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Homologies on Path Complexes

Bachelor's thesis in mathematical sciences
Supervisor: Abigail Linton
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Abstract. A path complex generalizes both simplicial complexes and directed graphs. We use the definition of path homology established by Grigor'yan, Lin, Muranov and Yau in [4], which grants a notion of homology on directed graphs. We then construct examples of path complexes that have torsion in their homology group with the goal of studying these homology groups geometrically and demonstrating the notion of torsion in path complexes.

Contents

1. Introduction

The goal of this thesis is to establish the notion of path complexes and study their homology groups. The thesis is heavily influenced by the work of Grigor'yan, Lin, Muranov and Yau in [4] and [6], and the book on algebraic topology by Hatcher [7]. We will consider some proofs and examples not covered in these sources, particularly in the final section where we discuss examples of complexes with torsion elements in their homology groups.

Path complexes were introduced by Grigor'yan, Lin, Muranov and Yau in [4]. A path complex is regarded as a generalization of a simplicial complex, as any simplicial complex is uniquely expressed as a path complex. By defining a boundary operator much like the boundary operator on simplicial complexes, the article establishes a homology and cohomology theory on path complexes that is consistent with the well established theory of homology on simplicial complexes.

The interest for path complexes mainly stem from their ability to easily represent any directed graph, giving a concept of path homology of directed graphs. Establishing a proper homology theory of directed graphs has been tried many times before with mixed success as explained in the introduction of [4]. There are however multiple advantages with path homology theory compared to these earlier attempts. Path homologies are easy to compute and allow non-trivial higher order homology groups when needed. With the homotopy theory for directed graphs established by Grigor'yan, Lin, Muranov and Yau in [5], we get homotopy invariance of path homologies of directed graphs. The homology theory is also proven to be dual to the cohomology theory of directed graphs established by Dimakis and Müller-Hoissen in [1] and [2]. This proof was performed by Grigor'yan and Muranov in [3].

Path complexes can concievably have many future applications in pure and applied mathematics. Problems involving coverage verification in sensor networks, such as the problem described by Tahbaz-Salehi and Jadbabaie in [8], is an example of where this theory could potentialy see future application. Path homologies were also used in a simple example of a graph coloring problem in [5].

The focus of this thesis will be computing homologies of path complexes. These results are applicable to any directed graph. The thesis starts with a general introduction to simplicial complexes and their homologies, as this theory has a lot of similarities to path homologies. In the next section we discuss paths and their boundaries. We then define path complexes and their homologies in the following two sections respectively. In the final section we discuss expressing the real projective plane as a path complex and calculate the corresponding homology groups. We will compute multiple examples throughout the thesis to give a better understanding of how these calculations are performed.

It could be of interest for future projects to explore the meaning of path homologies on directed graphs further. It would also be interesting to do further research into path cohomologies and their applications. This would involve defining an exterior differential dual to the boundary operator and exploring the concept of forms as defined in [4].

2. Simplicial Complexes

The theory of simplicial complexes and their homologies bear many similarities to the theory of path complexes. In this section we will establish some central concepts of this theory to have a point of reference when discussing path complexes.

Let S be a finite set of points in a Euclidean space. The elements of S are called vertices. We give every vertex in S a distinct natural number value i and denote the vertex by v_i.

Definition 2.1. Let $i_0 < i_1 < ... < i_n$ be natural numbers. Consider a set of $n+1$ vertices $v_{i_0}, v_{i_1}, ..., v_{i_n} \in S$, such that all the vectors $v_{i_1} - v_{i_0}, v_{i_2} - v_{i_0}, ..., v_{i_n} - v_{i_0}$ are linearly independent. The smallest convex subset that contain all these points is called an $n-$simplex.

We denote such an $n-$simplex by $[v_{i_0}, v_{i_1}, ..., v_{i_n}]$ as any $n-$simplex is uniquely determined by its vertices. We observe that a $0-$simplex is a point, a $1-$simplex is a line, a $2-$simplex is a filled triangle and a $3-$simplex is a filled tetrahedron. Any higher order simplex can be imagined as the $n-$dimensional analog of a triangle.

Consider a set of $n+1$ vertices $v_0, ..., v_n$ that define an $n-$simplex k. If we remove any m vertices from this set we get a generating set of an $(n-m)-$simplex that is completely contained in k. We refer to this $(n-m)-$simplex as the $(n-m)-$face of k.

Definition 2.2. A simplicial $n-$complex K is a set of simplices of order n or lower such that if $k_1, k_2 \in K$:

- Every face of a simplex k_1 in K is also in K.
- $k_1 \cap k_2 \in K$ if the intersection is non-empty.

We choose S to be all the $0-$simplices in K to not consider any unnecessary vertices. This also gives a well defined ordering of vertices for all simplices in K.

Example 2.3. Let K_1 be the simplicial complex in Figure 1.

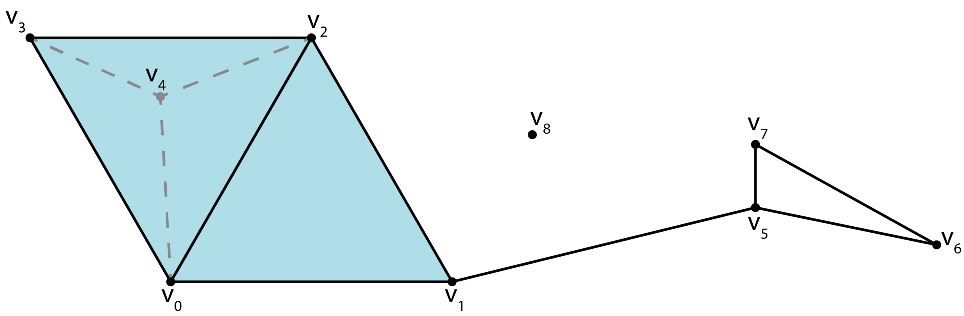

Figure 1. A simplicial complex.

