

Article

Remarks on Cosmological Bulk Viscosity in Different Epochs

Iver Brevik *  and Ben David Normann

Department of Energy and Process Engineering, Norwegian University of Science and Technology, N-7491 Trondheim, Norway; ben.d.normann@uis.no

* Correspondence: iver.h.brevik@ntnu.no

Received: 16 April 2020; Accepted: 16 June 2020; Published: 1 July 2020



Abstract: The intention of this paper is mainly two-fold. First, we point out a striking numerical agreement between the bulk viscosity in the lepton era calculated by Husdal (2016) and our own calculations of the present-day bulk viscosity when the functional form is $\zeta \sim \sqrt{\rho}$. From a phenomenological point of view, we thus seem to have an ansatz for the viscosity, which bridges the infancy of the Universe (~ 1 s) with the present. This can also be looked upon as a kind of symmetry between the early-time cosmology and the present-day cosmology: it is quite remarkable that the kinetic theory-based bulk viscosity in the early universe and the experimentally-based bulk viscosity in the present universe can be covered by the same simple analytical formula. Second, we consider the Kasner universe as a typical anisotropic model of Bianchi-Type I, investigating whether this geometrical model is compatible with constant viscosity coefficients in the fluid. Perhaps surprisingly, the existence of a shear viscosity turns out to be incompatible with the Kasner model. By contrast, a bulk viscosity is non-problematic in the isotropic version of the model. In the special case of a Zel'dovich (stiff) fluid, the three equal exponents in the Kasner metric are even determined by the bulk viscosity alone, independent of the value of the fluid energy density. We also give a brief comparison with some other recent approaches to viscous cosmology.

Keywords: viscous cosmology; kasner universe; anisotropic cosmology

1. Introduction

Recent years have witnessed an increased interest in bulk-viscous properties in the cosmic fluid. From a hydrodynamicist's point of view, it is almost surprising that this surge of interest has not occurred earlier. As is known, viscous effects are quite ubiquitous in ordinary hydrodynamics, and one should not expect the cosmic fluid to be an exception in that respect. There are in general two viscosity coefficients, the shear viscosity η and the bulk viscosity ζ , corresponding to first-order deviation from thermal equilibrium, although normally, the shear viscosity is omitted in cosmology because of the assumption about the spatial isotropy of the fluid. However, what happens if one takes away the assumption about isotropy? We demonstrate that in the Kasner universe, shear viscosity seems unphysical. Hence, we are left with bulk-viscous modifications to the equation of state. Moreover, we present in this paper a striking similarity between the results obtained for the bulk viscosity of the late Universe [1] based on a much-used phenomenological approach ($\zeta \sim \sqrt{\rho}$) and that calculated by Husdal and others (see below) for the early lepton era.

We will assume a spatially flat Friedmann–Lemaître–Robertson–Walker universe, where the metric is:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \quad (1)$$

and the energy-momentum tensor of the whole fluid is:

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - \theta\zeta)h_{\mu\nu}, \quad (2)$$

with $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ as the projection tensor. The scalar expansion is $\theta = U^\mu{}_{;\mu} = 3H$. In comoving coordinates, the components of the fluid four-velocity are $U^0 = 1, U^i = 0$.

The total energy density ρ of the cosmic fluid is taken to be composed of several parts,

$$\rho = \rho_{\text{DE}} + \rho_{\text{DM}} + \rho_{\text{B}} + \rho_{\text{R}}, \quad (3)$$

where subscripts DE, DM, B, and R refer to dark energy, dark matter, baryons, and radiation, respectively. Experiments show that the dark sector amounts to about 95% of the total energy content [2]. We will define ρ_{M} as the sum of the dark matter and baryons,

$$\rho_{\text{M}} = \rho_{\text{DM}} + \rho_{\text{B}}. \quad (4)$$

The contribution from radiation is negligible. Defining the usual density parameters $\Omega_i = \rho_i / \rho_c$ with ρ_c the critical density, we have:

$$\Omega_{\text{DE}} = \frac{\rho_{\text{DE}}}{\rho_c}, \quad \Omega_{\text{M}} = \frac{\rho_{\text{M}}}{\rho_c}, \quad \rho_c = \frac{3H^2}{8\pi G}. \quad (5)$$

At present (subscript 0), the Planck experiment finds ([2], Table 2). $\Omega_{\text{DE}} = 0.6847$, $\Omega_{0\text{M}} = 0.3153$, summing up to unity. It is also useful to note that $H_0 = 67.36 \text{ km s}^{-1} \text{ Mpc}^{-1} = 2.20 \times 10^{-18} \text{ s}^{-1}$, $\rho_{0c} = 3H_0^2 / 8\pi G = 8.5 \times 10^{-27} \text{ kg m}^{-3}$.

