ENTROPY OF TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract. We prove that all entire transcendental entire functions have infinite topological entropy.

1. Introduction

Topological entropy is a central property in dynamical systems and has been studied extensively, both in the complex setting and outside. More generally it was shown by Misiurewicz and Przytycki [MP77] that for smooth self-maps of compact manifolds of topological degree d the entropy is at least $\log(d)$. For polynomials and rational functions acting on the Riemann sphere, it was shown independently by Gromov (in a preprint from 1977, published in 2003 [Gro03]) and Lyubich [Lju83] that the topological degree is equal to $\log(d)$.

The goal in this paper is to determine the topological entropy of transcendental entire maps. Such maps have infinite topological degree, and hence one can expect that the topological entropy is also infinite. This is indeed the case, as we will prove here.

In [Ber00] Bergweiler proved that the Ahlfors Five Islands Property implies for any transcendental function f the existence of a bounded simply connected open set $D \subset \mathbb{C}$, and disjoint relatively compact subsets $U_1, U_2 \subset\subset D$ which are both being mapped univalently onto D by some iterate f^k. As was pointed out by Dujardin in [Duj04], an immediate consequence is that the topological entropy of a transcendental function is always strictly positive. Since no bound on k is given, the argument does not provide a definite lower bound on the entropy. The fact that the entropy is strictly positive follows also from the results by [CF96].

We will prove the following statement, which gives less information on the way the image covers the domain, but which does imply arbitrarily large lower bounds on the entropy.

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Theorem 1.1. *Let f be a transcendental entire function, and let $N \in \mathbb{N}$. There exists a non-empty bounded open set $V \subset \mathbb{C}$ so that $V \subset f(V)$, and such that any point in V has at least N preimages in V, counted with multiplicity.*

The fact that f has infinite entropy follows from the next statement. We refer to the Appendix for the proof and for the definition of entropy.

Theorem 1.2. *Let $V \subset \mathbb{C}$ be a bounded open set, and let $g : V \to \mathbb{C}$ be a holomorphic function, having a holomorphic continuation to a neighborhood of $\overline{V}$. Suppose that every $w \in V$ has at least N preimages in V, counted with multiplicity. Then the topological entropy of g is at least* $\log(N)$.

In the previous paper [BFP18] we treated the simpler case when the function f omits some value. In this case the domains V can be chosen equal to arbitrarily large annuli of fixed modulus. As will be pointed out in example 2.10, this cannot always be done for arbitrary transcendental functions. Instead, the domain V that we construct is either a simply connected subdomain of some annulus, or equals a large disk.

Acknowledgement: The fact that transcendental functions have infinite entropy was proved independently by Markus Wendt. His result from 2005, whose proof relies upon Ahlfors Five Island Theorem, was never made public but is mentioned in his PhD thesis [Wen05, Beispiel 4.7.3]. We are grateful to Walter Bergweiler for bringing the work of Wendt to our attention.

2. Proof of the main theorem

Notation. Throughout the paper we denote by $\Delta(z, r)$ the open Euclidean disk of radius $r > 0$ centered at $z \in \mathbb{C}$. For a set $C \subset \mathbb{C}$ we denote by $\operatorname{diam}_{\mathrm{Eucl}} C$ its Euclidean diameter.

For a hyperbolic domain $D \subset \mathbb{C}$ let us denote by $\rho_D(z)|dz|$ its Poincaré metric, where $\rho_D(z)$ is the hyperbolic density on D. For a subset $D' \subset D$, we denote by $\operatorname{diam}_D(D')$ the diameter of D' in the Poincaré metric of D. Following [Ahl] we will write $\Omega_{0,1}$ for the set $\mathbb{C} \setminus \{0, 1\}$.

Estimates in the hyperbolic metric. The following estimate on the density of the Poincaré metric of the twice puctured domain $\Omega_{0,1}$ is well known, see for example Theorem 1-12 in [Ahl].

Lemma 2.1. *The hyperbolic density satisfies*

$$\rho_{\Omega_{0,1}}(z) > \frac{1}{2|z| \ln |z|}$$

for $|z|$ sufficiently large.

In fact, more precise estimates by Hempel [Hem79] and Jenkins [Jen81] show that the above equation holds whenever

$$\ln |z| \geq K,$$

where

$$K = \frac{\Gamma^4(\frac{1}{4})}{4\pi^2} = 4.3768796\ldots,$$

hence is satisfied when $|z| > e^5$.

