

Since σ is symmetric and partial differentiation commutes, κ is symmetric. A simpler formula can be derived for κ if one chooses normal coordinates $(X^0, \ldots, X^n)$ around q to differentiate wrt. both q and p. From equation 4.8 and the symmetry of σ, $\overset{\text{right}}{\partial}_{\nu} \sigma(p,q) = -X_{\nu}(p) = -\eta_{\nu\alpha} X^{\alpha}(p)$ in the normal coordinates.

$$\begin{aligned}
\overset{\text{left}}{\partial}_{\mu} \overset{\text{right}}{\partial}_{\nu} \sigma(p,q) &= \frac{\partial}{\partial X^{\mu}}\bigg|_{p} - \eta_{\nu\alpha} X^{\alpha} \\
&= -\eta_{\nu\alpha} \delta^{\alpha}_{\mu} \\
&= -\eta_{\mu\nu}
\end{aligned}$$

This and the fact that $g_{\mu\nu}(q) = \eta_{\mu\nu}$ give an expression for the van Vleck determinant that is easier to compute.

$$\kappa(p,q) = |\det g(p)|^{-1/4}, \quad \text{where } g \text{ is computed in normal coordinates around } q. \tag{4.12}$$

Properties of the van Vleck determinant:

i)
$$\kappa(p,q) = \kappa(q,p) \tag{4.13}$$

ii)
$$\overset{\text{left}}{d}\kappa(p,q) = -\frac{1}{4}\frac{d\,(\det g)_p}{\det g(p)}\kappa(p,q), \quad \text{where } g \text{ is in normal coordinates around } q. \tag{4.14}$$

4.3 Parallel transport

Parallel transport is a linear isomorphism between tangent spaces determined by a curve joining them. Let $[0,1] \xrightarrow{\gamma} \mathcal{M}$ be a curve joining so that $q = \gamma(0)$ and $p = \gamma(1)$. The parallel transport

$$\begin{aligned} T_q\mathcal{M} &\xrightarrow{P} T_p\mathcal{M} \\ Z_0 &\mapsto Z(1) \end{aligned}$$

is defined by solving the IVP

$$\left\{\begin{matrix} \nabla_{\dot\gamma} Z = 0 \\ Z(0) = Z_0 \end{matrix}\right\}. \tag{4.15}$$

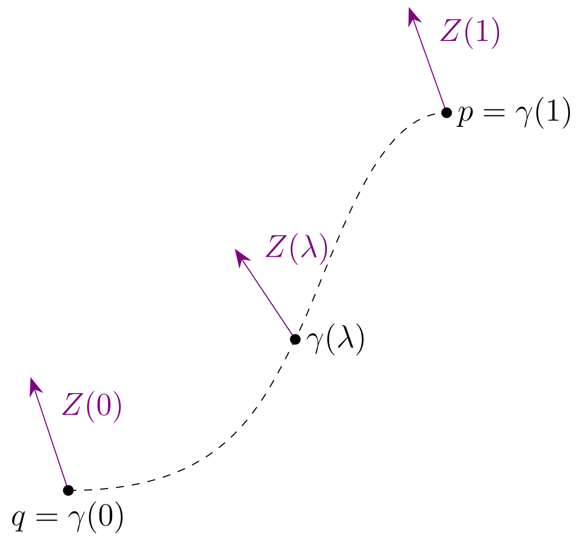

Figure 4.2: Parallel transport of vector Z along a curve γ provides a way to compare tangent vectors across different tangent planes. But the comparison is curve dependent.

Parallel transport does not depend on choice of parameterization of γ. A useful property of parallel transport is that it preserves the metric and time orientation. Preservation of the metric means that

$$\langle PZ_1, PZ_2 \rangle = \langle Z_1, Z_2 \rangle \qquad \forall Z_1, Z_2 \in T_q\mathcal{M}. \tag{4.16}$$

It is useful to describe parallel transport as a bitensor field. Again, when working in a geodesically convex domain $\Omega \subseteq \mathcal{M}$ there is a unique geodesic between any two points. Use parallel transport along the geodesics to define parallel transport as a bitensor field $P \in \Gamma\left(T\Omega \boxtimes T^*\Omega\right)$. Applying equation (4.16) on basis vectors in $T_q\mathcal{M}$ gives

$$\begin{aligned}
\langle Pe_\mu, Pe_\nu \rangle &= \langle e_\mu, e_\nu \rangle \\
\left\langle P^\alpha|_\mu(p,q)e_\mu, P^\beta|_\nu(p,q)e_\nu \right\rangle &= \langle e_\mu, e_\nu \rangle \\
P^\alpha|_\mu(p,q)P^\beta|_\nu(p,q)g_{\alpha\beta}(p) &= g_{\mu\nu}(q).
\end{aligned}$$

Since parallel transport is defined by an IVP, and IVPs have unique solutions, parallel transporting from q to p and then back to q again gives identity. That means

$$\begin{aligned}
P^\mu|_\alpha(q,p)P^\alpha|_\nu(p,q) &= \delta^\mu_\nu \\
P^\mu|_\alpha(q,p)\underbrace{P^\alpha|_\nu(p,q)P^\beta|^\nu(p,q)}_{g^{\alpha\beta}(p)} &= \delta^\mu_\nu P^\beta|^\nu(p,q) \\
P^\mu|^\beta(p,q) &= P^\beta|^\mu(q,p).
\end{aligned}$$

Also, directly from the definition of parallel transport

$$\begin{aligned}\overset{\text{left}}{\nabla}_{-Y} P(p,q) Z_0 &= \overset{\text{left}}{\nabla}_{\dot\gamma(1)} P(\gamma(1),q) Z_0 \\ &= \nabla_{\dot\gamma(1)} Z(1) \\ &= 0 \qquad\qquad\qquad \forall Z_0 \in T_q\mathcal{M}.\end{aligned}$$

Properties of parallel transport bitensor field:

i) $$P^{\mu}|^{\nu}(p,q) = P^{\nu}|^{\mu}(q,p) \tag{4.17}$$

ii) $$\overset{\text{left}}{\nabla}_{-Y} P(p,q) = 0, \qquad \text{where } q = \exp_p(Y). \tag{4.18}$$

4.4 Causality

Def 9
A tangent vector $X \in T\mathcal{M}$ is

1. *timelike if $\langle X\rangle^2 < 0$*
2. *lightlike if $\langle X\rangle^2 = 0$ and $X \neq 0$*
3. *spacelike if $\langle X\rangle^2 > 0$ or $X = 0$.*

Every tangent vector falls into exactly one of these three categories. A tangent vector is called causal if it is either timelike or lightlike.

Def 10
Let $p \in \mathcal{M}$. Two causal tangent vectors $X, Y \in T_p\mathcal{M}$ have same time orientation if

$$\langle X, Y\rangle < 0$$
$$\text{or}$$
$$X \text{ and } Y \text{ are both lightlike and } X = \lambda Y \text{ for a } \lambda > 0.$$

This is an equivalence relation on causal vectors in $T_p\mathcal{M}$. There are exactly two equivalence classes, and they are called future-directed and past-directed vectors.

This means that causal tangent vectors can be partitioned into future-directed and past-directed tangent vectors. Recall that $\mathcal{M}$ is assumed to be time oriented. That means $\mathcal{M}$ is equipped with a continuous designation of future-directed and past-directed tangent vectors among the causal tangent vectors all across itself. By continuity, parallel transport preserves time orientation.

$$Z \text{ is } \pm\text{-directed} \quad \Longleftrightarrow \quad PZ \text{ is } \pm\text{-directed} \qquad \forall Z \in T_q\mathcal{M}, \tag{4.19}$$

where "$+$" means future and "$-$" means past.

Def 11
Let $p \in \Omega$. Define the future and past light cone together with its inside and whole as

i) $J_\pm(p) \equiv \exp_p\left(\{X \in T_p\mathcal{M} \mid X \text{ is zero or } \pm\text{ directed}\}\right) \cap \Omega$

ii) $D_\pm(p) \equiv \exp_p\left(\{X \in T_p\mathcal{M} \mid X \text{ is timelike and } \pm\text{ directed}\}\right) \cap \Omega$

iii) $C_\pm(p) \equiv \exp_p\left(\{X \in T_p\mathcal{M} \mid X \text{ is lightlike and } \pm\text{ directed}\}\right) \cap \Omega$

iv) $J(p) \equiv J_-(p) \cup J_+(p)$

v) $D(p) \equiv D_-(p) \cup D_+(p)$

vi) $C(p) \equiv C_-(p) \cup C_+(p)$.

For all subsets $A \subseteq \Omega$ define

i) $J_\pm(A) \equiv \bigcup_{q\in A} J_\pm(q)$ *and* $J(A) \equiv \bigcup_{q\in A} J(q)$

ii) $D_\pm(A) \equiv \bigcup_{q\in A} D_\pm(q)$ *and* $D(A) \equiv \bigcup_{q\in A} D(q)$.

Note that $J_\pm(p)$ is closed in Ω, $D_\pm(p)$ is open and $C_\pm(p) \subseteq \Omega$ is a submanifold of codimension 1. It is because $\exp_p^{-1}(\Omega) \xrightarrow{\exp_p} \Omega$ is a diffeomorphism and the preimage theorem.

Proposition 1
Let $p \in \Omega$. Then

$$D_\pm(p) = D_\pm\left(J_\pm(p)\right) = J_\pm\left(D_\pm(p)\right) = D_\pm\left(D_\pm(p)\right) \tag{4.20}$$
$$J_\pm(p) = J_\pm\left(J_\pm(p)\right). \tag{4.21}$$

Proof is on p. 80.

Proposition 2

$$\Omega = \bigcup_{p \in \Omega} D_{\pm}(p). \tag{4.22}$$

Proof is on p. 81.

4.5 The advanced and retarded Green's function

The retarded and advanced Green's function that solve the Lorentz gauge conditional Gauss-Ampére equation (2.13) are constructed with reference to Friedlander's book [6]. Let $q \in \Omega$ and $j \in T_q\mathcal{M}$. The goal is to find $G_{\pm,j} \in \mathcal{D}'_{(1,0)}(\Omega)$ so that

$$\boxed{\mathcal{L}G_{\pm,j} = j\delta_q.} \tag{4.23}$$

Here, δ_q is the Dirac delta distribution on Ω. Start by writing $G_{\pm,j}$ as

$$G_{\pm,j} \equiv \frac{1}{4\pi} U_j \delta_{C_{\pm}(q)} + \frac{1}{4\pi} V_{\pm,j}, \tag{4.24}$$

where

$$\delta_{C_{\pm}(q)} \equiv \lim_{\epsilon \downarrow 0} \mathbf{1}_{J_{\pm}(q)} \cdot \delta\left(\sigma_q + \epsilon^2\right) \in \mathcal{D}'(\Omega), \tag{4.25}$$

$\mathbf{1}_{J_{\pm}(q)}$ is the indicator function on $J_{\pm}(q)$, $U_j \in \mathcal{T}_{(1,0)}(\Omega)$ and $V_{\pm,j} \in \mathcal{D}'_{(1,0)}(\Omega)$. The denominator 4π is explained later. It is the area of unit 3-sphere. Notice that $\delta\left(\sigma_q + \epsilon^2\right) \in \mathcal{D}'(\Omega)$ is not defined for $\epsilon = 0$ because $\operatorname{supp}\delta = \{0\} \nsubseteq \operatorname{Reg}(\sigma_q)$. The multiplication in (4.25) is defined because $\mathbf{1}_{J_{\pm}(q)}$ and $\delta\left(\sigma_q + \epsilon^2\right)$ have disjoint singular support. It is not obvious that the distribution limit in (4.25) exists, but it appears when we find a more computable expression for it.

The strategy is to apply the wave operator $\mathcal{L}$ on $G_{\pm,j}$ and see what choice of coefficients U_j and $V_{\pm,j}$

that makes $G_{\pm,j}$ a Green's function. By using Leibniz's rule and the chain rule distributions,

$$\begin{aligned}
\nabla_\alpha\left(U_j\delta_{C_\pm(q)}\right) =&\nabla_\alpha \lim_{\epsilon\to 0}\left(U_j\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)\right)\\
=&\lim_{\epsilon\to 0}\nabla_\alpha\left(U_j\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)\right) \qquad \text{by continuity}\\
=&\lim_{\epsilon\to 0}\nabla_\alpha U_j\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)+U_j\underbrace{\partial_\alpha\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)}_{\substack{=0\\ \text{because of disjoint supports}}}+U_j\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)\partial_\alpha\sigma_q
\end{aligned}$$

$$\begin{aligned}
\nabla^\alpha\nabla_\alpha\left(U_j\delta_{C_\pm(q)}\right) =&\nabla^\alpha\left(\lim_{\epsilon\to 0}\nabla_\alpha U_j\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)+U_j\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)\partial_\alpha\sigma_q\right)\\
=&\lim_{\epsilon\to 0}\nabla^\alpha\left(\nabla_\alpha U_j\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)+U_j\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)\partial_\alpha\sigma_q\right) \qquad \text{by continuity}\\
=&\lim_{\epsilon\to 0}\nabla^\alpha\nabla_\alpha U_j\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)+\nabla_\alpha U_j\underbrace{\partial^\alpha\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)}_{\substack{=0\\ \text{because of disjoint support}}}\\
&+\nabla_\alpha U_j\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)\partial^\alpha\sigma_q+\nabla^\alpha U_j\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)\partial_\alpha\sigma_q\\
&+U_j\underbrace{\partial^\alpha\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)}_{\substack{=0\\ \text{because of disjoint supports}}}\partial_\alpha\sigma_q+U_j\mathbf{1}_{J_\pm(q)}\delta''\left(\sigma_q+\epsilon^2\right)\underbrace{\partial^\alpha\sigma_q\partial_\alpha\sigma_q}_{=2\sigma_q}\\
&+U_j\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)\nabla^\alpha\partial_\alpha\sigma_q\\
=&\lim_{\epsilon\downarrow 0}\Box U_j\delta_{C_\pm(q)}+\left(2\nabla_\alpha U_j\partial^\alpha\sigma_q+U_j\Box\sigma_q\right)\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)\\
&+2U_j\sigma_q\mathbf{1}_{J_\pm(q)}\delta''\left(\sigma_q+\epsilon^2\right).
\end{aligned}$$

The Dirac delta distribution on $\mathbb{R}$ satisfies

$$t\delta(t)=0$$

because $\langle t\delta(t),\phi(t)\rangle=\langle\delta(t),t\phi(t)\rangle=0\cdot\phi(0)=0\ \forall\phi\in\mathcal{D}(\mathbb{R})$. Differentiate twice.

$$\begin{aligned}
\delta(t)+t\delta'(t)&=0\\
\delta'(t)+\delta'(t)+t\delta''(t)&=0\\
t\delta''(t)&=-2\delta'(t).
\end{aligned}$$

That means

$$\sigma_q\delta''\left(\sigma_q+\epsilon^2\right)=-\epsilon^2\delta''\left(\sigma_q+\epsilon^2\right)-2\delta'\left(\sigma_q+\epsilon^2\right). \tag{4.26}$$

Insert into $\Box\left(U_j\delta_{C_\pm(q)}\right)$.

$$\begin{aligned}
\Box\left(U_j\delta_{C_\pm(q)}\right)=&\lim_{\epsilon\downarrow 0}\Box U_j\delta_{C_\pm(q)}+\left(2\nabla_\alpha U_j\partial^\alpha\sigma_q+U_j\Box\sigma_q-4U_j\right)\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)\\
&-2U_j\mathbf{1}_{J_\pm(q)}\epsilon^2\delta''\left(\sigma_q+\epsilon^2\right),
\end{aligned}$$

so

$$\mathcal{L}\left(U_j\delta_{C_\pm(q)}\right) = \lim_{\epsilon\downarrow 0}\mathcal{L}U_j\delta_{C_\pm(q)} - \left(2\nabla_\alpha U_j\partial^\alpha\sigma_q + U_j\,\Box\,\sigma_q - 4U_j\right)\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right) \tag{4.27}$$
$$+ 2U_j\mathbf{1}_{J_\pm(q)}\epsilon^2\delta''\left(\sigma_q+\epsilon^2\right).$$

To compute $\lim_{\epsilon\to 0}\mathbf{1}_{J_\pm(q)}\epsilon^2\delta''\left(\sigma+\epsilon^2\right)$, let $\phi\in\mathcal{D}(\Omega)$ and use definition 3.13 of product of distributions.

$$\begin{aligned}
\left\langle \mathbf{1}_{J_\pm(q)}\cdot\epsilon^2\delta''\left(\sigma_q+\epsilon^2\right),\phi\right\rangle &= \epsilon^2\left\langle \delta''\left(\sigma_q+\epsilon^2\right),\underbrace{\rho_{\pm,\epsilon}\cdot\mathbf{1}_{J_\pm(q)}}_{\text{smooth}}\cdot\phi\right\rangle + \epsilon^2\left\langle \mathbf{1}_{J_\pm(q)},(1-\rho_\pm)\cdot 0\cdot\phi\right\rangle \\
&= \epsilon^2\left\langle \delta''(t),\int_{\sigma_q^{-1}(\{t-\epsilon^2\})}\rho_{\pm,\epsilon}\mathbf{1}_{J_\pm(q)}\phi\omega_{\sigma_q,t-\epsilon^2}\right\rangle \\
&= \epsilon^2\frac{d^2}{dt^2}\bigg|_0\int_{\sigma_q^{-1}(\{t-\epsilon^2\})}\rho_{\pm,\epsilon}\mathbf{1}_{J_\pm(q)}\phi\omega_{\sigma_q,t-\epsilon^2} \\
&= \epsilon^2\frac{d^2}{dt^2}\bigg|_{\epsilon^2}\int_{\sigma_q^{-1}(\{-t\})}\rho_{\pm,\epsilon}\mathbf{1}_{J_\pm(q)}\phi\omega_{\sigma_q,-t} = (*),
\end{aligned}$$

where $\rho_{\pm,\epsilon}\in C^\infty(\Omega)$ is 1 on a $\operatorname{singsupp}\delta\left(\sigma_q+\epsilon^2\right)=\sigma_q^{-1}\left(\{-\epsilon^2\}\right)$ and 0 on $\operatorname{singsupp}\mathbf{1}_{J_\pm(q)}=C_\pm(q)$.

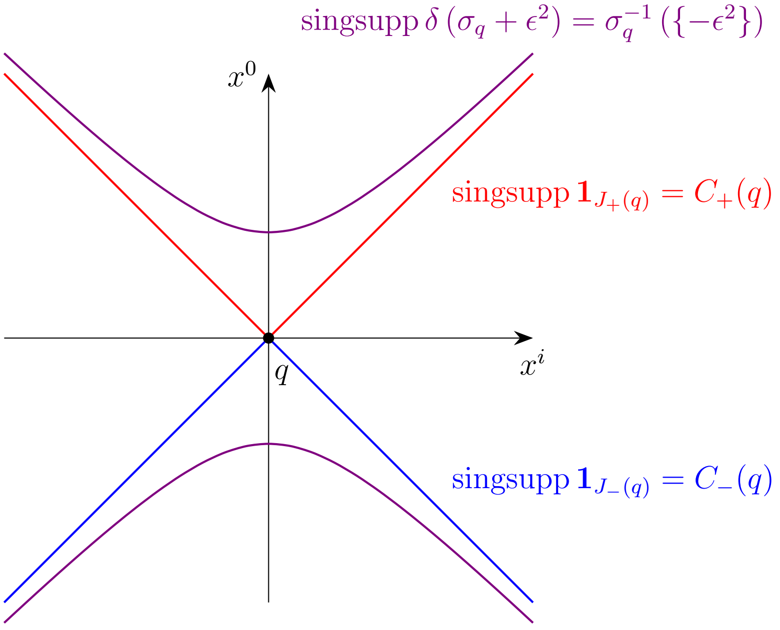

Figure 4.3: The drawing shows that singular support of $\mathbf{1}_{J_\pm(q)}$ and $\delta\left(\sigma_q+\epsilon^2\right)$ are disjoint as long as $\epsilon\neq 0$. As ϵ approaches zero, $\operatorname{singsupp}\delta\left(\sigma_q+\epsilon^2\right)$ approaches $\operatorname{singsupp}\mathbf{1}_{J_-(q)}\cup\operatorname{singsupp}\mathbf{1}_{J_+(q)}$.

To find the Leray form $\omega_{\sigma_q,-t} \in \Omega^3\left(\sigma_q^{-1}(\{-t\})\right)$, choose normal coordinates $\Omega \xrightarrow{(x^0,\ldots,x^3)\equiv(x^0,\mathbf{x})} \mathbb{R}^4$ around q. Then $\omega(x) = \sqrt{|\det g(x)|}dx^0 \wedge \cdots \wedge dx^3$. $d\sigma_q(x) = \partial_\mu \sigma_q(x) dx^\mu = \eta_{\mu\nu} x^\nu dx^\mu$.

$$\begin{aligned}
&d\sigma_q(x) \wedge \left(-\sqrt{|\det g(x)|}\frac{1}{x^0}dx^1 \wedge dx^2 \wedge dx^3\right) \\
=&\eta_{\mu\nu}x^\nu dx^\mu \wedge \left(-\sqrt{|\det g(x)|}\frac{1}{x^0}dx^1 \wedge dx^2 \wedge dx^3\right) \\
=&\eta_{00}x^0 dx^0 \wedge \left(-\sqrt{|\det g(x)|}\frac{1}{x^0}dx^1 \wedge dx^2 \wedge dx^3\right) \\
=&\sqrt{|\det g(x)|}dx^0 \wedge \cdots \wedge dx^3 \\
=&\omega(x).
\end{aligned}$$

On the submanifold $\sigma_q^{-1}(\{-t\})$, $x^0 = \pm\sqrt{|\mathbf{x}| + 2t}$. So the three spacial coordinates $\mathbf{x} = (x^1, x^2, x^3)$ work as a global coordinate system on $\sigma_q^{-1}(\{-t\})$. The unique Leray form is therefore

$$\omega_{\sigma_q,-t}\Big|_{\sigma_q^{-1}(\{-t\})\cap J_\pm(q)}(\mathbf{x}) = \mp\sqrt{\left|\det g\left(\pm\sqrt{|\mathbf{x}|^2+t}, \mathbf{x}\right)\right|}\frac{dx^1 \wedge dx^2 \wedge dx^3}{\sqrt{|\mathbf{x}|^2 + 2t}}, \qquad t > 0. \tag{4.28}$$

One can now convince oneself that the limit $\delta_{C_\pm(q)} = \lim_{\epsilon\downarrow 0} \mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q + \epsilon^2\right)$ exists by using the expression for the Leray form $\omega_{\sigma_q,-\epsilon^2}$. $\forall \phi \in \mathcal{D}(\Omega)$

$$\left\langle \mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right), \phi\right\rangle = \int_{\mathbb{R}^3} \phi\left(\pm\sqrt{|\mathbf{x}|^2+2\epsilon^2}, \mathbf{x}\right)\sqrt{\left|\det g\left(\pm\sqrt{|\mathbf{x}|^2+2\epsilon^2}, \mathbf{x}\right)\right|}\frac{d^3x}{\sqrt{|\mathbf{x}|^2+2\epsilon^2}} \tag{4.29}$$

$$\left\langle \delta_{C_\pm(q)}, \phi\right\rangle = \int_{\mathbb{R}^3} \phi\left(\pm|\mathbf{x}|, \mathbf{x}\right)\sqrt{\left|\det g\left(\pm|\mathbf{x}|, \mathbf{x}\right)\right|}\frac{d^3x}{|\mathbf{x}|}. \tag{4.30}$$

Insert (4.28) for the Leray form in $(*)$, and then go over to polar coordinates (r, θ, φ).

$$
\begin{aligned}
(*) =& \epsilon^2 \frac{d^2}{dt^2}\bigg|_{\epsilon^2} \int_{\mathbb{R}^3} \phi\left(\pm\sqrt{|\mathbf{x}|^2+2t}, \mathbf{x}\right) \sqrt{\left|\det g\left(\pm\sqrt{|\mathbf{x}|^2+2t}, \mathbf{x}\right)\right|} \frac{d^3x}{\sqrt{|\mathbf{x}|^2+2t}} \\
=& \epsilon^2 \frac{d^2}{dt^2}\bigg|_{\epsilon^2} \int_0^\infty \underbrace{\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \phi\left(\pm\sqrt{r^2+2t}, r, \theta, \varphi\right) \sqrt{\left|\det g\left(\pm\sqrt{r^2+2t}, r, \theta, \varphi\right)\right|} \cos\theta d\theta d\varphi}_{\equiv \psi\left(\pm\sqrt{r^2+2t}, r\right)} \frac{r^2 dr}{\sqrt{r^2+2t}} \\
=& \epsilon^2 \int_0^\infty \frac{\partial^2}{\partial t^2}\bigg|_{\epsilon^2} \psi\left(\pm\sqrt{r^2+2t}, r\right) \frac{r^2}{\sqrt{r^2+2t}} dr \\
=& \epsilon^2 \int_0^\infty \partial_0^2 \psi\left(\pm\sqrt{r^2+2\epsilon^2}, r\right) \frac{r^2}{\left(r^2+2\epsilon^2\right)^{3/2}} dr \\
& \mp 2\epsilon^2 \int_0^\infty \partial_0 \psi\left(\pm\sqrt{r^2+2\epsilon^2}, r\right) \frac{r^2}{\left(r^2+2\epsilon^2\right)^2} dr \\
& + 3\epsilon^2 \int_0^\infty \psi\left(\pm\sqrt{r^2+2\epsilon^2}, r\right) \frac{r^2}{\left(r^2+2\epsilon^2\right)^{5/2}} dr.
\end{aligned}
$$

Since ϕ has compact support, $(x^0, r)(\Omega) \ni (x^0, r) \mapsto \partial_0^2 \psi\left(x^0, r\right)$ also has compact support and therefore has a finite supremum. Denote the length (Lebesgue measure) of $r(\operatorname{supp} \partial_0^2 \psi) \subseteq \mathbb{R}$ by C. Since the radial coordinate $\Omega \xrightarrow{r} \mathbb{R}$ is continuous and $\operatorname{supp} \partial_0^2 \psi$ is compact, $r(\operatorname{supp} \partial_0^2 \psi)$ is compact and hence $C < \infty$. We also have the inequality $\frac{r^2}{\left(r^2+2\epsilon^2\right)^{3/2}} \leq \frac{2^{1/2}}{3^{3/2}} \frac{1}{\epsilon}$ $\forall r \geq 0$. That means the first integral can be bounded by $\frac{2^{1/2}}{3^{3/2}} C \sup_{(x^0, r) \in (x^0, r)(\Omega)} \left|\partial_0^2 \psi\left(x^0, r\right)\right| \frac{1}{\epsilon}$.

$$
\left|\epsilon^2 \int_0^\infty \partial_0^2 \psi\left(\pm\sqrt{r^2+2\epsilon^2}, r\right) \frac{r^2}{\left(r^2+2\epsilon^2\right)^{3/2}} dr\right| \leq \epsilon^2 \frac{2^{1/2}}{3^{3/2}} C \sup_{(x^0, r) \in (x^0, r)(\Omega)} \left|\partial_0^2 \psi\left(x^0, r\right)\right| \frac{1}{\epsilon},
$$

so

$$
\lim_{\epsilon \downarrow 0} \epsilon^2 \int_0^\infty \partial_0^2 \psi\left(\pm\sqrt{r^2+2\epsilon^2}, r\right) \frac{r^2}{\left(r^2+2\epsilon^2\right)^{3/2}} dr = 0.
$$

As for the second and third integral, do the substitution $s \equiv \frac{1}{\epsilon} r$.

$$
\begin{aligned}
\left|\epsilon^2 \int_0^\infty \partial_0 \psi\left(\pm\sqrt{r^2+2\epsilon^2}, r\right) \frac{r^2}{\left(r^2+2\epsilon^2\right)^2} dr\right| &= \left|\epsilon^2 \int_0^\infty \partial_0 \psi\left(\pm\sqrt{\epsilon^2 s^2+2\epsilon^2}, \epsilon s\right) \frac{\epsilon^2 s^2}{\left(\epsilon^2 s^2+2\epsilon^2\right)^2} \epsilon ds\right| \\
&= \left|\epsilon \int_0^\infty \partial_0 \psi\left(\pm\sqrt{\epsilon^2 s^2+2\epsilon^2}, \epsilon s\right) \frac{s^2}{\left(s^2+2\right)^2} ds\right| \\
&\leq \epsilon \sup_{r \geq 0} \left|\partial_0 \psi\left(\sqrt{r^2+2\epsilon^2}, r\right)\right| \underbrace{\int_0^\infty \frac{s^2}{\left(s^2+2\right)^2} ds}_{\text{finite}},
\end{aligned}
$$

so

$$\lim_{\epsilon\downarrow 0}\epsilon^2\int_0^\infty \partial_0\psi\left(\pm\sqrt{r^2+2\epsilon^2},r\right)\frac{r^2}{\left(r^2+2\epsilon^2\right)^2}dr=0.$$

The third integral is

$$\begin{aligned}\epsilon^2\int_0^\infty \psi\left(\pm\sqrt{r^2+2\epsilon^2},r\right)\frac{r^2}{\left(r^2+2\epsilon^2\right)^{5/2}}dr&=\epsilon^2\int_0^\infty \psi\left(\pm\sqrt{\epsilon^2 s^2+2\epsilon^2},\epsilon s\right)\frac{\epsilon^2 s^2}{\left(\epsilon^2 s^2+2\epsilon^2\right)^{5/2}}\epsilon ds\\&=\int_0^\infty \psi\left(\pm\epsilon\sqrt{s^2+2},\epsilon s\right)\frac{s^2}{\left(s^2+2\right)^{5/2}}ds.\end{aligned}$$

Take the limit $\epsilon\downarrow 0$.