The simplices of K_1 are

$$v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, [v_0, v_1], [v_0, v_2], [v_0, v_3], [v_0, v_4], [v_1, v_2],$$

$$[v_1, v_5], [v_2, v_3], [v_2, v_4], [v_3, v_4], [v_5, v_6], [v_5, v_7], [v_6, v_7], [v_0, v_1, v_2],$$

$$[v_0, v_2, v_3], [v_0, v_2, v_4], [v_0, v_3, v_4], [v_2, v_3, v_4], [v_0, v_2, v_3, v_4].$$

Note especially that every face of $[v_0, v_2, v_3, v_4]$ is a simplex in K_1. Note also that $[v_5, v_6, v_7]$ is not in K_1 although all its $1-$faces are in K_1.

A concept of boundary for an $n-$simplex is necessary when calculating homology groups. The sum of all $(n-1)-$faces is a natural choice for such a boundary. It turns out that it is convenient to change the sign of each term in the sum depending on which vertex we are removing because of the strictly increasing indices of the faces. This is tied to the orientation of simplices and how we want the boundary operator to act when calculating our homology groups.

Definition 2.4. Consider an $n-$simplex $[v_0, ..., v_n]$. The boundary of such a simplex is given by the boundary operator

$$\partial_s[v_0, ..., v_n] = \sum_{k=0}^{n}(-1)^k[v_0, ..., \hat{v_k}..., v_n] \tag{1}$$

The hat denotes the exclusion of that element from the simplex. For linear combinations of $n-$simplices we define the boundary operator linearly, giving us a homomorphism. For $n = 0$, we define the boundary to be zero.

We omit the proof of the following lemma. This is because it is completely analogous to the proof of Lemma 3.4. The proof can also be found in [7].

Lemma 2.5. $\partial_s^2 = 0$.

Consider the set of all linear combinations of $n-$simplices from some simplicial complex K with whole number coefficients. We denote this set $C_n = C_n(K, \mathbb{Z})$. We observe that the boundary of an $n-$simplex is an $(n-1)-$simplex itself, which gives rise to the inclusion $\partial_s|_{C_n} \subset C_{n-1}$. This, paired with Lemma 2.5, gives rise to the following chain complex:

$$0 \leftarrow C_0 \leftarrow ... \leftarrow C_{n-1} \leftarrow C_n \leftarrow ... \tag{2}$$

where the arrows represent the boundary operator on the simplices, ∂_s.

Definition 2.6. Let K be a simplicial complex. The nth homology group $H_n(K)$ is defined as

$$H_n(K) = H(C_n(K))) = Z_n/B_n = \text{Ker}\partial|_{C_n}/\text{Im}\partial|_{C_{n+1}} \tag{3}$$

The homology groups are defined for each set in the chain complex (2). The groups Z_n and B_n are referred to as the $n-$cycles and $n-$boundaries of our simplicial complex respectively.

Lemma 2.5 tells us that any boundary of an $(n + 1)$-simplex is also an $n-$cycle. We can therefore look at a simplicial complex and observe which cycles are boundaries, and then omit them from the generating sets. We denote a generating set by $\langle\cdot\rangle$.

Example 2.7. We want to find the homology of the simplicial complex K_1 in Figure 1. The boundary operator sends all $0-$simplices to 0, so $Z_0 = C_0$. Furthermore, we see that $\partial_s([v_1, v_2]) = \partial_s([v_0, v_2] - [v_0, v_1])$. We therefore include $[v_0, v_2]$ and $[v_0, v_1]$ in the generating set of B_0, but we do not include the $1-$simplex $[v_1, v_2]$ as it does not change the generated set. This same logic can be applied to many of the $1-$simplices in K_1, and we therefore choose to not include these elements in the generating set of B_0.

Adding a linear combination of elements from a generating set to another element in the same generating set does not change the generated set. This allows us to manipulate the generating sets of cycles and boundaries so that some of their generating elements cancel out.

$$\begin{aligned} H_0(K_1) &= \frac{\langle v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \rangle}{\langle v_1 - v_0, v_2 - v_0, v_3 - v_0, v_4 - v_0, v_5 - v_1, v_5 - v_6, v_5 - v_7 \rangle} \\ &= \frac{\langle v_0, v_1 - v_0, v_2 - v_0, v_3 - v_0, v_4 - v_0, v_5 - v_1, v_6 - v_5, v_7 - v_5, v_8 \rangle}{\langle v_1 - v_0, v_2 - v_0, v_3 - v_0, v_4 - v_0, v_5 - v_1, v_5 - v_6, v_5 - v_7 \rangle} \\ &= \langle v_0, v_8 \rangle \cong \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

We do the same process to find the higher order homology groups. We omit any cycle that corresponds to the boundary of a simplex as they will cancel out. The only $1-$cycle that is not removed is $[v_5, v_6] + [v_6, v_7] - [v_5, v_7]$, as $[v_5, v_6, v_7]$ is not a $2-$simplex. The only $2-$cycle is equivalent to the $2-$boundary generated by $[v_0, v_2, v_3, v_4]$ and is therefore omitted. Since there only exists one $3-$simplex and no higher order simplices there are no $n-$cycles for all $n \geq 3$. We therefore get

$$H_1(K_1) = \langle [v_5, v_6] + [v_6, v_7] - [v_5, v_7] \rangle \cong \mathbb{Z}$$

$$H_n(K_1) = 0 \text{ for all } n \geq 2$$

3. Paths

Before we are able to discuss path complexes in detail we have to first understand the concepts of paths and path boundaries. All of these defintions are established in [4]. They also bear a strong resemblance to equivalent definitions on simplicies which we discussed in the previous section.

Let V be a finite set of arbitrary points. This set differs from S since the points in V do not necessarily exist in a Euclidean space. The elements of V are also called vertices, but in the context of paths.

Definition 3.1. Any ordered set of vertices from V is called an elementary path on V. Such a set that contains $n + 1$ vertices is called an elementary $n-$path on V, or simply an elementary $n-$path when there is no ambiguity.