From now on we let $D \subset \mathbb{C}$ be a hyperbolic domain, let $d > 0$, and let $C \subset D$ with $\operatorname{diam}_D C < d/2$.

Lemma 2.2. *Let $\alpha \in \mathbb{C} \setminus \{0\}$. Let $f : D \to \mathbb{C} \setminus \{0, \alpha\}$ be holomorphic. Then there exists $k > 0$, depending only on d, such that the following holds:*

If there exists $w_M \in C$ with $|f(w_M)| > M > k|\alpha|$, then

$$|f(z)| > |\alpha|^{\frac{e^d - 1}{e^d}} \cdot M^{1/e^d}$$

for all $z \in C$.

Proof. Let us first suppose that $\alpha = 1$. Since $\Omega_{0,1}$ is a complete metric space, for any d there exists $k > 0$ such that if $|f(w_M)| > k$ then $f(C)$ is contained in the disk $|z| > e^5$. Since holomorphic maps are distance decreasing

$$\operatorname{diam}_{f(D)} f(C) < d/2,$$

and hence

$$\operatorname{diam}_{\mathbb{C}\setminus\{0,1\}} f(C) < d/2.$$

and in particular

$$\operatorname{dist}_{\mathbb{C}\setminus\{0,1\}}(f(z), f(w_M)) < d/2$$

for any $z \in C$. By Lemma 2.1 and the fact that $f(C)$ is contained in the disk $|z| > e^5$ it follows that

$$\begin{aligned} d/2 > \operatorname{dist}_{\mathbb{C}\setminus\{0,1\}}(f(z), f(w_M)) &\geq \int_{|f(z)|}^{|f(w_M)|} \frac{1}{2t \ln t} \\ &= \frac{1}{2}(\ln\ln |f(w_M)| - \ln\ln |f(z)|), \end{aligned}$$

which gives

$$|f(z)| > \exp(\exp(\ln\ln |f(w_M| - d)) \quad = \quad |f(w_M)|^{1/e^d} > |M|^{1/e^d}$$

When $\alpha \neq 1$ the result follows directly by considering the function $f(z)/\alpha$. □

From now on we define $k > 0$ as in the above lemma, depending on d.

Corollary 2.3. *Let $f : D \to \mathbb{C} \setminus \{0\}$ be holomorphic, let $w_M \in C$ and write $M = |f(w_M)|$. Let $|\alpha| < M/k$. If there is $z \in C$ so that $|f(z)| \leq |\alpha|^{1-\frac{1}{e^d}} M^{1/e^d}$, then there exists $z \in D$ so that $f(z) = \alpha$.*

Proof. If there is no $z \in D$ so that $f(z) = \alpha$, then $f : D \to \mathbb{C} \setminus \{0, \alpha\}$. Moreover $|f(w_M)| = M > k|\alpha|$. Hence $|f(z)| > |\alpha|^{1-1/e^d} M^{1/e^d}$ for all $z \in C$, a contradiction. □

The following covering lemma bears similarities with Theorem 2.2 in [RS15].

Lemma 2.4. *Let $f : D \to \mathbb{C} \setminus \{0\}$ be holomorphic. Let $0 \leq m < M$ be such that there exists $w_M, w_m \in C$ with $|f(w_m)| = m$ and $|f(w_M)| = M$. Then $f(D)$ contains the annulus*

$$A = \Big\{ \left(\frac{m^{e^d}}{M} \right)^{\frac{1}{e^d - 1}} < |z| < M/k \Big\}.$$

Proof. Let $\alpha \neq 0$ and suppose that $\alpha \notin f(D)$. By Corollary 2.3, if $|\alpha| < M/k$ we have

$$m \geq |\alpha|^{\frac{e^d - 1}{e^d}} M^{1/e^d}$$

which gives

$$|\alpha| \leq \left(\frac{m^{e^d}}{M} \right)^{\frac{1}{e^d - 1}},$$

which implies that $\alpha \notin A$. □

From now on we assume that the domain $D \subset \mathbb{C}$ is simply connected.

