Johan Vik Mathisen

An introduction to triangulated categories via the stable category of a Frobenius category

Bacheloroppgave i matematiske fag
Veileder: Johanne Haugland
Desember 2020

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Abstract

The goal of this thesis is to prove that the stable category of a Frobenius category is triangulated. This thesis is divided into two parts. Part one consists of an introduction to triangulated categories, with emphasis on intuition and motivation. Part two is an introduction to exact and quotient categories, leading up to the triangulation of the stable Frobenius category.

1 Introduction

The notion of a triangulated category was introduced independently in algebraic geometry by Jean-Louis Verdier [11], based on ideas on ideas of Alexander Groethendieck, and in algebraic topology by Dieter Puppe in the 1960's [8]. These constructions have since played prominent roles in algebraic topology, algebraic geometry and representation theory.

In the modern landscape of mathematics, triangulated categories mainly arise in two ways: either as stable homotopy categories of model categories, or as the stable category of a Frobenius category. These are called *topological* and *algebraic* triangulated categories respectively.

In this thesis we assume no prior knowledge of triangulated and exact categories. However, the reader should be familiar with additive and abelian categories, and elementary concepts from category theory. Some knowledge of general abstract algebra is of use, mostly to understand the motivation behind the theory presented. The text is written so that little is left to the reader, but it is nonetheless recommended to bring along pen and paper for the ride.

We begin with an elementary introduction, and give a thorough comparison of triangulated and abelian categories. From there we move on to the notion of exact categories and quotient categories, which enables us to define the stable Frobenius category. Finally, we show that the stable Frobenius category carries a triangulated structure.

Contents

2 Triangulated categories

In this section we give an introduction to the theory of *triangulated categories*, with emphasis on intuition and motivation. We present some rather remarkable consequences of the axioms, give an in depth proof of the fact that the category of vector spaces is triangulated and conclude with a comparison of triangulated and abelian categories.

Definition 2.1. A functor Σ between additive categories is called *additive* if for every pair of objects X, Y, the associated map $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(\Sigma X, \Sigma Y)$ is a homomorphism of abelian groups.

Let $\mathcal{T}$ be an additive category, and $\Sigma : \mathcal{T} \to \mathcal{T}$ be an additive autoequivalence. Being an additive autoequivalence means that there exists an additive functor $\Sigma^{-1} : \mathcal{T} \to \mathcal{T}$ such that $\Sigma^{-1} \circ \Sigma$ and $\Sigma \circ \Sigma^{-1}$ are naturally isomorphic to the identity functor on $\mathcal{T}$. A *triangle* is a sequence in $\mathcal{T}$ of the form

$$X \to Y \to Z \to \Sigma X.$$

A *morphism of triangles* is a triple (f, g, h) of morphisms, making the following diagram commutative in $\mathcal{T}$

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}$$

If f, g and h are all isomorphisms, then (f, g, h) is called an *isomorphism of triangles*.

Definition 2.2 (Triangulated category)**.** A *triangulated category* is a triple $(\mathcal{T}, \Sigma, \Delta)$ of an additive category $\mathcal{T}$, an additive autoequivalence Σ and a collection of distinguished triangles Δ satisfying the following axioms:

TR1
- Any triangle isomorphic to a distinguished triangle is distinguished.
- For every object X, the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow \Sigma X$$

  is distinguished.
- For every morphism $f : X \longrightarrow Y$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

TR2 (*Rotation axiom*)

For any $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in Δ, the triangles

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

and

$$\Sigma^{-1}Z \xrightarrow{-\Sigma^{-1}h} X \xrightarrow{f} Y \xrightarrow{g} Z$$

are also in Δ.

TR3 (*Morphism axiom*)
Given the solid part of the following diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dotted, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

where the rows are in Δ, there exists an h such that (f, g, h) is a morphism of triangles.

TR4 (*Octahedral axiom*)
Given distinguished triangles

$$X \xrightarrow{u} Y \to Z' \to \Sigma X,$$

$$Y \xrightarrow{v} Z \to X' \to \Sigma Y$$

and

$$X \xrightarrow{vu} Z \to Y' \to \Sigma X,$$

there exists a distinguished triangle

$$Z' \to Y' \to X' \to \Sigma Z',$$

making the following diagram commutative

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d, dotted] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "vu"] & Z \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d, dotted] & \Sigma X \arrow[d, "\Sigma u"] \\
 & X' \arrow[r, equal] \arrow[d] & X' \arrow[r] \arrow[d] & \Sigma Y \\
 & \Sigma Y \arrow[r] & \Sigma Z' &
\end{tikzcd}$$

Remark 2.3. The third object in a distinguished triangle is called the *cone* of the first morphism.

Remark 2.4. An additive category satisfying TR1 through TR3 is called *pre-triangulated*.

Remark 2.5. TR2 and TR3 give us a '2 out of 3'-property for morphisms of triangles. The reason for this is that by TR2 we can just rotate the triangle to get the missing morphism in the right spot, before using TR3. If we have 2 morphisms connecting two distinguished triangles, we always have the third.

Remark 2.6. One rather annoying remark is that the cone is not in general functorial, i.e the morphism h in TR3 is typically not unique. A quick example of this is the following

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[r, equal] \arrow[d, dotted, "h"] & \Sigma X \arrow[d] \\ 0 \arrow[r] & \Sigma X \arrow[r, equal] & \Sigma X \arrow[r] & 0 \end{tikzcd}$$

Both $h = 0$ and $h = \mathrm{id}$ makes the diagram commute. In fact, any morphism in $\mathrm{Hom}(\Sigma X, \Sigma X)$ will make the diagram commute.

The axioms presented here can be weakened. The morphism axiom has been shown to be redundant by J. P. May in [7]. Moreover, in the next section we will give a short proof that half of TR2 is sufficient. By half of TR2 we mean the following.

Definition 2.7 (TR2'). If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished triangle, then so is the triangle $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$.

When proving that a category is triangulated, the octahedral axiom is often the crux of the proof. Therefore the convenience of being able to use results for pre-triangulated categories to show TR4 overshadows the redundancies in the axioms presented.

2.1 Elementary properties

The definition of triangulated categories might at a first glance seem both unintuitive and unmotivated. By looking at some immediate consequences of the axioms, we get an impression of what these structures are. Later we devote a section to the comparison of triangulated and abelian categories, for the sake of intuition. For the reminder of this section, let $\mathcal{T}$ be a triangulated category with suspension functor Σ. Unless otherwise stated, all objects and morphisms come from $\mathcal{T}$.

Proposition 2.8 (Composition of morphisms). *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle. Then $vu = 0$ and $wv = 0$.*

Proof. By the rotation axiom, it is sufficient to show that $vu = 0$, since we can do the exact same argument on the rotated triangle to obtain $wv = 0$. By the rotation axiom (TR2), TR1 and the morphism axiom (TR3), the following diagram is a morphism of triangles

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "v"] & Z \arrow[r, "w"] \arrow[d, equal] & \Sigma X \arrow[r, "-\Sigma u"] \arrow[d] & \Sigma Y \arrow[d, "\Sigma v"] \\ Z \arrow[r, equal] & Z \arrow[r] & 0 \arrow[r] & \Sigma Z \end{tikzcd}$$

From the rightmost square we see that $0 = \Sigma v \circ -\Sigma u = -\Sigma(vu)$. As Σ is an autoequivalence, we obtain $vu = 0$, as desired. ☐

In other words, the above proposition says that the composition of any two consecutive morphisms in a distinguished triangle vanishes. In the following proposition, we see that that every distinguished triangle gives rise to a long exact sequence of abelian groups. Taken together, these propositions should give an impression of why the theory triangulated categories is a useful framework for doing homological algebra.

Proposition 2.9 (Long exact sequences)**.** *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle. For any object $T \in \mathcal{T}$, there is a long exact sequence of abelian groups*

$$\cdots \to \operatorname{Hom}_{\mathcal{T}}(T, \Sigma^i X) \xrightarrow{\Sigma^i u_*} \operatorname{Hom}_{\mathcal{T}}(T, \Sigma^i Y) \xrightarrow{\Sigma^i v_*} \operatorname{Hom}_{\mathcal{T}}(T, \Sigma^i Z)$$

$$\xrightarrow{\Sigma^i w_*} \operatorname{Hom}_{\mathcal{T}}(T, \Sigma^{i+1} X) \to \dots$$

Proof. By the rotation axiom, we only need to show that

$$\operatorname{Hom}_{\mathcal{T}}(T, X) \xrightarrow{u_*} \operatorname{Hom}_{\mathcal{T}}(T, Y) \xrightarrow{v_*} \operatorname{Hom}_{\mathcal{T}}(T, Z)$$

is exact. In other words we want to show that $\operatorname{Im} u_* = \operatorname{Ker} v_*$. Since we already know that the composition of two consecutive morphisms in a distinguished triangle vanish, we easily obtain one inclusion. By functoriality

$$v_* \circ u_* = (v \circ u)_* = 0,$$

hence $\operatorname{Im} u_* \subseteq \operatorname{Ker} v_*$. Let $f \in \operatorname{Ker} v_*$, i.e $v \circ f = 0$. We want to show that there exists a morphism g such that $f = u \circ g$. Consider the following diagram, whose rows are distinguished triangles

$$\begin{tikzcd}
T \arrow[r, equal] \arrow[d, dotted, "g"] & T \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma T \arrow[d, dotted, "\Sigma g"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & \Sigma X
\end{tikzcd}$$

By assumption, the middle square commutes, and by TR3 there exists a morphism g making the diagram a morphism of triangles. Now by looking at the leftmost square, we see that $f = u \circ g$. Hence, the sequence is exact. □

In the above proof, a particularly nice consequence of the rotation axiom was on display. In the infinite sequence of abelian groups, we only needed to consider the exactness in one degree to conclude that the entire sequence was exact. This is a property anyone who has worked with long exact sequences surely can appreciate.

Proposition 2.10 (Split triangles)**.** *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle where w is the zero morphism. Then u is a split monomorphism and v is a split epimorphism.*

Proof. We show that u is a split monomorphism by finding the left inverse of u. We have the following diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, dotted, "u'"] & Z \arrow[r, "w=0"] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & \Sigma X
\end{tikzcd},$$

which, by TR2, TR3 and out assumption on w, is a morphism of triangles. Call this completing morphism u'. We then have $u'u = \mathrm{id}$, and consequently u is a split monomorphism. The argument for v is similar. □

When working with abelian categories, and in particular problems in homological algebra, the 5-lemma is an indispensable tool. The same can be said for triangulated categories.

Proposition 2.11 (Triangulated 5-lemma)**.** *Let the following diagram be a morphism of triangles*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

If f and g are isomorphism, then so is h.

Proof. Assume that f and g in the diagram above are isomorphisms. We use Proposition 2.9 and apply $\mathrm{Hom}_{\mathcal{T}}(Z', -)$ to obtain the following exact sequences of abelian groups

$$\begin{tikzcd}
\mathrm{Hom}(Z',X) \arrow[r, "u_*"] \arrow[d, "f_*"] & \mathrm{Hom}(Z',Y) \arrow[r, "v_*"] \arrow[d, "g_*"] & \mathrm{Hom}(Z',Z) \arrow[r, "w_*"] \arrow[d, "h_*"] & \mathrm{Hom}(Z',\Sigma X) \arrow[r, "\Sigma u_*"] \arrow[d, "\Sigma f_*"] & \mathrm{Hom}(Z',\Sigma Y) \arrow[d, "\Sigma g_*"] \\
\mathrm{Hom}(Z',X') \arrow[r, "u'_*"] & \mathrm{Hom}(Z',Y') \arrow[r, "v'_*"] & \mathrm{Hom}(Z',Z') \arrow[r, "w'_*"] & \mathrm{Hom}(Z',\Sigma X') \arrow[r, "\Sigma u'_*"] & \mathrm{Hom}(V,\Sigma Y')
\end{tikzcd}$$

Since f and g are isomorphisms, f_*, g_* and $\Sigma f_*, \Sigma g_*$ are all isomorphisms. Hence, by the 5-lemma of abelian groups, we can conclude that h_* is an isomorphism. This gives preimage of $\mathrm{id}_{Z'}$ along h_*, i.e there exists a morphism $h' : Z' \to Z$ such that $\mathrm{id}_{Z'} = h \circ h'$. The morphism h' is a right inverse of h, and dually we find the left inverse by applying the contravariant Hom-functor and use the dual of Proposition 2.9. □

Remark 2.12. An immediate consequence of the triangulated 5-lemma is the unique completion of a morphism into a triangle. Consider the following commutative diagram, where both rows are distinguished

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, dotted, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X
\end{tikzcd}$$

By TR3 k exists, and by the triangulated 5-lemma it is an isomorphism. Thus any two distinguished triangles with the same starting morphism are isomorphic.

It is worth noting that the triangulated 5-lemma holds for pre-triangulated categories. With the 5-lemma at hand, we are able to prove that TR2' is sufficient.