For vertices $i_0, i_1, ..., i_n \in V$, let $i_0 i_1 ... i_n$ denote an elementary $n-$path in V. An elementary $0-$path is a vertex from V. The set of all linear combinations of elementary $n-$paths on V with coefficients in F is denoted $\Lambda_n = \Lambda_n(V, F)$. We consider $F = \mathbb{Z}$ unless otherwise specified. The elements of Λ_n are called the $n-$paths in V, and can be written in the form

$$v = \sum_{i_0 i_1 \dots i_n} v^{i_0 i_1 \dots i_n} i_0 i_1 \dots i_n \tag{4}$$

where $v^{i_0 i_1 \dots i_n} \in F$. The boundary operator on an elementary $n-$path is defined the same way as the boundary operator on an $n-$simplex. It naturally generalizes to any $n-$path as it is linear.

Definition 3.2. Consider an $n-$path v as in (4). Its boundary ∂v is

$$\begin{aligned}\partial v &= \sum_{i_0 i_1 \dots i_n} v^{i_0 i_1 \dots i_n} \partial(i_0 i_1 \dots i_n) \\ &= \sum_{i_0 i_1 \dots i_n} \sum_{q=0}^{n} (-1)^q v^{i_0 i_1 \dots i_n} i_0 i_1 \dots, \hat{i_q}, \dots, i_n \end{aligned} \tag{5}$$

where $\hat{i_q}$ denotes the exclusion of this element from the path. In this text the boundary of a $0-$path is defined to be 0.

Example 3.3. Consider a $2-$simplex $[v_0, v_1, v_2]$. Any ordered set of these vertices is considered an elementary path on the simplex. Instead consider only the ordered sets of vertices with strictly increasing indices. Only a finite set of paths fulfill this criteria:

$$\{v_0, v_1, v_2, v_0 v_1, v_0 v_2, v_1 v_2, v_0 v_1 v_2\}$$

We see that this is the set of all faces of $[v_0, v_1, v_2]$. We also observe that the boundary of any path from the set is also in the set.

Lemma 3.4. $\partial^2 = 0$

Proof. Let v be a path as in (4). We take the boundary of this path twice. As the boundary operator is linear, we can move the operator into our summation as in (5) and apply it to the elementary paths:

$$\begin{aligned}\partial^2 v &= \partial \left(\sum_{i_0 i_1 \dots i_n} \sum_{q=0}^{n} (-1)^q v^{i_0 i_1 \dots i_n} i_0 i_1 \dots \hat{i_q} \dots i_n \right) \\ &= \sum_{i_0 i_1 \dots i_n} \sum_{q=0}^{n} (-1)^q v^{i_0 i_1 \dots i_n} \partial(i_0 i_1 \dots \hat{i_q} \dots i_n) \end{aligned} \tag{6}$$

Consider the elementary path $i_0 i_1 \dots \hat{i_q} \dots i_n = i_0 i_1 \dots i_{q-1} i_{q+1} \dots i_n$. We give this path new indices saying

$$j_0 = i_0, j_1 = i_1, \dots, j_{q-1} = i_{q-1}, j_q = i_{q+1}, \dots, j_{n-1} = i_n.$$

This means that

$$\begin{aligned}
&\partial(i_0 i_1 ... \hat{i_q} ... i_n) = \partial(j_0 j_1 ... j_{n-1}) \\
&= \sum_{k=0}^{n-1} (-1)^k j_0 j_1 ... \hat{j_k} ... j_{n-1} \\
&= \sum_{k=0}^{q-1} (-1)^k j_0 j_1 ... \hat{j_k} ... j_{n-1} + \sum_{k=q}^{n-1} (-1)^k j_0 j_1 ... \hat{j_k} ... j_{n-1} \\
&= \sum_{k=0}^{q-1} (-1)^k i_0 i_1 ... \hat{i_k} ... \hat{i_q} ... i_n + \sum_{k=q}^{n-1} (-1)^k i_0 i_1 ... \hat{i_q} ... \widehat{i_{k+1}} ... i_n \\
&= \sum_{k=0}^{q-1} (-1)^k i_0 i_1 ... \hat{i_k} ... \hat{i_q} ... i_n + \sum_{k=q+1}^{n} (-1)^{k-1} i_0 i_1 ... \hat{i_q} ... \hat{i_k} ... i_n \qquad (7)
\end{aligned}$$

Combining (6) and (7) we get

$$\begin{aligned}
\partial^2 v = &\sum_{i_0 i_1 ... i_n} \sum_{q=0}^{n} \sum_{k=0}^{q-1} (-1)^{q+k} v^{i_0 i_1 ... i_n} i_0 i_1 ... \hat{i_k} ... \hat{i_q} ... i_n \\
&+ \sum_{i_0 i_1 ... i_n} \sum_{q=0}^{n} \sum_{k=q+1}^{n} (-1)^{q+k-1} v^{i_0 i_1 ... i_n} i_0 i_1 ... \hat{i_q} ... \hat{i_k} ... i_n \\
&= \sum_{i_0 i_1 ... i_n} \sum_{q=0}^{n} \sum_{k=0}^{q-1} (-1)^{q+k} v^{i_0 i_1 ... i_n} i_0 i_1 ... \hat{i_k} ... \hat{i_q} ... i_n \\
&+ \sum_{i_0 i_1 ... i_n} \sum_{k=0}^{n} \sum_{q=0}^{k-1} (-1)^{q+k-1} v^{i_0 i_1 ... i_n} i_0 i_1 ... \hat{i_q} ... \hat{i_k} ... i_n
\end{aligned}$$

In the final sum we have used that taking the sum over every q and demanding k greater than q gives the same terms as if we take the sum over every k demanding q less than k. Switching the index names k and q then gives

$$\begin{aligned}
\partial^2 v = &\sum_{i_0 i_1 ... i_n} \sum_{q=0}^{n} \sum_{k=0}^{q} (-1)^{q+k} v^{i_0 i_1 ... i_n} i_0 i_1 ... \hat{i_k} ... \hat{i_q} ... i_n \\
&+ \sum_{i_0 i_1 ... i_n} \sum_{q=0}^{n} \sum_{k=0}^{q} (-1)^{q+k-1} v^{i_0 i_1 ... i_n} i_0 i_1 ... \hat{i_k} ... \hat{i_q} ... i_n \\
&= \sum_{i_0 i_1 ... i_n} \sum_{q=0}^{n} \sum_{k=0}^{q} \left((-1)^{q+k} + (-1)^{q+k-1} \right) v^{i_0 i_1 ... i_n} i_0 i_1 ... \hat{i_k} ... \hat{i_q} ... i_n \\
&= 0.
\end{aligned}$$

