A Mathematics Teacher Educator's Feedback Affecting Teachers' Design of Hypothetical Learning Trajectories for Teaching Patterns

Melih Turgut
NTNU – Norwegian University of Science and Technology
melih.turgut@ntnu.no

Dilek Tanışlı
Anadolu University
dtanisli@anadolu.edu.tr

Nilüfer Köse
Anadolu University
nyavuzsoy@anadolu.edu.tr

This qualitative research presents how a mathematics teacher educator's feedbacks to two lower secondary school mathematics teachers affects their design of hypothetical learning trajectories regarding teaching patterns and their teaching practice. The data comes from a web portal, which was designed as a professional development tool for mathematics teachers. The teachers communicated with a mathematics teacher educator online, and prepared lesson plans, as well as receiving feedback. After implementing their designs, they also uploaded teaching episodes to the web portal, after which they received feedback. We triangulated data that came from the lesson plans, mathematics teacher educator's feedback and videos of teaching practice. The data was analysed within two lenses: the Knowledge Quartet and the Hypothetical Learning Trajectory. The findings reveal that through the feedback given by the mathematics teacher educator, the teachers substantially progressed in designing hypothetical learning trajectories and developed their professional knowledge, in particular regarding the foundation, transformation, and connection dimensions of the Knowledge Quartet.

Introduction

The professional development (PD) of mathematics teachers has recently received particular attention by researchers (e.g., Ball & Cohen, 2008; Rowland, 2013; Rowland, Huckstep & Thwaites, 2005; Turner & Rowland, 2011; Weston, 2018), most likely after the seminal work of Shulman (1986) that points out the core roles of subject matter knowledge, pedagogical content knowledge and curricular knowledge. Broadly speaking, the combination of these three can be summarised as subject matter knowledge with skills and competencies regarding the organisation of classroom activities including powerful, effective and alternative teaching strategies based on student knowledge. Along with improving teachers' PD regarding organising classroom activities, research has focused on teachers' awareness and skills concerning the development of mathematical thinking and student knowledge/learning. For example, specific PD activities on considering/noticing the development of student knowledge have improved teachers' design and implementation of teaching plans within the perspective of conceptualisation and problem-solving (Fennema et al., 1996; Franke et al., 2001; Kazemi & Franke, 2004). Therefore, teacher professional knowledge is necessary for the organisation of classroom activities which need to be built on students' existing learning/experiences and phenomenology (Ball & Cohen, 1999).

In Turkey, a number of studies have appeared pointing out deficiencies regarding teachers' pedagogical content knowledge (Işıksal & Çakıroğlu, 2011; Şen, 2021; Türnüklü et al., 2015; Ulusoy & Çakıroğlu, 2013). The PD activities of mathematics teachers are generally organised and held by public schools twice a year, with certain seminars on particular topics of teaching and learning not specifically for the content knowledge of a certain topic, such as algebra and/or geometry learning. The common view regarding these activities is a lack of sustainable PD programs, which could provide a particular reference to linking theory and practice (Yıldırım, 2013). Therefore, the design and implementation of sustainable programs to develop teachers' professional knowledge are still necessary to establish *collaboration* and *communication* between mathematics teachers and teacher educators.

The collaboration and communication between teachers and teacher educators have appeared as a core step to support the PD of mathematics teachers (Jaworski, 2001; Yang et al., 2015). For example, Jaworski (2001) addresses the notion of 'co-learning partnerships' between teachers and teacher educators regarding PD. Co-learning partnerships are indeed concerned about respecting for the roles and sharing responsibilities in the development process, and teachers' development as 'critical thinkers'. Co-learning partnerships could be developed through sustainable PD programs that provide teachers with 'a professional route' and by extending the partnerships to ones in classrooms between students and teachers (Jaworski, 2001) on a large scale.

Regarding sustainable PD programs, online mathematics teacher education has been an emerging paradigm in mathematics education to create a context from collaboration to individual PD (Borba & Llinares, 2012). This could enable the establish of co-learning partnerships. For example, Fernandez, Llinares and Rojas (2020) show that through an online teacher education program, sharing narratives and receiving feedback from colleagues and university tutors helps prospective teachers to enhance 'noticing' and leads to improvements in their practice. Recently, there has been an attempt regarding an exploration of teachers' professional learning in online platforms and noticing to consider student thinking (Beilstein et al., 2021) as well as the changing role of mathematics teacher educators as within new technological environments (Arzarello & Taranto, 2021). However, less attempt has been given to the link between mathematics teacher and teacher educator collaboration considering student thinking in lesson planning for practice. This has led us to consider the notion of Hypothetical Learning Trajectories (Simon, 1995, 2014) as a PD tool through a web-based platform to establish co-learning partnerships.

This paper is extracted from a large-scale project aimed at designing a web-based platform for mathematics teachers' PD, where the teachers collaborate with mathematics teacher educators to design and teach lower secondary school algebra. In the project, we refer to the notion of Hypothetical Learning Trajectory (Simon, 1995, 2014) and collaborate with teachers in this aspect to develop their competencies regarding consideration of the development of students' mathematical thinking. In this paper, we focus on a mathematics teacher educator's feedback, provided through a web-based portal with a lens of the Knowledge Quartet (Rowland et al., 2005), and how such feedback progressively affects two mathematics teachers' design and implementation of hypothetical learning trajectories for teaching patterns.

Conceptual Framework and the Project

Our conceptual framework consists of two major elements, the first is the knowledge quartet and the second is hypothetical learning trajectories. We briefly describe two dimensions and then end this section with a description of the project and two research questions.

Knowledge Quartet

The Knowledge Quartet (KQ) was developed by Rowland and colleagues aiming at elaborating an empirically based framework to identify how teachers' mathematics-related knowledge is enacted in teaching, building on Shulman's (1986) seminal work. The KQ model 'is designed to provide a guide to mathematical knowledge-in-use that is well suited to supporting teachers' professional reflection and learning' (Ruthven, 2011, p. 85). To analyse teaching practice, the KQ offers four dimensions of teachers' mathematics-related knowledge; Foundation, Transformation, Connection and Contingency (Rowland, 2013; Rowland et al., 2005). Foundation refers to the teacher's own educational/academic background and beliefs regarding a knowledge and understanding of mathematics per se, and how and why we learn and teach it systematically. In the foundation section, the teacher knows the conditions under which students could learn meaningful mathematics (Weston, 2018).

The second dimension, transformation, refers to the choice of different ways to represent mathematical ideas in order that they become available to students. These include a choice of examples, tasks, pedagogical strategies and the use of classroom materials for student activity. The connection dimension refers to an 'anticipation of complexity' (Rowland et al., 2005) and decisions with regard to sequencing planned activities regarding proposed concepts by making a connection between planned procedures and concepts. Here, the anticipation of complexity also refers to 'an awareness of possible difficulties and obstacles that students may have with different mathematical topics and tasks' (Petrou & Goulding, 2011, p. 18). However, the fourth dimension, contingency, refers to unanticipated moments in the classroom which are not elaborated on in the teaching plan. In other words, it corresponds to a 'deviation from agenda' (Rowland et al., 2005) and the provision of meaningful and helpful responses to students' ideas and answers.