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\epsilon^2\int_0^\infty \psi\left(\pm\sqrt{r^2+2\epsilon^2},r\right)\frac{r^2}{\left(r^2+2\epsilon^2\right)^{5/2}}dr\\=&\lim_{\epsilon\downarrow 0}\int_0^\infty \psi\left(\pm\epsilon\sqrt{s^2+2},\epsilon s\right)\frac{s^2}{\left(s^2+2\right)^{5/2}}ds\\=&\int_0^\infty \lim_{\epsilon\downarrow 0}\psi\left(\pm\epsilon\sqrt{s^2+2},\epsilon s\right)\frac{s^2}{\left(s^2+2\right)^{5/2}}ds\\=&\psi(0,0)\underbrace{\int_0^\infty \frac{s^2}{\left(s^2+2\right)^{5/2}}ds}_{=\frac{1}{6}}\\=&\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\underbrace{\phi\left(0,0,\theta,\varphi\right)}_{=\phi(q)}\underbrace{\sqrt{\left|\det g\left(0,0,\theta,\varphi\right)\right|}}_{=1}\sin^2\theta d\theta d\varphi\cdot\frac{1}{6}\\=&\frac{2\pi}{3}\phi(q).\end{aligned}$$

We get at last

$$\lim_{\epsilon\downarrow 0}\left\langle\epsilon^2\mathbf{1}_{J_\pm(q)}\cdot\delta''\left(\sigma_q+\epsilon^2\right),\phi\right\rangle=2\pi\phi(q)\qquad\forall\phi\in\mathcal{D}(\Omega).$$

$$\boxed{\lim_{\epsilon\downarrow 0}\epsilon^2\mathbf{1}_{J_\pm(q)}\cdot\delta''\left(\sigma_q+\epsilon^2\right)=2\pi\delta_q.\qquad(4.31)}$$

Now plug that into (4.27).

$$\mathcal{L}\left(U_j\delta_{C_\pm(q)}\right)=\lim_{\epsilon\downarrow 0}4\pi U_j\delta_q+\mathcal{L}U_j\delta_{C_\pm(q)}-\left(2\nabla_\alpha U_j\partial^\alpha\sigma_q+\left(\Box\,\sigma_q-4\right)U_j\right)\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right).\qquad(4.32)$$

Now apply $\mathcal{L}$ on $V_{\pm,j}$. But if $V_{\pm,j}$ shall cancel terms in $\mathcal{L}\left(U_j\delta_{C_\pm(q)}\right)$ that prevent $U_j\delta_{C_\pm(q)}$ from being a Green's function, $V_{\pm,j}$ must have a singularity at $C_\pm(q)$. Write therefore

$$V_{\pm,j}=\mathbf{1}_{J_\pm(q)}\cdot W_j,\qquad(4.33)$$

where $W_j \in \mathcal{T}_{(1,0)}(\Omega)$ is a smooth vectorfield. Now comes a trick. Note that

$$\mathbf{1}_{J_\pm(q)} = \lim_{\epsilon\downarrow 0} \mathbf{1}_{J_\pm(q)} \cdot \theta\left(-\sigma - \epsilon^2\right), \tag{4.34}$$

where $\theta \in \mathcal{D}'(\mathbb{R})$ is the Heaviside step function. Also, the Heaviside step function satisfies

$$\theta'(t) = \delta(t). \tag{4.35}$$

Apply $\mathcal{L}$ on $V_{\pm,j} = \lim_{\epsilon\downarrow 0} \mathbf{1}_{J_\pm(q)}\theta\left(-\sigma - \epsilon^2\right) W_j$.

$$\begin{aligned}
\nabla_\alpha V_{\pm,j} =& \nabla_\alpha \left(W_j \cdot \lim_{\epsilon\downarrow 0} \theta\left(-\sigma_q(p) - \epsilon^2\right) \mathbf{1}_{J_\pm(q)} \right) \\
=& \lim_{\epsilon\downarrow 0} \nabla_\alpha \left(W_j \cdot \theta\left(-\sigma_q - \epsilon^2\right) \mathbf{1}_{J_\pm(q)} \right) \qquad \text{by continuity} \\
=& \lim_{\epsilon\downarrow 0} \nabla_\alpha W_j \theta\left(-\sigma_q - \epsilon^2\right) \mathbf{1}_{J_\pm(q)} + W_j \underbrace{\theta'\left(-\sigma_q - \epsilon^2\right)}_{=\delta\left(-\sigma-\epsilon^2\right)=\delta\left(\sigma+\epsilon^2\right)} (-1)\partial_\alpha \sigma_q \mathbf{1}_{J_\pm(q)} \\
& + W_j \underbrace{\theta\left(-\sigma_q - \epsilon^2\right) \partial_\alpha \mathbf{1}_{J_\pm(q)}}_{=0 \text{ because of disjoint support}} \\
\nabla^\alpha \nabla_\alpha V_{\pm,j} =& \nabla^\alpha \left(\lim_{\epsilon\downarrow 0} \nabla_\alpha W_j \theta\left(-\sigma_q - \epsilon^2\right) \mathbf{1}_{J_\pm(q)} - W_j \delta\left(\sigma + \epsilon^2\right) \partial_\alpha \sigma_q \mathbf{1}_{J_\pm(q)} \right) \\
=& \lim_{\epsilon\downarrow 0} \nabla^\alpha \left(\nabla_\alpha W_j \theta\left(-\sigma_q - \epsilon^2\right) \mathbf{1}_{J_\pm(q)} - W_j \delta\left(\sigma + \epsilon^2\right) \partial_\alpha \sigma_q \mathbf{1}_{J_\pm(q)} \right) \\
=& \lim_{\epsilon\downarrow 0} \Box W_j \theta\left(-\sigma_q - \epsilon^2\right) \mathbf{1}_{J_\pm(q)} + \nabla_\alpha W_j \underbrace{\theta'\left(-\sigma_q - \epsilon^2\right)}_{=\delta(\sigma_q+\epsilon^2)} (-1)\partial^\alpha \sigma_q \mathbf{1}_{J_\pm(q)} \\
& + \nabla_\alpha W_j \underbrace{\theta\left(-\sigma_q - \epsilon^2\right) \partial^\alpha \mathbf{1}_{J_\pm(q)}}_{=0} - \nabla^\alpha W_j \delta\left(\sigma_q + \epsilon^2\right) \partial_\alpha \sigma_q \mathbf{1}_{J_\pm(q)} \\
& - W_j \delta'\left(\sigma_q + \epsilon^2\right) \underbrace{\partial^\alpha \sigma_q \partial_\alpha \sigma_q}_{=2\sigma_q} \mathbf{1}_{J_\pm(q)} - W_j \delta\left(\sigma_q + \epsilon^2\right) \Box \sigma_q \mathbf{1}_{J_\pm(q)} \\
& - W_j \underbrace{\delta\left(\sigma_q + \epsilon^2\right) \partial_\alpha \sigma_q \partial^\alpha \mathbf{1}_{J_\pm(q)}}_{=0} \\
=& \lim_{\epsilon\downarrow 0} \Box W_j \theta\left(-\sigma_q - \epsilon^2\right) \mathbf{1}_{J_\pm(q)} + \left(-2\partial^\alpha \sigma_q \nabla_\alpha W_j - \Box \sigma_q W_j\right) \mathbf{1}_{J_\pm(q)} \delta\left(\sigma_q + \epsilon^2\right) \\
& - 2 W_j \sigma \delta'\left(\sigma_q + \epsilon^2\right).
\end{aligned}$$

Since $t\delta'(t) = -\delta(t)$,

$$\sigma_q \delta'\left(\sigma_q + \epsilon^2\right) = -\delta\left(\sigma_q + \epsilon^2\right) - \epsilon^2 \delta'\left(\sigma_q + \epsilon^2\right). \tag{4.36}$$

Insert into $\Box V_{\pm,j}$.

$$\begin{aligned}\Box V_{\pm,j} =&\lim_{\epsilon\downarrow 0}\Box W_j H\left(-\sigma_q-\epsilon^2\right)\mathbf{1}_{J_\pm(q)}+\left(-2\partial^\alpha\sigma_q\nabla_\alpha W_j-\Box\sigma_q W_j+2W_j\right)\mathbf{1}_{J_\pm(q)}\delta\left(\sigma_q+\epsilon^2\right)\\&-2W_j\epsilon^2\delta'\left(\sigma_q+\epsilon^2\right)\\=&\Box W_j\mathbf{1}_{J_\pm(q)}+\left(-2\partial^\alpha\sigma_q\nabla_\alpha W_j-\left(\Box\sigma_q-2\right)W_j\right)\delta_\pm-2W_j\lim_{\epsilon\downarrow 0}\epsilon^2\delta'\left(\sigma_q+\epsilon^2\right).\end{aligned}$$

$\lim_{\epsilon\downarrow 0}\epsilon^2\delta'\left(\sigma_q+\epsilon^2\right)$ can be computed in similar fashion to $\lim_{\epsilon\downarrow 0}\epsilon^2\delta''\left(\sigma_q+\epsilon^2\right)$. Let $\phi\in\mathcal{D}(\Omega)$.

$$\begin{aligned}\left\langle\epsilon^2\delta'\left(\sigma_q+\epsilon^2\right),\phi\right\rangle&=\epsilon^2\left\langle\delta'(t+\epsilon^2),\int_{\sigma_q^{-1}(\{t\})}\rho_\pm\phi\omega_{\sigma_q,t}\right\rangle\\&=-\epsilon^2\frac{d}{dt}\bigg|_{\epsilon^2}\int_{\sigma_q^{-1}(\{-t\})}\rho_\pm\phi\omega_{\sigma_q,t}\\&=-\epsilon^2\frac{d}{dt}\bigg|_{\epsilon^2}\int_{\mathbb{R}^3}\phi\left(\pm\sqrt{|\mathbf{x}|^2+2t},\mathbf{x}\right)\sqrt{\left|\det g\left(\pm\sqrt{|\mathbf{x}|^2+2t},\mathbf{x}\right)\right|}\frac{d^3x}{\sqrt{|\mathbf{x}|^2+2t}}\\&=-\epsilon^2\frac{d}{dt}\bigg|_{\epsilon^2}\int_0^\infty\psi\left(\pm\sqrt{r^2+2t},r\right)\frac{r^2}{\sqrt{r^2+2t}}dr\\&=-\epsilon^2\int_0^\infty\frac{\partial}{\partial t}\bigg|_{\epsilon^2}\psi\left(\pm\sqrt{r^2+2t},r\right)\frac{r^2}{\sqrt{r^2+2t}}dr\\&=\underbrace{\mp\epsilon^2\int_0^\infty\partial_0\psi\left(\pm\sqrt{r^2+2\epsilon^2},r\right)\frac{r^2}{r^2+2\epsilon^2}dr}_{\longrightarrow 0\text{ since }\frac{r^2}{r^2+\epsilon^2}\leq 1}\\&\underbrace{+\epsilon^2\int_0^\infty\psi\left(\pm\sqrt{r^2+2\epsilon^2},r\right)\frac{r^2}{\left(r^2+2\epsilon^2\right)^{3/2}}dr}_{\longrightarrow 0\text{ by equation 4.5}}.\end{aligned}$$

$$\boxed{\lim_{\epsilon\downarrow 0}\epsilon^2\delta'\left(\sigma_q+\epsilon^2\right)=0.}\tag{4.37}$$

That means

$$\mathcal{L}V_{\pm,j}=\mathcal{L}W_j\mathbf{1}_{J_\pm(q)}+\left(2\partial^\alpha\sigma_q\nabla_\alpha W_j+\left(\Box\sigma_q-2\right)W_j\right)\delta_{C_\pm(q)}.\tag{4.38}$$

The wave operator applied on $G_{\pm,j}$ becomes

$$\begin{aligned}4\pi\mathcal{L}G_{\pm,j}=&4\pi U_j\delta_q+\left(\mathcal{L}U_j+2\partial^\alpha\sigma_q\nabla_\alpha W_j+\left(\Box\sigma_q-2\right)W_j\right)\delta_{C_\pm(q)}+\mathcal{L}W_j\mathbf{1}_{J_\pm(q)}\\&-\lim_{\epsilon\downarrow 0}\left(2\nabla_\alpha U_j\partial^\alpha\sigma_q+\left(\Box\sigma_q-4\right)U_j\right)\underbrace{\mathbf{1}_{J_\pm(q)}\delta'\left(\sigma_q+\epsilon^2\right)}_{\text{support within }D_\pm(q)}.\end{aligned}\tag{4.39}$$

If

$$\begin{aligned}
&& U_j(q) =& j && (4.40)\\
\text{Transport equation for } U_j && 2\partial^\alpha \sigma_q \nabla_\alpha U_j + (\Box\, \sigma_q - 4)\, U_j =& 0 & \text{on } D_\pm(q) & (4.41)\\
\text{Transport equation for } W_j && 2\partial^\alpha \sigma_q \nabla_\alpha W_j + (\Box\, \sigma_q - 2)\, W_j =& -\mathcal{L} U_j & \text{on } C_\pm(q) & (4.42)\\
\text{Wave equation in vacuum} && \mathcal{L} W_j =& 0 & \text{on } D_\pm(q), & (4.43)
\end{aligned}$$

$G_{\pm,j}$ satisfies (4.23) and hence is a Green's function. To solve (4.40 - 4.43), assume there are smooth solutions $U_j, W_j \in \mathcal{T}_{(1,0)}(\Omega)$. By restricting the transport equations to geodesics they become ODEs. Let $X \in T_q\mathcal{M}$ be causal and set $\gamma(\lambda) \equiv e^{\lambda X}$. Then evaluated on $\gamma(\lambda)$,

$$\partial^\alpha \sigma_q(\lambda) \nabla_\alpha U_j(\lambda) = \lambda \frac{DU_j}{d\lambda} \qquad \text{and} \qquad \partial^\alpha \sigma_q(\lambda) \nabla_\alpha W_j(\lambda) = \lambda \frac{DW_j}{d\lambda}.$$

Then

$$2\lambda \frac{DU_j}{d\lambda} + (\Box\, \sigma_q - 4)\, U_j = 0 \tag{4.44}$$

$$2\lambda \frac{DW_j}{d\lambda} + (\Box\, \sigma_q - 2)\, W_j = -\mathcal{L} U_j. \tag{4.45}$$

By (4.10), $\Box\, \sigma_q(\lambda) = 4 + \frac{1}{2} \frac{\lambda \frac{d}{d\lambda} \det g(\lambda)}{\det g(\lambda)}$. Plug in and solve the transport equation for U_j.

$$\begin{aligned}
2\lambda \frac{DU_j}{d\lambda} + \frac{1}{2} \frac{\lambda \frac{d}{d\lambda} \det g}{\det g} U_j =& 0\\
|\det g|^{1/4} \frac{DU_j}{d\lambda} + \frac{d}{d\lambda} \left(|\det g|^{1/4} \right) U_j =& 0\\
\underbrace{|\det g|^{1/4}\, P(q,\lambda) \frac{DU_j}{d\lambda}}_{\substack{= \frac{d}{d\lambda}(P(q,\lambda)U_j) \\ \text{by the definition of} \frac{D}{d\lambda}}} + \frac{d}{d\lambda} \left(|\det g|^{1/4} \right) P(q,\lambda) U_j =& 0\\
\frac{d}{d\lambda} \left(|\det g|^{1/4}\, P(q,\lambda) U_j \right) =& 0\\
|\det g|^{1/4}\, P(q,\lambda) U_j =& U_j(q) = j \qquad \text{by (4.40)}\\
U_j(\lambda) =& |\det g(\lambda)|^{-1/4}\, P(\lambda, q) j.
\end{aligned}$$

Solve the transport equation for W_j.

$$2\lambda \frac{DW_j}{d\lambda} + \bigg(\underbrace{\frac{1}{2}\frac{\lambda \frac{d}{d\lambda}\det g(\lambda)}{\det g(\lambda)} + 2}_{=2|\det g|^{-1/4}\frac{d}{d\lambda}\left(\lambda|\det g|^{1/4}\right)}\bigg) W_j = -\mathcal{L}U_j$$

$$2\lambda \left|\det g\right|^{1/4} \underbrace{P(q,\lambda)\frac{DW_j}{d\lambda}}_{=\frac{d}{d\lambda}(P(q,\lambda)W_j)} + 2\frac{d}{d\lambda}\left(\lambda \left|\det g\right|^{1/4}\right) P(q,\lambda)W_j = -\left|\det g\right|^{1/4} P(q,\lambda)\mathcal{L}U_j$$

$$\begin{aligned}
2\frac{d}{d\lambda}\left(\lambda \left|\det g\right|^{1/4} P(q,\lambda)W_j\right) &= -\left|\det g\right|^{1/4} P(q,\lambda)\mathcal{L}U_j \\
\lambda \left|\det g\right|^{1/4} P\left(q,\lambda\right) W_j &= -\frac{1}{2}\int_0^\lambda \left|\det g(\lambda')\right|^{1/4} P(q,\lambda')\mathcal{L}U_j(\lambda')d\lambda' \\
&\quad + \underbrace{Z}_{\text{integration constant}} \in T_q\mathcal{M} \\
W_j(\lambda) &= -\frac{1}{2\lambda}\int_0^\lambda \frac{\kappa(\lambda,q)}{\kappa(\lambda',q)} P(\lambda,\lambda')\mathcal{L}U_j(\lambda')d\lambda' \\
&\quad + \frac{1}{\lambda}\kappa(\lambda,q)P(\lambda,q)Z.
\end{aligned}$$

Since W_j is assumed to be smooth, the integration constant $Z=0$. The solutions are

$$\begin{aligned}
U_j(p) =& \kappa(p,q)P(p,q)j & \forall p \in J(q) \quad (4.46) \\
W_j\Big|_{C_\pm(q)}(p) =& -\frac{1}{2}\int_0^1 \frac{\kappa(p,q)}{\kappa(\gamma(\lambda),q)} P(p,\gamma(\lambda))\mathcal{L}U_j(\gamma(\lambda))d\lambda, & \forall p \in C_\pm(q), \quad (4.47) \\
& & \text{where } \gamma(\lambda) \equiv e^{\lambda X} \\
& & \text{and } p = \exp_q(X).
\end{aligned}$$

This can be confirmed by testing the solutions against (4.40 - 4.42). Determining W on all $J_\pm(q)$ can be done by solving the boundary value problem

$$\begin{Bmatrix} \mathcal{L}W_j(p) = 0 & \forall p \in D_\pm(q) \\ W_j(p) = W_j\Big|_{C_\pm(q)}(p) & \forall p \in C_\pm(q) \end{Bmatrix}. \tag{4.48}$$

According to theorem 4.5.1 in Friedlander's book [6], this boundary value problem has a solution if one requires Ω to be a "causal domain". $\Omega \subseteq \mathcal{M}$ is defined to be a causal domain if

i) Ω is geodesically convex

ii) $J_-(p) \cap J_+(q)$ is compact $\forall p, q \in \Omega$. (Causal diamonds are compact)

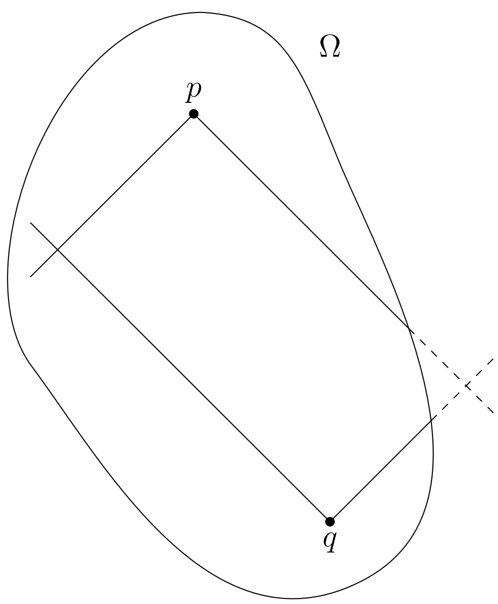

Figure 4.4: Causal diamonds may fail to be compact if for example part of them is "outside" the geodesically convex domain Ω.

That concludes the existence of a two Green's functions C and $G_{+,j}$ given by

$$G_{\pm,j} = \frac{1}{4\pi} U_j \delta_{C_\pm(q)} + \frac{1}{4\pi} W_j \mathbf{1}_{J_\pm(q)}, \qquad \text{where } U_j(p) = \kappa(p,q) P(p,q) j \text{ and } W_j \text{ is a solution to (4.48).} \tag{4.49}$$

With these two Green's functions we can prove the main result of this chapter.

Def 12
A subset $B \subseteq \Omega$ is

i) future-compact if $B \cap J_+(p)$ is compact $\forall p \in \Omega$

ii) past-compact if $B \cap J_-(p)$ is compact $\forall p \in \Omega$.

Note that a $\mp$-compact subset $B \subseteq \Omega$ is closed since

$$\begin{aligned}\Omega \setminus B &= \bigcup_{p\in\Omega} D_{\mp}(p) \setminus B \\ &= \bigcup_{p\in\Omega} D_{\mp}(p) \setminus \underbrace{(B \cap J_{\mp}(p))}_{\text{closed because it's compact}}\end{aligned}$$

is open.

Lemma 6
Let $B \subseteq \Omega$ be $\mp$-compact. Then $B \cap J_{\mp}(K)$ is compact $\forall$ compact $K \subseteq \Omega$.

Proof is on p. 82.

Lemma 7
Let Ω be a causal domain and let $B \subseteq \Omega$ be $\mp$-compact. Then $J_{\pm}(B)$ is $\mp$-compact.

Proof is on p. 82. Define

$$\mathcal{T}_{\pm,(r,s)}(\Omega) \equiv \{u \in \mathcal{T}_{(r,s)}(\Omega) \mid \operatorname{supp} u \text{ is } \mp\text{-compact}\} \tag{4.50}$$

$$\mathcal{D}'_{\pm,(r,s)}(\Omega) \equiv \{u \in \mathcal{D}'_{(r,s)}(\Omega) \mid \operatorname{supp} u \text{ is } \mp\text{-compact}\}\,. \tag{4.51}$$

Define U and W as bitensor fields in $\mathcal{T}_{(1,0)\boxtimes(0,1)}(\Omega)$ and bitensor distribution $V_{\pm} \in \mathcal{D}'_{(1,0)\boxtimes(0,1)}(\Omega)$.

$$U_j{}^{\mu}(p) \equiv U^{\mu}|_{\nu}(p,q)j^{\nu} \tag{4.52}$$

$$W_j{}^{\mu}(p) \equiv W^{\mu}|_{\nu}(p,q)j^{\nu} \tag{4.53}$$

$$V_{\pm,j}{}^{\mu}(p) \equiv V_{\pm}{}^{\mu}|_{\nu}(p,q)j^{\nu} \tag{4.54}$$

$\forall j \in T_q\Omega$. Here comes the main result of this chapter.

