Bachelor's project

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

Andreas Palm Sivertsen

Introduction to Commutative Ring Theory, from Localization to Complete Intersections

Bachelor's project in BMAT
Supervisor: Peder Thompson
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Introduction

This thesis will be an introduction to commutative ring theory, with an end goal of introducing complete intersection rings and reviewing some results about them. It will be written with the assumption that the reader is familiar with some basic algebraic concepts, such as groups, rings, and modules.

The first part is localisation of rings. It is important to have tools at hand to construct local rings in order to have a wider array of "nice" rings to work with. It is also important to know what properties such a construction will have. After that we will look at primary decomposition of ideals. This part consists of results about *primary ideals*, and how an intersection of them can be a way of representing an ideal, and that representation's properties. The theory of primary ideals also comes up when working with dimension theory as we will work with systems of parameters of local rings.

The next part will be about the $\mathfrak{a}$-adic completions of rings and modules, and the Artin-Rees lemma. This construction is complicated and is based on taking the inverse limit of an inverse system constructed from the ring and an ideal $\mathfrak{a}$. The last part of what we might call the preliminaries of this thesis is dimension theory. In this part we introduce the concept of graded rings and modules, and Hilbert functions, as well as proving some properties about dimensions specific for Noetherian local rings.

The last part will be about complete intersection rings, and some results regarding them. For example, that any C.I ring is of the form a regular local ring quotient with an ideal generated by a regular sequence. Here we will need all the previous parts to describe them sufficiently. We also need to introduce some new theory to be able to define them.

There is included an appendix on Category Theory and Homological Algebra as some of theory included relies on knowing some basic definitions from the fields.

Our book references will be [2] and [5], and for additional background, we refer the reader to [4].

We assume all rings we work with and define will be commutative and contain the multiplicative identify.

1 Localization

Localization can be intuitively understood as focusing on parts of a ring, in order to apply it some properties it previously lacked. Our main goal is to understand the theory of local rings. As an example, one can apply localization to $\mathbb{Z}$ to construct $\mathbb{Q}$. It will be shown that one can localize any domain to get a correspondent field.

Local rings is a concept that will show its use later on in this thesis. However, not every ring is local, of course, and not every localisation of a ring, even, is a local ring. This is why we need to introduce localization so that we later on can apply our theory to more rings, granted we localize them suitably. In this part we will mainly follow [2, Ch. 3]

Let A be a commutative ring and $S \subset A$ a multiplicativly closed subset of A. We construct the relation $\sim$ on $A \times S$ to be

$$(a, s) \sim (b, t) \iff (at - bs)u = 0$$

for some $u \in S$. It is easy to show that this is a equivalence relation:

It is obvious that $(a, s) \sim (a, s)$ as $as - as = 0$, and that if $(a, s) \sim (b, t)$ then $(b, t) \sim (a, s)$ as $(at - bs)u = 0 \implies (bs - as)u = 0$. Transitivity only holds if we either have the requirement of an $u \in S$ or if the ring A is a domain. Assume $(a, s) \sim (b, t)$ and $(b, t) \sim (c, r)$ then $\exists u, v \in S$ such that $(at - bs)u = (br - ct)v = 0$. We have $atu = bsu$ which leads us to

$$\begin{aligned} 0 &= (br - ct)vsu \\ &= (brvsu - ctvsu) \\ &= (aturv - ctvsu) \\ &= (ar - cs)tvu \implies (a, s) \sim (c, r) \end{aligned}$$

as $t, v, u \in S \implies tvu \in S$.

This can be intuitively understood by treating (a, s) as the fraction $\frac{a}{s}$. In fact we denote the equivalence class of (a, s) as $\frac{a}{s}$, and define the set of these equivalence classes $S^{-1}A$. The intuition of calling these elements fractions comes in handy when defining the ring structure of the set $S^{-1}A$.

We define additive and multiplicative binary operation on $S^{-1}A$ as

$$\frac{a}{s} + \frac{b}{t} = \frac{(at + bs)}{st}$$

and

$$\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$$

The ring $S^{-1}A$ that we have now defined, is what we call the *localization* of A with respect to S. There exists a homomorphism of rings $f : A \to S^{-1}A$, such that $f : a \mapsto \frac{a}{1}$. In general, f is not injective, but injectiveity holds if S contains no zero-divisors. It is easy to see that $(0,1) \sim (a,1)$ if a is a zero divisor, as $(0-a)u = 0$ if $au = 0$.

Theorem 1.1. Let $g : A \to B$ such that $\forall s \in S$, $g(s)$ is a unit in B. Then there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & B \\ & S^{-1}A \arrow[ur, "h"'] & \end{tikzcd}$$

Where f is as above.

Proof. Existence. We construct h to $h : (a/s) \mapsto g(a)g(s)^{-1}$. We check that this is in fact a ring homomorphism. Let $a, b \in A$ and $s, t \in S$. Multiplication preserving: $h((a/s)(b/t)) = h(ab/st) = g(ab)g(st)^{-1} = g(a)g(s)^{-1}g(b)g(t)^{-1} = h(a/s)h(b/t)$. Which we use to prove addition preserving $h(a/s+b/t) = h((at+bs)/st) = g(at+bs)g(st)^{-1} = g(at)g(st)^{-1} + g(bs)g(st)^{-1} = g(a)g(s)^{-1} + g(b)g(t)^{-1} = h(a/s) + h(b/t)$. Which leaves multiplicative identity preserving, $h(1/1) = g(1)g(1)^{-1} = 1_B$.

Uniqueness. Assume that there is an $h' : S^{-1}A \to B$, satisfying the conditions for h above.

$$h'(a/1) = h' \circ f(a) = g(a)$$

for all $a \in A$ and

$$h'(1/s) = h'((s/1)^{-1}) = h'(s/1)^{-1} = g(s)^{-1}$$

for all $s \in S$.
Putting this together we get:

$$h'(a/s) = h'((a/1)(1/s)) = h'(a/1)h'(1/s) = g(a)g(s)^{-1} = h(a/s)$$

. ∎

For the most part when one talks about localization, one means localization at a prime ideal $\mathfrak{p}$. What this means is that we choose $S = A - \mathfrak{p}$.

Lemma 1.2. Let $\mathfrak{p}$ be an ideal in A. $A - \mathfrak{p}$ is multiplicative closed $\iff$ $\mathfrak{p}$ is prime

Proof. $\Longleftarrow$:
We have that $\mathfrak{p}$ is prime. Let $a, b \in A - \mathfrak{p}$ and assume that $ab \notin A - \mathfrak{p}$. Then

$ab \in \mathfrak{p}$ but $a, b \notin \mathfrak{p}$ which is a contradiction, so $ab \in A - \mathfrak{p}$.
$\Longrightarrow$:
We have that $A - \mathfrak{p}$ is multiplicatively closed. Let $ab \in \mathfrak{p}$, assume $a \in A - \mathfrak{p}$ and $b \in A - \mathfrak{p}$. But $ab \in A - \mathfrak{p}$ as the set is multiplicatively closed, which is a contradiction so we have then that $a \in A - \mathfrak{p}$ and $b \in A - \mathfrak{p}$. $\square$

We denote the ring $S^{-1}A$, where $S = A - \mathfrak{p}$, by $A_{\mathfrak{p}}$. The elements p/s, where $p \in \mathfrak{p}$, and $s \in S$, form an ideal $\mathfrak{m}$ in $A_{\mathfrak{p}}$. As all elements of $A_{\mathfrak{p}}$ that are not in $\mathfrak{p}$ are units, and therefore generate the whole ring, or wil generate an ideal which is contained in $\mathfrak{m}$. $\mathfrak{m}$ is the only maximal ideal of $A_{\mathfrak{p}}$. This is equivalent with $A_{\mathfrak{p}}$ being a local ring.

1.1 Some Examples and Remarks

Remark 1.3. $S^{-1}A = 0 \iff 0 \in S$

It is easy to see that if $0 \in S$ then $\forall (a, s) \in S^{-1}A$, $(a, s) \sim (0, 1)$ as
$(1a - 0s)0 = 0$

Example 1.4. The localization of $\mathbb{Z}$ at the prime ideal (0) is $\mathbb{Q}$. In general, for an integral domain R, the localization $R_{(0)}$ is called the field of fractions of R.