To develop mathematics teachers' classroom practice, the KQ has been referred to as a heuristic tool to develop pre-service and in-service teacher knowledge, whereby a mentor or mathematics teacher educator can provide critical, constructive feedback (Turner, 2012; Livy, Herbert & Vale, 2019). For example, Livy, Herbert and Vale (2019) report using the KQ as a tool to support and promote two pre-service teachers' mathematical content knowledge over four years. The mentor teachers in their research provide critical feedback when pre-service teachers design lesson plans and reflections on observation. Their findings highlight that the KQ structure contributes to the development of the pre-service teachers' mathematical content knowledge. In summary, the KQ 'provides a means of reflecting on teaching and teaching knowledge, in order to develop both' (Turner & Rowland, 2011, p. 197). The KQ is not only a theoretical analysis tool to reflect on the classroom nature, but it also enables collaboration between teachers and teacher educators (Gumiero & Pazuch, 2021). Furthermore, reflecting on teaching practices could open a door to teacher noticing (Fernandez et al., 2020). In other words, a door to learning by noticing, which refers to '... foremost a systematic method for conducting research into one's own practice' (Mason, 2021, p. 231). We believe that reflecting on teaching practice framed by the KQ could provide an opportunity to revisit teachers' practice, in addition to reflecting on the main components of lesson design like goals, building on student thinking and learning.

In our research, we consider the central role of providing feedback in PD activities (Livy et al., 2019), and we take a combined view of 'student thinking' and the development of a mathematical thinking perspective (Fennema et al., 1996) to develop mathematics teachers' professional knowledge. Therefore, the feedback structure is organised around the targeted learning, design of a teaching plan and classroom activities, and students' possible learning paths, which is directly interconnected with the notion of the hypothetical learning trajectory.

Hypothetical Learning Trajectories

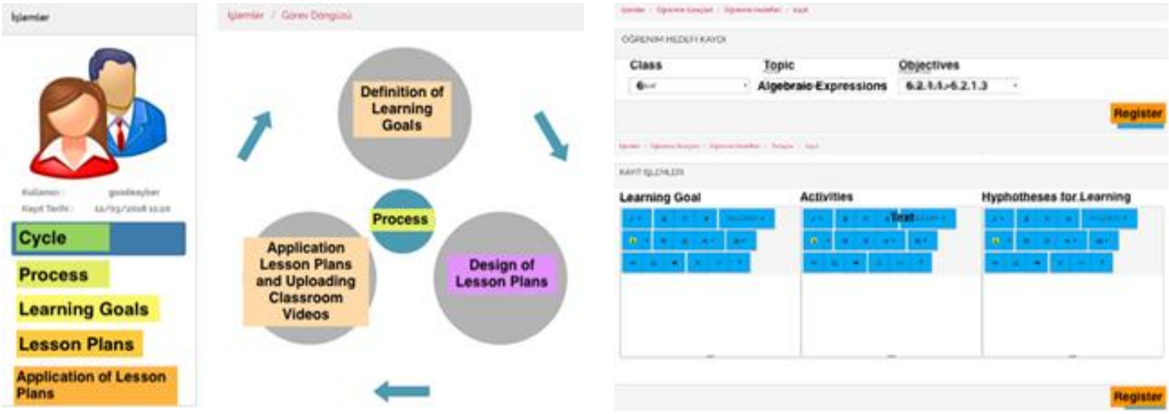
The notion of the hypothetical learning trajectory (HLT) was introduced by Simon (1995, 2014) as a teaching design model within a constructivist perspective. HLT puts student thinking at the centre and, along with this, it has three major components; a learning goal, design of teaching activities and tasks, and a hypothesized learning process. The first component refers to the teacher's determination of a learning goal to achieve desired student learning, while the second refers to designing a set of classroom activities that are in line with the learning goal. The third refers to an estimation regarding how students' thinking and meaning-making, step-by-step, will evolve through the learning activities.

The core idea in the HLT is an exemplification of a learning path regarding student thinking. This is because the design of classroom activities is heavily based on the prediction of the steps of student learning. The teacher should be aware of students' pre-knowledge and the order of topics and mathematical concepts. In other words, there exists a synergy between the trajectory of student learning and classroom activities, since 'the students' learning is significantly affected by the opportunities and constraints that are provided by the structure and content of the mathematics lessons' (Simon, 2014, p. 273). Here, the role of the teacher as a 'designer' is crucial. Therefore, the notion of HLT has been considered to be a PD tool in recent research (McCool, 2009; Wickstrom & Langrall, 2020; Wilson et al., 2013). For example, McCool (2009) refers to a PD program based on HLTs. The findings underline the teachers' ability to assess students within the perspective of mathematical thinking and the teachers managing to refer to their pedagogical knowledge effectively while guiding student learning in challenging topics. Similarly, Wilson et al. (2013) focus on both pre-service and in-service teachers with an HLT-based PD program, and how their progress in the development of pedagogical knowledge would occur, specifically on student thinking. The findings indicate that the HLT-based PD program develops both groups' professional knowledge regarding estimating and modelling student learning, linking teaching episodes and student thinking, and also understanding the taught concept mathematically.

From an epistemological point of view, we think that the findings of Wilson et al. (2013) are related to the foundation dimension of KQ, where the central notion relates to the teacher's background in (the nature of) mathematics. Moreover, the transformation dimension corresponds to the notions of a learning goal and design of classroom activities, while the connection dimension seems related to both the design of classroom activities and an estimation of student thinking in HLT. Therefore, we decided to take a combined view (i.e. KQ and HLT) for the PD of mathematics teachers.

The MEGEDEP Project and Research Questions

The MEGEDEP (which is an abbreviation of the project title in Turkish) project was designed to support mathematics teachers' PD in connecting theory and practice, and a collaboration and communicative context regarding teaching algebra. Why we focus on an *algebra* context was thanks to the background and experience of two researchers (in this project) in the teaching and learning of algebra. Regarding sustainability, we decided to design a specific web portal to establish (digital/virtual) collaboration between mathematics teachers and mathematics teacher educators. The web portal's main interface includes practical information regarding the project, a manual for using the website, introductory information about constructivism and HLT, in addition to exemplary teaching designs developed within the perspective of HLT. Teachers can register to the website, and they can then access a module page (Figure 1).



(a) (b)

Figure 1. (a) The user interface, (b) The interface regarding HLT components

Figure 1a illustrates the user interface, which shows the structure and cyclical process. For example, if the teacher clicks on the first cycle, then s/he needs to select a grade level, topic and associated curriculum competency. Thereafter, the teacher can formulate a learning goal, a brief description of planned classroom activities and hypotheses for student learning (see Figure 1b). The portal has an effective feedback structure, where it was followed by Khan (2005) to design an e-learning platform. As a result of this, when the teacher formulates HLT components, the assigned mathematics teacher educator (MTE) receives an email. The MTE reviews each part and provides feedback to improve the teacher's work. After this, the teacher revises his/her work and, if the MTE approves it, then s/he can design the plan in detail, step-by-step. The MTE reviews the plan and provides more feedback. The cyclical process ends when the teacher records her/his teaching episode and uploads it to the MEGEDEP portal and replies to a number of reflective questions. The MTE watches the episode and overviews the teacher's responses to reflective questions and provides feedback to the teacher before s/he prepares the next teaching episode. We present an overview of a pilot study, where teachers worked with the portal for three weeks and show how most of the eleven teachers progressed in designing HLTs regarding sixth-grade algebra (Tanışlı, Köse & Turgut, 2019).