□

By definition of the boundary operator, the boundary of an $n-$path is a $(n-1)-$path. Combined with Lemma 3.4 this gives rise to a chain complex of elementary $n-$paths on V:

$$0 \leftarrow \Lambda_0 \leftarrow ... \leftarrow \Lambda_{p-1} \leftarrow \Lambda_p \leftarrow ... \tag{8}$$

where the arrows represent the boundary operator ∂.

Definition 3.5. An elementary $n-$path $i_0 i_1 ... i_{n-1}$ is non-regular if $i_k = i_{k+1}$ for any $k = 0, 1, ..., n-2$. The elementary path is otherwise called regular.

A regular path is a path as in (4), but only consisting of regular elementary paths. The space of all such $n-$paths with coefficients in F is denoted by $R_n = R_n(V, F)$.

The boundary of a regular path is not necessarily regular.Take for example the path $i_0 i_1 i_0$ which has $i_0 i_0$ in its boundary despite being regular. We can however consider an alternate boundary operator that sets the coefficient of any non-regular term in the boundary to zero, giving us a regular path. We denote this boundary operator ∂^{reg} and name it the regular boundary operator. It gives rise to a chain complex similar to (8), but with regular spaces and the regular boundary operator.

4. Path Complexes

In this section we define the notion of path complexes and explore their relation to simplicial complexes and directed graphs. We also lay the groundwork for the next section, where we define the homology groups of a path complex.

Definition 4.1. A path complex P over V is a non-empty collection of elementary paths in V such that if $i_0 i_1 ... i_n \in P$, then $i_1 i_2 ... i_n \in P$ and $i_0 i_1 ... i_{n-1} \in P$.

The paths $i_1 i_2 ... i_n$ and $i_0 i_1 ... i_{n-1}$ are called the truncated paths of $i_0 i_1 ... i_n$. We say that elementary paths in P are allowed, while elementary paths not in P are non-allowed. Let P_n denote the set of allowed elementary $n-$paths. The elements of P_1 are called the edges of our path complex.

Consider a simplicial complex K with vertices $v_0, v_1, ... v_k \in S$. K can be expressed as a path complex by demanding that the path complex satisfy the following criteria.

- An $n-$path $v_{i_0} v_{i_1} ... v_{i_n}$ is only allowed if $i_0 < i_1 < ... < i_n$.
- An $n-$path is only allowed if its vertices are exactly the vertices of an $n-$simplex in K.

Example 4.2. We see that the simplicial complex K_1 in Figure 1 is uniquely identified by the path complex $P = P_0 \cup P_1 \cup P_2 \cup P_3$ where

$$P_0 = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$
$$P_1 = \{v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_1v_2, v_1v_5, v_2v_3, v_2v_4, v_3v_4, v_5v_6, v_5v_7, v_6v_7\}$$
$$P_2 = \{v_0v_1v_2, v_0v_2v_3, v_0v_2v_4, v_0v_3v_4, v_2v_3v_4\}$$
$$P_3 = \{v_0v_2v_3v_4\}$$

Note that $v_5v_6v_7$ is not in the path complex as $[v_5, v_6, v_7]$ is not a $2-$simplex in K_1.

Consider a path complex P. This path complex is uniquely associated with a simplicial complex if we can give every vertex a whole number value $i_0, i_1, ..., i_k$ such that the following two criteria are satisfied:

- It's perfect: Given an allowed elementary path, any subsequence of vertices from the path with the same ordering is allowed. This corresponds to the fact that every face of a simplex must be included in the simplicial complex.
- It's monotone: A path $i_{j_0}i_{j_1}...i_{j_n}$ is only allowed if $j_0 < j_1 < ... < j_n$.

We can also express any directed graph G as a path complex. Given a directed graph, allow any elementary path that strictly follows the directions of the graph, that is the path $i_0...i_n$ is only allowed if $i_{k-1}i_k$ is a directed edge in G for all $k = 1, ..., n$.

We denote the space that is spanned by all elementary $n-$paths in P by A_n. We call the elements of A_n allowed $n-$paths. Note that $\partial|_{A_n}$ is not necessarily a subset of A_{n-1}. As an example, consider a path $i_0i_1i_2$ in some path complex with boundary $\partial(i_0i_1i_2) = i_1i_2 - i_0i_2 + i_0i_1$. The truncated paths i_0i_1 and i_1i_2 have to be allowed by Definition 4.1, but i_0i_2 is not necessarily allowed. We define a subspace of allowed paths with this property.

Definition 4.3. For $n \geq 1$, define a subspace Ω_n of A_n as follows:

$$\Omega_n = \Omega_n(P) = \{v \in A_n | \partial v \in A_{n-1}\}$$

The elements of Ω_n are called ∂-invariant $n-$paths of V.

It is easy to see that $\Omega_0 = A_0$, $\Omega_1 = A_1$ and $\Omega_n \subset A_n$ for all $n \in \mathbb{N}$. Furthermore, we see that $\partial|_{\Omega_n} \subset \Omega_{n-1}$. Combined with Lemma 3.4 we therefore get a chain complex as in (8).

$$0 \leftarrow \Omega_0 \leftarrow ... \leftarrow \Omega_{p-1} \leftarrow \Omega_p \leftarrow ... \tag{9}$$

The boundary of an arbitrary elementary path is a linear combination of subsequences of the path with the same ordering as the original path. We therefore get that in a perfect path complex, the boundary of any allowed elementary path is an allowed path. Since the boundary operator is linear, the boundary of an allowed path from a perfect path complex is allowed. That is, if P is perfect, then $\Omega_n = A_n$ for all $n \in \mathbb{N}$. This is important as it saves us the trouble of having to check if allowed paths are $\partial-$invariant when working with perfect path complexes such as simplicial complexes.