Theorem 4 (Existence and uniqueness of solution)
Let Ω be a causal domain. Then

$$\mathcal{T}_{\pm,(1,0)}(\Omega) \xrightarrow{\mathcal{L}} \mathcal{T}_{\pm,(1,0)}(\Omega) \qquad \text{and} \qquad \mathcal{D}'_{\pm,(1,0)}(\Omega) \xrightarrow{\mathcal{L}} \mathcal{D}'_{\pm,(1,0)}(\Omega)$$

are bijective with inverses

$$\mathcal{T}_{\pm,(1,0)}(\Omega) \xrightarrow{G_{\pm}} \mathcal{T}_{\pm,(1,0)}(\Omega) \qquad \text{and} \qquad \mathcal{D}'_{\pm,(1,0)}(\Omega) \xrightarrow{G_{\pm}} \mathcal{D}'_{\pm,(1,0)}(\Omega)$$

given by

$$\langle G_{\pm}u, \phi\rangle = \langle u, G'_{\pm}\phi\rangle \qquad \forall \phi \in \mathcal{D}_{(1,0)}(\Omega),$$

where

$$G'_{\pm}\phi^{\nu}(q) = \frac{1}{4\pi}\int_{C_{\pm}(q)} U_{\mu}|^{\nu}(p,q)\phi^{\mu}(p)\omega_{\sigma_q,0}(p) + \frac{1}{4\pi}\int_{D_{\pm}(q)} W_{\mu}|^{\nu}(p,q)\phi^{\mu}(p)\omega(p). \tag{4.55}$$

Furthermore, $\forall j \in \mathcal{T}_{\pm,(1,0)}(\Omega)$

$$\boxed{G_{\pm}j^{\nu}(p) = \frac{1}{4\pi}\int_{C_{\mp}(p)} U^{\nu}|_{\mu}(p,q)j^{\mu}(q)\omega_{\sigma_p,0}(q) + \frac{1}{4\pi}\int_{D_{\mp}(p)} W^{\nu}|_{\mu}(p,q)j^{\mu}(q)\omega(q) \qquad \forall p \in \Omega.} \tag{4.56}$$

Proof is on p. 83.

4.6 Integral equation for W

It turns out there is an easier equation than the boundary value problem (4.48) to describe the bitensor field W. From equation (4.38), (4.42) and (4.43),

$$\mathcal{L}V_{\pm,j} = -\mathcal{L}U_j\delta_{C_{\pm}(q)}. \tag{4.57}$$

Since $V_{\pm,j} \in \mathcal{D}'_{\pm,(1,0)}(\Omega)$, theorem 4 says that

$$V_{\pm,j} = -G_{\pm}\left(\mathcal{L}U_j\delta_{C_{\pm}(q)}\right). \tag{4.58}$$

Let $\phi \in \mathcal{D}_{(1,0)}(\Omega)$ and apply $V_{\pm,j}$ on ϕ.

$$\begin{aligned}
\langle V_{\pm,j}, \phi\rangle &= \left\langle -G_{\pm}\left(\mathcal{L}U_j\delta_{C_{\pm}(q)}\right), \phi\right\rangle \\
&= -\left\langle \mathcal{L}U_j\delta_{C_{\pm}(q)}, G'_{\pm}\phi\right\rangle. \\
\int_{D_{\pm}(q)} W_j{}^{\mu}\phi_{\mu}\omega &= -\int_{C_{\pm}(q)} \mathcal{L}U_j{}^{\mu}G'_{\pm}\phi_{\mu}\omega_{\sigma_q,0} \\
&= -\frac{1}{4\pi}\int_{C_{\pm}(q)} \mathcal{L}U_j{}^{\mu}(p)\left(\int_{C_{\pm}(p)} U^{\alpha}|_{\mu}(p',p)\phi_{\alpha}(p')\omega_{\sigma_p,0}(p')\right)\omega_{\sigma_q,0}(p) \\
&\quad -\frac{1}{4\pi}\int_{C_{\pm}(q)} \mathcal{L}U_j{}^{\mu}(p)\left(\int_{D_{\pm}(p)} W^{\alpha}|_{\mu}(p',p)\phi_{\alpha}(p')\omega(p')\right)\omega_{\sigma_q,0}(p).
\end{aligned}$$

The first integration region is

$$\begin{aligned}
\Sigma_1 &\equiv \{(p,p') \in \Omega\times\Omega \mid p \in C_{\pm}(q) \text{ and } p' \in C_{\pm}(p)\cap \operatorname{supp}\phi \text{ and } p' \notin C_{\pm}(q)\} \\
&= \{(p,p') \in \Omega\times\Omega \mid p \in C_{\pm}(q)\cap C_{\mp}(p') \text{ and } p' \in D_{\pm}(q)\cap \operatorname{supp}\phi\}
\end{aligned}$$

because $C_\pm\left(C_\pm(q)\right)\setminus C_\pm(q)\subseteq D_\pm(q)$ (proposition 1). Note that requirement $p'\notin C_\pm(q)$ is forced, but doesn't affect the first integral because all $(p,p')\in\Omega\times\Omega$ for which $p'\in C_\pm(q)$ has measure zero. $C_\pm(q)\xrightarrow{\sigma_{p'}\big|_{C_\pm(q)}}\mathbb{R}$ is a submersion whenever $p'\notin C_\pm(q)$. In that case, the preimage theorem says that $\sigma_{p'}\big|_{C_\pm(q)}^{-1}(\{0\})=C_\mp(p')\cap C_\pm(q)$ is an orientable two-dimensional submanifold of Ω. Through a Leray form, factorize $\omega_{\sigma_q,0}\in\Omega^3\left(C_\pm(q)\right)$ as described in theorem 1.

$$\omega_{\sigma_q,0}=\omega_{(\sigma_q,\sigma_{p'}),0}\wedge d\sigma_{p'}$$

for a unique $\omega_{(\sigma_q,\sigma_{p'}),0}\in\Omega^2\left(C_\mp(p')\cap C_\pm(q)\right)$. So

$$\begin{aligned}
&\int_{C_\pm(q)}\mathcal{L}U_j{}^\mu(p)\left(\int_{C_\pm(p)}U^\alpha|_\mu(p',p)\phi_\alpha(p')\omega_{\sigma_p,0}(p')\right)\omega_{\sigma_q,0}(p)\\
&=\int_{\Sigma_1}\mathcal{L}U_j{}^\mu(p)U^\alpha|_\mu(p',p)\phi_\alpha(p')\omega_{\sigma_q,0}(p)\wedge\omega_{\sigma_p,0}(p')\\
&=\int_{\Sigma_1}\mathcal{L}U_j{}^\mu(p)U^\alpha|_\mu(p',p)\phi_\alpha(p')\omega_{(\sigma_q,\sigma_{p'}),0}(p)\wedge d\sigma_{p'}(p)\wedge\omega_{\sigma_p,0}(p')\\
&=\int_{\Sigma_1}\mathcal{L}U_j{}^\mu(p)U^\alpha|_\mu(p',p)\phi_\alpha(p')\omega_{(\sigma_q,\sigma_{p'}),0}(p)\wedge(-1)\underbrace{d\sigma_p(p')\wedge\omega_{\sigma_p,0}(p')}_{=\omega(p')}\\
&=\int_{D_\pm(q)}\left(\int_{C_\pm(q)\cap C_\mp(p')}\mathcal{L}U_j{}^\mu(p)U^\alpha|_\mu(p',p)\omega_{(\sigma_q,\sigma_{p'}),0}(p)\right)\phi_\alpha(p')\omega(p')
\end{aligned}$$

The orientation of the manifolds integrated over is chosen so that the differential forms are positive. The second integration region is

$$\begin{aligned}
\Sigma_2&\equiv\{(p,p')\in\Omega\times\Omega\mid p\in C_\pm(q)\text{ and }p'\in D_\pm(p)\cap\operatorname{supp}\phi\}\\
&=\{(p,p')\in\Omega\times\Omega\mid p\in C_\pm(q)\cap D_\mp(p')\text{ and }p'\in D_\pm(q)\cap\operatorname{supp}\phi\}
\end{aligned}$$

because $D_\pm\left(J_\pm(q)\right)=D_\pm(q)$ (proposition 1). So

$$\begin{aligned}
&\int_{C_\pm(q)}\mathcal{L}U_j{}^\mu(p)\left(\int_{D_\pm(p)}W^\alpha|_\mu(p',p)\phi_\alpha(p')\omega(p')\right)\omega_{\sigma_q,0}(p)\\
&=\int_{\Sigma_2}\mathcal{L}U_j{}^\mu(p)W^\alpha|_\mu(p',p)\phi_\alpha(p')\omega_{\sigma_q,0}(p)\wedge\omega(p')\\
&=\int_{D_\pm(q)}\left(\int_{C_\pm(q)\cap D_\mp(p')}\mathcal{L}U_j{}^\mu(p)W^\alpha|_\mu(p',p)\omega_{\sigma_q,0}(p)\right)\phi_\alpha(p')\omega(p').
\end{aligned}$$

Equation 4.58 written out becomes

$$\begin{aligned}
W_j{}^\alpha(p')=&-\frac{1}{4\pi}\int_{C_\pm(q)\cap D_\mp(p')}\mathcal{L}U_j{}^\mu(p)W^\alpha|_\mu(p',p)\omega_{\sigma_q,0}(p)\\
&-\frac{1}{4\pi}\int_{C_\pm(q)\cap C_\mp(p')}\mathcal{L}U_j{}^\mu(p)U^\alpha|_\mu(p',p)\omega_{(\sigma_q,\sigma_{p'}),0}(p).
\end{aligned}$$

Recall that $V_\pm(p,q) = \mathbf{1}_{J_\pm(q)}(p) \cdot W(p,q)$. The bitensor distribution $V_\mp \in \mathcal{D}'_{(1,0)\boxtimes(0,1)}(\Omega)$ satisfies

$$\begin{aligned} V_\mp{}^\mu|_\nu(p,q) = &-\frac{1}{4\pi}\int_{D_\pm(p)\cap C_\mp(q)} V_\mp{}^\mu|_\alpha(p,p')\overset{\text{left}}{\mathcal{L}}U^\alpha|_\nu(p',q)\omega_{\sigma_q,0}(p') \\ &-\frac{1}{4\pi}\int_{C_\pm(p)\cap C_\mp(q)} U^\mu|_\alpha(p,p')\overset{\text{left}}{\mathcal{L}}U^\alpha|_\nu(p',q)\omega_{(\sigma_p,\sigma_q),0}(p') \end{aligned} \tag{4.59}$$

$\forall p,q \in \Omega$, where $\omega_{(\sigma_p,\sigma_q),0} \in \Omega^2\left(C_\pm(p)\cap C_\mp(q)\right)$ is the Leray form satisfying

$$d\sigma_p(p') \wedge d\sigma_q(p') \wedge \omega_{(\sigma_p,\sigma_q),0}(p') = \omega(p') \qquad \forall p' \in C_\pm(p)\cap C_\mp(q).$$

Equation (4.59) has a physical interpretation. One can attempt solving it for $V_\mp$ with fixed point iteration, starting with $V_\mp = 0$. Assuming the iteration converges to the unique solution, the solution becomes a series expansion

$$\begin{aligned} &V_\mp{}^\mu|_\nu(p,q) \\ =&\sum_{k=1}^{\infty} \int\limits_{D_\pm(p)\cap C_\mp(q)} \frac{\omega_{\sigma_q,0}(p_k)}{-4\pi} \cdots \int\limits_{D_\pm(p)\cap C_\mp(p_3)} \frac{\omega_{\sigma_{p_3},0}(p_2)}{-4\pi} \int\limits_{C_\pm(p)\cap C_\mp(p_2)} \frac{\omega_{(\sigma_p,\sigma_{p_2}),0}(p_1)}{-4\pi} U^\mu|_{\alpha_1}(p,p_1)\overset{\text{left}}{\mathcal{L}}U^{\alpha_1}|_{\alpha_2}(p_1,p_2)\ldots\overset{\text{left}}{\mathcal{L}}U^{\alpha_k}|_\nu(p_k,q). \end{aligned} \tag{4.60}$$

The first term in the series can be interpreted as light moving along the future light cone $C_+(p)$ from a light source situated at p. Then it gets scattered into the inside of the light cone $D_+(p)$ to end up at the point q. The second term can be interpreted as light first moving along the light cone $C_+(p)$ to then get scattered into $D_+(p)$, and then get scattered a second time to finally end up at q. More generally, the k-th term represents light moving from p to q getting scattered k times during its journey. The bitensor fields U and $\overset{\text{left}}{\mathcal{L}}U$ evaluated at lightlike separated points completely describe the scattering process. In theory, gravity doesn't just bend light by bending geodesics. It also scatters it!

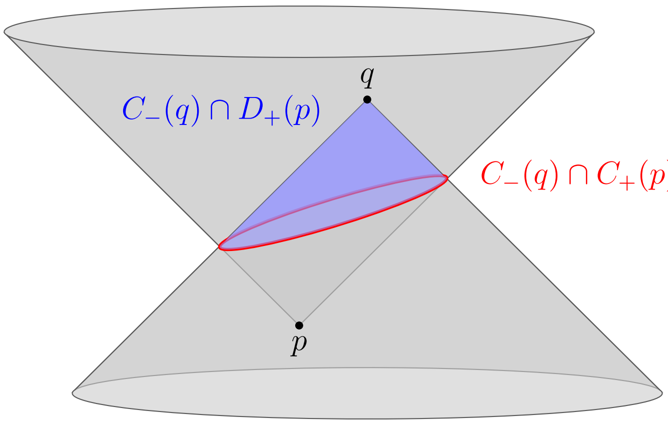

Figure 4.5: The integration regions in equation (4.59) showed in red and blue.

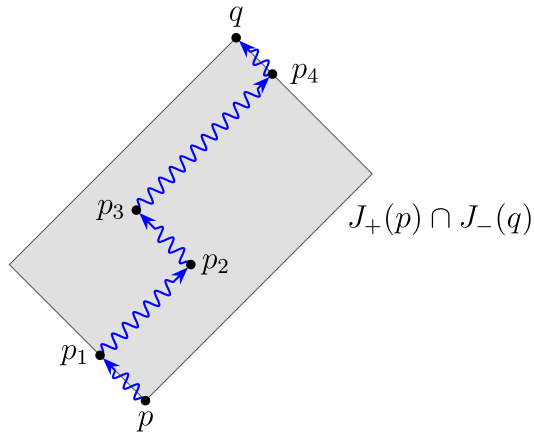

Figure 4.6: Illustration of the fourth term in the series expansion solution (4.60) for $V_\mp$. $p_1, \ldots, p_4$ are integrated over in the term. The term can be interpreted as light scattering four times from p to q on the causal diamond $J_+(p) \cap J_-(q)$.

4.7 Light source and luminous particles

The aim in this section is to use intrinsic properties of macroscopic light sources, such as type 1a supernovae, to describe light emission from the light sources. The idea is to divide a macroscopic light source into small compact pieces of matter that stays compact, also called particles. The purpose is that if the size of the particles are much less than the typical wavelengths that the particles emit, a useful approximation can be made. To model a luminous particle let $I_\tau \xrightarrow{\gamma} M$ be a timelike curve parametrized by proper time so that $\langle \dot{\gamma} \rangle^2 = -1$. γ describes the trajectory of the particle. For each $\tau \in I_\tau$ define the spacelike submanifolds

$$\Sigma(\tau) \equiv \exp_{\gamma(\tau)} \left((\operatorname{Span} \dot{\gamma}(\tau))^\perp \right) \cap \Omega. \tag{4.61}$$

These hypersurfaces are the spatial parts of the particle's local rest frames as it moves along γ. On $\Sigma(\tau)$ construct a volume form by contracting the 4-form ω induced by g against the coordinate vector field $\frac{\partial}{\partial y^0} \in \Gamma \left(T\Sigma(\tau) \right)$. $\omega_{\Sigma(\tau)} \equiv \frac{\partial}{\partial y^0} \lrcorner \omega$. Define the moments

$$K_r(\tau) \equiv \int_{\Sigma(\tau)} y^{\otimes r} \otimes d \left(\exp_{\gamma(\tau)} \right)_y^{-1} j \cdot \omega_{\Sigma(\tau)} \in T_{\gamma(\tau)} \Omega^{\otimes (r+1)}, \tag{4.62}$$

where $y(q) \equiv \exp^{-1}_{\gamma(\tau)}(q)$. Define

$$M^{\mu_1\cdots\mu_r|\mu\nu} \equiv K^{\mu_1\cdots\mu_r\nu|\mu} - K^{\mu_1\cdots\mu_r\mu|\nu}. \tag{4.63}$$

Note that $M^{\mu_1\cdots\mu_r|\mu\nu} = -M^{\mu_1\cdots\mu_r|\nu\mu}$. The moments $K_r(\tau)$ satisfy an approximate differential relation.

$$K^{\mu_1\cdots\mu_r|\mu}(\tau) \approx \frac{1}{r+1}\left(\frac{D}{d\tau}K^{\mu_1\cdots\mu_r\mu|0}(\tau) + \sum_{l=1}^{r} M^{\mu_1\cdots\hat{\mu}_l\cdots\mu_r|\mu\mu_l}(\tau)\right). \tag{4.64}$$

A hat on an index means that the index is absent. To prove (4.64), a definition and a lemma is needed.

Def 13 (The horizontal derivative of the exponential map)
Let $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$. The horizontal derivative $d_H \exp_X$ of the exponential map is defined by

$$\begin{aligned} T_p\mathcal{M} &\xrightarrow{d_H \exp_X} T_{e^X}\mathcal{M} \\ [\gamma] &\mapsto \left.\frac{d}{d\tau}\right|_0 \exp_{\gamma(\tau)}\left(P_\gamma^\tau X\right), \end{aligned}$$

where $T_{\gamma(0)}\mathcal{M} \xrightarrow{P_\gamma^\tau} T_{\gamma(\tau)}\mathcal{M}$ is parallel transport along the curve γ.

Lemma 8 (Total derivative is sum of vertical and horizontal derivative)
Let $I \xrightarrow{X} \mathcal{U} \subseteq T\mathcal{M}$ be a vector field along a curve $I \xrightarrow{\gamma} \mathcal{M}$. Then

$$\left.\frac{d}{d\tau}\right|_0 \exp(X(\tau)) = d\left(\exp_p\right)_{X(0)} \frac{DX}{d\tau}(0) + d_H \exp_{X(0)} \dot{\gamma}(0). \tag{4.65}$$

Proof is on p. 85.

Proof of approximation (4.64).
For each $\tau \in I_\tau$ extend $e_0(\tau) \equiv \dot{\gamma}(\tau)$ to an ordered orthonormal basis $(e_0(\tau), \ldots, e_3(\tau))$ in $T_{\gamma(\tau)}\Omega$ so that $e_\mu(\tau)$ is smooth wrt. τ. Let $\tau_0 \in I_\tau$. Construct normal coordinates $(y^0, \ldots, y^3)$ around $\gamma(\tau_0)$. Then $\exp_{\gamma(\tau_0)}(y^\mu(q)e_\mu) = q \quad \forall q \in \Omega$. In these normal coordinates, define $J^\nu \equiv \sqrt{|\det g|}j^\nu$. Then

from conservation of charge (equation (2.8)), $\partial_\nu J^\nu = \sqrt{|\det g|}\nabla_\nu j^\nu = 0$.

$$\begin{aligned}
K^{i_1\dots i_r|k} &= \int_{\mathbb{R}^3} y^{i_1}\dots y^{i_r} J^k d^3y \\
&= \int_{\mathbb{R}^3} \partial_l\left(y^k\right) y^{i_1}\dots y^{i_r} J^l d^3y \qquad \partial_l\left(y^k\right) = \delta_l^k \\
&= -\int_{\mathbb{R}^3} y^k \partial_l\left(y^{i_1}\dots y^{i_r} J^l\right) d^3y \\
&= -\sum_{m=1}^{r}\int_{\mathbb{R}^3} y^k y^{i_1}\dots \underbrace{\partial_l\left(y^{i_m}\right)}_{=\delta_l^{i_m}}\dots y^{i_r} J^l d^3y - \int_{\mathbb{R}^3} y^k y^{i_1}\dots y^{i_r} \underbrace{\partial_l J^l}_{=-\partial_0 J^0} d^3y \\
&= -\sum_{m=1}^{r}\int_{\mathbb{R}^3} y^{i_1}\dots \underbrace{y^{\hat{i}_m}}_{\text{not present}}\dots y^{i_r} y^k J^{i_m} d^3y + \underbrace{\int_{\mathbb{R}^3} y^{i_1}\dots y^{i_r} y^k \partial_0 J^0 d^3y}_{\equiv Q^{i_1\dots i_r k|0}} \\
&= -\sum_{m=1}^{r} K^{i_1\dots\hat{i}_m\dots i_r k|i_m} + Q^{i_1\dots i_r k|0} \\
&= -\sum_{m=1}^{r} M^{i_1\dots\hat{i}_m\dots i_r|i_m k} - \underbrace{\sum_{m=1}^{r} K^{i_1\dots\hat{i}_m\dots i_r i_m|k}}_{rK^{i_1\dots i_r|k}} + Q^{i_1\dots i_r k|0}.
\end{aligned}$$

$$(1+r)K^{i_1\dots i_r|k} = Q^{i_1\dots i_r k|0} + \sum_{m=1}^{r} M^{i_1\dots\hat{i}_m\dots i_r|k i_m}.$$

To relate $Q^{i_1\dots i_r k|0}$ with $\frac{D}{d\tau}K^{i_1\dots i_r k|0}$,

$$\begin{aligned}
&\left.\frac{D}{d\tau}\right|_{\tau_0} K^{|0}(\tau) \\
=&\left.\frac{D}{d\tau}\right|_{\tau_0}\int_{\mathbb{R}^3} y^{j_1}\dots y^{j_s} J^0 d^3y \cdot e_{j_1}\otimes\dots\otimes e_{j_s} \\
=&\int_{\mathbb{R}^3} y^{j_1}\dots y^{j_s}\left.\frac{\partial}{\partial\tau}\right|_{\tau_0} J^0\left(\exp_{\gamma(\tau)}\left(y^k e_k(\tau)\right)\right) d^3y \cdot e_{j_1}\otimes\dots\otimes e_{j_s} \\
&+\int_{\mathbb{R}^3} y^{j_1}\dots y^{j_s} J^0 d^3y \cdot \left.\frac{D}{d\tau}\right|_{\tau_0} e_{j_1}\otimes\dots\otimes e_{j_s}.
\end{aligned}$$

Define $z^\mu(\tau,\mathbf{y})$ as the solution to

$$\exp_{\gamma(\tau)}\left(y^k e_k(\tau)\right) = \exp_{\gamma(\tau_0)}\left(z^\mu(\tau,\mathbf{y}) e_\mu(\tau_0)\right).$$

Differentiate wrt. τ at $\tau = \tau_0$.

$$
\begin{aligned}
\left.\frac{\partial}{\partial \tau}\right|_{\tau_0} \exp_{\gamma(\tau)}\left(y^k e_k(\tau)\right) &= \left.\frac{\partial}{\partial \tau}\right|_{\tau_0} \exp_{\gamma(\tau_0)}\left(z^\mu(\tau, \mathbf{y}) e_\mu(\tau_0)\right) \\
d_H\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k} e_0 + d\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k}\left(y^i \frac{De_i}{d\tau}\right) &= d\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k}\left(\frac{\partial z^\mu}{\partial \tau} e_\mu\right) \\
d_H\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k} e_0 + y^i d\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k}\left(\frac{De_i}{d\tau}\right) &= \frac{\partial z^\mu}{\partial \tau} \partial_\mu.
\end{aligned}
$$

Apply dy^ν.

$$
\begin{aligned}
\frac{\partial z^0}{\partial \tau}(\tau_0, \mathbf{y}) &= \left\langle dy^0, d_H\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k} e_0\right\rangle + y^i \left\langle dy^0, d\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k}\left(\frac{De_i}{d\tau}\right)\right\rangle \\
&= \underbrace{\left\langle dy^0, d_H\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k} e_0\right\rangle}_{\approx 1 \text{ if particle small}} + y^i \left\langle e^0, \frac{De_i}{d\tau}\right\rangle \\
\frac{\partial z^j}{\partial \tau}(\tau_0, \mathbf{y}) &= \left\langle dy^j, d_H\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k} e_0\right\rangle + y^i \left\langle dy^j, d\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k}\left(\frac{De_i}{d\tau}\right)\right\rangle \\
&= \underbrace{\left\langle dy^j, d_H\left(\exp_{\gamma(\tau_0)}\right)_{y^k e_k} e_0\right\rangle}_{\approx 0 \text{ if particle small}} + y^i \left\langle e^j, \frac{De_i}{d\tau}\right\rangle,
\end{aligned}
$$

where $(e^0, \ldots, e^3)$ is the dual basis of $(e_0, \ldots, e_3)$.

$$
\begin{aligned}
\left.\frac{\partial}{\partial \tau}\right|_{\tau_0} J^0\left(\exp_{\gamma(\tau)}\left(y^k e_k(\tau)\right)\right) &= \left.\frac{\partial}{\partial \tau}\right|_{\tau_0} J^0\left(\exp_{\gamma(\tau_0)}\left(z^\mu(\tau, \mathbf{y}) e_\mu(\tau_0)\right)\right) \\
&= \frac{\partial z^\mu}{\partial \tau} \partial_\mu J^0(0, \mathbf{y}) \\
&\approx \partial_0 J^0(0, \mathbf{y}) + y^i \left\langle e^\mu, \frac{De_i}{d\tau}\right\rangle \partial_\mu J^0(0, \mathbf{y}).
\end{aligned}
$$

Insert into $\frac{D}{d\tau}K^{|0}$.

$$
\begin{aligned}
&\frac{D}{d\tau}K^{|0} \\
\approx &\int_{\mathbb{R}^3} y^{j_1}\dots y^{j_s}\left(\partial_0 J^0(0,\mathbf{y}) + y^i\left\langle e^\mu, \frac{De_i}{d\tau}\right\rangle \partial_\mu J^0(0,\mathbf{y})\right) d^3y \cdot e_{j_1}\otimes\dots\otimes e_{j_s} \\
&+ \int_{\mathbb{R}^3} y^{j_1}\dots y^{j_s} J^0 d^3y \cdot \frac{D}{d\tau} e_{j_1}\otimes\dots\otimes e_{j_s} \\
= &Q^{|0} \\
&\underbrace{+\left\langle e^\mu, \frac{De_i}{d\tau}\right\rangle \int_{\mathbb{R}^3} y^{j_1}\dots y^{j_s} y^i \partial_\mu J^0(0,\mathbf{y}) d^3y \cdot e_{j_1}\otimes\dots\otimes e_{j_s} + \int_{\mathbb{R}^3} y^{j_1}\dots y^{j_s} J^0 d^3y \cdot \frac{D}{d\tau}\left(e_{j_1}\otimes\dots\otimes e_{j_s}\right)}_{\approx 0 \text{ if the particle's acceleration is small compared to its interior oscillation of charges}}.
\end{aligned}
$$

So

$$
K^{i_1\dots i_r|k} \approx \frac{1}{1+r}\left(\frac{D}{d\tau}K^{i_1\dots i_r k|0} + \sum_{m=1}^{r} M^{i_1\dots\hat{i}_m\dots i_r|k i_m}\right).
$$

$\square$

Another intrinsic quantity of luminous particles is the direction in which they emit light. A direction can be described by a tangent vector which is normalized in some way. Let $p \in \Omega$ be the position of an observer. Define the direction vector from the particle towards the observer,

$$
\varsigma(\tau, p) \equiv -\frac{1}{\left\langle \dot{\gamma}(\tau), \exp^{-1}_{\gamma(\tau)}(p)\right\rangle} \exp^{-1}_{\gamma(\tau)}(p) \in T_{\gamma(\tau)}\Omega. \tag{4.66}
$$

Then $\langle \dot{\gamma}(\tau), \varsigma(\tau, p)\rangle = -1$, and ς is an *intrinsic* quantity of the luminous particle.