Theorem 2.5. *Let $f : D \to \mathbb{C} \setminus \{0\}$ be holomorphic in a neighborhood of D. Let $0 \leq m < M$ be such that there exists $w_M, w_m \in C$ with $|f(w_m)| = m$ and $|f(w_M)| = M$.*

Let $N \in \mathbb{N}$ and define

$$A_N = \Big\{ \left(\frac{m^{e^d}}{M} \right)^{\frac{1}{e^d - 1}} < |z| < M/k^N \Big\}.$$

Then every $\alpha \in A_N$ has at least N distinct preimages in D.

Proof. If A_N is empty there is nothing to prove. Otherwise, since f omits 0 and D is simply connected we can choose an N^{th}-root $g = f^{1/N}$. Observe that $|g(w_M)| = M^{1/N}$, and that $|g(w_m)| = m^{1/N}$. Let $\alpha \in A_N$. Let $\{\eta_j\}_{j=1\ldots N}$ be the N-th roots of α. Let

$$B = \Big\{ \left(\frac{m^{e^d/N}}{|g(w_M)|} \right)^{\frac{1}{e^d - 1}} < |z| < |g(w_M)|/k \Big\}.$$

Since $\alpha \in A_N$, $\eta_j \in B$ for all j. By Lemma 2.4 , for each $j = 1 \ldots N$ there is $z_j \in D$ so that $g(z_j) = \eta_j$. By definition of g, $f(z_j) = \alpha$. □

The following is immediate, replacing 0 by any complex number.

Theorem 2.6. *Let $f : D \to \mathbb{C} \setminus \{\alpha\}$ be holomorphic in a neighborhood of D with $\alpha \in \mathbb{C}$. Let $0 < m < M$ be such that there exists $w_m, w_M \in C$ with $|f(w_m)| = m$ and $|f(w_M)| = M$.*

Fix $N \in \mathbb{N}$. Let

$$A_N = \Big\{ \left(\frac{(m + |\alpha|)^{e^d}}{|M - |\alpha||} \right)^{\frac{1}{e^d - 1}} < |z - \alpha| < |M - |\alpha||/k^N \Big\}.$$

Then every point in A_N has at least N distinct preimages in D.

For $R > 0$ define the annulus

$$A_R := \{R/2 < |z| < 2R\}$$

Corollary 2.7. *Let $f : D \to \mathbb{C}$ be holomorphic in a neighborhood of D. Let $0 < m < M$ be such that there exists $w_m, w_M \in C$ with $|f(w_m)| = m$ and $|f(w_M)| = M$. Let $k = k(d)$ be as in Lemma 2.2.*

Fix $N \in \mathbb{N}$, and let R, j such that $k^N < R^{j/2}$. Suppose that m, M satisfy the conditions:

$$\frac{|M - 2R|}{k^N} > 4R$$

and

$$\left(\frac{(m + 2R)^{e^d}}{|M - 2R|} \right)^{\frac{1}{e^d - 1}} < \frac{1}{R^{j/2}}.$$

Then either $A_R \subset f(D)$, or else there exists $\alpha \in A_R \setminus f(D)$ so that

$$\left(A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}}) \right) \subset f(D).$$

In the latter case each $\beta \in A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}})$ has at least N distinct preimages in D.

Proof. If $f(D) \supset A_R$ there is nothing to prove. Otherwise there is $\alpha \in A_R \setminus f(D)$ and we are in the case that $f : D \to \mathbb{C} \setminus \{\alpha\}$, hence Theorem 2.6 applies, with $|\alpha| < 2R$. In particular $f(D)$ covers at least N times the annulus A_N defined in Theorem 2.6. The conclusion follows by observing that the conditions on m, M imply that $\left(A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}}) \right) \subset A_N$. $\square$

For $R > 0$ and $\theta \in [0, 2\pi]$ we define

$$D_R := \{R/2 + 1/9 < |z| < 2R - 1/9\,,\ |\mathrm{Arg}(z) - \theta| < 3\pi/4\}, \ \text{ and}$$
$$C_R := \{2R/3 < |z| < 3R/2\,,\ |\mathrm{Arg}(z) - \theta| < 2\pi/3\}.$$

Note that D_R is simply connected, that $C_R \subset D_R \subset A_R$, and that C_R has finite hyperbolic diameter in D_R, say $d/2$, which is independent from R and θ. From now on we let $k > 0$ be the corresponding constant found in Lemma 2.2.

