

Elias Klakken Angelsen

The K-theory and Morita equivalence classes of noncommutative tori

On algebraic and topological methods in operator algebras

Bachelor's project in mathematics

Supervisor: Franz Luef

May 2021

$$\begin{array}{ccccc}
 K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \xrightarrow{i_*} & K_0(A \times_\alpha \mathbb{Z}) \\
 \uparrow & & & & \downarrow \\
 K_1(A \times_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{1-\alpha_*} & K_1(A)
 \end{array}$$

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May 12, 2021

Abstract

When studying time-frequency analysis, one encounters unitary translation and modulation operators of great importance. These make up a framework for an operator algebraic approach to time-frequency analysis and can be studied through noncommutative tori, the universal C^* -algebras generated by two such operators. Noncommutative tori shows up as dynamical systems in terms of rotation algebras, in group theory as (completed) twisted group algebras, and even in theoretical physics, as interesting arenas for Yang-Mills theory on noncommutative spaces.

We develop tools originating from ideas in algebra and topology, such as Hilbert C^* -modules, which take us towards the operator algebraic formulation of Morita equivalence, and we generalize topological K -theory to noncommutative scenarios and present powerful consequences, such as the classification theorem of AF-algebras. Higher K -groups and computational tools are introduced, such as Bott periodicity, the six-term exact sequence and the Pimsner-Voiculescu sequence for crossed products. We attempt to apply Morita equivalences and K -theory to noncommutative tori through the work of Rieffel and Pimsner-Voiculescu, which yields a classification of isomorphism and Morita equivalence classes of noncommutative tori.

The thesis aims to give an overview of some of the beautiful theory and fruitful techniques coming from the interaction between several branches of mathematics, and hence the focus will lie on developing the theory.

Sammendrag

Når man studerer tidsfrekvensanalyse, møter man unitære translasjons- og modulasjonsoperatorer. Disse legger grunnlaget for en operatoralgebraisk tilnærming til emnet, og de studeres ofte gjennom ikke-kommutative tori, som er de universale C^* -algebraene generert av slike operatorer. Ikke-kommutative tori dukker blant annet opp som dynamiske systemer i form av rotasjonsalgebraer, i gruppeteori som (komplette) vridde gruppealgebraer og til og med i teoretisk fysikk, hvor de dukker opp som en interessant arena for Yang-Mills-teori i ikke-kommutative rom.

Vi utvikler verktøy fra ideer i algebra og topologi, slik som Hilbert C^* -moduler, som tar oss mot en operatoralgebraisk formulering av Morita-ekvivalens, og vi generaliserer topologisk K -teori til det ikke-kommutative tilfellet, hvilket gir sterke resultater, slik som klassifikasjonsresultatet for AF-algebraer. Høyere K -grupper og beregningsverktøy blir introdusert, slik som Bott-periodisitet, eksakte følger med seks ledd og Pimsner-Voiculescu-følgen for kryssede produkter. Vi forsøker å følge arbeidene til Rieffel og Pimsner-Voiculescu for å klassifisere isomorfi- og Morita-ekvivalensklasser av ikke-kommutative tori.

Målet med oppgaven er å gi et overblikk over noe av den vakre teorien og de nyttige teknikkene som oppstår når forskjellige felter i matematikken samarbeider. Dermed vil fokuset ligge på å utvikle teorien.

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Chapter 1

Introduction

This thesis can be viewed as the final product of my bachelors degree in mathematics at NTNU. Readers are not expected to have the same kind of background as myself, but will be encouraged to read the thesis nevertheless. Some background in analysis, algebra, and topology must be assumed to avoid vast amounts of pages concerning preliminaries, but we still include the most important concepts in the next section.

Why this immense amount of pages?

That is a good question, my friend. There are a lot of beautiful concepts to visit, most of which deserve more than a brief introduction, especially if we want to take the leap from a close-to-no prior knowledge of C^* -algebras to understanding the work by Pimsner-Voiculescu and Rieffel. To make sure we motivate, introduce and explore the concepts in sufficient depth, we gladly accept the fact that there is no page limit on such a thesis.

The author only knew the theory on C^* -algebras equivalent to a first course, for example as covered in the preliminaries, and the first basic definitions of operator K -theory together with a sufficient amount of homological algebra and algebraic topology. Therefore, most of what is written after the preliminaries, including the section on Gabor analysis in the preliminaries, was first introduced to the author throughout the spring of 2021, and is therefore included in-depth throughout the thesis.

Potential readers, beware! In fear of this thesis finding its place in some sort of “uncanny valley” between a bachelors thesis and a poorly written textbook, we will omit a bunch of technical proofs to prioritize giving an ambitious overview.

What are we studying?

There are some main players in this thesis, which deserve an honorable mention before we start the thesis in full. These are introduced below (perhaps in a too intense manner), and we include the topics we cover in each chapter.

Player 1 - A noncommutative torus.

The first player, given a fixed $\theta \in \mathbb{R}$, can be found in the corners of chapter 3 and 8. Through primary motivation from time-frequency analysis, it is the C^* -algebra generated by two unitary operators satisfying a given commutation relation. It has deep connections to Gabor frame theory, it can be realized both as twisted group algebras and as a dynamical system, it yields an interesting arena for Yang-Mills theory and is a brilliant example of a noncommutative space. Its name may scare away any topologically challenged analyst, but its theory lies deep in the heart of applied harmonic analysis. Certainly, we are talking about the noncommutative torus, A_θ .

Player 2 - A Hilbert C^* -module.

Undergraduates are scared of him (or her), algebraists think he (or she) is just a messy construction, your regular non-operator algebraist does not understand him (or her). It is the C^* -algebraic generalization of a Hilbert space and can be found in the vast plains of chapter 4. In its own right, it yields powerful tools to adapt Hilbert space theory to C^* -algebras. Given Morita equivalent C^* -algebras A and B , it works as the manifestation of the Morita equivalence, passing representations and ideal lattices back and forth through its powerful, A - and B -valued inner products. Without further ado, we are of course talking about the Hilbert C^* -module, ${}_A X_B$.

Player 3 - A pair of operator K -groups.

It is abstract, but immensely geometrical at the same time. Hiding behind chapters 5 and 7, it is vital to our understanding of AF-algebras in 6. Defined as analogues to the bundle-theoretic constructions in topological K -theory, they manifest as equivalence classes of projections and unitaries. Through Bott periodicity and techniques from homological algebra, they yield six-term exact sequences, which play prominent rôles in understanding noncommutative tori. They are functorial, they commute with colimits and they are universal. Given a C^* -algebra A , we are indeed talking about the K -groups, $K_0(A)$ and $K_1(A)$.

With the introduction of the players out of the way, we are soon ready to start the brawl.

Digressions

This thesis does not set out to do revolutionary things. The author does certainly believe that we can't have any more fun reading or writing than what we make for ourselves. As we are not doing research (yet), we get some freedom to just explore the world of mathematics, which we will certainly use for what it is worth. Therefore, we use the notion of a *Queequeg* to distinguish traditional remarks from purely exploratory digressions.

Queequeg 1.0.1. If the remarks are exploratory digressions that take us way outside the scope of the thesis, we will call them *Queequegs*. This word comes from the brilliant book "Moby Dick" ([20]) by Herman Melville, where Queequeg shows up as an easy-going son of a tribal chief leaving his island society to explore and experience the world, just out of pure curiosity. Therefore, when we encounter a *Queequeg*, readers should be aware that these are remarks meant to open doors to further exploration, deeper connections, and perhaps even to point readers to topics way beyond the authors knowledge.

Acknowledgements

This thesis could never have reached its length or quality if it were not for my supervisor, Franz Luef, which I owe for pushing me in directions I would not have explored myself and for helping me set the underlying structure of this thesis. Our discussions have been to much help, even if I needed some time for myself to understand what we were talking about.

The same can be said about Eduard Ortega, who sparked my interest in operator algebras and who has been to much help clearing up my broad and (perhaps) nonsensical questions the last year.

The students at Linjeforeningen Delta, Realfagsdagene, Realfagsreyven, and Studentrådet IE also deserve a special shoutout for letting me explore and develop my interests outside of academia while letting me get in touch with wonderful people, even during a pandemic.

Especially, I want to thank Thomas, Tallak, Nora, and the other students at "Mat-teland" for enlightening discussions the last year, both regarding academic and non-academic affairs. I also have to thank Johan for being as young, stupid, and excited as myself when we first started to study mathematics, leading us to learn way more about mathematics than we really should have.

My mother, my family, and my dog would certainly be angered if not mentioned here, as they have supported me for as long as I can remember.

Lastly, I want to thank myself. After all, I am the one who actually wrote this thesis.

A small poem

*Now my thesis has started to merge
And the timing is right on the verge
But I have delivered
You can say I have shivered
Because the number of pages diverge*

- The author

Chapter 2

Preliminaries

The field of operator algebras requires a lot of background to delve into. The reason for this, as we will see in this thesis, is that tools from algebra, topology, and analysis all come together to develop new theories.

If an optimal reader exists, this reader should have taken at least a first course in C^* -algebras, roughly similar to what is covered by the first chapter of Putnam ([29]). We shall mention the most important theorems, propositions, and definitions in this chapter, but we will assume that readers know basic functional analysis, some algebra (preferably basic representation theory and homological algebra), as well as some algebraic topology.

Readers with different backgrounds are also encouraged to read the thesis. Those should be aware that extensive googling, nlab'ing and wikipedia'ing may become necessary for certain parts of the thesis, as universal properties of kernels and cokernels, fun facts concerning short exact sequences or functional analysis and intuition from bundle theory may be thrown around as in a hurricane.

We introduce some important preliminaries in (a not necessarily meaningful) order.

2.1 Time-frequency analysis

Even though Gabor analysis will not be explicitly used in this thesis, there will be a few remarks about several topics from the field. Mostly, this will be references to papers by Luef ([14], [15], [16], [17]), where this branch of time-frequency analysis yields beautiful connections to topics such as projections in noncommutative tori and differential geometry. We therefore only mention the basic definitions and some beautiful results such that readers are able to understand some of these remarks. Be aware that this is not even close to a complete coverage of the basics, as we only skim over the main definitions without the motivation or depth they deserve. We refer to [7] and the references therein for a more thorough treatise.

To be able to do time-frequency analysis, we need a lattice to sample the time and frequency from.

Definition 2.1.1. A lattice $\Lambda \subseteq \mathbb{R}^2$ is a discrete subgroup of the form $\Lambda = A\mathbb{Z}^2$, where A is a real-valued, invertible 2×2 -matrix. Usually, we consider lattices of the form $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$. We define the volume of the lattice by $\text{vol}(\Lambda) = \det(A)$.

We will return to the following operators in chapter 3, but as they are quite important, we define the *translation* and *modulation* operators on $L^2(\mathbb{R})$ as

$$\begin{aligned} T_x f(t) &= f(t - x), \\ M_\omega f(t) &= e^{2\pi i \omega t} f(t), \end{aligned}$$

respectively.

For $\lambda = (x, \omega) \in \Lambda$, define $\pi(\lambda) = M_\omega T_x$ as the *time-frequency shift*.

Definition 2.1.2. The *Fourier transform* on $L^2(\mathbb{R})$ is given by

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt.$$

Definition 2.1.3. For a fixed window function $g \in L^2(\mathbb{R})$, define the *short-time Fourier transform (STFT)* with respect to g on $L^2(\mathbb{R})$ as

$$V_g f(x, \omega) = \langle f, \pi(x, \omega)g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt = \mathcal{F}(f \overline{T_x g})(\omega).$$

If we define a *weight function* to be a non-negative, continuous function $v : \mathbb{R}^2 \rightarrow \mathbb{R}$, we say a weight function $m : \mathbb{R}^2 \rightarrow \mathbb{R}$ is *v -moderate* if $m(x+y) \leq C v(x)m(y)$ for $x, y \in \mathbb{R}^2$.

With this, we can define modulation spaces.

Definition 2.1.4. Fix $p, q \in [1, \infty]$ and let m be a v -moderate weight. Then, *modulation spaces* are defined as all $f \in L^2(\mathbb{R})$ such that the STFT of f , $V_g f$, ends in the weighted, mixed-norm space $L_m^{p,q}(\mathbb{R}^2)$. More precisely, if $L_m^{p,q}(\mathbb{R}^2)$ denotes all $h : \mathbb{R}^2 \rightarrow \mathbb{C}$ such that

$$\|h\|_{L_m^{p,q}(\mathbb{R}^2)} := \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega < \infty,$$

we define modulation spaces as

$$M_m^{p,q}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \|f\|_{M_m^{p,q}} := \|(V_g f)\|_{L_m^{p,q}(\mathbb{R}^2)} < \infty \right\}.$$

There are a few special such spaces.

Definition 2.1.5. If we take $p = q = 1$ and we let $m_s(x, \omega) = (1 + |x|^2 + |\omega|^2)^{s/2}$ be a weight function, we can write $M_s^1(\mathbb{R}) := M_{m_s}^{1,1}(\mathbb{R})$.

If we take $m_s = 1$ to be the identity, we write $S_0(\mathbb{R}) := M_1^1$, which is called *Feichtinger's algebra*.

All of these can be shown to be invariant under the Fourier transform.

We can define Schwartz space, which is the space of functions with rapidly decaying derivatives, as an intersection of these modulation spaces. Indeed, this is equivalent to the original definition.

$$\mathcal{S}(\mathbb{R}) = \bigcap_{s \geq 0} M_s^1(\mathbb{R}).$$

By the inversion formula for the Fourier transform, the Fourier transform is a homeomorphism on $\mathcal{S}(\mathbb{R})$.

We also define frames in a Hilbert space \mathcal{H} , which intuitively are meant to be orthonormal bases with some slack.

Definition 2.1.6. A *frame* is a sequence $\{e_i\}$ indexed by $i \in I$ such that we can find constants $A, B \geq 0$ making the frame inequality hold, that is, for all $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_{i \in I} |\langle x, e_i \rangle|^2 \leq B\|x\|^2. \quad (2.1)$$

B is called the *Bessel bound*, as it implies all frames are Bessel sequences, and A is called the *redundancy* of the frame. If $A = B$, the frame is called *tight*, and if $A = B = 1$, the tight frame is called a *Parseval frame*.

The *dual frame* is any frame $\{e'_i\}$ satisfying

$$x = \sum_{i \in I} \langle x, e'_i \rangle e_i \quad \forall x \in \mathcal{H}. \quad (2.2)$$

To turn this theory into an operator-based theory, we define the following operators. Let $\{e_i\}$ be a frame for \mathcal{H} .

Definition 2.1.7. The *analysis operator* $C : \mathcal{H} \rightarrow \ell^2(I)$ is given by $x \mapsto \{\langle x, e_i \rangle\}_{i \in I}$. The *synthesis operator* $D : \ell^2(I) \rightarrow \mathcal{H}$ is given by $\{c_i\}_{i \in I} \mapsto \sum_i c_i e_i$. The *frame operator* $S = DC = C^*C = D^*D : \mathcal{H} \rightarrow \mathcal{H}$ is given by $x \mapsto \sum_i \langle x, e_i \rangle e_i$.

In fact, S is bounded, positive, invertible, and self-adjoint. The set $\{S^{-1}e_i\}_i$ turns out to be a dual frame to $\{e_i\}_i$. We call $\{S^{-1}e_i\}_i$ the *canonical dual frame*. By positivity, $S^{-1/2}$ is well-defined, and it turns out that $\{S^{-1/2}e_i\}_i$ is a tight frame, which we call the *canonical tight frame* associated to $\{e_i\}_i$.

Using $\pi(\lambda)$, we can define a special type of system, which is vital to time-frequency analysis.

Definition 2.1.8. A Gabor system with atom g is a set $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$. If this is a frame, we call it the Gabor frame, and an atom γ of any dual frame $\mathcal{G}(\gamma, \Lambda)$ of $\mathcal{G}(g, \Lambda)$ is called a *dual atom*.

Naturally, a *multi-window Gabor system* is a set $\{\pi(\lambda)g_j : \lambda \in \Lambda, 1 \leq j \leq n\}$ with a finite number of window functions $g_j \in L^2(\mathbb{R})$.

The (more general) form of the *mixed Gabor frame operator* in the single atom case, which is extended to multi-window frames by summation, is given by

$$S_{g,\gamma,\Lambda} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

$$f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\gamma. \quad (2.3)$$

It turns out $S_{g,\gamma,\Lambda} = id$ if and only if g and γ are dual Gabor atoms.

Given a lattice Λ , which we often consider to be a separable lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, it is interesting to know which requirements we need on the atom g or $\text{vol}(\Lambda)$ (e.g. $\theta = \alpha\beta$) for the system $\mathcal{G}(g, \Lambda)$ to be a frame.

We state the following important results.

Theorem 2.1.9. (Density)

If $\mathcal{G}(g, \Lambda)$ is a frame, then $\text{vol}(\Lambda) \leq 1$.

Theorem 2.1.10. (Balian-Low)

If $\mathcal{G}(g, \Lambda)$ is an orthonormal basis for $L^2(\mathbb{R})$, then we either have $tg(t) \notin L^2(\mathbb{R})$ or $\omega\hat{g}(\omega) \notin L^2(\mathbb{R})$.

2.2 Functional analysis

If readers have lacking knowledge of functional analysis, we may refer to [3] for a good introduction. Several of the results mentioned here can simply be done for Banach spaces, but since these results will mostly be applied to Hilbert spaces, we state definitions and theorems mostly in terms of Hilbert spaces.

Theorem 2.2.1. (Closed Graph Theorem) Let X and Y be Banach spaces and let $T : X \rightarrow Y$ be a linear map. If the graph of T is closed, then T is continuous.

Definition 2.2.2. Let \mathcal{H} be a Hilbert space and let T be a bounded, linear operator on \mathcal{H} , which we denote by $T \in B(\mathcal{H})$. We say T is compact if the image of the closed unit ball in \mathcal{H} under T is relatively compact in \mathcal{H} , that is, if its closure is compact in \mathcal{H} . We denote the subalgebra of $B(\mathcal{H})$ of compact operators as $\mathcal{K}(\mathcal{H})$, or sometimes just \mathcal{K} , if the Hilbert space does not matter.

Proposition 2.2.3. Let \mathcal{H} be a Hilbert space. All finite rank operators on \mathcal{H} are compact, and $\mathcal{K}(\mathcal{H})$ is the closed span of the rank-one operators $g \mapsto (g, k)h$ for $g, h, k \in \mathcal{H}$.

Proposition 2.2.4. $\mathcal{K}(\mathcal{H})$ is a closed, two-sided ideal of $B(\mathcal{H})$. In particular, it is a C^* -subalgebra of $B(\mathcal{H})$.

Since this is a proper ideal for infinite-dimensional Hilbert spaces, we surely expect the identity to not be a compact operator in this case. Indeed, it is possible to show that the identity operator is compact on a normed space X if and only if X is finite-dimensional.

To be able to understand the topology on character space in the next section, we need the following definition.

Definition 2.2.5. The *weak**-topology is the coarsest topology on X^* such that all representable functionals on X^* are continuous, that is, it is the topology generated by preimages of open balls under functionals $\phi_x : f \mapsto f(x)$ for $f \in X^*$.

2.3 A crash course on C^* -algebras

We refer to Putnam ([29]) and Murphy ([21]) for a treatise on C^* -algebras and operator theory, but most of the main results needed are stated in this section.

Definition 2.3.1. (C^* -algebra)

A C^* -algebra A is an algebra over \mathbb{C} with a norm $\|\cdot\|$ such that

1. we have an involution (a conjugate linear map $a \mapsto a^*$ satisfying $(a^*)^* = a$ and $(ab)^* = b^*a^*$),
2. the norm is submultiplicative ($\|ab\| \leq \|a\|\|b\|$),
3. A is complete in this norm,
4. the C^* -equality $\|a^*a\| = \|a\|^2$ holds.

The first three axioms ask for a Banach algebra, which is a norm complete, involutive \mathbb{C} -algebra with submultiplicative norm, meaning multiplication is continuous. In the last requirement, the C^* -equality, we are connecting the topological and algebraic properties, making C^* -algebras powerful tools.

Example 2.3.2. The following examples work as prototypes for C^* -algebras.

1. \mathbb{C} is a C^* -algebra with complex conjugation as involution.
2. $B(\mathcal{H})$ for a Hilbert space \mathcal{H} , where involution is given by taking adjoint operators.
3. $C(X) := C(X, \mathbb{C})$ on a compact Hausdorff space X with the supremum norm, where involution is given by pointwise complex conjugation.

We define morphisms between C^* -algebras the natural way, that is, we define $*$ -homomorphisms between C^* -algebras to be involution preserving algebra homomorphisms.

Definition 2.3.3. (Special elements) Some elements have certain properties worth naming.

1. An element a is self-adjoint if $a^* = a$.
2. An element p is a projection if $p^2 = p = p^*$.
3. An element a is normal if $a^*a = aa^*$.
4. If A is unital, an element u is unitary if $u^*u = 1 = uu^*$ (that is, if $u^{-1} = u^*$).
5. An element a is positive, written $a \geq 0$, if there exists some b such that $a = b^*b$.

What do C^* -algebras look like in general? There are some powerful results giving explicit structures for commutative C^* -algebras and finite-dimensional C^* -algebras.

Theorem 2.3.4. (Gelfand-Naimark) Let A be a C^* -algebra. Then there exists a Hilbert space \mathcal{H} and a C^* -subalgebra $B \subseteq B(\mathcal{H})$ such that $A \cong B$ as C^* -algebras.

Proposition 2.3.5. If A is finite-dimensional C^* -algebra, we can find integers K, N_1, \dots, N_K such that

$$A \cong \bigoplus_{k=1}^K M_{N_k}(\mathbb{C}).$$

This implies that all finite-dimensional C^* -algebras are unital.

Theorem 2.3.6. (Gelfand) Let A be a commutative, unital C^* -algebra, then there exists a compact Hausdorff space X such that $A \cong C(X)$ as C^* -algebras.

Remark 2.3.7. In fact, we can choose X to be the space of characters on A with the *weak**-topology and the isomorphism is given by the Gelfand transform, sending $a \in A$ to $\hat{a} = ev_a$, which is evaluation by a on the space of characters, $\mathcal{M}(A) = Hom_{C^*Alg_{Com}}(A, \mathbb{C}) \setminus \{0\}$.

Even better, these functors define inverse equivalences:

$$\begin{aligned} C(-) = Hom_{Top_{CptHD}}(-, \mathbb{C}) : Top_{CptHD} &\rightarrow C^*Alg_{Com} \\ \mathcal{M}(-) = Hom_{C^*Alg_{Com}}(-, \mathbb{C}) \setminus \{0\} : C^*Alg_{Com} &\rightarrow Top_{CptHD} \end{aligned}$$

To generalize to the nonunital case, take $C_0 : C^*Alg_{Com} \rightarrow Top_{LocCptHD}$ to get a similar result. The above equivalence is often called Gelfand duality.

As continuous functions and bounded operators make out the prototypes of C^* -algebras, we define a generalization of eigenvalues of an operator and the image of a continuous function, namely the spectrum.

Definition 2.3.8. Let A be a unital C^* -algebra and let $a \in A$. Define the *spectrum* of a as $\text{spec}(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible}\}$. The *spectral radius* $r(a)$ of a is the supremum of $|\lambda|$ for $\lambda \in \text{spec}(a)$.

Some authors, including the author of this thesis in sloppy or dark moments, denote the spectrum of a by $\sigma(a)$.

It is possible to show that the spectrum must be nonempty and compact. Also, taking the spectrum in a C^* -subalgebra $B \subseteq A$ yields the same results as in the ambient C^* -algebra A , which is called spectral permanence.

Proposition 2.3.9. Given a C^* -algebra A , we can find a unique (naive) unitization, that is, we can find a unital C^* -algebra \tilde{A} such that A is a closed two-sided ideal in \tilde{A} and $\tilde{A}/A \cong \mathbb{C}$.

\tilde{A} is explicitly constructed by letting $\tilde{A} = A \oplus \mathbb{C}$, the involution is given component-wise. The multiplication with unit $(0, 1)$ is given by $(a, \lambda)(b, \mu) = (\lambda b + \mu a + ab, \lambda\mu)$, but the norm is not canonical. See [29] for a brief explanation.

If we don't want to construct an explicit unit, it is always possible to find an approximate unit for a C^* -algebra.

Proposition 2.3.10. Every C^* -algebra A admits an approximate identity, that is, we can find an increasing net $(u_\lambda)_{\lambda \in \Lambda}$ such that $a = \lim u_\lambda a = \lim a u_\lambda$ for all $a \in A$.

This means that we can find approximate identities for closed ideals in C^* -algebras as well, since they themselves are C^* -algebras. Note that this could not possibly act as an approximate identity on the ambient C^* -algebra, if the ideal is proper.

As mentioned, positive elements are elements a such that we can find an element b with $a = b^*b$. There are useful results regarding positivity.

Proposition 2.3.11. Let \mathcal{H} be a Hilbert space and let T be a bounded operator on \mathcal{H} . The operator T is positive if and only if the sesquilinear form $(x, y) \mapsto \langle Tx, y \rangle$ is positive, where $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathcal{H} .

For self-adjoint elements, we get interesting properties.

Proposition 2.3.12. Let a be a self-adjoint element of a unital C^* -algebra. The following are equivalent.

1. $\text{spec}(a) \subseteq [0, \infty)$,
2. For all $t \geq \|a\|$, we have $\|t - a\| \leq t$,
3. For some $t \geq \|a\|$, we have $\|t - a\| \leq t$.

Proposition 2.3.13. Let A be a C^* -algebra and let $a, b \in A$ be self-adjoint elements. If $a \leq b$, then $x^*ax \leq x^*bx$ for all $x \in A$.

The inverse of the Gelfand transform turns out to be quite interesting and it unlocks a powerful tool in C^* -algebra theory, called *functional calculus*. Let a be a normal element in a C^* -algebra B and let f be in $C(\text{spec}(a))$. It is possible to show evaluation at a is a homeomorphism $\mathcal{M}(A) \rightarrow \text{spec}(a)$, where A denotes the C^* -algebra generated by a, a^* and 1 .

This implies we can find an element of A , denoted $f(a)$ such that the function is given by $f = \widehat{f(a)} \in C(\text{spec}(a))$. More precisely, if we let B be a unital C^* -algebra, a be a normal element of B , meaning $aa^* = a^*a$, and $A = C^*(a, 1) \subseteq B$ be the

C^* -algebra generated by 1 and a , then $f(a)$ is the unique element of A such that $\phi(f(a)) = f(\phi(a))$ for all $\phi \in \mathcal{M}(A)$.

Theorem 2.3.14. (Functional calculus) If B is a unital C^* -algebra and a is a normal element, then the map sending f to $f(a)$ is an isometric $*$ -isomorphism $C(\text{spec}(a)) \rightarrow C^*(a, 1) \subseteq B$. Furthermore, if $f(z) = \sum a_k z^m \bar{z}^n$ is a polynomial in z and \bar{z} , then $f(a) = \sum a_k a^m (a^*)^n$.

This allows us to consider elements in $C^*(a, 1)$ as complex-valued polynomials!

When studying C^* -algebras, there are many linear functionals, as C^* -algebras are Banach spaces, meaning we can apply the Hahn-Banach theorem. Characters are such linear functionals, but they are more rare, due to the fact that they respect lots of structure.

Some middle ground between being a linear functional and a character can be found in the definition of a trace. We say a functional ϕ is positive if $\phi(a^*a) \geq 0$.

Definition 2.3.15. A unit preserving, positive linear functional $\tau : A \rightarrow \mathbb{C}$ is called a *trace* if $\tau(ab) = \tau(ba)$. The trace is *faithful* if $\tau(a^*a) = 0$ implies $a = 0$.

On a finite-dimensional Hilbert space \mathcal{H} with orthonormal basis $\{\xi_1, \dots, \xi_n\}$, we can find a unique, normalized, faithful trace on $B(\mathcal{H})$ given by $\frac{1}{n} \sum \langle a \xi_i, \xi_i \rangle$. In fact, if p is a projection, then $\dim(p\mathcal{H}) = \tau(p) \dim(\mathcal{H})$.

2.4 Representations and group C^* -algebras

Much of the motivation of studying operator algebras came from $B(\mathcal{H})$ for some Hilbert space \mathcal{H} . Therefore it is natural to define a representation as a way of representing a C^* -algebra as bounded operators.

Definition 2.4.1. A *representation* of A is a pair (π, \mathcal{H}_π) such that $\pi : A \rightarrow B(\mathcal{H}_\pi)$ is a $*$ -homomorphism.

We say a representation π is *nondegenerate* if the only $\xi \in \mathcal{H}_\pi$ such that $\pi(A)\xi = 0$ is $\xi = 0$. More informally, the representation is nondegenerate if the Hilbert space is not too big.

A representation π is said to be *irreducible* if the only invariant, closed subspaces of π are 0 and \mathcal{H}_π , where a subspace N is invariant if $\pi(A)N \subseteq N$.

It is possible to show a representation is nondegenerate if and only if $\pi(1) = 1$ and that all representations can be built from nondegenerate representations and zero representations.

An element $\xi \in \mathcal{H}_\pi$ is *cyclic* if $\pi(A)\xi$ is dense in \mathcal{H}_π , and a representation is called *cyclic* if it has a cyclic vector. Nondegenerate representations are irreducible if and only if all nonzero vectors are cyclic.

Definition 2.4.2. A *state* on a unital C^* -algebra is a positive linear functional ϕ such that $\phi(1) = 1$.

Example 2.4.3. The only nonzero, nondegenerate representation of \mathbb{C} is $\pi(z) = z1$ since $\pi(1) = 1$ and Gelfand duality implies \mathbb{C} -linearity of π .

Theorem 2.4.4. If we are given a state ϕ on a unital C^* -algebra A , we can construct a representation, and if we are given a representation, we can construct a state.

In particular, it is possible to show that if a is a self-adjoint element, there exists an irreducible representation π of A such that $\|\pi(a)\| = \|a\|$.

This construction is called the *Gelfand-Naimark-Segal construction (GNS)* and can be found in any textbook on C^* -algebras.

Often the need to integrate on locally compact groups shows up. We can in fact find a suitable measure to do this, called the *Haar measure*. A (right) *Haar measure* is a (right) translation invariant, regular Borel measure on G which is finite on compact subsets.

Theorem 2.4.5. (Haar) If G is a locally compact group, there exists a left (and right) Haar measure on G . This measure is unique up to scalar multiples.

2.5 Short reminders from algebraic topology

Throughout the thesis, there will certainly be a need for homological algebra and algebraic topology. Much will be assumed to be known, and hence we refer readers to Rotman [37] for an introduction to homological algebra and to May [19] for a concise course in algebraic topology.

When we construct the definition of Morita equivalences through bimodules later, we need to be fluent in the language of tensor products. We define the algebraic tensor product of modules, as the case for algebras follows from this.

Definition 2.5.1. Let R be a ring, let A be an abelian group, and let M and N be R -modules. A morphism $\phi : M \times N \rightarrow A$ is R -balanced if it is bilinear and $\phi(mr, n) = \phi(m, rn)$ for $r \in R$.

A *tensor product* is the universal R -balanced abelian group in the sense that it is an abelian group $M \otimes_R N$ together with an R -balanced map $t : M \times N \rightarrow M \otimes_R N$ such that R -balanced morphisms $\phi : M \times N \rightarrow A$ factors uniquely through t .

Tensor products exist and are unique up to isomorphism.

Tensor products for C^* -algebras are not as well-behaved as the algebraic ones. If the C^* -algebras are *nuclear*, all completed tensor products coincide, but if not, we may have a vast collection of completed tensor products for C^* -algebras, making it hard to reason by categorical analogy in operator algebras. We refer to [21] for a more in-depth discussion of the topic.

To be able to adapt topological K -theory to operator algebras, it is certainly useful to understand vector bundles. We won't do that much topological K -theory, but (almost) everything we do in those chapters will be inspired from the topological case, meaning that facts and claims about vector bundles will be tossed around.

Note that if we replace the complex vector space structure with any other fiber, we obtain the definition of a more general construction called a *fiber bundle*.

Definition 2.5.2. Let B be a topological space, usually assumed to be compactly generated weak Hausdorff spaces ([19]). A (complex) n -dimensional *vector bundle* is a continuous map $p : E \rightarrow B$ such that $p^{-1}(b)$ has a (complex) vector space structure satisfying a trivialization condition. If U_α is a cover of the base space B , then $p^{-1}(U_\alpha)$ should be trivial in the sense that it is homeomorphic to $U_\alpha \times \mathbb{C}^n$.

Informally, a n -dimensional vector bundle is a construction where we glue on a n -dimensional vector space in each point in a way that looks locally trivial.

There are lots of reasons to study bundles, homological algebra, and functional analysis, but for now, we contain ourselves with the definition and leave the rest for another day.

Chapter 3

Motivation: Noncommutative tori

The main goal of the thesis is to study lots and lots of developed theory, but to not lose touch with reality, we need a motivating example we can come back to once in a while. For us, this will be the well-studied noncommutative C^* -algebras called noncommutative tori. We give a brief motivation for the structure and some explicit characterizations that will prove to be useful when attempting to classify these structures through the developed theory. Readers interested in deeper connections to time-frequency analysis should find other sources, such as [7].

3.1 A first date with noncommutative tori

Noncommutative tori arise quite naturally in time-frequency analysis and play a central rôle in the operator-based approach to the topic. For a good overview of time-frequency analysis, we refer readers to [7], but we will also follow [5] and [42] in this chapter.

We want to study functions $f \in L^2(\mathbb{R})$, as these can be thought of as signals. Intuitively, if we are given a signal and we can translate the signal and change its frequency, we have come a long way to be able to study all signals we care about. We recall the natural definitions of the translation and modulation operators.

Definition 3.1.1. Define the *translation operator*, T_x , as $T_x f(t) = f(t - x)$ and the *modulation operator*, M_ω , by $M_\omega f(t) = e^{2\pi i \omega t} f(t)$.

Two important things to note, which yield key points when defining noncommutative tori, is that these operators are unitary and obey the commutation relation $T_x M_\omega = e^{-2\pi i x \omega} M_\omega T_x$, which can be shown by a simple calculation.

We jump ahead to define the algebraic structure given through these operators, which yield our first definition of noncommutative tori.

Definition 3.1.2. Let U and V be unitaries such that $UV = e^{-2\pi i\theta} VU$. Define a *noncommutative torus* as the (universal) C^* -algebra generated by U and V and denote it A_θ .