Proposition 2.13. *Let $(\mathcal{T}, \Sigma, \Delta)$ be as in Definition 2.2, but assume that Δ satisfies* TR1, TR2$'$ *and* TR3. *Then* TR2 *is satisfied as well.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a distinguished triangle. By TR1, we have the following triangle in Δ

$$\Sigma^{-1}Z \xrightarrow{-\Sigma^{-1}h} X \xrightarrow{f'} Y' \xrightarrow{g'} Z.$$

Consider the following commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, equal] & \Sigma X \arrow[r, "-\Sigma f"] \arrow[d, equal] & \Sigma Y \arrow[r, "-\Sigma g"] \arrow[d, dotted, "k"] & \Sigma Z \arrow[d, equal] \\
Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f'"] & \Sigma Y' \arrow[r, "-\Sigma g'"] & \Sigma Z
\end{tikzcd}$$

where TR2' assures that the rows are distinguished, and by TR3 it is a morphism of triangles. The triangulated 5-lemma now applies, and we can conclude that k is an isomorphism. Consequently $Y \cong Y'$, which yields our desired conclusion. $\square$

2.2 A first example

Usually, examples of triangulated categories are quite advanced (for the undergraduate students at least), as the homotopy category of chain complexes over an abelian category or the stable Frobenius category. We on the other hand, wish to introduce the theory with a more elementary example. Our first example of a triangulated category will be the module category over a field, $\operatorname{Mod} K$, also known as the category of vector spaces. This is a way to get familiar with the definition, without using too advanced mathematics.

In order to understand the (yet to be defined) triangulated structure on $\operatorname{Mod} K$, we will make use of the following lemma. In particular, we will use an immediate corollary, namely that $\operatorname{Mod} K$ is a semisimple category.

Lemma 2.14. *Let R be a ring. A module P in $\operatorname{Mod} R$ is projective if and only if every short exact sequence of the form*

$$0 \to A \to B \to P \to 0$$

splits.

Proof. Let P be a projective R module and $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ a short exact sequence of R modules. Since P is projective and g is surjective, we get the following diagram

$$\begin{tikzcd}
 & & P \arrow[dl, dotted, "h"'] \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & P
\end{tikzcd}$$

We see that h is a left inverse of g, and hence injective. Since g is surjective and the sequence is exact, one obtains $B = \operatorname{Im} h \oplus \operatorname{Ker} g = \operatorname{Im} h \oplus \operatorname{Im} f \cong A \oplus P$. This means that the sequence splits.

For the converse, let P be a module such that every short exact sequence of the form above splits. As we know, every module is the image of a free module, in other words there exists a surjective map f from a free module F onto P. Considering the kernel of f, we get a short exact sequence

$$0 \to \operatorname{Ker} f \xrightarrow{\iota} F \xrightarrow{f} P \to 0$$

By assumption, this sequence splits, and $F \cong \operatorname{Ker} f \oplus P$. Consequently P is a direct summand of a free module. For the lifting property, we know that any free module is projective, so consider the following diagram where the bottom row is exact

$$\begin{tikzcd}
 & \operatorname{Ker} f \oplus P \arrow[dl, "h'"'] \arrow[d, bend right, "\pi"'] & \\
 & P \arrow[u, bend right, "\iota"'] \arrow[d, "k"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

If we now define $h := h'\iota$, we obtain our desired lifting map, since $gh = gh'\iota = k\pi\iota = k$. Therefore, P is projective. $\square$

Since all free modules are projective, every object in $\operatorname{Mod} K$ is projective. As an immediate consequence from the above lemma, every short exact sequence in $\operatorname{Mod} K$ splits.

From an algebraic point of view, vector spaces are quite simple structures, and the triangulated structure we are about to define somewhat reflects that. We let the additive autoequivalence on $\operatorname{Mod} K$ be given by the identity functor, and define the class of distinguished triangles to be

$$\Delta = \{X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X | X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X \xrightarrow{f} Y \text{ is exact}\},$$

which we see is closed under isomorphism. As a final preparation before the proof that this gives a triangulated structure, we find a particularly nice representative for each isomorphism class in Δ. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$$

be a distinguished triangle, and consider the short exact sequence

$$0 \to \operatorname{Ker} h \to Z \xrightarrow{h} \operatorname{Im} h \to 0.$$

Since $\operatorname{Im} h$ is projective, we know that $Z \cong \operatorname{Im} h \oplus \operatorname{Ker} h$. By exactness of our sequence, we obtain $\operatorname{Im} h = \operatorname{Ker} f$ and $\operatorname{Ker} h = \operatorname{Im} g$. The first isomorphism theorem yields

$$\operatorname{Im} g \cong Y/\operatorname{Ker} g = Y/\operatorname{Im} f = \operatorname{Cok} f,$$

and thus $Z \cong \operatorname{Ker} f \oplus \operatorname{Cok} f$. In particular, every distinguished triangle is isomorphic to a triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\left[\begin{smallmatrix} 0 \\ \pi \end{smallmatrix}\right]} \operatorname{Ker} f \oplus \operatorname{Cok} f \xrightarrow{[\iota\ 0]} X,$$

where π and ι are the cokernel and kernel maps respectively.

Proposition 2.15. *The triple* $(\operatorname{Mod} K, \operatorname{Id}, \Delta)$ *is a triangulated category.*

Proof. Every distinguished triangle is isomorphic to a triangle of the form

$$X \xrightarrow{f} Y \to \operatorname{Ker} f \oplus \operatorname{Cok} f \to X,$$

by the above discussion. Hence, it is sufficient to prove that these triangles satisfy the axioms.

TR1. We see that Δ is closed under isomorphisms, and clearly every linear transformation $f : X \to Y$ can be completed into a distinguished triangle. Since the identity on any object X is an isomorphism, the identity can always be completed into to a triangle

$$X = X \longrightarrow 0 \longrightarrow X.$$

TR2. The rotation axiom follows immediately from the requirements on Δ.

TR3. Consider the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "\left[\begin{smallmatrix} 0 \\ \pi \end{smallmatrix}\right]"] \arrow[d, "v"] & \operatorname{Ker} f \oplus \operatorname{Cok} f \arrow[r, "{[\iota\ 0]}"] & X \arrow[d, "u"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "\left[\begin{smallmatrix} 0 \\ \pi' \end{smallmatrix}\right]"] & \operatorname{Ker} f' \oplus \operatorname{Cok} f' \arrow[r, "{[\iota'\ 0]}"] & X' \end{tikzcd} \tag{2.1}$$

where the rows are distinguished and the left square commute. We split up the diagram and look at the kernel and cokernel separately. Consider the following commutative diagram

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, "\iota"] \arrow[d, dotted, "w_1"] & X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "v"] \\
\operatorname{Ker} f' \arrow[r, "\iota'"] & X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

By the universal property of $\operatorname{Ker} f'$, there exists a unique morphism w_1 making the diagram commute. Dually, we get a morphism w_2 between the cokernels. Since $\oplus$ is a coproduct, we can combine our maps

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r] \arrow[dr, "{[w_1,0]}"'] & \operatorname{Ker} f \oplus \operatorname{Cok} f \arrow[d, dotted, "\exists! w"] & \operatorname{Cok} f \arrow[l] \arrow[dl, "{[0,w_2]}"] \\
& \operatorname{Ker} f' \oplus \operatorname{Cok} f' &
\end{tikzcd}$$

where

$$w = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} : \operatorname{Ker} f \oplus \operatorname{Cok} f \longrightarrow \operatorname{Ker} f' \oplus \operatorname{Cok} f'$$

because, well, it fits. Now w completes our original diagram (2.1) making it commutative, since

$$\begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \begin{bmatrix} 0 \\ \pi \end{bmatrix} = \begin{bmatrix} 0 \\ w_2\pi \end{bmatrix} = \begin{bmatrix} 0 \\ \pi' v \end{bmatrix}$$

and

$$[\iota' \ 0] \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} = [\iota' w_1 \ 0] = [u\iota \ 0]$$

TR4. Let

$$X \xrightarrow{u} Y \to \operatorname{Ker} u \oplus \operatorname{Cok} u \to X,$$
$$Y \xrightarrow{v} Z \to \operatorname{Ker} v \oplus \operatorname{Cok} v \to Y$$

and

$$X \xrightarrow{vu} Z \to \operatorname{Ker} vu \oplus \operatorname{Cok} vu \to X$$

be distinguished triangles. We can construct the solid part of the following diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "{\left[\begin{smallmatrix} 0 \\ \pi_u \end{smallmatrix}\right]}"] \arrow[d, "v"] & \operatorname{Ker} u \oplus \operatorname{Cok} u \arrow[r, "{[\iota_u \ 0]}"] \arrow[d, dotted, "k"] & X \arrow[d, equal] \\
X \arrow[r, "vu"] & Z \arrow[r, "{\left[\begin{smallmatrix} 0 \\ \pi_{vu} \end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix} 0 \\ \pi_v \end{smallmatrix}\right]}"] & \operatorname{Ker} vu \oplus \operatorname{Cok} vu \arrow[r, "{[\iota_{vu} \ 0]}"] \arrow[d, dotted, "k'"] & X \arrow[d, "u"] \\
& \operatorname{Ker} v \oplus \operatorname{Cok} v \arrow[r, equal] \arrow[d, "{[\iota_v \ 0]}"] & \operatorname{Ker} v \oplus \operatorname{Cok} v \arrow[r, "{[\iota_v \ 0]}"] \arrow[d, "\varphi"] & Y \\
& Y \arrow[r, "{\left[\begin{smallmatrix} 0 \\ \pi_u \end{smallmatrix}\right]}"] & \operatorname{Ker} u \oplus \operatorname{Cok} u &
\end{tikzcd}$$

where $\varphi = \left[\begin{smallmatrix} 0 & 0 \\ \pi_u \iota_v & 0 \end{smallmatrix}\right]$. The goal now is to find k and k' which make the diagram commute, and the new triangle distinguished. The strategy is to use the universal properties exactly as we did in the proof for the morphism axiom, and then check exactness and commutativity. From the kernel and cokernel properties, we get both $k = \left[\begin{smallmatrix} k_1 & 0 \\ 0 & k_2 \end{smallmatrix}\right]$ and $k' = \left[\begin{smallmatrix} k_1' & 0 \\ 0 & k_2' \end{smallmatrix}\right]$. Since the construction of k is identical to the construction of w in the morphism axiom, we know that the top three squares commute. Similarly, the two remaining squares also commutes by construction of k'.

We now begin the process of showing exactness in the sequence. There are few surprises from here and out, but we do the rather laborious process of showing exactness, to maintain the introductory style of this section. We begin by showing that the triangle is exact at $\operatorname{Ker} vu \oplus \operatorname{Cok} vu$. The composition of k and k' is

$$\begin{bmatrix} k_1' & 0 \\ 0 & k_2' \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 k_1' & 0 \\ 0 & k_2 k_2' \end{bmatrix}$$

We must show that both compositions on the diagonal are exact. Consider the following diagram

$$\begin{tikzcd} \operatorname{Ker} u \arrow[r, "k_1"] & \operatorname{Ker} vu \arrow[r, "\iota_{vu}"] \arrow[d, "k_1'"] & X \arrow[d, "u"] \\ & \operatorname{Ker} v \arrow[r, "\iota_v"] & Y \end{tikzcd}$$

By commutativity, we get that $\iota_v k_1' k_1 = 0$, which implies that $k_1' k_1 = 0$, since ι_v is a monomorphism. The kernel of $u \iota_{vu}$ is $\operatorname{Ker} u$, because ι_{vu} is the inclusion of $\operatorname{Ker} vu$ into X and $\operatorname{Ker} vu$ contains $\operatorname{Ker} u$. By commutativity, the kernel of $\iota_v k_1'$ is $\operatorname{Ker} u$, and since ι_v is a monomorphism we get $\operatorname{Ker} k_1' = \operatorname{Ker} u = \operatorname{Im} k_1$. With the following diagram

$$\begin{tikzcd} Y \arrow[r, "\pi_u"] \arrow[d, "v"] & \operatorname{Cok} u \arrow[d, "k_2"] & \\ Z \arrow[r, "\pi_{vu}"] & \operatorname{Cok} vu \arrow[r, "k_2'"] & \operatorname{Cok} v \end{tikzcd}$$

and a dual argument to the one above, we get that the triangle is exact at $\operatorname{Ker} vu \oplus \operatorname{Cok} vu$. To show exactness at $\operatorname{Ker} v \oplus \operatorname{Cok} v$, we begin by composing the relevant morphisms. We have

$$\varphi k' = \begin{bmatrix} 0 & 0 \\ \pi_u \iota_v & 0 \end{bmatrix} \begin{bmatrix} k_1' & 0 \\ 0 & k_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \pi_u \iota_v k_1' & 0 \end{bmatrix}$$

Since ι_v is the inclusion map and the kernel of π_u is the image of u, we have that

$$\operatorname{Ker} \pi_u \iota_v = \operatorname{Ker} v \cap \operatorname{Im} u.$$

To determine the image of k_1', we use that $\operatorname{Im} k_1' = \operatorname{Im} \iota_v k_1'$, which holds as ι_v is just the inclusion map. By commutativity, we have $\operatorname{Im} \iota_v k_1' = \operatorname{Im} u \iota_{vu}$, which again is equal to $\operatorname{Ker} v \cap \operatorname{Im} u$.