Theorem 2.8. *Let f be a transcendental entire function. Let $N \in \mathbb{N}$. Then there exist arbitrarily large R and j large and $\theta \in [0, 2\pi]$ so that either $A_R \subset f(D_R)$ or else there exists $\alpha \in A_R \setminus f(D_R)$ so that $\left(A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}})\right) \subset f(D_R)$. In the latter case, each $\beta \in \left(A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}})\right)$ has at least N distinct preimages in D_R.*

Remark 2.9. We can moreover guarantee that C_R contains at least two points of maximum modulus which are at least $R/10$ apart from each other as well as from the boundary of C_R, and that there is a point $w_m \in C_R$ with $|f(w_m)| < 3$ and whose distance from ∂C_R is at least $R/10$.

Proof of Theorem 2.8 and Remark 2.9. Observe that the hypotheses on m and M in Corollary 2.7 are satisfied provided that there exists $w_m, w_M \in C_R$ such that $|f(w_M)| = M > R^j$ and $|f(w_m)| = m < 3R$ for large enough R, j. Since the maximum modulus of f on $\{|z| = R\}$ grows faster than any polynomial in R, for R large enough we can always assume that there is a point w_M with $|w_M| = R$ and $|f(w_M)| > R^j$.

By Picard's Theorem, f takes on every value infinitely many times except at most one value, so we can choose arbitrarily large R so that there also exists a point w_m with $|w_m| = R$ and $|f(w_m)| = m < 3R$. Since C_R contains strictly more than half the circle $|w| = R$, and that there are points of maximum modulus for every R, it follows that we can choose θ so that both w_m and w_M are contained in C_R, and such that C_R contains at least two points of maximum modulus which are at least $R/10$ apart from each other as well as from the boundary of C_R. The claim follows from Corollary 2.7. □

Example 2.10. It is not true in general that for any entire transcendental function f there exists $R > 0$ such that $f(A_R)$ covers A_R arbitrarily many times. Indeed, let

$$f(z) = \prod_{i=1}^{\infty} \frac{z_i - z}{z_i},$$

where $z_i \to \infty$ very rapidly. For any R the set $f(A_R)$ covers A_R at most once. To see this, notice that for $|z_{j-1}| << |z| << |z_{j+1}|$ one obtains

$$f(z) \sim c_j z^{j-1} \cdot \frac{z_j - z}{z_j},$$

where $c_j = (z_0 \cdot z_1 \cdot \ldots \cdot z_{j-1})^{-1}$. Let $w \in A_R$. By Rouché's Theorem the difference between the number of solutions to the equation $f(z) = w$ on the two disks $\Delta(0, R/2)$ and $\Delta(0, 2R)$ is at most 1, hence $f(A_R)$ covers A_R at most once.

We note however that this example does have infinite entropy. Indeed, for $|z_i| << R << |z_{i+1}|$ consider the image of the disk $\Delta(0, R)$. By the above estimates each point in $\Delta(0, R)$ will have exactly i preimages in $\Delta(0, R)$, counted with multiplicity. By Theorem 1.2 the entropy of f is at least $\log(i)$.

Final preparations. We will make a few elementary observations before we start the proof of our main result.

Lemma 2.11. *Let $N \in \mathbb{N}$, and let $D \subset \mathbb{C}$ be a bounded simply connected domain, let f be a holomorphic function defined in a neighborhood of $\overline{D}$, and suppose that there exists $r > 0$ such that $|f(z)| \geq r$ for all $z \in \partial D$. If there exists $\xi \in \Delta(0, r)$ with N preimages in D, then every point in $\Delta(0, r)$ has N preimages in D (counted with multiplicity).*

Proof. For $w \in \Delta(0, r)$ let $g_w = f - w$. We claim that g_w has the same number of zeroes (counted with multiplicity) as f. Observe that $|g_w - f| < |f|$ on ∂D because $|w| < r$ and $|f| \geq r$ on ∂D. The claim follows by Rouché's Theorem since the function g_ξ has N zeroes. □

Lemma 2.12. *Let $R, j > 4$. There exists $d > 0$ such that the following holds. Let $z_1, z_2 \in C_R$, $\alpha \in A_R$, and assume that $z_i \notin \Delta(\alpha, R/20)$ for $i = 1, 2$. Then there exists a simply connected open set $D \subset D_R \setminus \Delta(\alpha, 1/R^{j/2})$ with $z_1, z_2 \in D$ such that $\mathrm{dist}_D(z_1, z_2) < d/2$.*