Chapter 5

Electromagnetic waves in flat FRW universe

This chapter uses the mathematics developed in the two previous chapters to derive the relation between luminosity, redshift, observed light intensity and comoving distance implied by Maxwell's equations in curved spacetime.

5.1 The flat FRW universe and comoving coordinates

The flat FRW universe is the manifold

$$M \equiv I_t \times \mathbb{R}^3, \tag{5.1}$$

where $I_t \subseteq \mathbb{R}$ is an open interval. The canonical global coordinate system on M is denoted $(t, \chi^1, \chi^2, \chi^3)$. $M \xrightarrow{t} I_t$ is called *cosmic time*, and $M \xrightarrow{\boldsymbol{\chi}=(\chi^1,\chi^2,\chi^3)} \mathbb{R}^3$ are called *comoving coordinates.* M is equipped with the metric g given by

$$ds^2 = -dt^2 + a(t)^2 \left(\left(d\chi^1\right)^2 + \left(d\chi^2\right)^2 + \left(d\chi^3\right)^2 \right). \tag{5.2}$$

Sometimes the entire coordinate system $(t, \boldsymbol{\chi})$ is referred to as comoving coordinates. $I \xrightarrow{a} \langle 0, \infty \rangle$ is called the *scale factor* and is a function of cosmic time. The idea of this spacetime is that each time slice is homogeneous and isotropic in accordance with the cosmological principle. The galaxies are assumed to roughly be standing still in the comoving coordinate system. The scale factor then describes at what rate the galaxies are diverging from each other. Note that the scale factor is not physical since multiplying it with a positive number doesn't change physics. But the Hubble parameter, defined by

$$H(t) \equiv \frac{a'(t)}{a(t)}, \tag{5.3}$$

is physical and descibes the expansion rate of the universe over cosmic time.

5.2 The wave operator $\mathcal{L} = -\Box + R$

To compute the wave operator $\mathcal{T}_{+,(1,0)}(M) \xrightarrow{\mathcal{L}} \mathcal{T}_{+,(1,0)}(M)$ from the Lorentz gauge conditional Maxwell's equations (2.13), the Christoffel symbols and the Ricci tensor of (M, g) are needed. In comoving coordinates one gets

$$\Gamma^i{}_{i0} = \frac{a'}{a} \qquad \text{and} \qquad \Gamma^0{}_{ii} = aa' \qquad \text{and the rest} = 0,$$

$$R_{00} = -3\frac{a''}{a} \qquad \text{and} \qquad R_{ii} = aa'' + 2\,(a')^2 \qquad \text{and the rest} = 0.$$

i is NOT summed over. In any coordinate system the d'Alembertian of a vector field A is

$$\nabla^\alpha\nabla_\alpha A^\mu = \partial^\alpha\partial_\alpha A^\mu + 2\Gamma^{\mu\beta}{}_\gamma\partial_\beta A^\gamma - \Gamma^{\alpha\beta}{}_\beta\partial_\alpha A^\mu + \left(\partial^\delta\Gamma^\mu{}_{\delta\gamma} + \Gamma^{\mu\alpha}{}_\beta\Gamma^\beta{}_{\alpha\gamma} - \Gamma^\mu{}_{\gamma\beta}\Gamma^{\beta\delta}{}_\delta\right) A^\gamma. \tag{5.4}$$

In comoving coordinates the d'Alembertian of A is

$$\nabla^\alpha\nabla_\alpha A^0 = \partial^\alpha\partial_\alpha A^0 - 3\frac{a'}{a}\partial_0 A^0 + 2\frac{a'}{a}\partial_j A^j + 3\left(\frac{a'}{a}\right)^2 A^0 \tag{5.5}$$

$$\nabla^\alpha\nabla_\alpha A^i = \partial^\alpha\partial_\alpha A^i - 5\frac{a'}{a}\partial_0 A^i + 2\frac{a'}{a^3}\partial_i A^0 - \left(\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right) A^i, \tag{5.6}$$

and the Ricci tensor operating on A is

$$(RA)^0 \equiv R^0{}_\nu A^\nu = 3\frac{a''}{a} A^0 \tag{5.7}$$

$$(RA)^i \equiv R^i{}_\nu A^\nu = \left(\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right) A^i. \tag{5.8}$$

The wave operator in comoving coordinates becomes

$$\mathcal{L}A^0 = -\partial^\alpha\partial_\alpha A^0 + 3\frac{a'}{a}\partial_0 A^0 - 2\frac{a'}{a}\partial_j A^j + 3\left(\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right) A^0 \tag{5.9}$$

$$\mathcal{L}A^i = -\partial^\alpha\partial_\alpha A^i + 5\frac{a'}{a}\partial_0 A^i - 2\frac{a'}{a^3}\partial_i A^0 + 2\left(\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right) A^i. \tag{5.10}$$

I used Maxima CAS with the packages "ctensor" and "itensor" to carry out the calculations.

5.3 The exponential map

The exponential map

$$\begin{aligned} T_pM \supseteq \mathcal{U}_q &\xrightarrow{\exp_q} M \\ X &\mapsto \gamma(1) \end{aligned} \tag{5.11}$$

is computed by solving the geodesic equation

$$\ddot{\gamma}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta = 0 \tag{5.12}$$

from $\lambda = 0$ to $\lambda = 1$ with initial values

$$\left\{\begin{aligned} \gamma(0) &= p \\ \dot{\gamma}(0) &= X \end{aligned}\right\}.$$

$\mathcal{U}_q$ denotes the maximal domain of $\exp_q$. Since geodesics parallel transport their tangent vector along themselves

$$\frac{d}{d\lambda}\langle\dot{\gamma}(\lambda), \dot{\gamma}(\lambda)\rangle = 0. \tag{5.13}$$

Writing out the geodesic equation in comoving coordinates one gets

$$\ddot{t} = -aa' \left|\dot{\boldsymbol{\chi}}\right|^2 \tag{5.14}$$

$$\ddot{\chi}^i = -2\frac{a'}{a}\dot{t}\dot{\chi}^i. \tag{5.15}$$

$$\left\{\begin{aligned} t(0) &= t_0 \\ \dot{\chi}^i(0) &= \frac{1}{a_0}X^i \end{aligned}\right\}, \qquad a_0 \equiv a(t_0).$$

Multiply (5.15) $\dot{\chi}^i$ and integrate.

$$\begin{aligned} \dot{\chi}^i\ddot{\chi}^i &= -2\frac{a'}{a}\dot{t}\dot{\chi}^i\dot{\chi}^i \\ a^4\frac{d}{d\lambda}\left(\frac{1}{2}|\dot{\boldsymbol{\chi}}|^2\right) &= -2a^3a'\dot{t}|\dot{\boldsymbol{\chi}}|^2 \\ a^4\frac{d}{d\lambda}\left(\frac{1}{2}|\dot{\boldsymbol{\chi}}|^2\right) + \frac{d}{d\lambda}\left(\frac{1}{2}a^4\right)|\dot{\boldsymbol{\chi}}|^2 &= 0 \\ \frac{d}{d\lambda}\left(a^4|\dot{\boldsymbol{\chi}}|^2\right) &= 0 \\ |\dot{\boldsymbol{\chi}}| &= \frac{a_0}{a^2}|\mathbf{X}|. \end{aligned}$$

By equation (5.13),

$$\begin{aligned}
\frac{d}{d\lambda}\left(-\dot{t}^2 + a^2|\dot{\boldsymbol{\chi}}|^2\right) &= 0 \\
-\dot{t}^2 + \underbrace{a^2|\dot{\boldsymbol{\chi}}|^2}_{=\frac{a_0^2}{a^2}|\mathbf{X}|^2} &= \langle X\rangle^2 \\
\dot{t} &= \pm\sqrt{-\langle X\rangle^2 + \frac{a_0^2}{a^2}|\mathbf{X}|^2} \qquad (5.16) \\
\pm\int_0^1 \frac{\dot{t}(\lambda)}{\sqrt{-\langle X\rangle^2 + \frac{a_0^2}{a^2(t(\lambda))}|\mathbf{X}|^2}} d\lambda &= 1 \\
\pm\int_{t_0}^{t(1)} \frac{dt'}{\sqrt{-\langle X\rangle^2 + \frac{a_0^2}{a^2(t')}|\mathbf{X}|^2}} &= 1.
\end{aligned}$$

Equation (5.16) says that if X is causal (hence also non-zero), $t(\lambda)$ is invertible. Now integrate $|\dot{\boldsymbol{\chi}}|$.

$$\begin{aligned}
|\boldsymbol{\chi}(1)| &= |\mathbf{X}|\int_0^1 \frac{a_0}{a(t(\lambda))^2} d\lambda \\
&= |\mathbf{X}|\int_{t_0}^{t(1)} \frac{a_0}{a(t')^2}\frac{1}{\dot{t}(\lambda(t'))} dt' \qquad \text{assuming } X \text{ is causal} \\
&= \pm|\mathbf{X}|\int_{t_0}^{t(1)} \frac{a_0}{a(t')^2}\frac{dt'}{\sqrt{-\langle X\rangle^2 + \frac{a_0^2}{a(t')^2}|\mathbf{X}|^2}}.
\end{aligned}$$

This was not a complete derivation, but it suggests what the exponential map looks like. In order to prove the following formula one needs to check if the geodesic equation and initial condition are satisfied. The exponential map restricted on causal vectors X (that is $X \neq 0$ and $\langle X\rangle^2 \leq 0$) is

$$\exp_{(t_0,\boldsymbol{\chi}_0)}(X) = \begin{cases} \left(f_X^{-1}(1), \quad \boldsymbol{\chi}_0 + \operatorname{sgn}(X^0)\int\limits_{t_0}^{f_X^{-1}(1)} \frac{a(t_0)}{a(t)^2\sqrt{h_X(t)}} dt\mathbf{X}\right) & , \quad X \neq 0 \\ (t_0, \boldsymbol{\chi}_0) & , \quad X = 0 \end{cases}, \qquad (5.17)$$

where

$$\begin{aligned}
I_t &\xrightarrow{h_X} \mathbb{R} \\
t &\mapsto -\langle X\rangle^2 + \frac{a(t_0)^2}{a(t)^2}|\mathbf{X}|^2
\end{aligned}$$

and

$$
\begin{aligned}
I_t &\xrightarrow{f_X} \mathbb{R} \\
t &\mapsto \operatorname{sgn} X^0 \int_{t_0}^{t} \frac{dt}{\sqrt{h_X(t)}}.
\end{aligned}
$$

Note that f_X is not defined for $X = 0$.

5.4 Parallel transport along causal geodesics

Let $X \in T_pM$ be a causal vector and consider the geodesic $\gamma(\lambda) \equiv e^{\lambda X}$. Set $q \equiv e^X$. Then parallel transport $T_pM \xrightarrow{P(p,q)} T_qM$ in matrix form wrt. the orthonormal basis $(e_0, \dots, e_3)$ is given by

$$
Pe_\mu = P_\mu|^\nu(p,q)e_\nu. \tag{5.18}
$$

The parallel transport IVP (4.15) in the basis $(e_0, \dots, e_3)$ is

$$
\left\{
\begin{array}{l}
\dfrac{dZ^0}{d\lambda} = -\dfrac{a'}{a}\dot{\gamma}^j Z^j \\
\dfrac{dZ^i}{d\lambda} = -\dfrac{a'}{a}\dot{\gamma}^i Z^0 \\
Z(0) = Z_0 \in T_pM
\end{array}
\right\}. \tag{5.19}
$$

The solution is

$$
P_0|^0(p,q) = \frac{1}{2}\left(\frac{m_X(t_1)}{m_X(t_0)} + \frac{m_X(t_0)}{m_X(t_1)}\right) \tag{5.20}
$$

$$
P_0|^j(p,q) = \operatorname{sgn}(X^0)\frac{1}{2}\left(\frac{m_X(t_1)}{m_X(t_0)} - \frac{m_X(t_0)}{m_X(t_1)}\right)\hat{\mathbf{X}}^j \tag{5.21}
$$

$$
P_i|^0(p,q) = \operatorname{sgn}(X^0)\frac{1}{2}\left(\frac{m_X(t_1)}{m_X(t_0)} - \frac{m_X(t_0)}{m_X(t_1)}\right)\hat{\mathbf{X}}^i \tag{5.22}
$$

$$
P_i|^j(p,q) = \delta^i_j + \left(\frac{1}{2}\left(\frac{m_X(t_1)}{m_X(t_0)} + \frac{m_X(t_0)}{m_X(t_1)}\right) - 1\right)\hat{\mathbf{X}}^i\hat{\mathbf{X}}^j, \tag{5.23}
$$

where

$$
m_X(t) \equiv \sqrt{h_X(t)} + \frac{a(t_0)}{a(t)}|\mathbf{X}|
$$

and $p = (t_0, \boldsymbol{\chi}_0)$ and $q = (t_1, \boldsymbol{\chi}_1)$ in comoving coordinates.

5.5 The van Vleck determinant

From equation (4.12) the van Vleck determinant $\kappa(p,q)$ can be computed by first differentiate the exponential map to find the linear transform

$$T_pM \xrightarrow{d\left(\exp_p\right)_X} T_{e^X}M, \tag{5.24}$$

and then calculate the matrix of the metric tensor in normal coordinates,

$$g_{\mu\nu} \equiv \left\langle d\left(\exp_p\right)_X e_\mu, d\left(\exp_p\right)_X e_\nu \right\rangle. \tag{5.25}$$

The van Vleck determinant is then $\kappa(p,q) = |\det g|^{-1/4}$. Define the matrix $\xi(X)$ by

$$d\left(\exp_p\right)_X e_\mu \equiv \xi_\mu^\nu(X) e_\nu. \tag{5.26}$$

Then

$$\begin{aligned} g_{\mu\nu} &= \left\langle \xi_\mu^\alpha(X) e_\alpha, \xi_\nu^\beta(X) e_\beta \right\rangle \\ &= \xi_\mu^\alpha(X)\xi_\nu^\beta(X)\langle e_\alpha, e_\beta\rangle \\ &= \xi_\mu^\alpha(X)\xi_\nu^\beta(X)\eta_{\alpha\beta} \\ &= -\xi_\mu^0(X)\xi_\nu^0(X) + \xi_\mu^i(X)\xi_\nu^i(X). \end{aligned}$$

Differentiate $\exp_p$ at X to find $\xi(X)$ with help from the implicit function theorem and Leibniz's integral rule.

$$\xi_0^0(X) = A \tag{5.27}$$

$$\xi_j^0(X) = B\hat{\mathbf{X}}^j \tag{5.28}$$

$$\xi_0^i(X) = C\hat{\mathbf{X}}^i \tag{5.29}$$

$$\xi_j^i(X) = D\delta_j^i + E\hat{\mathbf{X}}^i\hat{\mathbf{X}}^j, \tag{5.30}$$

where

$$A = X^0\sqrt{h_X(t_1)}\int_{t_0}^{t_1}\frac{dt}{h_X(t)^{3/2}} \tag{5.31}$$

$$B = |\mathbf{X}|\sqrt{h_X(t_1)}\int_{t_0}^{t_1}\left(\frac{a(t_0)^2}{a(t)^2}-1\right)\frac{1}{h_X(t)^{3/2}}dt \tag{5.32}$$

$$C = \left|X^0\right||\mathbf{X}|\int_{t_0}^{t_1}\left(\frac{a(t_0)}{a(t_1)}-\frac{a(t_0)a(t_1)}{a(t)^2}\right)\frac{dt}{h_X(t)^{3/2}} \tag{5.33}$$

$$D = \operatorname{sgn} X^0\int_{t_0}^{t_1}\frac{a(t_0)a(t_1)}{a(t)^2}\frac{dt}{\sqrt{h_X(t)}} \tag{5.34}$$

$$E = |\mathbf{X}|^2\operatorname{sgn} X^0\int_{t_0}^{t_1}\left(\frac{a(t_0)}{a(t_1)}-\frac{a(t_0)a(t_1)}{a(t)^2}\right)\left(\frac{a(t_0)^2}{a(t)^2}-1\right)\frac{1}{h_X(t)^{3/2}}dt. \tag{5.35}$$

$$
\begin{aligned}
g_{00} &= -A^2 + C^2 \\
g_{0i} &= \left(-AB + C\left(D+E\right)\right)\hat{\mathbf{X}}^i \\
g_{ij} &= D^2\delta_{ij} + \left(-B^2 + 2DE + E^2\right)\hat{\mathbf{X}}^i\hat{\mathbf{X}}^j.
\end{aligned}
$$

Choose the basis vectors (e_1, e_2, e_3) so that $\hat{\mathbf{X}} = (1,0,0)$ for easier calculations without loss of generality because of spherical symmetry.

$$
\begin{aligned}
\det g &= \det \begin{bmatrix} -A^2+C^2 & -AB+C\left(D+E\right) & 0 & 0 \\ -AB+C\left(D+E\right) & -B^2+\left(D+E\right)^2 & 0 & 0 \\ 0 & 0 & D^2 & 0 \\ 0 & 0 & 0 & D^2 \end{bmatrix} \\
&= \left(\left(-A^2+C^2\right)\left(-B^2+\left(D+E\right)^2\right) - \left(-AB+C\left(D+E\right)\right)^2\right)\cdot D^4 \\
&= (*).
\end{aligned}
$$

I used Maxima to substitute the expressions for A, B, C, D and E and simplify. The Maxima code is listed in listing B.1 in appendix B.

$$
\begin{aligned}
(*) &= -\left|X^0\right|^2 h_X(t_1) \underbrace{\left(\int_{t_0}^{t_1} \frac{a_0 a_1}{a(t)^2}\frac{dt}{h_X(t)^{3/2}}\right)}_{\substack{=\frac{a_1}{a_0}\frac{\langle X\rangle^2}{|\mathbf{X}|^2}\int_{t_0}^{t_1}\frac{dt}{h_X(t)^{3/2}} \\ +\frac{a_1}{a_0}\frac{1}{|\mathbf{X}|^2}\int_{t_0}^{t_1}\frac{dt}{\sqrt{h_X(t)}}}}^2 \left(\int_{t_0}^{t_1} \frac{a_0 a_1}{a(t)^2}\frac{dt}{\sqrt{h_X(t)}}\right)^4 \\
&= -\left|X^0\right|^2 h_X(t_1)\frac{a_1}{a_0}\frac{1}{|\mathbf{X}|^2}\left(1 + \langle X\rangle^2 \int_{t_0}^{t_1}\frac{dt}{h_X(t)^{3/2}}\right)^2 \left(\int_{t_0}^{t_1} \frac{a_0 a_1}{a(t)^2}\frac{dt}{\sqrt{h_X(t)}}\right)^4.
\end{aligned}
$$

On the last equality, $f_X(t_1) = 1$ has been used. So

$$
\det g = -\frac{a_1^2}{a_0^2}\frac{\left|X^0\right|^2 h_X(t_1)}{|\mathbf{X}|^4}\left(1 + \langle X\rangle^2 \int_{t_0}^{t_1}\frac{dt}{h_X(t)^{3/2}}\right)^2 \left(\int_{t_0}^{t_1} \frac{a_0 a_1}{a(t)^2}\frac{dt}{\sqrt{h_X(t)}}\right)^4, \tag{5.36}
$$

and for $\langle X\rangle^2 = 0$

$$
\det g = -\left(\frac{1}{|\mathbf{X}|}\int_{t_0}^{t_1}\frac{a_1}{a(t)}dt\right)^4. \tag{5.37}
$$

5.6 The four-potential of j

Now we use theorem 4. Since the four-current j of a light source has past-compact support, the unique solution to the Lorentz gauge conditional Maxwell's equations $\mathcal{L}A = j$ that also has past-compact support is

$$A^{\mu}(p) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\kappa}\left(p;-|\mathbf{x}|,\mathbf{x}\right) P^{\mu}|_{\nu}\left(p;-|\mathbf{x}|,\mathbf{x}\right) j^{\nu}\left(p;-|\mathbf{x}|,\mathbf{x}\right) d^3x + \frac{1}{4\pi}\int_{D_-(p)} W^{\mu}|_{\nu}(p,q) j^{\nu}(q)\omega(q), \tag{5.38}$$

where $(x^0,\mathbf{x}) = (x^0,\ldots,x^3)$ are any normal coordinates around p. Substituting the expression for $\det g$ found in the previous section one gets

$$\begin{aligned} A^{\mu}(p) = &\int_{\mathbb{R}^3} P^{\mu}|_{\nu}(p;-|\mathbf{x}|,\mathbf{x}) j^{\nu}\left(-|\mathbf{x}|,\mathbf{x}\right) \frac{1}{|\mathbf{x}|}\left|\int_{t_0}^{t_1} \frac{a_1}{a(t)} dt\right| \frac{d^3x}{4\pi|\mathbf{x}|} \\ &+ \frac{1}{4\pi}\int_{D_-(p)} W^{\mu}|_{\nu}(p,q) j^{\nu}(q)\omega(q). \end{aligned} \tag{5.39}$$

The first term is unscattered light coming directly from the light source. The second term is called "the tail".

5.7 The luminosity light intensity relation from Maxwell's equations

Let $j \in \mathcal{T}_{+,(1,0)}(M)$ be the four-current of a luminous particle. The goal now is to calculate the light emitted by the particle in terms of intrinsic properties. The particle moves along a timelike curve $I_\tau \xrightarrow{\gamma} M$ parameterized by proper time τ so that $\langle\dot{\gamma}\rangle^2 = -1$. The Earth is assumed to be comoving. For each position $p \in M$ of the Earth there is a unique $\tau(p) \in I_\tau$ and $\mathbf{r}(p) \in \mathbb{R}^3$ so that $e^{\iota_p(\mathbf{r}(p))} = \gamma(\tau(p))$. Set $q(p) \equiv \gamma(\tau(p))$. Start with equation (5.39), but drop the tail.

$$A^{\mu}(p) = \int_{\mathbb{R}^3} \frac{1}{\kappa(p, e^{\iota_p(\mathbf{x})})} P^{\mu}|_{\nu}(p, e^{\iota_p(\mathbf{x})}) j^{\nu}\left(e^{\iota_p(\mathbf{x})}\right) \frac{d^3x}{4\pi|\mathbf{x}|}, \tag{5.40}$$

where ι_p is defined by

$$\begin{aligned} \mathbb{R}^3 &\xrightarrow{\iota_p} T_pM \\ \mathbf{x} &\mapsto -|\mathbf{x}|e_0 + x^j e_j. \end{aligned} \tag{5.41}$$

Define also

$$\begin{aligned} T_qM &\xrightarrow{\pi_V} \mathbb{R}^3 \\ y^{\mu}e'_{\mu} &\mapsto \mathbf{y} \end{aligned} \qquad \text{and} \qquad \begin{aligned} T_qM &\xrightarrow{p_V} \mathbb{R} \\ y^{\mu}e'_{\mu} &\mapsto y^0. \end{aligned} \tag{5.42}$$

Define the Lorentz transform matrix Λ by

$$e_{\mu} \equiv \Lambda^{\nu}_{\mu} e'_{\nu}. \tag{5.43}$$

Then Λ is of the form

$$\Lambda = \begin{bmatrix} \Lambda_0^0 & \Lambda_j^0 \\ \Lambda_0^i & \Lambda_j^i \end{bmatrix} = \begin{bmatrix} \Lambda_0^0 & \boldsymbol{\ell}^T \\ \boldsymbol{\lambda} & L \end{bmatrix}. \tag{5.44}$$

Since $\eta_{\mu\nu} = \Lambda_\alpha^\mu \Lambda_\beta^\nu \eta_{\alpha\beta}$, $\eta = \Lambda^T \eta \Lambda$. Notice that $\eta = \Lambda^T \eta \Lambda \Rightarrow \eta^{-1} = \left(\Lambda^T \eta \Lambda\right)^{-1} \Rightarrow \eta = \Lambda \eta \Lambda^T$. So both Λ and Λ^T are Lorentz transforms.

$$\eta = \Lambda^T \eta \Lambda \quad \Longrightarrow \quad L^T L = I + \boldsymbol{\ell}\boldsymbol{\ell}^T \quad \text{and} \quad \Lambda_0^0 \boldsymbol{\ell} = L^T \boldsymbol{\lambda} \tag{5.45}$$

$$\eta = \Lambda \eta \Lambda^T \quad \Longrightarrow \quad L L^T = I + \boldsymbol{\lambda}\boldsymbol{\lambda}^T \quad \text{and} \quad \Lambda_0^0 \boldsymbol{\lambda} = L\boldsymbol{\ell}. \tag{5.46}$$

We have

$$\begin{aligned}
& & e_0' = V = V^\mu e_\mu = V^\mu \Lambda_\mu^\nu e_\nu' \\
\Longrightarrow & & V^\mu \Lambda_\nu^0 = 1 \quad \text{and} \quad V^\mu \Lambda_\mu^i = 0 \\
\Longleftrightarrow & & \Lambda_0^0 V^0 + \boldsymbol{\ell} \cdot \mathbf{V} = 1 \quad \text{and} \quad \boldsymbol{\lambda} V^0 + L\mathbf{V} = 0 \\
\Longrightarrow & & \underbrace{L^T \boldsymbol{\lambda}}_{\Lambda_0^0 \boldsymbol{\ell}} V^0 + \underbrace{L^T L}_{=I+\boldsymbol{\ell}\boldsymbol{\ell}^T} \mathbf{V} = 0 \\
& & \underbrace{\left(\Lambda_0^0 V^0 + \boldsymbol{\ell} \cdot \mathbf{V}\right)}_{=1} \boldsymbol{\ell} + \mathbf{V} = \mathbf{0} \\
& & \mathbf{V} = -\boldsymbol{\ell}.
\end{aligned}$$

Do the coordinate transform

$$\begin{tikzcd}
\iota_p^{-1}\left(\exp_p^{-1}\left(\exp_q\left(\mathcal{N}_q\right)\right)\right) \arrow[r, "\iota_p"] \arrow[rrrr, bend left, "\phi"] & \exp_p^{-1}\left(\exp_q\left(\mathcal{N}_q\right)\right) \arrow[r, "\exp_p"] & \exp_q\left(\mathcal{N}_q\right) \arrow[r, "\exp_q^{-1}"] & \mathcal{N}_q \arrow[r, "\pi_V"] & \pi_V\left(\mathcal{N}_q\right).
\end{tikzcd}$$

ϕ is bijective. Taking the derivative at $\mathbf{x} \in \iota_p^{-1}\left(\exp_p^{-1}\left(\exp_q\left(\mathcal{N}_q\right)\right)\right)$ one gets

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "d(\iota_p)_{\mathbf{x}}"] \arrow[rrrr, bend left, "d\phi_{\mathbf{x}}"] & T_p M \arrow[r, "d\left(\exp_p\right)_{\iota_p(\mathbf{x})}"] & T_{e^{\iota_p(\mathbf{x})}} M \arrow[r, "d\left(\exp_q\right)^{-1}_{\exp_q^{-1}\left(e^{\iota_p(\mathbf{x})}\right)}"] & T_q M \arrow[r, "\pi_V"] & \mathbb{R}^3.
\end{tikzcd}$$

Define

$$\begin{tikzcd}
\pi_V\left(\mathcal{N}_q\right) \arrow[r, "\phi^{-1}"] \arrow[rrrrr, bend left, "y^0"] & \iota_p^{-1}\left(\exp_p^{-1}\left(\exp_q\left(\mathcal{N}_q\right)\right)\right) \arrow[r, "\iota_p"] & \exp_p^{-1}\left(\exp_q\left(\mathcal{N}_q\right)\right) \arrow[r, "\exp_p"] & T_qM \arrow[r, "\exp_q^{-1}"] & \mathcal{N}_q \arrow[r, "p_V"] & \mathbb{R}.
\end{tikzcd}$$

Differentiate y^0 at $\mathbf{y} = \mathbf{0}$.