In this paper, in order to report an in-depth analysis, we focus on one MTE’s feedback to two mathematics teachers’ teaching patterns at seventh grade, where their teaching designs based on HLT are entered into the MEGEDEP portal. We approach the main tenets of the MTE’s feedback through the KQ and how the teachers proceed in designing and implementing HLTs, as well as how they improve their teaching. Therefore, we focus on the following two interrelated research questions:

1. What are the main tenets of the MTE’s feedback in terms of the KQ for HLT-based lesson plans designed to teach the generalisation of patterns?
2. How does the MTE’s feedback through the MEDEGEP portal affect teachers’ design of HLT for teaching the generalisation of patterns?

Research Context and Methods

At the beginning of the project, the participating teachers received training on how to use the MEGEDEP, and numerous specific resources were provided through the system, through which the teachers could access. These resources include theoretical information that introduces noticing student thinking, student knowledge, predicting learning trajectories, exemplary learning trajectories, lesson plans, and videos of the implementation of particular HLT. The present paper reports a part of the PD activities carried out within the scope of the project. Our case study (where our theoretical stance is a qualitative paradigm) includes the progress of two lower secondary school mathematics teachers when they designed HLT-based plans to teach generalization of patterns at seventh grade. Within the scope of the research, feedback on the teaching plans and implementation processes of the two teachers was made by an MTE. The MTE here collaborated with other two project members (with PhD degrees) while guiding teachers' PD. In other words, the MTEs agreed on several joint decisions to ensure the provision of orientations needed by the teachers.

After the teachers completed their initial training regarding the project and the web portal, the MTE asked the teachers to prepare an estimated learning trajectory, based on the purpose of generalizing a pattern and finding its rule. The learning trajectories uploaded to the system by the teachers were evaluated by the MTE and feedback was given to the teachers through the system. Teachers who refined their HLTs were asked to prepare teaching activities, and the MTE also provided feedback on these activities. After this, the teachers carried out teaching practices and uploaded video recordings to the system. Feedback was provided with regard to teaching episodes by the MTE and, throughout the entire project, the process was continued cyclically as four cycles. In this paper, we focus on the generalisation of patterns, which is the second cycle. In the first cycle, all of the participating teachers designed HLT-based plans regarding addition and multiplication with algebraic expressions and implemented them after receiving feedback from the MTE.

Participants

The project description and invitations were sent to schools through administrative correspondences. Later, forty-two (twelve for the pilot study, and thirty for the main study) teachers (working different cities in Turkey) volunteered to participate in the project and later received training regarding the project and the MEGEDEP platform as described earlier. In this paper, we present a specific case including two teachers from the main study, Ada and Banu (pseudonyms), who work at public schools. Ada is a lower secondary school mathematics teacher with four years of professional experience and Banu who has nine years of professional experience. Banu has a master's degree in mathematics education, while Ada, with less professional experience, has not been involved in postgraduate education or any PD projects. Banu also has longer teaching experience in addition to her master's degree. Therefore, we decided to focus on the case of Ada and Banu to reflect how two teachers with different backgrounds would progress through collaboration with the MTE in designing HLT-based lessons. The aim here is also to explore how both teachers react to feedback given through a web-based system and to reveal how their PD has progressed through feedback. The MTE reported in this paper has a master's degree with fourteen years of teaching experience, is a PhD student in mathematics education and is interested in the topics of learning algebra and HLT.

Data Collection and Analysis

In the study, data was collected from two different sources. The first data source is the designed HLTs that were entered into the web portal. The second data source is video recordings of the seven-hour classroom teaching that teachers uploaded to the system. In an analysis of the collected data, we first focus on the MTE's feedback for the appropriate or inappropriate parts of the HLT-based on conceptual learning. Following this, we code the feedback through a list of twenty contributory codes (which refer to characterising the observed actions) (Table 1) proposed by Rowland (2013). After this, we identify the main tenet(s) of the feedback structure according to the dimensions of the KQ.

Table 1

Contributory codes associated with the KQ dimensions (Rowland, 2013, p. 25)

Dimension	Exemplary Contributory Codes
Foundation	• Awareness of purpose • Adheres to textbook • Concentration on procedures • Identifying errors • Overt display of subject knowledge • Theoretical underpinning of pedagogy • Use of mathematical terminology
Transformation	• Choice of examples • Choice of representation • Use of instructional materials • Teacher demonstration
Connection	• Anticipation of complexity • Decisions regarding sequencing • Recognition of conceptual appropriateness • Making connections between procedures • Making connections between concepts • Making connections between representations
Contingency	• Deviation from agenda • Responding to students' ideas • Teacher insight during instruction • Responding to the (un)availability of tools and resources

Following Table 1, for example, we coded the MTE's feedback regarding mathematical knowledge behind the pattern generalisation with 'overt display of subject knowledge', while we coded the feedback regarding formulation of an explicit learning goal as 'awareness of the purpose'. Following this, we identified the main tenet(s) of the feedback (concerning teaching pattern generalisation) which is in relation to an associated dimension of the KQ. We created tables showing the feedback and associated codes, and then exemplified the MTE's feedback and how the teachers progressed narratively in HLT-related steps.

Findings

The findings were explored by the focus of learning objectives, hypotheses for learning progression, course/activity plans and implementation processes within the scope of the estimated learning trajectory created by the teachers.

Hypotheses Regarding Learning Goal and Progression of Learning

Both teachers were asked to determine their hypotheses regarding the progress of learning with a learning goal and to upload them to the web portal. Ada expressed a learning goal as 'revealing a certain rule among all the steps of the pattern', while Banu expressed 'the relationship between the number of steps in the pattern and the number of terms corresponding to the number of steps'. The learning goal expressed by Banu indicates an emphasis on a functional relationship in pattern generalization, while Ada has a rule-oriented approach. In the context of foundation knowledge, we interpret that Banu has a theoretical infrastructure on the subject, while Ada considers the learning of the generalisation of the patterns superficial and does not have a theoretical infrastructure. The teachers' hypotheses regarding the progress of learning are presented in Table 2.

Table 2

The teachers' hypotheses regarding learning progression

Ada	Banu
• Finding the relationship between terms by using a trial-and-error strategy, like the following number pattern: 3, 7, 11, 15, ... • Write the constant difference between terms as the coefficient of the variable (for example, type 4*n*) and find it by trying the constant term to be added to this algebraic expression	• Find numerical relationships between the number of steps in a figure pattern and the number of terms in that step

As can be seen from Table 2, Banu prefers to start with a figure pattern, but focuses only on finding numerical relationships in the pattern generalization process. Ada starts with a number pattern and takes an approach to find the rule of the pattern through a trial-and-error strategy. Neither of the teachers correctly determine their hypotheses of learning progression in pattern generalization. For this reason, feedback is given to the teachers by the MTE. Table 3 overviews an analysis of feedback structure regarding learning goals and the hypothesized progression of student learning.