Example 1.5. The localization of $\mathbb{Z}$ at a prime ideal $(p) = \mathfrak{p}$, where p is prime, will be of the form $\mathbb{Z}_{\mathfrak{p}} = \{\frac{a}{s} \mid a \in \mathbb{Z}, p \nmid s\}$

The notation for localization at a prime ideal in the ring $\mathbb{Z}$ might be confused for the notation for the field $\mathbb{Z}/p\mathbb{Z}$. Therefore, we state that when referring to the field $\mathbb{Z}/p\mathbb{Z}$ or the ring $\mathbb{Z}/n\mathbb{Z}$, we will use this notation.

Example 1.6. $R = k[x_1, ..., x_n]$, $R_{(0)}$ is the construction of the field of rational functions.

Example 1.7. $R_{(x_1,..,x_n)} = \{f/g; f, g \in R, g(0, .., 0) \neq 0\}$

Example 1.8. If $S = \alpha^n; \alpha \in R, n \in \mathbb{N}$ then we denote the localization as R_α. If $R = k[x]$, then the localization R_x, (note, not at the ideal generated by x), is what we call the ring of Laurent Polynomials over k, $k[x, x^{-1}]$.

1.2 Localization is a covariant exact functor

We can apply the construction of localization on an A-module, say M. We define S as before, but construct another equivalence relation $\sim$ on M and S. Let $m, n \in M$, and $s, t \in S$, we define $\sim$ as:

$$(m, s) \sim (n, t) \iff \exists u \in S \text{ such that } (sn - tm)u = 0$$

We denote the equivalence classes of (m, s) as before with $\frac{m}{s}$. The localisation of M is denoted with $M_{\mathfrak{p}}$ if $S = A - \mathfrak{p}$ for some prime ideal of A, $\mathfrak{p}$. $S^{-1}M$ is a $S^{-1}A$-module, and there exists a canonical A-module homomorphism $u : M \to S^{-1}A$, such that $u(m) = m/1$.

Let $f : M \to N$ be an A-module homomorphism. The localization of M then induces a $S^{-1}A$-module homomorphism $S^{-1}f : S^{-1}M \to S^{-1}N$, defined to be $S^{-1}f(m/s) = f(m)/s$. With this, we now have that localization at S is a functor S^{-1} : Mod $A \to$ Mod $S^{-1}A$. We will now be show that S^{-1} is exact, and provide some results which are corollary to this fact.

Theorem 1.9. Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an exact sequence of A-modules. Then $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact.

Proof. As S^{-1} is a functor, $S^{-1}M' \to S^{-1}M \to S^{-1}M''$ is a complex, i.e Im $S^{-1}f \subseteq$ Ker $S^{-1}g$. It remains to prove that Ker $S^{-1}g \subseteq$ Im $S^{-1}f$. Let $m/s \in$ Ker $S^{-1}g$, then by definition of $\sim$ $\exists u \in S$ such that $0 = g(m)u = g(mu) \implies mu \in$ Ker g. By exactness, $mu \in \text{Im} f \implies \exists m' \in M'$ such that $f(m') = mu$. Therefore, we have that $m/s = f(m')/su$ in $S^{-1}M$, and $f(m')/su = (S^{-1}f)(m'/su) \in$ Im $S^{-1}f$. This was what we wanted since this implies that Ker $S^{-1}g \subseteq$ Im $S^{-1}f$. □

This result leads us to uncover many properties of localization. The first of which we will look at is that localization respects quotients of modules.

Corollary 1.10. Localization respect quotients of modules, i.e let $N \subset M$ be a submodule of an A-module M, then $S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N)$

Proof. Construct the exact sequence

$$0 \to N \to M \to M/N \to 0$$

If we apply localization to this we get

$$0 \to S^{-1}N \to S^{-1}M \to S^{-1}(M/N) \to 0$$

The corollary follows from this and the first(or third) isomorphism theorem for modules. □

1.3 Local properties

The rest of the results of exactness gives us will be what we call local properties. A property of a A-module M is said to be local if it is preserved by localization. I.e if M has property $\iff$ $M_{\mathfrak{p}}$ has property, for any prime ideal $\mathfrak{p}$.

The most immediate local property is if $M = 0 \iff M_{\mathfrak{p}} = 0$, this is obvious from the fact that localisation is a functor and Mod A is an Abelian category. Similarly, since S^{-1} is exact it preserves kernels and cokernels, which means it preserves injections, i.e let $\phi : M \to N$, then there exists an $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$
An important result for is that Noetherianess is preserved when localizing. To prove this, we first need the following lemma.

Lemma 1.11. All ideals of $S^{-1}A$ are of the form $f(I)S^{-1}A$, where I is an ideal in A and $f : A \to S^{-1}A$ as defined before.

Proof. Let J be an ideal of $S^{-1}A$. We construct $I = f^{-1}(J)$ as an ideal of A. Let $a/s \in J$, then $(a/s)f(s) = f(a) \in J \implies a \in f^{-1}(J)$. Then we have that $a/s = (1/s)(f(a)) \in f(I)S^{-1}A$ for any a/s ideal J in A. The convsere inclusion, $f(I)S^{-1}A \subseteq J$ is trivial so we have that $f(I)S^{-1}A = J$. $\square$

Theorem 1.12. If A is Noetherian, then so is $S^{-1}A$

Proof. Let $J_0 \subseteq J_1 \subseteq ... \subseteq J_n \subseteq ...$ be an ascending chain of ideals in $S^{-1}A$. Then, for any $J_i \in S^{-1}A$ we have from the last lemma that they are of the form $f(I_i)S^{-1}A$. Which means that the ascending chain can be written as $f(I_1)S^{-1}A \subseteq f(I_2)S^{-1}A \subseteq \dots$. But as A is Noetherian the chain $I_1 \subseteq I_2 \subseteq \dots$ must stabilize at some n which means that for that same n we have $\dots \subseteq f(I_n)S^{-1}A = f(I_{n+1})S^{-1}A = \dots$ which by definition gives us that $S^{-1}A$ is Noetherian. $\square$

Remark 1.13. The converse of this theorem does not hold so the property of Noetherianess is not fully "local", but the result is still important.

2 Primary Decomposition

We start the explanation of primary ideals and ideal structure of rings by looking at the definition for primary ideals. The reader may find this resembling the definition of prime ideals. For this section we refer to chapter 4 in [2].
This is really a generalisation of ideal factorisation in Dedekind domains, which of course again is a generalisation of the fundamental theorem of arithmetic. We will not look at Dedekind domains in this chapter as it is not trivial to expand the theory we look at here to their factorization of ideals [2, Ch. 9].

Definition 2.1. An ideal $\mathfrak{q} \neq A$ of a ring A, is a primary ideal if

$$xy \in \mathfrak{q} \implies x \in \mathfrak{q} \text{ or } y^n \in \mathfrak{q}$$

for some $n \geq 0$

The first result is key to be able to talk about primary decomposition as it allows us to categorize primary ideals by the smallest prime ideal which contains them.

Theorem 2.2. Let $\mathfrak{q}$ be a primary ideal in a ring A, then the radical of $\mathfrak{q}$, $r(\mathfrak{q})$ is the smallest prime ideal $\mathfrak{p}$, containing $\mathfrak{q}$. We say then that $\mathfrak{q}$ is $\mathfrak{p}$-primary.

Proof. We first prove that $r(\mathfrak{q})$ is prime. Let $xy \in r(\mathfrak{q})$, then $(xy)^m \in \mathfrak{q}$, by definition, for some $m \in \mathbb{N}$. From the definition of primary ideals, $x^m \in \mathfrak{q}$ or $y^{mn} \in \mathfrak{q}$, for some $n \in \mathfrak{q}$. Which by definintion gives us $x \in r(\mathfrak{q})$ or $y \in r(\mathfrak{q})$, which proves prime that $r(\mathfrak{q})$. That $r(\mathfrak{q})$ is the smallest prime ideal containing $\mathfrak{q}$ is obvious from the fact that $r(\mathfrak{q})$ can thought of as the intersection of all primes containing $\mathfrak{q}$. □

Example 2.3. The primary ideal (p^i), $i \in \mathbb{N}$ in $\mathbb{Z}$ has radical $r((p^i)) = (p)$. In fact the only primary ideals of $\mathbb{Z}$ are (0), (p^i),

An equivalent definition of primary ideals is as follows:

$$\mathfrak{q} \text{ is primary } \iff A/\mathfrak{q} \neq 0 \text{ and ever zero-divisor in } A/\mathfrak{q} \text{ is nilpotent}$$

This can be thought of as a primary ideal version of the equivalent definitions for prime ideals, $A/\mathfrak{p}$ is a domain $\iff$ $\mathfrak{p}$ prime, and for maximal ideals $A/\mathfrak{m}$ is a field $\iff$ $\mathfrak{m}$ maximal. We use this in the following examples taken from [2, p. 51].