Proof. We consider three cases:
(i) $\Delta(\alpha, R/20) \cap D_R = \emptyset$. Then we choose $D = D_R$.
(ii) $\Delta(\alpha, R/20) \cap C_R = \emptyset$. Let D be the tubular neighborhood of C_R with radius $R/20$.
(iii) $\Delta(\alpha, R/20) \cap C_R$ is nonempty. Let $I_1 \ldots I_4$ be four arcs starting at α, two radial segments and two circular arcs, ending when they hit the boundary of D_R (see Figure 1 for an illustration). Then we let $D = D_R \setminus (\Delta(\alpha, R/20) \cup I_i)$ for a suitable i depending on the position of the points z_1, z_2. It is clear that D is simply connected, and that i can be chosen to obtain a uniform bound on $\mathrm{dist}_D(z_1, z_2)$ not depending on the positions of z_1, z_2 and of α. □

Lemma 2.13. *Let $\varepsilon > 0$ and $\ell \in \mathbb{N}$. Let α, z_1, z_2 and $x_1, \ldots, x_\ell$ be points in the annulus*

$$A_R(\varepsilon) := \{R/2 + \varepsilon R \leq |z| \leq 2R - \varepsilon R\}$$

and assume that $|\alpha - z_j|$ and $|x_i - z_j| \geq \varepsilon R$ for all $i = 1, \ldots \ell$ and $j = 1, 2$. Then there exists $d > 0$, depending only on ε and ℓ, and a simply connected domain $D \subset A_R$ avoiding all the points x_i and satisfying

$$D \cap \Delta(\alpha, \frac{1}{R^{j/2}}) = \emptyset,$$

such that $\mathrm{dist}_D(z_1, z_2) < d/2$.

Proof. Up to rescaling we may assume that $R = 1$. Observe that for each choice of points α, z_1, z_2 and $x_1, \ldots, x_\ell$ we can find such a simply connected domain D containing z_1, z_2 by removing the disk $\Delta(\alpha, \frac{1}{R^{j/2}})$ and for each

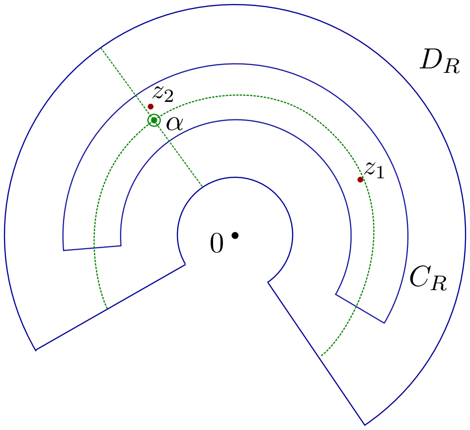

Figure 1. Illustration of the proof of Lemma 2.12. In green: the point α, the boundary of the disk $\Delta(\alpha, R/20)$ and the four arcs I_i.

point α or x_i a path connecting the point α or x_i to ∂A_R. Each path can be chosen to be either a radial interval, or a combination of a small circular interval and a radial interval.

Note that the construction also works when the points lie in the closed annulus $\overline{A_R(\varepsilon)}$ and that each construction gives uniform estimates on the hyperbolic distance between z_1 and z_2 for nearby locations of the points. Compactness of the initial conditions implies that the constant d depends only on ε and ℓ.

$\square$

Main statement and proof. Let us recall the statement of our main Theorem:

Theorem 1.1. *Let f be a transcendental entire function, and let $N \in \mathbb{N}$. There exists a non-empty bounded open set $V \subset \mathbb{C}$ so that $V \subset f(V)$ and such that any point in V has at least N preimages in V under f, counted with multiplicity.*

Proof. Fix $N \in \mathbb{N}$. Let $d/2$ be such that Lemma 2.12 and Lemma 2.13 hold for $\ell = N$ and $\varepsilon = \frac{R}{2N(N+2)}$. Observe that if Lemma 2.13 is satisfied for $d/2$ with $\ell = n$, it is also satisfied for all $\ell < n$. Let k be so that Lemma 2.2 holds for $d/2$. Let j and R large enough so that Corollary 2.7 holds for k. Choose R, θ, and j such that the hypotheses in Theorem 2.8 are satisfied.