When we say that this C^* -algebra is universal, we mean that any other structure A generated by two unitaries \tilde{U}, \tilde{V} such that this commutation relation holds, will have a uniquely induced $*$ -homomorphism $A_\theta \rightarrow A$ sending $U \mapsto \tilde{U}$ and $V \mapsto \tilde{V}$. A proof that A_θ is universal can be found in [5]. Note that noncommutative tori are unital.

The name comes from the fact that for $\theta = 0$ we actually get $A_0 \cong C(\mathbb{T}^2)$, which by Gelfand duality should amount to studying \mathbb{T}^2 . To see this, note that if $\theta = 0$, we actually have a commutative C^* -algebra, meaning it can be realized as $C(X)$ for some compact Hausdorff space X , due to results by Gelfand (2.3.6). The remaining task is to show that X is homeomorphic to \mathbb{T}^2 . By universality, we can send U, V to the coordinate functions z_1, z_2 on \mathbb{T}^2 , corresponding to time and frequency, if we stay time-frequency-minded. This will yield the homeomorphism. We refer to [42] for details.

It is useful to note that we can restrict ourselves to $\theta \in [0, 1]$, or even better, we can restrict to $\theta \in [0, 1/2]$ when working with noncommutative tori by the following result.

Proposition 3.1.3. A noncommutative torus A_η is isomorphic to a noncommutative torus A_θ for some $\theta \in [0, 1/2]$.

Proof. To restrict ourselves to $\theta \in [0, 1]$, note that this follows by universality if we use the fact that the commutation relation is unchanged under $\theta \rightarrow \theta + n$ for $n \in \mathbb{Z}$, which means $A_{\theta+n} \cong A_\theta$. Now, to see the restriction to $\theta \in [0, 1/2]$, note that modulo 1, we can consider A_η for an $\eta \in [-1/2, 0]$. By the $*$ -automorphism on A_θ sending $U \mapsto V$ and $V \mapsto U$, we obtain the same commutation relation as if we choose $\theta = -\eta$. This means that $A_\eta \cong A_\theta$ by universality, where $\theta \in [0, 1/2]$. \square

To understand the structure better, we look for a more explicit, time-frequency-related definition of A_θ . The first explicit forms of noncommutative tori are given through twisted group algebras. In more abstract applications, such as in connections between noncommutative geometry and Gabor analysis, this definition may be the most useful, as it allows us to treat more general examples along the lines of noncommutative tori. We will not use the following characterization further, but it is worth mentioning to observe the explicit connections to time-frequency analysis.

Recall that we define a lattice $\Lambda \subseteq \mathbb{R}^2$ to be a discrete subgroup of the form $\Lambda = A\mathbb{Z}^2$, where A is a real-valued, invertible 2×2 -matrix. If we sample x and ω from a lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, where we put reasonable requirements on the sampling density $\theta = \alpha\beta$, we can phrase much of time-frequency analysis through an operator π , defined by $\pi(\lambda) = M_x T_\omega$, where $\lambda = (x, \omega) \in \Lambda$.

Definition 3.1.4. Define $A_\theta^1 = \{a = \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda), (a_\lambda) \in \ell^1(\Lambda)\} \subseteq B(L^2(\mathbb{R}))$ with norm given by the ℓ^1 -norm on the coefficients.

Multiplication is given through twisted convolution in the sense that

$$\left(\sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda) \right) \left(\sum_{\lambda \in \Lambda} b_\lambda \pi(\lambda) \right) = \sum_{\lambda \in \Lambda} (a \natural b)_\lambda \pi(\lambda),$$

where the twisted convolution is defined for $\lambda = (x, \omega), \mu = (u, \eta)$ by

$$(a \natural b)_\lambda := \sum_{\mu \in \Lambda} a_\mu b_{\lambda - \mu} e^{-2\pi i \theta(x-u)\eta}.$$

The convolution is necessary to get a Banach algebra, while the twisted convolution ensures that we actually get back the noncommutative torus we are working with.

Now, this is not a C^* -algebra yet, but we can define A_θ as its enveloping C^* -algebra, that is, by completing A_θ^1 in the norm given by $\|a\| = \sup \|\rho(a)\|$, where the supremum is taken over all representations ρ of A_θ^1 . This is isomorphic to the universal definition by $U \mapsto M_\beta, V \mapsto T_\alpha$.

More generally, this construction of A_θ comes from considering the enveloping C^* -algebra of a twisted group algebra $\ell^1(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ with an associated 2-cocycle $c((x, \omega), (u, \eta)) = e^{2\pi i x \eta}$, which justifies the name *twisted group algebra* for the noncommutative tori.

Queequeg 3.1.5. There are lots of results and concepts that could have been mentioned to show the depth of studying noncommutative tori through operator algebras, but these should be saved for a thesis in its own right. Nevertheless, some of them are so motivating and beautiful they deserve a short mention.

If we consider the coefficients as coming from Schwartz functions on Λ or equivalently by intersecting such structures over weighted ℓ^1 -spaces, as in the case for modulation spaces, we get what we call a smooth noncommutative torus, A_θ^∞ , but this is not a Banach algebra.

It is possible to prove theorems in time-frequency analysis, such as the Balian-Low theorem from Gabor analysis, through techniques of noncommutative differential geometry on noncommutative tori. This can be found in [16], where the proof boils down to defining connections with constant curvature on Schwartz space, which can be realized as a finitely generated projective module over a smooth noncommutative torus, unlocking the tools of differential geometry as we can think of it as a vector bundle through the Serre-Swan theorem (5.1.1).

In [36], an attempt due to Rieffel and Connes at defining a Yang-Mills energy functional through such connections is explained, yielding noncommutative tori as interesting arenas for doing theoretical physics as well. We briefly return to this as a final digression.

It is also possible to define higher-dimensional analogues of noncommutative tori, but we restrict ourselves to the two-dimensional cases, as there are several open questions in the higher-dimensional cases that we will resolve later for the two-dimensional case. Interested readers should also be referred to [14] for a connection between multi-window Gabor frames and higher-dimensional noncommutative tori through the Waxler-Raz biorthogonality relations for multi-window Gabor frames.

Even though the previous definition of noncommutative tori through π is beautiful and extremely handy in time-frequency analysis, we will not use it much in the pages to come. On the other hand, understanding noncommutative tori as C^* -algebras coming from dynamical systems will be important later.

3.2 Crossed product C^* -algebras and rotation algebras

It is possible to realize noncommutative tori as crossed product algebras coming from dynamical systems, which will be vital when we want to apply the Pimsner-Voiculescu sequence to calculate the K -groups of A_θ . We start by recalling the definition of C^* -dynamical systems and crossed product C^* -algebras, most of which can be found in [42].

Definition 3.2.1. The triple (A, G, α) is called a C^* -dynamical system if A is a C^* -algebra, G is a discrete group (a group with the discrete topology), and the action $\alpha : G \rightarrow \text{Aut}(A)$ is a group homomorphism.

To get a structure from such a C^* -dynamical system, we consider finite formal sums.

Definition 3.2.2. Define AG as finite formal sums $\sum_{g \in G} a_g g$, where $a_g \in A$. Addition and scalar multiplication are defined the canonical way, but for multiplication, we require $\alpha_g(a) = g a g^{-1}$ whenever it shows up. More precisely,

$$\begin{aligned}
 \left(\sum_{g \in G} a_g g \right) \left(\sum_{h \in G} b_h h \right) &= \sum_{g \in G} \sum_{h \in G} a_g g b_h h \\
 &= \sum_{g \in G} \sum_{h \in G} a_g g b_h g^{-1} g h \\
 &= \sum_{g \in G} \sum_{h \in G} a_g \alpha_g(b_h) g h \\
 &= \sum_{h \in G} \left(\sum_{g \in G} a_g \alpha_g(b_{g^{-1}h}) \right) h,
 \end{aligned} \tag{3.1}$$

where we map $h \mapsto g^{-1}h$ for simplicity.

This structure can be given an involution by $(ag)^* = \alpha_{g^{-1}}(a^*)g^{-1}$, which boils down to $(ag)^* = g^{-1}a^*$.

Remark 3.2.3. By requiring the automorphism to be inner in the sense that it works as conjugation, we preserve the noncommutative structure from our C^* -algebras.

Given such a formal structure, it would be natural to consider the enveloping C^* -algebra.

Definition 3.2.4. We define the *crossed product algebra* $A \times_\alpha G$ as the enveloping C^* -algebra of AG , that is, as the completion of AG in the norm $\|f\| = \sup \|\rho(f)\|$, where the supremum is taken over all representations ρ of AG .

With the crossed product structure defined, we can define the rotation algebras, which turns out to be isomorphic to the noncommutative tori.

Definition 3.2.5. Fix a number $\theta \in \mathbb{R}$. Define an action of \mathbb{Z} on $C(S^1)$ by $n \mapsto \alpha^n$, where $\alpha(f) = f(t - \theta)$ is the action given by precomposing by rotation on the circle with an angle θ .

Define the rotation algebra as the crossed product $C(S^1) \times_\alpha \mathbb{Z}$.

Proposition 3.2.6. For a fixed θ , the rotation algebra $C(S^1) \times_\alpha \mathbb{Z}$ is isomorphic to the noncommutative torus A_θ .

Proof. We refer to [42] for a full proof, but the idea is that we can find two generators in $C(S^1) \times_\alpha \mathbb{Z}$. First, define $z \in C(S^1)$ by $z(t) = e^{2\pi it}$. Secondly, define w to be the element representing α by conjugation. Then, it is possible to show the commutation relation of the noncommutative torus holds, yielding a $*$ -homomorphism $U \mapsto z$ and $V \mapsto w$. By functional calculus, we can find an inverse. \square

Queequeg 3.2.7. It is possible to connect the twisted group algebra to the crossed product by a partial Fourier transform. Let $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ and let $a \in A_\theta^1$ be given by $a = \sum a_{k,l} \pi(\alpha k, \beta l)$. In Gabor analysis, the goal is to study what $\pi(\alpha k, \beta l)g$ looks like, while in operator algebras, we want to understand the operator $\sum a_{k,l} \pi(\alpha k, \beta l)$. If we compute

$$\begin{aligned} a &= \sum a_{k,l} \pi(\alpha k, \beta l) \\ &= \sum a_{k,l} M_{\beta l} T_{\alpha k} \\ &= \sum \tilde{a}(k, t) T_{\alpha k}, \end{aligned}$$

we have a partial Fourier transformation linking $a \in \ell^1$ and \tilde{a} . In fact, the representation of the form $\sum \tilde{a}(k, t) T_{\alpha k}$ shows that it is an element of $C(S^1) \times_\alpha \mathbb{Z}$, where α is the rotation action on the circle. Namely, that the composition of $\sum \tilde{a}(k, t) T_{\alpha k}$ and $\sum \tilde{b}(k, t) T_{\alpha k}$ yields an operator of the form $\sum_{k \in \mathbb{Z}} (\tilde{a} \star \tilde{b})(k, t) T_{\alpha k}$ for

$$(\tilde{a} \star \tilde{b})(k, t) = \sum_{l \in \mathbb{Z}} \tilde{a}(l, t) \tilde{b}(k-l, t - \alpha l)$$

and $(\sum \tilde{a}(k, t)T_{\alpha k})^* = \sum \tilde{a}^*(k, t)T_{\alpha k}$, where

$$\tilde{a}^*(k, t) = \tilde{a}(-k, t - \alpha k).$$

If we do Gabor analysis, the former corresponds to the Janssen representation of the Gabor frame operator, while the latter corresponds to the Walnut representation. More about this can be found in [7].

After we develop more theory, we will return to study noncommutative tori. First, we move on to study Morita equivalence for C^* -algebras.

Chapter 4

Hilbert C^* -modules and Morita equivalences

The study of representations is quite fruitful in the world of operator algebras and extremely important to be able to realize these structures in applications, as seen in the previous chapter. The notion of Morita equivalences between rings from representation theory has proven to be quite useful and can probably be adapted to operator algebras in the sense that Morita equivalent rings have equivalent module categories and thus yield equivalent representations. In this chapter, we seek to work out the technicalities that let us adapt Morita equivalences to the world of C^* -algebras, even though this may be a long and technical journey.

To construct our sense of Morita equivalences, we are inspired by the classical Morita theorem ([24]), which asserts that two rings are Morita equivalent if and only if there exists bimodules taking us between the module categories, and thus inducing representations. Seeking to connect the idea of bimodules and induced representations to C^* -algebras, we first need to develop some theory about Hilbert modules, which was introduced for commutative C^* -algebras by Kaplansky ([10]) and extended to the noncommutative case independently by Paschke ([25]) and Rieffel ([32]) in the '70s. These play the main rôle of the bimodules connected to our C^* -algebras.

After this, we induce representations back and forth, which leads to the concept of imprimitivity bimodules, which are the bimodules implementing our Morita equivalences. These play a crucial rôle, as their existence yields the explicit construction of ideal lattice isomorphisms and induced representations between C^* -algebras. We follow the presentation of the topic given by Raeburn and Williams ([31]), but interested readers should also be referred to the work of Lance ([13]). The first section will be quite pedantic, but for later sections, we skip some of the details and refer readers to [31].

Throughout this chapter, A and B will be C^* -algebras.

4.1 Hilbert modules

We follow [31] and define our (right) A -modules to be vector spaces X with a bilinear pairing $X \times A \rightarrow X$, $(x, a) \mapsto x \cdot a$ satisfying the usual conditions. Algebraists usually define modules over rings as additive abelian groups with an action from a ring, but as we try to generalize the behaviour of spaces over \mathbb{R} or \mathbb{C} and want to apply a lot of tools from analysis, it is natural to replace the assumption of additive abelian groups by assuming we have a vector space over \mathbb{C} (or \mathbb{R}).

To preserve the relevant structure, we want to generalize Hilbert spaces to modules. If this is supposed to work out with the actions given by modules, we first need to consider what inner product modules should look like. We define the right inner product module structure, even though an analogue definition could be given for left inner product module structures.

Definition 4.1.1. A (right) A -module X is called a (right) *inner product A -module* if it has a pairing $\langle \cdot, \cdot \rangle_A : X \times X \rightarrow A$ such that the following holds:

- a) $\langle x, \lambda y + \mu z \rangle_A = \lambda \langle x, y \rangle_A + \mu \langle x, z \rangle_A$,
- b) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A \cdot a$,
- c) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$,
- d) $\langle x, x \rangle_A \geq 0$,
- e) $\langle x, x \rangle_A = 0$ implies that $x = 0$.

We often write X_A to indicate that X is a right A -module.

Remark 4.1.2. Note that for condition (d), the inner product maps to A , which means that inequalities are phrased in terms of positivity of elements as these are inequalities in C^* -algebras. This will be the case for all inequalities arising in C^* -algebras.

We can easily show that this inner product is conjugate linear in the first variable by a standard calculation using (a) and (c):

$$\langle \lambda x + \mu y, z \rangle_A = \langle z, \lambda x + \mu y \rangle_A^* = (\lambda \langle z, x \rangle_A + \mu \langle z, y \rangle_A)^* = \bar{\lambda} \langle x, z \rangle_A + \bar{\mu} \langle y, z \rangle_A$$

Also, (b) and (c) imply that $\langle a \cdot x, y \rangle_A = a^* \cdot \langle x, y \rangle_A$.

Together, this implies that

$$\text{span} \{ \langle x, y \rangle_A : x, y \in X \}$$

is a two-sided ideal in A .

Let us consider some basic examples to see our definition in play.

Example 4.1.3. Inner product \mathbb{C} -modules are, as expected, the inner product spaces with a \mathbb{C} -valued inner product. This inner product should be conjugate linear in the first variable.

Example 4.1.4. Define an inner product A -module structure on A with inner product given by $\langle a, b \rangle_A = a^*b$. All axioms except the last one follow from computations. The last axiom follows from the C^* -equality.

The definition seems to make sense in this basic case. A natural question to ask is whether or not we can use this to define a norm in a canonical way. We postulate that

$$\|x\|_A := \|\langle x, x \rangle_A\|^{1/2}$$

gives a norm analogous to the \mathbb{C} -valued case. To prove this structure works out as wanted, we need some extra results. If these C^* -algebraic analogues of inner product spaces are nice generalizations of regular inner product spaces, we should also expect some of the key results in this theory to hold, such as the Cauchy-Schwarz inequality.

Proposition 4.1.5. (The Cauchy-Schwarz inequality) Let X be an inner product A -module and $x, y \in X$. Then

$$\langle x, y \rangle_A^* \langle x, y \rangle_A \leq \|\langle x, x \rangle_A\| \langle y, y \rangle_A$$

with the inequality interpreted in the sense of positivity, which is the standard way to interpret inequalities for C^* -algebras. Also, if X is an inner product module over a dense $*$ -subalgebra of A , this still holds if we interpret the inequality in the completion of A_0 , making it an inner product module over a C^* -algebra.

As this is not our usual Cauchy-Schwarz inequality, we need a sufficient condition for positivity to make our life easier. If we connect the notion of positivity in C^* -algebras to the normal notion of positivity in the ground field, we may be able to use common tools in the proof, such as the regular Cauchy-Schwarz inequality.

Lemma 4.1.6. Let a be an element of A . If $\rho(a) \geq 0$ for all states ρ of A , then a is positive.

Proof. We assume $\rho(a) \geq 0$ for all states ρ . To show $a \geq 0$, we want to construct a specific state ϕ such that $\phi(a) \geq 0$ implies $a \geq 0$. To do this, let π be a faithful representation of A on a Hilbert space \mathcal{H} , which we can choose from the GNS-construction of representations (2.3). This allows us to define a state $\phi(a) = (\pi(a)h, h)$ for all h of norm 1, where (\cdot, \cdot) denotes the inner product in the Hilbert space. As ϕ is a state, we have $(\pi(a)h, h) \geq 0$ for all h . This is the definition of $\pi(a)$ being a positive operator on our Hilbert space and we know these are the positive elements in the C^* -algebra $B(\mathcal{H})$ by 2.3.11. Now we can use 2.3.12 to get that $\pi(a) \geq 0 \iff \text{spec}_{B(\mathcal{H})}(\pi(a)) \subseteq [0, \infty)$. This can be pushed down to the $*$ -subalgebra $\pi(A)$ by spectral permanence, which in turn yields the results for a as π is faithful. Thus we have $\text{spec}_A(a) \subseteq [0, \infty)$, which means $a \geq 0$. \square

Equipped with our new lemma, we show 4.1.5.

Proof. (Proof of 4.1.5)

We want to show that

$$z = \|\langle x, x \rangle_A\| \langle y, y \rangle_A - \langle x, y \rangle_A^* \langle x, y \rangle_A \geq 0.$$

After our technical work, we want to use lemma 4.1.6. Take an arbitrary state ρ . If we show $\rho(z) \geq 0$, we are finished. Using the linearity of ρ and reordering, we see that it is enough to show

$$\rho(\langle x, y \rangle_A^* \langle x, y \rangle_A) \leq \|\langle x, x \rangle_A\| \rho(\langle y, y \rangle_A).$$

Looking to apply the usual Cauchy-Schwarz inequality, we consider the (semi-definite) positive sesquilinear form $(u, v) \mapsto \rho(\langle u, v \rangle_A)$. We apply the usual Cauchy-Schwarz inequality to this, giving

$$|\rho(\langle u, v \rangle_A)| \leq \rho(\langle u, u \rangle_A)^{1/2} \rho(\langle v, v \rangle_A)^{1/2}.$$

Inserting $u = x \langle x, y \rangle_A$ and $v = y$ yields the following:

$$\begin{aligned} \rho(\langle x, y \rangle_A^* \langle x, y \rangle_A) &= \rho(\langle x \langle x, y \rangle_A, y \rangle_A) \\ &\leq \rho(\langle x \langle x, y \rangle_A, x \langle x, y \rangle_A \rangle_A)^{1/2} \rho(\langle y, y \rangle_A)^{1/2} \\ &= \rho(\langle x, y \rangle_A^* \langle x, x \rangle_A \langle x, y \rangle_A)^{1/2} \rho(\langle y, y \rangle_A)^{1/2} \end{aligned}$$

From 2.3.13, we know that for $a \leq b$, we have the inequality $d^*ad \leq d^*bd$ for all d . By noting $a \leq \|a\|1$, we get $d^*ad \leq \|a\|d^*d$. The fact $a \leq \|a\|1$ comes from the equivalent requirements for positivity in terms of inequalities (2.3.12), first used on the positive element a and then to show $\|a\|1 - a \geq 0$.

Thus,

$$\rho(\langle x, y \rangle_A^* \langle x, y \rangle_A) \leq (\|\langle x, x \rangle_A\| \rho(\langle x, y \rangle_A^* \langle x, y \rangle_A))^{1/2} \rho(\langle y, y \rangle_A)^{1/2},$$

which yields our result after squaring and cancelling. \square

As intended, we get the following proposition from this lemma, with some extra work.

Proposition 4.1.7. Let X be an inner product A -module. The postulated norm,

$$\|x\|_A := \|\langle x, x \rangle_A\|^{1/2},$$

defines a norm on X which is submultiplicative with respect to the action from A . The normed module $(X_A, \|\cdot\|_A)$ is nondegenerate in the sense that the span of elements of the form $x \cdot a$ makes up a dense subset of X . Even better,

$$X \cdot \langle X, X \rangle_A := \text{span}\{x \cdot \langle y, z \rangle_A : x, y, z \in X\}$$

is dense in X_A with respect to the normed structure given by this inner product.

Proof. By the inner product structure of $\langle \cdot, \cdot \rangle$ and the axioms for the A -norm, we get all the requirements except the triangle inequality for free. We developed the Cauchy-Schwarz inequality (4.1.5) exactly for this purpose. This follows from

$$\begin{aligned} \|x + y\|_A^2 &\leq \|\langle x, x \rangle_A\| + \|\langle x, y \rangle_A\| + \|\langle y, x \rangle_A\| + \|\langle y, y \rangle_A\| \\ &\leq \|x\|_A^2 + 2\|x\|_A\|y\|_A + \|y\|_A^2 \\ &= (\|x\|_A + \|y\|_A)^2, \end{aligned}$$

where we used 4.1.5 in the last inequality.

To show submultiplicativity, consider the squared expression and apply submultiplicativity of $\|\cdot\|$.

$$\|x \cdot a\|_A^2 = \|a^* \langle x, x \rangle_A a\| \leq \|a^*\| \|x\|_A^2 \|a\| = \|a\|^2 \|x\|_A^2$$

To show that expressions on the form $x \cdot \langle y, z \rangle_A$ are dense, we need to consider an approximate identity $\{u_\lambda\}$ on the span of the inner products $\langle x, y \rangle_A$ to get our approximation game started. Let $\epsilon > 0$. We have that

$$\begin{aligned} \|x - x \cdot u_\lambda\|_A^2 &= \langle x - x \cdot u_\lambda, x - x \cdot u_\lambda \rangle_A \\ &= \|\langle x, x \rangle_A - \langle x, x \rangle_A u_\lambda - u_\lambda \langle x, x \rangle_A + u_\lambda \langle x, x \rangle_A u_\lambda\|. \end{aligned}$$

We can find a u_λ such that $\|x - x \cdot u_\lambda\|_A < \epsilon/2$. As the approximate identity lives on the span of the inner products, we find a linear combination of elements arbitrarily close to u_λ , that is, we can find $x_i, y_i \in X$ such that $\|\sum_i \langle x_i, y_i \rangle_A - u_\lambda\| < \epsilon/2\|x\|_A$, where the right hand side is chosen for later cancellation.

Now we have

$$\begin{aligned} \left\| x - x \cdot \left(\sum_i \langle x_i, y_i \rangle_A \right) \right\|_A &= \left\| x - x \cdot u_\lambda + x \cdot u_\lambda - x \cdot \left(\sum_i \langle x_i, y_i \rangle_A \right) \right\|_A \\ &\leq \|x - x \cdot u_\lambda\|_A + \left\| x \cdot u_\lambda - x \cdot \left(\sum_i \langle x_i, y_i \rangle_A \right) \right\|_A \\ &< \epsilon/2 + \|x\|_A \cdot \epsilon/2\|x\|_A = \epsilon. \end{aligned}$$

□

The inner product structure seems to work out. Looking to generalize Hilbert spaces, we add some requirements of completeness in this norm. This leads to the notion of a Hilbert C^* -module.

Definition 4.1.8. Let X be an inner product A -module. If X is complete in the norm $\|\cdot\|_A$, we call X a *Hilbert A -module*. Also, if the ideal

$$\text{span} \{ \langle x, y \rangle_A : x, y \in X \}$$

is dense in A , we call X a *full Hilbert A -module*.

Before we move on, we should examine some examples.

Example 4.1.9. A Hilbert \mathbb{C} -module is a Hilbert space over \mathbb{C} with the usual multiplication and inner product, which is defined to be conjugate linear in the first variable by setting $\langle x, y \rangle_{\mathbb{C}} = (y, x)$, where (\cdot, \cdot) denotes the regular inner product in \mathbb{C} , which is conjugate linear in the second variable.

Example 4.1.10. A_A is a full Hilbert module if we define multiplication as usual and $\langle a, b \rangle_A = a^*b$ as earlier. The A -module norm is the same as the normal norm in A by the C^* -equality, which implies completeness. In C^* -algebras, we can always find an approximate identity by 2.3.10, which means that we can approximate elements arbitrarily well by a product with some u_λ . Thus, by existence of approximate identities, this inner product will span a dense subset. We can also use this construction to find non-full Hilbert A -modules. Take any proper, closed two-sided ideal I of A . Then I_A will not be full, as the closure of $\text{span} \{ \langle i, j \rangle_A \}$ for $i, j \in I$ is indeed I instead of A , i.e. we miss elements of $A \setminus I$.

The following examples are a bit more exotic. Some of them are just beautiful examples and some of them do play an important role in Morita Equivalence of C^* -algebras, which we will return to in section 4.5.

Example 4.1.13 will certainly require some more topological and algebraic background to understand, but due to its beauty, as well as the fact that the Serre-Swan theorem can be considered to be the foundation of K -theory, we include it here so readers can get an impression of the immense relevance of operator algebras in other fields. Example 4.1.14 will certainly require more analytical background to understand thoroughly, but this is claimed to be the example that motivated the theory of Morita equivalence for C^* -algebras by Rieffel. Most of the later results on Morita equivalences can be applied to example 4.1.14, but this is outside the scope of this thesis. Therefore we only mention it here as a historical (and perhaps encouraging) example.

Example 4.1.11. Let \mathcal{H} be a Hilbert space and let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the C^* -algebra of compact operators on \mathcal{H} . If $h \otimes \bar{k}$ denotes the rank-one operator given by $l \mapsto h \cdot \langle k, l \rangle_{\mathbb{C}} = (l, k)h$, we can define a left Hilbert \mathcal{K} -module structure on \mathcal{H} by setting

$$T \cdot h = T(h) \quad \text{and} \quad \langle x, y \rangle_{\mathcal{K}} = x \otimes \bar{y}.$$

The operator norm of $h \otimes \bar{h}$ is given by $\|h\|^2$ by considering the supremum over norm one elements x of $h \otimes \bar{h}(x)$ and recalling that we have equality in 4.1.5 when the terms are linearly dependent. By choosing $x = h/\|h\|$, we attain the maximum

with a normalized element, which yields the result by writing out the terms. Then the \mathcal{K} -norm agrees with the regular norm on \mathcal{H} , making \mathcal{H} complete in the \mathcal{K} -norm. The fullness of $\text{span}\{\langle x, y \rangle\}_{\mathcal{K}}$ follows from the fact that finite rank operators are dense in \mathcal{K} .

Example 4.1.12. (Direct sums of Hilbert modules) The direct sum of two Hilbert A -modules X and Y can be made by giving the algebraic direct sum (right) A -module $X \oplus Y$ an A -valued inner product by

$$\langle (x, y), (x', y') \rangle_A := \langle x, x' \rangle_A + \langle y, y' \rangle_A,$$

which yields a complete structure by

$$\begin{aligned} \|x\|_A^2 &= \|\langle x, x \rangle_A\| \leq \|\langle x, x \rangle_A + \langle y, y \rangle_A\| \\ &= \|(x, y)\|_A^2 \leq \|x\|_A^2 + \|y\|_A^2. \end{aligned}$$

One should be aware of which norms have a subscripted A , as this is quite important to see which space we work in.

Example 4.1.13. In the '50s, Serre made some advances on the theory of vector bundles in the category of affine varieties. Later, in the early '60s, Swan developed the work of Serre further, which resulted in the celebrated Serre-Swan theorem ([40]), which asserts that the spaces of sections $\Gamma(X, E)$ of a vector bundle $p : E \rightarrow X$ over a compact Hausdorff space X are finitely generated projective $C(X, \mathbb{C})$ -modules. With some modifications, such as in [22], we can get a Hilbert module structure. Let E be a fiber bundle on X with fibers being Hilbert spaces over \mathbb{C} . Let $\langle \cdot, \cdot \rangle_x$ denote the inner product in the fiber with base point x . We define the $C(X)$ -valued inner product on $\Gamma(X, E)$ by $\langle \sigma_1, \sigma_2 \rangle(x) := \langle \sigma_1(x), \sigma_2(x) \rangle_x$. Then

$$\langle \sigma_1, \sigma_2 \rangle \in C(X) \quad \text{for } \sigma_1, \sigma_2 \in \Gamma(X, E), \quad x \in X.$$

If X is locally compact, we only have to replace $C(X)$ by $C_0(X)$ to get the same structure. It should be possible to show that every Hilbert $C_0(X)$ -module is isomorphic to a Hilbert module of this form ([22]).

Example 4.1.14. Let G be a locally compact group, which we assume to be unimodular just to simplify the expressions, and let H be a closed, unimodular subgroup of G . We want to construct a $C^*(H)$ -module X starting with $C_c(G)$, which is the complex-valued continuous functions on G with compact support. Consider $C_c(G)$ as a $C_c(H)$ -module by the action given by

$$f \cdot b(s) = \int_H f(st^{-1})b(t)dt,$$

where $f \in C_c(G)$. Define the inner product $C_c(G) \times C_c(G) \rightarrow C_c(H)$ by

$$\langle f, g \rangle_{C_c(H)}(s) = \int_G \overline{f(r)}g(rs)dr.$$

We reference the reader to page 15 and appendix C in [31] for details and proofs of the following claims:

1. $\langle f, g \rangle_{C_c(H)}$ is conjugate linear in the first variable,
2. $\langle f, g \cdot b \rangle_{C_c(H)} = \langle f, g \rangle_{C_c(H)} * b$ for $b \in C_c(H)$,
3. $\langle f, g \rangle_{C_c(H)}^* = \langle g, f \rangle_{C_c(H)}$,
4. $\langle f, f \rangle_{C_c(H)}$ is positive in the C^* -completion $C^*(H)$,
5. the range of $\langle \cdot, \cdot \rangle_{C_c(H)}$ spans a dense ideal in $C_c(H)$.

These claims justify our structure and give quite an interesting example, but $C_c(G)$ is unfortunately not a Hilbert $C^*(H)$ -module just yet. The first problem is the fact that $C_c(G)$ is not necessarily complete. This is not a major issue, and actually quite common when constructing Hilbert modules. The other problem is the fact that the completed group C^* -algebra $C^*(H)$ does not act on $C_c(G)$, as the action is only defined for $C_c(H)$.

The following lemma amends these problems. The proof is omitted, as we will not use it in this thesis, but we refer to lemma 2.16. in [31] for a proof.

Lemma 4.1.15. (Completions) Let A_0 be a dense $*$ -subalgebra of A , where A is a C^* -algebra. Assume X_0 is a right A_0 -module. We suppose that X_0 can be considered as a pre-inner product A_0 -module (i.e. non-complete) in the sense that we can find a form $\langle \cdot, \cdot \rangle_0 : X_0 \times X_0 \rightarrow A_0$ such that (a)-(d) of definition 4.1.1 is satisfied. The last condition is assumed to hold in the completion A of A_0 . Then there exists a Hilbert A -module X and a linear map $q : X_0 \rightarrow X$ such that the image of q is dense, $q(x) \cdot a = q(x \cdot a)$ for $x \in X_0, a \in A_0$ and $\langle q(x), q(y) \rangle_A = \langle x, y \rangle_0$. This Hilbert A -module X is called the completion of the pre-inner product A_0 -module X_0 .

This ends our brief study of the Hilbert modules themselves. As usual, the study of morphisms between these objects should be at least as interesting as the objects themselves, which leads us onwards in our study.

4.2 Adjointable operators on Hilbert modules

When we work with Hilbert spaces, an essential result of the Hilbert space theory is the fact that we can always find the adjoint of an operator. This is not necessarily the case in Hilbert modules, as we will see. To try to mimic Hilbert space theory once again, we define adjointable operators and delve into their properties.

Definition 4.2.1. Let X, Y be Hilbert A -modules. We call a function $T : X \rightarrow Y$ *adjointable* if there exists a function $T^* : Y \rightarrow X$ such that

$$\langle T(x), y \rangle_A = \langle x, T^*(y) \rangle_A$$

for all $x \in X, y \in Y$.

It turns out that these adjointable operators are reasonable to study.

Lemma 4.2.2. If $T : X \rightarrow Y$ is an adjointable map between Hilbert A -modules, then T is a bounded, linear map from X to Y as A -modules.

Proof. All of these claims can be shown for T^* as well, by similar approaches. By the Cauchy-Schwarz inequality (4.1.5), we have that for any $x \in X$,

$$\|x\|_A = \sup \{ \|\langle x, y \rangle_A\| : y \in X \text{ and } \|y\|_A \leq 1 \}.$$

This implies that $x = y \iff \langle x, z \rangle_A = \langle y, z \rangle_A$ for all $z \in X$ by considering the norm of the differences. Now we can use this, along with some standard inner product tricks from the Hilbert space theory, to show A -linearity. We only show the A -linearity of T as the claim for T^* is totally analogous. Take an arbitrary $y \in X$ and $a \in A$. By writing

$$\begin{aligned} \langle T(x \cdot a), y \rangle_A &= \langle x \cdot a, T^*(y) \rangle_A = a^* \langle x, T^*(y) \rangle_A \\ &= a^* \langle T(x), y \rangle_A = \langle T(x) \cdot a, y \rangle_A, \end{aligned}$$

we have shown $T(x \cdot a) = T(x) \cdot a$. If we do a similar calculation, but replace $x \cdot a$ with $x + x'$ for some arbitrary $x' \in X$, we get that T is A -linear.