Example 2.4. Let (x, y^n) be an ideal in $k[x, y]$, $n \in \mathbb{N}$ then $k(x, y)/(x, y^n) \cong k(y)/(y^n)$. The only zero-divisors in $k(y)/(y^n)$ are powers of y, so (x, y^n) is primary. $r(x, y^n) = (x, y)$ which is maximal, then prime, but (x, y^n) is not a power of (x, y).

Example 2.5. The converse is also not true, a power of a prime ideal need not be primary. Let$\mathfrak{a} = (xy - z^2)$ be an ideal in $k[x, y]$, and let $A = k[x, y]/\mathfrak{a}$. Then $\mathfrak{p} = (x + \mathfrak{a}, z + \mathfrak{a})$ is a prime ideal in A, as $A/\mathfrak{p} \cong k[y]$ is a integral domain. We have that $0 + \mathfrak{a} = xz - z^2 + \mathfrak{a} \in \mathfrak{p}^2$ but $x + \mathfrak{a} \notin \mathfrak{p}^2$ and $y + \mathfrak{a} \notin r(\mathfrak{p}^2) = \mathfrak{p}$, hence $\mathfrak{p}^2$ is not primary despite being a power of a prime.

However, the powers of a maximal ideal $\mathfrak{m}$ in a ring A, are $\mathfrak{m}$-primary ideals. The generalisation of this is our next theorem.

Theorem 2.6. If $r(\mathfrak{q}) = \mathfrak{m}$ is maximal, then $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal.

Proof. We define $r(\mathfrak{q}) = \mathfrak{m}$ as above. One can see that $\bar{\mathfrak{m}}$ is the nilradical in $A/\mathfrak{q}$. As the nilradical is the intersection of all prime ideals, $\mathfrak{m}$ is the only prime ideal. This means that every element of $A/\mathfrak{q}$ is either a unit or nilpotent, which again means that every zero-divisor is nilpotent, which concludes the proof. □

We can now begin the introduction of ideal representation as intersections of primary ideals. We set that a primary decomposition of an ideal $\mathfrak{a}$ in a ring A is of the form

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i \tag{2.7}$$

Where $\mathfrak{q}_i$ are primary ideals.
Primary decomposition of the form above need not exist for a given ideal, but we will only focus on the ideals for which the decomposition exists. For this construct to be what call a primary decomposition, we require two properties. Firstly that all $r(\mathfrak{q}_i)$ are distinct primes, and secondly that the decomposition is minimal, that is $\mathfrak{q}_j \notin \bigcap_{i=1}^{n} \mathfrak{q}_i$, $\forall j \leq n$. The first thing we need to cover to get this is quotient ideals.

Definition 2.8. Let I, J be ideals in a ring A, then $(I : J) = \{a \in A\ ;\ aJ \subseteq I\}$

Example 2.9. $I = 0$ gives us $Ann(J)$. If $J = 0$ we get $(I : 0) = A$

If $J = (x)$ is a principal ideal generated by an element x we write $(I : J) = (I : (x)) = (I : x)$ by convention.

Example 2.10. Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal in A. If $x \in \mathfrak{q}$ then $(\mathfrak{q} : x) = (1)$. If $x \notin \mathfrak{p}$, then $(\mathfrak{q} : x) = \mathfrak{q}$.

Lemma 2.11. Let $\{\mathfrak{q}_i\}_{i \leq n}$ be a set of $\mathfrak{p}$-primary ideals, $\mathfrak{p}$ some prime ideal. Then $\mathfrak{q} = \bigcap_{i=1}^{n}$ is also $\mathfrak{p}$-primary.

Proof. The proof is based on the fact that radical of rings commutes with intersections of ideals. That is $r(\mathfrak{q}) = \bigcap_{i=1}^{n} r(\mathfrak{q}_i) = \mathfrak{p}$. We now need to prove that $\mathfrak{q}$ is primary. Let $xy \in q$, then $xy \in \mathfrak{q}_i$ for all $i \leq n$. Let $y \notin \mathfrak{q}$, then $y \notin \mathfrak{q}_j$, for some $j \leq n$, but since $xy \in \mathfrak{q}_j$ and $\mathfrak{q}_j$ is primary, $x \in \mathfrak{p} \implies x^k \in \mathfrak{q}$ for some $k \in \mathbb{N}$, thus $\mathfrak{q}$ is primary □

Let $\bigcap_{i=0}^{n} \mathfrak{q}_i$ be the intersection of primary ideals that does not meet the requirement of a primary decomposition that that every $r(\mathfrak{q}_i) \neq r(\mathfrak{q}_j)$, for $i \neq j$. But from the last lemma, we can construct this set from the elements of the intersection

$$\{\mathfrak{q}_{\mathfrak{p}_i}; \bigcap_{r(\mathfrak{q}_j)=\mathfrak{p}_i} \mathfrak{q}_j\}$$

The intersection of all $\mathfrak{q}_{\mathfrak{p}_i}$'s will get us a decomposotion of $\mathfrak{a}$ that satisfies the first requirement of primary decompositions. By iterativly removing all superfluous ideals, i.e primary ideals in which $\mathfrak{q}_{\mathfrak{p}_j} \subseteq \bigcap_i \mathfrak{q}_{\mathfrak{p}_i}$ are removed from the set we intersect. We have now achieved an intersection of primary ideals that equals the ideal $\mathfrak{a}$, in which every ideal intersected is $\mathfrak{p}_i$-primary, where $\mathfrak{p}_i$ is unique for each, and which is minimal. We now call this intersection a primary decomposition of $\mathfrak{a}$.

2.1 Uniqueness of primary decompositions

Our next goals is to prove that primary decompositions are unique up to $\mathfrak{p}_i$'s. To do this, we need the following lemmas.

Lemma 2.12. Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal, $\mathfrak{p}$ prime ideal in A, and $x \in A$, $x \notin \mathfrak{q}$. Then $(\mathfrak{q} \ : \ x)$ is $\mathfrak{p}$-primary.

Proof. From example 2 we get that if $x \notin \mathfrak{p}$ then $(\mathfrak{q} : x) = \mathfrak{q}$ so we look at the case where $x \notin \mathfrak{q}$, but $x \in \mathfrak{p}$. Let $y \in (\mathfrak{q} : x)$ then we have that $xy \in \mathfrak{q}$, and the series of implications: $x \notin \mathfrak{q} \implies y^n \in \mathfrak{q} \implies y \in \mathfrak{p}$. From this we get that $r((\mathfrak{q} : x)) = \mathfrak{p}$. Now let some $ab \in (\mathfrak{q} : x)$ and suppose $b \notin \mathfrak{p}$, then $xab \in \mathfrak{q} \implies xa \in \mathfrak{p}$, hence $xa \in \mathfrak{p}$ then $a^n \in (\mathfrak{q} : x)$ for some n. ☐

Lemma 2.13. If $\mathfrak{p} = \bigcap_i \mathfrak{a}_i$, where $\mathfrak{p}$ is a prime ideal, and $\mathfrak{a}_i$ are ideals in a ring A, then $\mathfrak{a}_i = \mathfrak{p}$ for some i.

Proof. Assume that there is an $x_i \in \mathfrak{a}_i$ for all i's, where $x_i \notin \mathfrak{p}$, and have that atleast for on i and j, $x_i \neq x_j$ (as to maintain the equality of the intersection and $\mathfrak{p}$). Then we can construct $\prod_i x_i \in \prod \mathfrak{a}_i \subseteq \bigcap_i \mathfrak{a}_i$, but $\prod_i \notin \mathfrak{p}$ as $\mathfrak{p}$ is prime, contradiction. We now have that $\mathfrak{a}_i \subseteq \mathfrak{p}$ for some i. But as $\bigcap_i \mathfrak{a}_i = \mathfrak{p}$ we get the inverse inclusion $\mathfrak{p} \in \mathfrak{a}_i$. ☐

Theorem 2.14 (First uniqueness theorem). Let $\mathfrak{a}$ be a decomposable ideal in A, then $\mathfrak{a} = \bigcap_{i=0}^{n} \mathfrak{q}_i$ is a primary decomposition. Let $\mathfrak{q}_i = \mathfrak{p}_i$, for all $1 \leq i \leq n$. Then the set $\{\mathfrak{p}_i; 1 \leq i \leq n\} = \{r(\mathfrak{a} : x); x \in A\}$. This means that the set of prime ideals, $\mathfrak{p}_i$, of which are radicals of the primary ideals $\mathfrak{q}_i$ are independent on the choice of primary decomposition and are therefor unique for each ideal $\mathfrak{a}$.