Finally we show exactness at $\operatorname{Ker} u \oplus \operatorname{Cok} u$. We begin by calculating the composition of the relevant morphisms as usual

$$k\varphi = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \pi_u \iota_v & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k_2 \pi_u \iota_v & 0 \end{bmatrix}$$

We know what cokernels of linear transformations look like, thus consider the following commutative diagram

$$\begin{tikzcd}
 & & Z \arrow[dr, "\pi_{vu}"] & \\
\operatorname{Ker} v \arrow[r, "\iota_v"] & Y \arrow[ur, "v"] \arrow[dr, "\pi_u"'] & & Z/\operatorname{Im} vu \\
 & & Y/\operatorname{Im} u \arrow[ur, "k_2"'] &
\end{tikzcd}$$

We have that $v\iota_v = 0$ and therefore $k_2\pi_u\iota_v = \pi_{vu}v\iota_v = 0$, which gives us $\operatorname{Im} \pi_u\iota_v \subseteq \operatorname{Ker} k_2$. For the reversed inclusion, let $a \in \operatorname{Ker} k_2$. Since π_u is an epimorphism, a has a preimage along π_u, call it a'. Now by assumption, a' vanishes when we follow the lower part of the diagram, and by commutativity also along the upper part. This implies that a' is either in the kernel of v or in the kernel of π_{vu}. If $a' \in \operatorname{Ker} v$, then it lies in the image of ι_v, and consequently $a \in \operatorname{Im} \pi_u\iota_v$. If a' is not in the kernel of v, then $v(a') \in \operatorname{Im} vu$, which implies that $a' \in \operatorname{Im} u$. This again implies that $a' = 0$. We conclude that $\operatorname{Im} \pi_u\iota_v = \operatorname{Ker} k_2$ and

$$\operatorname{Ker} u \oplus \operatorname{Cok} u \xrightarrow{k} \operatorname{Ker} vu \oplus \operatorname{Cok} vu \xrightarrow{k'} \operatorname{Ker} v \oplus \operatorname{Cok} v \xrightarrow{\varphi} \operatorname{Ker} u \oplus \operatorname{Cok} u$$

is a distinguished triangle. This shows that $\operatorname{Mod} K$ carries a triangulated structure. $\square$

Remark 2.16. It should be noted that any semisimple abelian category can be given a triangulated structure in an analogous way. The proof is fairly similar to the one for $\operatorname{Mod} K$. The reason is that the cone of the first morphism is the sum of its kernel and cokernel, since all short exact sequences splits, and that the first isomorphism theorem holds in abelian categories.

2.3 Comparison of triangulated and abelian categories

A careful reader may have noticed that there are similarities between the theory of triangulated and abelian categories. In this section we discuss these similarities, and look at the abelian counterpart of the axioms of triangulated categories. It is a good exercise to do this comparison on your own. The octahedral axiom has a particularly neat counterpart in abelian categories. This section will be somewhat informal.

Both triangulated and abelian categories have an underlying additive structure, and from there their axioms take different paths, but the similarities do not end. Both in the theory of abelian and triangulated categories, a fundamental role is played by a distinguished class of 3-term sequences, namely the short exact sequences of an abelian category and the distinguished triangles of a triangulated category. We have notions of morphisms, and in particular isomorphisms, between both these classes of 3-term sequences. Identity morphisms give rise to triangles and short exact sequences, in their respective categories, by TR1 and the fact that the identity is an isomorphism.

The analogue of the morphism axiom in abelian categories is related to the fact that the third object in a triangle is both a weak kernel and a weak cokernel. If we look at the following morphism of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dotted, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X', \end{tikzcd}$$

we can think about the part to the left of h as the "cokernel part" and the one to the right as the "kernel part" of the diagram. To make it clear what we mean by this, consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dotted, "h"] & 0 \\ 0 \arrow[r] & X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r] & 0 \end{tikzcd}$$

in an abelian category and assume that the rows are short exact sequences. In this setup, we know that $Z \cong \operatorname{Cok} u$ and $Z' \cong \operatorname{Cok} u'$, and there will exist a map h by the universal property of the cokernel. Using the rotation axiom we get a similar setup using the kernel property.

Finally, we take a look at the octahedral axiom. The axiom states that given three distinguished triangles related by a composition, it guarantees the existence of a fourth. We have a similar result for abelian categories. Let $X \xrightarrow{u} Y \to Z'$, $Y \xrightarrow{v} Z \to X'$ and $X \xrightarrow{vu} Z \to Y'$ be short exact sequences. We can make a diagram as follows

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & X \arrow[r, "vu"] & Z \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d, dotted] & 0 \\
 & & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

The dotted arrows exist by the cokernel property of Z' and Y' respectively. The rightmost sequence is exact since the upper right square is a pushout and the cokernel is a right exact functor. For the sake of simplicity, assume the above diagram consists of objects in $\operatorname{Mod} R$ for some ring R. Then we know that the cokernels are quotients, so $Z' \cong Y/X$, $Y' \cong Z/X$ and $X' \cong Z/Y$ by the initial assumption. Exactness of the dotted sequence then gives us that X' is also isomorphic to Y'/Z', i.e we have

$$\frac{Z/X}{Y/X} \cong Z/Y,$$

which is known as the third isomorphism theorem.

Even though triangulated and abelian categories have many analogue properties, the following result shows that the concepts only overlap slightly.

Proposition 2.17. *Let $\mathcal{A}$ be a category which is both triangulated and abelian. Then every short exact sequence in $\mathcal{A}$ splits.*

Proof. Let $0 \to X \xrightarrow{f} Y \to Z \to 0$ be an exact sequence in $\mathcal{A}$. By TR1 and TR2, we get a distinguished triangle

$$\Sigma^{-1}Z' \xrightarrow{w} X \xrightarrow{f} Y \xrightarrow{g} Z'$$

In a distinguished triangle any consecutive morphisms vanish, so $fw = 0$. But f is a monomorphism, since it is the first morphism in a short exact sequence. Hence $w = 0$ and by Proposition 2.10 the sequence splits. $\square$

3 Exact categories

In this section we introduce the notion of exact categories, in the sense of Quillen [9]. Exact categories can be seen as a generalization of abelian categories. The definition of exact categories captures the essential properties of short exact sequences without assuming the existence of all kernels and cokernels. The main references for the following sections are [1] and [3].

3.1 Definition and examples

Definition 3.1. Let $\mathcal{A}$ be an additive category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called *exact* if f is a kernel of g, and g is a cokernel of f. The pair (f, g) is called a *kernel-cokernel pair*.

Definition 3.2. Let $\mathcal{E}$ be a family of kernel-cokernel pairs in an additive category $\mathcal{A}$. If $(f, g) \in \mathcal{E}$, we call f an *inflation* (*admissable monomorphism*), g a *deflation* (*admissable epimorphism*) and (f, g) a *conflation*.

Inflations and deflations will be denoted by $\rightarrowtail$ and $\twoheadrightarrow$, respectively. A morphism of exact sequences is a triple (φ, ψ, θ) of morphisms, making the following diagram commute in $\mathcal{A}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi"] & B \arrow[r, "g"] \arrow[d, "\psi"] & C \arrow[d, "\theta"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd}$$

The triple (φ, ψ, θ) is an isomorphism of exact sequences if all three morphisms are isomorphisms.

Definition 3.3 (Exact category)**.** Let $\mathcal{A}$ be an additive category and $\mathcal{E}$ a family of kernel-cokernel pairs in $\mathcal{A}$. Assume $\mathcal{E}$ is closed under isomorphisms and satisfies the following axioms:

Ex0 For any pair of objects A, B in $\mathcal{A}$, the canonical sequence

$$A \overset{\iota_A}{\rightarrowtail} A \oplus B \overset{\pi_B}{\twoheadrightarrow} B$$

is in $\mathcal{E}$.

Ex1 The composition of two deflations is again a deflation.

Ex1^{op} The composition of two inflations is again an inflation.

Ex2 If $f : A \twoheadrightarrow C$ is a deflation and $g : B \to C$ is any morphism, then the pullback

$$\begin{tikzcd} C' \arrow[r, two heads, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\text{PB}"] & B \arrow[d, "g"] \\ A \arrow[r, two heads, "f"] & C \end{tikzcd}$$

exists and f' is a deflation.

Ex2^{op} If $f : C \rightarrowtail A$ is a inflation and $g : C \to B$ is any morphism, then the pushout

$$\begin{tikzcd} C \arrow[r, tail, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\text{PO}"] & A \arrow[d, "g'"] \\ B \arrow[r, tail, "f'"] & C' \end{tikzcd}$$

exists and f' is an inflation.

Ex3 Let $f : B \to C$ be a morphism with a kernel. If there exists a morphism $g : A \to B$ such that $f \circ g$ is a deflation, then f is a deflation.

Ex3^{op} Let $f : A \to B$ be a morphism with a cokernel. If there exists a morphism $g : B \to C$ such that $g \circ f$ is a inflation, then f is a inflation.

Then $(\mathcal{A}, \mathcal{E})$ is an *exact category*, and $\mathcal{E}$ is an *exact structure* on $\mathcal{A}$.

Remark 3.4. We often say that $\mathcal{A}$ is exact, meaning that $(\mathcal{A}, \mathcal{E})$ is an exact category.

Remark 3.5. By the duality of the axioms, $\mathcal{E}$ is an exact structure on $\mathcal{A}$ if and only if $\mathcal{E}^{op}$ is an exact structure on $\mathcal{A}^{op}$.

Remark 3.6. The axioms presented above are due to Quillen [9]. Yoneda showed that Ex3 and Ex3^{op} was a consequence of the other axioms [12]. This fact was rediscovered by Keller 30 years later, and he also showed some redundancies in the remaining axioms as well [5]. The interested reader is encouraged to investigate this further in [2].

Note that since $A \oplus 0 \cong A$ in any additive category and by Ex0 both the canonical sequences are conflations, the identity morphism id_A is both an inflation and a deflation. Isomorphisms are also both inflations and deflations. To see this, let $f : A \to B$ be an isomorphism, and consider the following diagrams

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "f"] \\ 0 \arrow[r] & A \arrow[r, "f"] & B \end{tikzcd}$$

Since the collection of conflations is closed under isomorphisms, both bottom rows must be conflations.

A natural question to ask is what properties a subcategory of an exact category must possess in order to inherit the exact structure, and be an exact category in its own right. We next show that exact categories can be thought of as an axiomatization of what we call extension closed subcategories.

Definition 3.7 (Extension closed)**.** Let $(\mathcal{A}, \mathcal{E})$ be an exact category and consider a full subcategory $\mathcal{A}'$ of $\mathcal{A}$ containing the zero-object. The subcategory $\mathcal{A}'$ is *extension closed* if whenever $A \rightarrowtail B \twoheadrightarrow C$ is in $\mathcal{E}$, with $A, C \in \mathcal{A}'$, then $B \in \mathcal{A}'$.

Proposition 3.8. *An extension closed subcategory $\mathcal{A}'$ of an exact category $(\mathcal{A}, \mathcal{E})$ is exact, with exact structure $\mathcal{E}' = \{A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E} | A, C \in \mathcal{A}'\}$.*

Proof. We begin by showing that $\mathcal{A}'$ is additive. By assumption, we have $0 \in \mathcal{A}'$ and $\mathrm{Hom}_{\mathcal{A}'}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)$ for all $A, B \in \mathcal{A}'$, so the $\mathcal{A}'$ has a zero object and all hom-sets are abelian groups. To see that $\mathcal{A}'$ has all biproducts, let $A, B \in \mathcal{A}'$ and consider the conflation $A \rightarrowtail A \oplus B \twoheadrightarrow B \in \mathcal{E}$. As $\mathcal{A}'$ is extension closed, we see that $A \oplus B \in \mathcal{A}'$, and $\mathcal{A}'$ is consequently additive.

Let $A \cong A'$ in $\mathcal{A}$, with $A \in \mathcal{A}'$. Then $A \rightarrowtail A' \twoheadrightarrow 0 \in \mathcal{E}$, as isomorphisms are both inflations and deflations. Again, as $\mathcal{A}'$ is extension closed, $A' \in \mathcal{A}'$, which shows that $\mathcal{A}'$ is closed under isomorphisms. The axioms Ex0, Ex1 and Ex1^{op} are inherited from the exact structure of $\mathcal{A}$. The axioms Ex2 and Ex2^{op} follows from Lemma 3.16, which says that pullbacks (pushouts) along deflations (inflations) have isomorphic cokernels (kernels). For Ex3, let $f : B \to C$ be a morphism with a kernel in $\mathcal{A}'$. If there exists a $g : A \to B$ such that fg is a deflation in $\mathcal{A}$, then f is a deflation in $\mathcal{A}$. Hence, one obtain $\mathrm{Ker} f \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}$, but since $\mathrm{Ker} f$ and C lies in $\mathcal{A}'$, we also get that f is a deflation in $\mathcal{A}'$. The argument in the case of Ex3^{op} is dual. □

Remark 3.9. The exact structure on the subcategory above will be referred to as the *induced* exact structure.

Now we present some examples, to get an impression of what type of structures we are working with. It is worth noticing that all the examples are extension closed subcategories of an abelian category.

Example 3.10. Any abelian category is exact, with several possible exact structures. The largest class consists of all short exact sequences, and this will be referred to as the *standard exact structure*. The smallest class consists of all split-exact sequences. To argue briefly for the fact that an abelian category with the standard exact structure is in fact an exact category, we see that Ex0 - Ex2 holds immediately. For Ex3, let fg be an epimorphism and π_f the cokernel of f. Then $\pi_f fg = 0$, and since fg is an epimorphism, we get $\pi_f = 0$, which implies that f is an epimorphism. The argument in the case of Ex3^{op} is dual.

Example 3.11. Let R be a ring. The full subcategory of $\mathrm{Mod}\, R$, consisting of projective modules is extension closed. It is hence an exact category when equipped with the induced exact structure. Any short exact sequence ending with a projective module is split, by Lemma 2.14. Hence, the middle term is a direct sum of projective modules, and again projective. A similar argument shows that the full subcategory of injective modules is exact as well.

Example 3.12. The category $\mathbf{Ab}_{tf}$ of torsion free abelian groups is an extension closed subcategory of $\mathbf{Ab}$, and consequently inherits the exact structure. As a reminder, an abelian group is torsion free if no element has finite order. To see that $\mathbf{Ab}_{tf}$ is extension closed, let

$$0 \to A \to B \to C \to 0$$

be an exact sequence in $\mathbf{Ab}$, where A and C are torsion free. Assume that $b \in B$ is a torsion element. Without loss of generality assume $n \in \mathbb{Z}$ annihilates b. The image of a torsion element is again torsion, and hence its image in C must be zero, as C is torsion free. Since the sequence is exact, b must lie in the image of $A \to B$. Now let a be a preimage of b. As the arrows are group homomorphisms, we get that $na \mapsto nb = 0$. Since $A \to B$ is injective, this implies that $na = 0$. By assumption, A is torsion free, from which we obtain that $a = 0$. Consequently, B is torsion free.