It follows that either $A_R \subset f(D_R)$, or $f(D_R)$ covers $A_R \setminus \Delta(\alpha, 1/R^{j/2})$ at least N times for some $\alpha \in A_R \setminus f(D_R)$.

Case I. : $f(D_R) \not\supset A_R$.

In this case there exists $\alpha \in A_R \setminus f(D_R)$ such that $f(D_R)$ covers at least N times the set $A_R \setminus \Delta(\alpha, 1/R^{j/2})$. Let us look for a subset of D_R such that its image covers itself at least N times.

If

$$f(\Delta(\alpha, 1/R^{j/2}) \cap D_R) \cap D_R = \emptyset, \tag{2.1}$$

we can choose $V = D_R \setminus \Delta(\alpha, 1/R^{j/2})$ and the proof is complete. Hence we can assume that $f(\Delta(\alpha, 1/R^{j/2}) \cap D_R) \cap D_R \neq \emptyset$, and in particular that there exists a point $\xi \in \Delta(\alpha, 1/R^{j/2}) \cap D_R$ with $|f(\xi)| < 2R$.

We will also assume that $\Delta(\alpha, R/20) \subset\subset D_R$. Indeed, if this is not the case, the proof is completely analogues by replacing D_R by a slightly larger simply connected open set $\widetilde{D}_R \subset\subset A_R$ for which $\Delta(\alpha, R/20) \subset\subset \widetilde{D}_R$ is satisfied. In this case, if $f(\widetilde{D}_R)$ keeps omitting α we apply the proof of case I, otherwise we move to case II.

Let w_M be a point in $C_R \setminus \Delta(\alpha, R/20)$ for which $|f(w_M)| \geq R^j$. Recall that we may assume that such point exists, since by Remark 2.9 we can choose C_R to contain at least two points of maximum modulus of distance at least $R/10$ apart from each other.

We claim that there also exists a point $w_m \in C_R \setminus \Delta(\alpha, R/20)$ so that $|f(w_m)| < 3R$. Let $w_m \in C_R$ be as in Remark 2.9. If $\Delta(\alpha, R/20) \cap \partial C_R \neq \emptyset$, $w_m \in C_R \setminus \Delta(\alpha, R/20)$ as required. Otherwise $\Delta(\alpha, R/20) \subset\subset C_R$. In this case, let us assume by contradiction that $|f(z)| > 3R$ for all $z \in C_R \setminus \Delta(\alpha, R/20)$. Then we also have that $|f(z)| \geq 3R$ on $\partial\Delta(\alpha, R/20)$. By Lemma 2.11, since there is $\xi \in \Delta(\alpha, R/20)$ with $|f(\xi)| < 2R$, we have that $f(\Delta(\alpha, R/20)) \supset \Delta(0, 3R)$. This contradicts the fact that $\alpha \in \Delta(0, 3R)$ was assumed not to lie in $f(D_R)$.

Now let D be as in Lemma 2.12, where $z_1 := w_M$ and $z_2 := w_m$. Since A_R is not contained in $f(D)$, it follows by Corollary 2.7 that $f(D)$ covers $A_R \setminus \Delta(\alpha, R^{j/2})$ at least N times. Since D is contained in $A_R \setminus \Delta(\alpha, R^{j/2})$ this concludes the proof of case I.

Case II: $f(D_R) \supset A_R$**.**

Observe that for each fixed N, Theorem 2.8 holds for arbitrarily large radii R. If there is at least one of them for which case I holds, we are done. Otherwise, for every R given by Theorem 2.8 we have that $f(D_R) \supset A_R$ and hence that $f(A_R) \supset A_R$.

If there are arbitrarily large R for which $f(A_R)$ covers itself at least N times we are also done. Hence we may assume that there exists $1 \leq \ell < N$ such that for any of the R given by Theorem 2.8 there is a point $\alpha = \alpha(R) \in A_R$ which has at most ℓ preimages in A_R, counted with multiplicity. We can therefore find a sequence of values of R for which the maximum number of preimages in A_R of some point α is at most ℓ, and write $\zeta_1 = \zeta_1(R), \ldots, \zeta_\ell = \zeta_\ell(R) \in A_R$ for the preimages of α in A_R.