The only thing left to show is that T is bounded, but to show this, we have to use tools from functional analysis, such as the Closed Graph Theorem (theorem 2.2.1). This can be used since Hilbert A -modules are Banach spaces and since T was just shown to be linear. Assume we have a convergent sequence $x_n \rightarrow x$ in X and that $T(x_n) \rightarrow z$ for some $z \in Y$. We show $T(x) = z$. If we take an arbitrary $y \in Y$, we have both $\langle T(x_n), y \rangle_A \rightarrow \langle z, y \rangle_A$ and $\langle T(x_n), y \rangle_A = \langle x_n, T^*(y) \rangle_A \rightarrow \langle x, T^*(y) \rangle_A = \langle T(x), y \rangle_A$. Thus $\langle T(x), y \rangle_A = \langle z, y \rangle_A$ for arbitrary y , which yields $T(x) = z$. The graph of T is then closed, so boundedness follows from the Closed Graph Theorem. \square

A natural question to ask now is whether or not all bounded, linear A -module maps are adjointable. The following counterexample shows that this is in fact not the case. As usual, counterexamples work best in the world of analysis, not in the world of algebra.

Example 4.2.3. (Example of a bounded A -linear operator between Hilbert A -modules that are not adjointable)

Let $A = C([0, 1])$ and define $I = \{g \in A : g(0) = 0\}$. Then A and I are both Hilbert A -modules the usual way. Define the direct sum Hilbert A -module $X = A \oplus I$ and let $T : X \rightarrow X$ be defined by $(f, g) \mapsto (g, 0)$. T is bounded, A -linear and $\|T\| = 1$. Now, assume that T has an adjoint T^* and consider $(f, g) := T^*(0, 1)$. For all pairs $(h, k) \in X$, we write

$$\bar{k} = \langle T(h, k), (1, 0) \rangle_A = \langle (h, k), (f, g) \rangle = \bar{h}f + \bar{k}g,$$

which would imply $f \equiv 0, g \equiv 1$ as the pair (h, k) was arbitrary. This is a contradiction since g is supposed to be in I and hence satisfy $g(0) = 0$. Therefore T is a bounded, A -linear operator between Hilbert A -modules that is not adjointable.

Definition 4.2.4. Let X and Y be Hilbert A -modules. We let $\mathcal{L}(X, Y)$ denote the set of all adjointable operators $X \rightarrow Y$. In the case $Y = X$, we just write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$.

Remark 4.2.5. By using the same inner product tricks as in the proof of lemma 4.2.2, we can show that the adjoint of $T \in \mathcal{L}(X)$ is unique, adjointable, and that $T^{**} = T$. Taking the adjoint turns out to be an involution. Thus $\mathcal{L}(X)$ is a subalgebra of the Banach algebra $B(X)$ of bounded operators on X . It turns out, as we restrict ourselves to the operators that mimic the bounded operators on Hilbert spaces, that $\mathcal{L}(X)$ certainly is a C^* -algebra, as the C^* -equality still holds.

Proposition 4.2.6. If X is a Hilbert A -module, then $\mathcal{L}(X)$ is a C^* -algebra when given the operator norm.

To try to say more about $\mathcal{L}(X)$, we want a characterization of the positive elements in this C^* -algebra.

Lemma 4.2.7. If X is a Hilbert A -module and $T : X \rightarrow X$ is a linear operator, then T is a positive element of $\mathcal{L}(X)$ if and only if $\langle T(x), x \rangle_A \geq 0$ for all $x \in X$.

Using this lemma, we can deduce a nice bound for the operator norm of adjointable operators.

Corollary 4.2.8. If X is a Hilbert A -module and $T \in \mathcal{L}(X)$, then

$$\langle T(x), T(x) \rangle_A \leq \|T\|^2 \langle x, x \rangle_A.$$

Proof. A similar result applies to operators in $\mathcal{L}(X, Y)$. We want to show that $\|T\|^2 \langle x, x \rangle_A - \langle T(x), T(x) \rangle_A \geq 0$. Since T^*T is positive by definition, we apply lemma 4.2.7 to get that $\|T\|^2 I - T^*T$ is positive in $\mathcal{L}(X)$. Then we can find an $S \in \mathcal{L}(X)$ such that $\|T\|^2 I - T^*T = S^*S$, which means that

$$\|T\|^2 \langle x, x \rangle_A - \langle T(x), T(x) \rangle_A = \langle (\|T\|^2 I - T^*T)x, x \rangle_A = \langle S^*S(x), x \rangle_A \quad (4.1)$$

$$= \langle S(x), S(x) \rangle_A \geq 0. \quad (4.2)$$

□

Remark 4.2.9. If X is a Hilbert A -module and $T \in \mathcal{L}(X)$ is positive, we have $\|T\| = \sup \{ \|\langle T(x), x \rangle_A\| : \|x\|_A \leq 1 \}$ by the definition of positivity and the C^* -equality.

If $A = \mathbb{C}$, the Hilbert A -module structure on X is just a Hilbert space structure \mathcal{H} over \mathbb{C} , as mentioned earlier. In this case, every operator is adjointable, which means $\mathcal{L}(X) = B(\mathcal{H})$. In Hilbert space theory, both $B(X)$ and $\mathcal{K}(X)$ play a prominent rôle in the structure theory of Hilbert spaces, but if we consider $B(X)$ as a C^* -algebra, it is too big. For example, it would be nice if we could consider all operators as limits of finite-dimensional operators, which means $\mathcal{K}(X)$ plays the main rôle in

the structure theory of C^* -algebras. If $\mathcal{L}(X)$ is the analogue of $B(X)$ in the Hilbert module theory, perhaps it would be fruitful to find some corresponding version of $\mathcal{K}(\mathcal{H})$.

A natural question to ask is how we should define $\mathcal{K}(X, Y)$ and how it should be related to $\mathcal{L}(X, Y)$. Recalling that $\mathcal{K}(\mathcal{H})$ is an ideal in $B(\mathcal{H})$ (proposition 2.2.4) and that $\mathcal{K}(\mathcal{H})$ is the closed span of rank operators (proposition 2.2.3), we seek to define some analogue rank-one operators in $\mathcal{L}(X, Y)$ as a start.

Definition 4.2.10. Let X, Y be Hilbert A -modules. For $x \in X, y \in Y$, we define $\Theta_{y,x} : X \rightarrow Y$ by $\Theta_{y,x}(z) := y \cdot \langle x, z \rangle_A$.

Define $\mathcal{K}(X, Y)$ to be the closed, linear subspace of $\mathcal{L}(X, Y)$ given by the span of $\{\Theta_{y,x} : y \in Y, x \in X\}$. If $X = Y$, we only write this as $\mathcal{K}(X)$.

By analogy to the Hilbert space theory, we call this the *algebra of compact operators*, even though the operators are not necessarily compact in the usual case. An example, which may be outside of the scope of the thesis, is the Gabor frame operator, when viewed as a rank-one operator of Rieffel's Heisenberg modules [14]. In the context of Hilbert C^* -modules, it is in fact a rank-one operator, but it is not compact in the regular sense.

Remark 4.2.11. This is actually well defined in $\mathcal{L}(X, Y)$, by the following computation showing these operators are adjointable in a quite explicit sense.

Let $x, x' \in X$ and $y, y' \in Y$.

$$\langle \Theta_{y,x}(x'), y' \rangle_A = \langle y \cdot \langle x, x' \rangle_A, y' \rangle_A = \langle x, x' \rangle_A^* \langle y, y' \rangle_A = \langle x', x \rangle_A \langle y, y' \rangle_A \quad (4.3)$$

$$= \langle x', x \cdot \langle y, y' \rangle_A \rangle_A = \langle x', \Theta_{x,y}(y') \rangle_A \quad (4.4)$$

Therefore, $\Theta_{y,x}$ is adjointable and $\Theta_{y,x}^* = \Theta_{x,y}$.

This algebra of compact operators seems to work as the regular compacts on a Hilbert space, as we see in the following lemma.

Lemma 4.2.12. Let X be a Hilbert A -module. Then $\mathcal{K}(X)$ is a closed, two-sided ideal in $\mathcal{L}(X)$.

Proof. Take any $T \in \mathcal{L}(X)$. To show the absorption property, just note that

$$T\Theta_{x,y}(z) = T(x \cdot \langle y, z \rangle_A) = T(x) \cdot \langle y, z \rangle_A = \Theta_{T(x),y}(z).$$

By the earlier remark (4.3), $\mathcal{K}(X)$ is closed under involution. A similar calculation as the one above yields the fact that this is in fact a two-sided ideal. Closedness follows from the definition of $\mathcal{K}(X, Y)$. \square

It turns out that if X is a (right) Hilbert A -module, then it is also a full left Hilbert $\mathcal{K}(X)$ -module.

Theorem 4.2.13. Let X be a (right) Hilbert A -module. Then X is a full Hilbert $\mathcal{K}(X)$ -module with the action $T \cdot x := T(x)$ and inner product ${}_{\mathcal{K}(X)}\langle x, y \rangle := \Theta_{x,y}$. The norm induced by this inner product coincides with the norm induced by the A -valued inner product.

We end this subsection with a technical result, which may be interesting in its own right, as it enhances the decomposition we already know from proposition 4.1.7.

Proposition 4.2.14. Let X be a Hilbert A -module. For all $x \in X$, we can find a unique $y \in X$ such that x decomposes as $x = y \cdot \langle y, y \rangle_A$.

The proof can be found in [31], but we encourage interested readers to try to write out the decomposition in \mathbb{C} .

4.3 Multiplier algebras

A treatment of Hilbert modules will not be complete until we include a section on multiplier algebras, even though it is not at the core of this thesis. Multiplier algebras resemble a noncommutative analogue of Stone-Ćech compactification from topology, which in some sense is the maximal compactification of a space. For more thorough treatments of the topic, one should take a look at [31] or [13].

In some sense, adjoining a unit to a C^* -algebra resembles compactifying a space. We start by generalizing unitizations to see how the more algebraic theory of unitizations of C^* -algebras should work.

If we can embed the nonunital C^* -algebra as an ideal in a larger, unital C^* -algebra without giving it too much space, we should be close to a good definition. Therefore, by requiring the ideal to "see" all other ideals in the ambient C^* -algebra, we should have a useful place to start.

Definition 4.3.1. An ideal I in a C^* -algebra A is essential if I has nonzero intersection with each nonzero ideal of A .

With some clever tricks, we show an equivalent and more useful requirement for an ideal to be essential.

Lemma 4.3.2. Let I be a (closed) ideal of A . Then I is essential if and only if $aI = (0)$ means that $a = 0$.

Proof. Let $a \in A$. Assume I is an essential ideal. To show the right-hand side, algebraic experience (or some sixth sense) tells us it could help to study the ideal generated by a . Define the ideal generated by a by $(a) = \overline{\text{span}}\{AaA\}$. Now $aI = (0)$ is equivalent to $(a) \cdot I = (0)$. We know that since both of these are ideals in a C^* -algebra, $(a) \cap I = (a) \cdot I$ by the argument below, which implies $(a) \cap I = (0)$. Since I is essential, $(a) = (0)$, which means $a = 0$, as wanted. To show the converse, we assume that $a = 0$ if $aI = (0)$. Take a nonzero ideal J and consider a nonzero

$a \in J$. Then we have that $aI \neq (0) \implies (a) \cap I \neq (0)$, which then means $I \cap J \neq (0)$ as $(a) \subseteq J$. Therefore I is essential. \square

Remark 4.3.3. In the proof above, we claim that intersections of ideals are the same as products! In the well-behaved world of commutative algebra, we know that only coprime ideals have this property. This is certainly not commutative algebra, but it may still be surprising that this strong claim holds for all ideals in C^* -algebras!

Let us explain this more in-depth. If we have two (closed) ideals $I, J \subseteq A$, we know $I \cdot J \subseteq I \cap J$ by the absorption property and the definition of the product ideal. To show the other inclusion, we use some of the powerful tools in the theory of C^* -algebras. Let $x \in I \cap J$ and let $\{u_\lambda\}$ be an approximate identity for the ideal I , which exists by proposition 2.3.10 since the closed ideal I is a C^* -algebra in its own right. Now, we can consider x as an element in I , which means that for all $\epsilon > 0$, we can find u_λ such that $\|x - x \cdot u_\lambda\| < \epsilon$. Considering x as an element in both $I \cap J$ and J , we see that we can approximate elements in $I \cap J$ arbitrarily well by elements in $I \cdot J$ the exact same way. $\|x - x \cdot u_\lambda\| < \epsilon$, which takes an element x in $I \cap J$ and approximates it by some element $x \cdot u_\lambda$ in $I \cdot J$.

With that out of the way, we can define what we mean by a unitization of a C^* -algebra in a more formal way.

Definition 4.3.4. A *unitization* of A is a unital C^* -algebra B with an injective homomorphism $i : A \hookrightarrow B$ such that $i(A)$ is an essential ideal of B .

With new definitions, we should always ask pedantic questions to check our sanity. For example, does this generalize the regular way of adjoining a unit? What happens for C^* -algebras that already have a unit?

Remark 4.3.5. If A is already a unital C^* -algebra, with unit 1_A , then we can't find any other unitization of A than A itself. If A is embedded as an ideal in a C^* -algebra B , then it should be an essential ideal. Choosing a $b \in B \setminus A$, we should have that $b1_A \in A$ by absorption. Now $b - b1_A \neq 0$, but $(b - b1_A)A = 0$, which contradicts A being essential by lemma 4.3.2.

Example 4.3.6. Our normal way of adjoining a unit (proposition 2.3.9) is indeed a unitization. We consider the embedding $i : A \hookrightarrow \tilde{A} = A \oplus \mathbb{C}$ given the natural way. $i(A)$ is indeed an ideal in \tilde{A} . To check that it is essential, apply lemma 4.3.2 to the following setup. Set $(a, \lambda)i(A) = 0$, which means that $(a, \lambda)(b, 0) = 0$ for all $b \in A$. We want to show $(a, \lambda) = (0, 0)$. Writing out the multiplication and assuming for a contradiction that $\lambda \neq 0$, we get $-ab = \lambda b$, which gives $\frac{-ab}{\lambda} = b$ for all b . Then we also have $c((-1/\lambda)a)^* = c$ for all c by conjugation, which then implies $(-1/\lambda)a = ((-1/\lambda)a)^*$, and therefore $((-1/\lambda)a)$ is an identity on A . The pair (a, λ) was arbitrary, which is a contradiction. Therefore, $\lambda = 0$, but this implies that $(a, 0)(a^*, 0) = 0$. Now both $a = 0$ and $\lambda = 0$, which means $i(A)$ is an essential ideal. Therefore the pair (\tilde{A}, i) is a unitization.

The following example explains some of the relevance to Hilbert modules.

Example 4.3.7. Consider $L : A \rightarrow \mathcal{L}(A_A)$. This is the embedding of A sending $a \mapsto L_a$, which is the operator acting by left multiplication of a . It would be natural to ask if such an embedding yields a unitization of A . Since

$$\Theta_{a,b}(c) = a\langle b, c \rangle_A = ab^*c = L_{ab^*}(c),$$

we can show that the image of L is $\mathcal{K}(A_A)$. Recalling lemma 4.3.2, we assume $T \in \mathcal{L}(A_A)$ is such that $TK = 0$ for all $K \in \mathcal{K}(A_A)$. Then $T\Theta_{b,c} = \Theta_{Tb,c} = 0$ for all $b, c \in A$. Then $Tb = 0$ for all b , but this means precisely $T = 0$. Therefore $\mathcal{K}(A_A)$ is an essential ideal, so $(\mathcal{K}(A_A), L)$ is a unitization.

It turns out that this particular unitization plays a special rôle, since it ends up being our main definition of a multiplier algebra.

Before we define multiplier algebras, we take a short side-step to the topological compactification mentioned at the start of the section.

Example 4.3.8. We say a compact Hausdorff space Y is a *compactification* of a space X if there exists an injection $i : X \hookrightarrow Y$ such that i is a homeomorphism onto a dense subset of Y . One can show compactifications exist for all topological spaces. Connecting this to C^* -algebras, we know commutative C^* -algebras are spaces of continuous functions, and compactifying the space X makes even the identity vanish at infinity, which means we have adjoined a unit to our C^* -algebra. We can then find a unitization of $A = C_0(X)$ by the induced (and extended) injection $i_* : C_0(X) \rightarrow C(Y)$ which is given by

$$i_*(f)(y) = \begin{cases} 0 & \text{if } y \notin i(X), \\ f(x) & \text{if } y = i(x) \text{ for } x \in X. \end{cases}$$

In topology, we define some sort of maximal compactification βX . This is called the Stone-Ćech compactification of X , which is maximal in the sense that all continuous functions from X to a compact Hausdorff space factor uniquely through βX .

This motivation leads us to a clear problem. Can we find a maximal unitization? What do we even mean by a maximal unitization? Note that due to the contravariant nature of $C(-, \mathbb{C})$ in example 4.3.8, we flip the arrows in our definition.

Definition 4.3.9. If $i : A \hookrightarrow B$ is a unitization of A , we say it is *maximal* if for all other embeddings j_C of A as an essential ideal in a C^* -algebra C , we can find a homomorphism $\phi : C \rightarrow B$ such that

$$\begin{array}{ccc} & & B \\ & \nearrow i & \uparrow \phi \\ A & & \\ & \searrow j_C & \downarrow \\ & & C \end{array}$$

commutes.

Given this diagrammatic definition of *maximal unitizations*, we can compare the structure to other structures with similar properties, at least on a more philosophical level. Often these structures are unique up to isomorphism. If such a unique maximal unitization of A exists, it will be our definition for the multiplier algebra, $M(A)$, of A .

As mentioned earlier, the unitization $L : A \hookrightarrow \mathcal{L}(A_A)$ is important in constructing these algebras.

Theorem 4.3.10. The unitization $L : A \hookrightarrow \mathcal{L}(A_A)$ is maximal. It is unique in the sense that if $i : A \hookrightarrow B$ is another maximal unitization, the induced homomorphism $\phi : B \rightarrow \mathcal{L}(A_A)$ is an isomorphism.

From a purely algebraic point of view, this would not seem to be so hard to prove, but as we work in the world of analysis, we have a lot of structure to respect. Therefore we omit the proof, as we need to develop this theory a bit deeper to give a readable proof. Interested readers are referred to [31] and [13] for more depth and for the last proofs in this section.

Nevertheless, this leads us to a natural definition and interesting examples.

Definition 4.3.11. We define the *multiplier algebra* of A to be $\mathcal{L}(A_A)$, which we denote by $M(A)$.

Remark 4.3.12. Historically speaking, one defined $M(A)$ in terms of double centralizers, which are pairs of operators on A meant to resemble the centralizer of a ring. These pairs were called multipliers, which justifies the name. Theorem 4.3.10 makes the approach to multiplier algebras through $\mathcal{L}(A_A)$ a valid approach and hence motivating our definition.

We end the section with some examples of multiplier algebras to show that this seems to be a reasonable construction.

Proposition 4.3.13. Let X be a Hilbert A -module, let \mathcal{H} be a Hilbert space, and let T be a locally compact Hausdorff space. Then we have:

1. $\mathcal{L}(X) \cong M(\mathcal{K}(X))$
2. $B(\mathcal{H}) \cong M(\mathcal{K}(\mathcal{H}))$
3. $C_b(T) \cong M(C_0(T)) \cong C(\beta T)$

Here βT is the Stone-Ćech compactification from example 4.3.8 and $C_b(T)$ denotes the bounded, continuous functions on T .

With this, we end our brief discussion on multiplier algebras and move on towards the core of chapter 4, namely induced representations and Morita equivalence for C^* -algebras.

4.4 Induced representations

Aiming to define some sort of Morita equivalence for C^* -algebras, we should recall from ring theory that two rings are Morita equivalent if they give equivalent module categories and therefore equivalent categories of representations. If we can find a way to pass representations between C^* -algebras A and B , perhaps we can make sense of A and B being Morita equivalent.

To do this, we want to use our theory of Hilbert modules. Suppose A and B are C^* -algebras and let X be a (right) Hilbert B -module. Suppose we have a homomorphism $A \rightarrow \mathcal{L}_B(X)$, i.e. A acts as adjointable operators on X_B . Denote the action by $a \cdot x$ for $a \in A, x \in X$. We will show that we can use the bimodule ${}_A X_B$ to pass representations between A and B , but to construct representations, we will need to work with tensor products.

We state a lemma on tensor products of Hilbert spaces and refer to Chapter 2.4 of [31] for a proof, as it is more difficult to prove than imagined.

Lemma 4.4.1. If V, W are Hilbert spaces, we can define an inner product on the tensor product of vector spaces, $V \otimes W$, such that

$$(v_1 \otimes w_1, v_2 \otimes w_2) = (v_1, v_2)(w_1, w_2).$$

The completion of the vector space tensor product in the norm induced by this inner product is what we call the Hilbert space inner product.

Remark 4.4.2. There may be room for confusion below, so let us get things straight before we do more work on tensor products. We are working with one algebraic tensor product, which we from now on will denote by \odot , and complete tensor products, which we denote the usual way by \otimes . Elementary tensors will always be denoted with \otimes , but it should be clear from the context where they live. Tensor products for C^* -algebras are not the kind of tensor product one may know from more algebraic constructions, such as tensor products of abelian groups or modules. As mentioned (2.5.1), tensor products in these settings are certainly unique up to isomorphism. The problem is that when we work with C^* -algebras and Hilbert modules, we have a lot of structure to respect. This leads to us losing the regular universal property of $V \otimes W$ being the canonical "bilinear structure" in the sense that bounded bilinear operators on $V \times W$ give bounded linear operators on $V \otimes W$. This does not always happen. On the other hand, one can phrase some sort of universal property if we restrict to so-called weak Hilbert-Schmidt operators. We refer to [31] and [21] for more on the topic.

Our goal is to take a nondegenerate representation π of B on some Hilbert space \mathcal{H}_π and have A act as adjointable operators on a Hilbert B -module X , because then we can use this structure to move representations from B to A . Roughly speaking, we want to have \mathcal{H}_π to be the Hilbert space on which we represent B with π , then

take our bimodule X and define a representation on some completion of $X \otimes H_\pi$ to connect the A -related structure of X and the representation $\pi : B \rightarrow B(\mathcal{H}_\pi)$.

To make this construction, we first need an appropriate inner product to complete $X \otimes H_\pi$ in that connects the representation to the Hilbert space inner product (4.4.1).

Proposition 4.4.3. Let ${}_A X_B$ be a Hilbert B -module with an action of A as adjointable operators and let $\pi : B \rightarrow B(\mathcal{H}_\pi)$ be a nondegenerate representation. Then there exists a unique, positive, semi-definite inner product on the algebraic vector space tensor product, $X \otimes \mathcal{H}_\pi$, such that

$$(x \otimes h, y \otimes k) := (\pi(\langle y, x \rangle_B)h, k).$$

In the proof of proposition 4.4.3, we will need to check the positivity of the sesquilinear form we construct. At that point, we will naturally need to consider a family $(\langle x_i, x_j \rangle_B)$, which we can view as a $n \times n$ -matrix. Before we start the proof, we proactively state a lemma on the positivity of such elements and refer readers to [31] for a proof.

Lemma 4.4.4. Let X be a (right) Hilbert B -module and let $x_1, \dots, x_n \in X$. The matrix given by $(\langle x_i, x_j \rangle_B)_{i,j}$ is a positive element of $M_n(B)$.

With this lemma under our belt, we set out to prove proposition 4.4.3.

Proof. (*Proof of proposition 4.4.3*) To start off, we define a pairing satisfying the wanted relation. If we fix y, k in $(\pi(\langle y, x \rangle_B)h, k)$, we get a bilinear map in (x, h) , which induces a linear map $f_y \otimes f_k : X \otimes \mathcal{H}_\pi \rightarrow \mathbb{C}$. We want to define a sesquilinear form on $(X \otimes \mathcal{H}_\pi)$, but for now we can define a bilinear form $(y, k) \mapsto \overline{f_y \otimes f_k}$ from $X \times \mathcal{H}_\pi$ to the dual space $(X \otimes \mathcal{H}_\pi)^\sim$. This certainly induces a linear map L on $X \otimes \mathcal{H}_\pi$, which we can use to define the inner product. Define now $(a, b) := \overline{L_b(a)}$, which is a sesquilinear form on $X \otimes \mathcal{H}_\pi$. To check the positivity, we want to prove that

$$\left(\sum x_i \otimes h_i, \sum x_i \otimes h_i \right) = \sum_{i,j} (\pi(\langle x_i, x_j \rangle_B)h_j, h_i) \geq 0.$$

This is where our matrix positivity lemma comes in. By passing π to the matrix algebra representation π_n , we can consider this sum more easily and apply lemma 4.4.4. If we let $M = (\langle x_i, x_j \rangle_B)$, we can rewrite this inner product to

$$\sum_i \left(\sum_j \pi(\langle x_i, x_j \rangle_B)h_j \mid h_i \right) = \sum_i ((\pi_n(M)(h_k))_i \mid h_i) = (\pi_n(M)(h_k) \mid (h_k)).$$

Since M is positive, $\pi_n(M)$ is positive, so this last term is nonnegative. \square

Remark 4.4.5. We should note that this is only a semi-definite inner product, as elements of the form $(x \cdot b) \otimes h - x \otimes \pi(b)h$ are sent to zero by the inner product with any other element $y \otimes k$, but these are not necessarily zero. This is not a problem, as we mod out vectors of length zero when we complete the space. In fact, this yields some sort of B -balancedness in the completion, in the sense that $(x \cdot b) \otimes h = x \otimes \pi(b)h$. To emphasize this, we give the tensor product a subscript B .

Now, finally, we explain how to induce representations between C^* -algebras.

Proposition 4.4.6. Let X be a Hilbert B -module and assume that A acts as adjointable operators. If π is a nondegenerate representation of B on a Hilbert space \mathcal{H}_π , then we get an induced representation $\text{Ind } \pi$ of A on the Hilbert space $X \otimes_B \mathcal{H}_\pi$, which we obtain by completing the algebraic tensor product in the inner product from proposition 4.4.3. The induced representation is given by

$$\text{Ind } \pi(a)(x \otimes_B h) := (a \cdot x) \otimes_B h,$$

and if X is nondegenerate as an A -module in the sense that ${}_A \langle X, X \rangle \cdot X$ is dense in ${}_A X$, then this induced representation is nondegenerate as well. If we feel the need to emphasize the Hilbert module X or the C^* -algebras A and B , we write $X - \text{Ind}_B^A \pi$, $\text{Ind}_B^A \pi$ or $X - \text{Ind } \pi$.

Proof. The proof is a lengthy and technical endeavour giving little insight, and we omit the proof and refer interested readers to [31] for a proof, as our focus lies on giving an overview of the subject. \square

Before we move on to the next section where we define Morita equivalences through bimodules, we consider some basic examples of induced representations.

Example 4.4.7. Let $\pi_A : A \rightarrow B(\mathcal{H})$ be a representation. We know the nondegenerate representations of \mathbb{C} on a Hilbert space \mathcal{H}_π are $\pi_{\mathbb{C}}(z) = z1$, with 1 being the identity operator on \mathcal{H}_π . Consider ${}_A \mathcal{H}$ as a right Hilbert \mathbb{C} -module and consider the inner product, which gives

$$(x \otimes h \mid y \otimes k) := (\pi(\langle y, x \rangle_{\mathbb{C}})h \mid k) = (\langle y, x \rangle_{\mathbb{C}} h \mid k) = (x \mid y)(h \mid k).$$

Thus the tensor product on which we get the induced representation on A actually turns out to be the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{H}_\pi$. The induced representation on A coming from $\pi_{\mathbb{C}}$, is the tensor product representation $\text{Ind } \pi_{\mathbb{C}}$ is given by $\pi_A \otimes 1$ such that $a \mapsto \pi_A(a) \otimes 1$.

Example 4.4.8. Consider B_B as a left Hilbert module over its multiplier algebra $M(B) = \mathcal{L}(B_B)$ and let π be a nondegenerate representation of B . We know the formula and the space that yields an explicit description of the induced representation, but it would be interesting to build a representation on the same Hilbert space. This is what we aim to construct. If we consider the linear extension ϕ of

the map $b \otimes h \mapsto \pi(b)h$, we see the following calculation shows that this map is an isometry $B \otimes \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi$ where we use the Hilbert space inner product on the tensor product:

$$(\phi(b \otimes h) | \phi(c \otimes k)) = (\pi(b)h | \pi(c)k) = (\pi(c^*b)h | k) \quad (4.5)$$

$$= (\pi(\langle c, b \rangle_B)h | k) = (b \otimes h | c \otimes k) \quad (4.6)$$

From the definition of nondegeneracy, we get that the range of ϕ is dense. If we complete the tensor product, we get an induced, surjective unitary transformation $U : B \otimes_B \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi$. Fix an element $m \in M(B)$ and consider now the operator given by

$$U \text{Ind } \pi(m) U^* (\pi(b)h) = (U \text{Ind } \pi(m) U^*) U (b \otimes_B h) \quad (4.7)$$

$$= U \text{Ind } \pi(m) (b \otimes_B h) \quad (4.8)$$

$$= U (mb \otimes_B h) = \pi(mb)h. \quad (4.9)$$

This certainly extends to a representation of $M(B)$ on the same Hilbert space \mathcal{H}_π .

Note that unitary operators are a sort of intertwining operators. We state a proposition (without proof) from [31] showing the functoriality of $X - \text{Ind}_B^A$ from the category of nondegenerate representation of B and bounded intertwining operators to the similar category for A .

Proposition 4.4.9. Assume that A acts as adjointable operators on a Hilbert B -module X , π_i are nondegenerate representations of B on Hilbert spaces \mathcal{H}_i and $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded intertwining operator $T(\pi_1(b)h) = \pi_2(b)(Th)$. Then the transformation $1 \otimes T : X \otimes \mathcal{H}_1 \rightarrow X \otimes \mathcal{H}_2$ extends to a bounded operator $1 \otimes_B T : X \otimes_B \mathcal{H}_1 \rightarrow X \otimes_B \mathcal{H}_2$ intertwining the induced representations. Furthermore, the map $T \mapsto 1 \otimes_B T$ is $*$ -linear and if $S : \mathcal{H}_2 \rightarrow \mathcal{H}_3$ intertwines π_2 and π_3 , then we have that $1 \otimes_B (S \circ T) = (1 \otimes_B S) \circ (1 \otimes_B T)$. In particular, $X - \text{Ind}$ preserves unitary equivalence and direct sums.

With some more work on this theory, one can also show that induced representations play nice with ideals and faithfulness.

Proposition 4.4.10. Let X be an Hilbert $A - B$ -bimodule. Then

1. if π, ρ are representations of B with the same kernels, then $\ker(X - \text{Ind}_B^A \pi) = \ker(X - \text{Ind}_B^A \rho)$.
2. if π is a faithful representation of B and the action of A on X is faithful, the induced representation is a faithful representation of A .

4.5 Equivalence bimodules and Morita equivalence

As mentioned in the introduction of the chapter, the natural way to express Morita equivalences should be through bimodules in the operator algebraic context. Now,

after a bit of work, we have developed a framework in which we can construct bimodules between C^* -algebras that let us pass representations back and forth. There are yet some technicalities and unanswered questions connected to these induced representations, such as whether or not inducing a representation back and forth will yield the same representation, but for now, we postpone these technicalities to the next section. We first do some rewarding work on imprimitivity bimodules and Morita equivalence, which will be essential for the later applications to noncommutative tori.

Definition 4.5.1. An A – B -imprimitivity bimodule is an A – B -bimodule X such that the following hold:

1. X is a full left Hilbert A -module and a full right Hilbert B -module.
2. The actions from A and B work as adjointable operators in the sense that for all $x, y \in X, a \in A$ and $b \in B$

$$\langle a \cdot x, y \rangle_B = \langle x, a^* \cdot y \rangle_B \quad \text{and} \quad {}_A \langle x \cdot b, y \rangle = {}_A \langle x, y \cdot b^* \rangle$$

3. For all $x, y, z \in X$,

$${}_A \langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_B$$

Remark 4.5.2. This seems to be a reasonable definition, as we want the Hilbert bimodule structure and adjointable operators axioms to be able to pass representations between the C^* -algebras, as well as letting the inner products play nicely together. There is some redundancy in the definition, as the second axiom actually implies $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ and $(\lambda a) \cdot (x \cdot b) = a \cdot (x \cdot (\lambda b))$, that is, we do not need to assume X is a bimodule to start with. This follows by showing each side of the equality sign give the same inner product with all other elements in X .

The word imprimitivity bimodule comes from Rieffel’s original work on this topic, where this theory was connected to the imprimitivity theorem of Mackey. A perhaps more intuitive word for these modules would be ”equivalence bimodules”, as they certainly play the rôle as the manifestation of our Morita equivalence.

Let us consider some examples we already know to see this in play.

Example 4.5.3. We know Hilbert spaces \mathcal{H} are right Hilbert \mathbb{C} -modules and left Hilbert $\mathcal{K}(\mathcal{H})$ -modules, where the left inner product is given by ${}_{\mathcal{K}(\mathcal{H})} \langle h, k \rangle := h \otimes \bar{k}$. By earlier examples, we know the axioms hold and we see that the last axiom is certainly just the definition of the rank-one operator coming from the inner product. Thus \mathcal{H} is a $\mathcal{K}(\mathcal{H})$ – \mathbb{C} -imprimitivity bimodule.

Example 4.5.4. A is certainly an A – A -imprimitivity bimodule with the inner products defined earlier, by straightforward computations of the axioms.

Before we study more interesting examples and give a go at Morita equivalence, we can actually rephrase the definition with an equivalent version of the second axiom and define it for pre-inner product modules. The following lemma from [31] will justify the definition in the noncomplete case.

Lemma 4.5.5. Let X be an A – B -bimodule such that the first and last axiom in definition 4.5.1 holds. Then X is an A – B -imprimitivity bimodule (i.e. the second axiom holds) if and only if for all $a \in A, b \in B$ and $x \in X$,

$$\langle a \cdot x, a \cdot x \rangle_B \leq \|a\|^2 \langle x, x \rangle_B \quad \text{and} \quad \langle x \cdot b, x \cdot b \rangle \leq \|b\|^2 \langle x, x \rangle$$

holds.

Proof. If the regular axiom holds, that is, if A acts as adjointable operators, then we already have the result by proposition 4.2.8. Conversely, we can compute

$$\begin{aligned} \langle_A \langle x, y \rangle \cdot z, w \rangle_B &= \langle x \cdot \langle y, z \rangle_B, w \rangle_B = \langle z, y \rangle_B \langle x, w \rangle_B \\ &= \langle z, y \cdot \langle x, w \rangle_B \rangle_B = \langle z, {}_A \langle x, y \rangle^* \cdot w \rangle_B \end{aligned} \quad (4.10)$$

by using the first and last axiom. By fullness, the ideal ${}_A \langle X, X \rangle$ is dense in A , which means that $\langle a \cdot z, w \rangle_B = \langle z, a^* \cdot w \rangle_B$ holds for all a in this ideal and all $z, w \in X$. The new middle axiom implies boundedness, which means we can extend this to all of A by continuity. The new axiom implies that multiplication with A is an operator that has norm at most $\|a\|$, and therefore we get $\|\langle a \cdot z, w \rangle_B\| \leq \|a\| \|z\|_B \|w\|_B$ by the Cauchy-Schwarz inequality and the C^* -equality. A similar argument for B will yield the result. \square

Now we can define imprimitivity bimodules over pre- C^* -algebras, to extend our definition to more applicable cases. We want to mimic lemma 4.1.15.