Table 3

Analysis of feedback given to the teachers on learning goals and the progression of learning

Dimension	Associated Code	Feedback	The Main Tenet(s)
Foundation	Overt display of subject knowledge	Mathematical knowledge behind the pattern generalization: Functional relationship, figural reasoning, numerical reasoning	*Suggestions for mathematical infrastructure and its teaching*
	Awareness of purpose	Recommended learning goal: Development of functional thinking	
Transformation	Choice of examples	– Examples for numerical reasoning – Explore the growth in a linear pattern using figural reasoning: Analysis of the physical structure of the figure pattern	*Suggestions for planning and implementing teaching* (*framework for pedagogical approach*) *and the hypotheses for learning progression*
	Choice of representation	– Developing numerical relationships to generalise the pattern: Convert figural reasoning to numerical reasoning using table representation – Express the general rule of the pattern: Express the relationship	

	between varying quantities – Finding the desired term of the pattern: Calculate the desired term using the variable as varying quantities

As shown in Table 3, the initial feedback given to the teachers is for the foundation dimension of the KQ. In this context, the MTE provides feedback regarding the functional relationship between the steps of the pattern and the terms corresponding to the steps that are needed for students to generalise patterns. After this, the MTE considers the use of figure patterns in the generalisation process and suggests analysing the physical structure of the pattern since it is, most likely, a well-accepted strategy to link the situation with students' figural reasoning skills. Within this context, the MTE underlines that figural reasoning from a visual inference would be possible thanks to a set of specific situations. In addition, the MTE provides many examples for the development of numerical reasoning in pattern generalization. Secondly, it is emphasized that students' development of functional thinking should be in line with the scope of the learning goals that the teachers set. The teachers are then given feedback on how they needed to plan according to the hypothesized learning progress in the context of the transformation dimension of the KQ. Items (in the transformation part) presented in Table 3 are proposed as exemplary learning progress.

Feedback for the Prepared Teaching Activities

As a next step, the teachers are asked to prepare a teaching plan based on learning goals and estimated learning progress and to upload it to the web portal. In addition, the teachers are asked to express what the necessary pre-knowledge and possible misconceptions concerning the patterns are when designing their lessons. While Banu explains the pre-knowledge of students from primary school, Ada does not enter anything into the system regarding pre-knowledge. As for the misconceptions, Banu gives examples of a number of the misconceptions that are frequently found in the literature and encountered in her students. For example, Banu expresses that when the teacher writes a pattern like '3, 5, 7, 9, …', the students tend to think 'each number is two more than its predecessor, so the general term is $n + 2$'. However, Ada expresses a challenge rather than a misconception by stating that students have difficulty writing the general rule of the pattern algebraically.

Since the teachers are advised to use the figure pattern in feedback on the goal of learning and the progress of learning, a colourless, linear T pattern (which is made up of unit squares) is presented as teaching material and they are asked to prepare a task for this pattern and to upload it to the system. In this task, the teachers are expected to explain in detail what questions to ask students, how to reach the general rule by considering the student's thinking in depth; in other words, how to discover the functional relationship. Within this context, Ada includes questions such as, "*What do you think of this pattern?*" and "*How can we express the relationship between the steps of this pattern and the number of squares in each step?*". According to the MTE, Ada is unable to ask appropriate and provocative questions in line with the learning goal and learning progression that orient students for an examination of the relationship between the growth of the figure pattern and the steps. Furthermore, the MTE thinks that Ada could not generalize the given pattern using different strategies. However, Banu generalizes the T pattern using different strategies and states that she could ask the questions in Figure 2 to guide her students correctly and to ensure the emergence of figural reasoning.

- Let us find an L in each step.
- What is left after you draw this L?
- How do we find out how many squares are in this L-letter?
- What is the relationship between the number of squares outside the letter L and the number of steps?
- What is the relationship between the number of squares in the L and the number of steps?
- How about step 10?

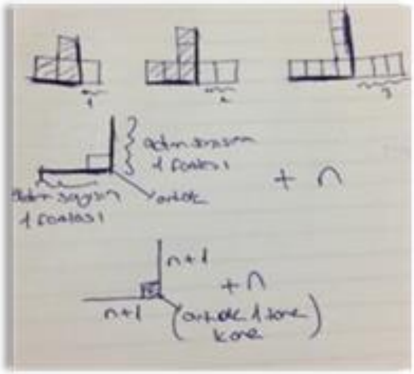


Figure 2. Banu’s exemplary questions and strategies

Figure 2 shows that Banu considers asking many questions, however she is not focusing on analysing the physical structure of the figural pattern and students' existing knowledge. Rather, Banu thinks she should orient her students within a certain strategy. Considering all of the information that the teachers uploaded to the web portal, the MTE provides the same feedback to each teacher. Table 4 summarizes the feedback structure regarding the teachers’ plans.

Table 4

Analysis of the feedback given to the teaching activities prepared by the teachers

Dimension	Associated Code	Feedback	The Main Tenet(s)
Foundation	Identifying errors	Explaining the difference between a misconception and a learning challenge	*Exemplifying misconceptions and student errors and underlining patterns in the mathematics curriculum*
		Providing exemplary misconceptions regarding learning patterns	
	Adheres to textbook	Underlining how patterns appear in the mathematics curriculum with respect to different grades	
Transformation	Choice of representation	Suggesting using table representation where input and output values are written that make numeric relationships visible	*Suggestions for teaching strategies and the use of different representations*
	Teacher demonstration	Explaining different strategies for analysing the physical structure of the given figure pattern	

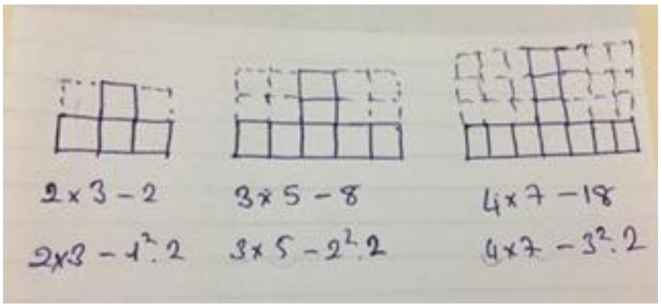
Connection	Decisions about sequencing	Asking questions that will reveal student thinking about generalising the given figure pattern	*Suggestions for the teaching sequence and building inter-concept relationships*
	Making connections between concepts	The transition from the functional relationship to the variable, i.e., defining a variable as a changing quantity	

First, as shown in Table 4, the MTE focuses on the difference between a misconception and a learning challenge/difficulty by explaining each of them, and furthermore, the MTE explains possible misconceptions regarding patterns and pattern generalisation of the students in detail. Regarding the pre-knowledge of the students, the MTE underlines how the patterns are elaborated on in the context of grade levels in the mathematics (national) curriculum.