Proof. For a given $x \in A$, $x \notin \mathfrak{a}$ we have the qoutient ideal $(\mathfrak{a} : x)$, one can see that intesections commute with the first term of qoutient ideals, which gives us $(\mathfrak{a} : x) = (\bigcap_i \mathfrak{q}_i : x) = \bigcap_i (\mathfrak{q}_\mathfrak{i} : x)$. This gives us $r(\mathfrak{a} : x) = r(\bigcap_i (\mathfrak{q}_i : x)) = \bigcap_i r(\mathfrak{q}_i : x) = \bigcap_i \mathfrak{p}_i$, which we get from 2.12. We have that $r(\mathfrak{a} : x)$ is prime, as $\{(\mathfrak{q}_i : x)\}_{1\leq i\leq n}$ is primary, and that its intersection is primary. we then use 2.13 to get that $r(\mathfrak{a} : x) = \mathfrak{p}_i$ for some i. Hence every ideal of the form $(r(\mathfrak{a} : x))$ is one of the p_is. Conversely, we have that for each $i \leq n$, there exists a $x_i \notin \mathfrak{q}$, but $x_i \in \bigcap_{j\neq i} \mathfrak{q}_j$, as the decomposition is minimal by assumption. This gives us $r(\mathfrak{a} : x_i) = \mathfrak{p}_i$, which concludes the proof. $\square$

Example 2.15. Let $\mathfrak{a} = (x^2, xy)$ be an ideal in $A = k[x, y]$. $\mathfrak{p}_1 = (x)$ and $\mathfrak{p}_2 = (x, y)$. We have that $\mathfrak{p}_2^2$ is primary by 2.6, and we can see that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2^2$. But, we also have $\mathfrak{a} = \mathfrak{p}_1^2 \cap \mathfrak{p}_2^2$, this is a also a primary decomposition of $\mathfrak{a}$, as (x^2) is primary as well. This coincides with the uniqueness theorem as both decompositions are of the same prime ideals, namely $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

2.2 Noetherian rings have only decomposable ideals

We naturally need $\mathfrak{a}$ to be a decomposeable ideal for us to apply this theory. This does not limit us much as the rings we will focus on are Noetherian. These will be shown to only have decomposable ideals. We follow [2, Ch. 7].

Lemma 2.16. In a Noetherian ring A every ideal is a finite intersection of irreducible ideals

Proof. We will prove this by contradiction. Assume that S is the set of ideals in A which are not a finite intersection of irreducible ideals. As A is Noetherian and S is not empty, S must have a maximal element. We call this ideal $\mathfrak{a}$. $\mathfrak{a}$ is reducible by assumption, so we have that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, where $\mathfrak{b} \supsetneq \mathfrak{a}$ and $\mathfrak{c} \supsetneq \mathfrak{a}$, but then $\mathfrak{b}, \mathfrak{c} \notin S$, hence $\mathfrak{b}, \mathfrak{c}$ must be a finite intersection of irreducible ideals and so is $\mathfrak{a}$: contradiction $\implies S = \emptyset$ $\square$

Definition 2.17. An ideal $\mathfrak{a}$ is irreducible if

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies (\mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c})$$

Lemma 2.18. In a Noetherian ring every irreducible ideal is primary

Proof. We prove this by proving that if the zero ideal in $A/\mathfrak{a}$ is irreducible then it is primary. Let $xy = 0$ and assume without loss of generality that $y \neq 0$. The ascending chain $Ann(x) \subseteq Ann(x^2) \subseteq \ldots$, must stabilize at some n, e.g. $Ann(x^n) = Ann(x^{n+1}) = \ldots$ as A is Noetherian. We have that $(x^n) \cap (y) = 0$, as for any $a \in (y) \implies a \in Ann(x) \subseteq Ann(x^n)$. For any $a \in (x^n)$, it is of the form $a = bx^n$, for which we get that $bx^{n+1} = ax = 0$, hence $b \in Ann(x^{n+1}) = Ann(x^n) \implies a = bx^n = 0$. As (0) is irreducible and $(y) \neq 0$, $x^n = 0$, which from definition of primary ideals means that (0) is primary, which concludes the proof. $\square$

It follows that

Theorem 2.19 (Noether-Laskar)**.** In a Noetherian ring A, every ideal $\mathfrak{a} \subseteq A$ has a primary decomposition.

3 Completion

Completion of groups, modules and rings can be understood as an algebraic equivalence to completion and completeness on topological structures. Just as with localization, one of our goals with completion is to "simplify" or "zoom in on" rings and apply properties while maintaining Noetherianess and exactness. However this requires more work to prove than with localization. The algebraic concept of completion differs from the topological concept the reader might already be familiar with. The method we will use to construct our completions is based on the inverse limit of inverse systems. The inverse systems can be understood as equivalence classes of Cauchy sequences, however we forgo this observation in our construction as superfluous for our purposes. We will follow [2, Ch. 10] and [5, Ch. 9].

This concept of completion is also different from, for example the completion that gets us from $\mathbb{Q}$ to $\mathbb{R}$. However, with the inverse limit method we can construct, for example, the p-adic integers, written $p\mathbb{Z}$ or $\mathbb{Z}_p$ from the integers $\mathbb{Z}$, and the power series ring over a field, k ,$k[[x_1, ..., x_n]]$ from the polynomial ring $k[x_1, ..., x_n]$.

In order to define equivalences of completions, we require some topological properties to our rings and modules. Which is why we will introduce the concept of topological groups and rings.

Definition 3.1. A *topological group* $(G, +, \tau)$ is a group $(G, +)$ with an assigned topological space τ, where the binary operation $+ : G \times G \to G$, $(x, y) \mapsto x + y$, and the inverse $-1 : G \to G$, $x \mapsto -x$, are both continuous in the given topology

Definition 3.2. Let X be topological space, a *system of neighborhoods* of an element $x \in X$ is all the open sets containing x.

Let $\mathcal{S} = \{G_\lambda\}_{\lambda\in\Lambda}$ be a family of subgroups of a topological abelian group G and $\{\Lambda, <\}$ a directed set. We define the topology on G as systems of neighborhoods of 0, which makes every subgroup G_λ open sets of the topology over G. We give each G/G_λ the quotient topology, and let $\phi_{\gamma\mu} : G/G_\mu \to G/G\gamma$, for $\gamma < \mu$, be the natural surjection from $g + G_\mu \mapsto g + G_\lambda$, $g \in G$. With this we can construct what we call an inverse system $\{G/G_\lambda, \phi_{\gamma\mu}\}$.
To take the inverse limit of our inverse system we will not use the categorical definition as the more specialised definition for groups suffices. However, the construction can be defined functorially, and a diagram explaining the

Definition 3.3 (Inverse limit of groups)**.** Let $\{I, <\}$ be a directed set, $\{X_{i\in I}, f_{kj}\}$ be an inverse system, that is, we must have that $f_{ik} \circ f_{kj} = f_{ij}$ for $i < k < j$ and $f_{jj} = id_{A_j}$, then $\lim_{\leftarrow} X_i = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i | f_{kj}(x_j) = x_k, \forall j < k\}$, where $x_i \in X_i$

We are now ready for a definition of completion of a group, which when one substitutes group for a module, and require the scalar multiplication on the module to be continuous one has completion of modules. It follows then that one can construct a completion of a ring by letting the module be equal to the ring it is over.

Definition 3.4 (Completion of a group)**.** Let $\{G_\lambda, \Lambda\}$ with $\phi_{\gamma\mu}$ be as above, then the completion $\hat{G}$ of G is defined as

$$\hat{G} = \lim_{\leftarrow_\lambda} G/G_\lambda$$

From the definition of inverse limit above it is clear to see that the topology on the completion is the subspace topology of the product topology of $\prod_\lambda G/G_\lambda$, which is enough for us to see that the completion $\hat{G}$ is uniquely determined up to isomorphisms of topological spaces.
Let $\mathcal{S}' = \{G'_{\upsilon\in\Upsilon}, \Upsilon\}$ be a different family of subgroups of G than $\mathcal{S}$, and $\{\Upsilon, <\}$ be another directed set. Then the topology G, given by $\mathcal{S}$ is the topology given by $\mathcal{S}'$ if and only if, every G_λ is a subgroup of a G'_υ, and every G'_κ is a subgroup of a G_γ. $\lim G/G_\lambda \cong \lim G/G_\upsilon$ as topological modules, thus the topology on $\hat{G}$ is uniquely determined by the topology of G.