Consider now the Friedmann equations:

$$\theta^2 = 24\pi G\rho, \quad (6)$$

$$\dot{\theta} + \frac{1}{2}\theta^2 = -12\pi G(p - \zeta\theta). \quad (7)$$

The obvious task is to figure out how to model the bulk viscosity (although not a focus in the present work, we also remark that a positive, non-vanishing bulk viscosity will generate entropy [3]). One obvious choice—and the one made in this paper—is to model the bulk viscosity as a function of the total energy density, $\zeta = \zeta(\rho)$. In [1,4], we advocated the power-law form:

$$\zeta(\rho) = \zeta_0 \left(\frac{\rho}{\rho_0} \right)^\lambda, \quad (8)$$

with λ a constant. Preference, although not a very strong one, was given to the case $\lambda = 1/2$, in agreement with several other investigators having compared with experiments.

Our motivations for undertaking the present investigation are the following:

(i) The phenomenological approach above was based on redshifts up to about $z = 2.3$. It is of interest to make a big jump in the cosmological scale, back to the lepton-photon universe, characterized by temperatures between $T = 10^{12} \text{ K}$ and 10^{10} K , where the Universe was populated by photons, neutrinos, electrons, and their antiparticles. Furthermore, under such circumstances, a bulk viscosity appears, explained in kinetic terms as a result of the imbalance between the free paths of neutrinos and the other particles. The maximum bulk viscosity occurs at the time of neutrino decoupling, $T = 10^{10} \text{ K}$. We will base our analysis on the recent papers of Husdal et al. [5–7]. Moreover, we will show that a bold extrapolation of the formula (8) with $\lambda = 1/2$ back to this very early instant brings surprisingly good agreement with the kinetic theory based result for ζ . It becomes suggestive to assume that Formula (8) holds for very longer times back than what is so far justified from observations. In other

words, there is apparently a kind of symmetry between the early-time cosmology and the present-day cosmology as far as the bulk viscosity is concerned.

(ii) Some effort ought to be made to clarify the reasons why there are apparently conflicting statements in the recent literature. The extensive analysis of Yang et al. [8] favors the form $\zeta \propto \rho^m$ with $m \approx -0.4$, thus of a generalized Chaplygin form, and quite different from what we stated above. As one might expect, this discrepancy is rooted in differences in the initial formalism. We consider this theme in some detail in Section 3.

(iii) The common omission of the shear viscosity in cosmology is not quite trivial, all the time that the shear viscosity is the dominant viscosity in ordinary fluid mechanics. One might suspect that even a slight anisotropy in the cosmic fluid could easily compensate for the bulk viscosity. We consider this point in Section 4, choosing the anisotropic Kasner universe as an example. It actually turns out that the Kasner model is not easily compatible with a shear viscosity. On the other hand, the model admits a bulk viscosity without any problems, in the degenerate case of spatial isotropy.

Readers interested in review articles on viscous cosmology may consult [3,9–12]. By now, the literature is rich with contributions on the topic of viscous cosmology. The contributions include investigations of the early universe [13–16], the late universe [17–19], the phantom divide [20–25], models for the dark sector [26–29], and others [30–39].

2. Possible Relationship to the Bulk Viscosity in the Lepton-Photon Epoch

On the basis of $H(z)$ measured for different moderate values of z ($0 < z < 2.3$), we estimated in [1,4] the present bulk viscosity ζ_0 to lie in the interval:

$$10^4 \text{ Pa s} \leq \zeta_0 \leq 10^6 \text{ Pa s}. \quad (9)$$

Although we refer the interested reader to the sources for detailed explanations, we will in the following clarify under what assumptions these results were obtained. First of all, we made use of the ansatz (8), testing three different values for the exponent λ , $\lambda = (0, 1/2, 1)$. The equation of state was assumed in the simple form $p = w\rho$ with w a constant. In view of the dominance of the dark energy component of the fluid, we took the value of w to lie closely to -1 when we investigated a one-component fluid model [4]. Mathematically, this means:

$$p = (-1 + \alpha)\rho, \quad |\alpha| \ll 1. \quad (10)$$

From the 2015 Planck data [40], $w = -1.019^{+0.075}_{-0.080}$. It corresponds to:

$$\alpha_{\min} = -0.099, \quad \alpha_{\max} = +0.056. \quad (11)$$

From the second Friedmann Equation (7) together with the ansatz (8), we see that one may introduce an effective pressure $p_{\text{eff}} = w_{\text{eff}}\rho$, with an effective equation of state parameter:

$$w_{\text{eff}} = -1 + \alpha - \frac{3H\zeta_0}{\rho_0} \left(\frac{\rho}{\rho_0} \right)^{\lambda-1}. \quad (12)$$

Thus, if $\lambda \neq 1$, w_{eff} is in general a function of ρ .