Definition 4.5.6. Let $A_0 \subset A$ and $B_0 \subset B$ be dense $*$ -subalgebras. An A_0 – B_0 -pre-imprimitivity bimodule is a vector space X_0 and an A_0 – B_0 -bimodule such that the following axioms hold:

1. X_0 is a left pre-inner product A_0 -module and a right pre-inner product B_0 -module.
2. ${}_A \langle X_0, X_0 \rangle$ and $\langle X_0, X_0 \rangle_{B_0}$ span dense ideals of A and B .
3. For all $a \in A_0, b \in B_0$ and $x \in X_0$, the following inequalities hold in the completions A and B :

$$\langle a \cdot x, a \cdot x \rangle_{B_0} \leq \|a\|^2 \langle x, x \rangle_{B_0} \quad \text{and} \quad {}_{A_0} \langle x \cdot b, x \cdot b \rangle \leq \|b\|^2 \langle x, x \rangle$$

4. For all $x, y, z \in X_0$,

$${}_{A_0} \langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_{B_0} \quad (4.11)$$

Wanting to complete the pre-imprimitivity bimodule, we need the two seminorms given by these inner products to agree.

Proposition 4.5.7. If X_0 is an A_0 – B_0 -pre-imprimitivity bimodule the same way as before, then for all $x \in X_0$,

$$\|x\|_A^2 = \|{}_A \langle x, x \rangle\| = \|\langle x, x \rangle_{B_0}\| = \|x\|_B^2.$$

Proof. For any fixed $x \in X$, we use the Cauchy-Schwarz inequality and the inequality from the pre-imprimitivity bimodule definition to deduce

$$\begin{aligned} \|\langle x, x \rangle_{B_0}\|^2 &= \|\langle x, x \rangle_{B_0} \langle x, x \rangle_{B_0}\| = \|\langle x, x \cdot \langle x, x \rangle_{B_0} \rangle_{B_0}\| \\ &= \|\langle x_{A_0} \langle x, x \rangle \cdot x \rangle_{B_0}\| \\ &\leq \|\langle x, x \rangle_{B_0}\|^{1/2} \|\langle x_{A_0} \langle x, x \rangle \cdot x_{A_0} \langle x, x \rangle \cdot x \rangle_{B_0}\|^{1/2} \\ &\leq \|\langle x, x \rangle_{B_0}\|^{1/2} \|x_{A_0} \langle x, x \rangle\| \|\langle x, x \rangle_{B_0}\|^{1/2}. \end{aligned}$$

Thus, this means $\|\langle x, x \rangle_{B_0}\| \leq \|x_{A_0} \langle x, x \rangle\|$, and as this situation is symmetric, we have the opposite inequality as well. \square

As the norms agree, we can use lemma 4.1.15 and lemma 4.5.5 to get the following proposition:

Proposition 4.5.8. (Completions of pre-imprimitivity bimodules) Given the setup as above, let X_0 be an A_0 – B_0 -pre-imprimitivity bimodule. Then there exists an A – B -imprimitivity bimodule X and an A_0 – B_0 -bimodule homomorphism $q : X_0 \rightarrow X$ such that the image of q is dense in X and for all $x, y \in X_0, a \in A_0, b \in B_0$,

$$\langle q(x), q(y) \rangle_B = \langle x, y \rangle_{B_0}, \quad {}_A \langle q(x), q(y) \rangle = {}_{A_0} \langle x, y \rangle$$

Enough with the technicalities for now. Let us see what we can say about A and B when we have an A – B -imprimitivity bimodule X between them. One (perhaps) unexpected result is the following:

Proposition 4.5.9. All full (right) Hilbert B -modules X are $\mathcal{K}(X)$ – B -imprimitivity bimodules. Conversely, if X is an A – B -imprimitivity bimodule, then there exists an isomorphism $\phi : A \rightarrow \mathcal{K}(X)$ preserving the inner product.

Proof. Suppose first we have a full (right) Hilbert B -module X . Then we know from earlier examples (4.1.11) that X is a full left Hilbert $\mathcal{K}(X)$ -module. We know that $\mathcal{K}(X)$ acts as adjointable operators, but for the second identity of the second axiom, we consider the following for $b \in B, x, y, z \in X$:

$$\kappa_{(X)} \langle x \cdot b, y \rangle(z) = (x \cdot b) \cdot \langle y, z \rangle_B = x \cdot \langle y \cdot b^*, z \rangle_B = \kappa_{(X)} \langle x, y \cdot b^* \rangle(z).$$

As mentioned earlier, the last axiom is just the definition of the rank-one operator coming from the $\mathcal{K}(X)$ -valued inner product.

For the converse, assume X is an A – B -imprimitivity bimodule. We know A acts as adjointable operators, meaning the regular map $\phi : A \rightarrow \mathcal{L}(X), \phi(a)(x) := a \cdot x$ is a homomorphism of C^* -algebras, which means that it has closed range. The last axiom, ${}_A \langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_B$, gives the definition of the stereotypical rank-one operator when put through ϕ . More precisely, we get

$$\phi(\langle x, y \rangle) = \kappa_{(X)} \langle x, y \rangle$$

for all $x, y \in X$, which means that the range of ϕ must be $\mathcal{K}(X)$. To show injectivity, we pick $a \in \ker \phi$. The fullness of X as an Hilbert A -module lets us approximate a by some element of the form $a \sum_i \langle x_i, y_i \rangle$. This element is certainly in the kernel of ϕ , as elements in the kernel of ϕ act as the zero operator, which means

$$a \sum_i \langle x_i, y_i \rangle = \sum_i \langle a \cdot x_i, y_i \rangle = \sum_i \langle 0, y_i \rangle = 0.$$

Therefore ϕ is an isomorphism. \square

Queequeg 4.5.10. Readers with a deeper background from algebraic representation theory may (and only may) have expected such a result due to similarities to the original Morita theorem. The Morita theorem ([24]) says that two rings S, R are Morita equivalent if and only if one of them can be realized as the endomorphism ring of a finitely generated projective generator in the module category of the other ring, where a generator in a category \mathcal{C} is an object G for which $\text{Hom}_{\mathcal{C}}(G, H) \neq 0$ for all nonzero objects H . By waving hands and crossing our fingers for deep connections, we can think about $\mathcal{K}(X)$ as the analogue to the endomorphism ring, as it takes care of a lot of the structure theory in our context.

Whether or not there is a solid connection to some Morita theorem for C^* -algebras, is unknown to the author at the current point, but it may be motivating to remark the similarities between the ring theoretic theory and the C^* -algebraic theory, even if such a connection does not exist.

Nevertheless, after all the technical work on Hilbert modules and induced representations, we are finally ready to define Morita equivalences.

Definition 4.5.11. (Morita equivalence) Let A and B be two C^* -algebras. We say A and B are *Morita equivalent* if there exists an A – B -imprimitivity bimodule X .

As mentioned, X is the manifestation of the Morita equivalence, and therefore we say X implements the Morita equivalence.

Remark 4.5.12. Morita equivalence is certainly weaker than isomorphism. Assume $\phi : A \rightarrow B$ is an isomorphism. Then we can choose $X = B$ and give it the following structure.

$$x \cdot b := xb, \quad a \cdot x := \phi(a)x, \quad \langle x, y \rangle_B := x^*y, \quad \text{and } {}_A \langle x, y \rangle = \phi^{-1}(xy^*)$$

The word "equivalence" can be justified, as one can show the following with a bit of technical work on tensor products of imprimitivity bimodules.

Theorem 4.5.13. Morita equivalence is an equivalence relation on C^* -algebras

Proof. The proof is omitted here, but readers interested in tensor product technicalities should certainly take a look at [31] for details. \square

With some further technical work, one can show an extremely useful characterization of Morita equivalent C^* -algebras in terms of complementary full corners, but first, we need to define and understand these full corners.

Example 4.5.14. Let $p \in A$ be a projection. We can realize pA as a full right Hilbert pAp -module with inner product $\langle ap, bp \rangle_{pAp} = pa^*bp$. This is the structure inherited when considering it as a subset of A . Fullness follows as products a^*b are dense in A . Similarly, pA is a full left Hilbert pAp -module, which means that we can realize pA as an pAp - pAp -imprimitivity bimodule.

Definition 4.5.15. A C^* -subalgebra B of a A is called a *corner* of A if there exists a projection $p \in A$ such that $B = pAp$. A corner is called *full* if $\text{span}ApA$ is dense in A , or equivalently, the corner is not contained in any proper, closed, two-sided ideal. A corner qAq is called *complementary* to pAp if $q = 1_A - p$.

Remark 4.5.16. Full corners of A are Morita equivalent to A via pAp , which means that any complementary full corners are Morita equivalent with qAq as the $qAq - pAp$ -imprimitivity bimodule by explicit calculation and transitivity of Morita equivalences.

Amazingly enough, one can show all Morita equivalences arise in a similar manner.

Theorem 4.5.17. Two C^* -algebras A and B are Morita equivalent if and only if there exists a third C^* -algebra C with complementary full corners isomorphic to A and B .

Proof. The proof is omitted, but readers wanting to learn a new bunch of magic tricks should take a look at [31] and the linking algebra theory behind the proof. \square

In the light of this theorem, we see that it would be interesting, from a purely algebraic point of view, for now, to be able to classify all projections. This will be extremely relevant when we return to K -theory later in the thesis with more geometric motivation, but this result will also be relevant when we want to classify noncommutative tori in the last chapter.

4.6 The Rieffel correspondence

In this last section, we will show that Morita equivalent C^* -algebras have the same ideal structure and representation theory, that is, the Ind -functors going each way are actually inverses up to unitary equivalence.

This section will not be as thorough as it deserves to be, since there is a lot of depth and beauty in this theory, but our motivation for working through this is for justification and motivation on why studying Morita equivalence classes should be interesting.

First, note that the set of closed, two-sided ideals of A , $\mathcal{I}(A)$, can be partially ordered by inclusion with a lattice structure, that is, for each pair of ideals $I, J \in \mathcal{I}(A)$, we can find a greatest lower bound and a least upper bound with respect to this ordering.

Morita equivalent C^* -algebras have isomorphic lattices of ideals in the sense that we can find an order-preserving bijection between the lattices. In fact, imprimitivity bimodules give us isomorphisms between the ideal lattices of A and B and the lattice of closed A – B -submodules of X by preimages and images under the inner products.

Theorem 4.6.1. Let X be an A – B -imprimitivity bimodule. Then there are lattice isomorphisms among $\mathcal{I}(A)$, $\mathcal{I}(B)$ and the lattice of closed A – B -submodules of X given by the following correspondences:

1. If $K \in \mathcal{I}(A)$ and $J \in \mathcal{I}(B)$, then the corresponding A – B -bimodules are given by

$$\begin{aligned} {}_K X &= \{y \in X : \langle y, x \rangle_A \in K \text{ for all } x \in X\} \\ X_J &= \{y \in X : \langle x, y \rangle_B \in J \text{ for all } x \in X\} \end{aligned}$$

2. If Y is a closed A – B -submodule, then the corresponding ideals are given by

$$\begin{aligned} {}_Y I &= \overline{\text{span}}\{\langle x, y \rangle_B : x \in X, y \in Y\} \in \mathcal{I}(B) \\ I_Y &= \overline{\text{span}}\{\langle y, x \rangle_A : x \in X, y \in Y\} \in \mathcal{I}(A) \end{aligned}$$

Those who are notationally challenged may, with every right, be confused by the subscripts and large letters, but if we interpret the subscript to tell us where the structure comes from and the large letters as which "side" of the bimodule it lives, it should be quite clear what this means. For example, I_Y is the ideal that lives in the left structure (in $\mathcal{I}(A)$) and comes from the closed A – B -submodule Y .

To prove this, we need a more explicit description of the induced bimodules.

Lemma 4.6.2. Assume X is an A – B -bimodule and J is an ideal of B . Then the induced submodule, $X_J = \{y \in X : \langle x, y \rangle_B \in J \text{ for all } x \in X\}$, is actually a closed A – B -submodule and can be realized as

$$\overline{X \cdot J} = X_J = \{y \in X : \langle y, y \rangle_B \in J\}.$$

Proof. We already know $\overline{X \cdot J} \subset X_J \subset \{y \in X : \langle y, y \rangle_B \in J\}$, since continuity of the inner product yields that X_J is closed in X and J is an ideal. Now, take any $y \in Y$ such that $\langle y, y \rangle_B \in J$ and take u_λ to be an approximate identity for J . Then for all $\epsilon > 0$, we can find an u_λ such that

$$\|y - y \cdot u_\lambda\|_B^2 = \|\langle y, y \rangle_B - \langle y, y \rangle_B u_\lambda - u_\lambda \langle y, y \rangle_B + u_\lambda \langle y, y \rangle_B u_\lambda\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore $y \in \overline{X \cdot J}$. □

Proof. (*Proof of theorem 4.6.1*) To see that we actually get isomorphisms, we want to show that the maps $J \mapsto X_J$ and $Y \mapsto Y_I$ are inverses of each other. Since they preserve containment, they preserve partial orders. If we can show they are inverses, these will yield lattice isomorphisms. We show this for $J \in \mathcal{I}(B)$, as the result for A is similar. If we actually untangle the definition, we easily see that $X_J I \subset J$. To show the reverse inclusion, we now know $X_J = \overline{X \cdot J}$, which means that we can write $J \langle X, X \rangle_B J = \langle X \cdot J, X \cdot J \rangle_B \subset X_J I$. By fullness, $\langle X, X \rangle_B$ is dense in B , which means that the left-hand side spans a dense ideal of J . Continuity yields that this is indeed closed, so we get $X_J I = J$. To show the converse, consider Y to be a closed $A - B$ -submodule of X . Then, as earlier, we see that $Y \subset X_{Y_I}$ by unraveling the definition. For the reverse inclusion, note that $X_{Y_I} = \overline{X \cdot Y_I}$ by lemma 4.6.2, which means that X_{Y_I} can be spanned by elements $x \cdot \langle y, z \rangle_B$ for $x, y \in X, z \in Y$. This is actually in Y since X is an $A - B$ -imprimitivity bimodule and $z \in Y$, by $x \cdot \langle y, z \rangle_B = {}_A \langle x, y \rangle \cdot z$. Therefore $X \cdot Y_I \subset Y$, and as Y is closed, we get equality. \square

This lattice isomorphism $\mathcal{I}(B) \rightarrow \mathcal{I}(A)$ is given a special name, namely the *Rieffel correspondence*. By the last result of the next proposition, we use the same notation as for the induced representations. We state some results on how the Rieffel correspondence works before we move on to the last technical part of the chapter.

Proposition 4.6.3. (The Rieffel Correspondence) Let X be an $A - B$ -imprimitivity bimodule. Then $X - \text{Ind}_B^A : \mathcal{I}(B) \rightarrow \mathcal{I}(A)$ is given by

$$X - \text{Ind}_B^A J = \overline{\text{span}} \{ {}_A \langle x \cdot b, y \rangle : x, y \in X, b \in J \}$$

and if $K = X - \text{Ind}_B^A J$, then $\overline{K \cdot X} = \overline{X \cdot J}$ is the corresponding closed submodule.

If we let π be a representation of B , then

$$X - \text{Ind}_B^A(\ker \pi) = \ker(X - \text{Ind}_B^A \pi).$$

Remark 4.6.4. To see the relevance of the last part of the proposition, note that we can realize any ideal $I \subset A$ as a kernel of a representation π . To do this, choose the representation given by the composition $A \rightarrow A/I \rightarrow B(\mathcal{H})$, where the first map is the quotient and the second is a faithful representation given by the GNS-construction (2.4.4).

Proof. Recall that the definition and lemma 4.6.2 actually yields $X - \text{Ind}_B^A J = I_{X_J} = I_{\overline{X \cdot J}}$. The middle submodule can be expressed by

$$X - \text{Ind}_B^A J = \overline{\text{span}} \{ {}_A \langle x, y \rangle : x \in X_J \text{ and } y \in X \},$$

which is exactly what we want since $X_J = \overline{X \cdot J}$, with limit points working as expected due to the continuity of the inner product.

If $K = X - \text{Ind}_B^A J$, we get $\overline{K \cdot X} = {}_K X = I_{X_J} X = X_J = \overline{X \cdot J}$ by repeated use of lemma 4.6.2 as well as theorem 4.6.1.

For the last part, let $J = \ker \pi$ and note that elements of the form ${}_A\langle x, y \rangle$ for $x \in X, y \in X_J$ spans $\text{Ind} J$, and

$$\begin{aligned} X - \text{Ind}_B^A \pi({}_A\langle x, y \rangle)(z \otimes_B h) &= ({}_A\langle x, y \rangle \cdot z) \otimes_B h \\ &= (x \cdot \langle y, z \rangle_B) \otimes_B h \\ &= x \otimes_B \pi(\langle y, z \rangle_B) h \\ &= 0, \end{aligned}$$

where the last line follows from the fact that $\langle y, z \rangle_B \in J$. Now, take $a \in \ker(\text{Ind}_B^A \pi)$, which means that $a \cdot x \otimes_B h = 0$ for all $x \in X, h \in \mathcal{H}_\pi$. Let $h, k \in \mathcal{H}_\pi$ and $x, y \in X$. Then

$$(a \cdot x \otimes_B h \mid y \otimes k) = (\pi(\langle y, a \cdot x \rangle_B) h \mid k) = 0$$

This implies $\langle y, a \cdot x \rangle_B \in J$ for all $x, y \in X$, which in turn gives $a \cdot x \in X_J$.

To conclude, note that we can approximate a by elements of the form

$$a \left(\sum_A \langle x_i, y_i \rangle \right) = \sum_A \langle a \cdot x_i, y_i \rangle$$

by fullness of X , which in turn means that a must be in $I_{X_{\ker \pi}} = I_{X_J} = \text{Ind}_B^A(\ker \pi)$ \square

An interesting and powerful property of the Rieffel correspondence, which can be found in [31], is the fact that the corresponding ideals and quotients are Morita equivalent as well.

Proposition 4.6.5. Let X is an $A - B$ -imprimitivity bimodule with Rieffel correspondence $X - \text{Ind} : \mathcal{I}(B) \rightarrow \mathcal{I}(A)$ and let $J \in \mathcal{I}(B)$ and $K = X - \text{Ind} J \in \mathcal{I}(A)$. Then X_J is a $K - J$ -imprimitivity bimodule and $X^J := X/X_J$ is an $A/K - B/J$ -imprimitivity bimodule such that the quotient norm on X^J coincides with the norm induced by the B/J -valued inner product.

To get the inversely induced representations, we first need to develop some notion of a dual module.

Example 4.6.6. To demonstrate conjugated vector spaces, take a (right) Hilbert space \mathcal{H} (a right Hilbert \mathbb{C} -module) and define $\overline{\mathcal{H}}$, which is just \mathcal{H} as an abelian group. Let $\flat : \mathcal{H} \rightarrow \overline{\mathcal{H}}$ be the identity map as abelian groups and define the action on $\overline{\mathcal{H}}$ by $\lambda \cdot \flat(h) = \flat(h \cdot \bar{\lambda})$, which yields a left Hilbert \mathbb{C} -module structure on $\overline{\mathcal{H}}$.

In the same manner, if X is an $A - B$ -imprimitivity bimodule, let \tilde{X} be the conjugate vector space over a field K such that there is a K -linear bijection $\flat : X \rightarrow \tilde{X}$.

Then \tilde{X} is a $B - A$ -imprimitivity bimodule with structure as

$$\begin{aligned} \flat \cdot \flat(x) &= \flat(x \cdot b^*) & \flat(x) \cdot a &= \flat(a^* \cdot x) \\ {}_B \langle \flat(x), \flat(y) \rangle &= \langle x, y \rangle_B & \langle \flat(x), \flat(y) \rangle_A &= \langle x, y \rangle \end{aligned}$$

for $x, y \in X, b \in B, a \in A$.

The inverse to $X - \text{Ind}$ will be given by $\tilde{X} - \text{Ind}$, but first, let us show that \tilde{X} is the inverse of X in another interesting sense.

First, recall that isomorphisms of $A - B$ -imprimitivity bimodules are bimodule isomorphism preserving both inner products. To prove a given linear map is an isomorphism, we only need to check that it preserves inner products and has dense range, since preserving inner products implies the map is isometric, which in turn means that it is an injection with closed range. We can also do some tricks with the inner products to show these maps are bilinear if they satisfy these two properties.

Proposition 4.6.7. If X is an $A - B$ -imprimitivity bimodule, we can find an isomorphism $\phi : {}_B(\tilde{X} \otimes_A X)_B \rightarrow {}_B B_B$ such that $\phi(\flat(x) \otimes_A y) = \langle x, y \rangle_B$. In the same way, we have an isomorphism $\psi : {}_A(X \otimes_B \tilde{X})_A$ such that $\psi(x \otimes_B \flat(y)) = {}_A \langle x, y \rangle$.

Proof. We have a bilinear, A -balanced map $(\flat(x), y) \mapsto \langle x, y \rangle_B$, which induces a linear map $\phi : \tilde{X} \otimes_A X \rightarrow B$. With some work on internal tensor products of imprimitivity bimodules, we can show by straightforward calculations that this preserves inner products. The same argument then applies to \tilde{X} since $\tilde{\tilde{X}} = X$. \square

Why should we expect that $\tilde{X} - \text{Ind}$ is the inverse to $X - \text{Ind}$? Well, the space on which $X - \text{Ind} \pi$ acts is $X \otimes_B \mathcal{H}_\pi$, which means that $\tilde{X} - \text{Ind}(X - \text{Ind} \pi)$ acts on $\tilde{X} \otimes_A (X \otimes_B \mathcal{H}_\pi)$. We come back to the original Hilbert space by

$$\tilde{X} \otimes_A (X \otimes_B \mathcal{H}_\pi) \cong (\tilde{X} \otimes_A X) \otimes_B \mathcal{H}_\pi \cong B \otimes_B \mathcal{H}_\pi \cong \mathcal{H}_\pi.$$

Theorem 4.6.8. Let X be an $A - B$ -imprimitivity bimodule and let π, ρ be nondegenerate representations of B and A . Then $\tilde{X} - \text{Ind}(X - \text{Ind} \pi)$ and $X - \text{Ind}(\tilde{X} - \text{Ind} \rho)$ are naturally unitarily equivalent to π and ρ , respectively.

Proof. Due to the long and perhaps not so enlightening nature of the proof, we only sketch the key ideas. We define $U : \tilde{X} \otimes_A (X \otimes_B \mathcal{H}_\pi) \rightarrow \mathcal{H}_\pi$, which is the unitary operator giving our unitary equivalence. Fix any $\flat(x) \in \tilde{X}$. We can consider the bilinear map $(y, h) \mapsto \pi(\langle x, y \rangle_B)h$, which induces a linear map $\beta_x : X \otimes_B \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi$. Now we can define a bilinear map when we let $\flat(x) \in \tilde{X}$ vary, which induces a map U such that $U(\flat(x) \otimes_A (y \otimes_B h)) = \pi(\langle x, y \rangle_B)h$.

Nondegeneracy of π means by definition that U has dense range. By lengthy calculations, U preserves inner products, which means that it extends to a unitary isomorphism. By several other calculations, we can show that U actually gives a unitary equivalence. Given two representations π_i for $i = 1, 2$, we consider an intertwining operator between the Hilbert spaces \mathcal{H}_i for those representations. If U_i are these unitaries as mentioned above for representation i , one can show $U_2 \circ ((\tilde{X} - \text{Ind}) \circ (X - \text{Ind})) = T \circ U_1$ by more calculations.

We show the result for the other representation by applying the first part to ${}_B \tilde{X}_A$ instead of ${}_A \tilde{X}_B$. \square

We end our theoretical story of Hilbert modules and Morita equivalences with some expected corollaries.

Corollary 4.6.9. Let X be an A – B -imprimitivity bimodule. The inverse of the Rieffel correspondence $X - \text{Ind} : \mathcal{I}(B) \rightarrow \mathcal{I}(A)$ is indeed $\tilde{X} - \text{Ind}$.

Proof. Since every ideal can be realized as the kernel of some representation, theorem 4.6.8 and the fact that $X - \text{Ind}(\text{Ker } \pi) = \text{Ker}(X - \text{Ind})$ from proposition 4.6.3 gives the wanted result. \square

Corollary 4.6.10. Let X be an A – B -imprimitivity bimodule and let π be a nondegenerate representation of B . Then the induced representation $X - \text{Ind } \pi$ is irreducible if and only if π is irreducible.

Proof. If either of these were reducible, then $\tilde{X} - \text{Ind}$ and $X - \text{Ind}$ would preserve such a direct sum decomposition by proposition 4.4.9. \square

Queequeg 4.6.11. Readers interested in applications of sheaf cohomology in analysis should take a look at the last chapters of [31], where lots of tools regarding classifying continuous-trace C^* -algebras and applications to group theory are studied.

After the technical work of fitting Morita equivalence to the world of C^* -algebras, we shift our focus towards adapting topological K -theory, which seems to be at the other side of the spectrum compared to Morita equivalence, as we can guess our approach, as well as the results of operator K -theory, based on a good overview of the topological case.

Chapter 5

First steps towards K -theory: The K_0 -theory

In this chapter we motivate, define and develop several results on the zeroth K -group, K_0 . Inspired by the Serre-Swan theorem, we adapt the construction of K -theory from topological K -theory through the Grothendieck group to define K_0 as a group of equivalence classes of projections. To do this, we first need to establish some notions of equivalence classes of projections, before we construct related algebraic structures. These structures are passed through the Grothendieck group to get an abelian group, becoming our premature definition of K_0 . After developing well-known results for K_0 , we need to amend some small problems for nonunital algebras in such a way that our results still hold.

In this section, we closely follow [35] and we will frequently refer there for lengthy, technical, or unenlightening proofs. It is also possible to follow other sources, such as the encyclopedic book by Blackadar ([1]) or Davidson ([5]).

5.1 Motivation: The Serre-Swan theorem, projective modules, and projections in C^* -algebras

When discussing Hilbert modules, we used modified vector bundles as an example (4.1.13). The power behind this example came from the celebrated Serre-Swan theorem ([40]).

Theorem 5.1.1. (Serre-Swan) Let X be a compact Hausdorff space, let $p : E \rightarrow X$ be a vector bundle, and let $\Gamma(E) := \Gamma(X, E)$ be the space of sections (splits to the epimorphism p) on the total space E . Then a $C(X)$ -module P is isomorphic to a module of the form $\Gamma(E)$ if and only if P is finitely generated and projective.

The functor $[\Gamma : VectBund(X) \rightarrow fg\text{-}proj(C(X))]$ yields an equivalence of categories.

One can study vector bundle theory, which is highly geometric in nature, by using the language of finitely generated projective modules. Ideas like this allowed algebraic geometers in the late '50s, such as Grothendieck, to develop new theories in algebraic geometry, e.g. in the context of coherent sheaves, through "classes" (and thus the name K -theory, by the German word for class, "Klasse"). Later, in the '60s, topologists like Atiyah and Hirzebruch applied these constructions to vector bundles to develop topological K -theory, which is a reduced cohomology theory in the sense of Eilenberg-Steenrod. For a brief introduction to the history on the topic, we refer to [11] and the references therein. Later, algebraists like Quillen ([30]) took up the story from a more algebraic point of view, and operator algebraists made their own variant of K -theory yielding interesting results such as the classification of AF-algebras.

Perhaps finitely generated projective modules seem a bit too far-fetched to adapt to operator algebras. Projective modules have a lot of definitions, and even though the definition involving exactness of the covariant Hom-functor may be the most usual, one can define an A -module P to be projective if and only if P is isomorphic to a summand of a free module, as this will surely be equivalent. In more operator algebraic terms, given a C^* -algebra A and an A -module P , we say P is projective if and only if there exists a projection $p \in A$ and a free module A^n such that $P \cong pA^n$.

Projections in C^* -algebras turns out to be hard to find and classify, and from an algebraic point of view, they should be! If we can find all (self-adjoint) idempotents of an algebra, we can say a lot about the algebraic structure. Perhaps the best example to give after a long chapter on Morita equivalences is the result on Morita equivalence and complementary full corners (4.5.17), where projections certainly play a key rôle.

Recall that commutative C^* -algebras are all of the form $C_0(X)$ for some locally compact Hausdorff space X by Gelfand duality (theorem 2.3.6). Projections in these commutative C^* -algebras should then correspond to finitely generated projective modules, which corresponds to vector bundles over X by Serre-Swan (5.1.1). Considering the same process of classifying projections in noncommutative C^* -algebras, we may think about the coming theory as an analogue of classifying "noncommutative vector bundles". This idea is of importance in noncommutative geometry and we can only refer interested readers to Connes ([4]) for a treatment of this topic.

Projections will be the foundation for the K_0 -theory, and it turns out we can formulate the K_1 -theory in terms of unitary elements in a similar manner, which we postpone to later chapters (7.1), but as unitaries are important elements in constructing K_0 as well, we include the necessities.

It turns out that the theory should be defined in terms of unital C^* -algebras, but through unitizations, there should be a way to link the nonunital case to the unital definition.

5.2 Equivalences of projections and homotopy

The main part of this section will be on developing equivalence relations compatible with a monoid structure on projections to construct our invariants. Perhaps inspired by topological K -theory ([43], [8]), we want to apply the Grothendieck construction, which involves abelian monoids and equivalence relations. Aiming to construct a relevant abelian monoid, we try out different types of equivalence relations of projections to see if we can get an abelian monoid up to some equivalence. In the end, all these equivalence relations give the same equivalence classes when we stabilize our construction by passing to matrices.

Let A be a C^* -algebra. To make sure we are all on the same page:

Definition 5.2.1. If A is unital, we denote the group of unitary elements in A by $\mathcal{U}(A)$. The set of all projections in A is denoted by $\mathcal{P}(A)$.

Definition 5.2.2. Let $a, b \in A$. We say a and b are **homotopic**, denoted $a \sim_h b$, if there exists a path between a and b . That is, if there exists a continuous function $v : [0, 1] \rightarrow A$ such that $v(0) = a$ and $v(1) = b$.

More generally, two elements a, b are homotopic if the constant maps $f_a : X \rightarrow A$, $x \mapsto a$ and $f_b : X \rightarrow A$, $x \mapsto b$ are homotopic in the usual sense, where X is some arbitrary C^* -algebra. Two maps $f, g : A \rightarrow B$ are homotopic if there exists a continuous function $v : [0, 1] \times A \rightarrow B$ such that $v(0, x) = f(x)$ and $v(1, x) = g(x)$ for $x \in A$.

We denote by $\mathcal{U}_0(A)$ by the set of all u in $\mathcal{U}(A)$ such that $u \sim_h 1$ in $\mathcal{U}(A)$.

For now, we claim that homotopy is the strongest of the equivalence relations we will consider. At a later stage, it turns out we can pass to matrices to make homotopy equivalent with the other equivalence relations we will consider, as well as to get rid of some dimensional problems with our construction. The preceding lemma originated as a technicality to be used in algebraic K -theory, but it is still highly relevant to the study of operator K -theory.

Lemma 5.2.3. (Whitehead)

Let $u, v \in \mathcal{U}(A)$. Then

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \quad \text{in } \mathcal{U}(M_2(A))$$

In particular, if $v = u^*$, it follows that

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{in } \mathcal{U}(M_2(A))$$

One can actually prove this with not too much work, but it is not necessary for our purposes. We refer to page 17 of [35] for a proof.

If one is interested to develop the study of K_1 -theory from scratch, this small study of unitary elements should be extended quite a bit, as unitary elements are one of the main players in this theory. We come back to this at a later point (section 7.1).

Definition 5.2.4. Let $p, q \in \mathcal{P}(A)$ for some arbitrary C^* -algebra. We define the following equivalence relations on $\mathcal{P}(A)$.

- $p \sim_h q$ if there exists a path from p to q as defined in 5.2.2.
- $p \sim_u q$ if $\exists u \in \mathcal{U}(\tilde{A})$ such that $q = upu^*$. We say p and q are *unitarily equivalent*.
- $p \sim q$ if $\exists v \in A$ with $p = v^*v$ and $q = vv^*$. We say p and q are *Murray-von Neumann equivalent*.

With our definitions ready to roll, we only need to check that they work out the way we want them to. We list some results on these matters, most of which have detailed proofs in chapter 2. of [35].

Proposition 5.2.5. The relations on $\mathcal{P}(A)$ as defined in 5.2.4, are actually equivalence relations.

Proof. The only part we need a clever trick to prove is the transitivity of the *Murray-von Neumann* equivalence, as the rest mainly follows from writing out the definitions. If v is a partial isometry, i.e. v^*v is a projection, then we have $v = vv^*v$. To show this, define $z = (1 - vv^*)v$ and calculate z^*z . By repeated applications of the fact that v^*v and vv^* are projections along with the C^* -equality, this is 0. If we now set $p = v^*v$ and $q = vv^*$, we can show that $v = qv = vp = qvp$. Consider projections p, q, r such that $p \sim q$ and $q \sim r$. Choose partial isometries v, w such that $p = v^*v, q = vv^* = w^*w$ and $r = ww^*$. Then $z = wv$ will be the partial isometry giving the equivalence $p \sim r$. \square

We further state some results from [35] on how the hierarchy of equivalence relations is ordered, but omit the proofs as we are more eager to move on to the K_0 -theory than to deal with technical lemmas.

Proposition 5.2.6. Let A be a unital C^* -algebra and let p, q be projections in A . Then the following are equivalent:

1. $p \sim_u q$,
2. $q = upu^*$ for some $u \in \mathcal{U}(A)$,
3. $p \sim q$ and $1_A - p \sim 1_A - q$.

In the light of this result, we see that *Murray-von Neumann* equivalence is a weaker form of equivalence than unitary equivalences for unital C^* -algebras. We will see that this holds for non-unital C^* -algebras as well in proposition 5.2.9.

Proposition 5.2.7. Let A be a C^* -algebra and let p, q be projections in A .

1. If $\|p - q\| < 1$, then $p \sim_h q$.
2. $p \sim_h q \iff \exists u \in \mathcal{U}_0(\tilde{A})$ such that $q = upu^*$

Remark 5.2.8. We can show a similar statement to the first claim for unitary elements u, v as well, but then we only require $\|u - v\| < 2$ to get the same result.