Let $W := A_R \cap \{z \in \mathbb{C} : |f(z)| < 2R\}$. For $i = 1 \dots \ell$ let W_i be the connected component of W which contains ζ_i (possibly, they are not all distinct). Now one of the two following cases occurs.

Case IIa: For arbitrarily large R there exists $R/2 < r < 2R$ such that the circle $\{|z| = r\}$ does not intersect the set W.

We claim that if R is chosen large enough then $f(\Delta(0, r))$ covers $\Delta(0, r)$ at least N times, giving the claim. Let $v \in \mathbb{C}$ be a non-exceptional value for f with $|v| < 1$. By Picard's theorem, f takes on the value v infinitely many times in any neighborhood of infinity, hence by choosing R sufficiently large we may assume that v has at least N preimages in the disk $\Delta(0, R/2) \subset \Delta(0, r)$. Since W does not intersect the circle $\partial\Delta(0, r)$, we have that $|f(z)| \geq 2R$ on $\partial\Delta(0, r)$. Hence by Lemma 2.11, in $\Delta(0, r)$, the function f takes on any value in $\Delta(0, 2R) \supset \Delta(0, r)$ at least N times, counted with multiplicity.

Case IIb: for arbitrarily large R the set W intersects all circles $\{|z| = r\}$ for $R/2 < r < 2R$.

Then there is some W_i, say W_0 up to relabeling, with diameter at least $\frac{3R}{2\ell}$ for arbitrarily large R.

We claim that there exist $w_m, w_M \in A_R$ with $|f(w_m)| < 2R$ and $|f(w_M)| > R^j$, and such that $|w_m - \zeta_i|, |w_M - \zeta_i| > \frac{R}{2\ell(\ell+2)}$ for $i = 1, \dots, \ell$, and $|w_m - \alpha|, |w_M - \alpha| > \frac{R}{2\ell(\ell+2)}$. We also claim that the distance between w_m, w_M and the boundary of A_R is at least $\frac{R}{2\ell(\ell+2)}$.

Indeed, there are at most $\ell + 1$ points in W_0 that need to be avoided (all of the ζ_i and α), so we can always find $w_m \in W_0$ which is at Euclidean distance at least $\frac{\operatorname{diam}_{\text{Eucl}} W_0}{2(\ell+2)} > \frac{3R}{4\ell(\ell+2)}$ from all of the ζ_i and from α, as well as from the boundary of A_R. By definition $|f(w_m)| < 2R$.

To find w_M it is enough to find a point of maximum modulus in A_R minus the set $U = \bigcup_i \Delta(\zeta_i, \frac{R}{2\ell(\ell+2)}) \cup \Delta(\alpha, \frac{R}{2\ell(\ell+2)})$, and which is at distance at least $\frac{R}{2\ell(\ell+2)}$ from ∂A_R. This means that we have to avoid at most $\ell + 2$ disks of diameter $\frac{R}{\ell(\ell+2)}$, hence there are circles in $A_R \setminus U$ in which we can choose a point of maximum modulus as required, which settles the claim.

By Lemma 2.13 and our choice of d in the beginning of the proof, we can find $D \subset A_R$ simply connected with $w_m, w_M \in D$ and $\zeta_i \notin D$ for $i = 1, \dots \ell$, and with $D \cap \Delta(\alpha, \frac{1}{R^{j/2}}) = \emptyset$, and such that $\operatorname{dist}_D(w_m, w_M) < d/2$. By our choice of the constants k, j and R Corollary 2.7 holds, and since $f(D)$ omits α by construction, we have that for R sufficiently large $f(D)$ covers at least N times the set

$$A_R \setminus \Delta(\alpha, 1/R^{j/2}) \supset D.$$

□

Let us observe that each of the three cases I, IIa and IIb can occur. Indeed, case I occurs when f has an omitted value [BFP18]; case IIa occurs in Example 2.10; and case IIb occurs for $f(z) = e^z$.

Appendix: Topological entropy on $\mathbb{C}$.