We know, however, that the cosmic fluid is not actually a one-component fluid, but rather consists of many different constituents. As mentioned earlier, the total energy density is assumed to be composed by a dark energy component ρ_{DE} and a matter component $\rho_{\text{M}} = \rho_{\text{DM}} + \rho_{\text{B}}$ satisfying $\Omega_{\text{DE}} + \Omega_{\text{M}} = 1$. Denoting the homogeneous solution corresponding to $\zeta = 0$ by $\rho_{\text{h}}(a)$, we can decompose:

$$\rho_{\text{h}}(a) = \sum_i \rho_{0i} a^{-3(w_i+1)}, \quad (13)$$

with $i = (\text{DE}, \text{M})$, $w_{\text{DE}} = -1$, $w_{\text{M}} = 0$. Moreover, for simplicity, we assume that the viscosity can be associated with the fluid as a whole. Going beyond a phenomenological approach, this seems to imply that the viscosity directly or indirectly is sourced by the interaction of the various components of the phenomenological one-component fluid. Consider [41,42] for interesting discussions. Furthermore, a bulk viscosity associated with the overall fluid would be most accurate if the dark sector interacts with the constituents of the standard model. Associating the viscosity with the fluid as a whole enables us to write:

$$\rho(a) = \rho_{\text{h}}(a)[1 + u(a)], \quad (14)$$

where $u(a)$ is determined from Friedmann's equations. It is useful to define the auxiliary quantity:

$$B_0 = 12\pi G\zeta_0, \quad (15)$$

whose value is about unity in astronomical units (note the subscript ₀ on B ; this subscript was not used in our previous works, but should have been there in order to indicate that this is a present-day value). We introduce the common notation $E = H/H_0$ and give the final formulas for the most actual option $\lambda = 1/2$ only. Then,

$$u(z, B) = (1 + z)^{-\frac{2B}{H_0}}, \quad (16)$$

$$E(z, B) = \sqrt{\Omega(z)}(1 + z)^{-\frac{B}{H_0}}. \quad (17)$$

We have thus delineated the assumptions that led us to the result (9), by comparison with the observations of $H(z)$. This is also largely in agreement with earlier investigators [31,43].

Let us now go back to the early Universe, the lepton-photon era, in which case the viscosity coefficients have to be calculated by kinetic theory. There are two factors that are important for the calculation, namely the state of the system (we assume it to be a pure lepton-photon mixture) and then the transport equations for the fluid. The free mean paths for the neutrinos are much larger than those of the electromagnetically interacting particles, thus building up a temperature difference between the fluid components. The electromagnetic particles will cool somewhat faster than the neutrinos. Both coefficients η and ζ can be evaluated via the Chapman–Enskog approximation [5–7,44,45], here given for ζ only,

$$\zeta(T) = nk_{\text{B}}T \sum_k a_k \alpha_k. \quad (18)$$

The n is the particle density; a_k are the coefficients for particle k for the linearized relativistic Boltzmann equation; and α_k are known state parameters. As mentioned above, the most significant instant is that of neutrino decoupling, $T = 10^{10}$ K, at which Husdal obtained [6]:

$$\zeta = 1.26 \times 10^{22} \text{ Pa s}, \quad \eta = 1.0 \times 10^{25} \text{ Pa s}. \quad (19)$$

This value of ζ can be compared with that obtained from the expression (8) extrapolated back to the instant of neutrino decoupling. The relation between ρ and T in the early Universe is [7] (in geometric units):

$$\rho = \frac{\pi^2}{30} g_*(T) (k_{\text{B}}T)^4, \quad (20)$$

with g_* denoting the effective degrees of freedom at temperature T . Table 1 shows the resulting estimates for ζ for the three actual parameter values $\lambda = (0, 1/2, 1)$. What is apparent is that, by choosing $\lambda = 1/2$ and also by taking ζ equal to the logarithmic mean of the interval (9), i.e.,

$$\zeta_0 = 10^5 \text{ Pa s}, \quad (21)$$

we find practically the same value of ζ as with (8).

Table 1. The table should be read horizontally and shows the numerical values of ζ calculated by Equation (8) for different parameter value inputs $\{\lambda, \zeta_0, \Omega_0\}$. The parameter values given in the three first columns are used as input when calculating the values for ζ in the rightmost column.

Table Showing ζ Estimates			
λ (—)	ζ_0 (Pa s)	Ω_0	ζ from Equation (8) (Pa s)
0	$\forall \zeta_0$	(0,1]	$\zeta = \zeta_0$
1/2	$\zeta_0 = 10^4$	[0.1,1]	$\sim 10^{21}$
1/2	$\zeta_0 = 10^5$	[0.1,1]	$\sim 10^{22}$
1/2	$\zeta_0 = 10^6$	[0.1,1]	$\sim 10^{23}$
1	$\zeta_0 = 10^4$	[0.9,1]	$\sim 10^{38}$
1	$\zeta_0 = 10^5$	[0.9,1]	$\sim 10^{39}$
1	$\zeta_0 = 10^6$	[0.9,1]	$\sim 10^{40}$

We find this coincidence striking. Of course, it is not a proof for the extended applicability of the formula (8), but we think it deserves attention. It suggests that the formula can be used beyond the interval where it was originally constructed. Furthermore, it is notable that the parameter value $\lambda = 1/2$ turns out to be the favorable choice, in agreement with other analyses as mentioned earlier.