The second part, on the other hand, tells us the following:

Proposition 5.2.9. Let A be a C^* -algebra (not necessarily unital) and p, q be projections in A .

1. If $p \sim_h q$, then $p \sim_u q$.
2. If $p \sim_u q$, then $p \sim q$.

Proof. By proposition 5.2.7, we need a unitary equivalence where our unitary element is homotopic to the identity in $\mathcal{U}(\tilde{A})$ for the unitary equivalence to imply homotopy. Thus homotopy is a stronger relation than unitary equivalence.

For the second part, assume $p = uqu^*$ for some unitary $u \in \mathcal{U}(\tilde{A})$. Then $v = uq$ is in A and $p = vv^*, q = v^*v$. \square

Example 5.2.10. (Partial counterexample to the converse of proposition 5.2.9)

A tempting question to ask is whether or not there was a need for this result in the first place? Can we find projections p, q that are *Murray-von Neumann* equivalent, but not unitarily equivalent?

From proposition 5.2.6, it would suffice to find p, q such that $p \sim q$, but $1_A - p \not\sim 1_A - q$. To construct this counterexample, we consider a non-unitary isometry s on, such as the unilateral shift S on $\ell^2(\mathbb{N})$ from page 25 of [35]. Recall isometries in unital C^* -algebras are elements s such that $s^*s = 1$. Consider such a non-unitary isometry s in a unital C^* -algebra A . Then we know, by the definition of *Murray-von Neumann* equivalence, that $s^*s \sim ss^*$ since $s \in A$.

Then $1 - s^*s = 0$ and $1 - ss^*$ is a nonzero projection. The latter cannot be equivalent to 0. If $v^*v = 0$, then $v = 0$ by the C^* -equality, which means $vv^* = 0$. Therefore, the only projection equivalent to the zero projection is the zero projection itself, meaning that the zero projection $1 - s^*s$ is not equivalent to the projection $1 - ss^*$.

Similar arguments can be made for the other converse of proposition 5.2.9, by using the Whitehead lemma (5.2.3)

Note that proposition 5.2.6 gives us an explicit requirement to be able to move upwards in the hierarchy (5.2.9).

It is natural to ask whether we can do some modifications or perhaps tweak our C^* -algebra to make *Murray-von Neumann* equivalence imply homotopy.

By the power of the Whitehead lemma 5.2.3, we can do this, given that we put our C^* -algebra in a matrix algebra.

Proposition 5.2.11. Let A be a C^* -algebra. Let p, q be projections in A .

Denote the matrices $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$ by p' and q' , respectively. Then

1. if $p \sim q$, then $p' \sim_u q'$ in $M_2(A)$
2. if $p \sim_u q$, then $p' \sim_h q'$ in $M_2(A)$

Proof. For the first part, find an element v in A such that $p = v^*v, q = vv^*$. Construct

$$u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}, \quad w = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}.$$

Both u and w are unitary elements in $M_2(\tilde{A})$ by using some of the identities mentioned in the sketch of the proof of proposition 5.2.5. Also, by computation, we can note that $wup'u^*w^* = q'$ and that wu is a unitary in $M_2(\tilde{A})$. Thus, since wu is in the unitization of $M_2(A)$, $\widetilde{M_2(A)}$, we will have an element of $\mathcal{U}(\widetilde{M_2(A)})$ to define our unitary equivalence in $M_2(A)$

For the second part, use the assumption that $p \sim_u q$ in A to find u in $\mathcal{U}(\tilde{A})$ such that $q = upu^*$. We know from the Whitehead lemma (5.2.3) that the identity in $\mathcal{U}(M_2(\tilde{A}))$ is homotopic to the matrix with u and u^* on the diagonal. Thus we can find a continuous path $v(t)$ in $\mathcal{U}(M_2(\tilde{A}))$ such that

$$v(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v(1) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}.$$

Define $e_t = v(t)p'v(t)^*$. Then we have that e_t belongs to $\mathcal{P}(M_2(A))$, where the map $t \rightarrow e_t$ is continuous and $e_0 = p', e_1 = q'$, meaning $p' \sim_h q'$. \square

5.3 A monoid of projections and the Grothendieck construction

One of the key ideas we have observed is that if we pass to matrices, the *Murray-von Neumann* equivalence becomes as powerful as homotopy, even though the former is a lot easier to work with. To be able to construct our invariants, we use this as our main motivation. Let A be a C^* -algebra.

Definition 5.3.1. We define $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$ and $\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A)$.

For $p \in \mathcal{P}_n(A), q \in \mathcal{P}_m(A)$, we define

$$p \oplus q = \text{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \quad \text{in } \mathcal{P}_{n+m}(A).$$

For such p, q , we say $p \sim_0 q$ if there exists a v in $M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$, were we recall that taking adjoints in the matrix algebras correspond to transposing and taking adjoints entrywise.

This *Murray-von Neumann*-like relation will be critical to defining the K_0 -group, which is why it is denoted with a subscripted 0. Actually, if p and q belong in the same level of $\mathcal{P}_\infty(A)$, then $p \sim_0 q \iff p \sim q$ by definition. Thus, we have made our *Murray-von Neumann* equivalence independent of the dimension of the matrix algebra our projections lie in, and the generalization to matrices preserves the idea of making this relation as powerful as homotopy.

One should also note that the structure $(\mathcal{P}_\infty(A), \oplus)$ actually gives us a monoid.

Remark 5.3.2. Our need for matrices was not only inspired our wish to reverse the equivalence hierarchy (5.2.9), but also to amend problems with the missing structure in $\mathcal{P}(A)$. Let p, q to be projections in A . We would want $p + q$ to be a projection, but by setting $p = q$, we see that $p + p = 2p \neq (2p)^2$ is not in $\mathcal{P}(A)$. By passing to \mathcal{P}_∞ , we can sum p and q without letting them interact.

We wanted an abelian monoid structure, at least up to some equivalence. To ensure that our generalization of the *Murray-von Neumann* equivalence does not admit immediate pathological behaviour, we state the following properties of our new equivalence relation and refer readers to [35] for the details.

Proposition 5.3.3. Let A be a C^* -algebra and let p, p', q, q', r be projections in $\mathcal{P}_\infty(A)$.

1. \sim_0 is actually an equivalence relation on $\mathcal{P}_\infty(A)$,
2. $p \sim_0 p \oplus 0_n$, where 0_n is the zero n -dimensional matrix over A ,
3. if $p \sim_0 p'$ and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$,
4. $p \oplus q \sim_0 q \oplus p$,
5. $(p \oplus q) \oplus r = p \oplus (q \oplus r)$,
6. if p and q are projections in $\mathcal{P}_n(A)$ such that $pq = 0$, that is, they are mutually orthogonal, then $p + q$ is a projection with $p + q \sim_0 p \oplus q$.

Definition 5.3.4. Let A be a C^* -algebra.

Define $V(A) = \mathcal{P}_\infty(A) / \sim_0$. Addition on $V(A)$ is defined as $[p]_V + [q]_V := [p \oplus q]_V$.

This is well defined by proposition 5.3.3 and it makes $(V(A), +)$ an abelian monoid.

Remark 5.3.5. One should note that at each step in this construction, we create new objects. The same can be done for morphisms, as a $*$ -homomorphism $\phi : A \rightarrow B$ of C^* -algebras induces a $*$ -homomorphism on the matrix algebras. This induces a monoid homomorphism $\mathcal{P}_\infty(A) \rightarrow \mathcal{P}_\infty(B)$, which in turn induces a homomorphism of abelian monoids $V(A) \rightarrow V(B)$. It is thus straight forward to show that $V : C^*Alg \rightarrow AbMon$ is a covariant functor from the category of C^* -algebras to the category of abelian monoids.

This is totally analogous to topological theory, and we adapt the letter V from topological K -theory, where the analogue of this monoid is the monoid of vector bundles under Whitney summation.

Remark 5.3.6. One can define this abelian monoid in other ways. We follow the structure of [35] quite closely, but if one follows [1], the definition is a bit different. That being said, the result will be the same. In [1], they show that every idempotent will be homotopic to a projection and then they define $M_\infty(A)$ to be the direct limit of $M_n(A)$ under the maps $a \mapsto \text{diag}(a, 0)$, which means that $M_\infty(A)$ consists of infinite matrices with finitely many non-zero entries. This may not be a C^* -algebra in its own right, but we can complete it to be one, which we call the stabilization of A . Then one can define $V(A)$ as equivalence classes of idempotents in $M_\infty(A)$, but this will give the same result as our, perhaps more explicit, approach.

With this abelian monoid describing equivalence classes of projections, we are ready to apply the Grothendieck construction to construct K_0 . We first explain this important construction. From now on, all monoids are abelian if not otherwise specified.

Construction 5.3.7. (The Grothendieck construction)

Let $(S, +)$ be an abelian monoid and define \sim on $S \times S$ by setting $(x_1, y_1) \sim (x_2, y_2)$ if we can find $z \in S$ such that $x_1 + y_2 + z = x_2 + y_1 + z$.

We can note that \sim actually defines an equivalence relation. The term $+z$ is added to ensure the transitivity of this relation. If we start with the relation without this term, obtained from wanting $x_1 - y_1$ to be equal $x_2 - y_2$ in the semigroup, it fails to be a transitive relation. Taking the transitive hull, we obtain the relation defined above.

As we are working with additive maps between the monoids, the terms $+z$ do not yet cancel, but as soon as we pass to a group structure, they disappear. Therefore, this term is only relevant in $AbMon$, where it ensures transitivity.

We define the quotient $G(S) = (S \times S) / \sim$.

If $[x_1, y_1], [x_2, y_2]$ denotes two elements in $G(S)$, we define addition component-wise by $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$.

Note that $[x, x] = 0$ and that $-[x, y] = [y, x]$.

Definition 5.3.8. The abelian group $G(S)$ associated to the abelian monoid S is called *the Grothendieck group of S* .

Define **the Grothendieck map**

$$\gamma_S : S \rightarrow G(S), \quad x \mapsto \gamma_S(x) = [x + y, y].$$

First, we should note that γ_S is additive since addition is done componentwise. By the definition of the equivalence relation in $G(S)$ we can see that it is independent

of choice of y as $[x + y', y'] = [x + y, y] \iff x + y' + y = x + y + y'$, which surely holds for all y' .

This construction has some useful properties.

Proposition 5.3.9. Let S be an abelian monoid. Then the following properties hold:

1. Universal property.

Let H be an abelian group and let $\phi : S \rightarrow H$ be an additive map. Then there exists a unique homomorphism $\tilde{\phi} : G(S) \rightarrow H$ of abelian groups such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & H \\ & \searrow \gamma_S & \nearrow \tilde{\phi} \\ & G(S) & \end{array}$$

commutes.

2. Functoriality.

We know the construction assigns an abelian group $G(S)$ to each abelian monoid S . For all abelian monoids S, T and for every additive map $\phi : S \rightarrow T$, there exists a unique group homomorphism $G(\phi) : G(S) \rightarrow G(T)$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & T \\ \downarrow \gamma_S & & \downarrow \gamma_T \\ G(S) & \xrightarrow{G(\phi)} & G(T) \end{array}$$

commutes.

3. $G(S) = \{\gamma_S(x) - \gamma_S(y) : x, y \in S\}$.
4. For $x, y \in S$, $\gamma_S(x) = \gamma_S(y) \iff x + z = y + z$ for some $z \in S$.
5. γ_S is injective if and only if S has the cancellation property, that is, if and only if $x + z = y + z \implies x = y$.
6. If we take $S \subseteq H$, where H is some abelian group and S is closed under addition, then $G(S) \cong S_H = \langle \{x - y : x, y \in S\} \rangle$, the subgroup of H generated by the differences in S .

Proof. We first prove 3., as the characterization as differences will yield uniqueness of our maps.

3. We rewrite the elements in $G(S)$ and use the definition of γ_S .

$[x, y] = [0, y - x] = [x + y, y] - [x + y, x] = \gamma_S(x) - \gamma_S(y)$, as wanted.

1. Assume we are given such H and ϕ . Note that $[x_1, y_1] = [x_2, y_2]$ implies

$$\phi(x_1) + \phi(y_2) + \phi(z) = \phi(x_2) + \phi(y_1) + \phi(z)$$

in H for some $z \in S$ by additivity of ϕ . This yields $\phi(x_1) - \phi(y_1) = \phi(x_2) - \phi(y_2)$, which means that a map $\tilde{\phi} : G(S) \rightarrow H$ defined by $[x, y] \mapsto \phi(x) -$

$\phi(y)$ would be well-defined, additive and make the diagram in 1. commute. Uniqueness follows from point 3. since any other additive map that fits here would need to send these differences to the same thing.

2. Apply 1. to $H = G(T)$ with the additive map $S \rightarrow H$ being the composition $S \rightarrow T \rightarrow G(T)$.
4. If $x + z = y + z$, the results follow from $G(S)$ being a group and γ_S being additive. If $\gamma_S(x) = \gamma_S(y)$, then $[x + y, y] = [y + x, x]$, which means $(x + y) + x + w = (y + x) + y + w$. Therefore, with $z = x + y + w$, we get the result.
5. Follows directly from 4.
6. We know S must be an abelian monoid with the cancellation property since it is closed under addition in the ambient abelian group H . Considering the additive inclusion $i : S \hookrightarrow H$ and the induced map $\psi = G(i) : G(S) \rightarrow H$, we must have $\psi(\gamma_S(x)) = x$ for all x in S . By 3., elements in $G(S)$ are differences, so the image of ψ is the subgroup of H generated by the differences. If $\psi(\gamma_S(x) - \gamma_S(y)) = 0$, $x - y = 0$, which means $\gamma_S(x) = \gamma_S(y)$, that is, ψ is injective.

□

Remark 5.3.10. The Grothendieck functor G can be realized as the left adjoint of the forgetful functor $U : Ab \rightarrow AbMon$, i.e. (G, U) is an adjoint pair. For our purposes, we must have that G preserves colimits, as it has a right adjoint. In fact, we will see this is vital when we work with AF-algebras in chapter 6, as AF-algebras are colimits of finite-dimensional C^* -algebras. We refer to [37] for details on category theory.

Queequeg 5.3.11. More categorically attracted readers may be interested to hear that there have been some developments on generalizing the definition of the Grothendieck group to n -exangulated categories. An extriangulated category is the unifying concept of a triangulated category and an exact category. By generalizing triangles to n -angles and working out the right definitions, an n -exangulated category is the unifying concept of a n -angulated category and an exact category. Several of the properties we have shown above can be carried over to n -exangulated categories. Interested readers are referred to [9].

We are finally ready to define K_0 .

5.4 The K_0 -group - the unital case

Definition 5.4.1. (K_0 for unital C^* -algebras) Let A be a unital C^* -algebra and let $(V(A), +)$ be the abelian monoid of projections defined in 5.3.4. Then $K_0(A)$ is defined to be the Grothendieck group of $V(A)$.

$$K_0(A) = G(V(A))$$

Define $[\cdot]_0 : \mathcal{P}_\infty(A) \rightarrow K_0(A)$ by $[p]_0 = \gamma([p]_V)$.

It turns out that there is another type of interesting equivalence on $\mathcal{P}_\infty(A)$ which is weaker than the generalized *Murray-von Neumann* equivalence.

Definition 5.4.2. (Stable equivalence) Let $p, q \in \mathcal{P}_\infty(A)$ and define $p \sim_s q$ if and only if there exists $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. This relation is called *stable equivalence*.

If A is unital and 1_n denotes the unit of $M_n(A)$, $p \sim_s q$ if and only if $p \oplus 1_n \sim_0 q \oplus 1_n$ for some n . All stable equivalences can be realized like this, since if $p \oplus r \sim_0 q \oplus r$ for $r \in \mathcal{P}(A)$, we have

$$p \oplus 1_n \sim_0 p \oplus r \oplus (1_n - r) \sim_0 q \oplus r \oplus (1_n - r) \sim_0 q \oplus 1_n$$

Let us examine how this structure looks.

Proposition 5.4.3. (Standard picture of K_0 for unital C^* -algebras)

Given a unital C^* -algebra A . Then the following hold:

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\} \quad (5.1)$$

$$= \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A), n \in \mathbb{N}\}. \quad (5.2)$$

1. $[p \oplus q]_0 = [p]_0 + [q]_0$ for all projections $p, q \in \mathcal{P}_\infty(A)$ and $[0_A]_0 = 0$.
2. If $p, q \in \mathcal{P}_n(A)$ and $p \sim_h q$, then $[p]_0 = [q]_0$.
3. If $p, q \in \mathcal{P}_n(A)$ are mutually orthogonal projections, then $[p + q]_0 = [p]_0 + [q]_0$.
4. For $p, q \in \mathcal{P}_\infty(A)$, we have $[p]_0 = [q]_0$ if and only if $p \sim_s q$.

Proof. The first equality follows from the properties of the Grothendieck construction (proposition 5.3.9). An element $g \in K_0(A)$ can hence be written $g = [p]_0 - [q]_0$ for some $p \in \mathcal{P}_k$ and $q \in \mathcal{P}_l$. If we choose $n \geq \max\{k, l\}$, we can define $p' = p \oplus 0_{n-k}$ and $q' = q \oplus 0_{n-l}$. By proposition 5.3.3, we have $p \sim_0 p'$ and $q \sim_0 q'$. Now $p', q' \in \mathcal{P}_n(A)$ and $g = [p']_0 - [q']_0$.

For 1., note that

$$[p \oplus q]_0 = \gamma([p \oplus q]_V) = \gamma([p]_V + [q]_V) = \gamma([p]_V) + \gamma([q]_V) = [p]_0 + [q]_0$$

and that this implies $[0_A]_0 + [0_A]_0 = [0_A]_0$.

For 2., just recall the hierarchy of relations (proposition 5.2.9) and that the regular *Murray-von Neumann* equivalence is at the bottom of the hierarchy. This is a special case of \sim_0 for when p, q are in the same degree of $\mathcal{P}_\infty(A)$. Therefore $p \sim_h q \implies p \sim_0 q \implies [p]_V = [q]_V \implies [p]_0 = [q]_0$ by the definition of $[\cdot]_0$ for $p, q \in \mathcal{P}_n(A)$.

For 3., recall from proposition 5.3.3 that if p and q are mutually orthogonal projections, $p + q \sim_0 p \oplus q$. Then the result follows from 1.

For 4., recall the properties of the Grothendieck map (proposition 5.3.9). Since $[p]_0 = [q]_0$, we can find a projection $r \in \mathcal{P}_\infty(A)$ such that $[p]_V + [r]_V = [q]_V + [r]_V$, which in turn means $[p \oplus r] = [q \oplus r]$. Then $p \oplus r \sim_0 q \oplus r$, which is the definition of $p \sim_s q$. Conversely, stable equivalence implies $p \oplus r \sim_0 q \oplus r$ for some r . This means $[p]_0 + [r]_0 = [q]_0 + [r]_0$. Recall that $K_0(A)$ is a group, which yields $[p]_0 = [q]_0$ by cancellation. \square

It would be natural for the universal property of the Grothendieck construction to be transferred to K_0 , which will ensure the functoriality of K_0 .

Proposition 5.4.4. (Universal property of K_0)

Let A be unital, let G be an abelian group, and let $v : \mathcal{P}_\infty(A) \rightarrow G$ be a map such that

1. $v(p \oplus q) = v(p) + v(q)$ for all projections $p, q \in \mathcal{P}_\infty(A)$
2. $v(0_A) = 0$
3. For all projections p, q that belong to the same degree of $\mathcal{P}_\infty(A)$ and $p \sim_h q$, then $v(p) = v(q)$.

If these hold, there exists a unique group homomorphism $f : K_0(A) \rightarrow G$ such that

$$\begin{array}{ccc} \mathcal{P}_\infty(A) & & \\ \downarrow [\cdot]_0 & \searrow v & \\ K_0(A) & \xrightarrow{f} & G \end{array}$$

commutes.

Remark 5.4.5. It is possible (but not necessarily easy) to show that the homotopy requirement in the third assumption on v can be replaced with \sim_u , \sim_0 and \sim_s to give equivalent results given the first two assumptions on v . Note that the unitary equivalence must happen in the same degree $\mathcal{P}_n(A)$, while the two last relations only need to hold for $p, q \in \mathcal{P}_\infty(A)$.

Proof. We sketch the proof. To construct the map, first show that for $p, q \in \mathcal{P}_\infty(A)$ such that $p \sim_0 q$, we have $v(p) = v(q)$. This implies that the map $f : V(A) \rightarrow G$ given by $f([p]_V) = v(p)$ is well defined, as v must factor through $V(A)$ due to the first statement in the proof. Additivity of f follows from additivity of v . We can then find a unique group homomorphism by the universal property of the Grothendieck group (proposition 5.3.9) such that the wanted diagram commutes, which completes the proof. \square

We now know this construction takes unital C^* -algebras to their K_0 -groups, but what happens with morphisms between C^* -algebras? Given a map between C^* -algebras, for example $A \rightarrow B$, we know we get an induced map $\phi : \mathcal{P}_\infty(A) \rightarrow$

$\mathcal{P}_\infty(B)$. Defining $\nu : \mathcal{P}_\infty(A) \rightarrow K_0(B)$ by $\nu(p) = [\phi(p)]_0$, we can see that this actually satisfies the necessary properties to invoke the universal property of K_0 given in proposition 5.4.4. This implies ν factors uniquely through a group homomorphism $K_0(\phi) : K_0(A) \rightarrow K_0(B)$ given by exactly $K_0(\phi)([p]_0) = [\phi(p)]_0$.

This actually makes it quite easy to see that K_0 defines a functor from the category of unital C^* -algebras to the category of abelian groups. It is usual to consider the zero C^* -algebra, $\{0\}$, as unital. We denote the zero morphism, $A \rightarrow B$, by $0_{B,A}$.

Proposition 5.4.6. (Functoriality of K_0 in the unital case)

1. If A is a unital C^* -algebra, then $K_0(Id_A) = Id_{K_0(A)}$, where Id_A is the identity map on A .
2. If A, B, C are unital C^* -algebras and $\phi : A \rightarrow B$, $\psi : B \rightarrow C$ are $*$ -homomorphisms, then K_0 preserves composition, that is, $K_0(\psi \circ \phi) = K_0(\psi) \circ K_0(\phi)$.
3. $K_0(\{0\}) = 0$ and $K_0(0_{B,A} : A \rightarrow B) = 0_{K_0(B), K_0(A)} : K_0(A) \rightarrow K_0(B)$.

The first two claims of this proposition check that K_0 is a functor, while the last claim ensures that the zero C^* -algebra plays nicely with K_0 .

Proof. By the definition of the maps induced by K_0 , we see $K_0(Id_A)([p]_0) = [p]_0$ and $K_0(\psi \circ \phi)([p]_0) = (K_0(\psi) \circ K_0(\phi))([p]_0)$, which means that we are finished by the standard picture of K_0 . The last claim follows by writing out the construction of $K_0(\{0\})$ and realizing $0_{B,A} = 0_{B,0} \circ 0_{0,A} : A \rightarrow \{0\} \rightarrow B$. \square

Remark 5.4.7. We have defined a functor, but only for unital C^* -algebras. Observant readers may have noted that this construction seems to work out for nonunital C^* -algebras as well. The problem, which we will soon return to, is that most of the computational powers are lost if we define K_0 this way for nonunital C^* -algebras. As we will see later, the same definition of K_0 for nonunital C^* -algebras will yield a functor that is not even half exact, making the usual computations through long exact sequences close to useless.

Remark 5.4.8. Readers familiar with topological K -theory may recall that topological K -theory is a reduced cohomology theory, implying that the K -functor should be contravariant (thus denoted K^0). In similar cases, there is usually some contravariant composition lurking backstage, flipping the arrows. Indeed, given a map $f : X \rightarrow Y$ of base spaces and a vector bundle $\pi : E \rightarrow Y$, the induced bundle is given by the pullback bundle $f^*\pi : f^*E \rightarrow X$, where f^*E denotes the pullback of the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \pi \\ X & \xrightarrow{f} & Y, \end{array}$$

which explains why K^0 is contravariant.

Since our approach to the theory is taken from algebraic topology and based on reversing the hierarchy of equivalence relations, we better expect that this functor is homotopy invariant!

Proposition 5.4.9. Assume A and B are unital C^* -algebras. Then

1. if $\psi, \phi : A \rightarrow B$ are homotopic $*$ -homomorphisms, $K_0(\psi) = K_0(\phi)$.
2. if A, B are homotopy equivalent through $\phi : A \rightarrow B$ and $\psi : B \rightarrow A$, we have $K_0(A) \cong K_0(B)$. In particular, $K_0(\phi), K_0(\psi)$ are isomorphisms with $K_0(\phi)^{-1} = K_0(\psi)$.

Proof. For 1., choose any pointwise continuous path of $*$ -homomorphisms, ϕ_t , connecting these homotopic maps. Extend this pointwise to a path in the matrix algebras $\phi_t : M_n(A) \rightarrow M_n(B)$ for each n . Now the path $t \mapsto \phi_t(p)$ is continuous for each $p \in \mathcal{P}_n(A)$. So $\phi(p) = \phi_0(p) \sim_h \phi_1(p) = \psi(p)$. Therefore we must have

$$K_0(\phi)([p]_0) = [\phi(p)]_0 = [\psi(p)]_0 = K_0(\psi)([p]_0),$$

which means we are done, since the standard picture of K_0 expresses $K_0(A)$ in terms of these classes $[p]_0$.

For 2., just note that the composition of ϕ and ψ , which is sent to the composition of the induced maps due to functoriality of K_0 , is homotopic to the identity, which is sent to the identity. \square

Our main way of computing K -groups will be through exact sequences. Before we show K_0 preserve simple split exact sequences, we prove a lemma, which may be interesting in its own right.

Lemma 5.4.10. Let $\phi, \psi : A \rightarrow B$ be mutually orthogonal $*$ -homomorphisms in the sense that $\phi(x)\psi(y) = 0$ for all $x, y \in A$. Then $\phi + \psi$ is a $*$ -homomorphism and $K_0(\phi + \psi) = K_0(\phi) + K_0(\psi)$.


Proof. By mutual orthogonality, it is straightforward to see $\phi + \psi$ is a $*$ -homomorphism that induces mutually orthogonal maps when extended to matrices. That is, we have $(\phi + \psi)_n = \phi_n + \psi_n$. Using the standard picture of K_0 given in proposition 5.4.3, we fix $p \in \mathcal{P}_n(A)$ and obtain

$$\begin{aligned} K_0(\phi + \psi)([p]_0) &= [(\phi + \psi)_n(p)]_0 = [\phi_n(p) + \psi_n(p)]_0 \\ &= [\phi_n(p)]_0 + [\psi_n(p)]_0 \\ &= K_0(\phi)([p]_0) + K_0(\psi)([p]_0), \end{aligned}$$

as wanted. \square

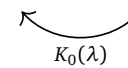
Lemma 5.4.11. Given a unital C^* -algebra A , the split exact sequence

$$0 \longrightarrow A \xrightarrow{i} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$



induces a split exact sequence

$$0 \longrightarrow K_0(A) \xrightarrow{K_0(i)} K_0(\tilde{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \longrightarrow 0$$



This split exact sequence may seem a bit weird to consider, as we assume A is unital already, but it will be vital when defining $K_0(A)$ for nonunital C^* -algebras, and certainly, we should know what the "easy" K -groups such as $K_0(\mathbb{C})$ are.

Proof. We keep it brief. Define $f = 1_{\tilde{A}} - 1_A$ and note that since A is already unital, we get $\tilde{A} = A + \mathbb{C}f$ and $af = fa = 0$ for all $a \in A$. We can define $*$ -homomorphisms $\mu : \tilde{A} \rightarrow A$ by $\mu(a + \alpha f) = a$ and $\lambda' : \mathbb{C} \rightarrow \tilde{A}$ by $\lambda'(\alpha) = \alpha f$.

By writing out expressions for $K_0(\text{id})$ in each term of the short exact sequence through μ and λ , noting that $i \circ \mu$ and $\lambda' \circ \pi$ are mutually orthogonal and applying lemma 5.4.10, one can deduce that this sequence is exact. \square

We consider some examples of K_0 -groups of known, unital C^* -algebras, and as we will see in chapter 8, traces play a key rôle in the classification of noncommutative tori.

Example 5.4.12. $K_0(M_n(\mathbb{C}))$ is isomorphic to \mathbb{Z} for all n and the isomorphism is given by $K_0(\text{Tr})$, where Tr denotes the standard trace on $M_n(\mathbb{C})$. Let e denote a one-dimensional projection in this matrix algebra, then the generator of $K_0(M_n(\mathbb{C}))$ as a cyclic group is given by $[e]_0$. For $n = 1$, we obtain $K_0(\mathbb{C}) \cong \mathbb{Z}$.

Proof. (Short explanation)

Consider an element $g \in K_0(M_n(\mathbb{C}))$. Then we may find k and $p, q \in M_k(M_n(\mathbb{C})) = M_{kn}(\mathbb{C})$ such that $g = [p]_0 - [q]_0$. This implies that

$$K_0(\text{Tr}(g)) = \text{Tr}(p) - \text{Tr}(q) = \dim(p(\mathbb{C}^{nk})) - \dim(q(\mathbb{C}^{nk})),$$

which means that $K_0(\text{Tr})(g)$ certainly is an integer. It is possible to show that $\text{Tr}(p) = \text{Tr}(q)$ is equivalent to $p \sim q$, which implies that $K_0(\text{Tr})$ is injective. Since the image of $K_0(\text{Tr})$ is a subgroup of \mathbb{Z} that contains $1 = K_0(\text{Tr})([e]_0)$, where e is a one dimensional projection, $K_0(\text{Tr})$ is surjective. \square

Example 5.4.13. Let \mathcal{H} be an infinite-dimensional, separable Hilbert space. Then $K_0(B(\mathcal{H})) = 0$.

Proof. Note that $M_n(B(\mathcal{H}))$ can be identified with $B(\mathcal{H}^n)$. The map $\dim : \mathcal{P}_\infty(B(\mathcal{H})) \rightarrow \{0, 1, 2, \dots, \infty\}$ given by $\dim(p) = \dim(p(\mathcal{H}^n))$ is surjective, since $\mathcal{P}_n(B(\mathcal{H})) = \mathcal{P}(B(\mathcal{H}^n))$. In a similar manner to example 5.4.12, we can show that two projections in the same degree of $\mathcal{P}_\infty(A)$ are equivalent if and only if they yield the same value under \dim , meaning that \dim is injective on each degree of $\mathcal{P}_\infty(A)$. Recall that dimension is additive and note that $\dim(p \oplus 0) = \dim(p)$, which means that this extends to all $p, q \in \mathcal{P}_\infty(A)$. Now the map $d([p]_V) = \dim(p)$ is a well defined monoid isomorphism $d : V(B(\mathcal{H})) \rightarrow \{0, 1, \dots, \infty\}$. Therefore $K_0(B(\mathcal{H})) \cong G(\{0, 1, \dots, \infty\}) \cong 0$. The Grothendieck group of such a group is 0 since all elements will be equivalent to 0 by the existence of ∞ . \square

Remark 5.4.14. The same result actually holds if \mathcal{H} is not separable, but we omit the proof.

Example 5.4.15. Let X be a contractible, compact Hausdorff space. The map $\dim : K_0(C(X)) \rightarrow \mathbb{Z}$ is an isomorphism, where $\dim([p]_0) = \text{Tr}(p(x))$ is independent of the choice of x with Tr denoting the standard trace on $M_n(\mathbb{C})$.

Proof. The idea is constructing a homotopy equivalence between $C(X)$ and \mathbb{C} , but we omit the proof and refer readers to [35]. Nevertheless, we want to explain briefly why \dim is well defined. Consider the map $x \mapsto \text{Tr}(p(x))$. This belongs to $C(X, \mathbb{Z})$. The connectedness of X implies that every function $C(X, \mathbb{Z})$ is constant since nonconstant functions would give an immediate separation of X by preimages. This implies $x \mapsto \text{Tr}(p(x))$ is constant and thus \dim is independent of $x \in X$. \square

As we can see, these computations require a bit of work. We will soon develop some computational tools, but first, we need to extend K_0 to nonunital C^* -algebras.

5.5 Extending K_0 to the nonunital framework

At the end of the previous section, we calculated some basic examples of K_0 -groups. If we were to continue in this manner, it would be natural to consider tweaking closely related topological spaces to see how the behaviour of K_0 would change. This leads to some problematic behaviour of K_0 regarding half exactness when we study nonunital algebras. Recall the following basic definitions.

Definition 5.5.1. A functor F is called *exact* if it preserves the exactness of all short exact sequences. If the functor only preserves all split exact sequences, we say that it is *split exact*. If the only exactness preserved is exactness in the middle object, we call the functor *half exact*.

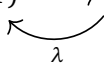
We come back to the half exactness of K_0 when we have amended these problems with a new definition of K_0 .

Example 5.5.2. (Problems with half exactness of K_0 for nonunital C^* -algebras)

We refer to [35] for the details, but we explain the idea behind finding the problems for nonunital C^* -algebras.

For example, if we let X be a connected, compact Hausdorff space and x_0 being any point in X , we can consider the split exact sequence given by

$$0 \longrightarrow C_0(X \setminus \{x_0\}) \xrightarrow{i} C(X) \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$



We have defined $\pi(f) = f(x_0)$ and λ to be the function lifting a scalar α to the constant function sending everything to α . With some work, one can show the following diagram commutes:

$$\begin{array}{ccc} K_0(C(X)) & & \\ K_0(\pi) \downarrow & \searrow \text{dim} & \\ K_0(\mathbb{C}) & \xrightarrow{K_0(Tr)} & \mathbb{Z} \end{array}$$


Since we know $K_0(Tr)$ is an isomorphism, we get $\text{Ker}(\text{dim}) = \text{Ker}(K_0(\pi))$, where dim is defined in example 5.4.15. With some more work, for example, showing the K_0 -group of the nonunital C^* -algebra $C_0(X \setminus \{x_0\})$ is zero, we can show that the induced sequence is exact in the middle object $K_0(C(X))$ if and only if the map $\text{dim} : K_0(C(X)) \rightarrow \mathbb{Z}$ is an isomorphism. With a lot more work, we can show that $K_0(C(S^2)) = \mathbb{Z} \oplus \mathbb{Z}$, which means that in this case, dim will not even be injective. In cases like this, we will not have exactness even in the middle term of the induced sequence.

If it is not even possible to guarantee some sort of exactness of K_0 for simple spaces such as $C(S^2)$, we lose up to all computational power from the analogue of the standard tools of algebraic topology through exact sequences.