Example 3.13. The category $\mathbf{Ab}_t$ of torsion abelian groups is an extension closed subcategory of $\mathbf{Ab}$. As for the torsion free case above, $\mathbf{Ab}_t$ is exact when equipped with the induced exact structure.

When exact categories were first introduced in [9], extension closed subcategories of abelian categories were the prototype example, and the motivation for the definition. It can in fact be shown that a small category $\mathcal{S}$ is exact if and only if it is an extension closed subcategory of an abelian category $\mathcal{A}$. Since any abelian category is exact, and extension closed subcategories of exact categories are again exact, the "if" part is immediate. For the converse, the idea is that the subcategory of the functor category $\mathrm{Fun}(\mathcal{S}, \mathbf{Ab})$, consisting of left exact functors is abelian. It can be shown that the Yoneda functor embeds $\mathcal{S}$ as a full, extension closed subcategory of this abelian category. The details can be found in Appendix A of [2].

3.2 Elementary properties

In this section we will present some important elementary properties of exact categories, and observe that in particular pushouts and pullbacks behave similarly as for abelian categories. For a reader who is solely interested in understanding why the stable Frobenius category is triangulated, Lemma 3.17 and Lemma 3.18 are the results from this section strictly necessary for the proof. We begin by introducing some terminology.

Definition 3.14. A commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is called *bicartesian* if it is both a pullback and a pushout.

In general additive categories, there is a clear connection between pushouts, pullbacks and exactness of a sequence. In particular, there is a correspondence between bicartesian squares and kernel-cokernel pairs.

Lemma 3.15. *In an additive category $\mathcal{A}$, consider* $A \xrightarrow{\begin{bmatrix} -f \\ g \end{bmatrix}} B \oplus C \xrightarrow{\begin{bmatrix} h & i \end{bmatrix}} D$ *and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$

where the square commutes, i.e, $\begin{bmatrix} h & i \end{bmatrix} \begin{bmatrix} -f \\ g \end{bmatrix} = 0$. *The following statements hold:*

(1) The square is a pullback $\iff$ $\begin{bmatrix} -f \\ g \end{bmatrix}$ *is a kernel of* $\begin{bmatrix} h & i \end{bmatrix}$.

(2) The square is a pushout $\iff$ $\begin{bmatrix} h & i \end{bmatrix}$ *is a cokernel of* $\begin{bmatrix} -f \\ g \end{bmatrix}$.

(3) The square is bicartesian $\iff$ $\left(\begin{bmatrix} -f \\ g \end{bmatrix}, \begin{bmatrix} h & i \end{bmatrix} \right)$ *is a kernel-cokernel pair.*

Proof. We see that (1) holds by looking at the following diagrams

$$\begin{tikzcd} T \arrow[drr, bend left, "t_1"] \arrow[dr, dotted, "\varphi"] \arrow[ddr, bend right, "t_2"'] & & \\ & A \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\text{PB}"] & B \arrow[d, "h"] \\ & C \arrow[r, "i"] & D \end{tikzcd} \qquad \begin{tikzcd} T \arrow[drr, bend left] \arrow[dr, dotted, "\varphi"] \arrow[ddr, bend right, "\begin{bmatrix} -t_1 \\ t_2 \end{bmatrix}"'] & & \\ & A \arrow[r] \arrow[d, "\begin{bmatrix} -f \\ g \end{bmatrix}"] & 0 \arrow[d] \\ & B \oplus C \arrow[r, "\begin{bmatrix} h & i \end{bmatrix}"'] & D \end{tikzcd}$$

Given the pullback (left), we have the kernel (right) and vice versa. The statement (2) is dual. One obtains (3) by combining (1) and (2). $\square$

In abelian categories it is a well-known fact that pushouts preserve cokernels, and that pullbacks preserve kernels. The following lemma shows that an analogue of this holds for exact categories.

Lemma 3.16. *Given the following diagram*

$$\begin{tikzcd} & & E \arrow[d, "k"] \\ A \arrow[r, tail, "f"] \arrow[d, "h"] & B \arrow[r, two heads, "g"] & C \\ D & & \end{tikzcd}$$

in an exact category, in which the middle row is a conflation. Then it can be completed into a commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, equal] & C' \arrow[r, two heads, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\text{PB}"] & E \arrow[d, "k"] \\ A \arrow[r, tail, "f"] \arrow[d, "h"] \arrow[dr, phantom, "\text{PO}"] & B \arrow[r, two heads, "g"] \arrow[d, "i_2"] & C \arrow[d, equal] \\ D \arrow[r, tail, "i_1"] & A' \arrow[r, two heads, "g'"] & C \end{tikzcd}$$

where all rows are conflations.

Proof. We consider the bottom part of the diagram. The argument for the upper part is dual. Since f is an inflation, the pushout of f and h exists. Using the notation above we have that i_1 is an inflation, and it's a part of a conflation (i_1, η), where $\eta : A' \to F$ is the cokernel of i_1. We know that (F, η) is unique up to isomorphism. This means that we need to show that g' exists and that (C, g') is the cokernel of (D, i_1).

Consider the morphsims $g : B \to C$ and $0 : D \to C$. We have $gf = 0 = 0h$, and thus by the universal property of the pushout, there exists a unique morphism $g' : A' \to C$ satisfying $g'i_1 = 0$ and $g'i_2 = g$. Now we check if the universal property holds for (C, g'). Suppose $\varphi : A' \to W$ is such that $\varphi i_1 = 0$. We know that g is the cokernel of f and $\varphi i_1 h = 0 = \varphi i_2 f$, and thus there exists a unique morphism $\alpha : C \to W$ with $\alpha g = \varphi i_2$. Our next step is to show $\varphi = \alpha g'$. Observe that $(\varphi - \alpha g')i_2 = 0$ by construction of g', and $(\varphi - \alpha g')i_1 = 0$ by assumption on φ. Hence, by the pushout property of A', we can conclude that $\varphi = \alpha g'$. This shows that φ factors uniquely through g', and consequently that (i_1, g') is a conflation. $\square$

By the nature of the axioms of an exact category, one often take pushouts and pullbacks along inflations and deflations. The following lemma shows us that in such a setting, we always get bicartesian squares.

Lemma 3.17. *Let*

$$\begin{tikzcd} B' \arrow[r, two heads, "g'"] \arrow[d, "h'"] & C' \arrow[d, "h"] \\ B \arrow[r, two heads, "g"] & C \end{tikzcd}$$

be a commutative square in an exact category. The following are are equivalent:

1. *The square is a pullback.*
2. *The sequence* $B' \xrightarrow{\left[\begin{smallmatrix} g' \\ -h \end{smallmatrix}\right]} C' \oplus B \xrightarrow{\left[\begin{smallmatrix} h & g \end{smallmatrix}\right]} C$ *is a conflation.*
3. *The square is bicartesian.*

4. The square is part of a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, equal] & B' \arrow[r, two heads, "g'"] \arrow[d, "h'"] & C' \arrow[d, "h"] \\ A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C \end{tikzcd}$$

Proof. (1) $\implies$ (2): From Lemma 3.15, we have that $\left[\begin{smallmatrix} g' \\ -h' \end{smallmatrix}\right]$ is a kernel of $[h\, g]$. Since we have a morphism $\left[\begin{smallmatrix} 0 \\ \iota_B \end{smallmatrix}\right]$ such that

$$\begin{tikzcd} B \arrow[r, "\left[\begin{smallmatrix} 0 \\ \iota_B \end{smallmatrix}\right]"] \arrow[rr, bend right, two heads, "g"'] & C' \oplus B \arrow[r, "{[h\, g]}"] & C \end{tikzcd}$$

commutes, axiom Ex3 gives us that $[h\, g]$ is a deflation. Hence, it is a part of a conflation $K \xrightarrow{\left[\begin{smallmatrix} k_1 \\ k_2 \end{smallmatrix}\right]} C' \oplus B \xrightarrow{[h\, g]} C$. Since the collection of conflations is closed under isomorphism, we get that $B' \xrightarrow{\left[\begin{smallmatrix} g' \\ -h \end{smallmatrix}\right]} C' \oplus B \xrightarrow{[h\, g]} C$ is a conflation.
(2) $\implies$ (3): Since conflations are kernel-cokernel pairs, we obtain from Lemma 3.15 that the square is bicartesian.
(3) $\implies$ (4): This follows from Lemma 3.16.
(4) $\implies$ (1): Assume we have the following diagram

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, equal] & B' \arrow[r, two heads, "g'"] \arrow[d, "h'"] & C' \arrow[d, "h"] \\ A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C \end{tikzcd}$$

We want to show that the right square is a pullback. By Lemma 3.15, we may show that

$$B \xrightarrow{\left[\begin{smallmatrix} -g' \\ h' \end{smallmatrix}\right]} C' \oplus B \xrightarrow{[h\, g]} C$$

is a conflation. From Lemma 3.16 we have the following commutative diagram, where all rows and columns are conflations

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, tail, "f"] \arrow[dr, phantom, "\text{PO}"] & B' \arrow[r, two heads, "g'"] \arrow[d, "j"] & C' \arrow[d, equal] \\ B \arrow[r, tail, "j'"] \arrow[d, two heads, "g"] & P \arrow[r, "e'"] \arrow[d, "e"] & C' \\ C \arrow[r, equal] & C & \end{tikzcd} .$$

From the pushout property we get a morphism $\alpha : P \to C' \oplus B$, as shown in the following diagram

$$
\begin{tikzcd}
A \arrow[r, tail, "f'"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\text{PO}"] & B' \arrow[d, tail, "j"] \arrow[ddr, bend left, "\left[\begin{smallmatrix} -g' \\ h' \end{smallmatrix}\right]"] & \\
B \arrow[r, tail, "j'"] \arrow[drr, bend right, "\left[\begin{smallmatrix} 0 \\ \mathrm{id}_B \end{smallmatrix}\right]"'] & P \arrow[dr, dotted, "\alpha"] & \\
& & C' \oplus B
\end{tikzcd}
.
$$

where $\alpha j' = \left[\begin{smallmatrix} 0 \\ \mathrm{id}_B \end{smallmatrix}\right]$ and $\alpha j = \left[\begin{smallmatrix} -g' \\ h' \end{smallmatrix}\right]$. Since $(j'h' - j)f' = 0$ and g' is a cokernel of f', there exists a unique γ such that $\gamma g' = j'h' - j$. The claim is now that α is an isomorphism, with inverse $\beta = [\gamma \ j]$. To see this, notice first that $\alpha\gamma g' = \left[\begin{smallmatrix} \mathrm{id}_C \\ 0 \end{smallmatrix}\right] g'$, which implies that $\alpha\gamma = \left[\begin{smallmatrix} \mathrm{id}_C \\ 0 \end{smallmatrix}\right]$, since g' is an epimorphism. Now $\alpha\beta = [\alpha\gamma \ \alpha j'] = \mathrm{id}_{C' \oplus B}$. For $\beta\alpha$ we observe that $\beta\alpha j' = \mathrm{id}_P j'$, and $\beta\alpha j = \mathrm{id}_P j$. Thus by the pushout property, we get $\beta\alpha = \mathrm{id}_P$. Hence, α is an isomorphism. Finally, we show that the following is an isomorphism of sequences

$$
\begin{tikzcd}
B' \arrow[r, "\left[\begin{smallmatrix} -g' \\ h' \end{smallmatrix}\right]"] \arrow[d, equal] & C' \oplus B \arrow[r, "{[h \ g]}"] \arrow[d, "\beta"] & C \arrow[d, equal] \\
B' \arrow[r, tail, "j"] & P \arrow[r, two heads, "e"] & C
\end{tikzcd}
$$

We observe that the left square commutes. Moreover, $e\gamma g' = gh' = hg'$, which implies that $e\gamma = h$, and consequently $e\beta = [e\gamma \ ej'] = [h \ g]$. Thus the diagram commutes and

$$
B \xrightarrow{\left[\begin{smallmatrix} -g' \\ h' \end{smallmatrix}\right]} C' \oplus B \xrightarrow{[h \ g]} C
$$

is a conflation. $\square$

We are familiar with the fact that in abelian categories, pushouts and pullbacks of monomorphisms and epimorphisms are again monomorphisms and epimorphisms. The situation is similar for exact categories.

Lemma 3.18. *Given a pushout*

$$
\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, two heads, "g"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, "g'"] \\
C \arrow[r, tail, "f'"] & D
\end{tikzcd}
$$

in an exact category, then g' is a deflation. Moreover, if g is an isomorphism, so is g'.