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "d\phi_{\mathbf{r}}{}^{-1}"] \arrow[rrrrr, bend left, "d\left(y^0\right)_{\mathbf{0}}"] & \mathbb{R}^3 \arrow[r, "d\left(\iota_p\right)_{\mathbf{r}}"] & T_pM \arrow[r, "d\left(\exp_p\right)_{\iota_p(\mathbf{r})}"] & T_qM \arrow[r, "d\left(\exp_q^{-1}\right)_q = \mathrm{id}_{T_qM}"] & T_qM \arrow[r, "p_V"] & \mathbb{R}.
\end{tikzcd}$$

$$\begin{aligned}
p_V e_\mu &= p_V \Lambda^\nu_\mu e'_\nu = \Lambda^0_\mu \\
\pi_V e_\mu &= \pi_V \Lambda^\nu_\mu e'_\nu = \Lambda^i_\mu e'_i \\
d\left(\iota_p\right)_{\mathbf{r}} \mathbf{u} &= \left.\frac{d}{d\lambda}\right|_0 -|\lambda \mathbf{u} + \mathbf{r}| e_0 + (\lambda \mathbf{u}^j + \mathbf{r}^j) e_j = -\hat{\mathbf{r}} \cdot \mathbf{u} e_0 + u^j e_j.
\end{aligned}$$

So in the orthonormal basis $(e_0, \ldots, e_3)$

$$[p_V] = \begin{bmatrix} V^0 & -\mathbf{V}^T \end{bmatrix} \qquad \text{and} \qquad [\pi_V] = \begin{bmatrix} \boldsymbol{\lambda} & L \end{bmatrix} \qquad \text{and} \qquad \left[d\left(\iota_p\right)_{\mathbf{r}}\right] = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$
\begin{aligned}
[d\phi_{\mathbf{r}}] &= [\pi_V]\left[d\left(\exp_p\right)_{\iota_p(\mathbf{r})}\right]\left[d\left(\iota_p\right)_{\mathbf{r}}\right] \\
&= \begin{bmatrix}\boldsymbol{\lambda} & L\end{bmatrix}\begin{bmatrix}A & B & 0 & 0\\ C & D+E & 0 & 0\\ 0 & 0 & D & 0\\ 0 & 0 & 0 & D\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \\
&= (B-A)\boldsymbol{\lambda}\hat{\mathbf{r}}^T + L\underbrace{\begin{bmatrix}E+D-C & 0 & 0\\ 0 & D & 0\\ 0 & 0 & D\end{bmatrix}}_{\equiv G}.
\end{aligned}
$$

By Woodbury matrix identity

$$
\begin{aligned}
[d\phi_{\mathbf{r}}]^{-1} &= (LG)^{-1} - \frac{1}{1+\hat{\mathbf{r}}^T(LG)^{-1}(B-A)\boldsymbol{\lambda}}(LG)^{-1}(B-A)\boldsymbol{\lambda}\hat{\mathbf{r}}^T(LG)^{-1} \\
&= G^{-1}L^{-1} - \frac{(B-A)}{1+(B-A)\underbrace{\hat{\mathbf{r}}^T G^{-1}}_{\frac{\hat{\mathbf{r}}^T}{E+D-C}}\underbrace{L^{-1}\boldsymbol{\lambda}}_{-\frac{\mathbf{V}}{V^0}}}G^{-1}\underbrace{L^{-1}\boldsymbol{\lambda}}_{-\frac{\mathbf{V}}{V^0}}\underbrace{\hat{\mathbf{r}}^T G^{-1}}_{\frac{\hat{\mathbf{r}}^{\mathbf{T}}}{E+D-C}}L^{-1} \\
&= G^{-1}\left(I_3 + \frac{\frac{B-A}{E+D-C}}{V^0 - \frac{B-A}{E+D-C}\hat{\mathbf{r}}\cdot\mathbf{V}}\mathbf{V}\hat{\mathbf{r}}^T\right)L^{-1}.
\end{aligned}
$$

Evaluating the equations (5.31) - (5.35) at $X = \iota_p(\mathbf{r})$ gives

$$B - A = -\frac{a_0}{a_1} \tag{5.47}$$

$$E + D - C = \frac{a_0}{a_1}. \tag{5.48}$$

We the get

$$[d\phi_{\mathbf{r}}]^{-1} = G^{-1}\left(I_3 - \frac{1}{V^0 + \hat{\mathbf{r}}\cdot\mathbf{V}}\mathbf{V}\hat{\mathbf{r}}^T\right)L^{-1}. \tag{5.49}$$

Now calculate the derivative of y^0 at $\mathbf{y} = \mathbf{0}$ in matrix form.

$$
\begin{aligned}
&\left[d\left(y^0\right)_{\mathbf{0}}\right] \\
&= [p_V]\left[d\left(\exp_p\right)_{\iota_p(\mathbf{r})}\right]\left[d\left(\iota_p\right)_{\mathbf{r}}\right][d\phi_{\mathbf{r}}]^{-1} \\
&= \begin{bmatrix} V^0 & -\mathbf{V} \end{bmatrix} \begin{bmatrix} A & B & 0 & 0 \\ C & D+E & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [d\phi_{\mathbf{r}}]^{-1} \\
&= \begin{bmatrix} V^0 & -\mathbf{V} \end{bmatrix} \begin{bmatrix} B-A & 0 & 0 \\ E+D-C & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix} \begin{bmatrix} \frac{1}{E+D-C} & 0 & 0 \\ 0 & \frac{1}{D} & 0 \\ 0 & 0 & \frac{1}{D} \end{bmatrix} \left(I_3 - \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \mathbf{V}\hat{\mathbf{r}}^T\right) L^{-1} \\
&= \begin{bmatrix} V^0 & -\mathbf{V} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(I_3 - \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \mathbf{V}\hat{\mathbf{r}}^T\right) L^{-1} \\
&= -\left(V^0 \hat{\mathbf{r}}^T + \mathbf{V}^T\right)\left(I_3 - \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \mathbf{V}\hat{\mathbf{r}}^T\right) L^{-1} \\
&= -\left(\frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \hat{\mathbf{r}}^T + \mathbf{V}^T\right) L^{-1}.
\end{aligned}
$$

Using the following identities of the Lorentz transform Λ:

$$
\begin{aligned}
L^T L &= I_3 + \boldsymbol{\ell} \underbrace{\boldsymbol{\ell}^T}_{=\frac{1}{\Lambda_0^0}\boldsymbol{\lambda}^T L} \\
\underbrace{\left(L^T - \frac{1}{\Lambda_0^0}\boldsymbol{\ell}\boldsymbol{\lambda}^T\right)}_{=L^{-1}} L &= I_3 \\
L^{-1} &= \left(I_3 - \frac{\mathbf{V}\mathbf{V}^T}{|V^0|^2}\right) L^T.
\end{aligned}
$$

We arrive at

$$
\left[d\left(y^0\right)_{\mathbf{0}}\right] = -\frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \left(\hat{\mathbf{r}}^T + \frac{1}{V^0}\mathbf{V}^T\right) L^T \tag{5.50}
$$

To calculate $\det d\phi_{\iota_p(\mathbf{r})}$ use the matrix determinant lemma.

$$
\begin{aligned}
\left|\det d\phi_{\iota_p(\mathbf{r})}\right| &= \left|\det\left(-\frac{a_0}{a_1}\boldsymbol{\lambda}\hat{\mathbf{r}}^T + L\begin{bmatrix}\frac{a_0}{a_1} & 0 & 0\\ 0 & D & 0\\ 0 & 0 & D\end{bmatrix}\right)\right| \\
&= \left|\left(1 - \frac{a_0}{a_1}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{a_1}{a_0} & 0 & 0\\ 0 & \frac{1}{D} & 0\\ 0 & 0 & \frac{1}{D}\end{bmatrix}\underbrace{L^{-1}\boldsymbol{\lambda}}_{=-\frac{1}{V^0}\mathbf{V}}\right)\right|\underbrace{|\det L|}_{=V^0}\frac{a_0}{a_1}D^2 \\
&= \frac{a_0}{a_1}D^2\left(V^0 + \hat{\mathbf{r}}\cdot\mathbf{V}\right) \\
&= \frac{a_0}{a_1}\frac{1}{\kappa^2}\left(V^0 + \hat{\mathbf{r}}\cdot\mathbf{V}\right).
\end{aligned}
$$

<u>Approximation 1: $f \gg H$</u>

$$
\begin{aligned}
A^\mu(p) &= \int_{\mathbb{R}^3}\frac{1}{\kappa(p, e^{y^0(\mathbf{y})e_0' + y^j e_j'})}P^\mu|_\nu(p, e^{y^0(\mathbf{y})e_0' + y^j e_j'})j^\nu\left(e^{y^0(\mathbf{y})e_0' + y^j e_j'}\right)\frac{1}{4\pi|\mathbf{x}(\mathbf{y})|}\frac{1}{\left|\det d\phi_{\mathbf{x}(\mathbf{y})}\right|}d^3y \\
&\approx \frac{1}{4\pi r(p)}\frac{P^\mu|_\nu(p, q(p))}{\kappa(p, q(p))\cdot\left|\det d\phi_{\mathbf{r}(p)}\right|}\int_{\mathbb{R}^3} j^\nu\left(y^0(\mathbf{y}), \mathbf{y}\right)d^3y.
\end{aligned}
$$

<u>Approximation 2: wavelength $\gg$ size of luminous particle</u>

$J^\nu\left(y^0, \mathbf{y}\right) \equiv \sqrt{|\det g\left(y^0, \mathbf{y}\right)|}j^\nu\left(y^0, \mathbf{y}\right)$. Assume $\lim_{k\to\infty}\left|\frac{\int_0^{y^0(\mathbf{y})}\partial_0^{k+1}J^\nu(t, \mathbf{y})\left(y^0(\mathbf{y}) - t\right)^k dt}{k!}\right| \ll \left|J^\nu\left(y^0(\mathbf{y}), \mathbf{y}\right)\right|$ $\forall\mathbf{y}$ within the region of the particle. Taylor's theorem says then that Taylor expanding wrt. y^0 about

0 should give a good approximation.

$$
\begin{aligned}
&\int_{\mathbb{R}^3} J^\nu\left(y^0(\mathbf{y}),\mathbf{y}\right) d^3y \\
\approx&\int_{\mathbb{R}^3}\sum_{n=0}^{\infty}\frac{1}{n!}\partial_0^n J^\nu(0,\mathbf{y})\left(y^0(\mathbf{y})\right)^n d^3y && \text{by the assumption} \\
=&\int_{\mathbb{R}^3}\sum_{n=0}^{\infty}\frac{1}{n!}\partial_0^n J^\nu(0,\mathbf{y})\left(\partial_i y^0(\mathbf{0})y^i\right)^n d^3y + O\left(\frac{1}{r^2}\right) \\
& && \text{by Fubini-Tonelli theorem if} \\
=&\sum_{n=0}^{\infty}\int_{\mathbb{R}^3}\frac{1}{n!}\partial_0^n J^\nu(0,\mathbf{y})\left(\partial_i y^0(\mathbf{0})y^i\right)^n d^3y + O\left(\frac{1}{r^2}\right) && \sum_{n=0}^{\infty}\frac{\int_{\mathbb{R}^3}\left|\partial_0^n j^\nu(0,\mathbf{y})\left(\partial_i y^0(\mathbf{0})y^i\right)^n\right| d^3y}{n!} \\
& && \text{converges,} \\
=&\sum_{n=0}^{\infty}\frac{\partial_{i_1}y^0(\mathbf{0})\dots\partial_{i_1}y^0(\mathbf{0})}{n!}\int_{\mathbb{R}^3} y^{i_1}\dots y^{i_n}\partial_0^n J^\nu(0,\mathbf{y})d^3y + O\left(\frac{1}{r^2}\right) \\
\approx&\sum_{n=0}^{\infty}\frac{\partial_{i_1}y^0(\mathbf{0})\dots\partial_{i_1}y^0(\mathbf{0})}{n!}\frac{D^n K^{i_1\dots i_n|\nu}}{d\tau^n}(\tau(p)) \\
\equiv&E^\nu\left(\partial_i y^0(\mathbf{0})e'^i\right).
\end{aligned}
\tag{5.51}
$$

Consider the pullback bundle $\gamma^*TM \xrightarrow{\pi_{\gamma^*TM}} I_\tau$. E is defined as the bundle morphism

$$
\begin{aligned}
\gamma^*TM &\xrightarrow{E} \gamma^*TM \\
\varsigma &\mapsto \sum_{n=0}^{\infty}\frac{\varsigma_{\mu_1}\dots\varsigma_{\mu_n}}{n!}\frac{D^n K^{\mu_1\dots\mu_n|}}{d\tau^n}\left(\pi_{\gamma^*TM}(\varsigma)\right).
\end{aligned}
$$

Define $E(\tau)\equiv E\left(\partial_i y^0(\mathbf{0})e'^i(\gamma(\tau))\right)\in T_{\gamma(\tau)}M$. Insert into $A^\mu(p)$.

$$
A_\mu(p)\approx\underbrace{\frac{1}{4\pi r(p)\kappa(p,q(p))\cdot\left|\det d\phi_{\mathbf{r}(p)}\right|}}_{\equiv Z(p)}P_\mu|^\nu(p,q(p))E_\nu\left(\tau(p)\right).
$$

$$
Z = 4\pi r\frac{a_0}{a_1}\frac{1}{\kappa}\left(V^0+\hat{\mathbf{r}}\cdot\mathbf{V}\right). \tag{5.52}
$$

To calculate $d\tau(p)\in T_pM$, use the implicit function theorem. The equation

$$
\gamma(\tau) = e^{\iota(t_0,\boldsymbol{\chi}_0)(\mathbf{r})}
$$

can be written as

$$
F^\mu\left(t_0,\boldsymbol{\chi}_0;\tau,\mathbf{r}\right)=0,
$$

where

$$
\begin{aligned}
F^0\left(t_0, \boldsymbol{\chi}_0; \tau, \mathbf{r}\right) &\equiv 1 - \frac{1}{r}\int_{\gamma^0(\tau)}^{t_0} \frac{a(t)}{a(t_0)} dt \\
F^i\left(t_0, \boldsymbol{\chi}_0; \tau, \mathbf{r}\right) &\equiv \gamma^i(\tau) - \chi_0^i - \hat{r}^i \int_{\gamma^0(\tau)}^{t_0} \frac{dt}{a(t)}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F^0}{\partial t_0} &= H_0 - \frac{1}{r} & \frac{\partial F^i}{\partial t_0} &= -\frac{1}{a_0}\hat{r}^i \\
\frac{\partial F^0}{\partial \chi_0^j} &= 0 & \frac{\partial F^i}{\partial \chi_0^j} &= -\delta_j^i \\
\frac{\partial F^0}{\partial \tau} &= \frac{a_1}{a_0}\frac{1}{r}\dot{\gamma}^0 & \frac{\partial F^i}{\partial \tau} &= \dot{\gamma}^i + \frac{1}{a_1}\dot{\gamma}^0\hat{r}^i \\
\frac{\partial F^0}{\partial r^j} &= \frac{\hat{r}^j}{r} & \frac{\partial F^i}{\partial r^j} &= \frac{1}{a_1\kappa}\cdot\left(-\delta_j^i + \hat{r}^i\hat{r}^j\right).
\end{aligned}
$$

Again, to simplify calculations, choose the comoving coordinates so that $\hat{\mathbf{r}} = (r, 0, 0)$.

$$
\begin{aligned}
\frac{\partial F}{\partial(\tau, \mathbf{r})} &= \begin{bmatrix} \frac{a_1}{a_0}\frac{1}{r}\dot{\gamma}^0 & \frac{\hat{r}^j}{r} \\ \dot{\gamma}^i + \frac{1}{a_1}\dot{\gamma}^0\hat{r}^i & \frac{1}{a_1\kappa}\cdot\left(-\delta_j^i + \hat{r}^i\hat{r}^j\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{a_1}{a_0}\frac{1}{r}\dot{\gamma}^0 & \frac{1}{r} & 0 & 0 \\ \dot{\gamma}^1 + \frac{1}{a_1}\dot{\gamma}^0 & 0 & 0 & 0 \\ \dot{\gamma}^2 & 0 & -\frac{1}{a_1\kappa} & 0 \\ \dot{\gamma}^3 & 0 & 0 & -\frac{1}{a_1\kappa} \end{bmatrix}.
\end{aligned}
$$

The implicit function theorem says

$$
\begin{aligned}
\frac{\partial(\tau, \mathbf{r})}{\partial(t_0, \boldsymbol{\chi}_0)} &= -\left[\frac{\partial F}{\partial(\tau, \mathbf{r})}\right]^{-1} \frac{\partial F}{\partial(t_0, \boldsymbol{\chi}_0)} \\
\begin{bmatrix} \frac{\partial \tau}{\partial t_0} & \frac{\partial \tau}{\partial \boldsymbol{\chi}_0} \\ \frac{\partial \mathbf{r}}{\partial t_0} & \frac{\partial \mathbf{r}}{\partial \boldsymbol{\chi}_0} \end{bmatrix} &= -\begin{bmatrix} \frac{a_1}{a_0}\frac{1}{r}\dot{\gamma}^0 & \frac{1}{r} & 0 & 0 \\ \dot{\gamma}^1 + \frac{1}{a_1}\dot{\gamma}^0 & 0 & 0 & 0 \\ \dot{\gamma}^2 & 0 & -\frac{1}{a_1\kappa} & 0 \\ \dot{\gamma}^3 & 0 & 0 & -\frac{1}{a_1\kappa} \end{bmatrix}^{-1} \begin{bmatrix} H_0 - \frac{1}{r} & 0 & 0 & 0 \\ -\frac{1}{a_0} & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
\begin{bmatrix} \frac{\partial \tau}{\partial t_0} & \frac{\partial \tau}{\partial \boldsymbol{\chi}_0} \\ \frac{\partial \mathbf{r}}{\partial t_0} & \frac{\partial \mathbf{r}}{\partial \boldsymbol{\chi}_0} \end{bmatrix} &= \begin{bmatrix} \frac{a_1}{a_0}\frac{1}{\dot{\gamma}^0 + a_1\dot{\gamma}^1} & a_1\frac{1}{\dot{\gamma}^0 + a_1\dot{\gamma}^1} & 0 & 0 \\ 1 - rH_0 - \frac{a_1^2}{a_0^2}\frac{\dot{\gamma}^0}{\dot{\gamma}^0 + a_1\dot{\gamma}^1} & -\frac{a_1^2}{a_0}\frac{\dot{\gamma}^0}{\dot{\gamma}^0 + a_1\dot{\gamma}^1} & 0 & 0 \\ \frac{a_1^2}{a_0}\frac{\kappa\dot{\gamma}^2}{\dot{\gamma}^0 + a_1\dot{\gamma}^1} & a_1^2\frac{\kappa\dot{\gamma}^2}{\dot{\gamma}^0 + a_1\dot{\gamma}^1} & -a_1\kappa & 0 \\ \frac{a_1^2}{a_0}\frac{\kappa\dot{\gamma}^3}{\dot{\gamma}^0 + a_1\dot{\gamma}^1} & a_1^2\frac{\kappa\dot{\gamma}^3}{\dot{\gamma}^0 + a_1\dot{\gamma}^1} & 0 & -a_1\kappa \end{bmatrix}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d\tau &= \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \left(\frac{a_1}{a_0} dt + a_1 \hat{r}_i d\chi^i \right) \\
&= \frac{a_1}{a_0} \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \left(e^0 + \hat{r}_i e^i \right).
\end{aligned}
$$

$$
\partial^\mu \tau = \frac{a_1}{a_0} \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \left(\iota_p(\hat{\mathbf{r}}) \right)^\mu , \tag{5.53}
$$

where $V(\tau) \equiv \dot{\gamma}(\tau)$. $d\tau$ is lightlike so

$$
\langle d\tau \rangle^2 = 0. \tag{5.54}
$$

From the definition of ς in equation (4.66),

$$
\begin{aligned}
[\varsigma] &= -\frac{1}{\left\langle V, \exp_q^{-1}(p) \right\rangle} \left[\exp_q^{-1}(p) \right] \\
&= -\frac{1}{\left\langle V, \lambda \exp_q^{-1}(p) \right\rangle} \lambda \left[\exp_q^{-1}(p) \right] \qquad \forall \lambda \in \mathbb{R} \\
&= \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{where } \lambda \text{ was picked so that } \exp_q^{-1}(p) = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
[P(q,p)] \left[g^{-1} d\tau \right] &= \begin{bmatrix} \frac{1}{2}\left(\frac{a_1}{a_0} + \frac{a_0}{a_1} \right) & \frac{1}{2}\left(\frac{a_1}{a_0} - \frac{a_0}{a_1} \right) & 0 & 0 \\ \frac{1}{2}\left(\frac{a_1}{a_0} - \frac{a_0}{a_1} \right) & \frac{1}{2}\left(\frac{a_1}{a_0} + \frac{a_0}{a_1} \right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{a_1}{a_0} \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

g^{-1} here is the inverse metric tensor that make covariant vectors contravariant. By inverting the parallel transport, we get

$$
\partial^\mu \tau = -P^\mu|_\nu(p,q) \varsigma^\nu . \tag{5.55}
$$

Write ς in the basis $(e'_0, \ldots, e'_3)$.

$$\begin{aligned}
\varsigma^\mu e_\mu &= \varsigma^\mu \Lambda^\nu_\mu e'_\nu \\
\varsigma'^\mu &= \Lambda^\mu_\nu \varsigma^\nu \\
&= \begin{bmatrix} V^0 & -\mathbf{V}^T \\ -\frac{1}{V^0} L\mathbf{V} & L \end{bmatrix} \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{V^0 + \hat{\mathbf{r}} \cdot \mathbf{V}} \begin{bmatrix} V^0 + \hat{\mathbf{r}} \cdot \mathbf{V} \\ -L\left(\hat{\mathbf{r}} + \frac{1}{V^0}\mathbf{V}\right) \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \left[d\left(y^0\right)_{\mathbf{0}}\right]^T \end{bmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
E^\nu \left(\partial^i y^0(\mathbf{0}) e'_i\right) &= \sum_{n=0}^{\infty} \frac{\partial_{i_1} y^0(\mathbf{0}) \ldots \partial_{i_1} y^0(\mathbf{0})}{n!} \frac{D^n K^{i_1 \ldots i_n|\nu}}{d\tau^n} \\
&= \sum_{n=0}^{\infty} \frac{\varsigma_{i_1} \ldots \varsigma_{i_n}}{n!} \frac{D^n K^{i_1 \ldots i_n|\nu}}{d\tau^n} \\
&= E(\varsigma).
\end{aligned}$$

$$\begin{aligned}
\varsigma_\mu E^\mu &= \underbrace{\varsigma_0}_{=-1} E^0 + \varsigma_i \sum_{n=0}^{\infty} \frac{\varsigma_{i_1} \ldots \varsigma_{i_n}}{n!} \frac{D^n K^{i_1 \ldots i_n|\mu}}{d\tau^n} \\
&\approx -E^0 + \varsigma_i \sum_{n=0}^{\infty} \frac{\varsigma_{i_1} \ldots \varsigma_{i_n}}{n!} \frac{1}{n+1} \frac{D^n}{d\tau^n} \left(\frac{DK^{i_1 \ldots i_n \nu|0}}{d\tau} + \sum_{l=1}^{n} M^{i_1 \ldots \hat{i_l} \ldots i_n|\nu i_l} \right) \quad \text{by approximation 4.64} \\
&= -E^0 \sum_{n=0}^{\infty} \frac{\varsigma_{i_1} \ldots \varsigma_{i_n} \varsigma_i}{(n+1)!} \frac{D^{n+1} K^{i_1 \ldots i_n i|0}}{d\tau^{n+1}} + \sum_{n=0}^{\infty} \sum_{l=1}^{n} \underbrace{\frac{\varsigma_{i_1} \ldots \varsigma_{i_n} \varsigma_i}{(n+1)!} \frac{D^n M^{i_1 \ldots \hat{i_l} \ldots i_n|ii_l}}{d\tau^n}}_{=0} \\
&= -E^0 + E^0 + K^{|0} \\
&= K^{|0}.
\end{aligned}$$

