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A homological approach to the Poincaré--Birkhoff--Witt theorem

Bachelor's project in Mathemathics
Supervisor: Markus Szymik
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1 Introduction

The goal of this thesis is to develop some tools including Hochschild cohomology, filtered and graded algebras and algebraic deformation theory in order to take a conceptual approach to proving the Poincare-Birkoff-Witt(PBW) theorem.
The thesis will follow the proof of a generalized version of the theorem given in [BG96] and [Wit19].
In order to read the thesis some knowledge of homological algebra is required. Throughout the thesis we will be working over an arbitrary field denoted K, elements of which will usually be referred to as k. All tensor products are taken over K unless otherwise specified. If tensor products are unfamiliar then the tensor product of vector spaces may simply be thought of as the vector space with the cartesian product of bases as basis.

Theorem 1.1 (The Classical PBW theorem)**.** *Given a Lie algebra L over a field K, the associated graded of its universal enveloping algebra denoted U(L) is isomorphic to the symmetric algebra generated by L denoted S(L).*

The new version of the proof provides some interesting context to the theorem. It shows that for finite dimensional Lie algebras that up to some classification of algebras the universal algebras are "close" to the symmetric algebra. With some more theory it is possible to prove that this closeness is equivalent to the bracket satisfying the Jacobi identity.
The thesis will be organized as follows: first the concepts used in the statement of the theorem will be introduced. Then we will introduce some new tools and or concepts among which are Hochschild cohomology, filtered and graded algebras and algebraic deformations, which we will then use to prove the theorem.

1.1 Tensor algebra

A tensor algebra of a vector space V, denoted T(V) is the space $\bigoplus_{i=0}^{\infty} V^{\otimes n}$ With multiplication given by concatenating elements with a tensor.

1.2 Symmetric algebra

The symmetric algebra of a k-vector space V, commonly denoted as S(V) is defined as follows.

$$S(V) = \frac{T(V)}{(x \otimes y - y \otimes x)}$$

Lemma 1.1. *Given a vector space V with basis B we have the following algebra isomorphism, which can be constructed by sending the basis elements of v to themselves and then extending multiplicitatively and additively.*

$$S(V) \simeq K[B]$$

1.3 Exterior algebra

The exterior algebra of a vector space V denoted $\Lambda(V)$ is defined as

$$\Lambda(V) = \frac{T(V)}{(x \otimes x)}$$

It is common to use $\wedge$ instead of tensors between the indices of the elements of this algebra.
The algebra is anticommutative, which means that $\bigwedge \sigma(v_i) = sign(\sigma) \bigwedge v_i$.
Where here σ is some permutation taken from the symmetric group on n letters.
Due to this a basis for this algebra is given by imposing some total order on the basis of V and then ordered sequences of basis elements of V becomes a basis for $\Lambda(V)$.
$\Lambda^n(V)$ denotes the vector subspace that is the image of $V^{\otimes n}$. That is wedges of length n.

1.4 Lie algebra

Definition 1.1 (Lie algebra)**.** *A Lie algebra over a field k is a k- vector space together with a multiplication (not necessarily associative) often denoted as a bracket* $[,]$ *such that the bracket is anti-symmetric and bilinear over k and satisfies the following permutation equation, known as the jacobi identity.*

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$$

The morphisms in the category of k-Lie algebras are k- homomorphisms that respect the bracket.

Example 1.1 (Trivial example)**.** *Given any k-vectorspace it can be made into a k-Lie algebra trivial by choosing the bracket* $[a, b] = 0$

Example 1.2 (Associative algebras)**.** *Given an assoicitive k-algebra A, We can make A into a k-Lie algebra by considering the bracket* $[a, b] = ab - ba$*, this gives us a forgetfull functor from the category of associative k-algebras to the category of k-Lie algebras.*

1.4.1 Universal enveloping algebra

We can find an adjoint to the forgetfull functor from the category of associative k-algebras to the category of Lie algebras. This functor called the universal enveloping algebra of a Lie algebra L denoted U(L) can be described explicitly as follows.

$$U(L) = \frac{T(L)}{(a \otimes b - b \otimes a - [a, b])}$$

With morphisms being given on the degree one element and extended multiplicatively to the rest of the algebra.
To see that this is an actual adjoint to the forgetfull functor we may observe the

existence of a unit counit adjunction with injection of L into Lie(U(L)), denoted η_L as unit and mapping the degree one elements to themselves with evaluation of multiplication on the higher degree elements from U(Lie(A)) to A denoted ϵ_A as co-unit. Note that the last map is well defined since the relations we have modded out of the tensor algebra also exists in A.
In order to show that this is a unit-co unit pair we need to show that $Id_{U(L)} = \epsilon_{U(L)} \circ U(\eta_L)$ and $Id_{Lie(A)} = Lie(\epsilon_A) \circ \eta_{Lie(A)}$.

$\epsilon_{U(L)} \circ U(\eta_L) : U(L) \xrightarrow{U(\eta_L)} U(Lie(U(L))) \xrightarrow{\epsilon_{U(L)}} U(L)$

Since this is a map on the universal enveloping algebra of L it is enough to see what happens on generators, that is, elements of L. It is immoderate to see that the elements of L are sent to themselves over this composition, since both maps are defined by sending L to itself and extending. This means that the composition is equal to the identity.

$$Lie(\epsilon_A) \circ \eta_{Lie(A)} : Lie(A) \xrightarrow{\eta_{Lie(A)}} Lie(U(Lie(A))) \xrightarrow{Lie(\epsilon_A)} Lie(A)$$

Again this can be seen to be the identity by sending the elements of A over the two morphisms, noting that neither changes the elements of A.

2 New tools/concepts

2.1 Filtered and graded algebras

2.1.1 Graded algebras

In this thesis we will only consider algebras graded by integers. A graded algebra A is then a k-algebra with a decomposition into a direct sum over the integers $A \simeq \bigoplus_{z \in \mathbb{Z}} A_z$ as a vector space with the additional requirement that $A_n \cdot A_m \subseteq A_{m+n}$. A homogeneous element is defined as an element that is contained in a single summand. The index of a summand is commonly refered to as the degree. The degree of an element is usually considered to be the index of the biggest summand on which the projection of the element is non-zero.
The canonical example of a graded algebra is a polynomial ring over a field. Another good example is a tensor algebra. A graded algebra modulo an ideal generated by a collection of homogeneous elements is again a graded algebra.

We can also have two simultaneous gradings on an algebra, this is called a bi-grading and there is nothing surprising about it. It consists of a decomposition of $A \simeq \bigoplus_{(z_1,z_2) \in \mathbb{Z} \times \mathbb{Z}} A_{(z_1,z_2)}$ such that $A_{(z_1,z_2)} \cdot A_{(z_3,z_4)} \subseteq A_{(z_1+z_3,z_2+z_4)}$.
Again a natural example is the polynomial algebra over a tensor algebra.

Definition 2.1 (Homogeneous ideal)**.** *A homogeneous ideal of a graded algebra A is an ideal such that every element of the ideal can be factored into a sum of homogeneous element.*
Note that this is equivalent to being generated by a set of homogeneous elements.

2.1.2 Filtered algebras

A filtered algebra is an algebra B with a set of sub-vectorspaces $\{U_z\}$ for each integer such that $U_n \subseteq U_{n+1}$ and $U_n \cdot U_m \subseteq U_{n+m}$ and $\cup_{z\in\mathbb{Z}} U_z = B$. A graded algebra can be considered as a filtered algebra by letting $U_n = \bigoplus_{i\in\mathbb{Z}|i\leq n} A_i$. A natural example of filtered algebras are graded algebras modulo a non-homogeneous ideal. These algebras will no longer be graded, however they will be filtered. With $U_n = p(\bigoplus_{i\in\mathbb{Z}|i\leq n} A_i)$ where p is the projection onto the quotient. Note that since p is surjective this is exhaustive and since p is a homomorphism it satisfies the multiplication criterion.

2.1.3 Associated graded algebra

Since we can go from graded algebras to filtered algebras it is natural to ask if we can go the other way. The answer to this question is the associated graded algebra. The associated graded algebra of a filtered algebra is defined as the vector space $\bigoplus_{z\in\mathbb{Z}} A_z$ where $A_z = \frac{U_z}{U_{z-1}}$ with multiplication given on homogeneous elements by $(u_z+U_{z-1})\cdot(u_n+U_{n-1}) = u_z\cdot u_n+u_z\cdot U_{n-1}+U_{z_1}\cdot u_n+U_{z-1}\cdot U_{n-1} = u_z \cdot u_n + U_{n+z-1}$.

2.1.4 Associated graded morphism

Given a morphism of filtered algebras $\phi : A \to B$ that respects the filter, there is a natural definition of the associated graded morphism. $Gr(\phi) : Gr(A) \to Gr(B)$ defined on homogeneous elements as $(\phi)(a_i + U_{i-1}) = \phi(a_i) + \phi(U_{i-1}))$ This is well defined since U_{i-1} maps into U'_{i-1}.

2.1.5 Graded modules

Given a graded algebra A, we can define graded modules over the algebra.

Definition 2.2 (Graded module)**.** *A graded module M over an algebra A is a A module with a decomposition $M \simeq \bigoplus_{z\in\mathbb{Z}} M_z$ as vector spaces and such that $A_n \cdot M_z \subseteq M_{n+z}$*

Given a grading on an algebra A we can define a grading on the tensor product $A^{\otimes n}$ by letting

$$A_z^{\otimes n} = \bigoplus_{(\alpha_j)_1^n|\sum_{i=1}^n \alpha_j=z} \bigotimes_{j=1}^n A_{\alpha_j}$$

This means that for a sequence of homogeneous elements $deg(\bigotimes_{j=1}^{n} a_j) = \sum_{j=1}^{n} deg(a_j)$ similar to how degrees are usually defined in multivariable polynomial rings.

Given a graded algebra A, we can define what is know as the category of graded modules where the objects are graded modules and the hom sets are defined as follows.

Definition 2.3 (Graded Hom)**.**

$$Hom_{A,Gr}(M,N) = \bigoplus_{z \in Z} Hom_A(M,N)_z = \bigoplus_{z \in Z} \{f \in Hom_A(M,N) \mid f(M)_i \subseteq A_{i+z} \forall i\}$$

Note that this usually isn't the same as $Hom_A(M,N)$ since it is a direct sum, not a direct product. It is only a very nice canonical subspace.
We can then define homogeneous morphisms of degree n to be morphisms contained in $Hom_A(M,N)_n$.

2.2 Hochschild cohomology

Note that this entire section about Hochschild cohomology will be very analogous to simplicial cohomology as usually taught in algebraic topology 1.