We amend these problems as fast as possible, without changing our successful approach for unital C^* -algebras.

Definition 5.5.3. Let A be a nonunital C^* -algebra and consider the associated short exact sequence

$$0 \longrightarrow A \xrightarrow{i} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$



Define $K_0(A) = \text{Ker}[K_0(\pi) : K_0(\tilde{A}) \rightarrow K_0(\mathbb{C})]$, for all C^* -algebras A , independently of whether it is unital or not.

Remark 5.5.4. Since $K_0(A)$ is a kernel in the category of abelian groups, it is certainly an abelian subgroup of $K_0(\tilde{A})$. We get an induced map $[\cdot]_0 : \mathcal{P}_\infty(A) \rightarrow K_0(A)$ by the universal property of the kernel, since all $p \in \mathcal{P}_\infty(A)$ induces a class $[p]_0 \in K_0(\tilde{A})$,

which is mapped to 0 by $K_0(\pi)$. Readers with experience from topological K -theory should note that this resembles reduced K -theory, which intuitively corresponds to K -theory modulo trivial bundles.

We still want our earlier results on functoriality and homotopy invariance to hold.

To get the induced maps in K_0 -theory for nonunital C^* -algebras, note that for all C^* -algebras A , we have the following short exact sequence,

$$0 \longrightarrow K_0(A) \xrightarrow{?} K_0(\tilde{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \longrightarrow 0.$$

\curvearrowright
 $K_0(\lambda)$

The first map is either $K_0(i)$ or the inclusion, depending on whether or not A is unital.

To get the induced map in the nonunital case, just note that if $\tilde{\phi}$ is the induced map on the unitizations coming from the map ϕ , we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{i_A} & \tilde{A} & \xrightarrow{\pi_A} & \mathbb{C} \longrightarrow 0 \\ \parallel & & \phi \downarrow & & \tilde{\phi} \downarrow & & \parallel \\ 0 & \longrightarrow & B & \xrightarrow{i_B} & \tilde{B} & \xrightarrow{\pi_B} & \mathbb{C} \longrightarrow 0. \end{array}$$

This induces a commutative diagram in K_0 -theory.

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(\tilde{A}) & \xrightarrow{K_0(\pi_A)} & K_0(\mathbb{C}) \longrightarrow 0 \\ \parallel & & K_0(\phi) \dashrightarrow & & K_0(\tilde{\phi}) \downarrow & & \parallel \\ 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(\tilde{B}) & \xrightarrow{K_0(\pi_B)} & K_0(\mathbb{C}) \longrightarrow 0. \end{array}$$

The dashed arrow, $K_0(\phi)$, exists and is unique by $K_0(B)$ being the kernel of $K_0(\pi_B)$ and the composition $K_0(A) \rightarrow K_0(\tilde{A}) \rightarrow K_0(\tilde{B}) \rightarrow K_0(\mathbb{C})$ being zero.

Since the morphism is unique and the map $[p]_0 \mapsto [\phi(p)]_0$ fits here, we still get $K_0(\phi)([p]_0) = [\phi(p)]_0$.

In the same way as earlier (proposition 5.4.6), we get that K_0 must be a homotopy invariant functor.

Proposition 5.5.5. (Functoriality of K_0)

1. If A is a C^* -algebra, then $K_0(Id_A) = Id_{K_0(A)}$, where Id_A is the identity map on A .
2. If A, B, C are C^* -algebras and $\phi : A \rightarrow B$, $\psi : B \rightarrow C$ are $*$ -homomorphisms, then K_0 preserves composition, that is, $K_0(\psi \circ \phi) = K_0(\psi) \circ K_0(\phi)$.

$$3. K_0(\{0\}) = 0 \text{ and } K_0(0_{B,A} : A \rightarrow B) = 0_{K_0(B), K_0(A)} : K_0(A) \rightarrow K_0(B).$$

Proof. The proof is extremely similar to proposition 5.4.6 and thus omitted. \square

Proposition 5.5.6. Assume A and B are C^* -algebras. Then

1. if $\psi, \phi : A \rightarrow B$ are homotopic $*$ -homomorphisms, then $K_0(\psi) = K_0(\phi)$.
2. if A, B are homotopy equivalent through $\phi : A \rightarrow B$ and $\psi : B \rightarrow A$, then $K_0(A) \cong K_0(B)$ and $K_0(\phi), K_0(\psi)$ are isomorphisms with $K_0(\phi)^{-1} = K_0(\psi)$.

Proof. The proof is extremely similar to the proof of proposition 5.4.9, and is thus omitted. \square

We still have functoriality, but we must also hope the powerful standard picture of $K_0(A)$ holds, at least in a similar manner.

First, consider the split exact sequence

$$0 \longrightarrow A \xrightarrow{i} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$

$\begin{array}{c} \longleftarrow \lambda \\ \text{---} \end{array}$

We can return to \tilde{A} by the composition $s = \lambda \circ \pi$, which we will call the *scalar mapping*, which will be defined by $s(a + z1) = z1$ for all $a \in A, z \in \mathbb{C}$. Then we can see that $\pi(s(x)) = \pi(x)$ and $x - s(x) \in A$ for all $x \in \tilde{A}$. One can note that the scalar mapping is natural in the sense that for a $*$ -homomorphism $\phi : A \rightarrow B$, we get a commutative diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{s} & \tilde{A} \\ \tilde{\phi} \downarrow & & \downarrow \tilde{\phi} \\ \tilde{B} & \xrightarrow{s} & \tilde{B}. \end{array}$$

We can extend this scalar mapping the standard way to $\mathcal{P}_\infty(\tilde{A})$. The standard picture of $K_0(A)$ is given in terms of the scalar mapping and scalar elements, i.e. in terms of elements such that $x = s(x)$.

Proposition 5.5.7. Given a C^* -algebra A ,

$$K_0(A) = \{[p]_0 - [s(p)]_0 : p \in \mathcal{P}_\infty(\tilde{A})\}.$$

In addition, the following hold:

1. For each pair of projections $p, q \in \mathcal{P}_\infty(\tilde{A})$, the following are equivalent:
 - $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$,
 - there exists k, l such that $p \oplus 1_k \sim_0 q \oplus 1_l$ in $\mathcal{P}_\infty(\tilde{A})$,

- there exists projections r_1, r_2 such that $p \oplus r_1 \sim_0 q \oplus r_2$, where r_1, r_2 are scalar elements.
2. If $p \in \mathcal{P}_\infty(\tilde{A})$ satisfies $[p]_0 - [s(p)]_0 = 0$, then we can find m such that $p \oplus 1_m \sim s(p) \oplus 1_m$.
 3. Given a $*$ -homomorphism ϕ ,

$$K_0(\phi)([p]_0 - [s(p)]_0) = [\tilde{\phi}(p)]_0 - [s(\tilde{\phi}(p))]_0.$$

Proof. This is quite a long proof. Interested readers are referred to [35], while we continue our quest towards proving the half exactness of K_0 . \square

Remark 5.5.8. Readers with experience from topological K -theory should note that these scalar elements work as an analogue of trivial bundles if we restrict ourselves to compact Hausdorff spaces. In this case, the standard picture in topological K -theory is given by a difference of equivalence classes of a vector bundle and a trivial bundle.

We borrow this slightly technical lemma from [35], which we will need when working with elements in a kernel to prove half exactness.

Lemma 5.5.9. Suppose $\phi : A \rightarrow B$ is a $*$ -homomorphism and suppose we choose $g \in \text{Ker } K_0(\phi)$. Then

1. there exists a natural number n , a projection p in $\mathcal{P}_n(\tilde{A})$ and a unitary $u \in M_n(\tilde{B})$ such that $u\tilde{\phi}(p)u^* = s(\tilde{\phi}(p))$ and $g = [p]_0 - [s(p)]_0$.
2. if ϕ is surjective, we can find a $p \in \mathcal{P}_\infty(\tilde{A})$ such that $\tilde{\phi}(p)$ is a scalar element and $g = [p]_0 - [s(p)]_0$.

The following lemma will be helpful as well.

Lemma 5.5.10. Given a short exact sequence of C^* -algebras,

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0,$$

the following hold:

1. $\tilde{\phi}_n : M_n(\tilde{I}) \rightarrow M_n(\tilde{A})$ is injective.
2. An element a belongs to $\text{Im}(\tilde{\phi}_n)$ if and only if $\tilde{\psi}_n(a)$ is a scalar element in $M_n(\tilde{B})$.

Proposition 5.5.11. (Half exactness of K_0) Given a short exact sequence of C^* -algebras,

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0,$$

we get an exact induced sequence,

$$K_0(I) \xrightarrow{K_0(\phi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B),$$

in the sense that $\text{Im}(K_0(\phi)) = \text{Ker}(K_0(\psi))$.

Proof. $\text{Im}(K_0(\phi))$ is certainly contained in $\text{Ker}(K_0(\psi))$ by functoriality, since

$$K_0(\psi) \circ K_0(\phi) = K_0(\psi \circ \phi) = K_0(0) = 0.$$

Conversely, take $g \in \text{Ker}(K_0(\psi))$ and apply lemma 5.5.9 to find a natural number n and a projection $p \in \mathcal{P}_n(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $\tilde{\psi}(p)$ is a scalar element. Now, by lemma 5.5.10, p must be in the image of $\tilde{\phi}$, which means we can find a preimage $e \in M_n(\tilde{I})$. This preimage is unique, which implies that e is a projection, by writing out the self-adjointness and idempotency of e . This lets us write

$$g = [\tilde{\phi}(e)]_0 - [s(\tilde{\phi}(e))]_0 = K_0(\phi)([e]_0 - [s(e)]_0) \in \text{Im}(K_0(\phi)).$$

Hence K_0 is half exact. □

It should be reasonable to expect only half-exactness, at least as an analogue to topological K -theory. In $hTop$, we can't make sense of short exact sequences. We have to work with pairs, yielding natural three-term sequences, but not in an exact sense. Usually, we can't just guarantee exactness at the endpoints of exact sequences coming from $hTop$, which means the natural notion of exactness to expect is half-exactness.

From half-exactness, it follows that this functor is split exact.

Proposition 5.5.12. (Split exactness of K_0)

For every split exact sequence of C^* -algebras,

$$0 \longrightarrow I \xrightarrow{\phi} A \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\psi} \end{array} B \longrightarrow 0,$$

we get an induced split exact sequence of abelian groups,

$$0 \longrightarrow K_0(I) \xrightarrow{K_0(\phi)} K_0(A) \begin{array}{c} \xleftarrow{K_0(\lambda)} \\ \xrightarrow{K_0(\psi)} \end{array} K_0(B) \longrightarrow 0.$$

Proof. Exactness in the middle term follows from proposition 5.5.11 and functoriality yields exactness in the last term, since

$$\text{Id}_{K_0(B)} = K_0(\text{Id}_B) = K_0(\psi) \circ K_0(\lambda).$$

For a proof of injectivity, readers are referred to [35]. □

Proposition 5.5.13. Given C^* -algebras A and B , we have

$$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B).$$

More specifically, the canonical inclusion maps $i_A, i_B : A, B \rightarrow A \oplus B$ will yield the isomorphism through $K_0(i_A) \oplus K_0(i_B) : K_0(A) \oplus K_0(B) \rightarrow K_0(A \oplus B)$.

Proof. Consider the diagram

$$\begin{array}{ccccccc}
 & & & K_0(A) \oplus K_0(B) & & & \\
 & & \nearrow & \downarrow & \searrow & & \\
 0 & \longrightarrow & K_0(A) & & K_0(B) & \longrightarrow & 0 \\
 & & \searrow & \downarrow & \nearrow & & \\
 & & & K_0(A \oplus B) & & &
 \end{array}$$

$K_0(i_A) \oplus K_0(i_B)$

and note that it commutes as \oplus defines a biproduct in the category of abelian groups. Therefore, by the five lemma or some standard diagram chase, we get that the wanted map is an isomorphism. \square

Remark 5.5.14. Note that this implies $K_0(\tilde{A}) \cong K_0(A) \oplus \mathbb{Z}$ since $K_0(\mathbb{C}) \cong \mathbb{Z}$.

Before we consider examples of when exactness fails at each end of the sequence, we state a useful stability property of K_0 from [35].

Proposition 5.5.15. (Stability) Fix a C^* -algebra A and a natural number n . Then the $*$ -homomorphism

$$\lambda_{n,A} : A \rightarrow M_n(A), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

induces an isomorphism $K_0(\lambda_{n,A}) : K_0(A) \rightarrow K_0(M_n(A))$.

More generally, let $i_{\mathcal{K}} : A \rightarrow \mathcal{K}A$ is the canonical inclusion of A into its stabilization, $\mathcal{K}A$, which can be shown to be $\mathcal{K} \otimes A$, where \mathcal{K} denotes the compact operators on some Hilbert space. Then the induced map $K_0(i_{\mathcal{K}}) : K_0(A) \rightarrow K_0(\mathcal{K}A)$ is an isomorphism.

Example 5.5.16. (Counterexample to right exactness)

Consider the short exact sequence

$$0 \longrightarrow C_0((0, 1)) \longrightarrow C([0, 1]) \xrightarrow{\phi} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0,$$

where $\phi(f) = (f(0), f(1))$.

By earlier computations (5.4.12 and 5.4.15), we know $K_0(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K_0(C([0, 1])) \cong \mathbb{Z}$. Therefore $K_0(\phi)$ is not a surjection, and thus we have a counterexample to right exactness.

Example 5.5.17. (Counterexample to right exactness)

Consider a separable, infinite dimensional Hilbert space \mathcal{H} , let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ denote the algebra of compact operators and let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ and $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ denote the algebra of bounded operators and the *Calkin algebra*, respectively.

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{B} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

We know from example 5.4.13 that $K_0(\mathcal{B}) = 0$. With some work on stabilizations of C^* -algebras as mentioned in proposition 5.5.15, we can show that $K_0(\mathcal{K}) \cong \mathbb{Z}$, which means we have a counterexample to left exactness, as the induced map cannot be injective.

This completes our first steps towards K -theory. We will soon enough (in chapter 7) develop higher K -groups and work out computational tools such as the Pimsner-Voiculescu sequence which we will later apply to noncommutative tori in chapter 8, but we first work through some important applications of the K_0 -theory.

Queequeg 5.5.18. Even though we only define the foundations of operator K -theory in this and the coming chapters, there is no need to stop there. It is possible to work out some sort of universal coefficient theorem, Künneth formula, an analogue to the Chern character, and more useful theory from algebraic topology and homological algebra. To do this, one would have to develop a bunch of new theories, such as extension theory in the sense of Brown-Douglas-Fillmore, KK-theory in the sense of Kasparov, and noncommutative geometry in the sense of Connes ([4]). Interested readers are referred to the encyclopedia-like book by Blackadar ([1]) for these constructions and further references.

Chapter 6

Important applications of K_0 -theory

Before we move on to higher K -theory, the computational tools and consequences for noncommutative tori, we briefly establish some important applications of the K_0 -theory. We will first explore the notions of finite, infinite, and stably finite algebras through projections and ordered groups, which lets us realize $K_0(A)$ and its cone as an ordered group structure. After a brief discussion of colimits and the continuity of K_0 , we move on to AF-algebras and the classification theorem by Elliott. This will be vital to classifying noncommutative tori, even though these are not AF-algebras. All stated propositions are to be found mainly in [35], but we also refer to [1].

6.1 Stably finite C^* -algebras and ordered K_0 -groups

Our aim is the classification theorem of Elliott on AF-algebras, which are certain C^* -algebras that arise as a limit of finite-dimensional algebras, as we will define later. To be able to study these in-depth through K_0 -theory, we need to go back a step to consider some questions related to how we constructed K_0 .

When defining K_0 of a C^* -algebra, we first reversed the hierarchy of equivalences (5.2.9) by passing to matrices (5.2.11). To fix dimensional issues, we considered projections in all matrix rings, which seemed to be justified by the stability property of K_0 (5.5.15).

We didn't really dwell on what the possible "infinite projections" would look like, but to be able to append more structure (and hence more computational power) to K_0 , we briefly return to the topic.

Definition 6.1.1. Let p be a projection in a C^* -algebra A . If p is equivalent to a proper subprojection of itself, we say p is infinite. That is, p is infinite if there exists

a projection q in A such that $p \sim q < p$. We say p is finite if it is not infinite.

We say a unital C^* -algebra A is finite if 1_A is a finite projection. If not, A is called infinite. A is called stably finite if it stays finite when we pass it to matrices, that is, if $M_n(A)$ is finite for all natural n .

For nonunital C^* -algebras, we consider finiteness in terms of their unitizations.

Finiteness of projections is the key property that lets us determine whether or not a C^* -algebra is finite. We borrow this presentation of equivalent statements for unital C^* -algebras from [35].

Lemma 6.1.2. If A is a unital C^* -algebra, the following are equivalent:

1. A is finite.
2. All isometries in A are unitary.
3. All projections in A are finite.
4. Every left-invertible element of A is invertible.
5. Every right-invertible element of A is invertible.

Remark 6.1.3. One should note that the requirement of A being unital cannot be loosened. We always have the implication [A is finite] \implies [All projections in A are finite], but for nonunital C^* -algebras, there are some examples where the converse do not hold, such as the Toeplitz algebra, which we leave to interested readers to investigate.

The fact that we need to define stably finite C^* -algebras suggests that there exist algebras that lose their finiteness when passed to a matrix ring of some dimension n . These certainly exist, but they are not the easiest to construct.

Before we consider the connection to $K_0(A)$, we need to define ordered groups.

Definition 6.1.4. Let G be a group. A pair (G, G^+) is called an *ordered abelian group* if G is abelian, G^+ is a subset of G and the following claims hold:

- $G^+ + G^+ \subseteq G^+$
- $G^+ \cap (-G^+) = 0$
- $G^+ - G^+ = G$

This lets us define a partial order on G by saying $x \leq y \iff y - x \in G^+$.

Let $u \in G^+$ be an element in an ordered abelian group (G, G^+) such that for all $g \in G$, we can find an n with $-nu \leq g \leq nu$. Then we call u an order unit and the triple (G, G^+, u) is called an ordered abelian group with a distinguished order unit.

Ordered abelian groups are simple if all nonzero elements are order units.

Example 6.1.5. Consider an abelian group G with an order relation satisfying $x + z \leq y + z$ whenever $x \leq y$ and $x, y, z \in G$. If we define $G^+ = \{x \in G : x \geq 0\}$,

then (G, G^+) will satisfy at least the two first axioms of being an ordered abelian group.

The intuitive example of an ordered abelian group is given by considering $(\mathbb{Z}, \mathbb{N}_0)$.

In the light of this definition, we recall the standard picture of K_0 (proposition 5.5.7), which makes it natural to define the following:

Definition 6.1.6. Let A be a C^* -algebra and define the *positive cone* of $K_0(A)$ to be the set

$$K_0(A)^+ = \{[p]_0 : p \in \mathcal{P}_\infty(A)\} \subseteq K_0(A).$$

As one would probably expect from a construction expected to tell us more about the structure, the positive cone plays nicely with biproducts.

Proposition 6.1.7. The positive cone of $K_0(A \oplus B)$ is given by

$$K_0(A \oplus B)^+ \cong K_0(A)^+ \oplus K_0(B)^+.$$

It is natural to ask which criteria we can put on A to interpret $(K_0(A), K_0(A)^+)$ as an ordered abelian group.

Proposition 6.1.8. Let A be a C^* -algebra.

1. $K_0(A)^+ + K_0(A)^+ \subseteq K_0(A)^+$
2. If A is unital, then $K_0(A)^+ - K_0(A)^+ = K_0(A)$ and $[1_A]$ is an order unit for $K_0(A)$.
3. If A is stably finite, then $K_0(A)^+ \cap (-K_0(A))^+ = 0$.

Therefore, if A is unital and stably finite, $(K_0(A), K_0(A)^+, [1_A]_0)$ is an ordered abelian group with distinguished order unit $[1_A]_0$.

To be able to talk about isomorphic order groups later on, we need to define some notions related to morphisms.

Definition 6.1.9. Let (G, G^+) and (H, H^+) be ordered abelian groups. A group homomorphism $\phi : G \rightarrow H$ is called *positive* if $\phi(G^+) \subseteq H^+$. If ϕ is an isomorphism $G \rightarrow H$ and $\phi(G^+) = H^+$, we call ϕ an *order isomorphism*. If u, v denotes the order units of these ordered abelian groups, respectively, we say ϕ is *order unit preserving* if $\phi(u) = v$.

(G, G^+, u) and (H, H^+, v) are said to be isomorphic if there exists an order unit preserving order isomorphism between G and H .

Remark 6.1.10. By the definition of the induced maps in K_0 -theory, we see that for a $*$ -homomorphism $\phi : A \rightarrow B$, $K_0(\phi) : K_0(A) \rightarrow K_0(B)$ is a positive group homomorphism.

Given an isomorphism ϕ , functoriality implies that $K_0(\phi)$ is an order isomorphism. If, in addition, the C^* -algebra is unital, $K_0(\phi)$ is order unit preserving.

The triple $(K_0(A), K_0(B)^+, [1_A]_0)$ is an isomorphism invariant of A .

Queequeg 6.1.11. We can develop this a bit further and show that all states on the ordered group $(K_0(A), K_0(A)^+, [1_A]_0)$ for unital C^* -algebras A can be realized as $K_0(\tau)$ for some quasi-trace τ on A . If in addition A is exact in the sense that the endofunctor $B \mapsto B \otimes A$ is exact, where \otimes denotes what's called the minimal tensor product of C^* -algebras, then τ can be chosen to be a trace and not just a quasi-trace. We refer to [35] and the references therein for further elaboration.

6.2 AF-algebras and classification

Before we are ready to define AF-algebras, we need a quick detour to category theory and the definition of a colimit.

Definition 6.2.1. An *inductive sequence* in a category is a sequence of objects $\{A_n\}_{n=1}^\infty$ with morphisms $\phi_n : A_n \rightarrow A_{n+1}$. These define connecting morphisms $\phi_{m,n} = \phi_{m-1} \circ \dots \circ \phi_n : A_n \rightarrow A_m$.

Definition 6.2.2. A *colimit* (or *inductive limit*) of an inductive sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \dots$$

is a system $(A, \{\mu_n\}_{n=1}^\infty)$ consisting of an object A and morphisms $\mu_n : A_n \rightarrow A$ such that the following hold:

1. We have a commutative diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\phi_n} & A_{n+1} \\ & \searrow \mu_n & \swarrow \mu_{n+1} \\ & A & \end{array}$$

2. If $(B, \{\lambda_n\}_{n=1}^\infty)$ is a system satisfying the same property, there exists a unique morphism $\lambda : A \rightarrow B$ such that

$$\begin{array}{ccc} & A_n & \\ & \swarrow \phi_n & \searrow \lambda_n \\ A & \xrightarrow{\lambda} & B \end{array}$$

commutes.

Remark 6.2.3. Indeed, one can define colimits in a category \mathcal{C} a more categorical way as the left adjoint to the constant functor, that is, if we let \mathcal{X} denote some (small) index category, we can define the colimit as the left adjoint to $\Delta(-) : \mathcal{C} \rightarrow \text{presh}_{\mathcal{C}} \mathcal{X}$. Limits are defined dually.

Note that due to the second property of the colimit, it is certainly unique up to isomorphism.

The colimit does not need to exist in a category. To see this, just consider the attempt at finding an inductive limit in the category of finite sets with $A_n = \{1, \dots, n\}$ and ϕ_n as the inclusion $A_n \hookrightarrow A_{n+1}$. The limit could not possibly be a finite set, and hence not in the category.

Therefore we have the following definition.

Definition 6.2.4. A category is called (co-) complete if it contains all of its (co-) limits.

Nevertheless, all the categories we will work with will be cocomplete and satisfy nice properties, some of which are mentioned here.

Proposition 6.2.5. The categories of C^* -algebras, abelian groups, and ordered abelian groups satisfy the following properties.

- C^*Alg is a cocomplete category and furthermore,
 1. $\|\mu_n(a)\| = \lim_{m \rightarrow \infty} \|\phi_{m,n}(a)\|$ for all natural numbers n and all $a \in A_n$,
 2. $A = \bigcup_{n=1}^{\infty} \mu_n(A_n)$,

where $\phi_{m,n}$ denotes the composition $\phi_{m,n} : A_n \rightarrow A_m$ for $m \geq n$ and $(A, \{\mu_n\}_{n=1}^{\infty})$ denotes the limit.

- Ab is a cocomplete category and furthermore,

1. $G = \bigcup_{n=1}^{\infty} \mu_n(G_n)$,
2. $\text{Ker}(\mu_n) = \bigcup_{m=n+1}^{\infty} \text{Ker}(\phi_{m,n})$,

where ϕ_n denotes the map $\phi_n : G_n \rightarrow G_{n+1}$ and $(G, \{\mu_n\}_{n=1}^{\infty})$ denotes the limit.

- $OrdAb$, the category of ordered abelian groups with positive group homomorphisms as morphisms, is a cocomplete category. Furthermore, the colimit is given by $((G, G^+), \{\mu_n\})$, where $(G, \{\mu_n\})$ denotes the colimit in Ab and $G^+ = \bigcup_{n=1}^{\infty} \mu_n(G_n^+)$.

With the knowledge of these colimits and cocomplete categories, it is now interesting (and well-defined) to consider the C^* -algebras arising as colimits of finite-dimensional algebras.

Definition 6.2.6. An *AF-algebra* (or *Approximately Finite-dimensional algebra*) is a C^* -algebra isomorphic to the colimit of a sequence of finite-dimensional C^* -algebras.

Before we move on to examples and connections to K_0 -theory, we present an equivalent statement for separable C^* -algebras from [35] due to Ola Bratteli.

Proposition 6.2.7. A separable C^* -algebra is an AF-algebra if and only if for every $\epsilon > 0$ and every finite subset $\{a_1, a_2, \dots, a_n\}$ of A , there exists a finite dimensional sub- C^* -algebra B of A and elements b_1, \dots, b_n in B such that $\|a_i - b_i\| < \epsilon$ for all i .

This justifies the name Approximately Finite-dimensional algebra, as it is certainly approximately finite-dimensional.

Now, let us consider some examples.

Example 6.2.8. A trivial example could be to take a finite-dimensional C^* -algebra A and let $A_n = A$ with the identity between A_n and A_{n+1} . Then, by magic, A becomes the colimit.

Example 6.2.9. We know the C^* -algebra of compact operators on a (separable) Hilbert space, $\mathcal{K}(\mathcal{H})$, comes from finite rank operators. It is certainly possible to realize this as an AF-algebra.

Example 6.2.10. Consider the sequence where $A_n = M_{2^n}(\mathbb{C})$ and the connecting homomorphism $\phi : A_n \rightarrow A_{n+1}$ is given by $\phi(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.

The inductive limit of this sequence is called the CAR-algebra, where CAR stands for "canonical anticommutation relations", which we explain in Queequeg 6.2.18.

This is a classical example of an AF-algebra ([5]), but its history and interest go a lot deeper than the author understands at the current point.

Queequeg 6.2.11. It is possible to show that commutative C^* -algebras are AF if their spectrum is totally disconnected. By "spectrum", we mean the spectrum of the C^* -algebra, which is defined in the appendix of [31]. This is defined in terms of the primitive spectrum of prime ideals with the hull-kernel topology, which is a noncommutative analogue of the Zariski topology. For commutative C^* -algebras, the spectrum is exactly the primitive spectrum. This holds for example for continuous functions a Cantor set, which is an extremely interesting C^* -algebra related to dynamical systems ([6]).

Since finite-dimensional algebras are direct sums of matrix algebras (proposition 2.3.5), we can wonder if K_0 respects colimits, as this would give us immense computational power for AF-algebras since the K_0 -group would be easy to study. Since the Grothendieck functor is a left adjoint, this does in fact hold.

Theorem 6.2.12. (Cocontinuity of K_0) Given an inductive sequence of C^* -algebras, $K_0(\lim_{\rightarrow} A_n) \cong \lim_{\rightarrow} K_0(A_n)$ as abelian groups. If each $(K_0(A_n), K_0(A_n)^+)$ are ordered abelian groups, then this isomorphism also holds in the category of ordered abelian groups. Furthermore,

1. $K_0(A) = \bigcup_{n=1}^{\infty} K_0(\mu_n)(K_0(A_n))$
2. $K_0(A)^+ = \bigcup_{n=1}^{\infty} K_0(\mu_n)(K_0(A_n)^+)$

Remark 6.2.13. If we are given an AF-algebra, we have an inductive sequence of C^* -algebras, which gives us an induced sequence of abelian groups by applying K_0 . Now we only need to take the limit of this sequence, which will be much easier, as we know a lot about finite-dimensional C^* -algebras and K_0 . For example, if A is finite-dimensional,

$$\begin{aligned}
K_0(A) &\cong K_0\left(\bigoplus_{k=1}^K M_{N_k}(\mathbb{C})\right) && \text{(since } A \text{ is finite-dimensional)} \\
&\cong \bigoplus_{k=1}^K K_0(M_{N_k}(\mathbb{C})) && \text{(since } K_0 \text{ preserves biproducts)} \\
&\cong \bigoplus_{k=1}^K K_0(\mathbb{C}) && \text{(by stability of } K_0) \\
&\cong \bigoplus_{k=1}^K \mathbb{Z} && \text{(since } K_0(\mathbb{C}) \cong \mathbb{Z} \text{ is known),}
\end{aligned}$$

which implies that we should need to consider sequences of abelian groups on the form $A_n = \mathbb{Z}^{K_n}$ when studying the K_0 -theory of AF-algebras.

This leads to the definition of a dimension group, which certainly is a step in the right direction of understanding the structure of AF-algebras.

Definition 6.2.14. A *dimension group* is an ordered abelian group isomorphic to the colimit of a sequence of ordered abelian groups of the form

$$\mathbb{Z}^{n_1} \xrightarrow{\alpha_1} \mathbb{Z}^{n_2} \xrightarrow{\alpha_2} \mathbb{Z}^{n_3} \xrightarrow{\alpha_3} \dots,$$

where n_i are positive integers, α_i are positive group homomorphisms and $(\mathbb{Z}^n)^+$ is given the usual way as n -tuples of nonnegative integers.

One can certainly ask how strong the connection between K_0 -groups and dimension groups is. There are, perhaps surprisingly, powerful and beautiful answers to questions like these.

First, note that all AF-algebras are stably finite since they are colimits of finite-dimensional C^* -algebras, which are direct sums of matrix algebras over \mathbb{C} . We know the identity is a finite projection in each of these matrix algebras since traces are constant on equivalence classes by the trace property and the definition of the Murray-von Neumann equivalence. Any proper subprojection of the identity would not yield the same value under the standard trace, and can hence not be equivalent to the identity. Therefore, each of these matrix algebras are finite, which means that each finite-dimensional C^* -algebra is finite in the sense of definition 6.1.1. Now, by proposition 6.2.5, the colimit must also be stably finite.

This implies that for an AF-algebra A , $(K_0(A), K_0(A)^+)$ is an ordered abelian group.

Proposition 6.2.15. If A is an AF-algebra, the ordered K_0 -group $(K_0(A), K_0(A)^+)$ is a dimension group. Conversely, every dimension group is order isomorphic to the ordered K_0 -group of some AF-algebra.

With some more work and a couple of technical lemmas, it is possible to prove Elliott's classification theorem through the power of these dimension groups.

Theorem 6.2.16. (Elliott) Two unital AF-algebras A and B are isomorphic if and only if the triples $(K_0(A), K_0(A)^+, [1_A]_0)$ and $(K_0(B), K_0(B)^+, [1_B]_0)$ are isomorphic.

Moreover, if there is an order unit preserving order isomorphism,

$$\alpha : (K_0(A), K_0(A)^+, [1_A]_0) \rightarrow (K_0(B), K_0(B)^+, [1_B]_0),$$

we can find a $*$ -isomorphism $\phi : A \rightarrow B$ satisfying $K_0(\phi) = \alpha$.

Remark 6.2.17. Even though Elliott's theorem (6.2.16) requires the AF-algebras to be unital, it is still possible to give such a classification result in the nonunital case. We have to change our positive cone

$$K_0(A)^+ = \{[p]_0 : p \in \mathcal{P}_\infty(A)\} \subseteq K_0(A)$$

and the order unit to something else. If we define the *dimension range* of A to be

$$\mathcal{D}_0(A) = \{[p]_0 : p \in \mathcal{P}(A)\} \subseteq K_0(A)^+,$$

a similar theorem also holds for nonunital AF-algebras.

Two AF-algebras A and B are isomorphic if and only if the pairs $(K_0(A), \mathcal{D}_0(A))$ and $(K_0(B), \mathcal{D}_0(B))$ are isomorphic in the sense that it exists a group isomorphism $\alpha : K_0(A) \rightarrow K_0(B)$ with $\alpha(\mathcal{D}_0(A)) = \mathcal{D}_0(B)$.

We end this brief section on applications of K_0 -theory with the following dense remark and continue on our journey towards the higher K -groups and the computational tools we have been dreaming about for the last $n + 1$ pages.

Queequeg 6.2.18. Interested readers should certainly consider learning about the Effros-Handelman-Shen theorem, which classifies exactly when ordered abelian groups are dimension groups, but we leave this excursion to [35] and [1].

AF-algebras and classifications are topics of wide interest when studying dynamical systems. Measurable dynamical systems that are irreducible, in the sense that they are ergodic, are closely linked to something called Cantor Minimal Systems. There are a lot of interesting AF-algebras, both connected to Cantor Minimal Systems and not. These are perhaps better explained diagrammatically in terms of Bratteli diagrams, named after the Norwegian mathematician Ola Bratteli. We refer readers to sources such as [6] for treatments of these topics.

The AF-algebra we saw in example 6.2.10 can actually be realized in some way as a model for fermionic systems in quantum mechanics. One usually considers the

space of quantum states to be some separable Hilbert space, which in a technical sense must be a so-called rigged space for us to be able to talk about the Dirac delta distribution as a potential function. We know there can only exist half-integer spin particles, called fermions, or integer spin particles, called bosons. Quantum states of fermions satisfy the canonical anticommutation relation and bosonic quantum states satisfy the canonical commutation relation. The algebras generated by these quantum states are called CAR- and CCR-algebras, respectively. For a treatise on these topics, we refer to [38] and [26].

With some poetic license, these canonical anticommutation relations are the reason for the Pauli principle, which in turn is the reason why we (or anything fun for that matter) exist.