Proof. By the dual of Lemma 3.17, the square is bicartesian, and $[g' \ f']$ is a deflation. We want to find a morphism h such that $g'h$ is a deflation. Since the identity on any object is a deflation, the morphism $\begin{bmatrix} \mathrm{id}_B & 0 \\ 0 & g \end{bmatrix}$ is a deflation. By Ex1, we get that

$$[g' \ f'] \begin{bmatrix} \mathrm{id}_B & 0 \\ 0 & g \end{bmatrix} = [g' \ f'g] = [g' \ g'f] = g'[\mathrm{id} \ f]$$

is a deflation. Now, by Ex3, g' is a deflation if and only if it has a kernel. Since g is a deflation, it is part of a conflation

$$A' \overset{a}{\rightarrowtail} A \overset{g}{\twoheadrightarrow} B.$$

By the kernel property of $A' \rightarrowtail A$ and pullback property of A, we get for all (K, ι) such that $g'\iota = 0$ the following diagram

$$\begin{tikzcd}
K \arrow[drrr, bend left, "\iota"] \arrow[drr, dotted, "\varphi"] \arrow[dr, dotted, "\varphi'"'] \arrow[ddrr, bend right, "0"'] & & & \\
& A' \arrow[r, tail, "a"] & A \arrow[r, tail, "f"] \arrow[d, two heads, "g"] \arrow[dr, phantom, "\text{BC}"] & B \arrow[d, "g'"] \\
& & C \arrow[r, tail, "f'"] & D
\end{tikzcd}$$

This shows that (A', fa) is a kernel of g', and we can conclude that g' is a deflation. If additionally g is an inflation, i.e an isomorphism, then from Ex2^{op} we get that g' is an inflation as well. $\square$

As a little fun fact, and to further point out the structural importance of the isomorphism theorems, we include the following result.

Proposition 3.19 (3. isomorphism theorem)**.** *Given the solid part of the following diagram in an exact category*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \arrow[d, tail, dotted] \\
A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads, dotted] \\
& Z \arrow[r, equal] & Z
\end{tikzcd}$$

then the diagram can uniquely be completed such that $X \rightarrowtail Y \twoheadrightarrow Z$ is a conflation, and all squares commute. Moreover, the upper right square is bicartesian.

Proof. The morphisms exist uniquely by the cokernel property of $B \twoheadrightarrow X$ and $C \twoheadrightarrow Y$. By Lemma 3.17 the upper right square is bicartesian, which implies that $X \rightarrowtail Y$ is an inflation. Now by the dual of Lemma 3.17 the rightmost sequence is a conflation. $\square$

For further information on exact categories see [2].

3.3 Projective and injective objects

In this section we present some elementary results on injective and projective objects. Readers who are familiar with these concepts may skim this section, but should keep in mind that objects are projective (injective) with respect to the exact structure. Different exact structures give rise to different collections of projective and injective objects.

When we encounter a new class of categories, a natural question to ask is what the corresponding notion of structure preserving functors between such categories is. For exact categories we have the following.

Definition 3.20 (Exact functor)**.** Let $(\mathcal{A}, \mathcal{E})$ and $(\mathcal{A}', \mathcal{E}')$ be exact categories. An *exact functor* $F : \mathcal{A} \to \mathcal{A}'$ is a functor which takes conflations in $\mathcal{A}$ to conflations in $\mathcal{A}'$.

Exact functors are necessarily additive. It is worth noticing that the definition above coincides with the definition of exact functors between abelian categories, when we think of abelian categories as exact categories with the standard exact structure.

Example 3.21. Let $\mathcal{A}'$ be an extension closed subcategory of an exact category $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}'$ is equipped with the induced exact structure. Then the inclusion functor is exact.

Lemma 3.22 (Hom is left exact)**.** *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence in an additive category $\mathcal{A}$. Then for any object D in $\mathcal{A}$, the sequences of abelian groups

$$0 \longrightarrow \mathrm{Hom}(D, A) \xrightarrow{f_*} \mathrm{Hom}(D, B) \xrightarrow{g_*} \mathrm{Hom}(D, C)$$

$$0 \longrightarrow \mathrm{Hom}(C, D) \xrightarrow{g^*} \mathrm{Hom}(B, D) \xrightarrow{f^*} \mathrm{Hom}(A, D)$$

are exact. If additionally g (f) is a split epimorphism (monomorphism), then g_* (f^*) is surjective.

Proof. We show that the covariant Hom is left exact, i.e that

$$0 \longrightarrow \mathrm{Hom}(D, A) \xrightarrow{f_*} \mathrm{Hom}(D, B) \xrightarrow{g_*} \mathrm{Hom}(D, C)$$

is exact. The other statement is dual. Consider a morphism $\varphi \in \mathrm{Hom}(D, A)$. As f is a monomorphism, we have that $f_*(\varphi) = f\varphi = 0$ if and only if $\varphi = 0$, and consequently f_* is injective. The image of f_* is contained in the kernel of g_* since $g_* f_*(-) = g \circ f \circ - = 0$. For the reverse inclusion, let $\psi \in \operatorname{Ker} g_*$, i.e $g\psi = 0$. Since $g\psi = 0$, we know that ψ must factor through the kernel of g. This gives a unique map h such that $\psi = fh = f_*(h)$. If g is a split epimorphism, then the original sequence splits. Since Hom is additive, it preserves split exact sequences, and thus g_* is surjective. $\square$

As we now have seen, mathematicians' favorite functor, Hom, is not in general exact. We can now do one of two things, either bury our heads in disbelief, never touching Hom again, or we can put some substantial effort into understanding how far Hom is from being exact, and why this is the case. As mathematicians, the canonical choice is of course to take the tough road. Not to surprisingly, the answer depends on what object you fix.

Definition 3.23. Let $\mathcal{A}$ be an exact category.

1. An object $I \in \mathcal{A}$ is injective if the functor
$$\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to \mathrm{Ab}$$
is exact. Here we equip Ab with the standard exact structure.
2. An object $P \in \mathcal{A}$ is projective if the functor
$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{Ab}$$
is exact.
3. $\mathcal{A}$ has enough injectives if any object A fits into a conflation
$$A \rightarrowtail I \twoheadrightarrow B$$
with I injective.
4. $\mathcal{A}$ has enough projectives if any object B fits into a conflation
$$A \rightarrowtail P \twoheadrightarrow B$$
with P projective.

Remark 3.24. The projective objects in $\mathcal{A}$ coincide with the injective objects in $\mathcal{A}^{op}$.

Digging deeper into the questions asked above naturally leads you to interesting topics, such as projective and injective resolutions, derived functors, and derived categories. We will not dwell further on these question, but rather explore what projective and injective object "are", and how they interact with the category they live in.

Definition 3.25. For $(\mathcal{A}, \mathcal{E})$ be an exact category. We define proj $\mathcal{A}$ and inj $\mathcal{A}$ to be the full subcategories of $\mathcal{A}$ consisting of all projective and injective objects, respectively.

The following proposition gives a characterization of projective and injective objects. When they are encountered in the wild, it is often more convenient to use the lifting property (2), as opposed to the definition in terms of exactness of Hom.

Proposition 3.26. *Let $\mathcal{A}$ be an exact category. The following are equivalent for an injective object $I \in \mathcal{A}$:*

1. *I is an injective object.*
2. *For any diagram*

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[dr, "h"'] & B \arrow[r, two heads, "g"] \arrow[d, dotted, "\exists h'"] & C \\ & I & \end{tikzcd}$$

   with the top row a conflation, there is a morphism $h' : B \to I$ such that $h'f = h$.

3. *Any conflation $I \rightarrowtail B \twoheadrightarrow C$ splits.*

Proof. (1) $\implies$ (2):
Let $A \rightarrowtail B \twoheadrightarrow C$ be a conflation and I an injective object. By definition, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, I) \xrightarrow{g^*} \mathrm{Hom}(B, I) \xrightarrow{f^*} \mathrm{Hom}(A, I) \to 0$$

In particular, this yields that f^* is surjective, i.e that any $h \in \mathrm{Hom}(A, I)$ has a preimage h' along f^*. Spelled out, we have $f^*(h') = h'f = h$.

(2) $\implies$ (3):
The following diagram commutes

$$\begin{tikzcd} I \arrow[r, tail, "f"] \arrow[dr, equal] & B \arrow[r, two heads, "g"] \arrow[d, dotted, "\exists h'"] & C. \\ & I & \end{tikzcd}$$

Thus, f is a split monomorphism, and the sequence consequently splits.

(3) $\implies$ (1):
Assume we have a conflation $A \rightarrowtail B \twoheadrightarrow C$ and a morphism $h : A \to I$ with I injective. By Lemma 3.16, we have the solid part of the following diagram

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, "h"] \arrow[dr, phantom, "\text{PO}"] & B \arrow[r, two heads, "g"] \arrow[d, "h'"] & C \arrow[d, equal] \\ I \arrow[r, tail, "f'"] & A' \arrow[r, two heads, "g'"] \arrow[l, dotted, bend left, "f''"] & C, \end{tikzcd}$$

where the left square is a pushout. Since the bottom row splits, there exists an f'' such that $f''f' = \mathrm{id}_I$. Define the morphism $\varphi : B \to I$ as $\varphi := f''h'$. We have $f'\varphi f = f'f''h'f = f'h$, which since f' is a monomorphism, implies that $\varphi f = h$. $\square$

The dual result holds for projective objects. Recall that we proved this in the case of modules in Lemma 2.14. As we have already seen, exact categories usually have several exact structures. The flexibility of this notion is analogous to that of a topological structure on a set. Given a set, the choice of topological structure on that set affects which maps in and out of the space that are continuous. The relationship between the exact structure and the collections of projective and injective objects resembles this in the sense that a smaller exact structure allows for more injective and projective objects, and vice versa. Intuitively, it should make sense, since with a smaller exact structure, there are fewer conflations Hom needs to preserve.

Example 3.27. In a module category with the standard exact structure, injective and projective objects are precisely the injective and projective modules. Module categories have enough injectives and enough projectives.

Example 3.28. In an exact category with the minimal exact structure, i.e the collection of all split-exact sequences, every object is both projective and injective. This can be seen from the following diagram

$$\begin{tikzcd}
& C \arrow[d, dotted, "{\begin{bmatrix}0\\h\end{bmatrix}}"] \arrow[dr, bend left, "h"] & \\
A \arrow[r, tail, "{\begin{bmatrix}\mathrm{id}_A\\0\end{bmatrix}}"] \arrow[dr, bend right, "h'"'] & A \oplus B \arrow[r, two heads, "{[0\ \mathrm{id}_B]}"] \arrow[d, dotted, "{[h'\ 0]}"] & B \\
& C &
\end{tikzcd}$$

Equivalently, we know that Hom is additive, so it preserves split exact sequences. Such a category obviously has enough projectives and enough injectives.

Example 3.29. Any initial object is projective and any terminal object is injective. The zero object is both projective and injective.

4 Frobenius categories and the associated stable category

In this section we build up to the definition of the *stable Frobenius category*, and conclude by showing that it canonically carries a triangulated structure. This is not to say that the proof is in any way obvious, but the autoequivalence and the distinguished triangles are defined in a natural way. In order to get there, we first need to define the concept of a *quotient category*.

4.1 Quotient categories

Given an arbitrary ring (with unity), it is possible to think of it as a pre-additive category with one object. In this way, pre-additive categories can be seen as a generalisation of rings. This section will revolve around another one of these generalized constructions, namely that of a categorical quotient. Recall that for rings we have to introduce the concept of a two-sided ideal in order to have a well-defined quotient ring, and for general pre-additive categories, it is not much different.

Definition 4.1. Let $\mathcal{A}$ be a pre-additive category and $\mathcal{I}$ a class of morphisms of $\mathcal{A}$. Denote $\mathcal{I}(A,B) = \mathcal{I} \cap \operatorname{Hom}_{\mathcal{A}}(A,B)$. We say that $\mathcal{I}$ is a *(two-sided) ideal* of $\mathcal{A}$ if:

1. For each pair of objects A, B in $\mathcal{A}$, $\mathcal{I}(A,B)$ is a subgroup of $\operatorname{Hom}_{\mathcal{A}}(A,B)$.
2. If $f \in \operatorname{Hom}_{\mathcal{A}}(A,B)$, $g \in \mathcal{I}(B,C)$ and $h \in \operatorname{Hom}_{\mathcal{A}}(C,D)$, then $hgf \in \mathcal{I}(A,D)$.

This definition coincides with our usual definition of a two-sided ideal in ring theory when $\mathcal{A}$ only has one object. As we have hinted at, given a pre-additive category and an ideal $\mathcal{I}$, we can now define a new category $\mathcal{A}/\mathcal{I}$. The objects of $\mathcal{A}/\mathcal{I}$ are just the objects of $\mathcal{A}$ and

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{I}}(A,B) := \frac{\operatorname{Hom}_{\mathcal{A}}(A,B)}{\mathcal{I}(A,B)}$$

For notational convenience, we write $\overline{f}$ for the equivalence class $f + \mathcal{I}(A,B)$ throughout this discussion. To show that this quotient indeed is a valid category, we begin by showing that the composition map $(\overline{f}, \overline{g}) \mapsto \overline{gf}$ is well-defined. Let $f - f' \in \mathcal{I}(A,B)$ and $g - g' \in \mathcal{I}(B,C)$. Note that

$$gf - g'f' = gf - g'f + g'f - g'f' = (g - g')f + g'(f - f') \in \mathcal{I}(A,C).$$

This implies that $\overline{gf} = \overline{g'f'}$. The composition is necessarily associative, since it is associative in $\mathcal{A}$. We immediately get that $\overline{\mathrm{id}_A}$ is the identity on A and $\overline{0}_{(A,B)}$ is the zero element in $\operatorname{Hom}_{\mathcal{A}/\mathcal{I}}(A,B)$. By definition of the group structure on $\operatorname{Hom}_{\mathcal{A}/\mathcal{I}}(A,B)$ and the composition, we have

$$\overline{h}(\overline{g}+\overline{g'})\overline{f} = \overline{h}(\overline{g+g'})\overline{f} = \overline{h(g+g')f} = \overline{hgf + hg'f} = \overline{hgf} + \overline{hg'f} = \overline{h}\overline{g}\overline{f} + \overline{h}\,\overline{g'}\,\overline{f},$$

which shows that $\mathcal{A}/\mathcal{I}$ is a pre-additive category. This also defines a functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{I}$, which is identity on objects and maps f to $\overline{f}$. It is additive and will be referred to as the *quotient functor*.