So

$$\varsigma_\mu \frac{DE^\mu}{d\tau} \approx \frac{d}{d\tau}\left(\varsigma_\mu E^\mu\right) = \frac{dK^{|0}}{d\tau} = 0$$

because $K^{|0}$ is the total electric charge of the particle, which is conserved. To find the electromagnetic field generated by j, exterior differentiate A.

$$
\begin{aligned}
F_{\mu\nu}(p) &= \partial_\mu A_\nu(p) - \partial_\nu A_\mu(p) \\
&= \partial_\mu \left(\frac{1}{Z} P_\nu|^\alpha E_\alpha \right) - \partial_\nu \left(\frac{1}{Z} P_\mu|^\alpha E_\alpha \right) \\
&\approx \frac{1}{Z} \left(P_\nu|^\alpha \frac{DE_\alpha}{d\tau} \partial_\mu \tau - P_\mu|^\alpha \frac{DE_\alpha}{d\tau} \partial_\nu \tau \right) \qquad \text{because wavelength} \ll \text{distance to observer,} \\
&= \frac{1}{Z} \frac{DE_\alpha}{d\tau} \left(P_\nu|^\alpha \partial_\mu \tau - P_\mu|^\alpha \partial_\nu \tau \right) \\
&= -\frac{1}{Z} \frac{DE_\alpha}{d\tau} \left(P_\nu|^\alpha P_\mu|^\beta \varsigma_\beta - P_\mu|^\alpha P_\nu|^\beta \varsigma_\beta \right) \\
&= -\frac{1}{Z} P_\mu|^\alpha P_\nu|^\beta \left(\varsigma_\alpha \frac{DE_\beta}{d\tau} - \varsigma_\beta \frac{DE_\alpha}{d\tau} \right).
\end{aligned}
$$

Define

$$
\mathcal{F}_{\alpha\beta}(\varsigma) \equiv \left(\frac{DE_\alpha}{d\tau}(\varsigma)\varsigma_\beta - \frac{DE_\beta}{d\tau}(\varsigma)\varsigma_\alpha \right). \tag{5.56}
$$

Then

$$
F_{\mu\nu}(p) \approx \frac{1}{Z(p)} \, P_\mu|^\alpha(p,q) P_\nu|^\beta(p,q) \mathcal{F}_{\alpha\beta}. \tag{5.57}
$$

Note that $\mathcal{F} \in T_qM \wedge T_qM$ is an intrinsic quantity of the luminous particle. Now calculate the contribution to $T_{\mu\nu}(p)$ from two luminous particles, (1) and (2).

$$
\begin{aligned}
T_{\mu\nu}(p) = {} & \left(\sum_n \frac{1}{\overset{(n)}{Z}} \overset{(n)}{P}_\mu|^\beta \overset{(n)}{P}{}^\alpha|^\epsilon \mathcal{F}_{\beta\epsilon}\right)\left(\sum_m \frac{1}{\overset{(m)}{Z}} \overset{(m)}{P}_\nu|^\delta \overset{(m)}{P}_\alpha|^\sigma \mathcal{F}_{\delta\sigma}\right) \\
& - \frac{1}{4} g_{\mu\nu}(p) \left(\sum_n \frac{1}{\overset{(n)}{Z}} \overset{(n)}{P}{}^\alpha|_\epsilon \overset{(n)}{P}{}^\beta|_\delta \mathcal{F}^{\epsilon\delta}\right)\left(\sum_m \frac{1}{\overset{(m)}{Z}} \overset{(m)}{P}_\alpha|_\gamma \overset{(m)}{P}_\beta|_\zeta \mathcal{F}^{\gamma\zeta}\right) \\
= {} & \sum_n \frac{1}{\overset{(n)}{Z}{}^2} \overset{(n)}{P}_\mu|^\beta \underbrace{\overset{(n)}{P}{}^\alpha|^\epsilon \overset{(n)}{P}_\alpha|^\sigma}_{=g^{\epsilon\sigma}\left(\overset{(n)}{q}\right)} \overset{(n)}{P}_\nu|^\delta \mathcal{F}_{\beta\epsilon}\mathcal{F}_{\delta\sigma} - \frac{1}{4} g_{\mu\nu}(p) \frac{1}{\overset{(n)}{Z}{}^2} \underbrace{\overset{(n)}{P}{}^\alpha|_\epsilon \overset{(n)}{P}_\alpha|_\gamma}_{=g_{\epsilon\gamma}\left(\overset{(n)}{q}\right)} \underbrace{\overset{(n)}{P}{}^\beta|_\delta \overset{(n)}{P}_\beta|_\zeta}_{=g_{\delta\zeta}\left(\overset{(n)}{q}\right)} \mathcal{F}^{\epsilon\delta}\mathcal{F}^{\gamma\zeta} \\
& + \underbrace{\sum_{n,m} \text{cross terms}}_{\approx 0} \\
\approx {} & \sum_n \frac{1}{\overset{(n)}{Z}{}^2} \overset{(n)}{P}{}^\alpha|_\mu \overset{(n)}{P}{}^\beta|_\nu \left(\overset{(n)}{\mathcal{F}}{}_\alpha{}^\sigma \overset{(n)}{\mathcal{F}}_{\beta\sigma} - \frac{1}{4} g_{\alpha\beta}\left(\overset{(n)}{q}\right) \overset{(n)}{\mathcal{F}}{}^{\sigma\delta} \overset{(n)}{\mathcal{F}}_{\sigma\delta}\right).
\end{aligned}
$$

$$
\begin{aligned}
\overset{(n)}{\mathcal{F}}{}_\mu{}^\alpha \overset{(n)}{\mathcal{F}}_{\nu\alpha} = {} & \frac{1}{\overset{(n)}{Z}{}^2} \left(\overset{(n)}{\varsigma}_\mu \frac{D\overset{(n)}{E}{}^\alpha}{d\tau} - \overset{(n)}{\varsigma}{}^\alpha \frac{D\overset{(n)}{E}_\mu}{d\tau}\right)\left(\overset{(n)}{\varsigma}_\nu \frac{D\overset{(n)}{E}_\alpha}{d\tau} - \overset{(n)}{\varsigma}_\alpha \frac{D\overset{(n)}{E}_\nu}{d\tau}\right) \\
= {} & \frac{1}{\overset{(n)}{Z}{}^2} \left(\overset{(n)}{\varsigma}_\mu \frac{D\overset{(n)}{E}{}^\alpha}{d\tau} \overset{(n)}{\varsigma}_\nu \frac{D\overset{(n)}{E}_\alpha}{d\tau} - \overset{(n)}{\varsigma}_\mu \underbrace{\frac{D\overset{(n)}{E}{}^\alpha}{d\tau} \overset{(n)}{\varsigma}_\alpha}_{\approx 0} \frac{D\overset{(n)}{E}_\nu}{d\tau} - \overset{(n)}{\varsigma}_\nu \underbrace{\frac{D\overset{(n)}{E}_\alpha}{d\tau} \overset{(n)}{\varsigma}{}^\alpha}_{\approx 0} \frac{D\overset{(n)}{E}_\mu}{d\tau} + \underbrace{\overset{(n)}{\varsigma}{}^\alpha \overset{(n)}{\varsigma}_\alpha}_{\approx 0} \frac{D\overset{(n)}{E}_\mu}{d\tau} \frac{D\overset{(n)}{E}_\nu}{d\tau}\right) \\
= {} & \frac{1}{\overset{(n)}{Z}{}^2} \frac{D\overset{(n)}{E}{}^\alpha}{d\tau} \frac{D\overset{(n)}{E}_\alpha}{d\tau} \overset{(n)}{\varsigma}_\mu \overset{(n)}{\varsigma}_\nu \\
\overset{(n)}{\mathcal{F}}{}^{\alpha\beta} \overset{(n)}{\mathcal{F}}_{\alpha\beta} \approx {} & \left(\overset{(n)}{\varsigma}{}^\alpha \frac{D\overset{(n)}{E}{}^\beta}{d\tau} - \overset{(n)}{\varsigma}{}^\beta \frac{D\overset{(n)}{E}{}^\alpha}{d\tau}\right)\left(\overset{(n)}{\varsigma}_\alpha \frac{D\overset{(n)}{E}_\beta}{d\tau} - \overset{(n)}{\varsigma}_\beta \frac{D\overset{(n)}{E}_\alpha}{d\tau}\right) \\
= {} & 2 \frac{D\overset{(n)}{E}{}^\beta}{d\tau} \frac{D\overset{(n)}{E}_\beta}{d\tau} \underbrace{\overset{(n)}{\varsigma}{}^\alpha \overset{(n)}{\varsigma}_\alpha}_{\approx 0} - 2 \underbrace{\overset{(n)}{\varsigma}{}^\alpha \frac{D\overset{(n)}{E}_\alpha}{d\tau}}_{\approx 0} \underbrace{\overset{(n)}{\varsigma}_\beta \frac{D\overset{(n)}{E}{}^\beta}{d\tau}}_{\approx 0} \\
\approx {} & 0.
\end{aligned}
$$

Define

$$
\overset{(n)}{\mathcal{T}}_{\mu\nu}(\varsigma) \equiv \frac{D\overset{(n)}{E}{}^\alpha}{d\tau}(\varsigma) \frac{D\overset{(n)}{E}_\alpha}{d\tau}(\varsigma) \varsigma_\mu \varsigma_\nu. \tag{5.58}
$$

This quantity is also intrinsic to the luminous particle. Summing over all ordered pair of particles $n \in \{1, \dots, N\}$ gives

$$T_{\mu\nu}(p) \approx \sum_n \frac{1}{\overset{(n)}{Z}{}^2} \overset{(n)}{P}_\mu|^\alpha \overset{(n)}{P}_\nu|^\beta \overset{(n)}{T}_{\alpha\beta}. \tag{5.59}$$

Define

$$L \equiv \frac{1}{4\pi} \sum_n \frac{D\overset{(n)}{E}{}^\alpha}{d\tau} \frac{D\overset{(n)}{E}_\alpha}{d\tau}. \tag{5.60}$$

This is an intrinsic quantity to the light source. A priori, it depends on the direction from the light source towards the observer. However, by assuming spherical symmetry of the light source, L is independent of the direction. It is then just a number intrinsic to the light source. The Poynting vector measured at $p \in M$ is

$$\begin{aligned}
\mathbf{S} &= T^{0i} e_i \\
&\approx \sum_n \frac{1}{\overset{(n)}{Z}{}^2} \overset{(n)}{P}{}^0|_\alpha \overset{(n)}{P}{}^i|_\beta \frac{D\overset{(n)}{E}{}^\alpha}{d\tau} \frac{D\overset{(n)}{E}_\alpha}{d\tau} \varsigma^\alpha \varsigma^\beta e_i \\
&= \sum_n \frac{1}{\overset{(n)}{Z}{}^2} \frac{D\overset{(n)}{E}{}^\alpha}{d\tau} \frac{D\overset{(n)}{E}_\alpha}{d\tau} \partial^0\tau \partial^i \tau e_i \\
&= -\sum_n \frac{1}{\left(4\pi r \frac{a_0}{a_1} \frac{1}{\kappa} \left(\overset{(n)}{V}{}^0 + \hat{\mathbf{r}} \cdot \overset{(n)}{\mathbf{V}}\right)\right)^2} \frac{D\overset{(n)}{E}{}^\alpha}{d\tau} \frac{D\overset{(n)}{E}_\alpha}{d\tau} \frac{a_1^2}{a_0^2} \left(\overset{(n)}{V}{}^0 + \hat{\mathbf{r}} \cdot \overset{(n)}{\mathbf{V}}\right)^2 \hat{\mathbf{r}} \\
&\approx -\frac{1}{4\pi \frac{a_0^4}{a_1^4} a_1^2 \left|\boldsymbol{\chi}_1 - \boldsymbol{\chi}_0\right|^2} \underbrace{\frac{1}{4\pi} \sum_n \frac{D\overset{(n)}{E}{}^\alpha}{d\tau} \frac{D\overset{(n)}{E}_\alpha}{d\tau}}_{=L} \hat{\mathbf{r}}.
\end{aligned}$$

The light intensity I is the length of the Poynting vector.

$$I = |\mathbf{S}| \approx \frac{L}{4\pi \frac{a_0^2}{a_1^2} a_0^2 \chi^2} = \frac{L}{4\pi (1+z)^2 a_0^2 \chi^2}.$$

The measured light intensity due to the first part of the Green's function is

$$I \approx \frac{L}{4\pi (1+z)^2 a_0^2 \chi^2}, \tag{5.61}$$

where L is intrinsic to the light source and can therefore be calibrated in the case of standard candles. What about the tail part of the Green's function? The tail term is difficult to calculate. This is discussed in the next chapter.

Chapter 6

Result and discussion

The result (5.61) and equation (1.1) that was derived in the introduction are identical. However, equation (5.61) involves only the contribution from the singular part of the retarded Green's function $G_{-,p}$ given in (4.49), that is, only the non-scattered light.

6.1 Limitations

6.1.1 The tail

Contribution from the tail term $V_{-,p} \in \mathcal{D}'_{(1,0)}(\Omega) \otimes T_pM$ involves an integral over both space and time, whereas the singular part $U_p \delta_{C(p)} \in \mathcal{D}'_{(1,0)}(\Omega) \otimes T_pM$ involves an integral over the three-dimensional past light cone. As mentioned at the end of section 4.5, the tail part can be the physically interpreted as light scattered by gravity, and the singular part describes the unscattered light. The components of $V_{-,p}$ is listed in appendix B. It's components contain terms proportional to one of the following four terms:

$$\frac{1}{a_0^2 \chi^2} \qquad \frac{H}{a_0 \chi} \qquad H^2.$$

All are of dimension $[\text{length}]^{-2}$. The contribution to $A(p)$ from the tail is

$$A^\mu_{\text{tail}}(p) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{I_t} V_{-,p}{}^\mu|_\nu(t, \boldsymbol{\chi}) j^\nu(t, \boldsymbol{\chi}) a(t)^3 dt d^3\chi. \tag{6.1}$$

Fourier transform $a^3 V_{-,p}$ and j.

$$j(\omega, \boldsymbol{\chi}) \equiv \int_{I_t} j(t, \boldsymbol{\chi}) e^{-i\omega t} \frac{dt}{\sqrt{2\pi}} \tag{6.2}$$

$$V_{-,p}(\omega, \boldsymbol{\chi}) \equiv \int_{I_t} V_{-,p}(t, \boldsymbol{\chi}) a(t)^3 e^{-i\omega t} \frac{dt}{\sqrt{2\pi}}. \tag{6.3}$$

Then by unitarity of Fourier transform,

$$\int_{I_t} V_{-,p}{}^{\mu}|_{\nu}(t, \boldsymbol{\chi}) j^{\nu}(t, \boldsymbol{\chi}) a(t)^3 dt = \int_{\mathbb{R}} \overline{V_{-,p}{}^{\mu}|_{\nu}}(\omega, \boldsymbol{\chi}) j^{\nu}(\omega, \boldsymbol{\chi}) d\omega. \tag{6.4}$$

In order for the time integral to not oscillate to zero, j and $V_{-,p}$ must contain frequencies of common magnitude as seen in equation (6.4). This seems unlikely because the frequencies of $V_{-,p}$ are probably of magnitude $\frac{1}{a_0\chi}$, $\sqrt{\frac{H}{a_0\chi}}$ and H. The magnitude of these parameters in the case of a type 1a supernova are shown in the following table:

Quantity	Order of magnitude
$\frac{1}{a_0\chi}$	$<10^{-11}\,\mathrm{s}^{-1}$ [5]
$\sqrt{\frac{H}{a_0\chi}}$	$<10^{-15}\,\mathrm{s}^{-1}$
H	$\approx 10^{-19}\,\mathrm{s}^{-1}$ [4]

Table 6.1: Typical order of magnitude of three quantities associated with the position of a type 1a supernova at cosmic time t_1 relative to Earth. a_0 is the scale factor evaluted at t_0 = now, χ is the comoving distance between Eart and the supernova and H is the typical value of the Hubble parameter on the cosmic time interval $[t_1, t_0]$.

In order to measure such low frequencies one has to detect light from the light source over a period of at least 7000 years. Such low frequencies do not contribute to the observed light intensity. Therefore, the tail term due to the gravity of the flat FRW universe doesn't seem to contribute to the observed light intensity.

This whole reasoning may suggest that gravity only scatters low frequencies, and the stronger the gravity is the higher frequencies can be scattered.

6.1.2 The Lorentz gauge

There is one possibility that renders almost everything from chapter 4 and beyond invalid. That is whether the Lorentz gauge (2.14) is satified. Four-potentials A produced by the retarded Green's function $G_{-,p}$ given in (4.49) satisfy equation (2.13). In the simpel case of Minkowski space, the Lorentz gauge is satisfied as a consequence of conservation of charge (2.8). But that doesn't mean A also satisfies the Lorentz gauge in the case of curved spacetime. However, there are two conditions that together with conservation of charge imply the Lorentz gauge condition. Consider κ, σ, U, $V_\pm$, $\mathbf{1}_\pm \equiv \mathbf{1}_{\{(p,q)\in\Omega^2 \ \mid \ p\in J_\pm(q)\}}$ and $\delta_\pm \equiv \lim_{\epsilon\downarrow 0} \mathbf{1}_\pm \delta\left(\sigma+\epsilon^2\right)$ all as bitensor distributions. The two conditions are

$$\overset{\text{left}}{\nabla}_\mu U^\mu|_\nu = -\overset{\text{right}}{\partial}_\nu \kappa \qquad \text{and} \qquad \overset{\text{right}}{\partial}_\mu \overset{\text{left}}{\nabla}_\alpha V_\pm{}^\alpha|_\nu - \overset{\text{right}}{\partial}_\nu \overset{\text{left}}{\nabla}_\alpha V_\pm{}^\alpha|_\mu = 0. \tag{6.5}$$

We then have

$$\overset{\text{left}}{\nabla}_\mu G_{\pm,p}{}^\mu|_\nu = \lim_{\epsilon\downarrow 0} \overset{\text{left}}{\nabla}_\mu U^\mu|_\nu \mathbf{1}_\pm \delta\left(\sigma_p+\epsilon^2\right) + U^\mu|_\nu \underbrace{\overset{\text{left}}{\partial}_\mu \left(\mathbf{1}_\pm \delta\left(\sigma+\epsilon^2\right)\right)}_{=\mathbf{1}_\pm \overset{\text{left}}{\partial}_\mu\left(\delta\left(\sigma+\epsilon^2\right)\right)} + \overset{\text{left}}{\nabla}_\mu V_\pm{}^\mu|_\nu.$$

$$\begin{aligned} U^\mu|_\nu \overset{\text{left}}{\partial}_\mu \left(\delta\left(\sigma+\epsilon^2\right)\right) &= \kappa P^\mu|_\nu \delta'\left(\sigma+\epsilon^2\right) \overset{\text{left}}{\partial}_\mu \sigma \\ &= -\kappa \overset{\text{right}}{\partial}_\nu \sigma \delta'\left(\sigma+\epsilon^2\right) \\ &= -\kappa \overset{\text{right}}{\partial}_\nu \left(\delta\left(\sigma+\epsilon^2\right)\right). \end{aligned}$$

Using the two conditions (6.5),

$$\begin{aligned} \overset{\text{left}}{\nabla}_\mu G_\pm{}^\mu|_\nu &= \lim_{\epsilon\downarrow 0} -\overset{\text{right}}{\partial}_\nu \kappa \mathbf{1}_\pm \delta\left(\sigma+\epsilon^2\right) - \kappa \underbrace{\mathbf{1}_\pm \overset{\text{right}}{\partial}_\nu \left(\delta\left(\sigma+\epsilon^2\right)\right)}_{=\overset{\text{right}}{\partial}_\nu\left(\mathbf{1}_\pm\delta\left(\sigma+\epsilon^2\right)\right)} + \overset{\text{left}}{\nabla}_\mu V_\pm{}^\mu|_\nu \\ &= -\overset{\text{right}}{\partial}_\nu \left(\kappa\delta_\pm\right) + \overset{\text{left}}{\nabla}_\mu V_\pm{}^\mu|_\nu. \end{aligned}$$

By the Poincaré lemma (lemma 1), if $\Omega \subseteq \mathcal{M}$ is contractible, there is a function $\Omega\times\Omega \xrightarrow{\Phi} \mathbb{R}$ so that $\overset{\text{left}}{\nabla}_\mu V_\pm{}^\mu|_\nu = \overset{\text{right}}{\partial}_\nu \Phi$.

$$\overset{\text{left}}{\nabla}_\mu G_\pm{}^\mu|_\nu = \overset{\text{right}}{\partial}_\nu \left(-\kappa\delta_\pm + \Phi\right).$$

This means

$$\begin{aligned} \nabla_\mu A_\pm{}^\mu(p) &= \left\langle \overset{\text{left}}{\nabla}_\mu G_\pm{}^\mu|_\nu(p,q), j^\nu(q) \right\rangle \\ &= \left\langle \overset{\text{right}}{\partial}_\nu \left(-\kappa\delta_\pm+\Phi\right), j^\nu \right\rangle \\ &= -\left\langle -\kappa\delta_\pm+\Phi, \underbrace{\nabla_\nu j^\nu}_{=0} \right\rangle \\ &= 0, \end{aligned}$$

so the Lorentz gauge is satisfied by the two conditions.

6.2 Comparison with standard theory

In this thesis I have given two derivations that by first glance don't seem to have anything to do with each other, mathematically speaking. But they imply the same equation

$$I \approx \frac{L}{4\pi(1+z)^2 a_0^2 \chi^2}.$$

Is that a coincidence? A sphere in a cosmic time slice $t^{-1}(\{t_0\})$ of comoving radius χ has area $4\pi a(t_0)^2\chi^2$. A naive guess of the luminosity light intensity relation would therefore be

$$I = \frac{L}{4\pi a_0^2 \chi^2}.$$

The two extra factors $(1+z)$ and $(1+z)$ in the denominator have different explanations in two derivations. In the derivation in the introduction, one factor $(1+z)$ is from parallel transport of photon four-momentum, and the other $(1+z)$ is from the fact that $\frac{dt_1}{dt_0} = \frac{a_1}{a_0}$. In the derivation given in chapter 5, both factors $(1+z)$ and $(1+z)$ are from parallel transport since the energy momentum tensor is a second order tensor.

Chapter 7

Conclusion

The purpose of this thesis was to test the hypothesis that Maxwell's equations in curved spacetime (2.6) and (2.7) imply a different relation between luminosity, measured light intensity, redshift and comoving distance of a light source than equation (1.1). With disregard to wether the solution for the four-potential constructed with the Green's function derived in section 4.5 satisfies the Lorentz gauge condition, this master thesis has falsified the hypothesis and therefore verified that the luminosity light intensity relation (1.1) used for standard candles is in accordance with Maxwell's equations in curved spacetime. Otherwise, the study is inconclusive.

7.1 Summary

The luminosity light intensity relation (1.1) was derived from a premise that a light source can be described by a function $n(t, k)$ that is the number of photons the light source has ever emitted after cosmic time t with four-momentum k, and that the four-momentum is parallel transported through spacetime. In the following two chapters, distribution theory on curved spacetime together with the construction of the advanced and retarded Green's function for solving Maxwell's equations for the four-potential was carried out based on the theory in Friedlander's book [6]. The two Green's functions revealed a different way gravity can interact with light in addition to bending it by bending geodesics. Gravity also scatters light. The scattering is described by the tail term $V_\pm$ of the Green's functions given in equation (4.49). The retarded Green's function with the tail term ignored was applied together with a type of multimoment expansion (5.51) of the four-current of individual particles. Then the contribution to the electromagnetic field from each particle was summed over. The corresponding electromagnetic energy momentum tensor was calculated from the electromagnetic field, which the Poynting vector/light intensity is part of. Everything intrinsic to the light source was absorbed into the parameter L, which is the luminosity of the light source. The derivation ended up with exact same result as equation (1.1).

7.2 Suggestion for further study

7.2.1 Investigate fulfillment of Lorentz gauge condition

Since the Green's functions ability to solve Maxwell's equations depends on whether they produce Lorentz gauged four-potentials out of conserved four-currents, it is worth investigating whether the Lorentz gauge actually is fulfilled. I recommend starting by checking if the two conditions (6.5) are true. They are sufficient conditions, but maybe not necessary.

7.2.2 FRW universe with $k = \pm 1$

In this master thesis, only the flat FRW universe ($k = 0$) was considered. One can do the same calculations for open ($k = +1$) and closed ($k = -1$) FRW universe. Maybe it gives a different result than equation (1.1).

7.2.3 Scattering by the tail term

Gravity scatters light according to Maxwell's equations. When light is scattered, objects appears dimmer because some light scatters away. But on the other hand, an observer sees noise that he/she would not see if it wasn't for the scattering. However, it seems unlikely that the gravity of the flat FRW universe contributes significant extra noise due to scattering from the discussion in section 6.1.1. The fact that objects appear dimmer because of scattering is automatically taken care of by the singular part of the Green's function (4.49). But it may be that strong local gravity which the FRW metric does not describe can influence the propagation of light and thereby change the LLR. This is a difficult problem since it is hard to do exact calculations. In an article by R. Mankin and T. Laas, R. Tammelo [7] a perturbative method of describing scattering by the tail term is developed. A suggestion for further study is to determine if the the tail term V_{-} can give significant contribution to observed light intensity due to strong local gravity the light encounters.