2.2.1 A^e

Given a k algebra A we define an algebra called the enveloping algebra by $A^e = A \otimes_k A$ with multiplication defined as $(a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (d \cdot b)$
The big advantage with this algebra is that it simplifies language as a left A^e module is equivalent to an A-bimodule. To see this take an A bi-module M and define M as a left A^e module by defining $(a \otimes b)m = a(mb) = (am)b = amb$. observe that this is a left A^e module since $(c \otimes d)\big((a \otimes b)m\big) = (c \otimes d)(amb) = cambd = (ca \otimes bd)m$
Given a left A^e module M define M as a a bi-module by defining $am = (a \otimes 1)m$ and $mb = (1 \otimes b)m$. To see that this is infact an A bi-module observe the following equation.

$$(am)b = (1 \otimes b)\big((a \otimes 1)m\big) = \big((1 \otimes b)(a \otimes 1)\big)m = (a \otimes b)m = (a \otimes 1)(1 \otimes b)m = a(mb)$$

Now we observe that these two processes are mutually inverse.
We now observe that A is an A^e module since it is a bimodule over itself. Further we can define $A^{\otimes_k n}$ to be an A^e module by defining multiplication as $(a \otimes b)(a_1 \otimes \cdots \otimes a_n) = (aa_1 \otimes \cdots \otimes a_n b)$

2.2.2 Definition

The n-th Hochschild cohomology of a k-algebra A is defined as $HH^n(A) = Ext^n_{A^e}(A,A)$

2.2.3 The bar complex

The bar complex is our complex of choice for calculating the Hochschild chomology of an algebra A. using this complex gives us clear interpertations of the Hochschild cohomology.
The bar complex is defined as follows $B_n = A^{\otimes n+2}$ and $d_B^n : B_n \to B_{n-1}$ with $d_B^n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n}(-1)^i(\bigotimes_{j=0}^{i-1} a_j) \otimes a_i \cdot a_{i+1} \otimes (\bigotimes_{j=i+2}^{n+1} a_j)$.

$$\begin{array}{ccccccc} \ldots & \xrightarrow{d_B^2} & A \otimes_k A \otimes_k A & \xrightarrow{d_B^1} & A \otimes_k A & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle \pi} & & \\ & & & & A & & \end{array}$$

In order to see that this is an exact sequence of projective A^e modules we first observe that $A^{\otimes n}$ is a free A^e module by first choosing a basis B for $A^{\otimes n-2}$ as a k vector space, and then observing that $\{1 \otimes B \otimes 1\}$ becomes a A^e basis of $A^{\otimes n}$.
Next we observe that

$$\begin{aligned} d_B^{i-1} d_B^i(a_1 \otimes \cdots \otimes a_{n+1}) &= d_B^{i-1}\left(\sum_{i=1}^{n-1}(-1)^i(\bigotimes_{j=1}^{i-1} a_j) \otimes a_i \cdot a_{i+1} \otimes (\bigotimes_{j=i+2}^{n} a_j)\right) \\ &= \sum_{k=1}^{n-2}\sum_{i=1}^{n-1}(-1)^{i+k}(\bigotimes_{j=1}^{i-1} a_j) \otimes a_i \cdot a_{i+1} \otimes (\bigotimes_{j=i+2}^{n} a_j) \\ &= \sum_{k=1}^{n-2}\sum_{i=1}^{n-1}(-1)^{i+k}(i,k)(\bigotimes_{j=1}^{n} a_j) \end{aligned}$$

Here the touple (i,k) refers to first collapsing the ith tensor, then the kth tensor of the resulting complex. Now we note that if $k < i$ first collapsing the ith tensor, then the kth tensor gives the same result as first collapsing the kth tensor then collapsing the i-1st tensor.The only difference is that these two will have opposite signs in the sum above, and therefore cancel. These pairs exhaust the sum, therefore $d^{i-1}d^i = 0$ and hence this is a complex.
To see that it is exact we note that calculating homology of the complex as a A^e complex is the same as calculating the homology as a k-complex. Then we see that the identity map on the complex is homotopic to the zero map as a k-complex by the homotopy $h_n : A^{\otimes n+2} \to A^{\otimes n+3}$.

$$\bigotimes_{j=0}^{n+1} a_j \longrightarrow 1 \otimes \bigotimes_{j=0}^{n+1} a_j$$

We verify that this is a nullhomotopy of the identity map with the following equation.

$$
\begin{aligned}
(d^{n+1}h_n + h_{n-1}d^n)(\bigotimes_{j=0}^{n+1} a_j) &= d^{n+1}(1 \otimes \bigotimes_{j=0}^{n+1} a_j) + 1 \otimes \Big(\sum_{i=0}^{n}(-1)^i(\bigotimes_{j=0}^{i-1} a_j) \otimes a_i \cdot a_{i+1} \otimes (\bigotimes_{j=i+2}^{n+1} a_j)\Big) \\
&= 1 \cdot a_0 \otimes \bigotimes_{j=1}^{n+1} a_j + \sum_{i=0}^{n}(-1)^{i+1}(1 \otimes \bigotimes_{j=0}^{i-1} a_j) \otimes a_i \cdot a_{i+1} \otimes (\bigotimes_{j=i+2}^{n+1} a_j) \\
&+ \sum_{i=0}^{n}(-1)^i(1 \otimes \bigotimes_{j=0}^{i-1} a_j) \otimes a_i \cdot a_{i+1} \otimes (\bigotimes_{j=i+2}^{n+1} a_j) \\
&= \bigotimes_{j=0}^{n+1} a_j
\end{aligned}
$$

2.2.4 Rewriting the Hochschild cohomology/bar complex

2.2.5 Writing out the Hochschild cohomology

Note first that $Hom_{A^e}(A^{\otimes n+2}, A) \simeq Hom_k(A^{\otimes n}, A)$ as k modules by the two mutually inverse morphisms given below

$$
\begin{aligned}
f_n : Hom_{A^e}(A^{\otimes n+2}, A) &\longrightarrow Hom_k(A^{\otimes n}, A) \\
\phi &\longmapsto \big[(a_1 \otimes \cdots \otimes a_n) \to \phi(1 \otimes a_1 \otimes \ldots a_n \otimes 1)\big]
\end{aligned}
$$

$$
\begin{aligned}
f_n^{-1} : Hom_k(A^{\otimes n}, A) &\longrightarrow Hom_{A^e}(A^{\otimes n+2}, A) \\
\phi &\longmapsto \big[(a_0 \otimes \cdots \otimes a_{n+1}) \to (a_0 \otimes a_{n+1})\phi(a_2 \otimes \ldots a_{n-1})\big]
\end{aligned}
$$

Using these isomorphisms the differentials we also get a new version of the differentials $f_{n-1} \circ d^{n*} \circ f_n^{-1} = d_k^{n*} : Hom_k(A^{\otimes n}, A) \to Hom_k(A^{\otimes n-1}, A)$
Here d^{n*} refers to precomposition with d^n.
By calculating concretely we get the following description.

$$
d_k^{n*}(f)(\bigotimes_{i=1}^{n} a_i) = a_1 \cdot f(\bigotimes_{j=1}^{n} a_i) + \sum_{i=1}^{n-1}(-1)^i f(\bigotimes_{j=1}^{i} a_j{}_i \cdot a_{i+1} \bigotimes_{j=i+2}^{n} a_i) + (-1)^n f(\bigotimes_{i=1}^{n-1} a_i) \cdot a_n
$$

From now on this is the way we will study the Hochschild cohomology. The differential will be referred to as d^* instead of d_k^{n*} for simplicity reasons. The degree will always be implicit.
This way of looking at Hochschild cohomology simplifies some calculation and also makes it possible to find some very concrete realisations of Hochschild cohomology. However, the original formulation in terms of bimodule homology really is the one that links this construction to the structure on A in a meaning full way.

2.2.6 Gerstenhaber bracket

This section presents selected concepts from [**M**; Ger63]

Definition 2.4 (I-th composition). *Given $f \in Hom_k(A^{\otimes n}, A)$ and $g \in Hom_k(A^{\otimes m}, A)$ with $i \leq n$ the ith composition is defined as follows.*

$$f \circ_i g(\bigotimes_{j=1}^{n+m-1} a_j) = f(\bigotimes_{j=1}^{i-1} a_j \otimes g(\bigotimes_{j=i}^{i+m-1} a_j) \otimes \bigotimes_{j=i+m}^{n+m-1} a_j)$$

Definition 2.5 (Ring product). *Given $f \in Hom_k(A^{\otimes n}, A)$ and $g \in Hom_k(A^{\otimes m}, A)$ the ring product is defined as follows.*

$$f \circ g(\bigotimes_{j=1}^{n+m-1} a_j) = \sum_{i=1}^{m} (-1)^{(m-1)(i-1)} f \circ_i g(\bigotimes_{j=1}^{n+m-1} a_j) = \sum_{i=1}^{m} (-1)^{(m-1)(i-1)} f(\bigotimes_{j=1}^{i-1} a_j \otimes g(\bigotimes_{j=i}^{i+m-1} a_j) \otimes \bigotimes_{j=i+m}^{n+m-1} a_j)$$

The ring product distributes over addition since the ith-composition does so due to distrubutivuty of tensor products. The ring product is however not associative.
Note that the notation is well defined since in the case that the two morphisms are composable the ring product is the composition of the morphisms.

Definition 2.6 (Gerstenhaber bracket). *Given $f \in Hom_k(A^{\otimes n}, A)$ and $g \in Hom_k(A^{\otimes m}, A)$ the Gerstenhaber bracket is defined as follows.*

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

A useful consequence of this definition is that the differential d^* be realised as $[-, \pi]$

Lemma 2.1. *Given $f \in Hom_k(A^{\otimes n}, A)$ and $g \in Hom_k(A^{\otimes m}, A)$ then*

$$d^*(f \circ g) = (-1)^{m-1} d^*(f) \circ g + f \circ d^*(g) + (-1)^{m-1}((-1)^{nm} f \cdot g - g \cdot f)$$

Proof.