Secondly, the teachers are given detailed feedback on the use of the teaching approach, selection of representations, the transition between representations and establishing inter-concept relationships. Examples of how teachers can ask questions within their (own) examples are provided by the MTE. For example, the MTE arranges a set of questions considering Banu's example and questions (Table 5).

Table 5

The feedback that is given in the context of decisions regarding sequencing

Banu's Strategy	
Questions prepared by Banu	• Can you form each figure into a rectangle? • What are the lengths of the resulting rectangles at each step? • How many squares did you add to complete a rectangle?

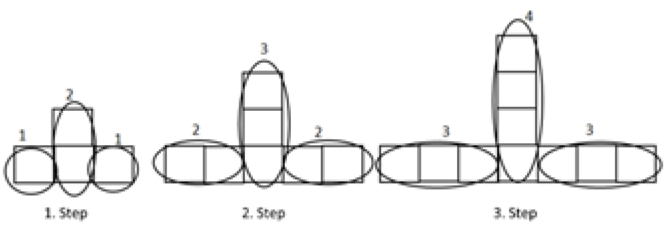
Feedback	• How can we complete each figure into a rectangle? • What are the lengths of the resulting rectangles in each step? • How can we find the total number of squares per step by taking advantage of the area of the rectangle? • How can we find the number of squares that make up the T at each step based on the total number of squares? • What is the relationship between the number of squares that make up the T and the number of steps in each stage? (In terms of the previous question) • How many squares are there in the 15 steps of the T figure? (In terms of the previous question)

In addition, the MTE exemplifies how different strategies would work in the process of generalising the pattern, and how it can be a heuristic tool for the transition from figure representation to table representation. The MTE underlines the importance of analysing the physical structure of the figure in different ways, particularly for Ada, who is substantially lacking in the context of presenting different strategies. The MTE provides many examples of movement between different strategies (Table 6).

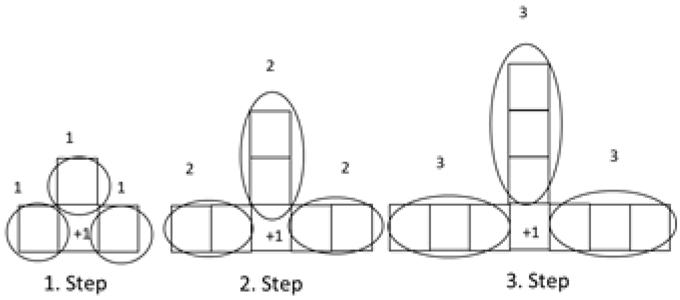
Table 6

Feedback for the use of different strategies and representations

Proposed Strategies | The transition from figure representation to table representation



Step No	Relationship	Total Number of Squares
1	1+2+1	4
2	2+3+2	7
3	3+4+3	10
…	…	…
A	A+(A+1) +A	3A+1



Step No	Relationship	Total Number of Squares
1	1+3.1	4
2	1+3.2	7
3	1+3.3	10
…	…	…
A	1+3.A	3A+1

In this process, in the context of establishing inter-conceptual relationships, Ada was advised to explain the concept of the variable as a changing quantity, especially after the discovery of a functional relationship, and to emphasize the equivalence of functional equations obtained as a result of different generalizations. Following the feedback regarding the use of different representations, the teachers were asked to prepare teaching materials including seven lesson hours of different figure patterns and to implement them in their classrooms.

Teachers' Practice and Feedback on the Process

After the teachers uploaded video recordings of the teaching practices to the system, the MTE watched and noted that Banu carried out successful teaching considering all the feedback given to her before. However, Banu had a lack of knowledge regarding unexpected moments in the classroom. On the other hand, Ada ignored most of the feedback given during the teaching process and had deficiencies. Table 7 presents the main points of the feedback given by the MTE through the system regarding deficiencies that have been identified as a result of analysing the teachers' teaching.

Table 7

Feedback on the teachers' practice

Teacher	Dimension	Associated Code	Feedback	The Main Tenet(s)
Ada	*Transformation*	Choice of examples	Providing new examples	*Suggestions for using the different strategies, representations, and interplay between them*
		Choice of representation	The use of different representations	
		Teacher demonstration	The need to analyse the physical structure of the pattern	
	Connection	Anticipation of complexity	Reminding (her) about the complexity and the difficulties	*Predicting the complex structure of teaching*

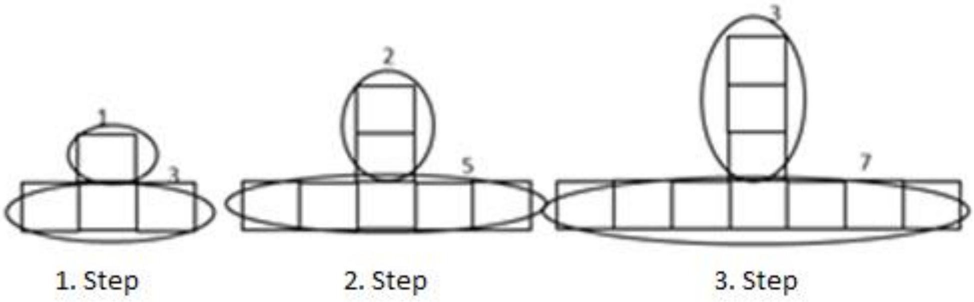
			of teaching patterns	*and establishing inter-relationship(s) between concepts*
		Making connections between concepts	Defining a variable as a changing quantity (aimed at functional thinking)	
			Reminding (her) to express the general rule in terms of a variable	
	Contingency	Responding to students' ideas	Listening to student ideas and opinions	*Suggestions for considering student queries*
			Suggesting a consideration of student thinking	
		Deviation from agenda	Spare time for student queries	
Banu	*Transformation*	Teacher demonstration	The need to analyse the physical structure of the pattern	*Highlighting the importance of analysing the physical structure of the pattern*
	Contingency	Responding to students' ideas	Considering all voices in the classroom	*Reacting to student queries*

Both teachers implemented the T-figure pattern task in the classroom. Banu guided her students to analyse the physical structure of the pattern using four different strategies. Banu, in particular, asked questions about linking the parts to the number of steps by disintegrating the squares in each step of the pattern, when the students had no idea how to proceed. Then Banu asked two questions: "*How do you draw step 35 of the pattern? Then how do you proceed?*". This allowed students to think about their (own) drawings, analyse the structure with different strategies and share their solutions. The MTE thought that Banu was proficient in her demonstrations and wrote:

> *It was very good to guide the students to analyse the physical structure of the pattern, to examine each step, to question the distant step, and to emphasize the total or multiplicative relationships in the rule for the students who generalised it based on the physical structure of the pattern. ... Congratulations...*

In addition, Banu listened to all of her students' opinions and reflected on them before creating an environment to discuss and establish argumentation. However, in this process, Banu often ignored the students who responded incorrectly at unexpected moments, rather focusing on correct answers. For example, one student stated, "*I multiply the fixed difference between the terms in the pattern by the number of steps, that is, 35.3=105*". The MTE reflected on this point (within the context of the contingency dimension) and reminded the teacher of the importance of student thinking, advising her to consider all voices while orchestrating classroom discussions.