If $\mathcal{A}$ is additive, $\mathcal{A}/\mathcal{I}$ is also additive for any ideal $\mathcal{I}$. The zero-object in $\mathcal{A}$ is a a zero-object in $\mathcal{A}/\mathcal{I}$, since $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(0, A)$ and $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A, 0)$ still only contains one element each. Let $A \oplus B$ be the biproduct of A and B, and π_A, π_B, ι_A and ι_B be the projections and inclusions. Since the quotient functor is additive, we have

$$\overline{\pi_i}\,\overline{\iota_i} = \overline{\pi_i \iota_i} = \overline{\mathrm{id}_i} \quad \text{for } i = A, B$$

$$\overline{\iota_i}\,\overline{\pi_j} = 0 \quad \text{for } i \neq j$$

$$\overline{\iota_A \pi_A} + \overline{\iota_B \pi_B} = \overline{\iota_A \pi_A + \iota_B \pi_B} = \overline{\mathrm{id}}_{A \oplus B}$$

This together with the fact that the quotient functor is identity on objects gives us a biproduct in $\mathcal{A}/\mathcal{I}$. This discussion proves the following proposition.

Proposition 4.2. *Let $\mathcal{A}$ be a (pre)additive category and $\mathcal{I}$ a two-sided ideal in $\mathcal{A}$. Then the quotient category $\mathcal{A}/\mathcal{I}$ is (pre)additive and the quotient functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is additive.*

Let $\mathcal{A}$ be a preadditive category, and $\mathcal{N} \subseteq \mathcal{A}$ a full subcategory closed under finite direct sums. For all pairs of objects A, B in $\mathcal{A}$, define $\mathcal{I}(A, B) \subseteq \mathrm{Hom}_{\mathcal{A}}(A, B)$ to consist of all morphisms $f : A \to B$ that fit into a diagram of the form

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & B \\
& N \arrow[ur, "\psi"'] &
\end{tikzcd}$$

with $N \in \mathcal{N}$. In this case, f is said to *factor through* N. We claim that $\mathcal{I}$, the union of all $\mathcal{I}(A, B)$, is an ideal. The zero morphism factors through any object, in particular it lies in $\mathcal{I}(A, B)$ for all $A, B \in \mathcal{A}$. If $f, g \in \mathcal{I}(A, B)$, then $f = \psi\varphi$ for some $\varphi : A \to N$ and $\psi : N \to B$, where N is in $\mathcal{N}$. Similarly g factors through some N', and we can write $g = \psi'\varphi'$. Since $\mathcal{N}$ is closed under finite direct sums, we now have the following commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f-g"] \arrow[dr, "\tilde{\varphi}"'] & & B \\
& N \oplus N' \arrow[ur, "\tilde{\psi}"'] &
\end{tikzcd}$$

where $\tilde{\varphi} = \left[\begin{smallmatrix} \varphi \\ -\varphi' \end{smallmatrix}\right]$ and $\tilde{\psi} = [\psi\ \psi']$, which shows that $f - g \in \mathcal{I}(A, B)$. Thus we have shown that $\mathcal{I}(A, B)$ is a subgroup of $\mathrm{Hom}_{\mathcal{A}}(A, B)$ for all $A, B \in \mathcal{A}$. Now let $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, $g \in \mathcal{I}(B, C)$ and $h \in \mathrm{Hom}_{\mathcal{A}}(C, D)$. The following diagram shows that hgf factors through an object in $\mathcal{N}$

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[rr, "g"] \arrow[dr, "\varphi"'] & & C \arrow[r, "h"] & D \\
& & N \arrow[ur, "\psi"'] & &
\end{tikzcd}
$$

Hence $\mathcal{I}$ is an ideal. Given an exact category $\mathcal{A}$, we denote the ideal of morphisms that factor through an injective by $\mathscr{I}$. Similarly, we denote the ideal of morphisms that factor through a projective by $\mathscr{P}$.

Definition 4.3. Let $\mathcal{A}$ be an exact category. The *injectively stable category* $\overline{\mathcal{A}}$ is the quotient category $\mathcal{A}/\mathscr{I}$, and the *projectively stable category*, $\underline{\mathcal{A}}$ is the quotient category $\mathcal{A}/\mathscr{P}$.

We use the notation $\mathrm{Hom}_{\overline{\mathcal{A}}}(A, B) := \overline{\mathrm{Hom}}_{\mathcal{A}}(A, B)$ and $\mathrm{Hom}_{\underline{\mathcal{A}}}(A, B) := \underline{\mathrm{Hom}}_{\mathcal{A}}(A, B)$. Moreover we denote representatives of the equivalence classes by $\overline{f}$ and $\underline{f}$, respectively.

Remark 4.4. Assume that $f : X \to B$ factors through some injective object J, as shown in the diagram below. If $\mu : X \rightarrowtail I$ is an inflation and I is injective, then there exists a morphism $\alpha : I \to B$ such that $\alpha\mu = f$.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, tail, "\mu"] \arrow[ddr, tail, "g"'] & & B \\
& I \arrow[ur, dotted, "\alpha"] \arrow[d, dotted, tail, "\beta"] & \\
& J \arrow[uur, "h"'] &
\end{tikzcd}
$$

To see this, notice that the morphism β exists, since J is injective. If we define $\alpha := h\beta$, then $\alpha\mu = h\beta\mu = hg = f$.

4.2 Definition and examples

Above we have seen that it makes sense to define both the projectively and injectively stable category of an exact category. In this section we study what happens when the projective and injective objects coincide.

Definition 4.5 (Frobenius category)**.** A *Frobenius category* is an exact category $(\mathscr{F}, \mathcal{E})$ such that:

1. $\mathscr{F}$ has enough injectives.
2. $\mathscr{F}$ has enough projectives.
3. An object is injective if and only if it is projective.

For a reader with some experience with representation theory of finite dimensional algebras, the idea of projective and injective modules coinciding is not too obscure.

Example 4.6. Let F be a Frobenius algebra. Then mod F, the category of finitely generated F modules, is a Frobenius category.

Example 4.7. Let $K[G]$ be the group algebra of a finite group G over a field K. Then mod $K[G]$ is a Frobenius category.

Example 4.8. Let $C(\mathcal{A})$ denote the category of chain complexes over an abelian category. If we equip $C(\mathcal{A})$ with the exact structure

$$\mathcal{E} = \{\text{All degree-wise split short exact sequences}\},$$

then $C(\mathcal{A})$ is a Frobenius category. A short exact sequence of complexes

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0,$$

is degree-wise split if in each degree we have the sequence $A^i \to A^i \oplus C^i \to C^i$, where the maps are the obvious ones. The injective and projective objects with respect to this exact structure coincides, and are given by all complexes of the form

$$\cdots \to C^{i-1} \oplus C^{i-2} \to C^i \oplus C^{i-1} \to C^{i+1} \oplus C^i \to \dots,$$

with differential $\left[\begin{smallmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{smallmatrix}\right]$. By a slight abuse of notation, we denote the complex above by $C^\bullet[1] \oplus C^\bullet$. That these complexes are in fact injective and projective, can be seen by considering the following diagram

$$\begin{tikzcd}
& C^\bullet[1] \oplus C^\bullet \arrow[d, dotted, "{\left[\begin{smallmatrix} 0 & 0 \\ h_1^\bullet & h_2^\bullet \end{smallmatrix}\right]}"] \arrow[dr, bend left, "{[h_1^\bullet\ h_2^\bullet]}"] & \\
A^\bullet \arrow[r, tail, "{\left[\begin{smallmatrix} \mathrm{id}_A^\bullet \\ 0 \end{smallmatrix}\right]}"] \arrow[dr, bend right, "{\left[\begin{smallmatrix} g_1^\bullet \\ g_2^\bullet \end{smallmatrix}\right]}"] & A^\bullet \oplus B^\bullet \arrow[r, two heads, "{[0\ \mathrm{id}_B^\bullet]}"] \arrow[d, dotted, "{\left[\begin{smallmatrix} g_1^\bullet & 0 \\ g_2^\bullet & 0 \end{smallmatrix}\right]}"] & B^\bullet \\
& C^\bullet[1] \oplus C^\bullet &
\end{tikzcd}$$

These projective and injective complexes are called *contractible*, and one can show that they are homotopy equivalent to the zero complex.

Definition 4.9 (The stable Frobenius category)**.** Let $\mathscr{F}$ be a Frobenius category. The *stable Frobenius category* $\underline{\mathscr{F}}$ is the quotient category $\mathscr{F}/\mathscr{P}$.

Remark 4.10. Since the projective and injective objects coincide, we have $\mathscr{F}/\mathscr{I} = \mathscr{F}/\mathscr{P}$. Thus we have to make a choice on notation. From here and out, we will denote $\mathrm{Hom}_{\underline{\mathscr{F}}}(A, B)$ by $\underline{\mathrm{Hom}}(A, B)$, and $\underline{f} \in \underline{\mathrm{Hom}}(A, B)$. There will be little room for confusion, as we solely consider Frobenius categories and their associated stable category in the remainder of the text.

4.3 Triangulation

At last we arrive at the golden nugget if the thesis, or main result, as some might call it. The somewhat abundant structure of Frobenius categories allows us to construct a triangulated structure in a rather natural way. The first time encountering these constructions might be a little overwhelming, which probably is a result of the intertwined nature of the whole thing. Throughout this section, assume that we work in a Frobenius category $\mathscr{F}$, with exact structure $\mathcal{E}$.

We initiate this section with three lemmas, which more or less is the backbone of the entire section, and will be referred to regularly. Finally we present a proof showing that the stable Frobenius category carries a triangulated structure.

Lemma 4.11. *Given two conflations $X \rightarrowtail I \twoheadrightarrow X'$ and $Y \rightarrowtail I' \twoheadrightarrow Y'$ with I' injective, and a morphism $f : X \to Y$, we get the following commutative diagram.*

$$\begin{tikzcd} X \arrow[r, tail, "\mu"] \arrow[d, "f"] & I \arrow[r, two heads, "\pi"] \arrow[d, dotted, "f'"] & X' \arrow[d, dotted, "g"] \\ Y \arrow[r, tail, "\mu'"] & I' \arrow[r, two heads, "\pi'"] & Y' \end{tikzcd}$$

Proof. Since $\mu' f : X \to I'$, $\mu : X \rightarrowtail I$ is an inflation and I' is injective, there exists a morphism $f' : I \to I'$ such that $\mu' f = f' \mu$. As (X', π) is a cokernel of μ and $\pi' f' \mu = \pi \mu' f = 0$, there exists a unique $g : X' \to Y$ such that $g\pi = \pi' f'$ $\square$

For a reader familiar with syzygies, it is known that taking (co)syzygies does not in general give a well-defined functor. But in the projectively (injectively) stable category, the following discussion shows that the (co)syzygy functor is in fact well-defined. In the case where the projective and injective objects coincide, it defines an autoequivalence.

Lemma 4.12. *Let $X \rightarrowtail I \twoheadrightarrow X'$ and $X \rightarrowtail I' \twoheadrightarrow X''$ be two conflations, with I and I' injective. Then X' and X'' are isomorphic in $\underline{\mathscr{F}}$.*

Proof. From Lemma 4.11 we can create the following commutative diagram

$$\begin{tikzcd} X \arrow[r, tail, "\mu"] \arrow[d, equal] & I \arrow[r, two heads, "\pi"] \arrow[d, "f'"] & X' \arrow[d, "g"] \\ X \arrow[r, tail, "\mu'"] \arrow[d, equal] & I' \arrow[r, two heads, "\pi'"] \arrow[d, "f''"] & X'' \arrow[d, "g'"] \\ X \arrow[r, tail, "\mu"] & I \arrow[r, two heads, "\pi"] & X' \end{tikzcd}$$

We have

$$f'' f' \mu = \mu \iff (f' f'' - \mathrm{id}_I)\mu = 0$$

Hence $f'f'' - \mathrm{id}_I$ factors uniquely through π, i.e there exists an $h : X' \to I$ such that $f'f'' - \mathrm{id}_I = h\pi$. This implies that

$$\pi h\pi = \pi f'f'' - \pi = g'g\pi - \pi = (g'g - \mathrm{id}_{X'})\pi$$

Since π is an epimorphism, one obtains $\pi h = g'g - \mathrm{id}_{X'}$, and consequently $\underline{g'g} = \underline{\mathrm{id}}_{X'}$. The argument for $\underline{gg'} = \underline{\mathrm{id}}_{X''}$ is similar. Hence, X' and X'' are isomorphic in $\underline{\mathscr{F}}$. $\square$

Let $[X]$ denote the isomorphism class of X in $\underline{\mathscr{F}}$. For all $X \in \mathscr{F}$ we have an inflation $X \rightarrowtail I$ where I is injective, which again is a part of a conflation $X \rightarrowtail I \twoheadrightarrow X'$. The previous lemma gives us that $[X']$ is independent of the choice of $X \rightarrowtail I \twoheadrightarrow X'$.

For each $X \in \mathscr{F}$ choose a conflation

$$X \xrightarrow{\mu(X)} I(X) \xrightarrow{\pi(X)} TX$$

with $I(X)$ injective. We want to construct an equivalence $T : \underline{\mathscr{F}} \to \underline{\mathscr{F}}$. On objects we define $T(X) := TX$, which is well-defined by our argument above. Given $f : X \to Y$, we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, tail, "\mu(X)"] \arrow[d, "f"] & I(X) \arrow[r, two heads, "\pi(X)"] \arrow[d, "I(f)"] & TX \arrow[d, "T(f)"] \\ Y \arrow[r, tail, "\mu(Y)"] & I(Y) \arrow[r, two heads, "\pi(Y)"] & TY \end{tikzcd}$$

The following lemma shows that $\underline{T(f)}$ is independent of choice of $I(f)$.