$$d_k^*(f\circ g)(\bigotimes_{j=1}^{m+n} a_j) = a_1(\sum_{i=1}^{m}(-1)^{(m-1)(i-1)} f(\bigotimes_{j=1}^{i-1} a_{j+1}\otimes g(\bigotimes_{j=i}^{i+m-1} a_{j+1})\otimes \bigotimes_{j=i+m}^{n+m-1} a_{j+1})) \tag{1}$$

$$+\sum_{i=1}^{m}(-1)^{(m-1)(i-1)}\sum_{s=1}^{i-1}(-1)^s f(\bigotimes_{j=1}^{s-1} a_j\otimes a_s\cdot a_{s+1}\otimes \bigotimes_{j=s+2}^{i-1} a_{j+1}\otimes g(\bigotimes_{j=i}^{i+m-1} a_{j+1})\otimes \bigotimes_{j=i+m}^{n+m-1} a_{j+1}) \tag{2}$$

$$+\sum_{i=1}^{m}(-1)^{(m-1)(i-1)}\sum_{s=i}^{i+m-1}(-1)^s f(\bigotimes_{j=1}^{i-1} a_j\otimes g(\bigotimes_{j=i}^{s-1} a_j\otimes a_s\cdot a_{s+1}\otimes \bigotimes_{s+1}^{i+m-1} a_{j+1})\otimes \bigotimes_{j=i+m}^{n+m-1} a_{j+1}) \tag{3}$$

$$+\sum_{i=1}^{m}(-1)^{(m-1)(i-1)}\sum_{s=i+m}^{n+m-1}(-1)^s f(\bigotimes_{j=1}^{i-1} a_j\otimes g(\bigotimes_{j=i}^{i+m-1} a_j)\otimes \bigotimes_{j=i+m}^{s}\otimes a_s\cdot a_{s+1}\otimes \bigotimes_{j=s+1}^{n+m-1} a_{j+1}) \tag{4}$$

$$+(-1)^{n+m}\sum_{i=1}^{m}(-1)^{(m-1)(i-1)} f(\bigotimes_{j=1}^{i-1} a_j\otimes g(\bigotimes_{j=i}^{i+m-1} a_j)\otimes \bigotimes_{j=i+m}^{n+m-1} a_j)\cdot a_{n+m} \tag{5}$$

$$f\circ d^*(g)(\bigotimes_{j=1}^{m+n} a_j) = \sum_{i=1}^{n}(-1)^{(m)(i-1)} f(\bigotimes_{j=1}^{i-2} a_j\otimes a_{i-1}\cdot g(\bigotimes_{j=i}^{i+m-1} a_j)\otimes \bigotimes_{j=i+m}^{n+m} a_j) \tag{6}$$

$$+\sum_{i=1}^{n}(-1)^{(m)(i-1)}\sum_{s=i}^{i+n-1}(-1)^{s-i+1} f(\bigotimes_{j=1}^{i-1} a_j\otimes g(\bigotimes_{j=i}^{s-1} a_j\otimes a_s\cdot a_{s+1}\otimes \bigotimes_{s+1}^{i+m-1} a_{j+1})\otimes \bigotimes_{j=i+m}^{n+m-1} a_{j+1}) \tag{7}$$

$$+\sum_{i=1}^{n}(-1)^{(m)(i-1)}(-1)^i f(\bigotimes_{j=1}^{i-1} a_j\otimes g(\bigotimes_{j=i}^{i+m-1} a_j)\cdot a_{i+m}\bigotimes_{j=i+m+1}^{n+m} a_j) \tag{8}$$

$$d^*(f)\circ g(\bigotimes_{j=1}^{m+n} a_j) \tag{9}$$

$$= g(\bigotimes_{j=1}^{n} a_j) f(\bigotimes_{j=n+1}^{i+m} a_j) \tag{10}$$

$$+ a_1(\sum_{i=1}^{m}(-1)^{(m-1)(i)} f(\bigotimes_{j=1}^{i-1} a_{j+1} \otimes g(\bigotimes_{j=i}^{i+m-1} a_{j+1}) \otimes \bigotimes_{j=i+m}^{n+m-1} a_{j+1})) \tag{11}$$

$$+ \sum_{i=1}^{m}(-1)^{(m-1)(i)} \sum_{s=1}^{i-1}(-1)^s f(\bigotimes_{j=1}^{s-1} a_j \otimes a_s \cdot a_{s+1} \otimes \bigotimes_{j=s+2}^{i-1} a_{j+1} \otimes g(\bigotimes_{j=i}^{i+m-1} a_{j+1}) \otimes \bigotimes_{j=i+m}^{n+m-1} a_{j+1}) \tag{12}$$

$$+ \sum_{i=1}^{m}(-1)^{(m-1)(i)}(-1)^{i-1} f(\bigotimes_{j=1}^{i-2} a_j \otimes a_{i-1} \cdot g(\bigotimes_{j=i}^{i+m-1} a_j) \otimes \bigotimes_{j=i+m}^{n+m} a_j) \tag{13}$$

$$+ \sum_{i=1}^{m}(-1)^{(m-1)(i-1)}(-1)^{i} f(\bigotimes_{j=1}^{i-1} a_j \otimes g(\bigotimes_{j=i}^{i+m-1} a_j) \cdot a_{i+m} \otimes \bigotimes_{j=i+m}^{n+m} a_{j+1}) \tag{14}$$

$$+ \sum_{i=1}^{m}(-1)^{(m-1)(i-1)} \sum_{s=i+m}^{n+m-1}(-1)^{s-m+1} f(\bigotimes_{j=1}^{i-1} a_j \otimes g(\bigotimes_{j=i}^{i+m-1} a_j) \otimes \bigotimes_{j=i+m}^{s} \otimes a_s \cdot a_{s+1} \otimes \bigotimes_{j=s+1}^{n+m-1} a_{j+1}) \tag{15}$$

$$+ (-1)^n \sum_{i=1}^{m}(-1)^{(m-1)(i-1)} f(\bigotimes_{j=1}^{i-1} a_j \otimes g(\bigotimes_{j=i}^{i+m-1} a_j) \otimes \bigotimes_{j=i+m}^{n+m-1} a_j) \cdot a_{n+m} \tag{16}$$

$$+ (-1)^{(n+1)+(n+1)(m-1)} f(\bigotimes_{j=1}^{n} a_j) g(\bigotimes_{j=n+1}^{n+m} a_j) \tag{17}$$

Now we note that
$(1) = (-1)^{m-1}(11)$
$(2) = (-1)^{m-1}(12)$
$(4) = (-1)^{(m-1)}(15)$
$(5) = (-1)^{(m-1)}(16)$
$(3) = (7)$.

Further we see that the rest cancel out, ie.
$(-1)^{m-1}(13) + (6) = 0$
$(-1)^{m-1}(14) + (7) = 0$
$(-1)^{m-1}(10) - (-1)^{m-1} g \cdot f = 0$
$(-1)^{m-1}(17) + (-1)^{m-1(nm)} f \cdot g = 0$
Verifying this comes down to counting signs since the expressions are otherwise

identical.

∎

Lemma 2.2. *Given $f \in Hom_k(A^{\otimes n}, A)$ and $g \in Hom_k(A^{\otimes m}, A)$ then*

$$d_k^*[f,g] = (-1)^{m-1}[d^*(f), g] + [f, d^*(g)]$$

Proof.

$$\begin{aligned} d_k^*[f,g] &= d^*(f \circ g) - (-1)^{(m-1)(n-1)} d^*(g \circ f) \\ &= (-1)^{m-1} d^*(f) \circ g + f \circ d^*(g) + (-1)^{m-1}((-1)^{nm} f \cdot g - g \cdot f) \\ &- (-1)^{(m-1)(n-1)}((-1)^{n-1} d^*(g) \circ f + g \circ d^*(f) + (-1)^{n-1}((-1)^{nm} g \cdot f - f \cdot g)) \\ &= (-1)^{m-1} d^*(f) \circ g + f \circ d^*(g)) \\ &- (-1)^{(m-1)(n-1)}((-1)^{n-1} d^*(g) \circ f + g \circ d^*(f)) \\ &= (-1)^{m-1}[d^*(f), g] + [f, d^*(g)] \end{aligned}$$

∎

2.2.7 Graded cohomology

Given that A is a graded algebra we can define a grading on A^e and $A^{\otimes n+2}$ by the tensor product grading. This makes $A^{\otimes n+2}$ a graded module over A^e. We can then define $Hom_{Gr,A^e}(A^{\otimes n+2}, A)$ and $Hom_{A^e}(A^{\otimes n+2}, A)_z$ which we can translate to defining $Hom_k(A^{\otimes n}, A)_z$.
Note that $Hom_{Gr,A^e}(A^{\otimes n+2}, A)$ will only be a subspace of $Hom_{A^e}(A^{\otimes n+2}, A)$.

Lemma 2.3. *Given two homogeneous morphisms $f \in Hom_k(A^{\otimes n}, A)_l$ $g \in Hom_k(A^{\otimes m}, A)_p$ then $f \circ g \in Hom_k(A^{\otimes n+m-1}, A)_{l+p}$*

Proof. It is enough to show $f \circ_i g \in Hom_k(A^{\otimes n+m-1}, A)_{l+p} \forall i$ since $Hom_k(A^{\otimes n+m-1}, A)_{l+p}$ is closed under addition and multiplication from k.

$$\begin{aligned} f \circ_i g(A_z^{\otimes n+m-1}) &= f \circ_i g\Big(\bigoplus_{(\alpha_j)_1^{n+m-1} | \sum_{j=1}^{n+m-1} \alpha_j = z} \bigotimes_{j=1}^{n+m-1} A_{\alpha_j}\Big) \\ &= \sum_{(\alpha_j)_1^{n+m-1} | \sum_{j=1}^{n+m-1} \alpha_j = z} f \circ_i g\Big(\bigotimes_{j=1}^{n+m-1} A_{\alpha_j}\Big) \\ &= \sum_{(\alpha_j)_1^{n+m-1} | \sum_{j=1}^{n+m-1} \alpha_j = z} f\Big(\bigotimes_{j=1}^{i-1} A_{\alpha_j} \otimes g\Big(\bigotimes_{j=i}^{i+m-1} A_{\alpha_j}\Big) \otimes \bigotimes_{j=i+m}^{n+m-1} A_{\alpha_j}\Big) \\ &\subseteq \sum_{(\alpha_j)_1^{n+m-1} | \sum_{j=1}^{n+m-1} \alpha_j = z} f\Big(\bigotimes_{j=1}^{i-1} A_{\alpha_j} \otimes A_{p+\sum_{j=i}^{i+m-1} \alpha_j} \otimes \bigotimes_{j=i+m}^{n+m-1} A_{\alpha_j}\Big) \\ &\subseteq \sum_{(\alpha_j)_1^{n+m-1} | \sum_{j=1}^{n+m-1} \alpha_j = z} A_{l+p+\sum j=1^{n+m-1}} \\ &= A_{l+p+z} \end{aligned}$$

$\square$

2.3 Koszul complex and symmetric algebra

In this section we want to construct a nice resolution of S(L). This will be fairly easy when L is finite dimensional, to consider the infinite dimensional case we will need some technical homological lemmas.
The complex constructed in the finite case is an application of the more general theory of Koszul complexes, however knowledge of this theory is not necessary for doing this construction.