Ada started the teaching activity with the T pattern and tried to analyse the physical structure of the pattern using two different strategies. Ada considered a strategy to discover the functional relationship from the analysis of the figure (see Figure 3), however, she did not provide her students with sufficient time to think. Accordingly, the task proved difficult for the students and the teacher could not foresee the complex structure of her strategy in the context of the connection. In fact, during the teaching process, the students had difficulty associating the square numbers in the base with the number of steps and tried to find a rule using a trial-and-error strategy. In this process, it was also determined that the teacher did not use the table representation, which indeed would help to see the numerical relationships in the pattern. A section of the teacher's in-class practice is presented below in Figure 3).



(1.2+1)+1 (2.2+1)+2 (3.2+1)+3

(x.2+1)+x (x.2+1)+x (x.2+1)+x

Figure 3. Ada’s initial strategy

While Ada was presenting the initial strategy, the following classroom discussion took place:

> *Teacher: Look, there are seven here (square numbers at the base in step 3) below. There are five here (horizontal square numbers in step 2), here (square numbers at the base in step 1) there are three. So, how do we relate these three steps to the number of squares? The number of steps. And which step is that? (By pointing to step 1) Step one? How do we get the third step in terms of one? How can we relate it? Yes…*
>
> *Student: Multiply by two and add one.*
>
> *Teacher: Hmm, multiplying by two and adding one... Does that provide it with all?*
>
> *Student: Yes.*
>
> *Teacher: Look, it's two times two.*
>
> *Student: Four.*
>
> *Teacher: Add one.*
>
> *Student: Five.*
>
> *Teacher: Well, we have to calculate all the squares in the figure. And what do we have to add to that?*

As can be seen from the classroom dialogue, when using this strategy, the teacher asked guiding questions that gave clues to the correct answer rather than questions that would make the student think. Through these questions, the teacher switched to direct algebraic representation without establishing a relationship between representations and wrote the general rule without making sense of it. In the context of establishing inter-concept relationships, Ada never mentioned the concept of a variable as a changing quantity. In addition, a number of students explained the physical structure of the pattern with a different strategy, but Ada ignored their thoughts and continued her teaching without deviating from her plan. As a second strategy, Ada referred to the square in the middle of the T pattern as a constant, and then expressed 3, 6, 9, … as the number of other squares. After this, Ada wrote 3+1, 6+1, 9+1 and asked how the number of steps and the total number of squares were interrelated. Regardless of the physical structure of the shape, the teacher, who focused solely on numerical relationships, caused the students difficulty in discovering the functional relationships. Therefore, the MTE provided feedback to the teacher, orchestrating the classroom discussion. Within this context, the MTE underlined that the students and their learning should be at the centre of the teaching. This was explained to Ada:

One student focused on analysing the physical structure of a pattern, but you directed your students towards numerical relationships that are in line with your (own) strategy, regardless of what the student said. However, it might have been more useful to listen to and question what the student was saying…

Moreover, the MTE pointed out that in the second strategy, Ada did not focus on one student's generalisation of the T pattern as $3n+1$ and commented:

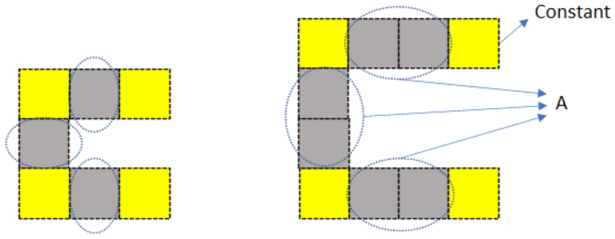
You could have questioned how the student found this rule. You could ask students to describe the fixed and changing squares from the given figure's physical structure through this rule. In addition, it is a significant deficiency that you do not mention the concept of a variable through the general rule of $3x+1$.

The MTE then decided to provide a new figure pattern and explained how to perform teaching through this pattern to underline the importance of the use of representations and the transition between them:

Dear Ada,

When analysing the physical structure of a shape pattern, it is necessary to determine which parts change and which parts remain constant at each step of the pattern. Then it is important to show the varying numbers with a table representation to see the numerical relationships more clearly. After this, the general rule of the pattern should be written verbally and algebraically to discover the functional relationship between the number of steps and the number of terms corresponding to the number of steps. Immediately after that, you can ask students for other thoughts and move forward in the same order, showing the equivalence of the general rule reached as a result of different analyses. I present as an example the analysis and table representation of the C pattern (see Figure 4) *for this conceptual progression:*

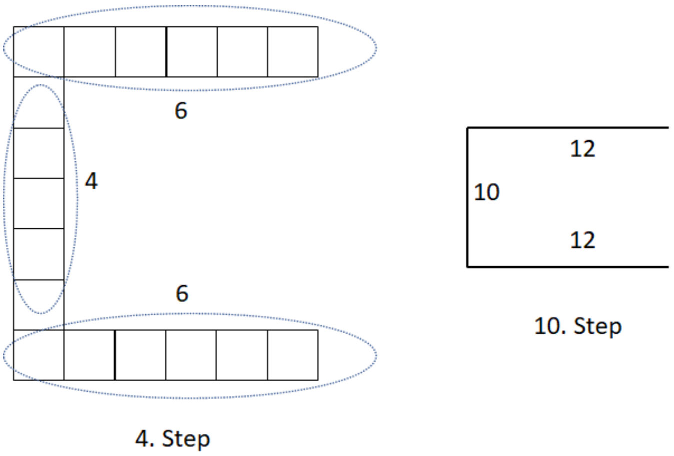
Let A be the number of steps in the given pattern and K be the number of squares. If the yellow squares are constant and grey squares represent the number of steps, then the general rule is 3A+4 = K.



Step No	Process	Total Number of Squares
1	3.1 + 4 = 7	7
2	3.2 + 4 = 10	10
3	3.3 + 4 = 13	13
.	.	.
.	.	.
.	.	.
A	3A + 4	...

Figure 4. Feedback given to Ada via the C pattern

In line with existing deficiencies and the MTE's comments, the teachers were asked to prepare and implement a new teaching plan based on the estimated learning trajectory. Then the teaching plans and practice videos were examined by the MTE again. As a result of this review, it was seen that Banu prepared an original example, while Ada prepared the C pattern following the previous feedback. When the implementation processes were monitored, it was seen that Banu had considered previous feedback carefully and performed better. In addition, even if it is limited, Ada was also observed to have been more successful than in the previous teaching with regard to asking the right questions and considering student queries. The MTE reflected on her final teaching through a direct example from Ada's own teaching (Figure 5).



Step No	Process	Total Number of Squares
1	3 + 3 + 1	7
2	4 + 4 + 2	10
3	5 + 5 + 3	13
4	6 + 6 + 4	16

$n+2+n+2+n$

$3n + 4$

Figure 5. An example from Ada's final practice

Regarding Figure 5, the MTE stated, "*I think it is quite a nice approach to use different representations like the following* (*i.e. Figure 5*) *and to associate them with each other*", which is positive feedback. This can be considered as progress for Ada, where she referred to the transition between the representations.