Chapter 7

Higher K -groups and tools in operator K -theory

In this chapter, we develop the computational tools that let us compute several K_0 -groups. To do this, we are still inspired by topological K -theory and the computational methods in algebraic topology through exact sequences. As we will see, the topological approach boils down to defining the K_1 -groups in an analogous way to the K_0 -group. This is done quite briefly, as many of the ideas are found in chapter 5. We refer to [35] for missing details.

This definition of K_1 can be connected to K_0 through an index map, which yields the first steps towards the long exact sequence in K -theory. We find an isomorphism $K_0(SA) \cong K_1(A)$, where S denotes the suspension endofunctor in the category of C^* -algebras. This inspires us to define higher K -groups and lift the index map to obtain the long exact sequence in K -theory, the beautiful Bott periodicity theorem, and more. For these first parts, we continue to follow [35] and [1]. To converge the thesis slowly towards the classification of noncommutative tori, we develop the computational tools further by adapting them to crossed product C^* -algebras and we derive the Pimsner-Voiculescu exact sequence.

7.1 The K_1 -theory and the index map

It is natural to try to define the K_0 -theory in terms of projections since these projections show up directly from the finitely generated projective modules corresponding to vector bundles in the Serre-Swan theorem (5.1.1). In topological K -theory, $K_1(X)$ is defined similarly to $K_0(X)$, but in terms of the suspension of the compact Hausdorff space X . This does not necessarily tell us anything interesting about our C^* -algebras at first glance, but we try to bend this towards a more operator algebraic approach to K_1 .

We refer readers to [43] for the following proposition.

Proposition 7.1.1. There is a bijection between k -dimensional vector bundles on the suspension of X and the homotopy class of maps from a compact Hausdorff space X into the group of invertible elements in $M_k(\mathbb{C})$, called clutching functions. More precisely, if $GL(A)$ denotes the invertible elements in A and $GL_k(A) = GL(M_k(A))$,

$$\text{Vect}_k(SX) \cong [X, GL_k(\mathbb{C})] \quad \text{as sets.}$$

If we forget the contravariant structure and just consider $GL(A)$, we turn the contravariant topological K^1 -functor to a covariant one, which explains why the approach we take yields a covariant functor.

Queequeg 7.1.2. The result above can probably be considered as a special case of the classification of principal G -bundles in algebraic topology. We refer to [18] for a topological treatise of bundle theory and homotopy theory.

The following proposition connects $GL(A)$ to the unitary elements by saying $\mathcal{U}(A)$ is a retract of $GL(A)$ that preserves homotopies coming from $GL(A)$. The element $|a|$ is defined as $|a| = (a^*a)^{1/2}$, where the square root makes sense due to the functional calculus (proposition 2.3.14).

Proposition 7.1.3. Let A be a unital C^* -algebra.

1. If z is an invertible element in A , then so is $|z|$, and $\omega(z) = z|z|^{-1}$ ends in $\mathcal{U}(A)$. Note that this implies the polar decomposition of elements in A , $z = \omega(z)|z|$.
2. The map $\omega : GL(A) \rightarrow \mathcal{U}(A)$ is continuous, $\omega(u) = u$ for $u \in \mathcal{U}(A)$, and $\omega(z) \sim_h z$ in $GL(A)$ for $z \in GL(A)$.
3. If $u, v \in \mathcal{U}(A)$ and $u \sim_h v$ in $GL(A)$, then $u \sim_h v$ in $\mathcal{U}(A)$ as well.

Proof. We prioritize giving a proof of this proposition, as it justifies our unitary-based approach to K_1 .

1. If z is invertible, then z^*z is invertible, and hence $|z| = (z^*z)^{1/2}$ is invertible with inverse explicitly given by $((z^*z)^{-1})^{1/2}$. Now define $\omega(z) = z|z|^{-1}$ and compute.

$$\omega(z)^* \omega(z) = |z|^{-1} z^* z |z|^{-1} = |z|^{-1} |z|^2 |z|^{-1} = 1,$$

as wanted.

2. We know multiplication in C^* -algebras is a continuous operation, meaning that the map $z \mapsto z^{-1}$ for $z \in GL(A)$ is continuous as well. If we want to show ω is continuous, we only need the fact that taking absolute values yield a continuous function. By considering the decomposition of the absolute value function, we see that it is the composition of continuous functions. Given an unitary u , we have $|u| = 1$, meaning $\omega(u) = u$ by definition.

For a fixed $z \in GL(A)$, define the continuous path $t \rightarrow z_t := \omega(z)(t|z| - (1-t) \cdot 1_A)$. We only need to show $z_t \in GL(A)$ for all $t \in [0, 1]$. Note that since

$|z|$ is positive and invertible, there exists a $\lambda \in (0, 1]$ such that $|z| \geq \lambda \cdot 1_A$ since the spectrum of $|z|$ is closed and bounded. Recall that we work in a C^* -algebra, meaning this inequality is phrased in terms of positivity. That is, λ exists since $0 \notin \text{spec}(|z|)$. This means that for each $t \in [0, 1]$, we have $t|z| + (1-t) \cdot 1_A \geq \lambda \cdot 1_A$. Hence z_t is invertible, so the continuous path $t \mapsto z_t$ is a homotopy $\omega(z) \sim_h z$ in $GL(A)$.

3. If we are given a continuous path $t \mapsto z_t$ between u, v in $GL(A)$, then $t \mapsto \omega(z_t)$ gives the wanted continuous path in $\mathcal{U}(A)$.

□

This allows us to take the geometric idea of defining K_1 in terms of vector bundles over suspended spaces and rather focus on attacking the problem through unitary elements in A . The idea of using $GL(A)$ to define K_1 turns out to be the right way to do it, as this is the theoretical basis for K_1 in algebraic K -theory, where K_1 is defined in terms of the abelianization of GL_∞ ([43]). We closely mimic the successful approach for projections in the K_0 -theory.

Definition 7.1.4. Let A be a unital C^* -algebra and let $\mathcal{U}(A)$ denote the groups of unitary elements. Define the following structures and binary operation.

$$\mathcal{U}_n(A) = \mathcal{U}(M_n(A)), \quad \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$$

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_{n+m}(A), \quad u \in \mathcal{U}_n(A), v \in \mathcal{U}_m(A)$$

Define a relation \sim_1 on $\mathcal{U}_\infty(A)$ by saying $u \sim_1 v$ for $u \in \mathcal{U}_n(A), v \in \mathcal{U}_m(A)$ if there exists a $k \geq \max\{m, n\}$ such that $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$ in $\mathcal{U}_k(A)$.

We refer to [35] for the following lemma on the properties of this construction. Note the similarity to the case with projections.

Lemma 7.1.5. If A is a unital C^* -algebra, then

1. \sim_1 is an equivalence relation on $\mathcal{U}_\infty(A)$.
2. $u \sim_1 u \oplus 1_n$ for all $u \in \mathcal{U}_\infty(A)$ and natural n .
3. $u \oplus v \sim_1 v \oplus u$ for all $u, v \in \mathcal{U}_\infty(A)$.
4. $u \sim_1 u'$ and $v \sim_1 v'$ implies $u \oplus v \sim_1 u' \oplus v'$ for all $u, u', v, v' \in \mathcal{U}_\infty(A)$.
5. if $u, v \in \mathcal{U}_n(A)$ for some n , then $uv \sim_1 vu \sim_1 u \oplus v$.
6. $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ in $\mathcal{U}_\infty(A)$.

With these properties, we can define the K_1 -group of a C^* -algebra, which by the previous lemma should be well-defined and abelian.

Definition 7.1.6. Let A be a C^* -algebra. Define the K_1 -group of A by

$$K_1(A) = \mathcal{U}_\infty(\tilde{A}) / \sim_1.$$

Let $[u]_1$ denote the equivalence class of $u \in \mathcal{U}_\infty(\tilde{A})$ and define addition on $K_1(A)$ by $[u]_1 + [v]_1 = [u \oplus v]_1$.

We do not need to pass through the Grothendieck group, as we did for K_0 . This is because $\mathcal{U}_\infty(A)$ already has a group structure, whereas for projections, we didn't even have a binary operation that made sense, as $p + p = 2p \neq (2p)^2$ for nonzero projections p .

By the fifth property mentioned in proposition 7.1.5, which follows from Whiteheads lemma (5.2.3), products in A and addition in $K_1(A)$ correspond quite nicely. This implies that $0 = [1_n]_1 = [uu^*]_1 = [u]_1 + [u^*]_1$, and hence $-[u]_1 = [u^*]_1$, which means that we preserve the unitary structure.

These remarks surely make $K_1(A)$ into an abelian group, and as we will see, most of the properties of $K_0(A)$ hold for K_1 , but more simply, as we didn't need to pass through the Grothendieck group to achieve an abelian group structure. Actually, compared to the K_0 -theory, it becomes so simple that the standard picture becomes more like a restatement of the definition and lemma 7.1.5 rather than a proposition, but we state it nevertheless.

Proposition 7.1.7. (Standard picture of K_1) Let A be a C^* -algebra. Then

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}_\infty(\tilde{A})\}$$

Furthermore, the map $[\cdot]_1 : \mathcal{U}_\infty(\tilde{A}) \rightarrow K_1(A)$ satisfies

1. $[u \oplus v]_1 = [u]_1 + [v]_1$,
2. $[1]_1 = 0$,
3. if $u, v \in \mathcal{U}_n(\tilde{A})$ with $u \sim_h v$, then $[u]_1 = [v]_1$.

In addition to the standard picture, we also get a universal property for K_1 , totally analogous to the property for K_0 .

Proposition 7.1.8. (Universal property of K_1) Assume A is a C^* -algebra, G is an abelian group and $v : \mathcal{U}_\infty(\tilde{A}) \rightarrow G$ satisfy the following properties:

1. $v(u \oplus w) = v(u) + v(w)$,
2. $v(1) = 0$,
3. if $u, w \in \mathcal{U}_n(\tilde{A})$ with $u \sim_h w$, then $v(u) = v(w)$.

Then there exists a group homomorphism $\alpha : K_1(A) \rightarrow G$ such that

$$\begin{array}{ccc} \mathcal{U}_\infty(\tilde{A}) & & \\ \downarrow [\cdot]_1 & \searrow v & \\ K_1(A) & \xrightarrow{\alpha} & G \end{array}$$

commutes.

Before we state more similarities between K_1 and K_0 , we should stop to ponder what this definition actually looks like for unital C^* -algebras. We are defining $K_1(A)$ in terms of the (naive) unitization \tilde{A} , but for unital C^* -algebras, we would want to define $K_1(A)$ as $\mathcal{U}_\infty(A)$. In fact, the following proposition from [35] shows us that it doesn't matter how we define it, as the two constructions are isomorphic as groups.

Proposition 7.1.9. Let A be a unital C^* -algebra. Then there exists an isomorphism $\rho : K_1(A) \rightarrow \mathcal{U}_\infty(A)/\sim_1$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{U}_\infty(\tilde{A}) & \xrightarrow{\mu} & \mathcal{U}_\infty(A) \\ [\cdot]_1 \downarrow & & \downarrow \\ K_1(A) & \xrightarrow{\rho} & \mathcal{U}_\infty(A)/\sim_1 \end{array}$$

Here μ denotes the lifted projection $\pi : \tilde{A} = A \oplus \mathbb{C} \rightarrow A$.

With this out of the way, how does K_1 lift morphisms? Well, the canonical way is by lifting to unitizations, lifting to matrices, and then restricting to unitary elements. As expected, K_1 is a functor.

Proposition 7.1.10. (Functoriality of K_1 and more) Let A, B and C be C^* -algebras. Then

1. $K_1(\text{Id}_A) = \text{Id}_{K_1(A)}$.
2. $K_1(\psi \circ \phi) = K_1(\psi) \circ K_1(\phi)$ for composable morphisms ψ, ϕ .
3. $K_1(\{0\}) = \{0\}$.
4. $K_1(0_{B,A}) = 0_{K_1(B), K_1(A)}$.
5. if $\psi, \phi : A \rightarrow B$ are homotopic $*$ -homomorphisms, then $K_1(\psi) = K_1(\phi)$.
6. if A and B are homotopy equivalent through $\phi : A \rightarrow B$ and $\psi : B \rightarrow A$, then $K_1(A) \cong K_1(B)$ and $K_1(\phi), K_1(\psi)$ are isomorphisms with $K_1(\phi)^{-1} = K_1(\psi)$.

There are more properties of K_1 that are so similar to the case of K_0 that we summarize them in the following proposition.

Proposition 7.1.11. (Further properties of K_1)

1. K_1 is half exact in the sense of proposition 5.5.11.
2. K_1 is split exact in the sense of proposition 5.5.12.
3. K_1 preserves biproducts in the sense of proposition 5.5.13.
4. K_1 is cocontinuous in the sense of theorem 6.2.12.
5. K_1 is stable in the sense of proposition 5.5.15.

Proof. The proofs can be found in [35], but note that they are extremely similar to the cases we proved earlier in the K_0 -theory. These will easily be verified when we show that K_1 can be realized as the composition of K_0 and the suspension functor S in theorem 7.2.5. \square

The construction has a lot of wanted properties, but we still need to develop it further to be close to the power of topological K -theory. Before we do this, we state some examples from [35] and [1] to get a feel on what different K_1 -groups look like. Some of them will be revisited later when we have developed our computational tools.

Example 7.1.12. (Some examples of K_1 -groups)

1. $K_1(M_n(\mathbb{C})) = K_1(\mathbb{C}) = 0$.
2. Let \mathcal{H} be a separable Hilbert space. Then $K_1(\mathcal{K}(\mathcal{H})) = K_1(B(\mathcal{H})) = 0$.
3. Let A be an AF-algebra. Then $K_1(A) = 0$.
4. $K_1(\mathcal{Q}(\mathcal{H})) \cong \mathbb{Z}$, where $\mathcal{Q}(\mathcal{H})$ denotes the Calkin algebra.
5. If n is natural, then $K_1(C(S^{2n})) = 0$.
6. If n is natural, then $K_1(C(S^{2n+1})) \cong \mathbb{Z}$.

It is time to start our hunt for computational tools. We know that if we are given a short exact sequence, we get an induced sequence in K_0 and K_1 . For computational reasons, we want to find a map that fits in, linking K_1 to K_0 . That is, given

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0,$$

we get induced sequences such that

$$K_1(I) \longrightarrow K_1(A) \longrightarrow K_1(B)$$

$$K_0(B) \longleftarrow K_0(A) \longleftarrow K_0(I).$$

Hence, if we can find a map $\delta_1 : K_1(B) \rightarrow K_0(I)$ linking these together to a diagram

$$\begin{array}{ccccc} K_1(I) & \longrightarrow & K_1(A) & \longrightarrow & K_1(B) \\ & & & & \vdots \\ & & & & \downarrow \\ K_0(B) & \longleftarrow & K_0(A) & \longleftarrow & K_0(I), \end{array}$$

we would certainly progress in finding a long exact sequence in K -theory.

Remark 7.1.13. Note that such a map would measure the obstruction to lifting unitaries from a matrix algebra over B to a matrix algebra over A , that is, it measures how much the map $K_1(A) \rightarrow K_1(B)$ fails to be surjective.

The main problem comes when we want to understand how this map is constructed. We need to take an equivalence class of unitary elements in $K_1(B)$ and assign to them an equivalence class of projections in $K_0(I)$ in a way that connects the exactness of the two sequences.

The following lemma borrowed from [35] takes care of this. We assume we are given a short exact sequence of the form

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0.$$

Lemma 7.1.14. Let $u \in \mathcal{U}_n(\tilde{B})$.

1. There exists a unitary $v \in \mathcal{U}_{2n}(\tilde{A})$ and a projection $p \in \mathcal{P}_{2n}(\tilde{I})$ such that

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\phi}(p) = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad s(p) = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}.$$

2. If $w \in \mathcal{U}_{2n}(\tilde{A})$ and $q \in \mathcal{P}_{2n}(\tilde{I})$ satisfy

$$\tilde{\psi}(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\phi}(q) = w \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} w^*,$$

then $s(q) = \text{diag}(1_n, 0_n)$ and $p \sim_u q$ in $\mathcal{P}_{2n}(\tilde{I})$.

In short, this lemma lets us take a unitary in B , assign a unitary $v \in A$, which is a step back in the sequence, and then construct a projection with this unitary. This projection can be lifted to a projection in I , which is well defined, as this projection is unique up to unitary equivalence.

Proof. The proof can be found in [35]. With more work on unitaries, this is not necessarily a challenging proof, if we are given the right lemmas. We leave it to the encouraged reader to investigate. \square

This lets us define a map with the following properties, as shown in [35].

Proposition 7.1.15. Define $\nu : \mathcal{U}_\infty(\tilde{B}) \rightarrow K_0(I)$ given by $\nu(u) = [p]_0 - [s(p)]_0$, where p corresponds to u as in lemma 7.1.14. This map has the following properties:

1. $\nu(u_1 \oplus u_2) = \nu(u_1) + \nu(u_2)$ for all $u_1, u_2 \in \mathcal{U}_\infty(\tilde{B})$,
2. $\nu(1) = 0$,
3. if $u_1 \sim_h u_2$ in $\mathcal{U}_n(\tilde{B})$, then $\nu(u_1) = \nu(u_2)$,
4. $\nu(\tilde{\psi}(u)) = 0$ for all $u \in \mathcal{U}_\infty(\tilde{A})$
5. $K_0(\phi)(\nu(u)) = 0$ for all $u \in \mathcal{U}_\infty(\tilde{B})$.

The first three properties are enough to invoke the universal property of K_1 and the last two will ensure that we actually get a complex. Exactness is yet to be shown.

Definition 7.1.16. The unique group homomorphism $\delta_1 : K_1(B) \rightarrow K_0(I)$ induced by the universal property of K_1 applied to the map $\nu : \mathcal{U}_\infty(\tilde{B}) \rightarrow K_0(I)$, is called the *index map* associated to the short exact sequence.

Remark 7.1.17. Indeed, the induced index map has the same properties as ν in the sense that $\delta_1([u]_1) = [p]_0 - [s(p)]_0$, $\delta_1 \circ K_1(\psi) = 0$ and $K_1(\phi) \circ \delta_1 = 0$.

By dense computations, one can actually show that the index map is natural.

Proposition 7.1.18. Assume we are given a commutative diagram of C^* -algebras with short exact rows and vertical $*$ -homomorphisms,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\phi} & A & \xrightarrow{\psi} & B & \longrightarrow & 0 \\ & & \gamma \downarrow & & \alpha \downarrow & & \beta \downarrow & & \\ 0 & \longrightarrow & I' & \xrightarrow{\phi'} & A' & \xrightarrow{\psi'} & B' & \longrightarrow & 0. \end{array}$$

If $\delta_1 : K_1(B) \rightarrow K_0(I)$ and $\delta'_1 : K_1(B') \rightarrow K_0(I')$ denote the index maps of the upper and lower short exact sequences, respectively, the following diagram commutes:

$$\begin{array}{ccc} K_1(B) & \xrightarrow{\delta_1} & K_0(I) \\ K_1(B) \downarrow & & \downarrow K_0(\gamma) \\ K_1(B') & \xrightarrow{\delta'_1} & K_0(I'). \end{array}$$

Queequeg 7.1.19. The name "index map" comes from the study of Fredholm operators and the Fredholm index. The index map is closely connected to this study. Given a Fredholm operator T on a Hilbert space \mathcal{H} , it is possible to show that

$$\text{index}(T) = (K_0(Tr) \circ \delta_1)([\pi(T)]_1),$$

where π is the quotient map $B(\mathcal{H}) \rightarrow \mathcal{Q}(\mathcal{H})$ and $K_0(Tr)$ is an isomorphism $K_0(\mathcal{K}) \rightarrow \mathbb{Z}$ induced from a suitable trace.

We do not intend to develop all of the details around the index map, but it is certainly a necessary component to create a long exact sequence in K -groups.

There are a lot of details to consider to do this thoroughly, but we only mark the end result.

Proposition 7.1.20. For all short exact sequences

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

of C^* -algebras, we get an induced exact sequence

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{K_1(\phi)} & K_1(A) & \xrightarrow{K_1(\psi)} & K_1(B) \\ & & & & \downarrow \delta_1 \\ K_0(B) & \xleftarrow{K_0(\phi)} & K_0(A) & \xleftarrow{K_0(\psi)} & K_0(I). \end{array}$$

Before we move on to the higher K -groups, we consider an easy example of a calculation, namely the K_1 -group of the Calkin algebra.

Example 7.1.21. Consider the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{B} \longrightarrow \mathcal{Q} \longrightarrow 0$$

of compact operators, bounded operators, and the Calkin algebra on a Hilbert space \mathcal{H} , respectively. This induces an exact sequence

$$\begin{array}{ccccc} K_1(\mathcal{K}) & \longrightarrow & K_1(\mathcal{B}) & \longrightarrow & K_1(\mathcal{Q}) \\ & & & & \downarrow \delta_1 \\ K_0(\mathcal{Q}) & \longleftarrow & K_0(\mathcal{B}) & \longleftarrow & K_0(\mathcal{K}). \end{array}$$

As we know $K_1(\mathcal{B}) = K_0(\mathcal{B}) = 0$, we do not even need to consider the maps in the sequence, as this immediately yields $K_1(\mathcal{Q}) \cong K_0(\mathcal{K}) \cong \mathbb{Z}$, where the last isomorphism can be shown by more work on stabilizations as in proposition 5.5.15.

7.2 Higher K -groups

To define the higher K -groups K_n , we must verify that $K_1(A) \cong K_0(SA)$ as in the topological case. This allows us to recursively define $K_{n+1}(A) := K_n(SA)$ and lift the index map to these higher K -groups and extend our exact sequence to a worthy long exact sequence in K -theory.

We start off with some definitions to understand how suspensions of C^* -algebras actually work.

Definition 7.2.1. The suspension SA and the cone CA of a C^* -algebra A is given by

$$\begin{aligned} SA &= \{f \in C([0, 1], A) : f(0) = f(1) = 0\} = C_0((0, 1), A) \\ CA &= \{f \in C([0, 1], A) : f(0) = 0\} \end{aligned}$$

These fit in a short exact sequence

$$0 \longrightarrow SA \longrightarrow CA \longrightarrow A \longrightarrow 0,$$

where the last morphism is given by $f \mapsto f(1)$.

Remark 7.2.2. Indeed, the cone is contractible. Let $\phi_t : CA \rightarrow CA$ be given by $\phi_t(f)(s) = f(st)$ for $f \in CA$ and $s, t \in [0, 1]$. This is continuous, $\phi_0 = 0$ and $\phi_1 = id$. Hence $K_0(CA) = K_1(CA) = 0$.

To see that S is functorial, we need a more explicit description of the suspension, which we borrow from [35].

Lemma 7.2.3. Let X be a locally compact Hausdorff space. If $f \in C_0(X)$ and $a \in A$, let $fa \in C_0(X, A)$ denote the element given by $(fa)(x) = f(x)a$. Then the set $\text{span}\{fa : f \in C_0(X), a \in A\}$ is dense in $C_0(X, A)$.

This allows us to understand the lifting of morphisms, as we only need to define the morphism on this dense subset. Let $\phi : A \rightarrow B$. Then $S\phi : SA \rightarrow SB$ is given by $S\phi(af) = \phi(a)f \in SB$. The following proposition is direct from the above lemma.

Proposition 7.2.4. The functor S is exact.

As mentioned, the higher K -groups in topological K -theory are defined in terms of iterated suspensions. For K_1 , we managed to tweak this description to a more operator algebraic approach, namely in terms of unitary elements. We have not lost the underlying isomorphism. In fact, it turns out to have quite a concrete description.

Theorem 7.2.5. The groups $K_0(SA)$ and $K_1(A)$ are isomorphic. Moreover, the collection of isomorphisms $\theta_A : K_1(A) \rightarrow K_0(SA)$ is natural in the sense that for every $*$ -homomorphism $\phi : A \rightarrow B$ between two C^* -algebras A and B , the diagram

$$\begin{array}{ccc} K_1(A) & \xrightarrow{K_1(\phi)} & K_1(B) \\ \theta_A \downarrow & & \downarrow \theta_B \\ K_0(SA) & \xrightarrow{K_0(S\phi)} & K_0(SB) \end{array}$$

is commutative. In fact, the isomorphisms have concrete descriptions. If $u \in \mathcal{U}_n(\tilde{A})$ satisfies $s(u) = 1_n$, and $v \in C([0, 1], \mathcal{U}_{2n}(\tilde{A}))$ satisfy $v(0) = 1_{2n}$, $v(1) = \text{diag}(u, u^*)$ and $s(v(t)) = 1_{2n}$ for all t , we can construct a projection $p = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^*$ in $\mathcal{P}_{2n}(\tilde{SA})$. This projection satisfies $s(p) = \text{diag}(1_n, 0_n)$ and $\theta_A([u]_1) = [p]_0 - [s(p)]_0$.

Proof. We keep it brief. Since CA is contractible, the exact sequence (proposition 7.1.20) applied to the short exact sequence (7.2.1), yields that the index map $\delta_1 : K_1(A) \rightarrow K_0(SA)$ is an isomorphism. Hence we can choose $\theta_A = \delta_1$. Naturality follows from the fact that morphisms $A \rightarrow B$ induce morphisms between the short exact sequences. The wanted naturality is just the naturality of the index map. To consider the explicit description, note that a function $f \in C([0, 1], M_{2n}(\tilde{A}))$ belongs to $M_{2n}(\tilde{CA})$ if and only if $s(f(t)) = f(0)$ for all t and that f belongs to $M_{2n}(\tilde{SA})$ if and only if $s(f(t)) = f(0) = f(1)$. With these identifications, it is possible to tweak the explicit form of the index map δ_1 to obtain the wanted description. \square

Due to this isomorphism, it would seem natural to define the higher K -groups inductively in a similar manner.

Definition 7.2.6. For $n \geq 2$, define a functor K_n from the category of C^* -algebras to the category of abelian groups by $K_n = K_{n-1} \circ S$, where S is the suspension endofunctor on the category of C^* -algebras.

Corollary 7.2.7. $K_n(A) := K_1(S^{n-1}A) \cong K_0(S^n A)$.

Since S is exact and the base cases K_0 and K_1 are half-exact, K_n is half-exact for all n .

Now, what about the index maps? Are we able to lift the index map δ_1 to higher analogues δ_n ?

Let

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

be a short exact sequence of C^* -algebras. Then, by exactness of S ,

$$0 \longrightarrow S^n I \xrightarrow{S^n \phi} S^n A \xrightarrow{S^n \psi} S^n B \longrightarrow 0$$

is also exact. Let δ_1 denote the index map of this short exact sequence.

The isomorphism $\theta_{S^{n-1}I} : K_n(I) \rightarrow K_0(S^n I)$ fits into the diagram

$$\begin{array}{ccc} K_{n+1}(B) & \xrightarrow{\dots \delta_{n+1} \dots} & K_n(I) \\ \parallel & & \downarrow \theta_{S^{n-1}I} \\ K_1(S^n B) & \xrightarrow{\delta_1} & K_0(S^n I), \end{array}$$

where the dotted map, δ_{n+1} , exists uniquely as the diagram commutes.

These higher index maps are natural in a similar manner to the original index map δ_1 , which can be proved by just moving the argument from $K_1(B)$ and $K_0(I)$ to $K_{n+1}(B)$ and $K_n(I)$.

This makes it possible to move on to the computational tools in K -theory.

7.3 Standard computational tools in K -theory

In this section, we give a brief overview of the main computational tools and important results in K -theory, such as the long exact sequence, Bott periodicity, and the six-term sequence.

We have already lifted our construction of the index map to higher K -groups, which means much of the groundwork is done towards constructing the long exact sequence.

Proposition 7.3.1. (Long exact sequence in K -theory)

All short exact sequences

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

of C^* -algebras induce long exact sequences in K -theory

$$\begin{aligned} \dots &\xrightarrow{\delta_{n+1}} K_n(I) \xrightarrow{K_n(\phi)} K_n(A) \xrightarrow{K_n(\psi)} K_n(B) \xrightarrow{\delta_n} K_{n-1}(I) \xrightarrow{K_{n-1}(\phi)} \dots \\ &\dots \xrightarrow{\delta_1} K_0(I) \xrightarrow{K_0(\phi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B), \end{aligned}$$

where δ_1 is the index map and δ_n denotes the higher analogues for $n \geq 2$.

Proof. We let δ'_n denote the n 'th index map associated to the sequence

$$0 \longrightarrow SI \xrightarrow{S\phi} SA \xrightarrow{S\psi} SB \longrightarrow 0.$$

We want to do an inductive argument. The diagrams

$$\begin{array}{ccccccccc} K_2(I) & \longrightarrow & K_2(A) & \longrightarrow & K_2(B) & \xrightarrow{\delta_2} & K_1(I) & \longrightarrow & K_1(A) & \longrightarrow & K_1(B) \\ \parallel & & \parallel & & \parallel & & \downarrow \theta_I & & \downarrow \theta_A & & \downarrow \theta_B \\ K_1(SI) & \longrightarrow & K_1(SA) & \longrightarrow & K_1(SB) & \xrightarrow{\delta'_1} & K_0(SI) & \longrightarrow & K_0(SA) & \longrightarrow & K_0(SB) \end{array}$$

and

$$\begin{array}{ccccccccc} \dots & \longrightarrow & K_n(B) & \xrightarrow{\delta_n} & K_{n-1}(I) & \longrightarrow & K_{n-1}(A) & \longrightarrow & K_{n-1}(B) & \longrightarrow & \dots \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \\ \dots & \longrightarrow & K_{n-1}(SB) & \xrightarrow{\delta'_{n-1}} & K_{n-2}(SI) & \longrightarrow & K_{n-2}(SA) & \longrightarrow & K_{n-2}(SB) & \longrightarrow & \dots \end{array}$$

commute. We already know the lower row in the first diagram is exact, which implies that the upper row is exact. The induction step now follows from the last diagram. \square

It is possible to say something stronger about this sequence than just exactness. Similar to the topological case, the sequence collapses.

Given a projection $p \in \mathcal{P}_n(A)$ for a unital C^* -algebra A , we can define *the projection loop* $f_p : \mathbb{T} \rightarrow \mathcal{U}_n(A)$ by $f_p(z) = zp + (1_n - p)$. If we use an equivalent definition of the suspension SA as $SA = \{f \in C(\mathbb{T}, A) : f(1) = 0\}$, it is possible to identify f_p with some function $f_p \in \mathcal{U}_n(\tilde{S}A)$. This allows us to define a map taking projections to continuous functions mapping into unitaries, that is, we can send projections to unitaries on the suspension of A .

Definition 7.3.2. (The Bott map) Define the *Bott map* $\beta_A : K_0(A) \rightarrow K_1(SA)$ by $\beta_A([p]_0) = [f_p]_1$ for $p \in \mathcal{P}_\infty(A)$. We refer to [35] for the fact that this is well defined.

We state the celebrated Bott periodicity theorem, which yields an amazing understanding of how the higher K -groups are connected.

Theorem 7.3.3. (Bott periodicity) The Bott map $\beta_A : K_0(A) \rightarrow K_1(SA)$ is an isomorphism for all C^* -algebras A .

The proof can be found in [35] or [1]. Note that this means all the K -theoretic information in a C^* -algebra is contained in its projections and unitary elements, or more precisely, in K_0 and K_1 , respectively. This means the only K -groups we need to calculate, are $K_0(A)$ and $K_1(A)$.

By induction, Bott periodicity implies the following corollary.

Corollary 7.3.4. For all C^* -algebras A and $n \geq 0$, $K_{n+2}(A) \cong K_n(A)$.

This should collapse our long exact sequence to a shorter, perhaps even cyclic, exact sequence, as we can just start over at K_0 when we study K_{2n} and K_1 when we study K_{2n+1} .

Definition 7.3.5. Let

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

be a short exact sequence of C^* -algebras and define the *exponential map* $\delta_0 : K_0(B) \rightarrow K_1(I)$ by the composition

$$K_0(B) \xrightarrow{\beta_B} K_2(B) \xrightarrow{\delta_2} K_1(I).$$

We then have the following powerful proposition.

Proposition 7.3.6. Let

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

be a short exact sequence of C^* -algebras. For all such short exact sequences, the six-term sequence

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K_0(\phi)} & K_0(A) & \xrightarrow{K_0(\psi)} & K_0(B) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(B) & \xleftarrow{K_1(\psi)} & K_1(A) & \xleftarrow{K_1(\phi)} & K_1(I) \end{array}$$

is exact.

This is indeed an important tool when doing computations. To observe the power of this six-term exact sequence, let us calculate $K_0(\mathcal{Q}(\mathcal{H}))$ for some separable Hilbert space \mathcal{H} .

Example 7.3.7. Let \mathcal{H} be a separable Hilbert space and let \mathcal{Q} denote the Calkin algebra, as earlier. Then

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(B) & \longrightarrow & K_0(\mathcal{Q}) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(\mathcal{Q}) & \longleftarrow & K_1(B) & \longleftarrow & K_1(\mathcal{K}) \end{array}$$

is exact. We know $K_0(\mathcal{B}) = K_1(\mathcal{B}) = 0$, yielding the exponential map to be an isomorphism $K_0(\mathcal{Q}) = K_1(\mathcal{K}) = 0$. The last equality follows from the fact that \mathcal{K} is an AF-algebra. We will soon return to the computation of K_1 for AF-algebras.

It is possible to show that the exponential map is natural in the same sense as the index maps. In addition, it has quite an explicit description, justifying the name. We briefly mention this here and refer to [35] for a proof.

Proposition 7.3.8. (Explicit description of the exponential map) Let

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

be a short exact sequence of C^* -algebras. Let $\delta_0 : K_0(B) \rightarrow K_1(I)$ be the associated exponential map and let $g \in K_0(B)$.

We can calculate $\delta_0(g)$ the following way: Find a projection $p \in \mathcal{P}_n(B)$ such that $g = [p]_0 - [s(p)]_0$. Lift this projection to an element $a \in M_n(\tilde{A})$ such that $\tilde{\psi}(a) = p$. There exists a unique unitary $u \in \mathcal{U}_n(\tilde{I})$ such that $\tilde{\phi}(u) = \exp(2\pi i a)$. Then $\delta_0(g) = -[u]_1$.

To conclude our general overview of K -theory, we use our computational tools to justify some of the K -groups claimed earlier.

Example 7.3.9. By stability, $K_1(M_n(\mathbb{C})) = K_1(\mathbb{C})$. It is possible to show that if u is a unitary element such that $\sigma(u) \neq \mathbb{T}$, then u must be homotopic to the identity (see chapter 2 of [35]). Any given unitary element in $M_n(\mathbb{C})$ has finite spectrum, which means that it must be homotopic to the identity. Hence, the unitary group of $M_k(M_n(\mathbb{C})) = M_{nk}(\mathbb{C})$ is connected, which implies that $\mathcal{U}_\infty(M_n(\mathbb{C})) / \sim_1$ must be the trivial group, that is, $K_1(M_n(\mathbb{C})) = 0$.