3. Discussion

We shall show that the results obtained in this paper are actually not surprising at all, when considered from a phenomenological point of view, where the fluid has an overall viscosity. Consider the Bianchi identity for the overall cosmological fluid ρ :

$$\dot{\rho} + \theta(1 + w)\rho - \zeta\theta^2 = 0. \quad (22)$$

This equation may be rewritten as:

$$\dot{\rho} = -\theta(1 + w(1 - r))\rho, \quad (23)$$

where we have defined:

$$r \equiv \frac{\theta}{w\rho}\zeta \quad (24)$$

Thus, r is the ratio between the viscous pressure $\zeta\theta$ and the equilibrium pressure $p = w\rho$. In the Eckart formalism, the viscous pressure exerted by ζ should remain a first order modification to the equilibrium pressure $w\rho$ throughout the history of the Universe. This translates into the requirement that $0 \leq r \ll 1$. Then, under the weak assumption that ζ is a monotonic function, we find:

$$r \approx \text{const.} \quad (25)$$

Detailed analysis of the dynamical behavior of the cosmological fluid could of course reveal small variations in the ratio r , so this relationship represents an approximation. Moreover, the variation in r could very well prove to be important in a variety of contexts (Such as that of understanding the (microscopic) mechanism that gives rise to the viscosity, or structure formation), but in this phenomenological analysis, we are nevertheless more interested in large-scale variations. Thus, putting $r = \text{const.}$ translates into $\zeta \sim \rho/\theta$. For a one-component fluid ρ , this is, by the application of the first Friedmann equation, just the same as ansatz (8). As such, our result is trivial, yet obviously worth mentioning, considering the wide application of other functional forms for the viscosity, also in the case of one-component fluids. The fact that the results of Husdal actually agree with our own calculations, as inferred from supernova observations, shows that our theoretical prejudice seems to

be confirmed by observation. This is, of course, not a trivial point. In more general terms, consider a viscosity $\zeta \sim \rho^m$ and an equilibrium pressure $w\rho$. Then, the ratio r defined in (24) becomes:

$$r \sim \rho^{m-1/2}. \quad (26)$$

The choice $m = 1/2$ now gives the condition (25). One may also observe that any $m < 1/2$ will cause the ratio to grow if ρ decreases. Consequently, unless ρ asymptotically approaches a constant value, the ratio must therefore eventually grow out of bounds of the first-order thermodynamic (Eckart) formalism.

Comparison with the Result $\zeta_D \sim \rho^{-0.4}$ Obtained by Yang et al.

As mentioned in the Introduction, Yang et al. [8] suggested a model where the effective pressure ρ_{eff} of the dark fluid $\rho_D = \rho_{\text{DE}} + \rho_{\text{DM}}$ is:

$$p_{\text{eff}} = w\rho_D + \sqrt{3}\alpha\rho \cdot \rho_D^{m-1}, \quad (27)$$

where w is the equation of state parameter, α and m are parameters of the theory, and ρ is the total energy density. The above equation makes sense only if p_{eff} and w are (respectively) the effective pressure and equation of state parameter for the unified dark fluid only, and not the overall fluid ρ . We therefore henceforth make this assumption. Thus, interpreting their results, we reach the conclusion that they worked with a different theory from our own. While they attributed the viscosity to the dark fluid only, we attribute the viscosity to the overall fluid. Perhaps more importantly, we fix the dark matter and dark energy components at present when obtaining our estimates for the present-day viscosity. As far as we understand, this is different from Yang et al., who (more appropriately) avoided such an a priori fixing of parameters. To sum up, the discrepancy derives from the difference in the theories. Without further examination, one may not conclude that the two approaches are in disagreement, per say. We shall repeat from the preceding subsection, however, that r in Equation (26) will eventually grow out of bounds for $m = -0.4$, since $-0.4 < 1/2$. Hence, a viscosity $\zeta \sim \rho^{-0.4}$ cannot be considered as a bulk-viscous modification in the ordinary sense, unless $\lim_{t \rightarrow \infty} \rho = \text{const.} \neq 0$. As a general dynamical modification to a homogeneous equation of state, the results of Yang et al. seem however to be valid.

4. The Viscous Kasner Universe

It is of interest to consider the anisotropic universe. The reason why the Universe is usually considered to be spatially isotropic is that observation strongly indicates such behavior. However, there is also evidence, in cosmology, as well as in ordinary fluid mechanics, that the shear viscosity grossly dominates over the bulk viscosity in magnitude. Thus, it might be possible that the combination of a slight anisotropy with a dominant shear viscosity leads to physically detectable consequences after all. This is the motivation for the analysis in the present section. We will focus on the anisotropic Kasner universe, as a typical example of an anisotropic space. It belongs to Bianchi-Type I. An introduction to this model can be found, for instance, in [46].

Consider the Kasner universe:

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \quad (28)$$

where the three p_i are constants, in the original formulation, which refers to a vacuum. Einstein's equations can be written:

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha), \quad (29)$$

and the non-vanishing Christoffel symbols are (no sum over i):

$$\Gamma_{ii}^0 = p_i t^{2p_i-1}, \quad \Gamma_{i0}^i = \Gamma_{0i}^i = \frac{p_i}{t}. \quad (30)$$

Allowing for both shear and bulk viscosities, we write the energy-momentum tensor as:

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - \zeta\theta)h_{\mu\nu} - 2\eta\sigma_{\mu\nu}, \quad (31)$$

where the scalar expansion $\theta = \theta^\alpha_{;\alpha} = U^\alpha_{;\alpha}$ is the trace of the expansion tensor:

$$\theta_{\mu\nu} = \frac{1}{2}(U_{\mu;\alpha}h^\alpha_\nu + U_{\nu;\alpha}h^\alpha_\mu), \quad (32)$$

and $\sigma_{\mu\nu}$ is the shear tensor:

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3}h_{\mu\nu}\theta. \quad (33)$$

Defining the numbers S and Q as:

$$S = \sum_{i=1}^3 p_i, \quad Q = \sum_{i=1}^3 p_i^2, \quad (34)$$

we can then write:

$$\theta = \frac{S}{t}, \quad \sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = -\frac{1}{2t^2}\left(\frac{1}{3}S^2 - Q\right). \quad (35)$$

With $R_i = t^{p_i}$ being the expansion factors of the metric, the directional Hubble parameters become $H_i = \dot{R}_i/R_i = p_i/t$, and the average Hubble parameter becomes:

$$H = \frac{1}{3} \sum_{i=1}^3 H_i = \frac{S}{3t} = \frac{\theta}{3}. \quad (36)$$

Let now $w = \text{constant}$ be the thermodynamic parameter,

$$p = w\rho. \quad (37)$$

With $\kappa = 8\pi G$, we can then write the Einstein Equation (29) as:

$$\begin{aligned} S - Q + \frac{3}{2}\kappa t \zeta S &= \frac{1}{2}\kappa t^2(1 + 3w)\rho, \\ p_i(1 - S - 2\kappa t \eta) + \frac{1}{2}\kappa t \left(\zeta + \frac{4}{3}\eta \right) S &= \frac{1}{2}\kappa t^2(w - 1)\rho. \end{aligned} \quad (38)$$

The basic formalism given here is as in our earlier works [47,48]. We are thus considering an isotropic fluid in an anisotropic space (what we mean by this is that the same fluid in an isotropic background would not possess shear viscosity). The fluid itself is modeled as a usual fluid with density ρ and scalar pressure p . Our model is thus different from one in which the fluid is taken to have anisotropic properties; cf., for instance, [49]. We will assume that the physical properties of the fluid are given at some initial time called t_{in} and investigate if these initial conditions are compatible with constant values of S and Q . In the vacuum case (no fluid at all), one has $S = Q = 1$ [46]. We expect the Kasner model to be appropriate for the early Universe and will naturally choose the instant of neutrino decoupling, $T_{\text{in}} = 10^{10}$ K, $t_{\text{in}} = 1$ s, as mentioned above.

We consider the development of the fluid from $t = t_{\text{in}}$ onwards; the initial energy and pressure being ρ_{in} and p_{in} . It is notable that the governing Equation (38) actually fix the later time dependence to be:

$$\zeta = \zeta_{\text{in}} \left(\frac{t}{t_{\text{in}}} \right)^{-1}, \quad \eta = \eta_{\text{in}} \left(\frac{t}{t_{\text{in}}} \right)^{-1}, \quad (39)$$

$$\rho = \rho_{\text{in}} \left(\frac{t}{t_{\text{in}}} \right)^{-2}, \quad p = p_{\text{in}} \left(\frac{t}{t_{\text{in}}} \right)^{-2}, \quad (40)$$

whereby we obtain a time independent set of equations,

$$\begin{aligned} S - Q + \frac{3}{2} \kappa t_{\text{in}} \zeta_{\text{in}} S &= \frac{1}{2} \kappa t_{\text{in}}^2 (1 + 3w) \rho_{\text{in}}, \\ p_i (1 - S - 2\kappa t_{\text{in}} \eta_{\text{in}}) + \frac{1}{2} \kappa t_{\text{in}} (\zeta_{\text{in}} + \frac{4}{3} \eta_{\text{in}}) S &= -\frac{1}{2} \kappa t_{\text{in}}^2 (1 - w) \rho_{\text{in}}, \end{aligned} \quad (41)$$

(note that in fundamental units where the basic unit is $[t] = \text{cm}$, one has $[\rho] = [p] = \text{cm}^{-4}$, $[\zeta] = \text{cm}^{-3}$). It is also convenient to note the following equations derived from those above,

$$2S(S - 1) = 3\kappa t_{\text{in}} \zeta_{\text{in}} S + 3\kappa t_{\text{in}}^2 (1 - w) \rho_{\text{in}}, \quad (42)$$

$$S^2 - Q = 2\kappa t_{\text{in}}^2 \rho_{\text{in}}. \quad (43)$$

The basic formalism outlined so far is essentially as in our earlier papers [47,48]. We assume now that the physical quantities $\{\rho_{\text{in}}, p_{\text{in}}, \zeta_{\text{in}}, \eta_{\text{in}}\}$ are given at $t = t_{\text{in}}$ and investigate if these initial conditions lead to acceptable values for the coefficients p_i in an anisotropic Kasner universe. From Equation (41), it follows that if the three p_i are to be unequal, the multiplying factor of p_i has to be zero,

$$S = 1 - 2\kappa t_{\text{in}} \eta_{\text{in}}. \quad (44)$$

This is thus an algebraic restriction on the p_i . If the fluid is nonviscous, then $S = 1$, in accordance with the original Kasner model in a vacuum. Once S is known, Q follows at once from Equation (43) as:

$$Q = S^2 - 2\kappa t_{\text{in}}^2 \rho_{\text{in}}. \quad (45)$$

Note that the bulk viscosity does not appear in the last two equations. This is as should be expected physically: anisotropy is caused by the shear only.