Our next goal is to show that $\hat{\hat{G}} \cong \hat{G}$, and to use this property to define completeness. Let $\psi : G \to \hat{G}$ and $\pi_\lambda : \hat{G} \to G/G_\lambda$. We define the new family of subgroups $\hat{\mathcal{S}}$ to construct the topology on $\hat{G}$. The elements of $\hat{\mathcal{S}}$ is defined to be $\hat{G_\lambda} = \ker \pi_\lambda$. As π_λ is a surjection we have by the third isomorphism theorem of group-homomorphisms that

$$\hat{G}/\hat{G_\lambda} \cong G/G_\lambda \tag{3.5}$$

And taking the inverse limit of this we get

$$\hat{\hat{G}} \cong \hat{G} \tag{3.6}$$

This is the property we check in order to define it as *complete*.

3.1 $\mathfrak{a}$-adic topolgies

Until now we have not stated how we choose the inverse system we use in our completion. One can of course take the inverse limit of any inverse system of subgroups, as described before. However for our purposes we will only look at the topologies described by the inverse system of the form

$$\{A/\mathfrak{a}^n\ \phi_{nm}\}$$

Where A is a ring, and $\mathfrak{a}$ is an ideal of A, and $\phi_{nm} : A/\mathfrak{a}^n \to A/\mathfrak{a}^m$, for $n \geq m$. We call this the $\mathfrak{a}$-adic topology on A.

$$\lim_{\longleftarrow} A/\mathfrak{a}^n = \hat{A}$$

We call this construction the $\mathfrak{a}$-adic completion of A. Similarly we can construct the $\mathfrak{a}$-adic completion of an A-module M with the following inverse limit.

$$\varprojlim M/\mathfrak{a}^n M = \hat{M}$$

Where $\hat{M} \in Mod\ \hat{A}$.
Completion of modules is a functor, Mod $A \to$ Mod $\hat{A}$, however, it is neither right- nor left-exact, generally[6, Tag 05JF]. We will look further in to completion as a functor on module categories in the next subsection.
This commutative diagram is the functoral definition of the $\mathfrak{a}$-adic completion of A, in where we apply the definition of inverse limit from category theory. $(m \leq n)$

$$\begin{tikzcd}
& A \arrow[dl, "f_n"'] \arrow[dr, "f_m"] \arrow[d, dashed, "\exists! \psi"] & \\
& \hat{A} \arrow[dl, "p_n"] \arrow[dr, "p_m"'] & \\
A/\mathfrak{a}^n \arrow[rr, "\phi_{nm}"'] & & A/\mathfrak{a}^m
\end{tikzcd}$$

Example 3.7. Let $\mathfrak{a} = (x_1, ..., x_n)$ be an ideal in $A = k[x_1, ..., x_n]$. $\mathfrak{a}$ is maximal, and the $\mathfrak{a}$-adic completion is the local ring, $k[[x_1, ..., x_n]]$. It will be shown that it is generally the case that an $\mathfrak{m}$-adic completion of a ring, where $\mathfrak{m}$ is a maximal ideal in the ring, is necessarily local.

Example 3.8. Let $(p) \in \mathbb{Z}$ be a prime ideal. The (p)-adic completion of $\mathbb{Z}$, $\mathbb{Z}_p$, is what we call the p-adic numbers. The elements in $\mathbb{Z}_p$ are of the form $\sum_{n=1}^{\infty} a_n p^n$, $a_n \in \mathbb{Z}$

3.2 Artin-Rees lemma

Definition 3.9. Let $M \in$ Mod A, and $\mathfrak{a}$ an ideal in A, and construct a descending chain of inclusions,
$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq \ldots$. This is called a *filtration* of M and is denoted as (M_i). It is called an $\mathfrak{a}$-*filtration* if, $\mathfrak{a}M_i \subseteq M_{i+1}$. And finally, it is called a *stable* $\mathfrak{a}$-*filtration* if for some n, all $i \geq n$ we have that $\mathfrak{a}M_i = \mathfrak{a}M_{i+1}$.

By this definition the filtration $(\mathfrak{a}^n M)$, $M \supseteq \mathfrak{a}M \supseteq \mathfrak{a}^2 M \supseteq \ldots$ is a stable $\mathfrak{a}$-filtration for any ideal $\mathfrak{a}$. It can be shown that every stable $\mathfrak{a}$-filtration (M_n) will give the same $\mathfrak{a}$-adic topology on M [5, 10.6].

Let A be a ring, and $\mathfrak{a}$ an ideal of A. We construct the graded ring $A^* = \bigoplus_n^{\infty} \mathfrak{a}^n$. Similarly, for $M \in$ Mod A, and an $\mathfrak{a}$-filtration of M, (M_n), we construct $M^* = \bigoplus_n^{\infty} M_n \in$ Mod A^* and is graded. If A is Noetherian, then $\mathfrak{a}$ is finitely generated, $(x_1, ..., x_s) = \mathfrak{a}$, then $A^* = A[x_1, ..., x_s]$ and is Noetherian by Hilbert's Basis Theorem [2, 7.5] We will look further into graded rings in the next section.

Lemma 3.10. Let A be a Noetherian ring, $M \in \text{mod } A$, (M_n) an $\mathfrak{a}$-filtration. Then

$$M^* \in \text{mod } A^* \iff (M_n) \text{ is stable}$$

Proof. We construct the graded module $Q_n = \bigoplus_{r=0}^{n} M_r$. Q_n is finitely generated as each M_n is finitely generated. Q_n is a subgroup of M^*, but is generally not a A^*-submodule. However, we can construct one:

$$M_n^* = M_0 \oplus M_1 \oplus \cdots \oplus M_n \oplus \mathfrak{a}M_n \oplus \mathfrak{a}^2 M_n \oplus \cdots \oplus \mathfrak{a}^r M_n \oplus \ldots$$

As Q_n is finitely generated as an Amodule, M_n^* is finitely generated as an A^*module. We have the ascneidng chain:

$$M_0^* \subseteq M_1^* \subseteq \cdots \subseteq M_n^* \subseteq \ldots$$

For which we have $M^* = \bigcup_{n=0}^{\infty} M_n^*$. As A^* is Noetherian, (M^* is finitely generated as as A^*-module) $\iff$ the ascending chain stabilizes at some t, . At this t, we have that $M^* = M_t^*$. So we have the equivelance as follows: $M^* = M_t^*$ for some $t \iff M_{t+r} = \mathfrak{a}^r M_t$ for all $r \in \mathbb{N}$ (which is the definition of a stable $\mathfrak{a}$-filtration). □

Theorem 3.11 (Artin-Rees Lemma)**.** Let $\mathfrak{a}$ be an ideal of a Noetherian ring A. And let M be a finitely generated A-module, and (M_n) a stable $\mathfrak{a}$-filtration. If $M' \subseteq M$, then $(M' \cap M_n)$ is a stable $\mathfrak{a}$-filtration.

Proof. We have that $\mathfrak{a}(M' \cap M_n) \subseteq \mathfrak{a}M' \cap \mathfrak{a}M_n \subseteq M' \cap M_{n+1}$, which is gives us that $(M' \cap M_n)$ is an $\mathfrak{a}$ filtration. Now we prove that it is stable. As $(M' \cap M_n)$ is an $\mathfrak{a}$-filtration, it defines an A^*-module, which is a submodule of M^*, which is generated by (M_n), and as M^* is finitely generated then so must the module generated by $(M' \cap M_n)$ also be, as A^* is Noetherian. We now apply 3.10. and get that $(M' \cap M_n)$ is stable which was what we wanted to prove. □

3.3 Some additional results on completion

As stated earlier $\mathfrak{a}$-adic completion of a module is not generally exact. However, it will can be shown, with the Artin-Rees lemma 3.11, that $\mathfrak{a}$-adic completion of M, is an exact functor **mod** $A \to$ **mod** $\hat{A}$, if A is Noetherian [2, 10.12]. The next theorem shows how this functor is defined.