Discussion and Conclusions

In this paper, we focus on two interrelated research questions: 'What are the main tenets of the MTE's feedback in terms of KQ for HLT-based lesson plans designed to teach the generalisation of patterns?', and 'How does the MTE's feedback through the MEDEGEP portal affect the teachers' design of the HLT for teaching generalisation patterns?' One MTE evaluated two teachers' HLT based lesson plans and provided feedback for the step-by-step lesson design and implementation process. Since the teacher's professional knowledge is necessary and acts as a locomotive for designing the HLT (Simon, 1995, 2014), we considered the KQ (Rowland et al., 2005) as a lens to approach teacher knowledge. In other words, we considered the combination of the KQ and HLT perspectives as a tool for supporting the PD of mathematics teachers. Regarding the first research question, we address that the main tenets of the MTE's feedback are based on *figural* and *numerical* reasoning for generalising patterns, referring to different *representations* and the *transition* between them, and *noticing*/considering student thinking and misconceptions and/or responding to student queries. As a general conclusion, such feedback structure progressively improved the teachers' skills, not only in designing HLTs for teaching patterns, but also by contributing to the teachers' knowledge, particularly regarding knowledge of the foundation, transformation and connection dimensions of the KQ.

In the beginning, while determining the learning goal for generalising patterns, one teacher referred to a functional relationship, while the other referred to a more rule-oriented approach, which is a particular reference for pre-service and in-service teachers as addressed by the researchers (Rivera & Becker, 2007; Zazkis & Liljedahl, 2002). Both teachers could not predict the possible learning paths of the students and ignored pre-knowledge for generalising patterns. Therefore, feedback structure was first based on the foundation dimension by addressing the functional relationship, and the figural and numerical reasoning, which are the core elements in pattern generalisation (Rivera & Becker, 2011). This follows guiding the teachers to consider an analysis of the physical structure of figure patterns in the classroom, which may be considered as a point of departure to discover numerical (and therefore functional) relationships. The latter point is based on improving the estimation of student thinking and learning, and on designing classroom activities within the HLT context. Regarding the transformation dimension of the KQ model, feedback for the transition between figure representation into table representation to establish the variation/covariation relationships for algebraic representation was also provided. This is because in the HLT, hypotheses for the progress of learning and expressing learning goals directly affect the preparation of teaching activities. Teachers must consider their students' pre-knowledge (their needs too) and misconceptions when preparing teaching activities (Simon, 1995, 2014). Therefore, the teacher's transformation and connection knowledge, which were at the centre of the MTE's feedback in this research, are of crucial importance while preparing teaching activities.

In implementing the designed activities, the role of asking provocative and guiding questions is crucial. In our case, one teacher had shortcomings in asking provocative and guiding questions, possibly due to her having had less teaching experience. Even as the teacher (Ada) proceeds in asking questions in her final episode, we are aware that pre-service and in-service teachers have difficulty in asking questions and revealing student opinions as underlined in recent research (Tanışlı, 2013; Tanışlı, Ayber & Kuzu, 2018). As Jacobs, Lamb and Philipp (2010) have stated, the development of the ability to ask questions and to respond to student queries takes time. In line with this, as has been shown in numerous studies (Turner, 2012; Turner & Rowland; 2011), teachers have issues in overcoming unexpected situations in the classroom. Similarly, in our case, both teachers either focused solely on a particular strategy ignoring student thinking and their needs or did not hear all of the students' queries. Through the feedback, one teacher substantially progressed, while the other still had issues in orchestrating classroom discussion to respond to all of her students' needs.

A perspective of student thinking/noticing has helped and guided teachers to understand how to design a lesson that enables conceptual learning (Kazemi & Franke, 2004; Wickstrom & Langrall, 2020). Important evidence has been provided in understanding learning objectives, designing hypotheses and teaching activities for learning progression, as well as enabling teachers to make changes to their activities during teaching, which is interconnected with contingency. In our case, awareness of *student thinking* within the scope of the HLT is positively reflected in the teachers' professional knowledge (Wilson et al. 2013). In this context, through the feedback provided online, teachers' professional knowledge, such as elaborating learning objectives/goals and student pre-knowledge, predicting misconceptions and estimation of learning in generalising patterns have developed. Within the lens of the KQ, both teachers progressed, particularly in the foundation, transformation, and connection dimensions, which is in line with (Livy et al., 2019). All these together imply how a combined HLT perspective helped the MTE to provide in-time feedback to contribute to teachers' PD regarding teaching patterns.

Research results have shown that teachers' PD can be achieved through a web-based education portal. Although there are studies (Francis & Jacobsen, 2013; Matranga & Silverman, 2020) aimed at supporting PD through distance education, an important element that distinguishes this work from other studies is the feedback structure given to teachers at every stage of the design and implementation process. Through the feedback provided, the teachers noticed their missing points and mistakes by communicating with the MTE, and such a PD program through a web portal contributed to teacher professional knowledge. In addition, the interrelation between the HLT and dimensions of the KQ model in structuring feedback could lead to revealing a theoretical/conceptual framework contributing to mathematics teachers PD. However, we are aware that such a fresh perspective needs more elaboration; for example, PD regarding teaching geometry with a combined view of the HLT and the KQ may be considered as a direction for future research.

Acknowledgement

The authors would like to thank Iveta Kohanova and Tim Rowland for reading, commenting on, and editing this paper, which have led to an improvement in its presentation. We are grateful to the reviewers for their constructive criticism and for providing us with helpful comments.

Funding

This research was supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under the SOBAG 1001 programme. The first author was affiliated with the Eskisehir Osmangazi University of Turkey when the data was collected.

References

Arzarello, F., & Taranto, E. (2021). Mathematics teacher educators within new technological environments: changing the perspective. In M. Goos & K. Beswick (Eds.), *The Learning and Development of Mathematics Teacher Educators* (pp. 383–400). Springer. https://doi.org/10.1007/978-3-030-62408-8_20

Ball, D.L. & Cohen, D.K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the Learning Profession* (pp. 3–31). Jossey-Bass.

Beilstein, S.O., Henricks, G.M., Jay, V., Perry, M., Bates, M.S., Moran, C.G., & Cimpian, J.R. (2021). Teacher voices from an online elementary mathematics community: examining perceptions of professional learning. *Journal of Mathematics Teacher Education*, *24*, 283–308. https://doi.org/10.1007/s10857-020-09459-z

Borba, M.C. & Llinares, S. (2012). Online mathematics teacher education: overview of an emergent field of research. *ZDM Mathematics Education*, *44*, 697–704. https://doi.org/10.1007/s11858-012-0457-3

Fennema, E., Carpenter, T.P., Franke, M.L., Levi, L., Jacobs, V.R., & Empson, S.B. (1996). A longitudinal study of learning to use children’s thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*(4), 403–434. https://doi.org/10.2307/749875

Fernandez, C., Llinares, S., & Rojas, Y. (2020). Prospective mathematics teachers’ development of noticing in an online teacher education program. *ZDM Mathematics Education*, *52*, 959–972. https://doi.org/10.1007/s11858-020-01149-7