Now, going over to dimensional units, we first note the useful relations $\kappa = 8\pi G/c^2 = 1.866 \times 10^{-26} \text{ m kg}^{-1}$, $1 \text{ MeV}^4 = 2.085 \times 10^{25} \text{ J m}^{-3}$. In the radiation dominated era:

$$a(t) = 2.2 \times 10^{-10} t^{1/2}, \quad T(t) = 10^{10} t^{-1/2} \text{ K}. \quad (46)$$

With $\eta_{\text{in}} = 1.0 \times 10^{25} \text{ Pa s}$, we then obtain from Equation (44):

$$S = 0.627. \quad (47)$$

The energy density in this region can be roughly estimated from $\rho_{\text{in}} c^2 = a_{\text{rad}} T^4$, where a_{rad} is the radiation constant:

$$a_{\text{rad}} = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} = 7.56 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}. \quad (48)$$

Here, the degeneracy factor is omitted (cf., for instance, [13]). Then, $\rho_{\text{in}} \sim 10^{25} \text{ J m}^{-3}$. Alternatively, we may use the equation $H_{\text{in}}^2 = (\kappa c^2/3) \rho_{\text{in}}$ to get somewhat more accurately:

$$\rho_{\text{in}} = 4.47 \times 10^8 \text{ kg m}^{-3}, \quad (49)$$

or $\rho_{\text{in}} = 4.02 \times 10^{25} \text{ J m}^{-3}$. With the latter values, we calculate:

$$Q = 1 - 2\kappa c^2 t_{\text{in}}^2 \rho_{\text{in}} = -1.11. \quad (50)$$

This is a non-acceptable result, as Q is a sum of quadratic numbers. We conclude that the Kasner model does not appear to be compatible with a shear viscosity.

There is also another reason why anisotropy ($p_i \neq 0$) is problematic in the Kasner model. From Equation (38), it follows that if $w < 1$, which is the case for the usual fluids, the combination $(\zeta + 4\eta/3)$ becomes negative. Although negative viscosities are occasionally considered in cosmology (cf. for instance, [50]), such a case is physically not very natural. This point was first noticed by Cataldo and Campo [51].

It is worthwhile to notice that the isotropic Kasner geometry easily allows for a viscosity. That means only the bulk viscosity comes into question, as this viscosity concept goes along with spatial isotropy. Let us consider this case in some more detail, defining a by $p_1 = p_2 = p_3 \equiv a$. From Equation (41), we obtain, still in dimensional units,

$$a = \frac{1}{6} \left[1 + \frac{3}{2} \kappa t_{\text{in}} \zeta_{\text{in}} + \sqrt{\left(1 + \frac{3}{2} \kappa t_{\text{in}} \zeta_{\text{in}} \right)^2 + 6\kappa c^2 t_{\text{in}}^2 (1-w) \rho_{\text{in}}} \right]. \quad (51)$$

This equation shows that if the physical quantities ρ_{in} and p_{in} , as well as the bulk viscosity ζ_{in} are given at $t = t_{\text{in}}$, then the isotropic version of the Kasner metric (28) is determined.

There is finally one exceptional case that should be noticed, namely a Zel'dovich fluid for which $w = 1$, the velocity of sound being equal to c . In that case,

$$a = \frac{1}{3} \left[1 + \frac{3}{2} \kappa t_{\text{in}} \zeta_{\text{in}} \right]. \quad (52)$$

The metric turns out to be determined by the bulk viscosity as the only physical parameter; the actual value of ρ_{in} being irrelevant. If the fluid is nonviscous, $\zeta_{\text{in}} = 0$, then $a = 1/3$. This corresponds to the metric coefficients in Equation (28) being $t^{2a} = t^{2/3}$.

5. Conclusions

In this work, we pointed out an agreement between the bulk-viscosity calculated by Husdal et al. [5–7] to be present in the lepton-photon era and the present-day phenomenological viscosity calculated by ourselves in previous works based on the evolution $\zeta \sim \sqrt{\rho} \sim H$ of the viscosity in isotropic cosmology. As mentioned above, this points towards a symmetry between early-time and present-time viscous cosmology. It is instructive to note that viscosities such as $\zeta \sim \rho$ or constant ζ do not provide the same agreement.

Our analysis of the Kasner model showed the inapplicability of the shear viscosity concept in the anisotropic version of this model (meaning that at least two of the three exponents p_i in the metric (28) are unequal). The isotropic Kasner model is however easily compatible with a bulk viscosity. Then, the Kasner model itself becomes isotropic, so that $p_1 = p_2 = p_3$. It is noteworthy that for the special case of a Zel'dovich fluid, the Kasner exponents are determined exclusively by the bulk viscosity alone; cf. Equation (52).