Theorem 3.12. Let $M \in$ **mod** A, $\mathfrak{a}$ an ideal in A, A Noetherian, and $\hat{A}$, $\hat{M}$ the $\mathfrak{a}$-adic completion of A, M, respectively. Then $\hat{A} \otimes_A M \cong \hat{M}$

Proof. Finite direct sums commute with tensor product [6, Tag 0CYG]. Hence, let $F = A^n$, then $\hat{A} \otimes_A F \cong \bigoplus^n \hat{A} = \hat{F}$. By assumption, M is finitely generated, we then have an exact sequence:

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

We have that tensor is a right exact functor, so applying $\hat{A}\otimes_A -$, to the sequence we get this commutative diagram, where to top row is exact:

$$\begin{tikzcd}
 & \hat{A}\otimes_A N \arrow[r] \arrow[d, "\gamma"] & \hat{A}\otimes_A F \arrow[r] \arrow[d, "\beta"] & \hat{A}\otimes_A M \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \hat{N} \arrow[r] & \hat{F} \arrow[r, "\delta"] & \hat{M} \arrow[r] & 0
\end{tikzcd}$$

β is an isomorphism, so α is a surjection, we then have that for any $\hat{A}\otimes_A M \to \hat{M}$ is a surjection as long as M is finitely generated. As A is Noetherian, we get that N is finitely generated. As stated earlier, the fact that the bottom row is exact is corallary from the Artin-Rees lemma if all the entries are finitely generated and A is Noetherian [2, 10.12]. From this we state that γ is a surjection. We now apply the snake lemma to this diagram and get the following map d:

$$\begin{tikzcd}
 & \ker\gamma \arrow[r] \arrow[d] & \ker\beta = 0 \arrow[r] \arrow[d] & \ker\alpha \arrow[d] \arrow[ddll, dashed, bend left, "d"] & \\
 & \hat{A}\otimes_A N \arrow[r] \arrow[d, "\gamma"] & \hat{A}\otimes_A F \arrow[r] \arrow[d, "\beta"] & \hat{A}\otimes_A M \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \hat{N} \arrow[r] \arrow[d] & \hat{F} \arrow[r, "\delta"] \arrow[d] & \hat{M} \arrow[r] \arrow[d] & 0 \\
 & \text{coker }\gamma \arrow[r] & \text{coker }\beta = 0 \arrow[r] & \text{coker }\alpha = 0 &
\end{tikzcd}$$

which, by the exactness of $0 \to \ker\alpha \xrightarrow{d} \text{coker }\gamma \to 0$, gives us that $\ker\alpha \cong \text{coker }\gamma$. But we have that γ is surjective, so $\text{coker }\gamma = 0 \cong \ker\alpha$, and we get that α is injective, which finally gives us that α is an isomorphism as it is also surjective. $\square$

Some use full attributes of completion arises from this theorem.

Theorem 3.13. Let A be a Noetherian ring, $\hat{A}$ its $\mathfrak{a}$-adic completion, then we have:

1. $\hat{\mathfrak{a}} = \hat{A}\mathfrak{a} \cong \hat{A}\otimes\mathfrak{a}$
2. $(\hat{\mathfrak{a}^n}) = (\hat{\mathfrak{a}})^n$
3. $\mathfrak{a}^n/\mathfrak{a}^{n+1} \cong \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}$

4. $\hat{\mathfrak{a}}$ is contained in the Jacobson radical of $\hat{A}$

Proof. 1.) A is Noetherian by assumption, therefore $\mathfrak{a}$ is finitely generated, and is, of course, also an A module. Therefore by 3.12 we have 1.
2.) from 1., we have

$$(\hat{\mathfrak{a}^n}) = \hat{A}(\hat{\mathfrak{a}^n}) = (\hat{A}\mathfrak{a})^n = (\hat{\mathfrak{a}})^n$$

3.) From 3.5 we have

$$\hat{A}/\hat{\mathfrak{a}}^{n+1} \cong A/\mathfrak{a}^{n+1}$$

the result follows by taking quotients.
4.) From 2. and 3.6 we can explicitly see that $\hat{A}$ is $\hat{\mathfrak{a}}$-adically complete. Hence, for any $x \in \hat{\mathfrak{a}}$

$$(1-x)^{-1} = 1 + x + x^2 + \dots$$

converges in $\hat{A}$. This gives us that $1-x$ is a unit, which implies that every $x \in \hat{\mathfrak{a}}$ is in the Jacobson radical which concludes the proof. □

Theorem 3.14. Let A be Noetherian ring, and $\mathfrak{m}$ a maximal ideal, then the $\mathfrak{m}$-adic completion $\hat{A}$ of A, is the local ring $(\hat{A}, \hat{\mathfrak{m}})$.

Proof. By 3.13 3.) we have $\hat{A}/\hat{\mathfrak{m}} \cong A/\mathfrak{m}$, $(\mathfrak{m}^0 = A)$, which means that $\hat{A}/\hat{\mathfrak{m}}$ is a field and that $\hat{\mathfrak{m}}$ is maximal. It now remains to show that $\hat{\mathfrak{m}}$ is the only maximal ideal. As the Jacobson radical is the intersection of all maximal ideals, it is is included in $\hat{\mathfrak{m}}$, but the inverse inclusion comes from 3.13 4., which means that $\hat{\mathfrak{m}}$ must be the only maximal ideal.

□

It can also be proven that the $\mathfrak{a}$-adic completion $\hat{A}$ of A is Noetherian if A is Noetherian [2, 10.26]. This means that we can add that the local ring $(\hat{A}, \hat{\mathfrak{m}})$ also is Noetherian in the last theorem.

Example 3.15. Lets look at $\mathbb{Z}$. The p-adic completion $\mathbb{Z}_p$ of $\mathbb{Z}$ is a localization of at a prime ideal, but as $\mathbb{Z}$ is a PID, (p) is also maximal and $\mathbb{Z}$ is also Noetherian, which means that $\mathbb{Z}_p$ is a local Noetherian ring. This also means that $\mathbb{Z}$ completed at another, non prime ideal, say (n), need not be local.

4 Dimension Theory

In this part we will first look at graded rings and modules. Graded rings can intuitively be understood as a deconstruction of rings into groups of homogeneous elements. In $k[x, y]$, an additive subgroup $A_n \subset k[x, y]$ would be the group of all homogeneous polynomial of some degree n. These are just tools for us to find work with the different ways of defining the dimension of a ring. Usually when talking about the dimension of a ring in commutative algebra we talk about the Krull dimension, namely the longest chain of prime ideals in a ring. This is why when we write $\dim A$, A a ring, we refer to the Krull dimension, unless otherwise specified. In this section we will also look into other ways of define thinking of dimension of a ring and see the relation between them.

4.1 Graded Rings and Modules and Hilbert Functions

Definition 4.1. A *graded ring* is a ring A of the form

$$A = \bigoplus_{n \in \mathbb{N}_0} A_n$$

Such that $(A_n)_{n \geq 0}$ is a family of additive subgroups of A as an additive group, and $A_i A_j \subseteq A_{i+j}$

We denote $A_+ = \bigoplus_{n>0} A_n$, and A_+ is an ideal of A.

Definition 4.2. A *graded module* is a module M of the form

$$M = \bigoplus_{n \in \mathbb{N}_0} M_n$$

Where (M_n) is a family of sub-modules of M where $A_m M_n \subseteq M_{m+n}$ for all $m, n \geq 0$, especially, $A_0 M_n \subseteq M_n$ which gives us that for all $n \geq 0$, $M_n \in$ Mod A_0.

Definition 4.3. An *associated graded ring* of A and $\mathfrak{a}$ is a graded ring $G_{\mathfrak{a}}(A)$ where the A_i's are of the form $A_i = \mathfrak{a}_i / \mathfrak{a}_{i+1}$, $\mathfrak{a}$ an ideal of A and $\mathfrak{a}^0 = A$, i.e

$$G_{\mathfrak{a}}(A) = \bigoplus_{n \in \mathbb{N}} \mathfrak{a}^n / \mathfrak{a}^{n+1}$$

The multiplication in the group is defined as $x_i \in \mathfrak{a}^i$ and $x_j \in \mathfrak{a}^j$, let $x_i + \mathfrak{a}^{i+1} \in \mathfrak{a}^i / \mathfrak{a}^{i+1}$ and $x_j + \mathfrak{a}^{j+1} \in \mathfrak{a}^j / \mathfrak{a}^{j+1}$ then $(x_i + \mathfrak{a}^{i+1})(x_j + \mathfrak{a}^{j+1}) = x_i x_j + \mathfrak{a}^{i+j+1} \in \mathfrak{a}^{i+j} / \mathfrak{a}^{i+j+1}$

Definition 4.4. Let A be a graded Noetherian ring. A *Poincaré Series* $P(M, t)$ of an A-module M, with respect to an additive function $\lambda :\ \text{mod } A \to \mathbb{Z}$, is an element of the power series ring over the integers $\mathbb{Z}[[t]]$ of the form

$$P(M, t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n$$

This means that the choice of the additive function λ is really the choice of the coefficients of each degree of the power series. For our purposes we will most for the time default this to be the length of the module, denoted $l(M)$. Which is defined as the longest chain of proper sub-modules of M.
This leads us directly to the Hilbert-Serre Theorem which states that the power series for some modules can be written as a rational function. More specifically

Theorem 4.5 (Hilbert-Serre)**.** Let A be a Noetherian graded ring, which means $A_0 \cong A/A_+$ is Noetherian, and let M be a finitely generated A-module, and $x_1, ..., x_s$ homogeneous elements generate A as an A_0-algebra, with respectivly degree $k_1, ..., k_s$. Then the Poincaré Series $P(M,t) = f(t)/\prod_{i=1}^{s}(1-t^{k_s})$, with $f(t) \in \mathbf{Z}[t]$.