2.3.1 Some homological algebra on tensor product of complexes

Given two complexes P and Q of A modules we define the tensor product bi-complex as follows.

$$\begin{array}{ccccccccccc}
\dots \longrightarrow & P_4 \otimes_A Q_3 & \xrightarrow{d_P^3\otimes 1} & P_3 \otimes_A Q_3 & \xrightarrow{d_P^2\otimes 1} & P_2 \otimes_A Q_3 & \xrightarrow{d_P^1\otimes 1} & P_1 \otimes_A Q_3 & \xrightarrow{d_P^0\otimes 1} & P_0 \otimes_A Q_3 \\
& \downarrow{\scriptstyle 1\otimes d_Q^2} & & \downarrow{\scriptstyle 1\otimes d_Q^2} & & \downarrow{\scriptstyle 1\otimes d_Q^2} & & \downarrow{\scriptstyle 1\otimes d_Q^2} & & \downarrow{\scriptstyle 1_Q^i} \\
\dots \longrightarrow & P_4 \otimes_A Q_2 & \xrightarrow{d_P^3\otimes 1} & P_3 \otimes_A Q_2 & \xrightarrow{d_P^2\otimes 1} & P_2 \otimes_A Q_2 & \xrightarrow{d_P^1\otimes 1} & P_1 \otimes_A Q_2 & \xrightarrow{d_P^0\otimes 1} & P_0 \otimes_A Q_2 \\
& \downarrow{\scriptstyle 1\otimes d_Q^1} & & \downarrow{\scriptstyle 1\otimes d_Q^1} & & \downarrow{\scriptstyle 1\otimes d_Q^1} & & \downarrow{\scriptstyle 1\otimes d_Q^1} & & \downarrow{\scriptstyle 1_Q^i} \\
\dots \longrightarrow & P_4 \otimes_A Q_1 & \xrightarrow{d_P^3\otimes 1} & P_3 \otimes_A Q_1 & \xrightarrow{d_P^2\otimes 1} & P_2 \otimes_A Q_1 & \xrightarrow{d_P^1\otimes 1} & P_1 \otimes_A Q_1 & \xrightarrow{d_P^0\otimes 1} & P_0 \otimes_A Q_1 \\
& \downarrow{\scriptstyle 1\otimes d_Q^0} & & \downarrow{\scriptstyle 1\otimes d_Q^0} & & \downarrow{\scriptstyle 1\otimes d_Q^0} & & \downarrow{\scriptstyle 1\otimes d_Q^0} & & \downarrow{\scriptstyle 1_Q^i} \\
\dots \longrightarrow & P_4 \otimes_A Q_0 & \xrightarrow{d_P^3\otimes 1} & P_3 \otimes_A Q_0 & \xrightarrow{d_P^2\otimes 1} & P_2 \otimes_A Q_0 & \xrightarrow{d_P^1\otimes 1} & P_1 \otimes_A Q_0 & \xrightarrow{d_P^0\otimes 1} & P_0 \otimes_A Q_0
\end{array}$$

We want to extend this to defining a tensorproduct n- complex and its total complex. The tensorproduct n-complex $\bigotimes_{j=1}^n P^j$ is then defined by associating to every element $(a_i)_{i=1}^n \in \mathbb{Z}^{\ltimes}$ the module $\bigotimes_{i=1}^n P_{a_i}^j$ with maps given by tensoring the identity map with the differential in the direction j for every direction.
The Total complex of the n-complex can then defined as

$$ToT(\bigotimes_{j=1}^n P^j)_m = \bigoplus_{(a_i)_{i=1}^n \in \mathbb{Z}^n | \sum a_i = m} \bigotimes_{j=1}^n P_{a_i}^j$$

with differential given on components as

$$\bigotimes_{j=1}^{n} P^j_{a_i} \to \bigotimes_{j=1}^{k-1} P^j_{a_i} \otimes P^k_{a_k-1} \otimes \bigotimes_{j=k+1}^{n} P^j_{a_i} = \prod_{j=k+1}^{n} (-1)^{a_i} \bigotimes_{j=1}^{k-1} 1^j_{a_i} \otimes d^k_{a_k} \otimes \bigotimes_{j=k+1}^{n} 1^j_{a_j}$$

Lemma 2.4. *Given n complexes then*

$$Tot(\bigotimes_{j=1}^{n} P^j) = Tot(Tot(\bigotimes_{j=1}^{n-1} P^j) \otimes P^n)$$

Proof.

$$\begin{aligned} Tot(\bigotimes_{j=1}^{n} P^j)_m &= \bigoplus_{(a_i)_{i=1}^n \in \mathbb{Z}^n | \sum a_i = m} \bigotimes_{j=1}^{n} P^j_{a_i} \\ &= \bigoplus_{k+s=m} \bigoplus_{(a_i)_{i=1}^{n-1} \in \mathbb{Z}^{n-1} | \sum a_i = k} \bigotimes_{j=1}^{n-1} P^j_{a_i} \otimes P^n_s \\ &= Tot(Tot(\bigotimes_{j=1}^{n-1} P^j) \otimes P^n) \end{aligned}$$

The differential given on components can similarly be seen to be identical. □

Lemma 2.5 (colimits of chain complexes)**.** *Given a direct system of Chain complexes C_j, if there exists a colimit of the induced direct system on the ith component of the chain complexes for every i then there exists a colimit of the direct system of chain complexes given by the point-wise direct limit with morphisms uniquely induced by the universal property of the colimit.*

Proof. First we need to see that this is in fact a chain complex. The morphisms $\varinjlim (d_j)_i : \varinjlim(C_j)_i \to \varinjlim(C_j)_{i-1}$ are uniquely induced as factoring the collection of maps $(d_j)_i : (C_j)_i \to (C_j)_{i-1} \to \varinjlim_{j \in J}(C_j)_{i-1}$ trough $\varinjlim_{j \in J}(C_j)_i$.
Therefore the composition $\varinjlim d_{i-1} \circ \varinjlim d_i$ has to be given by the composition of these two collection of maps, but this composition is 0 on every term so the uneqely induced map has to be the zero map. Hence it is a chain complex. Now we have to verify that for any other chain complex C such that we have a system of morphisms for every C_j we have a unique factoring trough the proposed colimit.
Suppose we have a chain complex C together with a collection of chain morphisms $\phi_j : C_j \to C$ that commute with the inclusions. Then by considering the ith term of each complex we get a collection of morphisms $(\phi_j)_i : (C_j)_i \to C_i$ which then factors trough $\varinjlim_{j \in J}(C_j)_i$.
To see that this is a chain homomorphism we need to demonstrate that

$$d^C_i \circ \varinjlim_{j \in J}(\phi_j)_i = \varinjlim_{j \in J}(\phi_j)_{i-1} \circ \varinjlim_{j \in J}(j)_i$$

$$d_i^C \circ \varinjlim_{j\in J}(\phi_j)_i = \varinjlim_{j\in J}(d_i^C \circ (\phi_j)_i)$$

But $(d_i^C \circ (\phi_j)_i) = ((\phi_j)_{i-1} \circ d_i^j)$ for every j which by using the uniqueness of the factorization trough the limit and the functoriality of colim gives us the following equality.

$$\varinjlim_{j\in J}(d_i^C \circ (\phi_j)_i) = \varinjlim_{j\in J}((\phi_j)_{i-1} \circ d_i^j) = \varinjlim_{j\in J}(\phi_j)_{i-1} \circ \varinjlim_{j\in J}(j)_i$$

The only thing left to prove is that this factorization is unique, however this follows from that any factorization would induce a factorization on terms which then have to be unique. ☐

Definition 2.7. *A poset-diagram here taken to be a diagram looking like the diagram of inclusions of subsets in a set. That is it is a diagram consisting of monomorphisms where every two objects have an object they both map into and there is at most one map between each object.*

This definition might or might not coincide with what is usually defined as a direct system. However I could not find a precise definition of a direct system.

Lemma 2.6. *Colimits commute with colimits.*

Proof. The proof goes by using universal properties. Assume we have a diagram J consisting of diagrams I_j. Then the object $\varinjlim_{j\in J}(\varinjlim I_j)$ satisfies the universal property of $\varinjlim(\varinjlim_{j\in J} I_j)$. This can be seen by reducing to a colimit of all objects in the diagram of diagrams. ☐

Lemma 2.7 (Kernels commute with colimits on poset diagrams)**.** *Kernels commute with colimits on poset diagrams in vec(K).*

Proof. We start by assuming we have a poset diagram J of $\phi_j : A_j \to B_j$. We have a injective map denoted f, from the colimit of the kernels to the kernel in the colimit induced by the universal property of the colimit of the kernel. We just need to show that this is an isomorphism. Colimits in vec(K) can be identified as direct sum modulo the relations induced by the morphisms. Then given an element a in $Ker(\varinjlim_{j\in J}\phi_j)$ it is an element in $\bigoplus A_j/(i(b)-b)$ so we can find a representative contained in a finite number of the summands. Since we have a poset diagram we have a object denoted s in the diagram containing all of these summands. Then letting i denote the inclusion in the comlimit since $i(\phi_s(a)) = \varinjlim_{j\in J}\phi_j(i(a)) = 0$ and i is injective we have $\phi_s(a) = 0$. Then a can be seen as an element in the colimit of the kernels and hence the map f is surjective, hence an isomorphism. ☐

Lemma 2.8 (Commutativity of homology)**.** *Homology commutes with direct limit on poset diagrams.*

Proof. Homology is defined by kernels and images and since kernels and images commute with direct limit homology commutes with direct limit. □

2.3.2 Construction of the resolution

Definition 2.8. *Given a totally ordered set B we define the direct system of $k[B_i]$ where B_i are all finite subsets of B with the inherited total order. The morphisms in this direct system are defined to be the extension of the inclusions on the sets B_i.*

Consider the complex of vector spaces over K given below

$$K[x] \otimes K[x] \xrightarrow{x\otimes 1 - 1\otimes x} K[x] \otimes K[x] \longrightarrow K[x] \longrightarrow 0$$

This complex is exact by the following null homotopy.

$$s_0(x^i \otimes x^j) = -\sum_{l=1}^{j} x^{i+j-l} \otimes x^{l-1}$$

$$s_{-1}(x^i) = x^i \otimes 1$$

Then we have the following resolution of $K[x]$ denoted by $P[x]$

$$K[x] \otimes K[x] \xrightarrow{(x\otimes 1 - 1\otimes x)\cdot} K[x] \otimes K[x] \longrightarrow 0$$

Now we will construct the complex we wanted. Let L be our Lie algebra. Let B be a set of basis vectors as a K- vector space with some total order. Now we may consider the complex $M = Tot(\bigotimes_{b\in B_i} P[b])$ with tensoring over K for every finite subset B_i of B. Then we may construct a direct system of these by extending the inclusions $B' \subset \tilde{B}$ as $i : Tot(\bigotimes_{b\in B'} K[b])_n \to Tot(\bigotimes_{b\in \tilde{B}} K[b])_n$ given on components as $\bigotimes_{b\in B'} P[b]_{i_b} \to \bigotimes_{b\in B'} P[b]_{i_b} \otimes \bigotimes_{b\in \tilde{B}\setminus B'} 1_{P[b]_0}$. is a chain complex homomorphism by construction of the total complex. In addition the composition of two of these inclusions is again such an inclusion, and since we can always take the union of two finite sets this gives us a direct system of cochain complexes.
Note also that each summand of each term in the complex can be considered to be isomorphic to $K[B']^e$ by reordering terms in the tensor product. Further each

the summands ("directions") are each multiplication by an element of $K[B']^e$, making it a $K[B']^e$ module homomorphism. and since sum of homomorphisms are again homomorphisms the complex is a free $K[B']^e$ complex.