Author Contributions: The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Normann, B.D.; Brevik, I. General bulk-viscous solutions and estimates of bulk viscosity in the cosmic fluid. *Entropy* **2016**, *18*, 215. [\[CrossRef\]](#)
2. Planck Collaboration. Planck 2018 results VI. *arXiv* **2018**, arXiv:1807.06209v2.
3. Weinberg, S. Entropy generation and the survival of protogalaxies in an expanding universe. *Astrophys. J.* **1971**, *68*, 175. [\[CrossRef\]](#)
4. Normann, B.D.; Brevik, I. Characteristic properties of two different viscous cosmology models for the future universe. *Mod. Phys. Lett. A* **2017**, *32*, 1750026. [\[CrossRef\]](#)
5. Husdal, L. On effective degrees of freedom in the early universe. *Galaxies* **2016**, *4*, 78. [\[CrossRef\]](#)
6. Husdal, L. Viscosity in a lepton-photon universe. *Astrophys. Space Sci.* **2016**, *361*, 263. [\[CrossRef\]](#)
7. Husdal, L.; Brevik, I. Entropy production in a lepton-photon universe. *Astrophys. Space Sci.* **2017**, *362*, 39. [\[CrossRef\]](#)
8. Yang, W.; Pan, S.; Di Valentino, E.; Paliathanasis, A.; Lu, J. Challenging bulk viscous scenarios with cosmological observations. *Phys. Rev. D* **2019**, *100*, 103518. [\[CrossRef\]](#)
9. Grøn, Ø. Viscous inflationary universe models. *Astrophys. Space Sci.* **1990**, *173*, 191.
10. Brevik, I.; Grøn, Ø. Relativistic viscous universe models. In *Recent Advances in Cosmology*; Travena, A., Sørensen, B., Eds.; Nova Scientific Publications: New York, NY, USA, 2013; pp. 99–127.
11. Zimdahl, W. ‘Understanding’ cosmological bulk viscosity. *Mon. Not. R. Astron. Soc.* **1996**, *280*, 1239. [\[CrossRef\]](#)
12. Brevik, I.; Grøn, Ø.; de Haro, J.; Odintsov, S.D.; Saridakis, E.N. Viscous cosmology for early- and late-time universe. *Int. J. Mod. Phys. D* **2017**, *26*, 1730024. [\[CrossRef\]](#)
13. Brevik, I. Brane viscous cosmology in the plasma era. *Astrophys. Space Sci.* **2015**, *355*, 179. [\[CrossRef\]](#)
14. Brevik, I.; Timoshkin, A.V. Viscous coupled fluids in inflationary cosmology. *JETP* **2016**, *122*, 679–684. [\[CrossRef\]](#)
15. Bamba, K.; Odintsov, S.D. Inflation in a viscous fluid model. *Eur. Phys. J. C* **2016**, *76*, 18. [\[CrossRef\]](#)
16. Del Campo, S.; Herrera, R.; Pavon, D. Cosmological perturbations in warm inflationary models with viscous pressure. *Phys. Rev. D* **2007**, *75*, 083518. [\[CrossRef\]](#)
17. Brevik, I.; Elizalde, E.; Nojiri, S.; Odintsov, S.D. Viscous little rip cosmology. *Phys. Rev. D* **2011**, *84*, 103508. [\[CrossRef\]](#)
18. Brevik, I.; Myrzakulov, R.; Nojiri, S.; Odintsov, S.D. Turbulence and little rip cosmology. *Phys. Rev. D* **2012**, *86*, 063007. [\[CrossRef\]](#)
19. Brevik, I. Temperature variation in the dark cosmic fluid in the late universe. *Mod. Phys. Lett. A* **2016**, *31*, 8. [\[CrossRef\]](#)
20. Brevik, I. Viscosity-induced crossing of the phantom divide in the dark cosmic fluid. *Front. Phys.* **2013**, *1*, 27. [\[CrossRef\]](#)
21. Brevik, I. Crossing of the $w = -1$ barrier in viscous modified gravity. *Int. J. Mod. Phys. D* **2006**, *15*, 767. [\[CrossRef\]](#)
22. Velten, H.; Wang, J.; Meng, X. Phantom dark energy as an effect of bulk viscosity. *Phys. Rev. D* **2013**, *88*, 123504. [\[CrossRef\]](#)
23. Brevik, I. Viscosity-induced crossing of the phantom barrier. *Entropy* **2015**, *17*, 6318. [\[CrossRef\]](#)
24. Nojiri, S.; Odintsov, S.D.; Tsujikawa, S. Properties of singularities in the (phantom) dark energy universe. *Phys. Rev. D* **2005**, *71*, 063004. [\[CrossRef\]](#)
25. Disconzi, M.M.; Kephart, T.W.; Scherrer, R.J. A new approach to cosmological bulk viscosity. *Phys. Rev. D* **2015**, *91*, 04532. [\[CrossRef\]](#)
26. Elizalde, E.; Obukhov, V.V.; Timoshkin, A.V. Inhomogeneous viscous dark fluid coupled with dark matter in the FRW universe. *Mod. Phys. Lett. A* **2014**, *29*, 1450132. [\[CrossRef\]](#)
27. Brevik, I.; Gorbunova, O. Dark energy and viscous cosmology. *Gen. Relativ. Gravit.* **2005**, *37*, 2039–2045. [\[CrossRef\]](#)
28. Bamba, K.; Capozziello, S.; Nojiri, S.; Odintsov, S.D. Dark energy cosmology: The equivalent description via different theoretical models and cosmography tests. *Astrophys. Space Sci.* **2012**, *342*, 155–228. [\[CrossRef\]](#)
29. Brevik, I.; Obukhov, V.V.; Timoshkin, A.V. Dark energy coupled with dark matter in viscous fluid cosmology. *Astrophys. Space Sci.* **2015**, *355*, 399–403. [\[CrossRef\]](#)