Theorem 4.4. For any path complex P, if $\Omega_n = 0$, then $\Omega_p = 0$ for all $p > n$.

Proof. We only need to prove that if $\Omega_n = 0$, then $\Omega_{n+1} = 0$. The rest of the proof follows by induction.

The case where $A_n = 0$ is trivial, as this gives $A_p = 0$ for all $p > n$ by the definition of a path complex. We therefore assume $A_n \neq 0$ and want to show that $\Omega_{n+1} = 0$. Assume that there exists some $v \in A_{n+1}, v \neq 0$. We write

$$v = \sum_{i_0 i_1 ... i_{n+1}} v^{i_0 i_1 ... i_{n+1}} i_0 i_1 ... i_{n+1}$$

where at least one coefficient $v^{i_0 i_1 ... i_{n+1}}$ is non-zero. By the definition of a path complex, the $n-$path

$$v' = \sum_{i_0 i_1 ... i_{n+1}} v^{i_0 i_1 ... i_{n+1}} i_0 i_1 ... i_n$$

is allowed as it is only built of allowed elementary $n-$paths, and it is not equal to zero as at least one coefficient is non-zero. Since $\Omega_n = 0$, the boundary $\partial v'$ is not in A_{n-1}. This means that at least one elementary path in $\partial v'$ is non-allowed, has non-zero coefficient and cannot be cancelled by any other term of $\partial v'$. Such an elementary path can be written $i_0 i_1 ... \hat{i_q} ... i_n$ for some allowed $n-$path $i_0 i_1 ... i_n$. Since we chose our v arbitrarily from A_{n+1}, such a non-allowed $(n-1)-$path exists in $\partial v'$ for every v' constructed as above.

Assume that $v \in \Omega_{n+1}$, that is $\partial v \in A_n$. Since $i_0 i_1 ... \hat{i_q} ... i_n$ is a non-allowed $(n-1)-$path with non-zero coefficient in $\partial v'$, $i_0 i_1 ... \hat{i_q} ... i_n i_{n+1}$ must be a non-allowed $n-$path with the same non-zero coefficient in ∂v. This summand has to be eliminated by other summands in the boundary of v if $v \in \Omega_{n+1}$. Any such summand that helps eliminate the non-allowed path above must be on the form

$$i_0 ... i_{q-1} i_{q+1} ... i_k J i_{k+1} ... i_{n+1} \text{ or } i_0 ... i_k J i_{k+1} ... i_{q-1} i_{q+1} ... i_{n+1} \tag{10}$$

where J is a vertex such that the $(n+1)-$path is allowed. We also see that the coefficients of v in front of the terms in (10) exist in v' in front of the elementary paths

$$i_0 ... i_{q-1} i_{q+1} ... i_k J i_{k+1} ... i_n \text{ or } i_0 ... i_k J i_{k+1} ... i_{q-1} i_{q+1} ... i_n. \tag{11}$$

Since the indices of the elementary $n-$paths in (11) correspond directly to the first $n+1$ indices of the elementary $(n+1)-$paths in (10) and their coefficients are the same, the boundaries will cancel in the same manner. This means that $v \in \Omega_{n+1} \Rightarrow v' \in \Omega_n$. This is a contradicition since $\Omega_n = 0$, so $v \notin \Omega_{n+1}$. Thus, $\Omega_{n+1} = 0$. □

5. Homologies on Path Complexes

We are finally ready to define homologies on path complexes.

Definition 5.1. Let P be a path complex. The nth path homology group $H_n(P)$ is defined as

$$H_n(P) = H(\Omega_n(P))) = Z_n^p / B_n^p = \text{Ker}\partial|_{\Omega_n} / \text{Im}\partial|_{\Omega_{n+1}} \tag{12}$$

The path homology groups are defined for each set in the chain complex (9).

We observe that $\text{Ker}\partial|_{\Omega_0} = \Omega_0$ since $\partial v = 0$ for all $v \in \Omega_0$. Furthermore, we observe that both $\text{Ker}\partial|_{\Omega_n}$ and $\text{Im}\partial|_{\Omega_{n+1}}$ are $F-$linear spaces generated by $n-$paths, and $\text{Im}\partial|_{\Omega_{n+1}} \subseteq \text{Ker}\partial|_{\Omega_n}$.

Example 5.2. Consider a path complex P connected to a simplicial complex K. Any $n-$path $i_0 i_1 ... i_n \in P$ corresponds directly to an $n-$simplex $[v_{i_0}, v_{i_1}, ..., v_{i_n}] \in K$. The spaces are therefore isomorphic, and the two boundary operators give equal results up to isomorphism. As the boundary operator and elements interact the same way both in K and P we get that the generating sets in the homology groups are equal, thereby giving the same homology groups. Homology theory on path complexes is therefore consistent with simplicial homology theory.

Lemma 5.3. For any path complex P, if $\Omega_n = 0$, then $H_n(P) = 0$.

Proof. We see that $\mathrm{Im}\partial|_{\Omega_{n+1}} \subseteq \mathrm{Ker}\partial|_{\Omega_n} \subseteq \Omega_n = 0$. Thus,

$$H_n(P) = \mathrm{Ker}\partial|_{\Omega_n} / \mathrm{Im}\partial|_{\Omega_{n+1}} = 0.$$

□

The following important result follows directly from Theorem 4.4 and Lemma 5.3.

Proposition 5.4. Consider any path complex P. If $\Omega_n = 0$, $H_p(P) = 0$ for every $p \geq n$.

This means that if we discover a ∂-invariant space Ω_n that is trivial we don't need to check any higher order ∂-invariant spaces or path homology groups.

In the following example we consider the difference between using the standard boundary operator ∂ and the regular boundary operator ∂^{reg}. Recall that the regular boundary operator sets the coefficient of any non-regular term in the boundary to zero.