Lemma 4.13. *Given a commutative diagram*

$$\begin{tikzcd} X \arrow[r, tail, "\mu(X)"] \arrow[d, "f"] & I(X) \arrow[r, two heads, "\pi(X)"] \arrow[d, "I_i(f)"] & TX \arrow[d, "T_i(f)"] \\ Y \arrow[r, tail, "\mu(Y)"] & I(Y) \arrow[r, two heads, "\pi(Y)"] & TY \end{tikzcd}$$

for $i \in \{1, 2\}$*, then* $\underline{T_1(f)} = \underline{T_2(f)}$ *in* $\underline{\mathscr{F}}$*.*

Proof. We want to show that $T_1(f) - T_2(f)$ factors through an injective object. The injective $I(Y)$ is a good candidate. The proof is essentially the same as that of Lemma 4.12. We have that $(I_1(f) - I_2(f))\mu(X) = 0$ by commutativity. This gives a unique $\varphi : TX \to I(Y)$ such that $\varphi\pi(X) = I_1(f) - I_2(f)$. Post-composing with $\pi(Y)$ gives

$$\pi(Y)\varphi\pi(X) = \pi(Y)(I_1(f) - I_2(f)) = (T_1(f) - T_2(f))\pi(X),$$

which implies that $\pi(Y)\varphi = T_1(f) - T_2(f)$ since $\pi(X)$ is an epimorphism. Hence, one obtains $\underline{T_1(f)} = \underline{T_2(f)}$ in $\underline{\mathscr{F}}$. $\square$

With this result, we can conclude that $T : \underline{\mathscr{F}} \to \underline{\mathscr{F}}$ is a well-defined functor. Our next aim is to prove that T is in fact an autoequivalence. It is worth noticing that up until this point we have not used that proj $\mathscr{F} =$ inj $\mathscr{F}$. This reflects the comment that taking (co)syzygy defines a functor when factoring out projective (injective) objects, as the same argument as above works in the projectively (injectively) stable category. In order for an inverse to exist, it is crucial that the projectives and injectives coincide, as will be seen in the proof below.

Theorem 4.14. *The functor $T : \underline{\mathscr{F}} \to \underline{\mathscr{F}}$ constructed above is an autoequivalence. If $T : [X] \to [X']$ is a bijection for all X, then T is an automorphism.*

Proof. We show that T is full, faithful and dense.

Dense: We want to show that T is surjective on isomorphism classes of objects, i.e that for any $Y \in \underline{\mathscr{F}}$ we have that $Y \cong TX$ for some $X \in \underline{\mathscr{F}}$. Since $\underline{\mathscr{F}}$ has enough projectives, we have a deflation $I \twoheadrightarrow Y$, where I is projective, which again is a part of a conflation $X \rightarrowtail I \twoheadrightarrow Y$. By Lemma 4.12, one obtains $Y \cong TX$. If $T : [X] \to [Y]$ is a bijection, then there exists a unique $X' \in [X]$ such that $T(X') = Y$.

Full: We want to show that for any $\underline{g} \in \underline{\mathrm{Hom}}(TX, TY)$, there exists an $f \in \underline{\mathrm{Hom}}(X, Y)$ such that $\underline{g} = T(\underline{f})$. Let $\underline{g} \in \underline{\mathrm{Hom}}(TX, TY)$. For projective objects $I(X)$ and $I(Y)$ we have a diagram

$$\begin{tikzcd} X \arrow[r, tail, "\mu(X)"] \arrow[d, dotted, "f"] & I(X) \arrow[r, two heads, "\pi(X)"] \arrow[d, dotted, "\varphi"] & TX \arrow[d, "g"] \\ Y \arrow[r, tail, "\mu(Y)"] & I(Y) \arrow[r, two heads, "\pi(Y)"] & TY \end{tikzcd}$$

by the dual of Lemma 4.11 and the fact that projective and injective objects coincide. Now by Lemma 4.13, one obtains $T(\underline{f}) = \underline{g}$, which implies that T is full.

Faithful: Assume $T(f) = T(g)$, we want to show that $\underline{f} = \underline{g}$. We have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, tail, "\mu(X)"] \arrow[d, "f-g"'] & I(X) \arrow[r, two heads, "\pi(X)"] \arrow[d, "I(f)-I(g)"] \arrow[dl, dotted, "h"'] & TX \arrow[d, "T(f)-T(g)=0"] \\ Y \arrow[r, tail, "\mu(Y)"] & I(Y) \arrow[r, two heads, "\pi(Y)"] & TY \end{tikzcd}$$

By assumption, we have

$$(T(f) - T(g))\pi(X) = \pi(Y)(I(f) - I(g)) = 0,$$

hence $I(f) - I(g)$ factors through the kernel $(Y, \mu(Y))$. We get a unique map h such that $\mu(Y)h = I(f) - I(g)$. Furthermore, we see that

$$\mu(Y)(f - g) = (I(f) - I(g))\mu(X) = \mu(Y)h\mu(X),$$

which since $\mu(Y)$ is a monomorphism, implies that $h\mu(X) = f - g$. Now, since $I(X)$ is injective, we conclude that $\underline{f} = \underline{g}$. $\square$

Remark 4.15. The construction of T involved making a choice for each object. It can be shown that two functors constructed in this manner, with different choices, will be naturally isomorphic. See for example [4] Section 2.2 for details.

Throughout the rest of this thesis we will write

$$X \overset{\mu(X)}{\rightarrowtail} I \overset{\pi(X)}{\twoheadrightarrow} TX \qquad \text{as} \qquad X \overset{x}{\rightarrowtail} I \overset{\overline{x}}{\twoheadrightarrow} TX,$$

to simplify the notation.

Let $u : X \to Y$ be a morhpism in $\mathscr{F}$, and $X \rightarrowtail I \twoheadrightarrow X'$ be a conflation with I injective. By Lemma 3.16, we have the solid part of the following diagram

$$\begin{tikzcd} X \arrow[r, tail, "\mu"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{PO}"] & I \arrow[r, two heads, "\pi"] \arrow[d, "f'"] & X' \arrow[d, equal] & \\ Y \arrow[r, tail, "v"] & C \arrow[r, two heads, "w'"] \arrow[rr, dotted, bend right, "w:=g'w'"'] & X' \arrow[r, dotted, "g'"] & TX \end{tikzcd},$$

where the bottom row is a conflation. By Lemma 4.12, there exists a morphism $g' : X' \to TX$ such that $\underline{g'}$ is an isomorphism. Define $w := g'w'$. The sequences

$$X \xrightarrow{f} Y \xrightarrow{v} C \xrightarrow{w} TX \ ,$$

and their images in $\underline{\mathscr{F}}$ will be called *standard triangles*.

Definition 4.16. Let Δ be the collection of all triangles in $\underline{\mathscr{F}}$ isomorphic to a standard triangle. A triangle in Δ will be called a *distinguished triangle*.

We are now almost ready to prove that $\underline{\mathscr{F}}$ is a triangulated category. In order to show the octahedral axiom, we will have to make use of the triangulated 5-lemma (2.11), which holds for pre-triangulated categories. As we mentioned in Section 2, a category is pre-triangulated if it satisfies TR1 - TR3. Hence, the plan is to first show that $\underline{\mathscr{F}}$ is pre-triangulated, for then to use the available machinery to tackle the octahedral axiom.

Theorem 4.17. *The triple* $(\underline{\mathscr{F}}, T, \Delta)$ *is a pre-triangulated category.*

Proof. Throughout this proof we only consider standard triangles, which is sufficient since every triangle is isomorphic to a standard triangle. Note that we have already proven that $\underline{\mathscr{F}}$ is additive, and that T is an additive autoequivalence.

TR1: From the construction of Δ it is clear that it is closed under isomorphisms, and that every morphism is part of a triangle. Now look at the triangle defined by the following diagram

$$\begin{tikzcd}
X \arrow[r, tail, "\mu"] \arrow[d, equal] \arrow[dr, phantom, "\text{PO}"] & I \arrow[r, two heads, "\pi"] \arrow[d, "f'"] & X' \arrow[d, equal] \\
X \arrow[r, tail, "v"] & C \arrow[r, two heads, "w'"] \arrow[rr, dotted, bend right, "w:=g'w'"'] & X' \arrow[r, "g'"] & TX
\end{tikzcd}$$

By Lemma 3.18, the morphism f' is an isomorphism, which implies that C is injective. In $\underline{\mathscr{F}}$ we have the following isomorphism of triangles

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "v"] \arrow[d, equal] & C \arrow[r, "w"] \arrow[d] & TX \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & TX
\end{tikzcd}$$

since any injective object is isomorphic to 0 in $\underline{\mathscr{F}}$. The top row is in Δ, and thus the bottom is as well.

TR2': Let $X \xrightarrow{u} Y \xrightarrow{v} C \xrightarrow{w} TX$ be a standard triangle given by the following diagram.

$$\begin{tikzcd}
X \arrow[r, tail, "\mu"] \arrow[d, "u"] \arrow[dr, phantom, "\text{PO}"] & I \arrow[r, two heads, "\pi"] \arrow[d, "\overline{u}"] & X' \arrow[d, equal] \\
Y \arrow[r, tail, "v"] & C \arrow[r, two heads, "w'"] \arrow[rr, bend right, "w:=g'w'"'] & X' \arrow[r, "g'"] & TX
\end{tikzcd}$$

By Lemma 4.11, we also have the following diagram.

$$\begin{tikzcd}
X \arrow[r, tail, "\mu"] \arrow[d, "u"] & I \arrow[r, two heads, "\pi"] \arrow[d, "u'"] & X' \arrow[d, "u''"] \\
Y \arrow[r, tail, "y"] & I(Y) \arrow[r, two heads, "\overline{y}"] & TY
\end{tikzcd}$$

By the pushout property of C, we get a unique morphism $\varphi : C \to I(Y)$ such that $\varphi\overline{u} = u'$ and $\varphi v = y$. Now consider the following diagram

$$\begin{tikzcd}
X \arrow[r, tail, "\mu"] \arrow[d, "u"] & I \arrow[d, "\overline{u}"] \arrow[dr, bend left, "u'"] & \\
Y \arrow[r, tail, "v"] \arrow[dr, bend right, "0"] & C \arrow[r, "\varphi"] \arrow[d, two heads, "w'"] \arrow[dr, dotted] & I(Y) \arrow[d, two heads, "\overline{y}"] \\
 & X' \arrow[r, "u''"] & TY
\end{tikzcd}$$

Since the upper left square is a pushout, there exists a unique morphism indicated by the dotted arrow. Both $u''w'$ and $\overline{y}\varphi$ fits, i.e the lower square commutes. Now consider

$$\begin{tikzcd}
Y \arrow[r, tail, "v"] \arrow[d, tail, "y"] & C \arrow[r, two heads, "w'"] \arrow[d, "\begin{bmatrix} \varphi \\ w' \end{bmatrix}"] & X' \arrow[d, equal] \\
I(Y) \arrow[r, tail, "\begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}"] \arrow[d, two heads, "\overline{y}"] & I(Y) \oplus X' \arrow[r, two heads, "{[0 \ \ \mathrm{id}]}"] \arrow[d, "{[\overline{y} \ \ -u'']}"] & X' \\
TY \arrow[r, equal] & TY &
\end{tikzcd},$$

which commutes by our discussion above. The middle row is a conflation by Ex0. It then follows by (the dual of) Lemma 3.17 that the upper left square is a pushout. Using this we get that

$$Y \xrightarrow{\underline{v}} C \xrightarrow{\begin{bmatrix} \underline{\varphi} \\ \underline{w'} \end{bmatrix}} I(Y) \oplus X' \xrightarrow{[\underline{\overline{y}} \ \ -\underline{u''}]} TY$$

is a distinguished triangle. The claim now is that the following is an isomorphism of triangles in $\underline{\mathscr{F}}$

$$\begin{tikzcd}
Y \arrow[r, "\underline{v}"] \arrow[d, equal] & C \arrow[r, "\begin{bmatrix} \underline{\varphi} \\ \underline{w'} \end{bmatrix}"] \arrow[d, equal] & I(Y) \oplus X' \arrow[r, "{[\underline{\overline{y}} \ \ -\underline{u''}]}"] \arrow[d, "{[0 \ \ \underline{g'}]}"] & TY \arrow[d, equal] \\
Y \arrow[r, "\underline{v}"] & C \arrow[r, "\underline{w}"] & TX \arrow[r, "-T\underline{u}"] & TY
\end{tikzcd}$$

By construction $\underline{g'}$ is an isomorphism, and injective objects are 0 in $\underline{\mathscr{F}}$. Thus all vertical morphisms are isomorphisms. By the definition of w, we see that the middle square commutes. It remains to show that the right square commutes. Since $\overline{y}$ is a morphism from an injective, it is zero in $\underline{\mathscr{F}}$. If the square is to commute, the morphisms $T\underline{u} \circ \underline{g'}$ and $\underline{u''}$ must be equal. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, equal] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "\overline{h}"] & TX \arrow[d, "h"'] \arrow[dd, bend left, "Tu"] \\
X \arrow[r, "\mu"] \arrow[d, "u"] & I \arrow[r, "\pi"] \arrow[d, "\overline{u''}"] & X' \arrow[d, "u''"'] \\
Y \arrow[r, "y"] & I(Y) \arrow[r, "\overline{y}"] & TY
\end{tikzcd}$$

where h be a representative of $\underline{g'}^{-1}$. From Lemma 4.13, we know that $T\underline{u} = \underline{u''}\underline{h}$, and consequently $T\underline{u} \circ \underline{g'} = \underline{u''}$. Thus the triangles are isomorphic in $\underline{\mathscr{F}}$ and

$$Y \xrightarrow{\underline{v}} C \xrightarrow{\underline{w}} TX \xrightarrow{-T\underline{u}} TY ,$$

is a distinguished triangle.