Francis, K. & Jacobsen, M. (2013). Synchronous online collaborative professional development for elementary mathematics teachers. *International Review of Research in Open and Distance Learning*, *14*(3), 319–343. https://doi.org/10.19173/irrodl.v14I3.1460

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers’ generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal*, *38*(3), 653–689. https://doi.org/10.3102%2F00028312038003653

Gumiero, B. S., & Pazuch, V. (2021). Teachers’ knowledge mobilized in geometry lessons and contingency situations. *International Electronic Journal of Mathematics Education, 16*(1), https://doi.org/10.29333/iejme/9371

Işıksal, M., & Çakıroğlu, E. (2011). The nature of prospective mathematics teachers’ pedagogical content knowledge: The case of multiplication of fractions. *Journal of Mathematics Teacher Education*, *14*(3), 213–230. https://doi.org/10.1007/s10857-010-9160-x

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. https://doi.org/10.5951/jresematheduc.41.2.0169

Jaworski, B. (2001). Developing mathematics teaching: Teachers, teacher educators, and researchers as co-learners. In F.-L.- Lin & T.J. Cooney (Eds.), *Making Sense of Mathematics Teacher Education* (pp. 295–320). Kluwer Academic Publishers. https://doi.org/10.1007/978-94-010-0828-0_14

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203–235. https://doi.org/10.1023/B:JMTE.0000033084.26326.19

Khan, B. (2005). *Managing e-Learning: Design, Delivery, Implementation and Evaluation*. IGI Global. http://doi:10.4018/978-1-59140-634-1

Livy, S., Herbert, S., & Vale, C. (2019). Developing primary pre-service teachers’ mathematical content knowledge: opportunities and influences. *Mathematics Education Research Journal*, *31*, 279–299. https://doi.org/10.1007/s13394-018-0252-8

Mason, J. (2021). Learning about noticing, by, and through noticing. *ZDM Mathematics Education*, *53*, 231–243. https://doi.org/10.1007/s11858-020-01192-4

Matranga, A., & Silverman, J. (2020). An emerging community in online mathematics teacher professional development: an interactional perspective. *Journal of Mathematics Teacher Education*, https://doi.org/10.1007/s10857-020-09480-2

McCool, J. K. (2009). *Measurement learning trajectories: A tool for professional development* (Order No. 3399297). Available from Publicly Available Content Database; Social Science Premium Collection. (205426397). https://www.proquest.com/dissertations-theses/measurement-learning-trajectories-tool/docview/205426397/se-2?accountid=12870

Petrou, M., & Goulding, M. (2011). Conceptualising teachers' mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 9–25). Springer. https://doi.org/10.1007/978-90-481-9766-8_2

Rivera, F., & Becker, J.R. (2011). Formation of pattern generalization involving linear figural patterns among middle school students: Results of a three-year study. In J. Cai & E. Knuth (Eds.), *Early Algebraization* (pp. 323–363). Springer. https://doi.org/10.1007/978-3-642-17735-4_18

Rowland, T. (2013). The Knowledge Quartet: the genesis and application of a framework for analysing mathematics teaching and deepening teachers' mathematics knowledge. *SISYPHUS Journal of Education*, *1*(3),15–43. https://doi.org/10.25749/sis.3705

Rivera, F., & Becker, J.R. (2007). Abduction–induction (generalization) processes of elementary majors on figural patterns in algebra. *The Journal of Mathematical Behavior*, *26*, 140–155. https://doi.org/10.1016/j.jmathb.2007.05.001

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8, 255–281. https://doi.org/10.1007/s10857-005-0853-5

Ruthven, K. (2011). Conceptualising mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 83–96). Springer. https://doi.org/10.1007/978-90-481-9766-8_6

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.3102%2F0013189X015002004

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114–145. https://doi.org/10.2307/749205

Simon, M.A. (2014). Hypothetical learning trajectories in mathematics education. In S. Lerman (Ed.), *Encyclopaedia of Mathematics Education*, (pp. 272–275). Springer. https://doi.org/10.1007/978-94-007-4978-8_72

Şen, C. (2021). Assessment of a middle school mathematics teacher's knowledge for teaching the 5th grade subject of fractions. *Turkish Journal of Computer and Mathematics Education*, *12*(1), 96–138. https://doi.org/10.17762/turcomat.v12i1.271

Tanışlı, D. (2013). Preservice primary school mathematics teachers' questioning skills and knowledge of students in terms of pedagogical content knowledge. *Education and Science*, *38*(169), 80–95.

Tanışlı, D., Köse, N., & Turgut, M. (2019). Professional development through a web-based portal: The progress of mathematics teachers teaching algebra based on hypothetical learning trajectories. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 2741–2748). Utrecht University and ERME.

Tanışlı, D., Ayber, G. & Karakuzu, B. (2018). Lesson design integration of the middle school mathematics teachers into teaching. *Anadolu Journal of Educational Sciences International*, *8*(2), 514–567. https://doi.org/10.18039/ajesi.454588

Turner, F. (2012). Using the knowledge quartet to develop mathematics content knowledge: the role of reflection on professional development. *Research in Mathematics Education*, *14*(3), 253–271. https://doi.org/10.1080/14794802.2012.734972

Turner, F., & Rowland, T. (2011). The knowledge quartet as an organising framework for developing and deepening teachers’ mathematical knowledge. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 195–212). Springer. https://doi.org/10.1007/978-90-481-9766-8_12

Türnüklü, E., Akkaş, E.N., & Gündoğdu Alaylı, F. (2015). Mathematics teachers’ perceptions of quadrilaterals and understanding the inclusion relations. In B. Ubuz, Ç. Haser & M.A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 705–714). Middle East Technical University and ERME.

Ulusoy, F. & Çakıroğlu, E. (2013). In-service elementary mathematics teachers’ conceptions of histogram and difficulties about its teaching process. *Elementary Education Online*, *12*(4), 1141–1156.

Yang, K.-L., Hsu, H.-Y., Lin, F.-L., Chen, J.-C., & Cheng, Y.-H. (2015). Exploring the educative power of an experienced mathematics teacher educator-researcher. *Educational Studies in Mathematics*, *89*, 19–39. https://doi.org/10.1007/s10649-014-9589-4

Yıldırım, A. (2013). Teacher education research in Turkey: Trends, issues and priority areas. *Education and Science*, *38*(169), 175–191.

Weston, T. L. (2018). Using the knowledge quartet to support prospective teacher development during methods coursework. In S. E. Kastberg, A. M. Tyminski, A. E. Lischka, & W. B. Sanchez (Eds.), *Building Support for Scholarly Practices in Mathematics Methods* (pp. 69–83). Information Age Publishing.

Wickstrom, M.H. & Langrall, C.W. (2020). The case of Mrs Purl: Using a learning trajectory as a tool for teaching. *Journal of Mathematics Teacher Education*, *23*, 97–125. https://doi.org/10.1007/s10857-018-9412-8

Wilson P. H., Mojica, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers’ understandings of students’ mathematical thinking. *Journal of Mathematical Behavior*, 32, 103–121. https://doi.org/10.1016/j.jmathb.2012.12.003

Zaskis, R. & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, *49*, 379–402. https://doi.org/10.1023/A:1020291317178