30. Nojiri, S.; Odintsov, S.D. Quantum de Sitter cosmology and phantom matter. *Phys. Lett. B* **2003**, *562*, 147–152. [[CrossRef](#)]
31. Wang, J.; Meng, X. Effects of new viscosity model on cosmological evolution. *Mod. Phys. Lett. A* **2014**, *29*, 1450009. [[CrossRef](#)]
32. Floerchinger, S.; Tetradis, N.; Wiedemann, U.A. Accelerating cosmological expansion from shear and bulk viscosity. *Phys. Rev. Lett.* **2015**, *114*, 091301. [[CrossRef](#)]
33. Brevik, I.; Gorbunova, O.; Nojiri, S.; Odintsov, S.D. On isotropic turbulence in the dark fluid universe. *Eur. Phys. J. C* **2011**, *71*, 1629. [[CrossRef](#)]
34. Barrow, J.D. The deflationary universe: An instability of the de Sitter universe. *Phys. Lett. B* **1986**, *180*, 4. [[CrossRef](#)]
35. Li, W.J.; Ling, Y.; Wu, J.P.; Kuang, X.M. Thermal fluctuations in viscous cosmology. *Phys. Lett. B* **2010**, *687*, 1–5. [[CrossRef](#)]
36. Cardenas, V.H.; Cruz, N.; Villanueva, J.R. Testing a dissipative kinetic k-essence model. *Eur. Phys. J. C* **2015**, *75*, 148. [[CrossRef](#)]
37. Paolis, d. F.; Jamil, M.; Quadir, A. Black holes in bulk viscous cosmology. *Int. J. Theor. Phys.* **2010**, *49*, 621–632. [[CrossRef](#)]
38. Van den Horn, L.J.; Salvati, G.A.Q. Cosmological two-fluid bulk viscosity. *Mon. Not. R. Astron. Soc.* **2016**, *457*, 1878–1887. [[CrossRef](#)]
39. Nojiri, S.; Odintsov, S.D. Inhomogeneous equation of state of the universe: Phantom era, future singularity, and crossing the phantom barrier. *Phys. Rev. D* **2005**, *72*, 023003. [[CrossRef](#)]
40. Planck Collaboration. Planck 2015 results XIII. *arXiv* **2015**, arXiv:1502.01589.
41. Colistete, R.; Fabris, J.C.; Tossa, J.; Zimdahl, W. Bulk viscous cosmology. *Phys. Rev. D* **2007**, *76*, 103516. [[CrossRef](#)]
42. Oliver F. Piattella, O.F.; Fabris J.C.; Zimdahl, W. Bulk viscous cosmology with causal transport theory. *J. Cosmol. Astropart. Phys.* **2011**, *2011*, 029.
43. Velten, H.; Schwarz, D.J. Dissipation of dark matter. *Phys. Rev. D* **2012**, *86*, 083501. [[CrossRef](#)]
44. Hogeveen, F.; Leeuwen, W.A.V.; Salvati, G.A.Q.; Schilling, E.E. Viscous phenomena in cosmology. I. Lepton era. *Phys. A Stat. Mech. Appl.* **1986**, *134*, 458. [[CrossRef](#)]
45. de Groot, S.R.; Leeuwen, W.A.; van Weert, C.G. *Relativistic Kinetic Theory: Principles and Applications*; Elsevier Science Ltd.: Amsterdam, The Netherlands, 1980.
46. Landau, L.D.; Lifshitz, E.M. *The Classical Theory of Fields*, 4th ed.; Pergamon Press: Oxford, UK, 1975.
47. Brevik, I.; Pettersen, S.V. Viscous cosmology in the Kasner metric. *Phys. Rev. D* **1997**, *56*, 3322. [[CrossRef](#)]
48. Brevik, I.; Pettersen, S.V. Can a Kasner universe with a viscous fluid be anisotropic? *Phys. Rev. D* **2000**, *61*, 127305. [[CrossRef](#)]
49. Cadoni, M.; Sanna, A.P.; Tuveri, M. Anisotropic fluid cosmology: An alternative to dark matter? *arXiv* **2020**, arXiv:2002.06988.
50. Brevik, I.; Grøn, Ø. Universe models with negative bulk viscosity. *Astrophys. Space Sci.* **2013**, *347*, 399. [[CrossRef](#)]
51. Cataldo, M.; del Campo, S. Comment on “Viscous cosmology in the Kasner metric”. *Phys. Rev. D* **2000**, *61*, 128301. [[CrossRef](#)]