Theorem 2.1 (Kunneth theorem)**.** *Given a field K and two chain complexes of vector spaces then for each $n \in \mathbb{N}$ there is an isomorphism*

$$\bigoplus_{m\in\mathbb{Z}} H_m(C) \otimes H_{n-m}(C') \simeq H_n(Tot(C \otimes C'))$$

Proof. Assume we have two complexes C and C'.
We first prove the theorem in the case that one of the complexes has zero differentials.
Let C_i denote the complex with C_i in the ith position and zero elsewhere then $H_n(Tot(C_i \otimes C')) \simeq H_n(C_i \otimes C') \simeq C_i \otimes H_{n-i}(C')$.
Where the secound term refers to tensoring evry component with C_i
Since the all differentials are zero we may decompose $C \simeq \bigoplus_i C_i$ then using the above formula

$$\begin{aligned} H_n(Tot(C \otimes C')) &\simeq H_n(Tot((\bigoplus_i C_i) \otimes C')) \\ &\simeq H_n(Tot(\bigoplus_i C_i \otimes C')) \\ &\simeq \bigoplus_i H_n(Tot(C_i \otimes C')) \\ &\simeq \bigoplus_i C_i \otimes H_{n-i}(C') \end{aligned}$$

Since we are working over vectorspaces, everything splits this means we can decompose $C_n \simeq im(d_n) \oplus H_n \oplus im(d_{n-1})$ then since the differentials do not interact with H_n for any n we may factor out this complex leaving us with a direct sum of complexes H which consists of the nth homology with zero differenetials and C/H wich is an exact complex consisting of the remaining terms. We do the same decomposition for C'.

$$\begin{aligned} H_n(Tot(C \otimes C')) &\simeq H_n(Tot((H \oplus C/H) \otimes (H' \oplus C'/H'))) \\ &\simeq H_n(Tot(H \otimes H' \oplus H \otimes C/H' \oplus C/H \otimes H' \oplus C/H \otimes C'/H')) \\ &\simeq H_n(Tot(H \otimes H')) \oplus H_n(Tot(H \otimes C/H')) \\ &\oplus H_n(Tot(C/H \otimes H')) \oplus H_n(Tot(C/H \otimes C'/H')) \\ &\simeq H_n(Tot(H \otimes H')) \\ &\simeq \bigoplus_{m\in\mathbb{Z}} H_m(C) \otimes H_{n-m}(C') \end{aligned}$$

Here the second to last isomorphism is archived by using that C/H and C'/H' are zero on homology and that tensoring with zero is zero. □

In addition in the case of the direct system described above the only nonzero homology becomes the 0th homology which then is $\bigotimes_{b\in B'} K[b]$ with the morphisms on homology induced from the inclusion morphism $B' \subseteq \tilde{B}$ as
$id \otimes \bigotimes_{b\in \tilde{B}\setminus B'} 1_b : \bigotimes_{b\in B'} K[b] \to \bigotimes_{b\in B'} K[b] \otimes \bigotimes_{b\in \tilde{B}\setminus B'} K[b]$. That this is the induced morphism follows from that the isomorphism of homologies proved in the kunneth theorem i acheived by direct sum decomposistion, therefore the induced morphism on hommology is the restriction of the inclusion morphism on the complex.

But the colimit along these inclusion morphisms on homology becomes $\underset{B'\in B}{\underrightarrow{\mathbf{Lim}}} H_0(P[B']) = \underset{B'\in B}{\underrightarrow{\mathbf{Lim}}} K[B'] = K[B]$ since we have a cannonical inclusion map of each $K[B']$ into $K[B]$ and because every element of $K[B]$ lies in some $k[B']$ since an element can only use a finite number of variables knowing what a morphism is on every $K[B']$ is equivalent to knowing what it does on every element of $K[B]$.

Therefore since

$$H_0(\underset{B'\in B}{\underrightarrow{\mathbf{Lim}}} P[B']) \simeq \underset{B'\in B}{\underrightarrow{\mathbf{Lim}}} H_0(P[B']) \simeq K[B]$$

and the rest of the homologies become zero by the same argument we have that $\underset{B'\in B}{\underrightarrow{\mathbf{Lim}}} P[B']$ provides a free resolution over K of $S(V) \simeq K[B]$.
We will now rewrite this resolution so that it resembles the bar resolution.
Given a finite totally ordered set B and V as the K-vectorspace with B as basis let $E_n = K[B] \otimes \Lambda^n(V) \otimes K[B]$. Then we have the following isomorphism denoted ϕ.

$$E_n \simeq_{K[B]^e} Tot(\bigotimes_{b\in B} K[b])_n = \bigoplus_{(a_b)_{b\in B} | a_b \in \{0,1\}} \bigotimes_{b\in B} (K[b] \otimes K[b])_{a_b}$$

Which is given by sending $1 \otimes \bigwedge_{b\in B'} b \otimes 1$ to $\bigotimes_{b\in B}(1 \otimes 1)_{a_b}$ for any subset B' of B of order n, where the wedge is taken with the order induced on B' by the order on B; and a_b is 1 if $b \in B'$ and 0 otherwise. This map is then both injective and surjective since we are sending basis elements to basis elements of free $K[B]^e$ modules in a 1 to 1 correspondence.

The differential under this isomorphism is then given by

$$
\begin{aligned}
d(1\otimes \bigwedge_{b\in B'} b\otimes 1) &= \phi d\phi^{-1}(1\otimes \bigwedge_{b\in B'} b\otimes 1) \\
&= \phi d(\bigotimes_{b\in B}(1\otimes 1)_{a_b}) \\
&= \phi \sum_{b'\in B} \bigotimes_{b\in B\setminus b'} (1\otimes 1)_{a_b} \otimes d^{b'}_{a_{b'}}(1\otimes 1) \prod_{b<b'|b\in B} (-1)^{a_b} \\
&= \phi \sum_{b'\in B'} \bigotimes_{b\in B\setminus b'} (1\otimes 1)_{a_b} \otimes (b'\otimes 1 - 1\otimes b') \prod_{b<b'|b\in B} (-1)^{a_b} \\
&+ \phi \sum_{b'\in B\setminus B'} \bigotimes_{b\in B\setminus b'} (1\otimes 1)_{a_b} \otimes \prod_{b<b'|b\in B} (-1)^{a_b} 0 \\
&= \phi \sum_{b'\in B'} \bigotimes_{b\in B\setminus b'} (1\otimes 1)_{a_b} \otimes (b'\otimes 1 - 1\otimes b') \cdot \prod_{b<b'|b\in B'} (-1) \\
&= \sum_{b'\in B'} \prod_{b<b'|b\in B'} (-1)\phi(\bigotimes_{b\in B\setminus b'} (1\otimes 1)_{a_b} \otimes (b'\otimes 1 - 1\otimes b')) \\
&= \sum_{b'\in B'} \prod_{b<b'|b\in B'} (-1)\phi\Big(\bigotimes_{b\in B\setminus b'} (1\otimes 1)_{a_b} \otimes (b'\otimes 1) \\
&- \bigotimes_{b\in B\setminus b'} (1\otimes 1)_{a_b} \otimes (1\otimes b')\Big) \\
&= \sum_{b'\in B'} \prod_{b<b'|b\in B'} (-1)\big(b'\otimes (\bigwedge_{b\in B'\setminus b'} b\big) \otimes 1 - 1\otimes (\bigwedge_{b\in B'\setminus b'} b\big) \otimes b')
\end{aligned}
$$

2.3.3 The colimit complex

The inclusion morphisms in the direct system coming from $\tilde{B} \subseteq B'$ then becomes the cannoncical inclusions $= K[\tilde{B}]\otimes \Lambda^n(V(\tilde{B}))\otimes K[\tilde{B}] \to K[B']\otimes \Lambda^n(V(B'))\otimes K[B']$ under ϕ.
Then the colimit of these for the direct system of subsets of B can be seen to be $K[B]\otimes \Lambda^n(V(B))\otimes K[B]$ due to the existence of a natural inclusion morphism and every element being described by only a finite number of elements in B. This means that we may identify the colimit complex with an identical complex where the differential is still identical as earlier since whenever we apply it on an element we are essentially applying it in the module described by some finite subset of B.

2.3.4 As a subcomplex of the bar complex

Now we will inject this resolution in the bar resolution. We will do this by a map $\psi : K[B]\otimes \bigwedge^n(V(B))\otimes K[B] \to K[B]^{\otimes n+2}$ wich acts on basis elementes as $\psi(1\otimes (\bigwedge_{b\in B'} b)\otimes 1) = \sum_{\sigma\in S_n} sign(\sigma)\cdot 1\otimes \bigotimes_{b\in B'} \sigma(b)\otimes 1$. Here we by S_n

mean the symmetric group on n letters. We need to verify that this is a chain map, that it is injective and identify its image.
To demonstrate that it is injective we recall that then tesor product of a collection of vectorspaces can be identified with the vectorspace with the cartesean product as of the bases of the vectorspaces as a basis. Then by the following calculation we see that a $K[B]^e$ linear combination with nonzero coefficents of the basis elements our domain is sent to a $K[B]^e$ linear combination of basis elements of our target, meaning that they are nonzero, making the map incjective.

$$\begin{aligned}\psi(\sum_{i=1}^{m}(p_i(B)\otimes q_i(B))\cdot(1\otimes(\bigwedge_{b\in B_i} b)\otimes 1)) &= \sum_{i=1}^{m}(p_i(B)\otimes q_i(B))\cdot\psi((1\otimes(\bigwedge_{b\in B_i} b)\otimes 1))\\ &= \sum_{i=1}^{m}(p_i(B)\otimes q_i(B))\sum_{\sigma\in S_n} sign(\sigma)\cdot 1\otimes\bigotimes_{b\in B_i}\sigma(b)\otimes 1)\\ &= \sum_{i=1}^{m}\sum_{\sigma\in S_n}(p_i(B)\otimes q_i(B))\cdot sign(\sigma)\cdot 1\otimes\bigotimes_{b\in B_i}\sigma(b)\otimes 1)\end{aligned}$$