Example 5.5. Consider a path complex P associated with the directed graph in Figure 2.

Figure 2

We construct A_n as the set of all regular $n-$paths on the directed graph in Figure 2 for every n. ´

$$\begin{aligned}
A_0 &= \langle 0, 1 \rangle \\
A_1 &= \langle 01, 10 \rangle \\
A_2 &= \langle 010, 101 \rangle \\
A_3 &= \langle 0101, 1010 \rangle \\
&\vdots
\end{aligned}$$

We observe that with the non-regular boundary operator we get $\partial(010) = 10 - 00 + 01$, $\partial(101) = 01 - 11 + 10$. These paths are not $\partial-$invariant as $00, 11 \notin A_1$, and we get

$$\Omega_0 = A_0 = \langle 0, 1 \rangle \ , \ \Omega_1 = A_1 = \langle 01, 10 \rangle \ , \ \Omega_n = 0 \text{ for every } n \geq 2$$

This gives

$$\begin{aligned}
&H_0(P) = \langle 0,1\rangle/\langle 1-0, 0-1\rangle = \langle 0, 1-0\rangle/\langle 1-0\rangle = \langle 0\rangle \cong \mathbb{Z}\\
&H_1(P) = \langle 01+10\rangle \cong \mathbb{Z}\\
&H_n(P) = 0 \text{ for every } n \geq 2
\end{aligned}$$

On the other hand, let us consider the same path complex with a regular boundary operator. This way $\partial^{reg}(010) = 10 + 01$ and $\partial^{reg}(101) = 01 + 10$. All allowed $2-$paths are therefore $\partial^{reg}-$invariant. More generally we see that omitting any vertex that is not the first or the last in an allowed elementary path would give a non-regular elementary path. The coefficients of these terms are all set to zero by ∂^{reg}, making any boundary a linear combination of truncated paths. As any truncated path is allowed by the definition of a path complex, we get that $\Omega_n = A_n$ for every n.

We observe that only multiples of one linear combination of two allowed elementary $n-$paths give an element in Z_n^p.

$$\begin{aligned}
Z_1 = \langle 10+01\rangle \ &, \ Z_2 = \langle 101-010\rangle \ ,\\
Z_3 = \langle 1010+0101\rangle \ &, \ Z_4 = \langle 10101-01010\rangle \ ,\\
Z_5 = \langle 101010+010101\rangle \ &, \ Z_6 = \langle 1010101-0101010\rangle \ ,\\
&\vdots
\end{aligned}$$

All these cycles are also boundaries. As $B_n^p \subset Z_n^p$, $B_n^p = Z_n^p$. Hence,

$$H_0(P) = \mathbb{Z} \ , \ H_n(P) = 0 \text{ for all } n \geq 1$$

We see that ∂ and ∂^{reg} give rise to different 1st homology groups $H_1(P)$. This difference stems from the fact that the standard boundary operator ∂ counts such a two-directional edge as a hole, illustrated in Figure 3, while the regular boundary operator ∂^{reg} does not. Whether or not such an edge should qualify as a hole is a matter of convention.

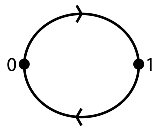

FIGURE 3

Example 5.6. Consider a path complex P associated with the directed graph in Figure 4.

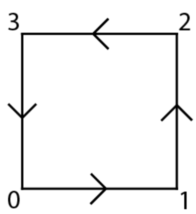

FIGURE 4

We allow all regular elementary paths along the directed edges in Figure 4.

$$\begin{aligned}
A_0 &= \langle 0,1,2,3\rangle\\
A_1 &= \langle 01,12,23,30\rangle
\end{aligned}$$

$$A_2 = \langle 012, 123, 230, 301\rangle$$
$$\vdots$$

Both boundary operators, ∂ and ∂^{reg}, give the same results in this example. Let us look at the boundaries of all the allowed elementary $2-$paths.

$$\begin{aligned}\partial(012) &= 12 - 02 + 01\\ \partial(123) &= 23 - 13 + 12\\ \partial(230) &= 30 - 20 + 23\\ \partial(301) &= 01 - 31 + 30\end{aligned}$$

Each of these boundaries contain one of the non-allowed $1-$paths $02, 13, 20$ and 31 each. No non-zero linear combination of these boundaries could therefore give an allowed $1-$path as the non-allowed terms will not cancel. We therefore get

$$\Omega_0 = A_0 = \langle 0, 1, 2, 3\rangle \;\;,\;\; \Omega_1 = A_1 = \langle 01, 12, 23, 30\rangle \text{ and}$$
$$\Omega_n = 0 \text{ for every } n \geq 2.$$

We get the following homology groups:

$$\begin{aligned}H_0(P) &= \langle 0, 1, 2, 3\rangle/\langle 1-0, 2-1, 3-2, 0-3\rangle\\ &= \langle 0, 1-0, 2-1, 3-2\rangle/\langle 1-0, 2-1, 3-2\rangle = \langle 0\rangle \cong \mathbb{Z}\\ H_1(P) &= \langle 01 + 12 + 23 + 30\rangle \cong \mathbb{Z}\\ H_n(P) &= 0 \text{ for every } n \geq 2.\end{aligned}$$

We often proceed as in Examples 5.5 and 5.6 by reducing the homology groups to their generating elements. If we do not care about the generating elements themselves we could instead discuss the dimension of the homology groups.

Proposition 5.7. Consider a path complex P.

$$\dim(H_n(P)) = \dim(\Omega_n) - \dim(\mathrm{Im}\partial|_{\Omega_n}) - \dim(\mathrm{Im}\partial|_{\Omega_{n+1}})$$

Proof. Since $H_n(P)$ is a quotient space,

$$\dim(H_n(P)) = \dim(\mathrm{Ker}\partial|_{\Omega_n}/\mathrm{Im}\partial|_{\Omega_{n+1}}) = \dim(\mathrm{Ker}\partial|_{\Omega_n}) - \dim(\mathrm{Im}\partial|_{\Omega_{n+1}})$$

Since the boundary operator is linear, we use the rank-nullity theorem.

$$\dim(\Omega_n) = \dim(\mathrm{Im}\partial|_{\Omega_n}) + \dim(\mathrm{Ker}\partial|_{\Omega_n}).$$

Hence,

$$\dim(H_n(P)) = \dim(\Omega_n) - \dim(\mathrm{Im}\partial|_{\Omega_n}) - \dim(\mathrm{Im}\partial|_{\Omega_{n+1}})$$

□

Proposition 5.7 can be useful, but we need to be careful when using it as it can lose some information that the generating elements give us. As an example this proposition fails to discover torsion in some non-orientable manifolds. We will therefore mostly avoid using this result.