Appendix A

Tail of Green's function restricted on light cone

Recall equation (4.47).

$$W_0{}^{\mu}|^{\nu}(p,q) \equiv -\frac{1}{2}\int_0^1 \frac{\kappa(p,q)}{\kappa(p,\gamma(\lambda))} P^{\mu}|_{\alpha}(p,\gamma(\lambda)) \overset{\text{left}}{\mathcal{L}} U^{\alpha}|^{\nu}(\gamma(\lambda),q)d\lambda$$

This is the tail part of the Green's function restricted on the light cone $C_{\pm}(q)$. The formula is explicit, but hard to calculate by hand. I used Maxima to calculate it. The Maxima commands I wrote is contained in a .wxmx file that can be downloaded here:
https://github.com/Vikhamar/The-Hubble-tension-and-electromagnetic-waves-in-flat-FRW-universe/blob/main/calculationofW0.wxmx
It can be opened in the free Maxima interface wxMaxima. W_0 depends only on cosmic time, so we can write $W_0(t_0,t_1)$ instead of $W_0(q,p)$. Here is the result:

$$\begin{aligned} W_0{}^0|^0(t_0,t_1) = &-\frac{\ln\left(\frac{a(t_1)}{a(t_0)}\right)}{2r^2a\left(t_1\right)^2} - \frac{3\left|X_0\right|H\left(t_1\right)}{4X_0ra\left(t_1\right)} + \frac{\left|X_0\right|\left(\int_{t_0}^{t_1}\frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^3}\,dt\right)a\left(t_0\right)^2}{4X_0ra\left(t_1\right)^2} \\ &-\frac{\left|X_0\right|\left(\int_{t_0}^{t_1}\frac{1}{a(t)^3}\,dt\right)a\left(t_0\right)^2}{4X_0r^3a\left(t_1\right)^2} + \frac{\left|X_0\right|H\left(t_0\right)a\left(t_0\right)}{4X_0ra\left(t_1\right)^2} + \frac{\left|X_0\right|\int_{t_0}^{t_1}\frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)}\,dt}{2X_0ra\left(t_1\right)^2} \\ &+\frac{1}{4r^2a\left(t_1\right)^2} - \frac{\left|X_0\right|a\left(t_0\right)^2H\left(t_1\right)}{4X_0ra\left(t_1\right)^3} - \frac{a\left(t_0\right)^2}{8r^2a\left(t_1\right)^4} - \frac{\left|X_0\right|\left(\int_{t_0}^{t_1}\frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^5}\,dt\right)a\left(t_0\right)^2}{2X_0r} \\ &+\frac{\left|X_0\right|\left(\int_{t_0}^{t_1}\frac{1}{a(t)^5}\,dt\right)a\left(t_0\right)^2}{4X_0r^3} + \frac{3\left|X_0\right|H\left(t_0\right)}{4X_0ra\left(t_0\right)} + \frac{1}{8r^2a\left(t_0\right)^2} \end{aligned} \tag{A.1}$$

$$
- \frac{3\left|X_0\right| \int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^3}\, dt}{4X_0 r} - \frac{\left|X_0\right| \int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt}{4X_0 r^3}
$$

$$
\begin{aligned}
W_0{}^0|^j(t_0,t_1) = \Bigg(& - \frac{\left|X_0\right| \ln\left(\frac{a(t_1)}{a(t_0)}\right)}{2X_0 r^2 a\left(t_1\right)^2} - \frac{3H\left(t_1\right)}{4ra\left(t_1\right)} - \frac{\left(\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^3}\, dt\right) a\left(t_0\right)^2}{4ra\left(t_1\right)^2} \\
& + \frac{3\left(\int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt\right) a\left(t_0\right)^2}{4r^3 a\left(t_1\right)^2} + \frac{H\left(t_0\right) a\left(t_0\right)}{4ra\left(t_1\right)^2} - \frac{\left(\int_{t_0}^{t_1} \frac{1}{a(t)^2}\, dt\right) a\left(t_0\right)}{r^3 a\left(t_1\right)^2} + \frac{\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)}\, dt}{2ra\left(t_1\right)^2} \\
& + \frac{\left|X_0\right|}{X_0 r^2 a\left(t_1\right)^2} + \frac{X_0}{4\left|X_0\right| r^2 a\left(t_1\right)^2} + \frac{a\left(t_0\right)^2 H\left(t_1\right)}{4ra\left(t_1\right)^3} - \frac{4\left|X_0\right| a\left(t_0\right)}{3X_0 r^2 a\left(t_1\right)^3} + \frac{\left|X_0\right| a\left(t_0\right)^2}{8X_0 r^2 a\left(t_1\right)^4} \\
& + \frac{\left(\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^5}\, dt\right) a\left(t_0\right)^2}{2r} - \frac{\left(\int_{t_0}^{t_1} \frac{1}{a(t)^5}\, dt\right) a\left(t_0\right)^2}{4r^3} + \frac{\left(\int_{t_0}^{t_1} \frac{1}{a(t)^4}\, dt\right) a\left(t_0\right)}{r^3} + \frac{H\left(t_0\right)}{4ra\left(t_0\right)} \\
& + \frac{5\left|X_0\right|}{24X_0 r^2 a\left(t_0\right)^2} - \frac{3\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^3}\, dt}{4r} - \frac{3\int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt}{4r^3} \Bigg) \hat{X}^j
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
W_0{}^i|^0(t_0,t_1) = \Bigg(& - \frac{\left|X_0\right| \ln\left(\frac{a(t_1)}{a(t_0)}\right)}{2X_0 r^2 a\left(t_1\right)^2} + \frac{H\left(t_1\right)}{4ra\left(t_1\right)} + \frac{\left(\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^3}\, dt\right) a\left(t_0\right)^2}{4ra\left(t_1\right)^2} \\
& - \frac{\left(\int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt\right) a\left(t_0\right)^2}{4r^3 a\left(t_1\right)^2} + \frac{H\left(t_0\right) a\left(t_0\right)}{4ra\left(t_1\right)^2} + \frac{\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)}\, dt}{2ra\left(t_1\right)^2} + \frac{X_0}{4\left|X_0\right| r^2 a\left(t_1\right)^2} \\
& + \frac{a\left(t_0\right)^2 H\left(t_1\right)}{4ra\left(t_1\right)^3} + \frac{\left|X_0\right| a\left(t_0\right)^2}{8X_0 r^2 a\left(t_1\right)^4} + \frac{\left(\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^5}\, dt\right) a\left(t_0\right)^2}{2r} - \frac{\left(\int_{t_0}^{t_1} \frac{1}{a(t)^5}\, dt\right) a\left(t_0\right)^2}{4r^3} \\
& - \frac{3H\left(t_0\right)}{4ra\left(t_0\right)} - \frac{\left|X_0\right|}{8X_0 r^2 a\left(t_0\right)^2} + \frac{3\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt}a(t)\right)^2}{a(t)^3}\, dt}{4r} + \frac{\int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt}{4r^3} \Bigg) \hat{X}^i
\end{aligned} \tag{A.3}
$$

$$
W_0^i|^j(t_0,t_1) = \Bigg(- \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{1}{a(t)^4}\, dt\right) a\left(t_0\right)^2}{2X_0 r^3 a\left(t_1\right)} + \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt\right) a\left(t_0\right)}{X_0 r^3 a\left(t_1\right)} + \frac{1}{3r^2 a\left(t_0\right) a\left(t_1\right)} \tag{A.4}
$$

$$
\begin{aligned}
&- \frac{\left|X_0\right| \int_{t_0}^{t_1} \frac{1}{a(t)^2}\, dt}{2X_0 r^3 a\left(t_1\right)} - \frac{1}{2r^2 a\left(t_1\right)^2} + \frac{a\left(t_0\right)^2}{6r^2 a\left(t_1\right)^4}\Bigg)\delta^{ij} \\
&+ \Bigg(- \frac{\ln\left(\frac{a(t_1)}{a(t_0)}\right)}{2r^2 a\left(t_1\right)^2} + \frac{\left|X_0\right| H\left(t_1\right)}{4X_0 r a\left(t_1\right)} + \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{1}{a(t)^4}\, dt\right) a\left(t_0\right)^2}{2X_0 r^3 a\left(t_1\right)} \\
&- \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt\right) a\left(t_0\right)}{X_0 r^3 a\left(t_1\right)} - \frac{1}{3r^2 a\left(t_0\right) a\left(t_1\right)} + \frac{\left|X_0\right| \int_{t_0}^{t_1} \frac{1}{a(t)^2}\, dt}{2X_0 r^3 a\left(t_1\right)} \\
&- \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt} a(t)\right)^2}{a(t)^3}\, dt\right) a\left(t_0\right)^2}{4X_0 r a\left(t_1\right)^2} + \frac{3\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt\right) a\left(t_0\right)^2}{4X_0 r^3 a\left(t_1\right)^2} + \frac{\left|X_0\right| H\left(t_0\right) a\left(t_0\right)}{4X_0 r a\left(t_1\right)^2} \\
&- \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{1}{a(t)^2}\, dt\right) a\left(t_0\right)}{X_0 r^3 a\left(t_1\right)^2} + \frac{\left|X_0\right| \int_{t_0}^{t_1} \frac{\left(\frac{d}{dt} a(t)\right)^2}{a(t)}\, dt}{2X_0 r a\left(t_1\right)^2} + \frac{7}{4r^2 a\left(t_1\right)^2} \\
&- \frac{\left|X_0\right| a\left(t_0\right)^2 H\left(t_1\right)}{4X_0 r a\left(t_1\right)^3} - \frac{2a\left(t_0\right)}{3r^2 a\left(t_1\right)^3} - \frac{7a\left(t_0\right)^2}{24r^2 a\left(t_1\right)^4} - \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{\left(\frac{d}{dt} a(t)\right)^2}{a(t)^5}\, dt\right) a\left(t_0\right)^2}{2X_0 r} \\
&+ \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{1}{a(t)^5}\, dt\right) a\left(t_0\right)^2}{4X_0 r^3} - \frac{\left|X_0\right| \left(\int_{t_0}^{t_1} \frac{1}{a(t)^4}\, dt\right) a\left(t_0\right)}{X_0 r^3} - \frac{\left|X_0\right| H\left(t_0\right)}{4X_0 r a\left(t_0\right)} - \frac{5}{24r^2 a\left(t_0\right)^2} \\
&+ \frac{3\left|X_0\right| \int_{t_0}^{t_1} \frac{\left(\frac{d}{dt} a(t)\right)^2}{a(t)^3}\, dt}{4X_0 r} + \frac{3\left|X_0\right| \int_{t_0}^{t_1} \frac{1}{a(t)^3}\, dt}{4X_0 r^3}\Bigg)\hat{X}^i \hat{X}^j
\end{aligned}
$$

Appendix B

Maxima command

Listing B.1: Maxima code

```
(%i1)  declare(integrate, linear);
(%i2)  hXt1:X0^2-X^2+a0^2/a1^2*X^2;
       /* Substitute the integrals */;
(%i3)  A:subst(
           integrate(hX(t)^(-3/2),t,t0,t1)=g2,
           expand(X0*hXt1^(1/2)*integrate(hX(t)^(-3/2), t, t0,t1)));
(%i4)  B:subst([
           integrate(hX(t)^(-3/2),t,t0,t1)=g2,
           integrate(1/a(t)^2*hX(t)^(-3/2),t,t0,t1)=g3],
           expand(X*hXt1^(1/2)*integrate((a0^2/a(t)^2-1)*hX(t)^(-3/2),t,t0,t1)));
(%i5)  C:subst([
           integrate(hX(t)^(-3/2),t,t0,t1)=g2,
           integrate(1/a(t)^2*hX(t)^(-3/2),t,t0,t1)=g3],
           expand(abs(X0)*X*integrate((a0/a1-a0*a1/a(t)^2)*hX(t)^(-3/2), t, t0,t1)));
(%i6)  D:subst([
           integrate(1/a(t)^2*hX(t)^(-1/2),t,t0,t1)=g1],
           expand(X0/abs(X0)*integrate(a0*a1/a(t)^2*hX(t)^(-1/2),t,t0,t1)));
(%i7)  E:subst([
           integrate(1/a(t)^2*hX(t)^(-1/2),t,t0,t1)=g1,
           integrate(hX(t)^(-3/2),t,t0,t1)=g2,
           integrate(1/a(t)^2*hX(t)^(-3/2),t,t0,t1)=g3,
           integrate(1/a(t)^4*hX(t)^(-3/2),t,t0,t1)=g4],
           expand(X0/abs(X0)*X^2*integrate((a0/a1-a0*a1/a(t)^2)*(a0^2/a(t)^2-1)*hX(t)^(-3/2),t,t0,t1)));
       /* The substitutions g1, g2, g3 and g4 are not linearly independent. Substiture relations. */
(%i8)  detg:radcan(subst([
           g4=1/a0^2/X^2*(g1+(-X0^2+X^2)*g3),
           g2=1/(X0^2-X^2)*(X0/abs(X0) - a0^2*X^2*g3)],
           expand(((-A^2+C^2)*(-B^2+(D+E)^2) - (-A*B+C*(D+E))^2)*D^4)));
       /* Substitute integrals back */
(%i9)  detg:subst([
           g1=integrate(1/a(t)^2*hX(t)^(-1/2),t,t0,t1),
           g3=integrate(1/a(t)^2*hX(t)^(-3/2),t,t0,t1)],
           detg);
```

This code can be find in the file vanvleck.wxmx in the github repository
https://github.com/Vikhamar/The-Hubble-tension-and-electromagnetic-waves-in-flat-FRW-universe/blob/main/vanvleck.wxmx.

Appendix C

Proofs

Proposition 3 (Maximum over compact space preserves continuity)
Let X and K be topological spaces and K be non-empty and compact. Let $X \times K \xrightarrow{f} \mathbb{R}$ be continuous. Then

$$\begin{aligned} X &\xrightarrow{f_{\max}} \mathbb{R} \\ x &\mapsto \max_{k \in K} f(x, k) \end{aligned}$$

is continuous.

Proof.
First note that the maxima exist because K is non-empty and compact, and f is continuous wrt. K. To prove that $f_{\max}$ is continuous, let $x_0 \in X$ and let $\epsilon > 0$. Need to show $\exists U \in \tau_X$ so that $x_0 \in U \subseteq f_{\max}^{-1}(\langle f_{\max}(x_0) - \epsilon, f_{\max}(x_0) + \epsilon \rangle)$. The maxima of f exist, so $\exists k_0 \in K$ so that $f(x_0, k_0) = f_{\max}(x_0)$. Since $f_{\max}(x_0) \geq f(x_0, k) \quad \forall k \in K$, the slice $\{x_0\} \times K \subseteq f^{-1}(\langle -\infty, f_{\max}(x_0) + \epsilon \rangle)$. Since f is continuous on $\{x_0\} \times K$, $\forall k \in K\ \exists U_k \in \tau_X$ and $\exists V_k \in \tau_K$ so that $(x_0, k) \in U_k \times V_k \subseteq f^{-1}(\langle -\infty, f_{\max}(x_0) + \epsilon \rangle)$. $\{V_k \mid k \in K\}$ is an open cover of K. By compactness, there is a finite subcover $\{V_{k_1}, \ldots, V_{k_N}\}$. Since f is continuous wrt. X at x_0, $\exists U_0 \in \tau_X$ so that $x_0 \in U_0$ and $U_0 \times \{k_0\} \subseteq f^{-1}(\langle f_{\max}(x_0) - \epsilon, f_{\max}(x_0) + \epsilon \rangle)$. Set $U \equiv U_0 \cap \bigcap_{n=1}^{N} U_{k_n}$. Then $x_0 \in U$ and $U \times K \subseteq f^{-1}(\langle -\infty, f_{\max}(x_0) + \epsilon \rangle)$. Let now $x \in U$. Since $x \in \bigcap_{n=1}^{N} U_{k_n}$, $f(x, k) \in \langle -\infty, f_{\max}(x_0) + \epsilon \rangle \quad \forall k \in K$, and since $f_{\max}$ is a maximum (not just a supremum), $f_{\max}(x) \in \langle -\infty, f_{\max}(x_0) + \epsilon \rangle$. Since $x \in U_0$, $f(x, k_0) \in \langle f_{\max}(x_0) - \epsilon, f_{\max}(x_0) + \epsilon \rangle$, and since $f_{\max}(x) \geq f(x, k_0)$, $f_{\max}(x) \in \langle f_{\max}(x_0) - \epsilon, \infty \rangle$. We have $x_0 \in U \subseteq f_{\max}^{-1}(\langle -\infty, f_{\max}(x_0) + \epsilon \rangle)$. $f_{\max}$ is continuous. $\square$

Proof of lemma 4.
Evaluation is continuous by the universal property of product topology as seen from the diagram

$$
\begin{tikzcd}
V' \arrow[d, hook] \arrow[dr, "{\langle \cdot , v \rangle}"] & \\
\mathbb{R}^V \arrow[r, "\pi_v"] & \mathbb{R}
\end{tikzcd}
.
$$

It says namely that $V' \hookrightarrow \mathbb{R}^V$ is continuous $\iff \langle \cdot, v\rangle$ is continuous $\forall v \in V$. Continuity of scalar multiplication and vector addition follow from the universal property of product topology and subspace topology, from that multiplication and addition of real numbers are continuous and from that the following two diagrams

$$
\begin{tikzcd}
 & \mathbb{R} \times V' \arrow[dl, "\cdot"'] \arrow[d] \arrow[r, "{\mathrm{id} \times \langle \cdot , v \rangle}"] & \mathbb{R} \times \mathbb{R} \arrow[d, "\cdot"] \\
V' \arrow[r, hook] & \mathbb{R}^V \arrow[r, "\pi_v"] & \mathbb{R}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & V' \times V' \arrow[dl, "+"'] \arrow[d] \arrow[r, "{\langle \cdot , v \rangle \times \langle \cdot , v \rangle}"] & \mathbb{R} \times \mathbb{R} \arrow[d, "+"] \\
V' \arrow[r, hook] & \mathbb{R}^V \arrow[r, "\pi_v"] & \mathbb{R}
\end{tikzcd}
$$

commute $\forall v \in V$. Since $\mathbb{R}$ is Hausdorff, and any product of Hausdorff spaces is Hausdorff, and any subspace of a Hausdorff space is Hausdorff, V' is Hausdorff. $\square$

Proof of lemma 5.
Since $\langle T' \cdot, v\rangle = \langle \cdot, Tv\rangle$ and evaluation $\langle \cdot, w\rangle$ is continuous $\forall w \in W'$, $\langle T' \cdot, v\rangle$ is continuous $\forall v \in V$. From the universal property of product topology and subspace topology

$$
\begin{tikzcd}
 & W' \arrow[dl, "T'"'] \arrow[d] \arrow[dr, "{\langle T' \cdot , v \rangle}"] & \\
V' \arrow[r, hook] & \mathbb{R}^V \arrow[r, "\pi_v"] & \mathbb{R}
\end{tikzcd}
$$

T' is continuous $\iff \langle T' \cdot, v\rangle$ is continuous $\forall v \in V$. So T' is continuous. $\square$

Proof of well-definedness in definition 4.
<u>Existence of $\tilde{\phi}$:</u>
Since $\operatorname{supp} \phi \cap \operatorname{supp} u$ is compact and U_ϕ is locally compact, $\exists V \in \tau_{U_\phi}$ so that $\operatorname{supp} \phi \cap \operatorname{supp} u \subseteq V$ and $\overline{V}$ is compact. And since $\overline{V}$ is compact, $\exists W \in \tau_{U_\phi}$ so that $\overline{V} \subseteq W$ and $\overline{W}$ is compact. Choose a

partition of unity $\{\rho_i\}_{i\in I}$ subordinate to the open cover $\left\{V, \mathcal{M}\setminus\overline{W}\right\}$. Set

$$\rho \equiv \sum_{\substack{i\in I\\ \operatorname{supp}\rho_i\subseteq V}} \rho_i \qquad \text{and} \qquad \tilde{\phi}(p) \equiv \begin{cases} \rho(p)\cdot\phi(p), & p\in U_\phi \\ 0, & p\in\mathcal{M}\setminus\overline{W}. \end{cases}$$

Since $\rho\big|_{\overline{V}} = 1$ and $\rho\big|_{\mathcal{M}\setminus\overline{W}} = 0$, the two expressions for $\tilde{\phi}$ agree on the value zero where their domains overlap. Since U_ϕ and $\mathcal{M}\setminus\overline{W}$ are both open, and both expressions are smooth, the gluing lemma for smooth functions says that $\tilde{\phi}$ is smooth. Since $\overline{W}$ is compact and $\operatorname{supp}\tilde{\phi}\subseteq\operatorname{supp}\rho\subseteq W$, $\operatorname{supp}\tilde{\phi}$ is compact. We conclude that $\tilde{\phi}\in\mathcal{D}_{(r,s)}(\mathcal{M})$, $\phi=\tilde{\phi}$ on V which is a neighbourhood of $\operatorname{supp}u\cap\operatorname{supp}\phi$ and $\operatorname{supp}\tilde{\phi}\subseteq\operatorname{supp}\phi$.

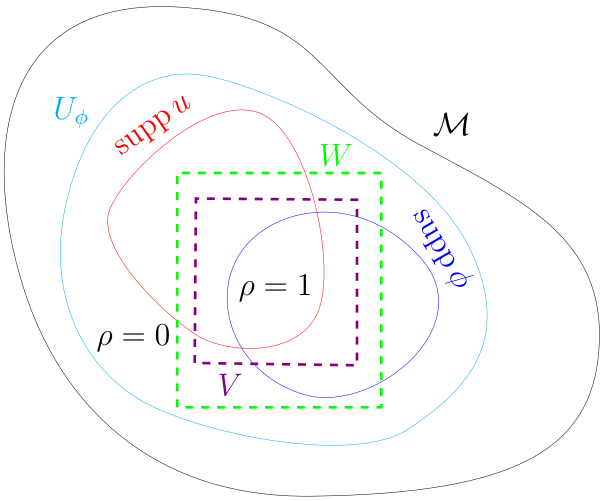

Figure C.1: Overview over which sets containing/intersecting which sets.

Independence of choice of $\tilde{\phi}$:
Let $\tilde{\phi}_1, \tilde{\phi}_1 \in \mathcal{D}_{(r,s)}(\mathcal{M})$ be two test functions so that $\tilde{\phi}_1 = \tilde{\phi}_2$ on a neighbourhood V of $\operatorname{supp}u\cap\operatorname{supp}\phi$ and $\operatorname{supp}\tilde{\phi}_1\cup\operatorname{supp}\tilde{\phi}_2\subseteq\operatorname{supp}\phi$. Need to show that $\left\langle u,\tilde{\phi}_1\right\rangle = \left\langle u,\tilde{\phi}_2\right\rangle$. Set $\psi\equiv\tilde{\phi}_1-\tilde{\phi}_2$. Since ψ is zero on V and V is open, $\operatorname{supp}\psi\cap V=\emptyset$. Since $\operatorname{supp}\tilde{\phi}_1\cup\operatorname{supp}\tilde{\phi}_2\subseteq\operatorname{supp}\phi$, $\operatorname{supp}\psi\subseteq\operatorname{supp}\phi\setminus V$. That means $\operatorname{supp}\psi\cap\operatorname{supp}u=\emptyset$ and $\psi\big|_{\mathcal{M}\setminus\operatorname{supp}u}\in\mathcal{D}_{(r,s)}\left(\mathcal{M}\setminus\operatorname{supp}u\right)$. From the definition support of u, $u\big|_{\mathcal{M}\setminus\operatorname{supp}u}=0$. That means $\langle u,\psi\rangle = \left\langle u\big|_{\mathcal{M}\setminus\operatorname{supp}u}, \psi\big|_{\mathcal{M}\setminus\operatorname{supp}u}\right\rangle = 0$. So $\left\langle u,\tilde{\phi}_1\right\rangle = \left\langle u,\tilde{\phi}_2\right\rangle$. $\square$

Proof for that definition (3.13) is well-defined.
Existence of ρ:
Since manifolds are normal Hausdorff, and $\operatorname{singsupp}u$ and $\operatorname{singsupp}v$ both are closed, $\exists$ open $U\subseteq\mathcal{M}$

containing $\operatorname{singsupp} v$ so that $\operatorname{singsupp} u \cap U = \emptyset$. ρ can be constructed by a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to the open cover $\{U, \mathcal{M} \setminus \operatorname{singsupp} v\}$ of $\mathcal{M}$. Set

$$\rho \equiv \sum_{\substack{i\in I \\ \operatorname{supp} \rho_i \subseteq U}} \rho_i .$$

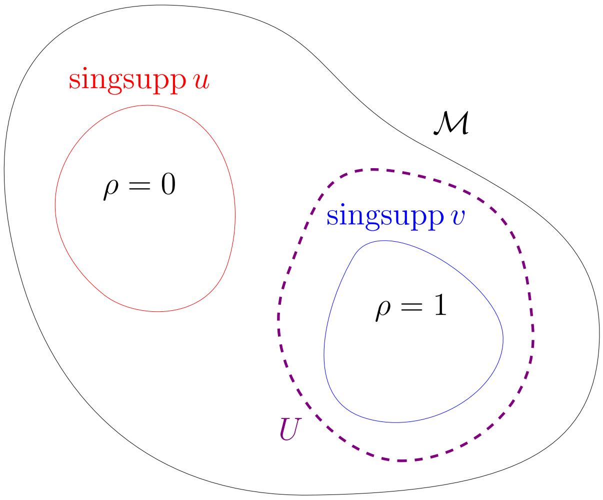

Figure C.2: Overview over which sets containing/being disjoint from which sets.