We now demonstrate that it is a chain map

$$\begin{aligned}&\psi(d(\otimes(\bigwedge_{b\in B'} b)\otimes 1))\\ &=\psi(\sum_{b'\in B'}\prod_{b<b'|b\in B'}(-1)\big(b'\otimes\big(\bigwedge_{b\in B'\setminus b'} b\big)\otimes 1-1\otimes\big(\bigwedge_{b\in B'\setminus b'} b\big)\otimes b'))\\ &=\sum_{b'\in B'}\prod_{b<b'|b\in B'}(-1)\big(\psi(b'\otimes\big(\bigwedge_{b\in B'\setminus b'} b\big)\otimes 1-1\otimes\big(\bigwedge_{b\in B'\setminus b'} b\big)\otimes b')\big)\\ &=\sum_{b'\in B'}\prod_{b<b'|b\in B'}(-1)\big(\sum_{\sigma\in S_{n-1}} sign(\sigma)(\cdot b'\otimes\bigotimes_{b\in B'\setminus b'}\sigma(b)\otimes 1-1\otimes\bigotimes_{b\in B'\setminus b'}\sigma(b)\otimes b')\big)\end{aligned}$$

$$\begin{aligned}d(\psi(\otimes(\bigwedge_{b\in B'} b)\otimes 1)) &= d(\sum_{\sigma\in S_n} sign(\sigma)\cdot 1\otimes\bigotimes_{b\in B'}\sigma(b)\otimes 1)\\ &=\sum_{j=1}^{n+1}(-1)^j\partial_j(\sum_{\sigma\in S_n} sign(\sigma)\cdot 1\otimes\bigotimes_{b\in B'}\sigma(b)\otimes 1))\\ &=\sum_{\sigma\in S_n} sign(\sigma)\cdot\bigotimes_{b\in B'}\sigma(b)\otimes 1\\ &+\sum_{j=2}^{n}(-1)^j\partial_j(\sum_{\sigma\in S_n} sign(\sigma)\cdot 1\otimes\bigotimes_{b\in B'}\sigma(b)\otimes 1))\\ &+(-1)^n\sum_{\sigma\in S_n} sign(\sigma)\cdot\bigotimes_{b\in B'}\sigma(b)\end{aligned}$$

Here ∂_j is the map collapsing the j'th tensor.
We may decompose S_n into two disjoint sets S_n^+ andS_n^-, the set of positive sign

elements and the set of negative sign elements. Then for any i between 1 and $n-1$ we may find a bijection between the two disjoint sets given by postcomposing a permutation with the permutation $(i, i+1)$ that is the permutation that switches the ith and the i+1th component. This is a bijection whose inverse is itself.

$$\sum_{j=2}^{n}(-1)^j \partial_j(\sum_{\sigma \in S_n} sign(\sigma) \cdot 1 \otimes \bigotimes_{b \in B'} \sigma(b) \otimes 1))$$

$$= \sum_{j=2}^{n}(-1)^j \partial_j(\sum_{\sigma \in S_n^+} 1 \otimes \bigotimes_{b \in B'} \sigma(b) \otimes 1)) - 1 \otimes \bigotimes_{b \in B'} (j-1, j) \circ \sigma(b) \otimes 1))$$

$$= \sum_{j=2}^{n}(-1)^j (\sum_{\sigma \in S_n^+} \partial_j(1 \otimes \bigotimes_{b \in B'} \sigma(b) \otimes 1) - \partial_j(1 \otimes \bigotimes_{b \in B'} ((j-1, j) \circ \sigma)(b) \otimes 1))$$

$$= \sum_{j=2}^{n}(-1)^j (\sum_{\sigma \in S_n^+} 0)$$

Where the cancellation in the last step occurs due to collapsing the jth tensor that is the j-1th tensor in the of the terms we permute, and since $K[B]$ is commutative we then get that the first and second summand become identical except for the sign.

Next we may decompose S_n into n disjoint sets, each consisting of the permutations that sends the ith element to the first position, we denote these by S_n^i. We may then by composing with the permutation $(1, i)$ identify each of these sets with the subgroup given by keeping the first element fixed, this subgroup is then canonically isomorphic to S_{n-1}. However the identification of each set with the subgroup will not respect the sign, it will be of by the sign

$$sign(1, i) = \prod_{0<j<i} sign(j, j+1) = \prod_{0<j<i} (-1) = (-1)^i$$

We may decompose similarly into the sets sending the ith element to the last position the sign the idetification of these sets to S_{n-1} then becomes

$$sign(1, n) = \prod_{i \leq j < n} sign(j, j+1) = \prod_{i \leq j < n} (-1) = (-1)^{n-i}$$

$$\sum_{\sigma\in S_n} sign(\sigma)\cdot \bigotimes_{b\in B'} \sigma(b)\otimes 1(-1)^n \sum_{\sigma\in S_n} sign(\sigma)\cdot \bigotimes_{b\in B'} \sigma(b)$$

$$= \sum_{b'\in B'} \Bigg(\prod_{b<b'|b\in B'} (-1) \sum_{\sigma\in S_{n-1}} sign(\sigma)\cdot b'\otimes \bigotimes_{b\in B'\setminus b'} \sigma(b)\otimes 1$$

$$(-1)^n \prod_{b'<b|b\in B'} (-1)\cdot \sum_{\sigma\in S_{n-1}} sign(\sigma)\cdot \bigotimes_{b\in B'\setminus b'} \sigma(b)\otimes b'\Bigg)$$

$$= \sum_{b'\in B'} \Bigg(\prod_{b<b'|b\in B'} (-1) \sum_{\sigma\in S_{n-1}} sign(\sigma)\cdot b'\otimes \bigotimes_{b\in B'\setminus b'} \sigma(b)\otimes 1$$

$$(-1)^n \prod_{b\leq b'|b\in B'} (-1)\cdot \prod_{b\leq b'|b\in B'} (-1)\cdot \prod_{b'<b|b\in B'} (-1)\cdot\cdot \sum_{\sigma\in S_{n-1}} sign(\sigma)\cdot \bigotimes_{b\in B'\setminus b'} \sigma(b)\otimes b'\Bigg)$$

$$= \sum_{b'\in B'} \Bigg(\prod_{b<b'|b\in B'} (-1) \sum_{\sigma\in S_{n-1}} sign(\sigma)\cdot b'\otimes \bigotimes_{b\in B'\setminus b'} \sigma(b)\otimes 1$$

$$(-1)^n\cdot(-1)^n \prod_{b\leq b'|b\in B'} (-1)\cdot \sum_{\sigma\in S_{n-1}} sign(\sigma)\cdot \bigotimes_{b\in B'\setminus b'} \sigma(b)\otimes b'\Bigg)$$

$$= \sum_{b'\in B'} \Bigg(\prod_{b<b'|b\in B'} (-1) \sum_{\sigma\in S_{n-1}} sign(\sigma)\cdot b'\otimes \bigotimes_{b\in B'\setminus b'} \sigma(b)\otimes 1$$

$$- \prod_{b<b'|b\in B'} (-1)\cdot \sum_{\sigma\in S_{n-1}} sign(\sigma)\cdot \bigotimes_{b\in B'\setminus b'} \sigma(b)\otimes b'\Bigg)$$

We may now combine all of this to find that ψ is a chain complex homomorphism.

$$
\begin{aligned}
&\psi(d(\otimes(\bigwedge_{b\in B'} b)\otimes 1))\\
&=\sum_{b'\in B'}\prod_{b<b'|b\in B'}(-1)\Big(\sum_{\sigma\in S_{n-1}} sign(\sigma)(\cdot b'\otimes\bigotimes_{b\in B'\setminus b'}\sigma(b)\otimes 1-1\otimes\bigotimes_{b\in B'\setminus b'}\sigma(b)\otimes b')\Big)\\
&=\sum_{\sigma\in S_n} sign(\sigma)\cdot\bigotimes_{b\in B'}\sigma(b)\otimes 1(-1)^n\sum_{\sigma\in S_n} sign(\sigma)\cdot\bigotimes_{b\in B'}\sigma(b)\\
&=\sum_{\sigma\in S_n} sign(\sigma)\cdot\bigotimes_{b\in B'}\sigma(b)\otimes 1\\
&+\sum_{j=2}^{n}(-1)^j\partial_j(\sum_{\sigma\in S_n} sign(\sigma)\cdot 1\otimes\bigotimes_{b\in B'}\sigma(b)\otimes 1))\\
&+(-1)^n\sum_{\sigma\in S_n} sign(\sigma)\cdot\bigotimes_{b\in B'}\sigma(b)\\
&=d(\psi(\otimes(\bigwedge_{b\in B'} b)\otimes 1))
\end{aligned}
$$

2.3.5 The image of the morphism

Let $A = K[B] \simeq \frac{T(V(B))}{(x\otimes y - y\otimes x)}$ and $R = (a\otimes b - b\otimes a) \subseteq V\otimes V$. We may then denote the image over ψ of the complex constructed in the previous section as

$$
\ldots \xrightarrow{d_B^4} A\otimes_k K_3'\otimes_k A \xrightarrow{d_B^3} A\otimes_k K_2'\otimes_k A \xrightarrow{d_B^2} A\otimes_k V\otimes_k A \xrightarrow{d_B^1} A\otimes_k A \longrightarrow 0
$$

We denote $A\otimes K_n'\otimes A$ by K_n. We immediately observe that $K_2' \simeq R$ by the definition of ψ.

2.3.6 Elements of K_3'

Suppose we have an element in $R\otimes V\cap V\otimes R$, it can be written as

$$
\begin{aligned}
\sum(x\otimes y - x\otimes y)\otimes z &= \sum x\otimes y\otimes z - x\otimes y\otimes z\\
&= \sum c\otimes(a\otimes b - b\otimes a)\\
&= \sum c\otimes a\otimes b - c\otimes b\otimes a
\end{aligned}
$$

Then

$$
\begin{aligned}
&\forall x \otimes y \otimes z \; \exists \, c \otimes a \otimes b \;\; s.t \;\; x \otimes y \otimes z = c \otimes a \otimes b \\
&\quad \implies \exists c' \otimes a' \otimes b' = -a \otimes c \otimes b \\
&\quad \implies \sum c \otimes a \otimes b - c \otimes b \otimes a \\
&\;= \sum e \otimes f \otimes g - f \otimes e \otimes g - e \otimes g \otimes f + f \otimes g \otimes e + g \otimes e \otimes f - g \otimes f \otimes e
\end{aligned}
$$

This is precisely an element of K_3'. Similar identifications may be made in higher degrees, but they are not necessary for this thesis.