6. Path Complexes with Torsion

The goal of this section is to construct a path complex with torsion in its homology group. We do this to get a better understanding of how such torsion elements appear in path homology groups and to explore how path complexes interact with cellular homology theory.

A CW-complex, as described in [7], is a more general complex than a simplicial complex. A CW-complex consists of cells instead of simplices. Any simplex can be regarded as a cell, but a cell is not required to be uniquely defined by its vertices. This allows for some computations that would be harder to do on simplicial complexes, such as finding the homology groups of the real projective plane. Figure 5 visualizes a CW-complex C that represents the real projective plane by gluing the edges a and b to their counterpart in coherence with their direction.

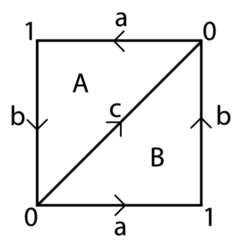

Figure 5

The CW-complex in Figure 5 has vertices $\{0, 1\}$, edges $\{a, b, c\}$ and $2-$cells $\{A, B\}$. The cellular homology groups of this CW-complex are calculated in [7] and shown to be

$$H_0(C) \cong \mathbb{Z} \ \ , \ \ H_1(C) \cong \mathbb{Z}/2\mathbb{Z} \ \ , \ \ H_n(C) = 0 \text{ for every } n \geq 2$$

We see that the first homology group $H_1(C)$ is isomorphic to a quotient group, meaning that C has torsion. This quotient group appears because the $1-$boundaries, that is the boundaries of A and B, are different although they have the same edges:

$$\partial A = a + b + c \ \ , \ \ \partial B = a + b - c$$

Combined with the $1-$cycles $\langle a + b, c\rangle$, the generating sets in the first homology group reduce to

$$H_1(C) = \langle c\rangle/\langle 2c\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

We will now see what happens if we try to create a path complex out of Figure 5 as if it was a simplicial complex.

Example 6.1. Consider a path complex P based on Figure 6 with the non-regular boundary operator.

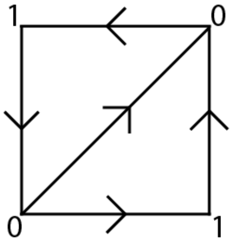

FIGURE 6

We allow the following paths:

$$A_0 = \langle 0, 1\rangle$$
$$A_1 = \langle 01, 10, 00\rangle$$
$$A_2 = \langle 001, 010\rangle$$

All of these elementary paths are $\partial-$invariant. As an example,

$$\partial(001) = 01 - 01 + 00 = 00$$
$$\partial(010) = 10 - 00 + 01$$

are both allowed. This gives $\Omega_n = A_n$ for all n. It is clear that no combination of the allowed elementary 2$-$paths can give a vanishing boundary. We therefore get

$$\begin{aligned}
H_0(P) &= \langle 0, 1\rangle/\langle 1-0, 0-1, 0-0\rangle = \langle 0, 1-0\rangle/\langle 1-0\rangle = \langle 0\rangle = \mathbb{Z}\\
H_1(P) &= \langle 01+10, 00\rangle/\langle 00, 10-00+01\rangle\\
&= \langle 01+10-00, 00\rangle/\langle 00, 01+10-00\rangle = 0\\
H_n(P) &= 0 \text{ for every } n \geq 2
\end{aligned}$$

Hence, $H_1(P)$ is different from the first homology group $H_1(C)$ of the CW-complex in Figure 5. This difference occur because the boundary $\partial(001) = 00$ does not represent the boundary of A, $\partial A = a + b + c$, properly. The only way an elementary 2$-$path v could represent A properly is if $\partial v = 01 + 10 + 00$, thereby giving rise to the path homology group $H_1(P) = \langle 00\rangle/\langle 2(00)\rangle$. Unfortunately no such 2$-$path exist because of the way the boundary operator is defined.

This shows that the simple way of creating a path complex corresponding to a simplicial complex does not immediately generalize to CW-complexes. The real projective plane can however be triangulated until it is a valid simplicial complex, as in Figure 7. Let us show that the homology groups of the path complex related to such a triangulation gives the homology of the real projective plane.

Example 6.2. Consider a path complex P derived from Figure 7 as if it was a simplicial complex consisting of 10 2$-$simplices. This gives

$$A_0 = \langle 0, 1, 2, 3, 4, 5\rangle$$

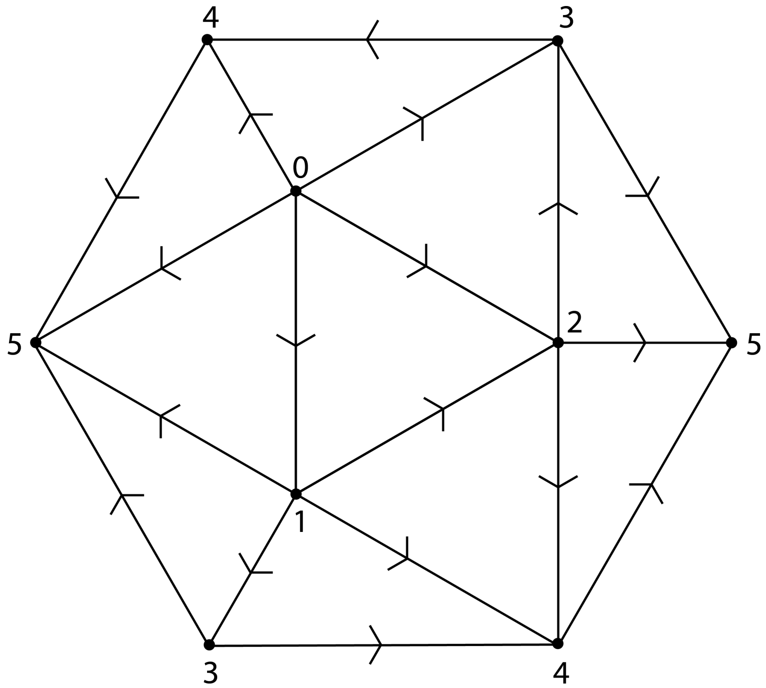

FIGURE 7

$$A_1 = \langle 01, 02, 03, 04, 05, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45\rangle$$