Proof. This is proven by induction on s. We then start by checking if the theorem holds for $s = 0$. This gives us $A_n = 0, \forall n > 0 \implies A = A_0 \implies M$ is a finitely generated A_0 module $\implies M_n = 0$ for sufficiently large n. $\implies P(M,t)$ is a polynomial, say $f(t) \in \mathbb{Z}$.
Now, let $s > 0$, and let the induction hypothesis be that the theorem holds for $s - 1$. We define $x_s : M_n \to M_{n+k_s}$, and K_n to be the kernel of $\{x_s : M_n \to M_{n+k_s}\}$ and L_{n+k_s} to be the cokernel. This gives us the exact sequence

$$0 \to K_n \to M_n \to^{x_s} M_{n+k_s} \to L_{n+k_s}$$

which by applying the additive function λ on each term of the sequences we get from [2.11 [2]]

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$$

Now we multiply the equation above with t^{n+k_s} and summing over n. This looks like

$$t^{k_s}\left(\sum_{n=0}^{\infty}\lambda(K_n)t^n - \sum_{n=0}^{\infty}\lambda(M_n)t^n + \sum_{n=0}^{\infty}\lambda(M_{n+k_s})t^n - \sum_{n=0}^{\infty}\lambda(L_{n+k_s})t^n\right) = 0$$

$$t^{k_s}\left(\sum_{n=0}^{\infty}\lambda(K_n)t^n - \sum_{n=0}^{\infty}\lambda(M_n)t^n\right) + \sum_{n=0}^{\infty}\lambda(M_{n+k_s})t^{n+k_s} - \sum_{n=0}^{\infty}\lambda(L_{n+k_s})t^{n+k_s} = 0$$

$$t^{k_s}(P(K,t) - P(M,t)) + (P(M,t) - g_1(t)) - (P(L,t) - g_2(t)) = 0$$

Where $g_1(t)$ and $g_2(t)$ are polynomials of degree k_s in $\mathbb{Z}[t]$

$$P(M,t) = \frac{t^{k_s}P(K,t) - P(L,t) - g_3(t)}{1 - t^{k_s}}$$

Where $g_3(t) \in \mathbb{Z}[t]$. □

We denote the pole of $P(M,t)$ in $t = 1$ as $d(M)$. For our purposes this is nothing more than the multiplicity of the term $(1-t)$ in the denominator of $P(M,t)$.

Example 4.6. Let $A = k[[x,y]]/(x^2, xy)$. This ring is local with the maximal ideal $\mathfrak{m} = (x,y)$. We want to find the Poincaré series $P(G_{\mathfrak{m}}(A), t)$. We choose our additive function to be $l(\mathfrak{m}^n/\mathfrak{m}^{n+1})$, the length of a module. We have that $G_{\mathfrak{m}}(A) = k \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \dots$. It is not hard to see that $l(k) = 1$, $l(\mathfrak{m}/\mathfrak{m}^2) = 2$ and for any $n > 1$ we have that $l(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$. So our Poincaré series is $1 + 2t + t^2 + t^3 + \dots$ which as $\mathbb{Z}[[t]]$ is (t)-adically complete gives us that our Poincaré Series is equal to $-\frac{t^2}{t-1} + 2t + 1$. Which gives us that $d(G_{\mathfrak{m}}(A)) = 1$.

4.2 Noetherian Local rings

It is now our goal to show equality between $d(A)$, the Krull dimension of A, and the least number of generators of an $\mathfrak{m}$-primary ideal of in a local Noetherian ring $(A, \mathfrak{m}, k)$, which we denote $\delta(A)$. The set $\{x_1, ..., x_{\delta(A)}\}$ is called the system of parameters of A, and, of course, generate an $\mathfrak{m}$-primary ideal.

This is proven by showing the inequalities as such $\delta(A) \geq d(A) \geq \dim A \geq \delta(A)$. We will prove the first step in this thesis, $\delta(A) \geq d(A)$. To do so we require some lemmas.

Theorem 4.7. For all large n, the length $l(A/\mathfrak{q}^n)$,$(A, \mathfrak{m})$ a local Neotherian ring, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, is of finite length, and is a polynomial for sufficently large n, denoted $\mathfrak{X}_{\mathfrak{q}}(n)$, of degree $\leq s$, where s is the minimal number of generators of $\mathfrak{q}$.

Proof. We must first prove that $l(A/\mathfrak{q}^n)$ is finite for all $n \in \mathbb{N}$. It can be proven that $A/\mathfrak{q}$ is Artin [2, 8.5]. Every $A/\mathfrak{q}^n$ is a Noetherian A module, and is therefore finitely generated, and is annihilated by $\mathfrak{q}$, thus it is an $A/\mathfrak{q}$ module and as $A/\mathfrak{q}$ is Artin, all it is modules is of finite length.
Now to prove that $\mathfrak{X}_{\mathfrak{q}}(n)$ is a polynomial. By assumption $(x_1, ..., x_s) = \mathfrak{q}$, we have that $(x_1 + \mathfrak{q}^2, \dots, x_s + \mathfrak{q}^2)$ generate $G_{\mathfrak{q}}(A) = A/\mathfrak{q}[x_1 + \mathfrak{q}^2, \dots, x_s + \mathfrak{q}^2]$ as an $A/\mathfrak{q}$-algebra, and each $x_i + \mathfrak{q}^2$ has degree 1. From [2, 11.2], we then have that each $l(\mathfrak{q}^n/\mathfrak{q}^{n+1})$ is a polynomial, say $f(n)$, of degree $\leq s-1$ for n sufficiently large. Lastly, we have that

$$l(A/\mathfrak{q}^n) = \sum_{i=1}^{n} l(\mathfrak{q}^i/\mathfrak{q}^{i+1})$$

which we use to say that

$$l(A/\mathfrak{q}^{n+1}) - l(A/\mathfrak{q}^n) = l(\mathfrak{q}^n/\mathfrak{q}^{n+1})$$

from which it follows that $l(A/\mathfrak{q}^n)$ is a polynomial of degree one plus the degree $f(n)$. In other words, of degree $\leq s$

□

Lemma 4.8. Let $(A, \mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{q}$ be a $\mathfrak{m}$-primary then

$$\deg \mathfrak{X}_{\mathfrak{q}}(n) = \deg \mathfrak{X}_{\mathfrak{m}}(n)$$

Proof. From [2, 7.16] we have that $\mathfrak{m}^r \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ for some $r \in \mathbb{N}$. Which means, $\mathfrak{m}^n \supseteq \mathfrak{q}^n \supseteq \mathfrak{m}^{rn}$, which immediately gives us

$$\mathfrak{X}_{\mathfrak{m}}(n) \leq \mathfrak{X}_{\mathfrak{q}}(n) \leq \mathfrak{X}_{\mathfrak{m}}(rn)$$

Let $n \to \infty$, we get that the equality of the degree holds asymptotically. □

When talking about local rings, say $(A, \mathfrak{m})$, we denote $d(G_{\mathfrak{m}}(A))$ as just $d(A)$. It can be proven that the degree of $\mathfrak{X}_{\mathfrak{m}}(n)$ is equal to $d(A)$ [2, 11.2]. Now by 4.7 and 4.8 we get the result:

Theorem 4.9. Let $(A, \mathfrak{m})$ be a Noetherian local ring, then $\delta(A) \geq d(A)$

Proof. From 4.7 we get that $d(A) \leq s$, where s is the least number of generators of $\mathfrak{m}$. As $\delta(A)$ is just the minimal number of generators of some $\mathfrak{m}$-primary ideal, it is obvious to see that $\delta(A) \geq d(A)$. □

The rest results which states $d(A) \geq \dim A$ and $\dim A \geq \delta(A)$ are proved in [2, Ch. 11] and [5, Ch. 13] and will be assumed from now on.