2.3.7 A projection commuting with the differential

Since K_n is a free A^e submodule of B_n we may factor $B_n \simeq K_n \oplus C_n$ where C_n is some free A^e module. We may then lift the identity map on $A \otimes A$ to the identity on K_n and some map c_n on C_n giving us a chain map $_n = [c_n, 1] : B \to K$. The reason we may do this lift is that K is exact, meaning each differential is epi on the kernel of the next and that C_n is free and therefore projective. This then becomes a chain map.

$$
\begin{array}{ccccccc}
C_3 \oplus A \otimes_k K_3' \otimes_k A & \xrightarrow{d_B^3} & C_2 \oplus A \otimes_k K_2' \otimes_k A & \xrightarrow{d_B^2} & C_1 \oplus A \otimes_k V \otimes_k A & \xrightarrow{d_B^1} & A \otimes_k A \\
\downarrow{\scriptstyle [c_3, id]} & & \downarrow{\scriptstyle [c_2, id]} & & \downarrow{\scriptstyle [c_1, id]} & & \downarrow{\scriptstyle 1} \\
A \otimes_k K_3' \otimes_k A & \xrightarrow{d_B^3} & A \otimes_k K_2' \otimes_k A & \xrightarrow{d_B^2} & A \otimes_k V \otimes_k A & \xrightarrow{d_B^1} & A \otimes_k A
\end{array}
$$

On B_2 this gives us a map $p : A \otimes A \to R$ such that $p \circ i = id_R$ for i the inclusion $i : R \to A \otimes A$

2.4 Ith order Algebraic deformations

Definition 2.9 (algebraic deformation)**.** *An i-th order algebraic deformation of a algebra A is an associative algebra structure on the $K[t]$ module $A[t]$ such that t is in the center and $A[t]/(t) \simeq A$.*

Definition 2.10 (I-th order algebraic deformation)**.** *An i-th order algebraic deformation of a algebra A is an associative algebra structure on the $\frac{k[t]}{(t^i)}$ module $\frac{A[t]}{(t^n)}$ such that t is in the center and $\frac{A[t]}{(t)} \simeq A$.*

There is a more general notion of an degree n algebraic deformation. What here is referred to as a I th order algebraic deformation is in fact a degree 2- ith order algebraic deformation. For example the algebraic deformations used to define the tangent space in algebraic geometry are 1.st order degree 1 deformations.

2.4.1 Observations about algebraic deformations

We note that for any deformation $(\frac{A[t]}{t^n}, \star)$ the multiplication is entirely determined by its value on the elements of A due to the following equation.

$$(\sum_{i=0}^{m} a_i t^i) \star (\sum_{j=0}^{s} b_j t^j) = \sum_{i=0,j=0}^{(n,s)} a_i \star b_j \star t^{i+j}$$

Further we note that the multiplication is a k-linear map $\star : A \otimes A \longrightarrow A[t]/t(n) = \sum_{i=0}^{n-1} At^i$. So that we may decompose it into n maps called μ_i for i between 0 and n-1. Then the requirement that $\frac{A[t]}{(t)} \simeq A$ implies that μ_0 is just the multiplication in A.
All of this also holds for an algebraic deformation.

2.4.2 Requirements on μ_i

The associativity of the (ith order) algebraic deformation imposes some restriction on the choice of μ_i, we will now see a reformulation of this requirement.

$$a \star (b \star c) = \sum_{i=0}^{\infty} a \star \mu_i(b \otimes c) t^i = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \mu_j(a \otimes \mu_i(b \otimes c)) t^{i+j}$$

$$= (a \star b) \star c = \sum_{i=0}^{\infty} \mu_i(a \otimes b) \star ct^i = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \mu_j(\mu_i(a \otimes b) \otimes c)) t^{i+j}$$

Note that summing to infinity is well defined even in the case of an algebraic deformation since only finitely many of the terms will be nonzero due to the assumption that the multiplication is well defined on $A[t]$.
Now separating powers of t and using that multiplication by t is injective for all elements of degree less than n we get that the following holding for all n is

equivalent.

$$\sum_{i=0}^{n} \mu_i(\mu_{n-i}(a \otimes b) \otimes c)) = \sum_{i=0}^{n} \mu_i(a \otimes \mu_{n-i}(b \otimes c))$$

$$\Updownarrow$$

$$\begin{aligned}
&\sum_{i=1}^{n-1} \mu_i(\mu_{n-i}(a \otimes b) \otimes c)) - \mu_i(a \otimes \mu_{n-i}(b \otimes c)) \\
&= \mu_0(a \otimes \mu_n(b \otimes c)) - \mu_0(\mu_n(a \otimes b) \otimes c)) + \mu_n(a \otimes \mu_0(b \otimes c)) - \mu_n(\mu_0(a \otimes b) \otimes c)) \\
&= a \cdot \mu_n(b \otimes c) - \mu_n(a \otimes b) \cdot c + \mu_n(a \otimes b \cdot c) - \mu_n(a \cdot b \otimes c)) \\
&= d_k^{3*}(\mu_n)(a \otimes b \otimes c)
\end{aligned}$$

Note that every step of this argument is an equivalence and that the associativty is the only problem we have to solve if we want to extend a order (n-1) deformation to a order n deformation by adding a μ_n. We therefore name $\sum_{i=1}^{n-1} \mu_i(\mu_{n-i}(a \otimes b) \otimes c)) = \sum_{i=1}^{n-1} \mu_i \circ \mu_{n-i}$ the (n-1)-th obstruction.
If we apply this formula to n=1 we immediately get

$$0 = \sum_{i=1}^{1-1} \mu_i \circ \mu_{n-i} = d^*(\mu_1)$$

Implying that μ_1 is a Hochschild 2- cocycle.

Lemma 2.9. *The (n-1)th obstruction is always a Hochschild 3- cocycle.*

Proof. We start by assuming that we have a n-1st order deformation. That is, we have $\sum_{i=1}^{k-1} \mu_i \circ \mu_{k-i} = d^*(\mu_k)$ for all $k < n$ and want to extend one more step.

$$\begin{aligned}
d_k^*(\sum_{i=1}^{n-1} \mu_i \circ \mu_{n-i}) &= \sum_{i=1}^{n-1} d_k^*(\mu_i \circ \mu_{n-i}) \\
&= \sum_{i=1}^{n-1} d_k^*(\mu_i) \circ \mu_{n-i} - \mu_i \circ d_k^*(\mu_{n-i}) - (\mu_i \cdot \mu_{n-i} - \mu_{n-i} \cdot \mu_i) \\
&= \sum_{i=1}^{n-1} (\sum_{j=1}^{i-1} \mu_j \circ \mu_{i-j}) \circ \mu_{n-i} - \mu_i \circ \sum_{j=1}^{n-i-1} \mu_j \circ \mu_{n-i-j} \\
&= \sum_{0<i,j,k<n \mid i+j+k=n-1} (\mu_i \circ \mu_j) \circ \mu_k - \mu_i \circ (\mu_j \circ \mu_k)
\end{aligned}$$

We now write out each summand.

$$\begin{aligned}
(\mu_i \circ \mu_j) \circ \mu_k(a \otimes b \otimes c \otimes d) = {}& \mu_i(\mu_j(\mu_k(a,b),c),d - \mu_j(a,\mu_k(b,c)),d + \mu_j(a,b)\mu_k(c,d) \\
&- \mu_k(a,b)\mu_j(c,d) + a, \mu_j(\mu_k(b,c),d) - a, \mu_j(b,\mu_k(c,d)))
\end{aligned}$$

$$\mu_i \circ (\mu_j \circ \mu_k)(a \otimes b \otimes c \otimes d) = \mu_i(\mu_j(\mu_k(a,b),c),d - \mu_j(a,\mu_k(b,c)),d \\ + a,\mu_j(\mu_k(b,c),d) - a,\mu_j(b,\mu_k(c,d)))$$

We notice that most of the terms cancel out leaving us with.

$$(\mu_i \circ \mu_j) \circ \mu_k - \mu_i \circ (\mu_j \circ \mu_k) = \mu_i(\mu_j, \mu_k - \mu_k, \mu_j)$$

This allows us to conclude

$$\begin{aligned} d_k^*(\sum_{i=1}^{n-1} \mu_i \circ \mu_{n-i}) &= \sum_{0<i,j,k<n \mid i+j+k=n-1} (\mu_i \circ \mu_j) \circ \mu_k - \mu_i \circ (\mu_j \circ \mu_k) \\ &= \sum_{0<i,j,k<n \mid i+j+k=n-1} \mu_i(\mu_j, \mu_k - \mu_k, \mu_j) \\ &= 0 \end{aligned}$$

□

Definition 2.11 (Specification to a value of a deformation)**.** *Given a (graded) algebraic deformation we can define a specification to a value in the field k of the deformation as the k-algebra given by. This is defined as the (filtered) algebra* $\frac{A[t]}{(t-k)}$.

There is a canonical way to identify this as a vector space with A. This is done by the evaluation morphism $eval_k : A[t] \to A$ which has kernel $(t-k)$. Under this identification we can view this as the algebra $(A, \star)$ where $a \star b = a \cdot b + \sum_{i=1}^{\infty} \mu_i(a \otimes b)t^i \mid_{t=k} = a \cdot b + \sum_{i=1}^{\infty} \mu_i(a \otimes b)k^i$.
Usually we will use the value 1 for k and refer to the specification to a value of the deformation as A'.
We could not define this for a nth order deformation, since t would then be nilpotent and the ideal would then contain 1 by multiplying $(k-t)$ by $k^{-1} \cdot (1 + \sum_{i=1}^{n} k^{-i} t^i)$.

2.4.3 Graded deformation

Definition 2.12. *An (ith order) graded gebraic deformation of an algebra A is an (ith order) algebraic deformation of A where* $\frac{A[t]}{(t^n)}$ *is considered as a graded algebra wih the additional requirement that the multiplication has degree 0, i.e* $\left(\star : \frac{A[t]}{(t^n)} \otimes \frac{A[t]}{(t^n)} \longrightarrow \frac{A[t]}{(t^n)}\right) \in Hom_{\frac{A[t]}{(t^n)}}\left(\frac{A[t]}{(t^n)} \otimes \frac{A[t]}{(t^n)}, \frac{A[t]}{(t^n)}\right)_0$
The grading is induced from A by assuming that t has degree one. That means the decomposition is given as ${\frac{A[t]}{(t^n)}}_i \simeq \bigoplus_{s+j=i} A_s t^j$.

The assumption that the multiplication has degree zero forces μ_i to have degree -i since t^i has degree i.

Lemma 2.10. *The associated graded of a specification to a value k of a deformation of A (Gr A') is isomorphic to A.*

Proof. We use the evaluation morphism in k to describe the ring. Since k is a member of the underlying field it necessarily is homogeneous of degree zero. This means that $eval_k(A_j t^i) = A_j$.