For maps acting on compact spaces the concept of topological entropy has been introduced in [AKM65]. In the literature there are several non-equivalent natural generalizations for the definition of topological entropy on non-compact spaces (see for example [Bow73b], [Bow71], [Bow73a], [Hof74], and more recently [HNP08]). We will use the following:

Definition 2.14 (Definition of topological entropy)**.** Let $f : Y \to Y$ be a self-map of a metric space (Y, d). Let X be a compact subset of Y. Let $n \in \mathbb{N}$ and $\delta > 0$. A set $E \subset X$ is called *(n, δ)-separated* if

- for any $z \in E$, its orbit $\{z, f(z), \ldots, f^{n-1}(z)\} \subset X$;
- for any $z \neq w \in E$ there exists $k \leq n-1$ such that $d(f^k(z), f^k(w)) > \delta$.

Let $K(n, \delta)$ be the maximal cardinality of an (n, δ)-separated set. Then the *topological entropy* $h_{\text{top}}(X, f)$ is defined as

$$h_{\text{top}}(X, f) := \sup_{\delta > 0} \left\{ \limsup_{n \to \infty} \frac{1}{n} \log K(n, \delta) \right\}.$$

We define the topological entropy $h_{\text{top}}(f)$ of f on Y as the supremum of $h_{\text{top}}(X, f)$ over all compact subsets $X \subset Y$.

When Y is compact the definition coincides with the usual definition. In the literature the finite orbits $\{z, f(z), \ldots, f^{n-1}(z)\}$ are often not required to remain in X. A disadvantage of this definition is that the entropy is then dependent on the metric; for example, the entropy of a polynomial acting on the complex plane is then infinite with respect to the Euclidean metric. Our definition above, which may give a smaller value for the entropy, is independent of the metric inducing the topology, and is invariant under topological conjugation.

We are now ready to prove Theorem 1.2. The ideas of the proof are similar to the ideas used in [MP77].

Proof of Theorem 1.2. Denote the set of critical points of g in $\overline{V}$ by $\mathcal{C}$. Note that $\mathcal{C}$ is finite. Let $\mathcal{C}_0 \subset \mathcal{C}$ contain only those critical points that are not periodic. Write D for the product of the local degrees of g at the critical points in $\mathcal{C}_0$.

Fix a point $w \in V$, not contained in a periodic cycle containing a critical point. It follows that all inverse orbits of w avoid a sufficiently small neighborhood of each super-attracting periodic cycle. Let us denote the complement of these neighborhoods in V by V'.

Let $m \in \mathbb{N}$, and let $\rho = \rho(m) > 0$ be such that for every $x \in \mathcal{C}_0$ and every $n = 1, \dots, m$ we have

$$g^n(\Delta(x,\rho)) \cap \Delta(x,\rho) = \emptyset.$$

Such ρ can be chosen by finiteness of $\mathcal{C}_0$, and since the points x are not periodic. By decreasing $\rho > 0$ if necessary we may assume that the disks $\Delta(x,\rho)$ are pairwise disjoint.

There exists an $\varepsilon = \varepsilon(m) > 0$ such that the following two properties hold: For each $y \in V' \setminus \bigcup_{x \in \mathcal{C}_0} g(\Delta(x,\rho))$ there are at least N preimages of y that are ε-separated. On the other hand, if $y \in g(\Delta(x,\rho))$ then the number of preimages (counted with multiplicity) near x that are not ε-separated is at most the local degree of g at x, and the other preimages have distance at least ε to the preimages near x.

Consider a finite inverse orbit y_0, y_{-1}, y_{-m} of a point $y_0 \in V'$. By the estimates on the number of preimages that may not be separated, and by the fact that any inverse orbit of length m enters each disk $\Delta(x,\rho)$ at most once, it follows that there are at most $D - 1$ other inverse orbits of y_0 of length m that are not ε-separated from y_0. Thus, a lower bound for the number of ε-separated backwards m-orbits of y_0 is given by $\frac{N^m}{D}$.

Since the lower estimate holds for any $y \in V'$, it holds in particular for any point in $f^{-km}(w)$. Hence for any $k \in \mathbb{N}$, the number of ε-separated backwards orbits of w of length km is at least

$$\left(\frac{N^m}{D}\right)^k = \left(\frac{N}{D^{1/m}}\right)^{km},$$

which is therefore a lower bound for $K(km, \varepsilon)$, the maximal cardinality of a (km, ε)-separated set. Since D is a fixed constant and we can let m converge to infinity as $\varepsilon \to 0$, it follows that the topological entropy is at least $\log(N)$. □

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