TR3: Consider two distinguished triangles constructed from the following diagrams

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[d, "u"] & I \arrow[r, "\pi"] \arrow[d, "\overline{u}"] & X' \arrow[d, equal] & \\
Y \arrow[r, "v"] & Z \arrow[r, "s"] \arrow[rr, bend right, "w:=\tau s"'] & X' \arrow[r, "\tau"] & TX
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "\mu'"] \arrow[d, "u'"] & I' \arrow[r, "\pi'"] \arrow[d, "\overline{u}'"] & A' \arrow[d, equal] & \\
B \arrow[r, "v'"] & C \arrow[r, "s'"] \arrow[rr, bend right, "w':=\lambda s'"'] & A' \arrow[r, "\lambda"] & TA
\end{tikzcd}$$

Assume we have the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\underline{u}"] \arrow[d, "\underline{\varphi}"] & Y \arrow[r, "\underline{v}"] \arrow[d, "\underline{\psi}"] & Z \arrow[r, "\underline{w}"] & TX \arrow[d, "T\underline{\varphi}"] \\
A \arrow[r, "\underline{u'}"] & B \arrow[r, "\underline{v'}"] & C \arrow[r, "\underline{w'}"] & TA
\end{tikzcd} \tag{4.1}$$

in $\underline{\mathscr{F}}$. Since $\underline{\tau}$ is an isomorphism, we can construct the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, tail, "x"] \arrow[d, equal] & I(X) \arrow[r, two heads, "\overline{x}"] \arrow[d, "\nu'"] & TX \arrow[d, "\nu"] \arrow[ddd, bend left, "T\varphi"] \\
X \arrow[r, tail, "\mu"] \arrow[d, "\varphi"] & I \arrow[r, two heads, "\pi"] \arrow[d, "\varphi'"] & X' \arrow[d, "\varphi''"] \\
A \arrow[r, tail, "\mu'"] \arrow[d, equal] & I' \arrow[r, two heads, "\pi'"] \arrow[d, "\lambda'"] & A' \arrow[d, "\lambda"] \\
A \arrow[r, tail, "a"] & I(A) \arrow[r, two heads, "\overline{a}"] & TA
\end{tikzcd}$$

by Lemma 4.11 and 4.12, where $\nu : TX \to X'$ is a representative of the equivalence class of $\underline{\tau}^{-1}$. By Lemma 4.13, we get that $T\underline{\varphi} = \underline{\lambda\varphi''\nu}$. Now the strategy is to find a morphism $i : I \to C$, such that $i\mu = v'\psi u$, for then to use the pushout property of Z to get our desired morphism. By assumption, we have $\underline{\psi u} = \underline{u'\varphi}$, i.e $\psi u - u'\varphi = \alpha\mu$ for some $\alpha : I \to B$ (Remark 4.4). The following calculation gives us the i mentioned above:

$$v'\psi u = v'(\alpha\mu + u'\varphi) = v'\alpha\mu + v'u'\varphi = v'\alpha\mu + \overline{u}'\mu'\varphi = v'\alpha\mu + \overline{u}'\varphi'\mu = (v'\alpha + \overline{u}'\varphi')\mu$$

Thus we let $i = v'\alpha + \overline{u}'\varphi'$. The following diagram might be illuminating in the upcoming arguments

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[d, "u"] & I \arrow[d, "\overline{u}"] \arrow[ddr, bend left, "v'\alpha+\overline{u}'\varphi'"] & \\
Y \arrow[r, "v"] \arrow[drr, bend right, "v'\psi"'] & Z \arrow[dr, dotted, "\theta"] & \\
 & & C
\end{tikzcd}$$

Now by the pushout property of Z, there exists a unique $\theta : Z \to C$, such that $\theta\overline{u} = v'\alpha + \overline{u}'\varphi'$ and $\theta v = v'\psi$. This θ makes diagram (4.1) commute. The middle square commutes by the construction of θ, while to see that the right square commutes requires a little more work. We use the pushout property of Z once more. If we in the pushout diagram above extend the bent arrows by post-composing with s', then the following calculations show that $s'\theta$ must be equal to $\varphi''s$:

$$(s'\theta - \varphi''s)\overline{u} = s'\theta\overline{u} - \varphi''s\overline{u} = s'(v'\alpha + \overline{u}'\varphi') - \varphi''\pi = 0 + \pi'\varphi' - \varphi''\pi = 0$$

and

$$(s'\theta - \varphi''s)v = s'v'\psi - \varphi''sv = 0 - 0 = 0$$

Thus $s'\theta = \varphi''s$, and

$$\underline{w'\theta} = \underline{\lambda s'\theta} = \underline{\lambda\varphi''\nu\tau s} = \underline{T\varphi\tau s} = \underline{T\varphi w}$$

We have hence shown that $(\underline{\mathscr{F}}, T, \Delta)$ is a pre-triangulated category. $\square$

An immediate consequence of the result above is that distinguished triangles in $\underline{\mathscr{F}}$ are uniquely determined by their starting morphism, as explained in Remark 2.12. In particular every distinguished triangle is isomorphic to a triangle constructed in the following manner

$$\begin{tikzcd}
X \arrow[r, tail, "x"] \arrow[d, "u"] \arrow[dr, phantom, "\mathrm{PO}"] & I(X) \arrow[r, two heads, "\overline{x}"] \arrow[d, "\overline{u}"] & TX \arrow[d, equal] \\
Y \arrow[r, tail, "v"] & C \arrow[r, two heads, "w"] & TX
\end{tikzcd}$$

Such triangles will be called *strictly standard*.

Theorem 4.18. *The triple* $(\underline{\mathscr{F}}, T, \Delta)$ *is a triangulated category.*

Proof. As we now know, the category $\underline{\mathscr{F}}$ is pre-triangulated, and the only part remaining is to show the octahedral axiom.

TR4: Assume that we have three distinguished triangles, defined by the following diagrams:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "x"] & Y \arrow[d, "i"] \\
I(X) \arrow[r, "\overline{u}"] \arrow[d, "\overline{x}"] & Z' \arrow[d, "i'"] \\
TX \arrow[r, equal] & TX
\end{tikzcd}
,
\quad
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "zi"] & Z \arrow[d, "j"] \\
I(Z') \arrow[r, "\overline{v}"] \arrow[d, "\pi"] & X' \arrow[d, "j'"] \\
\tilde{Y} \arrow[r, equal] & \tilde{Y} \arrow[d, "\nu"] \\
 & TY
\end{tikzcd}
\text{ and }
\begin{tikzcd}
X \arrow[r, "w:=vu"] \arrow[d, "x"] & Z \arrow[d, "k"] \\
I(X) \arrow[r, "\overline{w}"] \arrow[d, "\overline{x}"] & Y' \arrow[d, "k'"] \\
TX \arrow[r, equal] & TX
\end{tikzcd}
$$

The morphism z comes from the conflation

$$Z' \overset{z}{\rightarrowtail} I(Z') \overset{\overline{z}}{\twoheadrightarrow} TZ'.$$

Our goal is to find morphisms f, g and g' such that the triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{g'} TZ'$$

is distinguished, and makes the following diagram commute in $\underline{\mathscr{F}}$

$$
\begin{tikzcd}
X \arrow[r, "\underline{u}"] \arrow[d, equal] & Y \arrow[r, "\underline{i}"] \arrow[d, "\underline{v}"] & Z' \arrow[r, "\underline{i'}"] \arrow[d, dotted, "\underline{f}"] & TX \arrow[d, equal] \\
X \arrow[r, "\underline{w}:=\underline{vu}"] & Z \arrow[r, "\underline{k}"] \arrow[d, "\underline{j}"] & Y' \arrow[r, "\underline{k'}"] \arrow[d, dotted, "\underline{g}"] & TX \arrow[d, "\underline{Tu}"] \\
 & X' \arrow[r, equal] \arrow[d, "\underline{\nu j'}"] & X' \arrow[r, "\underline{\nu j'}"] \arrow[d, "\underline{g'}"] & TY \\
 & TY \arrow[r, "\underline{Ti}"] & TZ' &
\end{tikzcd}
\tag{4.2}
$$

By browsing through our rather large set of available morphisms, we are able to obtain f and g from the pushout property of Z' and Y' respectively

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "x"] & Y \arrow[d, "i"] \arrow[ddr, bend left, "kv"] & \\
I(X) \arrow[r, "\overline{u}"] \arrow[drr, bend right, "\overline{w}"'] & Z' \arrow[dr, dotted, "f"] & \\
 & & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "w=vu"] \arrow[d, "x"] & Z \arrow[d, "k"] \arrow[ddr, bend left, "j"] & \\
I(X) \arrow[r, "\overline{w}"] \arrow[drr, bend right, "\overline{v}z\overline{u}"'] & Y' \arrow[dr, dotted, "g"] & \\
 & & X'
\end{tikzcd}
$$

The construction f and g makes the squares to their left in 4.2 commute. With f and g on our hands, we are ready to construct the distinguished triangle we need. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "u"] & I(X) \arrow[d, "\overline{u}"] & \\
Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r, "z"] \arrow[d, "f"] & I(Z') \arrow[d, "\overline{v}"] \\
Z \arrow[r, "k"] & Y' \arrow[r, "g"] & X'
\end{tikzcd}$$

The lower right square commutes by the pushout property of Z', since

$$(\overline{v}z - gf)i = jv - gkv = 0$$

and

$$(\overline{v}z - gf)\overline{u} = 0.$$

Both the upper square and the tall rectangle are pushouts by construction, consequently the lower left square is a pushout. Then, since the flat rectangle and the lower left square are pushouts, the lower right square is also a pushout. Consequently, we can set up the following diagram

$$\begin{tikzcd}
Z' \arrow[r, "z"] \arrow[d, "f"] & I(Z') \arrow[r, "\overline{z}"] \arrow[d, "\overline{v}"] & TZ' \arrow[d, equal] \\
Y' \arrow[r, "g"] & X' \arrow[r, "g'"] & TZ'
\end{tikzcd}$$

which since the leftmost square is a pushout, defines a strictly standard triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{g'} TZ'.$$

Now we find fitting representatives for $\underline{Tu}$ and $\underline{Ti}$. As usual, by Lemma 4.11 and 4.12, we have the following commutative diagrams

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "u"] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "z\overline{u}"] & TX \arrow[d, "\varphi"] \\
Y \arrow[r, "zi"] \arrow[d, equal] & I(Z') \arrow[r, "\pi"] \arrow[d, "\overline{\nu}"] & \tilde{Y} \arrow[d, "\nu"] \\
Y \arrow[r, "y"] & I(Y) \arrow[r, "\overline{y}"] & TY
\end{tikzcd}$$

$$\begin{array}{ccccc}
Y & \xrightarrow{y} & I(Y) & \xrightarrow{\overline{y}} & TY \\
\| & & \downarrow{\scriptstyle \overline{u}} & & \downarrow{\scriptstyle \mu} \\
Y & \xrightarrow{zi} & I(Z') & \xrightarrow{\pi} & \tilde{Y} \\
\downarrow{\scriptstyle i} & & \| & & \downarrow{\scriptstyle \psi} \\
Z' & \xrightarrow{z} & I(Z') & \xrightarrow{\overline{z}} & TZ'
\end{array}$$

where μ is a representative of $\underline{\nu}^{-1}$. By Lemma 4.13, we can assume $\nu\varphi = Tu$ and $\psi\mu = Ti$. What now remains is to show that 4.2 indeed commutes. We already know that the upper left, upper middle and centre square commutes. For the two rightmost squares we have

$$(k'f - i')\overline{u} = k'\overline{w} - \overline{x} = 0$$

and

$$(Tuk' - \nu j'g)\overline{w} = \nu(\varphi k'\overline{w} - j'g\overline{w}) = \nu(\varphi\overline{x} - j'\overline{v}z\overline{u}) = \nu(\varphi\overline{x} - \pi z\overline{u}) = 0,$$

which by the pushout property of Z' and Y', implies that $k'f = i'$ and $Tuk' = \nu j'g$. Hence, the squares commute. For the last remaining square, we first note that $\psi j' = g'$, since

$$(\psi j' - g')j = -g'j = -g'gk = 0$$

and

$$(\psi j' - g')\overline{v} = \psi\pi - \overline{z} = 0$$

and X' is a pushout. Now at last we have

$$\underline{Ti\nu j'} = \underline{\psi\mu\nu j'} = \underline{\psi j'} = \underline{g'}$$

and can conclude that $\underline{\mathscr{F}}$ is a triangulated category. □

When an algebraist goes to work and stumbles across a triangulated category, chances are high that it is in fact the stable category of some Frobenius category.

Definition 4.19. A triangulated category is called *algebraic* if it is equivalent to the stable category of a Frobenius category.

For more discussion on algebraic triangulated categories see [10] and [6].

Example 4.20. We saw in 4.8 that $(C(\mathcal{A}), \mathcal{E})$ is a Frobenius category. The associated stable category $\underline{C(\mathcal{A})}$ is triangulated, and in fact equivalent to the homotopy category $K(\mathcal{A})$.

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