<u>Independence of choice of ρ:</u>
Let $\rho_1, \rho_2 \in C^\infty(\mathcal{M})$ be two functions that are both 0 on $\operatorname{singsupp} u$ and 1 on $\operatorname{singsupp} v$. Then

$$\begin{aligned}
&\left(\langle v, \rho_1 u\phi\rangle + \langle u, (1-\rho_1)v\phi\rangle\right) - \left(\langle v, \rho_2 u\phi\rangle + \langle u, (1-\rho_2)v\phi\rangle\right)\\
=&\langle v, \underbrace{(\rho_1-\rho_2)}_{\substack{0 \text{ on} \\ \operatorname{singsupp} u \,\cup\, \operatorname{singsupp} v}} u\phi\rangle + \langle u, \underbrace{(\rho_2-\rho_1)}_{\substack{0 \text{ on} \\ \operatorname{singsupp} u \,\cup\, \operatorname{singsupp} v}} v\phi\rangle\\
=&\int_V v(\rho_1-\rho_2)u\phi\omega - \int_V u(\rho_2-\rho_1)v\phi\omega\\
=&0, \qquad \text{where } V \equiv \mathcal{M} \setminus \left(\operatorname{singsupp} u \cup \operatorname{singsupp} v\right).
\end{aligned}$$

$\square$

Proof of proposition 1.
<u>$D_\pm\left(J_\pm(p)\right) = D_\pm(p)$:</u>
<u>$D_\pm\left(J_\pm(p)\right) \supseteq D_\pm(p)$</u> holds because $p \in J_\pm(p)$. To show $D_\pm\left(J_\pm(p)\right) \subseteq D_\pm(p)$, let $p_0 \in J_\pm(p)$ and

$p_1 \in D_\pm(p_0)$. Need to show that $p_1 \in D_\pm(p)$. If $p_0 = p$, there is nothing to show. Let therefore $p_0 \neq p$. Let $[0,1] \xrightarrow{\gamma} \Omega$ be the geodesic from $p_0 = \gamma(0)$ to $p_1 = \gamma(1)$. Define

$$[0,1] \xrightarrow{f} \mathbb{R}$$
$$\lambda \mapsto \sigma(p, \gamma(\lambda)).$$

Then $f(0) = 0$ and $f'(0) = \langle \underbrace{d\sigma_p(\gamma(0))}_{\pm\text{-directed}}, \overbrace{\dot\gamma(0)}^{\text{timelike and }\pm\text{-directed}} \rangle < 0$, since $p_0 \in J_\pm(p)$ and $p_1 \in D_\pm(p_0)$. If we can show that $f\big|_{\langle 0,1]} < 0$, then $\gamma(\langle 0,1]) \subseteq D(p)$. And since $p_0 \in J_\pm(p) \setminus \{p\}$ together with that the path-connected components of $J(p) \setminus \{p\}$ are $J_-(p) \setminus \{p\}$ and $J_+(p) \setminus \{p\}$, we must have that $\gamma([0,1]) \subseteq J_\pm(p) \setminus \{p\}$, and hence $p_1 \in D_\pm(p)$. Assume therefore $f\big|_{\langle 0,1]} \not< 0$. Then the set $\{\lambda \in \langle 0,1] \mid f(\lambda) \geq 0\}$ is non-empty. Define $\lambda_2 \equiv \inf\{\lambda \in \langle 0,1] \mid f(\lambda) \geq 0\}$. Now, since γ is C^1, f' is continuous and since $f'(0) < 0$, $\exists \lambda_0 \in \langle 0,1]$ so that $f'\big|_{[0,\lambda_0\rangle} < 0$. Since $f(0) = 0$, $f\big|_{\langle 0,\lambda_0]} < 0$. That means $\lambda_0 < \lambda_2$. The mean value theorem says that $\exists \lambda_1 \in \langle \lambda_0, \lambda_2\rangle$ so that $f(\lambda_2) - f(\lambda_0) = (\lambda_2 - \lambda_0) \cdot f'(\lambda_1)$. By the definition of λ_2, $f(\lambda_2) \geq 0$. That means $f(\lambda_2) - f(\lambda_0) > 0$ and hence $f'(\lambda_1) > 0$ $(*)$. From the definition of λ_2, $f\big|_{[0,\lambda_2]} \leq 0$. That means $\gamma([0,\lambda_2]) \subseteq J(p) \setminus \{p\}$. Again, since the path-connected components of $J(p) \setminus \{p\}$ are $J_-(p) \setminus \{p\}$ and $J_+(p) \setminus \{p\}$, and $\gamma(0) \in J_\pm(p) \setminus \{p\}$, $\gamma([0,\lambda_2]) \subseteq J_\pm(p)$. Since $\gamma(\lambda_1) \in J_\pm(p)$ and parallel transport preserves time orientation (4.19), $d\sigma_p(\gamma(\lambda_1)) = P(\gamma(\lambda_1), p) \exp_p^{-1}(\gamma(\lambda_1))$ is causal and $\pm$-directed. Since $d\sigma_p(\gamma(\lambda_1))$ and $\dot\gamma(\lambda_1)$ have same time orientation and one is causal and one is timelike, $f'(\lambda_1) = \langle d\sigma_p(\gamma(\lambda_1)), \dot\gamma(\lambda_1)\rangle < 0$, which is *contradiction* to $(*)$. That concludes $D_\pm(J_\pm(p)) \subseteq D_\pm(p)$.

<u>$J_\pm(D_\pm(p)) = D_\pm(p)$:</u>
<u>$J_\pm(D_\pm(p)) \supseteq D_\pm(p)$</u> holds because $p' \in J_\pm(p') \quad \forall p' \in \Omega$. To show $J_\pm(D_\pm(p)) \subseteq D_\pm(p)$, let $p_0 \in D_\pm(p)$ and $p_1 \in J_\pm(p_0)$. Need to show that $p_1 \in D_\pm(p)$. We have that $p \in D_\mp(p_0)$ and $p_0 \in J_\mp(p_1)$. So $p \in D_\mp(J_\mp(p_1))$. Has already shown that $D_\pm(J_\pm(p')) \subseteq D_\pm(p') \quad \forall p' \in \Omega$. By time reversing it, $D_\mp(J_\mp(p_1)) \subseteq D_\mp(p_1)$. So $p \in D_\mp(p_1)$. That means $p_1 \in D_\pm(p)$.

<u>$D_\pm(D_\pm(p)) = D_\pm(p)$:</u>
Has already shown $J_\pm(D_\pm(p)) \subseteq D_\pm(p)$, so $D_\pm(D_\pm(p)) \subseteq D_\pm(p)$. To show $D_\pm(D_\pm(p)) \supseteq D_\pm(p)$, let $p_1 \in D_\pm(p)$. Set $X \equiv \exp_p^{-1}(p_1)$. Set $p_0 \equiv e^{X/2}$. Then $p_0 \in D_\pm(p)$ and $p_1 \in D_\pm(p_0)$, so $p_1 \in D_\pm(D_\pm(p))$.

<u>$J_\pm(p) = J_\pm(J_\pm(p))$:</u>
Since $p' \in J_\pm(p') \quad \forall p' \in \Omega$, $J_\pm(p) \subseteq J_\pm(J_\pm(p))$. □

Proof of proposition 2.
By definition of $D_\pm(p)$, $\Omega = \bigcup_{p\in\Omega} D_\pm(p) \subseteq \Omega$. To show inclusion the other way, let $p \in \Omega$. Need to show $\exists p' \in \Omega$ so that $p \in D_\pm(p')$. Since $\exp_p$ is defined on a neighbourhood of $0 \in T_p\Omega$, $\exists$ $\mp$-directed timelike $X \in T_p\Omega$ so that $e^X \in \Omega$. Set $p' \equiv e^X$. Then $p' \in D_\mp(p)$ and hence $p \in D_\pm(p')$. □

Proof of lemma 6.
Let $K \subseteq \Omega$ be compact. Since $\Omega = \bigcup_{q\in\Omega} D_\mp(q)$ (proposition 2), $\forall k \in K\ \exists q_k \in \Omega$ so that $k \in D_\mp(q_k)$. This means $\{D_\mp(q_k) \cap K \mid k \in K\}$ is an open cover of K. Since K is compact, $\exists$ finitely many $p_1, \dots, p_N \in \Omega$ so that $K \subseteq \bigcup_{i=1}^k D_\mp(p_i)$. Then

$$J_\mp(K) \subseteq J_\mp\left(\bigcup_{i=1}^N D_\mp(p_i)\right) = \bigcup_{i=1}^N J_\mp\left(D_\mp(p_i)\right) \subseteq \bigcup_{i=1}^N J_\mp(p_i),$$

where the last inclusion is from proposition 1. $B \cap J_\mp(K) \subseteq B \cap \bigcup_{i=1}^N J_\mp(p_i) = \bigcup_{i=1}^N B \cap J_\mp(p_i)$ which is a finite union of compact sets and hence is compact. To show that $B \cap J_\mp(K)$ is compact, it remains to show that both B and $J_\mp(K)$ are closed in Ω. B is closed in Ω since it is $\mp$-compact. Now we show that $J_\mp(K)$ is closed. Since Ω is time orientable, $\exists$ timelike $\pm$-directed continuous vector field $T \in \Gamma(T\Omega)$. If $K = \emptyset$, $J_\mp(K) = \emptyset$ is closed. Let $K \neq \emptyset$. Notice that $p \in J_\mp(K)$ precisely when $\sigma(p,k) \leq 0$ and $\left\langle T(p), \exp_p^{-1}(k)\right\rangle \leq 0$ for a $k \in K$. Define therefore the map

$$\begin{aligned} \Omega &\xrightarrow{f} \mathbb{R} \\ p &\mapsto \min_{k\in K} \max\left\{\langle \sigma(p,k), \left\langle T, \exp_p^{-1}(k)\right\rangle\right\}. \end{aligned}$$

Since

$$\begin{aligned} \Omega \times \Omega &\xrightarrow{\exp^{-1}} T\Omega \\ (p,q) &\mapsto \exp_p^{-1}(q) \end{aligned}$$

is continuous wrt. the product topology on $\Omega \times \Omega$, f is continuous (see prop. 3 in append C). Note that the minima exist because $\exp^{-1}$ is continuous in second argument and K is compact. We have that $J_\mp(K) = f^{-1}(\langle -\infty, 0])$. Since f is continuous, $J_\mp(K)$ is closed. $\square$

Proof of lemma 7.
Let $p \in \Omega$. Need to show that $J_\pm(B) \cap J_\mp(p)$ is compact. Because $J_\pm\left(J_\pm(p')\right) = J_\pm(p') \quad \forall p' \in \Omega$, whenever $p' \notin J_\mp(p)$, $J_\pm(p') \cap J_\mp(p) = \emptyset$. This means

$$\begin{aligned} J_\pm(B) \cap J_\mp(p) &= \bigcup_{p'\in B} J_\pm(p') \cap J_\mp(p) \\ &= \bigcup_{p'\in B\cap J_\mp(p)} J_\pm(p') \cap J_\mp(p) \\ &= J_\mp(p) \cap J_\pm\left(B \cap J_\mp(p)\right). \end{aligned}$$

Since Ω is a causal domain, $J_\mp(p)$ is $\pm$-compact. Since B is $\mp$-compact, $K \equiv B \cap J_\mp(p)$ is compact. By lemma 6, $J_\pm(B) \cap J_\mp(p) = J_\mp(p) \cap J_\pm(K)$ is compact. $\square$

Proof of theorem 4.56.
To prove the theorem, it suffices show the following:

1. $\mathcal{D}'_{\pm,(1,0)}(\Omega) \xrightarrow{G_\pm} \mathcal{D}'_{\pm,(1,0)}(\Omega)$ is well-defined on its domain.

2. $\mathcal{T}_{\pm,(1,0)}(\Omega) \xrightarrow{G_\pm} \mathcal{T}_{\pm,(1,0)}(\Omega)$ and $\mathcal{D}'_{\pm,(1,0)}(\Omega) \xrightarrow{G_\pm} \mathcal{D}'_{\pm,(1,0)}(\Omega)$ hit within their codomain, and equation (4.56) holds.

3. $\begin{tikzcd} \mathcal{D}'_{\pm,(1,0)}(\Omega) \arrow[r, "G_\pm"] \arrow[rr, bend left, "\mathrm{id}"] & \mathcal{D}'_{\pm,(1,0)}(\Omega) \arrow[r, "\mathcal{L}"] & \mathcal{D}'_{\pm,(1,0)}(\Omega) \end{tikzcd}$ commutes.

4. $\mathcal{D}'_{\pm,(1,0)}(\Omega) \xrightarrow{\mathcal{L}} \mathcal{D}'_{\pm,(1,0)}(\Omega)$ is injective.

<u>1.</u>
Let $u \in \mathcal{D}'_{\pm,(1,0)}(\Omega)$. To show that $G_\pm u$ is well-defined, we need to show that $\langle G_\pm u, \phi\rangle = \left\langle u, G'_\pm \phi\right\rangle$ is a well-defined number $\forall \phi \in \mathcal{D}_{(1,0)}(\Omega)$ and that $G_\pm u$ is linear in ϕ. A priori, u is not defined on $G'_\pm \phi$ because it is not a test function. But following definition 4, $\left\langle u, G'_\pm \phi\right\rangle$ is a well-defined number if $G'_\mp \phi$ is smooth and $\operatorname{supp} u \cap \operatorname{supp} G'_\pm \phi$ is compact $\forall \phi \in \mathcal{D}_{(1,0)}(\Omega)$. To see that $G'_\mp \phi$ is smooth, pick a local orthonormal frame to write the integrals in equation (4.55) in normal coordinates wrt. the frame. Then one can see that smoothness follows from that the bitensors fields U, W and the exponential map are smooth. Will now show that $\operatorname{supp} G'_\pm \phi \subseteq J_\mp(\operatorname{supp}\phi)$. Since $\operatorname{supp}\phi$ is compact, $J_\mp(\operatorname{supp}\phi)$ is closed by lemma 7. It therefore suffices to show $\left(\operatorname{supp} G'_\pm \phi\right)^\circ \subseteq J_\mp(\operatorname{supp}\phi)$. Let $q \in (\operatorname{supp} G'_\pm \phi)^\circ$. Then $G'_\pm \phi(q) \neq 0$. If $q \notin J_\mp(\operatorname{supp}\phi)$, $\operatorname{supp}\phi \cap J_\pm(q) = \emptyset$. But $G'_\pm \phi(q)$ is an integral with integrand linear in ϕ only over the region $J_\pm(q)$. Therefore $q \in J_\mp(\operatorname{supp}\phi)$. We then have $\operatorname{supp} u \cap \operatorname{supp} G'_\pm \phi \subseteq \operatorname{supp} u \cap J_\mp(\operatorname{supp}\phi)$. Since $\operatorname{supp} u$ is $\mp$-compact and $\operatorname{supp}\phi$ is compact, by lemma 6 $\operatorname{supp} u \cap J_\mp(\operatorname{supp}\phi)$ is compact, and hence $\operatorname{supp} u \cap \operatorname{supp} G'_\pm \phi$ is compact. It remains to check if $G_\pm u$ is linear in ϕ. We have

$$\langle G_\pm u, \phi\rangle = \langle u, \psi\rangle\,,$$

where $\psi \in \mathcal{D}_{(1,0)}(\Omega)$ is any test function that equals $G'_\pm \phi$ on a neighbourhood of $\operatorname{supp} u \cap \operatorname{supp} G'_\pm \phi$. That confirms linearity, and $\mathcal{D}'_{\pm,(1,0)}(\Omega) \xrightarrow{G_\pm} \mathcal{D}'_{\pm,(1,0)}(\Omega)$ is well-defined.

<u>2.</u>
Will first show that $\mathcal{D}'_{\pm,(1,0)}(\Omega) \xrightarrow{G_\pm} \mathcal{D}'_{\pm,(1,0)}(\Omega)$ hits within its codomain. Let $u \in \mathcal{D}'_{\pm,(1,0)}(\Omega)$. Need to show that $\operatorname{supp} G_\pm u$ is $\mp$-compact, that is, $\operatorname{supp} G_\pm u \cap J_\mp(q)$ is compact $\forall q \in \Omega$. Will first show that $\operatorname{supp} G_\pm u \subseteq J_\pm(\operatorname{supp} u)$. Let $p \in \Omega \setminus J_\pm(\operatorname{supp} u)$. Need to show $p \notin \operatorname{supp} G_\pm u$. Since $\operatorname{supp} u$ is $\mp$-compact, by lemma 7 $J_\pm(\operatorname{supp} u)$ is also $\mp$-compact and hence closed. That means $\exists U \in \tau_\Omega$ so that $p \in U \nsubseteq J_\pm(\operatorname{supp} u)$. Let $\phi \in \mathcal{D}_{(1,0)}(\Omega)$ with $\operatorname{supp}\phi \subseteq U$. Since $U \nsubseteq J_\pm(\operatorname{supp} u)$, $\operatorname{supp} u \cap J_\mp(U) = \emptyset$.

We have already shown in part 1. that $\operatorname{supp} G'_{\pm}\phi \subseteq J_{\mp}(\operatorname{supp}\phi)$. So $\operatorname{supp} u \cap \operatorname{supp} G'_{\pm}\phi = \emptyset$. That means $0 = \langle u, G'_{\pm}\phi\rangle = \langle G_{\pm}u, \phi\rangle$. From definition 3 of support of distributions,

$$\operatorname{supp} G_{\pm}u = \Omega \setminus \left\{p' \in \Omega \;\middle|\; \exists U \in \tau_{\Omega} \text{ so that } p' \in U \text{ and } G_{\pm}u\big|_{U} = 0\right\}.$$

That means $p \notin \operatorname{supp} G_{\pm}u$. We have shown $\operatorname{supp} G_{\pm}u \subseteq J_{\pm}(\operatorname{supp} u)$, and since $J_{\pm}(\operatorname{supp} u)$ is $\mp$-compact and $\operatorname{supp} G_{\pm}u$ is closed, $\operatorname{supp} G_{\pm}u$ is $\mp$-compact, which finally means that $\mathcal{D}'_{\pm,(1,0)}(\Omega) \xrightarrow{G_{\pm}} \mathcal{D}'_{\pm,(1,0)}(\Omega)$ hits within its codomain. Next, we show that $\mathcal{T}_{\pm,(1,0)}(\Omega) \xrightarrow{G_{\pm}} \mathcal{T}_{\pm,(1,0)}(\Omega)$ hits within its codomain. We have already shown that $G_{\pm}$ preserves $\mp$-compact support. It remains to show that $G_{\pm}j$ is smooth $\forall j \in \mathcal{T}_{\pm,(1,0)}(\Omega)$. Let $j \in \mathcal{T}_{\pm,(1,0)}(\Omega)$ and $\phi \in \mathcal{D}_{\pm,(1,0)}(\Omega)$.

$$\begin{aligned}\langle G_{\pm}j, \phi\rangle =& \left\langle j, G'_{\pm}\phi\right\rangle \\ =& \frac{1}{4\pi}\int_{\Omega} j^{\mu}(q)\left(\int_{C_{\pm}(q)} U^{\nu}|_{\mu}(p,q)\phi_{\nu}(p)\omega_{\sigma_q,0}(p)\right)\omega(q) \\ &+ \frac{1}{4\pi}\int_{\Omega} j^{\mu}(q)\left(\int_{D_{\pm}(q)} W^{\nu}|_{\mu}(p,q)\phi_{\nu}(p)\omega(p)\right)\omega(q).\end{aligned}$$

We have

$$\int_{\Omega} j^{\mu}(q)\left(\int_{C_{\pm}(q)} U^{\nu}|_{\mu}(p,q)\phi_{\nu}(p)\omega_{\sigma_q,0}(p)\right)\omega(q) = \int_{\Omega}\left(\int_{C_{\mp}(p)} j^{\mu}(q)U^{\nu}|_{\mu}(p,q)\omega_{\sigma_p,0}(q)\right)\phi_{\nu}(p)\omega(p)$$
$$\int_{\Omega} j^{\mu}(q)\left(\int_{D_{\pm}(q)} W^{\nu}|_{\mu}(p,q)\phi_{\nu}(p)\omega(p)\right)\omega(q) = \int_{\Omega}\left(\int_{D_{\mp}(p)} j^{\mu}(q)W^{\nu}|_{\mu}(p,q)\omega(q)\right)\phi_{\nu}(p)\omega(p)$$

because integration regions are

$$\begin{aligned}\{(p,q) \in \Omega\times\Omega \mid p \in C_{\pm}(q)\} =& \{(p,q) \in \Omega\times\Omega \mid q \in C_{\mp}(p)\} \\ \{(p,q) \in \Omega\times\Omega \mid p \in D_{\pm}(q)\} =& \{(p,q) \in \Omega\times\Omega \mid q \in D_{\mp}(p)\}\end{aligned}$$

and from the defining property of Leray forms we have

$$\begin{aligned}d\sigma_q(p) \wedge \omega_{\sigma_q,0}(p) \wedge \omega(q) =& \omega(p) \wedge \omega(q) \\ =& \omega(p) \wedge d\sigma_p(q) \wedge \omega_{\sigma_p,0}(q) \\ =& d\sigma_p(q) \wedge \omega_{\sigma_p,0}(q) \wedge \omega(p).\end{aligned}$$

That means

$$G_{\pm}j(p) = \frac{1}{4\pi}\int_{C_{\mp}(p)} j^{\mu}(q)U^{\nu}|_{\mu}(p,q)\omega_{\sigma_p,0}(q) + \frac{1}{4\pi}\int_{D_{\mp}(p)} j^{\mu}(q)W^{\nu}|_{\mu}(p,q)\omega(q).$$

Equation (4.55) is proved. Smoothness of $G_{\pm}j$ follows then from the same reason the operator $G'_{\pm}$ produces smooth vector fields.

3.
Let $u \in \mathcal{D}'_{\pm,(1,0)}(\Omega)$. Need to show that $\mathcal{L}G_\pm u = u$. This holds true because $G_{\pm,j}$ is a Green's function $\forall j \in T\Omega$. To prove it, let $\phi \in \mathcal{D}_{(1,0)}(\Omega)$.

$$\begin{aligned}\langle \mathcal{L}G_\pm u, \phi\rangle &= \langle G_\pm u, \mathcal{L}\phi\rangle \\ &= \left\langle u, G'_\mp \mathcal{L}\phi\right\rangle \\ &= \left\langle u^\mu(q), \left\langle \mathcal{L}G_{\mp,q,\mu}, \phi\right\rangle\right\rangle \\ &= \left\langle u^\mu(q), \phi_\mu(q)\right\rangle .\end{aligned}$$

4.
Let $u \in \mathcal{D}'_{\pm,(1,0)}(\Omega)$ and $\mathcal{L}u = 0$. Need to show $u = 0$. We have shown that

$$\begin{tikzcd} \mathcal{D}'_{\pm,(1,0)}(\Omega) \arrow[r, "G_\pm"] \arrow[rr, bend left, "\mathrm{id}"] & \mathcal{D}'_{\pm,(1,0)}(\Omega) \arrow[r, "\mathcal{L}"] & \mathcal{D}'_{\pm,(1,0)}(\Omega), \end{tikzcd}$$

and since $\mathcal{T}_{\pm,(1,0)}(\Omega) \xrightarrow{G_\pm} \mathcal{T}_{\pm,(1,0)}(\Omega)$ hits within its image,

$$\begin{tikzcd} \mathcal{T}_{\pm,(1,0)}(\Omega) \arrow[r, "G_\pm"] \arrow[rr, bend left, "\mathrm{id}"] & \mathcal{T}_{\pm,(1,0)}(\Omega) \arrow[r, "\mathcal{L}"] & \mathcal{T}_{\pm,(1,0)}(\Omega). \end{tikzcd}$$

That means $\mathcal{L}G_\pm\phi = \phi \quad \forall\phi \in \mathcal{D}_{(1,0)}(\Omega)$, since $\mathcal{D}_{(1,0)}(\Omega) \subseteq \mathcal{T}_{\pm,(1,0)}(\Omega)$. We get

$$\begin{aligned}\langle u, \phi\rangle =&\langle u, \mathcal{L}G_\pm\phi\rangle \\ =&\langle \mathcal{L}u, G_\pm\phi\rangle \\ =&0.\end{aligned}$$

□

Proof of lemma 8.
Both $d\left(\exp_p\right)_{X(0)}$ and $d_H \exp_{X(0)}$ can be described by the Jacobi equation. Set $Y(\tau) \equiv d\left(\exp_p\right)_{\tau X(0)}\left(\tau \frac{DX}{d\tau}(0)\right)$ and $Z(\tau) \equiv d_H \exp_{\tau X(0)} \dot\gamma(0)$. Then $d\left(\exp_p\right)_{X(0)} \frac{DX}{d\tau}(0) = Y(1)$ and

$d_H \exp_{X(0)} \dot{\gamma}(0) = Z(1)$ and

$$\left\{ \begin{array}{l} \dfrac{D^2 Y}{d\tau^2} = -R\left(Y, \dfrac{d}{d\tau} e^{\tau X(0)}\right) \dfrac{d}{d\tau} e^{\tau X(0)} \\ Y(0) = 0 \\ \dfrac{DY}{d\tau}(0) = \dfrac{DX}{d\tau}(0) \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} \dfrac{D^2 Z}{d\tau^2} = -R\left(Z, \dfrac{d}{d\tau} e^{\tau X(0)}\right) \dfrac{d}{d\tau} e^{\tau X(0)} \\ Z(0) = \dot{\gamma}(0) \\ \dfrac{DZ}{d\tau}(0) = 0 \end{array} \right\}.$$

Set $Æ(\tau) \equiv \frac{d}{d\tau'}\big|_0 \exp(\tau X(\tau'))$. Then

$$\left\{ \begin{array}{l} \dfrac{D^2 Æ}{d\tau^2} = -R\left(Æ, \dfrac{d}{d\tau} e^{\tau X(0)}\right) \dfrac{d}{d\tau} e^{\tau X(0)} \\ Æ(0) = \dot{\gamma}(0) \\ \dfrac{DÆ}{d\tau}(0) = \dfrac{DX}{d\tau}(0) \end{array} \right\}.$$

By uniqueness of solution, $Æ(\tau) = Y(\tau) + Z(\tau)$. So

$$\left.\frac{d}{d\tau}\right|_0 \exp(X(\tau)) = Æ(1) = Y(1) + Z(1) = d\left(\exp_p\right)_{X(0)} \frac{DX}{d\tau}(0) + d_H \exp_{X(0)} \dot{\gamma}(0).$$

$\square$

Bibliography

[1] Sean M. Carrol. Lecture notes on general relativity. https://arxiv.org/abs/gr-qc/9712019, 1997.

[2] Planck Collaboratio. Planck 2018 results. vi. cosmological parameters. https://arxiv.org/abs/1807.06209, 2021.

[3] Adam G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. https://www.stsci.edu/~ariess/documents/1998.pdf, 1998.

[4] Adam G. Riess et al. A Comprehensive Measurement of the Local Value of the Hubble Constant with $1\,\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$ Uncertainty from the Hubble Space Telescope and the SH0ES Team. https://arxiv.org/abs/2112.04510, 2022.

[5] P. Frank Winkler et al. The sn 1006 remnant: Optical proper motions, deep imaging, distance, and brightness at maximum. https://arxiv.org/abs/astro-ph/0208415, 2002.

[6] F. G. Friedlander. *The wave equation on a curved space-time*. Cambridge University Press, 1975.

[7] R. Mankin and R. Tammelo T. Laas. A new approach to electromagnetic wave tails on a curved spacetime. https://arxiv.org/abs/gr-qc/0008047, 2000.

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