$$A_2 = \langle 012, 015, 023, 034, 045, 124, 134, 135, 235, 245\rangle$$

$$A_n = 0 \text{ for every } n \geq 3.$$

Since we are considering a path complex connected to a simplicial complex, $\Omega_n = A_n$ for every n.

As always, Z_0^p is equal to Ω_0. We find B_0^p by taking the boundary of every allowed elementary $1-$path, but remove any boundary that does not add to the generated set. As an example $\partial(15) = 5 - 1 = 5 - 0 + 0 - 1 = \partial(05) - \partial(01)$, so $\partial(15)$ does not have to be included if $\partial(01)$ and $\partial(05)$ are in the generating set. The generating set of the $0-$boundaries therefore reduce to $B_0^p = \langle 1-0, 2-0, 3-0, 4-0, 5-0\rangle$. This gives

$$\begin{aligned} H_0(P) &= \langle 0,1,2,3,4,5\rangle / \langle 1-0, 2-0, 3-0, 4-0, 5-0\rangle \\ &= \langle 0, 1-0, 2-0, 3-0, 4-0, 5-0\rangle / \langle 1-0, 2-0, 3-0, 4-0, 5-0\rangle \\ &= \langle 0\rangle \cong \mathbb{Z}. \end{aligned}$$

Since $\partial^2 = 0$, all $1-$boundaries are also $1-$cycles. We also observe that $\partial(34 + 45 - 35) = 0$, making $34 + 45 - 35$ a $1-$cycle. Any other $1-$cycle is generated by the set of these $1-$cycles. As an example, consider $02+25-05$. We can express this $1-$cycle as

$$02 + 25 - 05 = \partial(023) + \partial(045) + \partial(034) - \partial(235) - (34 + 45 - 35).$$

Figure 8 helps illustrate the choice of signs in the expression above.

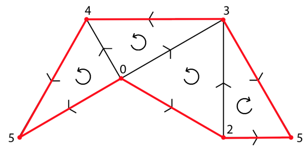

FIGURE 8

We have found a generating set for the 1−cycles. We however observe that this set is not linearly independent, as

$$\begin{aligned} &\partial(012)-\partial(015)+\partial(023)+\partial(034)+\partial(045)-\partial(124)\\ &+\partial(134)-\partial(135)-\partial(235)+\partial(245)=2(34+45-35) \end{aligned} \tag{13}$$

We can therefore remove one of the boundaries from the generating set, for example $\partial(245)$, as

$$\begin{aligned} \partial(245) &= 2(34+45-35)-2(34+45+35)+\partial(245)\\ &= 2(34+45-35)-(\partial(012)-\partial(015)+\partial(023)+\partial(034)\\ &+\partial(045)-\partial(124)+\partial(134)-\partial(135)-\partial(235)) \end{aligned}$$

We therefore remove this element from the generating set of our 1−cycles. In the generating set of our 1−boundaries we add the linear combination of 1−boundaries

$$\begin{gathered} \partial(012)-\partial(015)+\partial(023)+\partial(034)+\partial(045)\\ -\partial(124)+\partial(134)-\partial(135)-\partial(235) \end{gathered}$$

to the element $\partial(245)$ in the generating set. This element is therefore equal to (13).

With all this in mind we write out the first homology group omitting the remaining elements that correspons to the boundaries of 2−paths, as these elements are both 1−boundaries and 1−cycles. This gives

$$H_1(P)=\langle 34+45-35\rangle/\langle 2(34+45-35)\rangle\cong\mathbb{Z}/2\mathbb{Z}$$

This is equal to the first homology group $H_1(C)$ of the CW-complex in Figure 5. The element $34+45-35$ corresponds to the edge c in the CW-complex.

Clearly $B_2^p=0$ as there are no allowed 3−paths. To see that $Z_2^p=0$, consider first that every allowed elementary 1−path only exist in the boundary of exactly two allowed elementary 2−paths. An example is 01 which only exists in the boundary of 012 and 015, or 02 which only exists in the boundary of 012 and 023. This means that any allowed path v will have 01 in its boundary unless $v^{012}=v^{015}$. Similarly 02 will exist in the boundary unless $v^{012}=-v^{023}$. Similar results hold for every allowed elementary 1−path.

Now consider the following linear combination of every allowed 2−path.

$$012-015+023+034+045-124+134-135-235+245$$

The boundary of this linear combination corresponds to (13). The multiples of this linear combination are the only allowed non-zero 2−paths where every term contaning $01, 02, 03, 04, 05, 12, 13, 14, 15, 23, 24$ or 25 are all cancelled in the boundary. Changing any coefficient without changing all the other coefficients correspondingly, as described in the previous paragraph, will lead to one of these term not being equal to zero. Since a multiple of (13) has a non-zero boundary unless we multiply it by zero, there exist no non-zero allowed 2−path with vanishing boundary. Thus,

$$H_2(P) = 0 \text{ and } H_n(P) = 0 \text{ for every } n \geq 3.$$

The path homology of P is therefore the same as the homology of the real projective plane.

The torsion in Example 6.2 occur because we are able to construct an element in the generating set of the 1−boundaries that loops over an element in the generating set of the 1−cycles twice. With this in mind we should be able to create other path complexes with torsion elements in their homology groups that are not based on known triangulations of non-orientable surfaces.

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