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
A_{-1} & A_{-1}t & A_{-1}t^2 & A_{-1}t^3 & A_{-1}t^4 & \dots \\
A_0 & A_0t & A_0t^2 & A_0t^3 & A_0t^4 & \dots \\
A_1 & A_1t & A_1t^2 & A_1t^3 & A_1t^4 & \dots \\
A_2 & A_2t & A_2t^2 & A_2t^3 & A_2t^4 & \dots \\
A_3 & A_3t & A_3t^2 & A_3t^3 & A_3t^4 & \dots \\
A_4 & A_4t & A_4t^2 & A_4t^3 & A_4t^4 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Now using the decomposition above where each degree of $A[t]$ corresponds to a diagonal, we identify $U_z = eval_k(\bigoplus_{i \leq z} A[t]_i) = eval_k(\bigoplus_{i+j \leq z} A_i t^j) = \bigoplus_{i \leq z} A_i$
This means we can canonically consider $Gr(A')_n = \frac{U_z}{U_{z-1}} = \frac{\bigoplus_{i \leq z} A_i}{\bigoplus_{i \leq z-1} A_i} \simeq A_i$ as a vector space.
Now we want to see that the multiplication match, it is enough to show this on homogeneous elements. Denote by $\phi : \bigoplus_{\digamma} U_z \to A$ the bijection given by the above correspondence. Note that we will use the Canonical choice of representative.

$$\forall a \in Gr(A')_n, b \in Gr(A')_m \ \ \phi(a \star b) = \phi(a \cdot b + \sum_{i=1}^{\infty} \mu_i(a \otimes b)k^i)$$

However since $deg(\mu_i) = -i$ we have that $\deg(\mu_i(a \otimes b)k^i) = n + m - i < n + m - 1 \Rightarrow \sum_{i=1}^{\infty} \mu_i(a \otimes b)k^i \in Gr(A')_{n+m-1}$ this means that we have

$$\phi(a \star b) = \phi(a \cdot b + \sum_{i=1}^{\infty} \mu_i(a \otimes b)k^i) = \phi(a \cdot b) = a \cdot b$$

□

3 Proving the result

Theorem 3.1 (The Classical PBW theorem)**.** *Given a Lie algebra L over a field K, the associated graded of its universal enveloping algebra denoted U(L) is isomorphic to the symmetric algebra generated by L denoted S(L).*

Proof. In order to prove this we will demonstrate that there exists an algebraic deformation from S(L) to some module whose associated graded is isomorphic to that of U(L)

Recall from the section on the kozul complex that we have the vector subspace $K_1' = (x \otimes y - y \otimes x)$ of $V \otimes V$ which again is a subspace of $S(L)$ with an injection i and and a non-canonical projection p for each K_n' such that $p \circ i = id_{K_n'}$ and such that p and i commute with the differentials.

We begin by constructing a order 1 (infitesimal) deformation of S(A) with $\mu_1(a \otimes b) = [-,-] \circ p(a \otimes b) = p^*([-,-])$. Now we want to confirm that this is a Hochschild 2- cocycle of degree -1.

$$\begin{aligned} d^*(\mu_1)(a \otimes b \otimes c) &= d^*([-,-] \circ p)(a \otimes b \otimes c) \\ &= d^*(p^*([-,-]))(a \otimes b \otimes c) \\ &= p^*(d^*([-,-]))(a \otimes b \otimes c) \\ &= (d^*([-,-]))(p(a \otimes b \otimes c)) \end{aligned}$$

$$\begin{aligned} &d^*([-,-])(e \otimes f \otimes g - f \otimes e \otimes g - e \otimes g \otimes f + f \otimes g \otimes e + g \otimes e \otimes f - g \otimes f \otimes e) \\ &\quad = e[f,g] + f[g,e] + g[e,f] - ([f,g]e + [g,e]f + [e,f]g) \\ &\quad = e[f,g] - [f,g]e + f[g,e] - [g,e]f + g[e,f] - [e,f]g \in R \end{aligned}$$

Now since p has degree 0 and $[,]$ has degree -1 μ_1 has degree -1.
The next step will be to lift this infinitesimal deformation to a formal deformation. This will be done one step at a time.
We have seen in the section on algebraic deformations that the obstruction to lifting an ith-order graded deformation to a (i+1)th-order graded deformation is a Hochschild 3- cocycle of degree -(i+1).
The obstruction to lifting to a 2nd order graded deformation is then $\mu_1 \circ \mu_1$
Using that i induces an isomorphism on cohomology we get

$$\begin{aligned} &\mu_1 \circ \mu_1 \circ i(e \otimes f \otimes g - f \otimes e \otimes g - e \otimes g \otimes f + f \otimes g \otimes e + g \otimes e \otimes f - g \otimes f \otimes e) \\ &= \mu_1(e \otimes [f,g] - [f,g] \otimes e + f[g,e] - [g,e] \otimes f + g[e,f] - [e,f] \otimes g) \\ &= [e,[f,g]] + [f,[g,e] + [g,[e,f]] = jacobi = 0 \end{aligned}$$

Giving that $i^*(\mu_1 \circ \mu_1)$ is zero in cohomology on the kozul complex, implying that $(\mu_1 \circ \mu_1)$ is zero in cohomology on the bar complex, since i^* induces a graded isomorphism on cohomology, meaning that there exists a morphism $\mu_2' : A \otimes A$ such that $d(\mu_2') = \mu_1 \circ \mu_1$ wwich necessarily has degree -2. Now we need to alter this so that we get a μ_2 satisfying $i^*(\mu_2) = 0$.

Since $d^*(i^*(\mu_2')) = i^*(d^*(\mu_2')) = i^*(\mu_1 \circ \mu_1) = 0$ Meaning that $i^*(\mu_2')$ is a cocycle on the kozul resolution. Then again since i^* induces a graded isomorphism on cohomology there must be a corresponding cocycle on the bar complex $\tilde{\mu_2}$ such that $i^*(\tilde{\mu_2}) = i^*(\mu'_2)$. Then we can define $\mu_2 = \mu_2^{'} - \tilde{\mu_2}$. Now we have $d(\mu_2) = d^*(\mu_2^{'} - \tilde{\mu_2}) = \mu_1 \circ \mu_1 - 0$ and $i^*(\mu_2) = i^*(\mu_2^{'} - \tilde{\mu_2}) = 0$.
This deformation now has the properties we need, the only thing that remains is to extend it indefinitely. The 2.nd obstruction is $[\mu_2, mu_1]$ but $i * [\mu_2, mu_1] = 0$ since $i^*(\mu_2) = 0$, which means that it is a coboundary on the bar complex, so we can find a μ_3 which extends the deformation.
The remaining obstructions all have degree -4 or lower. Meaning they on the Kozul complex must lay in $Hom_k(K'(3), A)_z \mid z < -3$ but these spaces are all zero since elements of $K'(3)V \otimes V \otimes V$ all have degree 3 and elements of $A = \frac{T(V)}{R}$ all have degree 0 or greater. This means that all higher obstructions vanish on cohomology and we may lift this deformation indefinitly. We can also since the obstructions are zero on $K'(3)$ we may ensure that $i^*(\mu_i) = 0$ by the same argument as when we lifted to a order 2 deformation
All that remains is to show that this deformation with a specification to a value gives us $U(L)$, this value will be 1.
Consider the inclusion $\phi : L \to \frac{T(L)}{(x \otimes y - y \otimes x)}[t]/(t-1)$ it is injective and surjective on its image. It can then be extended to a algebra homomorphism $\phi : T(L) \to \frac{T(L)}{(x \otimes y - y \otimes x)}[t]/(t-1)$ wich is surjective. We want to show that elements of the form $x \otimes y - y - [x, y]$ map to zero over this morphism.

$$
\begin{aligned}
\phi(x \otimes y - y \otimes x - [x, y]) &= \phi(x) \star \phi(y) - \phi(y) \star \phi(x) - \phi([x, y]) \\
&= x \otimes y + \sum_{i=1}^{\infty} \mu_i(x \otimes y)t^i - (y \otimes x + \sum_{i=1}^{\infty} \mu_i(y \otimes x)t^i) - [x, y] \\
&= x \otimes y - y \otimes x + \sum_{i=1}^{\infty} \mu_i(x \otimes y - y \otimes x) - [x, y] \\
&= \mu_1(x \otimes y - y \otimes x) + \sum_{i=2}^{\infty} \mu_i(x \otimes y - y \otimes x) - [x, y] \\
&= [x, y] - [x, y] \\
&= 0
\end{aligned}
$$

Here we have used that $x \otimes y - y \otimes x = 0$ since we are working in the symmetric algebra and that $\mu_i(x \otimes y - y \otimes x) = 0$ for $1 < i$ by definition and that

$\mu_1(x \otimes y - y \otimes x) = [x, y]$ again by definition.
This implies that $\phi : T(L) \to \frac{T(L)}{(x\otimes y - y \otimes x)}[t]/(t-1)$ factors uniquely trough $\tilde{\phi} : \frac{T(L)}{(x\otimes y - y\otimes x - [x,y])} \to \frac{T(L)}{(x\otimes y - y\otimes x)}[t]/(t-1)$
this morphism we can then pass to a morphism $Gr(\tilde{\phi}) : Gr\left(\frac{T(L)}{(x\otimes y - y\otimes x - [x,y])}\right) \to Gr\left(\frac{T(L)}{(x\otimes y - y\otimes x)}[t]/(t-1)\right)$ again this sends the elements of L to themselves and extends.

Construct a morphism $\psi : T(L) \to Gr\left(\frac{T(L)}{(x\otimes y - y\otimes x - [x,y])}\right)$ by sending elements of L to themselves and extending multiplicatively and additivly. Let $R = (x \otimes y - y \otimes x - [x, y])$ Then $\psi(x \otimes y - y \otimes x) = (x \otimes y - y \otimes x + R \cap U_2) + U_1 = ([x, y] + R \cap U_2) + U_1 = 0$. Therefore we may factor the morphism trough $\frac{T(L)}{(x\otimes y - y\otimes x)}$.
This gives a surjective morphism $\frac{T(L)}{(x\otimes y - y\otimes x)} \to Gr\left(\frac{T(L)}{(x\otimes y - y\otimes x - [x,y])}\right)$ defined by sending generators to themselves. Now we put this together which gives us the following sequence:

$$\frac{T(L)}{(x\otimes y - y\otimes x)} \to Gr\left(\frac{T(L)}{(x\otimes y - y\otimes x - [x,y])}\right) \to Gr\left(\frac{T(L)}{(x\otimes y - y\otimes x)}[t]/(t-1)\right) \to \frac{T(L)}{(x\otimes y - y\otimes x)}$$

Now since all the maps send the generating set, L to itself the composition of all these maps sends L to itself and hence is the identity on $\frac{T(L)}{(x\otimes y - y\otimes x)}$. Since The composition of all these maps is injective, the first map has to be injective.
Hence

$$\frac{T(L)}{(x\otimes y - y\otimes x)} \simeq Gr\left(\frac{T(L)}{(x\otimes y - y\otimes x - [x,y])}\right)$$

An alternative proof is given by showing that ϕ is an isomorphism. For finite dimensional Lie algebras this follows by that ϕ is surjective and by counting the dimensions of the domain and co-domain and showing them to be equal. This version is interesting since it proves that the universal enveloping algebra is in fact a deformation of the symmetric algebra.
However extending this proof to arbitrary dimensions is hard, but seems like something that would be worthwhile. □

Another interesting fact about this theorem is that it is an equivalence. That is $T(V)/(x \otimes y - y \otimes x) \simeq Gr(T(V)/(x \otimes y - y \otimes x - \mu(x \otimes y - y \otimes x))$if and only if mu is given by a bracket. That is, it satisfies the Jacobi identity. This follows readily from the version of the theorem proved in [Wit19] section 5.5.

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