Note that this implies $K_1(A) = 0$ for all AF-algebras A , by cocontinuity of K_1 .

It is possible to show that $K_1(\mathcal{K}(\mathcal{H})) = K_1(B(\mathcal{H})) = 0$ for separable Hilbert spaces \mathcal{H} , in a similar manner. The first K -group being 0 follows from the compacts being an AF-algebra. This would imply the K -groups of the Calkin algebra, as done in earlier examples.

Example 7.3.10. Since \mathbb{R} is homeomorphic to $(0, 1)$, we see that the suspension SA of a C^* -algebra A is isomorphic to $C_0(\mathbb{R}, A)$. For a pair X, Y of locally compact Hausdorff spaces, we have $C_0(X, C_0(Y)) \cong C_0(X \times Y)$. More generally, similar results hold for pairs of compactly generated weak Hausdorff spaces ([19]). This implies that $S^n \mathbb{C} \cong C_0(\mathbb{R}^n)$, which means that

$$K_0(C_0(\mathbb{R}^n)) \cong K_n(\mathbb{C}) \quad \text{and} \quad K_1(C_0(\mathbb{R}^n)) \cong K_{n+1}(\mathbb{C}),$$

for all $n \geq 1$.

By Bott periodicity, we get

$$K_0(C_0(\mathbb{R}^n)) \cong K_n(\mathbb{C}) \cong \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

and

$$K_1(C_0(\mathbb{R}^n)) \cong K_{n+1}(\mathbb{C}) \cong \begin{cases} 0, & n \text{ even} \\ \mathbb{Z}, & n \text{ odd.} \end{cases}$$

Example 7.3.11. Since the one-point compactification of \mathbb{R}^n is homeomorphic to S^n for $n \geq 1$, $C(S^n)$ is isomorphic to the naive unitization of $C_0(\mathbb{R}^n)$. For K_1 , this means that

$$K_1(C(S^n)) \cong K_1(C_0(\mathbb{R}^n)) \cong \begin{cases} 0, & n \text{ even} \\ \mathbb{Z}, & n \text{ odd,} \end{cases}$$

while for K_0 , we split over the direct sum to obtain

$$K_0(C(S^n)) \cong K_0(C_0(\mathbb{R}^n)) \oplus K_0(\mathbb{C}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n \text{ even} \\ \mathbb{Z}, & n \text{ odd.} \end{cases}$$

Queequeg 7.3.12. In algebraic topology ([19]), cohomology theories can often be represented by a spectrum due to the Brown representability theorem ([39]). The motivational example is the Eilenberg-MacLane spaces $K(G, n)$, which represent singular cohomology with coefficients in a group G in the sense that $H^n(X; G)$ is in bijection with $[X, K(G, n)]$.

Operator K-theory is not a reduced cohomology theory, but a reduced homology theory. The problem with this, if we for example consider homology, is that $[-, -]$ preserves products, while homology does not. Is it even possible to find a way to represent the K-groups?

There are in fact results on the topic if we do lots (and lots) of work on KK -theory. Take for example proposition 4.2. in [23].

We can find a functor $\mathbb{K}(-) : C^*Alg \rightarrow Ho(Spectra)$ such that $\pi_n(\mathbb{K}(A)) \cong K_n(A)$.

What does this mean? We take a C^* -algebra A and get a spectrum $\mathbb{K}(A)$, which lies in the stable homotopy category, where we can define some sort of homotopy theory, giving back the K-functors.

For such spectra, we can always define homotopy groups. Define the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$, where Σ denotes the suspension of topological spaces. Let X be an object in $Ho(Spectra)$.

Now, define

$$\pi_n(X) = [\mathbb{S}, X]_n = [\Sigma^n \mathbb{S}, X].$$

Proposition 4.2. in [23] says that K_n can be realized as the composition

$$C^*Alg \xrightarrow{\mathbb{K}(-)} Ho(Spectra) \xrightarrow{\pi_n(-)} Ab.$$

Recall that our goal is to study noncommutative tori through K -theory. We move on to study a six-term sequence suited for crossed product C^* -algebras.

7.4 The Pimsner-Voiculescu exact sequence

We give a short introduction of the Pimsner-Voiculescu exact sequence, a six-term exact sequence suited for studying crossed product C^* -algebras.

There are several proofs of the following theorem. The original proof was by Pimsner and Voiculescu in [27], where they developed the sequence in a technical manner through so-called Toeplitz extensions. Our brief justification will be through more modern tools and is based on [1] and [36], even though it is historically in the totally wrong order. Therefore, the following statements and results are only included to display some beautiful tools.

Theorem 7.4.1. Let A be a C^* -algebra and let $\alpha \in \text{Aut}(A)$. If α_* denotes the induced morphism in K -theory, there exists a six-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \xrightarrow{i_*} & K_0(A \times_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \times_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{1-\alpha_*} & K_1(A). \end{array}$$

Proof. We sketch the main ideas found in [1] and [36], but we make several claims and omit details as the proof requires theory we have not encountered in this thesis.

We first assume we know Connes' Thom isomorphism theorem, saying that for an $\alpha : \mathbb{R} \rightarrow \text{Aut}(A)$, we have an isomorphism $K_i(A \times_\alpha \mathbb{R}) \cong K_{1-i}(A)$ for $i = 0, 1$. Secondly, we need the Takai duality theorem, stating that $(A \times_\alpha G) \times_{\hat{\alpha}} \hat{G} \cong A \otimes \mathcal{K}(L^2(G))$, where \hat{G} denotes the Pontryagin dual of continuous group homomorphisms into \mathbb{T} and $\hat{\alpha}$ denotes the induced action on the crossed product. The (completed) tensor product is not necessary to specify as \mathcal{K} is a nuclear C^* -algebra, meaning all completed tensor products coincide.

If we define the mapping torus M_α of α to be

$$M_\alpha = \{f : [0, 1] \rightarrow A : f(1) = \alpha(f(0))\},$$

we get a short exact sequence

$$0 \longrightarrow SA \longrightarrow M_\alpha \longrightarrow A \longrightarrow 0.$$

More precisely, the mapping torus of α is the pushout of the following diagram,

which is similar to the definition of a mapping cylinder from homotopy theory.

$$\begin{array}{ccc} A & \longrightarrow & A \times S^1 \\ \alpha \downarrow & & \\ A & & \end{array}$$

It is possible to realize the crossed product $B \times_{\alpha} \mathbb{R}$ as a mapping torus on $B \times_{\alpha} \mathbb{T}$ and $B \times_{\alpha} \mathbb{Z}$ as a mapping torus on $B \times_{\alpha} \mathbb{Z}_n$.

If we define $B = A \times_{\alpha} \mathbb{Z}$, the action $\beta = \hat{\alpha}$ of \mathbb{T} on B can be regarded as an action from \mathbb{R} where the integers acts trivially. Now, by Takai duality, the crossed product $B \times_{\beta} \mathbb{T}$ is isomorphic to $A \otimes \mathcal{K}$, and $B \times_{\beta} \mathbb{R}$ can be realized as a mapping torus, which we will denote by M_{α} when passed through K -theory.

Now these fit into the short exact sequence

$$0 \longrightarrow S(A \otimes \mathcal{K}) \longrightarrow B \times_{\beta} \mathbb{R} \longrightarrow A \otimes \mathcal{K} \longrightarrow 0.$$

If we had developed the theory of stabilizations sufficiently, then it would be well known that $K_i(A \otimes \mathcal{K}) \cong K_i(A)$, but we take this for granted. After applying the six-term exact sequence, Connes' Thom isomorphism, Bott periodicity, and the necessary shifts due to the suspension in the first term, we get the following diagram,

$$\begin{array}{ccccc} K_1(A) & \longrightarrow & K_0(M_{\alpha}) & \longrightarrow & K_0(A) \\ \partial \uparrow & & & & \partial \downarrow \\ K_1(A) & \longleftarrow & K_1(M_{\alpha}) & \longleftarrow & K_0(A). \end{array}$$

We can realize $A \times_{\alpha} \mathbb{Z}$ as a mapping torus, implying we only need to show that the connecting maps ∂ are of the form $1 - \alpha_*$, but this argument can be found in [1]. \square

With this powerful computational tool in our toolbox, we are ready to face the computation of K -groups of noncommutative tori.

Chapter 8

The case of noncommutative tori

In this final section, we will apply K -theory and Morita equivalence to noncommutative tori. Most of our results will be due to [33], but we are also inspired by [5]. The aim of the chapter is not to rewrite the research in [33], but to give an overview of some of the main results from the '80s on noncommutative tori, where K -theory and Morita equivalence played a prominent rôle. We start by computing the K -groups of noncommutative tori using the Pimsner-Voiculescu sequence.

8.1 K -groups of noncommutative tori

Consider A_θ as $C(S^1) \rtimes_\alpha \mathbb{Z}$, where the action $\alpha \in \text{Aut}(C(S^1))$ is given by $\alpha(f)(x) = f(e^{-2\pi i\theta} x)$, as explained in chapter 3. Note that rotating a point on the circle is homotopic to the identity by finding the canonical path rotating it back to the starting point.

The Pimsner-Voiculescu exact sequence then yields

$$\begin{array}{ccccc} K_0(C(S^1)) & \xrightarrow{1-\alpha_*} & K_0(C(S^1)) & \xrightarrow{i_*} & K_0(A_\theta) \\ \uparrow & & & & \downarrow \\ K_1(A_\theta) & \xleftarrow{i_*} & K_1(C(S^1)) & \xleftarrow{1-\alpha_*} & K_1(C(S^1)). \end{array}$$

Since α works as the identity up to homotopy, we get that $1 - \alpha_* = 0$, as the K -functors are homotopy invariant. This lets us split the Pimsner-Voiculescu exact sequence to the following short exact sequences:

$$0 \longrightarrow K_1(C(S^1)) \longrightarrow K_1(A_\theta) \longrightarrow K_0(C(S^1)) \longrightarrow 0$$

$$0 \longrightarrow K_0(C(S^1)) \longrightarrow K_0(A_\theta) \longrightarrow K_1(C(S^1)) \longrightarrow 0.$$

We know that $K_i(C(S^1)) \cong \mathbb{Z}$ for $i = 0, 1$. Since \mathbb{Z} is free in the category of abelian groups, as abelian groups are \mathbb{Z} -modules, the following sequences split.

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_1(A_\theta) \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(A_\theta) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

This implies $K_0(A_\theta) \cong K_1(A_\theta) \cong \mathbb{Z}^2$.

We now have a starting point for classification, but unfortunately, this is not enough to be able to classify noncommutative tori. With these K -groups, we recall the theory of AF-algebras and the classification theorem of Elliott (6.2.16) to find another approach.

8.2 Isomorphism classification and the unique trace

Not only will the K -groups themselves be too weak to classify noncommutative tori, but they are also independent of θ , since for all θ , we have $K_i(A_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$ as abelian groups. If we consider ordered abelian groups, this is no longer the case, as we can find irrational numbers θ and η such that there is no order isomorphism (or even a positive order homomorphism) $\mathbb{Z} + \eta\mathbb{Z} \rightarrow \mathbb{Z} + \theta\mathbb{Z}$.

From now on, let us assume $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is irrational. One reason why such an assumption could make things simpler can be found in ergodic theory ([41]), where irrational rotations yield an ergodic action on the circle, leaving no invariant subset of significant measure under the transformation. Due to this, we would expect the C^* -algebra to be simple. We return to this shortly.

Recall that AF-algebras can be classified through K_0 due to their structure as dimension groups. A hope could be to utilize this powerful theorem in the work on classification. In fact, in 1980, Pimsner and Voiculescu ([28]), along with other mathematicians at the time, showed that we can embed A_θ in an AF-algebra preserving the K_0 -group. The noncommutative tori are not AF-algebras themselves due to their nonzero K_1 -groups, but it is possible to approximate the generators U, V by finite-dimensional analogues through a continuous fraction expansion of θ , yielding a way of embedding A_θ in an AF-algebra. A crash course on the topic can be found in [5], which we will borrow some results from at a later stage. If this means we can apply the classification theorem of AF-algebras (6.2.16), the problem amounts to studying an ordered group structure on A_θ .

To construct the ordered structure, we need to realize what we are looking for. We have $K_0(A_\theta) \cong \mathbb{Z}^2$, which tells us that we have two equivalence classes of projections generating the K_0 -group. To understand this more thoroughly, we inspect what these projections look like for the simplest case, $\theta = 0$. In this case,

we know the noncommutative torus is $*$ -isomorphic to $C(\mathbb{T}^2)$. A crucial fact is that \mathbb{T}^2 is connected, implying $C(\mathbb{T}^2)$ is connected. If we have a projection $f \in C(\mathbb{T}^2)$, then $f^2 - f = 0$, meaning $f^2(x) - f(x) = 0$ for all x , which implies $f(x) = 0$ or $f(x) = 1$. Since $C(\mathbb{T}^2)$ is connected, f must map all x to either 0 or 1, as any mix would yield a nontrivial separation of $C(\mathbb{T}^2)$ by preimages under evaluation. Hence $f = 1$ or $f = 0$, yielding only trivial projections in A_θ .

If θ is nonzero, it is natural to ask whether or not we have any nontrivial projections. In the irrational case, it is possible to find a powerful approach to answering this question.

We will construct a unique normalized trace on A_θ . This trace will, at the level of K -theory, be central in proving most of our results.

Remark 8.2.1. First, we need to understand why we would care about such traces. In fact, their image on projections give an isomorphism invariant of C^* -algebras.

Assume we are given a separable, unital C^* -algebra B with a unique, normalized trace $\tau : B \rightarrow \mathbb{C}$. The values of τ on projections are quite interesting to study. When studying the trace evaluated on projections, we note that we end in \mathbb{R}^+ , meaning $\tau : B \rightarrow \mathbb{R}^+$. There are mainly three key points to notice to see that this is an isomorphism invariant.

1. Given a projection $p \in B$, we have $p \leq 1_B$, implying $0 \leq \tau(p) \leq \tau(1_B) = 1$ by positivity. The image of the projections end up in $[0, 1]$.
2. If two projections p and q are *Murray-von Neumann* equivalent, they give the same value under τ , because we can find a $v \in B$ such that $p = vv^*$ and $q = v^*v$. This implies $\tau(p) = \tau(vv^*) = \tau(v^*v) = \tau(q)$.
3. The image of τ on projections is countable. Denote this image by $\tau(\mathcal{P}(B))$ and let p_{λ_j} denote elements in $\mathcal{P}(B)$. Assume $\tau(\mathcal{P}(B))$ is uncountable. Then, since B is separable, we can find a dense, countable subset $\{b_i\}_{i=1}^\infty$. Now, define $B_i = \{b \in B : \|b - b_i\| < 1/2\}$. These are open sets and the union of the B_i is B . Observe that if we take two projections $p_{\lambda_1}, p_{\lambda_2} \in B_i$, we have $\|p_{\lambda_1} - p_{\lambda_2}\| < 1$, which in turn implies $p_{\lambda_1} \sim_h p_{\lambda_2}$. Since homotopic elements are *Murray-von Neumann* equivalent, $\tau(p_{\lambda_1}) = \tau(p_{\lambda_2})$. We have countably many B_i , meaning $\tau(\mathcal{P}(B))$ must be countable as well.

Finally, let B and B' be $*$ -isomorphic C^* -algebras through a $*$ -isomorphism $\phi : B \rightarrow B'$ and let τ, τ' denote the unique normalized traces on B and B' , respectively. Then $\tau' \circ \phi$ is also a normalized trace on B , meaning $\tau' \circ \phi = \tau$.

This means the image of projections under such a trace is an isomorphism invariant, due to

$$\{\tau'(p) : p \in B'\} = \{\tau'(\phi(\phi^{-1}(p))) : p \in B'\} = \{\tau'(\phi(q)) : q \in B\} = \{\tau(q) : q \in B\}.$$

Traces certainly deserve a place in operator K -theory, but how can we find such a

normalized trace and why is it unique? We refer to [5] for a deeper treatise on the concepts visited here.

It is time to start our construction of the normalized trace. First note that if we scale our generators (U, V) to $(\lambda U, \mu V)$, where λ, μ are both on the unit circle, the pair $(\lambda U, \mu V)$ still satisfies the commutation relation for (U, V) , yielding a $*$ -homomorphism $\rho_{\lambda, \mu}$ sending $U \mapsto \lambda U, V \mapsto \mu V$. This is clearly an automorphism, since the $*$ -homomorphism $\rho_{\bar{\lambda}, \bar{\mu}}$ is the inverse.

If we fix an $A \in A_\theta$, it is possible to show the morphism $f : \mathbb{T}^2 \rightarrow A_\theta$ sending (λ, μ) to $\rho_{\lambda, \mu}(A)$ is continuous. Then we can define the following endomorphisms on A_θ .

Define

$$\Psi_1(A) = \int_0^1 \rho_{1, e^{2\pi i t}}(A) dt \quad \text{and} \quad \Psi_2(A) = \int_0^1 \rho_{e^{2\pi i t}, 1}(A) dt,$$

where the integrals make sense as Riemann sums.

Proposition 8.2.2. Ψ_1 is a positive, contractive, faithful, idempotent surjection and Ψ_1 takes A_θ to $C^*(U)$.

Moreover, for a finite linear combination of $U^k V^l$ for $k, l \in \mathbb{Z}$, we have

$$\Psi_1\left(\sum a_{k,l} U^k V^l\right) = \sum a_{k,0} U^k \quad \text{and} \quad \Psi_1(A) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{-n}^n U^j A U^{-j}.$$

Similar results hold for $\Psi_2 : A_\theta \rightarrow C^*(V)$.

Theorem 8.2.3. The map $\tau = \Psi_1 \Psi_2 = \Psi_2 \Psi_1$ is a unique, faithful, unital, scalar valued trace on A_θ .

Proof. We show uniqueness and leave the rest to [5] after noting that $\tau(U^k V^l) = \delta_{k,0} \delta_{0,l}$ on each monomial $U^k V^l$, which means that τ picks out the coefficient of $U^0 V^0$.

If σ is another normalized trace on A_θ , we can note that by splitting σ over the sum and applying the trace property to the following expression, we find

$$\sigma(A) = \lim_{n \rightarrow \infty} \sigma\left(\frac{1}{2n+1} \sum_{-n}^n U^j A U^{-j}\right).$$

We know the latter part is equal to $\sigma(\Psi_1(A))$. Similar arguments for Ψ_2 show $\sigma(A) = \sigma(\Psi_2(A))$.

In total, we $\sigma(A) = \sigma(\Psi_2(A)) = \sigma(\Psi_1 \Psi_2(A)) = \sigma(\tau(A)) = \tau(A)$, where the last equality follows from the fact that $\tau(A)$ is a scalar. \square

The observant reader may have noted that we have not explicitly used $\theta \notin \mathbb{Q}$ to get the unique trace, but in fact, we have. It can be shown that the limit formula for the trace comes from a Birkhoff type-ergodic theorem ([41]) for C^* -algebras ([2]), which we can apply as irrational rotations yield ergodic transformations on the circle.

As expected, if θ is irrational, then A_θ is simple.

Theorem 8.2.4. A_θ is simple, which implies $C^*(\tilde{U}, \tilde{V})$ is canonically isomorphic to A_θ for any \tilde{U}, \tilde{V} satisfying the commutation relation.

Proof. Let I be a nonzero ideal of A_θ . We can find a positive, nonzero element x in I . The limit formula for Ψ_i shows the ideals map into themselves since $U^i x U^{-i}$ belongs to I , meaning that $\tau(x)$ belongs to I . Now, τ is faithful, meaning nonzero elements have a nonzero trace. Hence $\tau(x) = \lambda 1$ for some nonzero λ , but this means $I = A_\theta$.

By universality of A_θ , we can find a surjective $*$ -homomorphism from A_θ . This must then be an isomorphism since A_θ is simple. \square

Let us return to projections. The algebras $C^*(U)$ and $C^*(V)$ are both isomorphic to $C(\mathbb{T})$, which only contains trivial projections. We should not expect to find any nontrivial projections in A_θ , but the following result by Rieffel ([33]) tells us we could not be further away from the truth.

Theorem 8.2.5. For all $\alpha \in (\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$, there is a projection $p \in A_\theta$ such that $\tau(p) = \alpha$.

The proof can be found in [33] and [5], but it boils down to constructing projections on the form $p = T_g V + T_f + T_h V^*$, where T_x denotes the translation operator on $C(\mathbb{T})$, which is the rotation action on the circle. By computing the requirements $p^2 = p = p^*$, it is possible to find functions f, g and h such that $\tau(p) = \alpha$. This projection is called the Rieffel-Powers projection.

Queequeg 8.2.6. In [15], Luef creates projections in both weighted and smooth noncommutative tori by developing equivalent criteria in terms of Gabor frame of functions in weighted modulation spaces or Schwartz space, respectively. If we take $g \in M_s^1(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$, which can be realized as equivalence bimodules over certain noncommutative tori, the left inner product

$${}_\theta \langle g, g \rangle(l, t) = \sum_{k \in \mathbb{Z}} g(t - \alpha k) g(t - \alpha k - l)$$

is a projection in A_θ if and only if $\mathcal{G}(g, \Lambda)$ is a tight Gabor frame for $L^2(\mathbb{R})$. By the Morita equivalence between A_θ and $A_{-1/\theta}$, which we will soon return to, this is equivalent to

$$\langle g, g \rangle_{1/\theta} = \sum_{k \in \mathbb{Z}} g(t - \frac{k}{\alpha}) g(t - \frac{k}{\alpha} - l) = 1.$$

Rieffel choose an appropriate g such that just the case $k = 0$ is non-zero. This case is also known in Gabor analysis as *painless non-orthogonal Gabor expansions*. This means we can interpret the Rieffel projection ([33]) as coming from a tight Gabor frame, which gives a deeper reason for why such a projection should exist.

Through the embedding of A_θ in an AF-algebra, we obtain the following theorem, which can be found in [5].

Theorem 8.2.7. (Embedding noncommutative tori in AF-algebras)

There exists a $*$ -monomorphism $\rho : A_\theta \rightarrow \mathcal{A}_\theta$, where \mathcal{A}_θ is an AF-algebra, such that $\rho_* : K_0(A_\theta) \rightarrow K_0(\mathcal{A}_\theta)$ is a surjective $*$ -homomorphism. Moreover, if we let τ and σ denote the unique traces on A_θ and \mathcal{A}_θ , respectively, then $\tau_* = \sigma_* \rho_*$ is a surjective order homomorphism $K_0(A_\theta) \rightarrow \mathbb{Z} + \mathbb{Z}\theta$.

Remark 8.2.8. It is possible to find another approach to see that the AF-algebra we embed A_θ in has a unique trace. In [21], results on nuclear C^* -algebras are proved, such as the facts that all AF-algebras are nuclear and that the endofunctor $- \otimes A$ is exact for all nuclear C^* -algebras A . By 6.1.11, if we consider the unique trace on A_θ , we can consider the unique state in ordered K -theory. When embedding A_θ in an AF-algebra preserving the ordered K -theory, we can pull the state back uniquely to a trace on the AF-algebra, as AF-algebras are nuclear.

This leads to the following isomorphism classification. In fact, τ_* is an isomorphism, but we already have enough information to classify irrational rotation algebras.

Corollary 8.2.9. (Isomorphism classification)

Two irrational noncommutative tori A_θ and A_η are isomorphic if and only if $\eta \equiv \pm\theta \pmod{\mathbb{Z}}$.

Proof. If $\eta \equiv \pm\theta \pmod{\mathbb{Z}}$, these are certainly isomorphic as they can be made to obtain the same commutation relation, which means we can find a nonzero $*$ -homomorphism by universality. Due to the simplicity of irrational noncommutative tori, this must be an isomorphism.

Conversely, if A_θ and A_η are isomorphic, they have the same K_0 -group and the same unique trace, yielding an order isomorphism between $\mathbb{Z} + \mathbb{Z}\theta$ and $\mathbb{Z} + \mathbb{Z}\eta$. The above statement is an equivalence, due to the AF-algebra embedding. We also have $\mathbb{Z} + \mathbb{Z}\theta \cap [0, 1] = \mathbb{Z} + \mathbb{Z}\eta \cap [0, 1]$ as sets.

If these two are equal, we can observe that

$$\{e\theta \pmod{\mathbb{Z}} : e \in \mathbb{Z}\} = \{n + e\theta : n, e \in \mathbb{Z}, n + e\theta \in [0, 1]\} = \mathbb{Z} + \mathbb{Z}\theta \cap [0, 1].$$

Now, if we have $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1] = (\mathbb{Z} + \mathbb{Z}\eta) \cap [0, 1]$, we can find $a, a', b, b' \in \mathbb{Z}$ such that $e\theta = a + b\eta$ and $\theta = a' + b'\eta$. Observe that if $a + b\eta = c + d\eta$, we must have $a = c$ and $b = d$. If we work in \mathbb{R} , this implies that $\theta = a' + b'\eta = a/e + b/e\eta$, meaning $a' = a/e$ and $b' = b/e$, or equivalently, $e|a$ and $e|b$. By reversing the argument, we can find $b|a$ and $b|e$. If $e|b$ and $b|e$, we must have $e = \pm 1$ and

$b = \pm 1$, hence giving $\theta = a \pm \eta$. Since $a \in \mathbb{Z}$, we now have $\theta = \pm \eta \text{ mod } \mathbb{Z}$, as wanted. \square

8.3 Morita equivalence classification

This unique trace actually gives rise to the Morita equivalence classification as well.

The main result is the following, which can be found in [33] and [36].

Theorem 8.3.1. (Morita equivalence classification) Given two irrational numbers θ and η , A_θ and A_η are Morita equivalent if and only if θ and η are in the same orbit under the action of $GL(2, \mathbb{Z})$ on the irrational numbers as a linear fractional transformation.

That is, $GL(2, \mathbb{Z})$ acts as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \theta := \frac{a\theta + b}{c\theta + d}.$$

Proof. To show that they are isomorphic implies they are in the same orbit, we need more work, but for the converse, note that $GL(2, \mathbb{Z})$ is generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These send θ to $1/\theta$ and $\theta + 1$, respectively. The latter yield isomorphic noncommutative tori, and we claim that A_θ is Morita equivalent to $A_{1/\theta}$. This claim will be justified by the following lemma from [33]. \square

Lemma 8.3.2. Let G be a locally compact group and let H and K be closed subgroups of G . G acts by left and right translation on $C(G/H)$ and $C(G/K)$, respectively. We can restrict the action to K and H such that K acts on $C(G/H)$ and H acts on $C(G/K)$ with actions α_K and α_H , respectively.

Then the corresponding crossed product C^* -algebras $C(G/H) \times_{\alpha_K} K$ and $C(G/K) \times_{\alpha_H} H$ are Morita equivalent.

The claim now follows by taking $G = \mathbb{R}$, $H = \mathbb{Z}$ and $K = \mathbb{Z}\theta$. Then $C(\mathbb{R}/\mathbb{Z}) \times_{\alpha_{\mathbb{Z}\theta}} \mathbb{Z}\theta$ and $C(\mathbb{R}/\mathbb{Z}\theta) \times_{\alpha_{\mathbb{Z}}} \mathbb{Z}$ are Morita equivalent. The latter of these is equal to A_θ , while the first is isomorphic to $C(\mathbb{R}/\mathbb{Z}\theta^{-1}) \times_{\alpha_{\mathbb{Z}}} \mathbb{Z}$ by the homeomorphism $t \mapsto t\theta^{-1}$. Hence, A_θ is Morita equivalent to $A_{\theta^{-1}}$, meaning A_θ is Morita equivalent to A_η if they are in the same orbit of $GL(2, \mathbb{Z})$ by linear fractional transformations.

To show the converse statement, we need to utilize the unique traces.

Proposition 8.3.3. Given two C^* -algebras A and B with an $A - B$ -imprimitivity bimodule X , there is a bijection between the nonnormalized, finite traces on A and B . Given a trace τ on A , there exists a trace τ_X on B such that

$$\tau_X(\langle x, y \rangle_B) = \tau(\langle x, y \rangle_A) \quad \text{for all } x, y \in X.$$

The proof can be found in [33] along with the construction of τ_X .

The induced trace in K -theory that did most of the work for us. Linking traces on Morita equivalent C^* -algebras could be a fruitful approach to proving the theorem.

Recall theorem 4.5.17 that connects complementary full corners and Morita equivalence. Together with the following result from [33], we see that K -theory is an invariant for Morita equivalence.

Proposition 8.3.4. Let A be a unital C^* -algebra, let pAp be a full corner of A and let $i : pAp \rightarrow A$ be the injection. Then $i_* : K_0(pAp) \rightarrow K_0(A)$ is an isomorphism.

We also have the following proposition, which will be important.

Proposition 8.3.5. Assume we have two C^* -algebras A and B with an $A - B$ -imprimitivity bimodule X and a finite trace τ on A , with τ_X as the induced nonnormalized trace on B . Let ϕ_X denote the isomorphism $K_0(A) \rightarrow K_0(B)$ induced through the bimodule X and $\tau_*, \tau_{X,*}$ the induced traces in K -theory. Then $\tau_{X,*} \circ \phi_X = \tau_*$. In fact, the ranges of τ_* and $\tau_{X,*}$ are equal.

With these results, we actually have enough to prove the last part of the Morita equivalence classification.

Theorem 8.3.6. (Revisiting Morita equivalence classification)

Given two irrational numbers θ and η , A_θ and A_η are Morita equivalent if and only if θ and η are in the same orbit under the action of $GL(2, \mathbb{Z})$ on the irrational numbers as a linear fractional transformation.

Proof. We only need to prove that if A_θ and A_η are Morita equivalent, we will find θ and η in the same orbit of $GL(2, \mathbb{Z})$. Now, assume we have an equivalence bimodule X between A_θ and A_η and let τ be the unique normalized trace on A_θ . Then we can induce the nonnormalized trace τ_X on A_η such that $\tau_{X,*}(K_0(A_\eta)) = \tau_*(K_0(A_\theta))$. Since there is a unique trace on A_η , τ_X must differ from this trace by a scalar r , meaning that the equality $\tau_{X,*}(K_0(A_\eta)) = \tau_*(K_0(A_\theta))$ must be given by multiplication with r . More explicitly, $\mathbb{Z} + \mathbb{Z}\theta = r(\mathbb{Z} + \mathbb{Z}\eta)$. Now, we can find $a, b, c, d \in \mathbb{Z}$ such that $1 = r(a + b\eta)$ and $r = c + d\theta$, meaning that $\frac{1}{c+d\theta} = a + b\eta$. By rearranging, we have $\eta = \frac{1}{b}(\frac{1}{c+d\theta} - a) = \frac{1-ac-bd\theta}{bc+bd\theta}$, which we recognize as a linear fractional transformation. By symmetry of Morita equivalence, it must be invertible, which means the action is indeed from $GL(2, \mathbb{Z})$. \square

8.4 Rational noncommutative tori and construction of projections

A natural question now is to ask what happens if we do not assume θ to be rational.

In fact, Rieffel shows the following result by Høegh-Krohn and Skjelbred in [34].

Theorem 8.4.1. Given two rational numbers α and β in $[0, 1/2]$, then $A_\alpha \cong A_\beta$ if and only if $\alpha = \beta$.

This means, in the search of isomorphism classification, we have found one unifying result, namely $A_\alpha \cong A_\beta$ if and only if $\alpha = \beta$ for $\alpha, \beta \in [0, 1/2]$. Even though this holds both in the rational and irrational case, the rational case requires much more work to prove, and the proofs are not even close to similar.

For Morita equivalence, the rational case turns out to be quite simple.

Theorem 8.4.2. Let $\theta \in \mathbb{Q}$. Then A_θ is Morita equivalent to $C(\mathbb{T}^2)$.

This was shown by Rieffel in [34], and the argument goes along the same lines as when we argued A_θ and $A_{\theta^{-1}}$ are Morita equivalent.

We follow Rieffel ([33]) and end with a curiosity, linking equivalence bimodules to creating projections. This does for example mean that we can create projections in noncommutative tori directly by considering functions in time-frequency related spaces, such as Schwarz space and Feichtinger's algebra, as these are realized as equivalence bimodules between certain noncommutative tori. This can be found in [17].

Proposition 8.4.3. Let X be an $A - B$ -imprimitivity bimodule. For $x \in X$, the element ${}_A\langle x, x \rangle$ is a projection if and only if $x\langle x, x \rangle_B = x$.

Proof. If the latter holds, then ${}_A\langle x, x \rangle x = x\langle x, x \rangle_B$ implies ${}_A\langle x, x \rangle$ is idempotent. We already know ${}_A\langle x, x \rangle$ is self-adjoint. If the first holds, then

$${}_A\langle x\langle x, x \rangle_B - x, x\langle x, x \rangle_B - x \rangle = 0,$$

yielding $x\langle x, x \rangle_B = x$. □

8.5 Closing remarks and final digressions

There are a lot of concepts, theories, and beautiful rabbit holes that the author would love to discover. The small digressions to Gabor analysis and time-frequency analysis could certainly be extended, but due to the authors lack of knowledge in this area (or perhaps time and space), we settle for taunting hints at deeper connections. To recycle references, we can refer to [14], [15], [16], [17] and of course [7] more depth on this topic.

There are lots and lots of extensions (literally) to K -theory and Morita equivalences, but to avoid forgetting some of them, we simply refer to [1] and the references therein.

The excursion to AF-algebras could have been extended, but perhaps from a different point of view. Interested readers are referred to [6] for a treatise.

Without too much (formal) knowledge from theoretical physics, the author is amazed at the attempts at doing Yang-Mills theory on noncommutative tori. We attempt to explain it in the following manner, and we refer to [12] and [36] to correct any mistakes or lead to other references.

Connes showed all finitely generated projective modules V over noncommutative tori admit a constant curvature connection ∇ . With this connection, Rieffel and Connes defined a Yang-Mills energy functional by $YM(\nabla) = -\tau_{\text{End}(V)}(\{\Theta_\nabla, \Theta_\nabla\})$, where Θ_∇ denotes the curvature of ∇ and $\{\cdot, \cdot\}$ denotes an analogue of the Poisson bracket. They also showed constant curvature connections yield minima of the Yang-Mills functional, and hence they will be the YM -connections, as these are the ones minimizing the energy functional.

Of course, there are many other results and interesting topics we could delve into, but for now, we part ways with a poem.

A parting poem

*With new concepts now stuck in my head
as I lay down to sleep in my bed
my brain will be jamming
the digressions it's cramming
but to more mathematics it lead*

- *The author*

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