Example 4.10. We revisit the local ring $k[[x, y]]/(x^2, xy)$. We have the $\mathfrak{m}$-primary ideal (y^2), it is generated by one element so we have $\delta(A) \leq 1$. We also have a chain of inclusions of prime ideal $(x, y) \supset 0$, so we have $\dim A \geq 1$. We now have that $1 \geq \delta(A) = \dim A \geq 1$, and we conclude that $\dim A = 1 = \delta(A)$. We confirm this by remembering that $d(A) = 1$ from the last example with the same ring.

5 Complete Intersection Rings

In this part we will introduce and work with a special collection of Noetherian local rings, namely Complete Intersection rings, denoted C.I's or C.I rings for short. The class of local Noetherian rings have a similar chain of inclusions as different commutative domains, namely

$$\text{Cohen Macaulay} \supset \text{Gorenstein} \supset \text{C.I} \supset \text{Regular Local rings}$$

All these are very interesting classes of rings, however we will focus on only complete intersection rings in this thesis. C.I rings are the "nicest" local Noetherian rings which are not regular. The definition of these rings is involved and require introductions to some new theory. For this, section we follow [5], chapters 16 and 21. We will start with the Koszul complex.

5.1 The Koszul Complex and Regular Sequences

Definition 5.1. Let A be a ring and $x_1, ..., x_n \in A$ be a sequence of elements. We define the complex $K_\bullet$ as: $K_0 = A$ and $K_p = 0$ for all $p > n$. For any $0 \leq p \leq n$ we have $K_p = A_{e_{i_1 \dots i_p}}$, a free A-module of rank $\binom{n}{p}$, and where $\{e_{i_1 \dots i_p}; 1 < i_1 \leq ... \leq i_p n\}$ is the basis. This notation can be understood as $K_p = A^{\bigoplus \binom{n}{p}} = \bigoplus_{i=1}^{\binom{n}{p}} A$. The differential $d : K_p \to K_{p-1}$ is defined as

$$d(e_{i_1 \dots i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} e_{i_1 \dots \hat{i_r} \dots i_p}$$

For $p \geq 2$, and for $p = 1$ we have $d(e_i) = x_i$. The complex we now have defined is what we call the Koszul complex and is denoted along with the sequence at the start of the definition as $K_\bullet(x_1, .., x_n)$ [1].

We will now check that this is a complex by checking that $dd = 0$.

Example 5.2. Let $A = k$, a field, and $0, 1 \in k$ be a sequence. We consturct the Koszul complex $K(0, 1)$ as follow:

$$\begin{tikzcd}
0 \arrow[dr, "d_3"] & & K_1 = k \bigoplus k \arrow[dr, "d_1"] & \\
& K_2 = k \arrow[ur, "d_2"] & & K_0 = k
\end{tikzcd}$$

And the differentials are as follow

$$d_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$d_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

[1] The Koszul ccomplex can be equivalently described as an exterior algebra as well [5, Ch. 21]

$$d_3 = 0$$

We can see that $d_i \circ d_{i+1} = 0$ for any $i > 0$.

We will now introduce some notational conventions of Koszul complexes. Let $x_1, ..., x_n$ be a sequence in a ring A, and the Koszul complex of that sequence as defined before can be written as $K_\bullet(x_1, ..., x_n)$, $K_\bullet(\underline{x})$ or $K_{\bullet x,1...n}$. Let $M \in \text{Mod } A$, then $K_\bullet(\underline{x}) \otimes_A M = K_\bullet(\underline{x}, M)$. Let $C_\bullet$ be a complex of A-modules, then $C_\bullet \otimes_A K_\bullet(\underline{x}) = C_\bullet(\underline{x})$. Furthermore, $K_\bullet(\underline{x}) = K_\bullet(x_1) \otimes_A ... \otimes_A K_\bullet(x_n)$. And lastly, the homology group of the complex $H_p(K_\bullet(\underline{x}, M))$, is written as $H_p(\underline{x}, M)$.

The next step in order to define C.I rings is to look at what are called regular sequences in a ring A or an A-module, M, sometimes called A-sequences and M-sequences respectively.

Definition 5.3. Let $(x_i)_{n \in \mathbb{N}} \in A$, $M \in \text{Mod } A$. x is said to be a M-regular element if x is not a zero-divisor in M. A sequence of M-regular elements, $x_1, ..., x_n$ is said to be a regular M-sequence if: 1) x_1 is M-regular, x_2 is $M/(x_1)M$-regular, x_3 is $M/(x_1, x_2)M$-regular, ... , x_n is $M/(x_1, ..., x_{n-1})M$-regular. And 2), that $M/(x_1, ..., x_i)M \neq 0$ for all $1 \leq i \leq n$.

Permutations of regular sequences need not be regular.

Example 5.4. Let $A = k[x, y, z]$. The sequence $x, y - xy, z - zx$ is a regular A-sequence, as x is not a zero-divisor in A, and $y - xy$ is not a zero divisor in $A/(x)$, and $z - zx$ is not a zero-divisor in $A/(x, y - yx)$. However, a permutation of this sequence, say $y - yx, z - zx, x$ is not regular, as $z - zx$ is a zero divisor in $k[x, y, z]/(y - yx)$.

Lastly, we need to define the *embedding dimension* of a local ring.

Definition 5.5. Let $(A, \mathfrak{m}, k)$ be a local ring. The embedding dimension of A is defined as

$$\text{emd dim } A = \dim_k \ (\mathfrak{m}/\mathfrak{m}^2)$$

5.2 Defining Complete Intersections

We are now able to look at the definition of Complete intersection rings.

Definition 5.6. Let (A,$\mathfrak{m}$,k) be a Noetherian local ring. Let $x_1, ..., x_n$ be a minimal generating set for $\mathfrak{m}$. By Nakayamas lemma, we have that n is equal to emb dim A [5, Ch. 21]. Let $E_\bullet = K(x_1, ..., x_n)$ be the Koszul complex of $(x_1, ..., x_n)$. As $\mathfrak{m}H_p(E_\bullet) = 0$ [2], we have that $H_p(E_\bullet)$ is a k vector space. Let

$$\epsilon_n(A) = \dim_k \ H_n(E_\bullet)$$

A is a *Complete Intersection* if $\epsilon_1 = \text{emb dim } A - \dim A$.

[2] This is not a trivial result, the proof of this can be found in [5, 16.4]

Definition 5.7. A *regular local ring* A is a Noetherian local ring in which

$$\text{emb dim A} = \dim A$$

where $\dim A$ is the Krull dimension.

An equivalent definition of complete intersection ring is as follows

Definition 5.8. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. A is a C.I $\iff$ $\hat{A} \cong R/(\mathfrak{a})$, where $\hat{A}$ is the $\mathfrak{m}$-adic completion of A, and R is an $\mathfrak{m}$-adically complete regular local ring, and $\mathfrak{a}$ is generated by a regular R-sequence

Theorem 5.9. 5.6 is equivalent to 5.8

Proof. The proof of this is out of the scope of this thesis. □

From 5.8 it is easy to that if $(A, \mathfrak{m})$ is a regular local ring then it is a complete intersection, as $0 \in A$ is a regular A-sequence.

Example 5.10. Let $A = k[x, y]$. (x, y) is a maximal ideal of A. We construct the (x, y)-adic completion of A, and get the power series ring over k, $k[[x, y]]$ which by 3.14 is a local Noetherian ring, so we can apply the theory from section 4.2. We have that $k[[x, y]] = R$ is a regular local ring, to show this we look at $d(R)$. Each $l((x, y)^n/(x, y)^{n+1}) = n$, this gives is the Poincaré series

$$1 + 2t + 3t^2 + \cdots = \frac{1}{(t-1)^2}$$

which gives us $d(R) = 2 = \dim R$. As (x, y) is the maximal ideal in R and it is generated by 2 elements, we have that emb dim $R = 2 = \dim R$, which satisfies the necessary and sufficient conditions for R to be a regular local ring.
R is also (x, y)-adically complete. We have the R-regular sequence, $x^2 \in R$. We take the quotient of R by (x^2) and get $C = k[[x, y]]/(x^2)$. Our goal is to confirm that C is a complete intersection. We have by 2.6 that $(x + y^2)$ is (x, y)-primary as (x, y) is maximal and $r((x + y^2)) = (x, y)$. We have then that $\delta(C) \leq 1$. But we can construct a chain of prime ideals as such $(x, y) \supset 0$ which is obviously of length 1, therefore $\dim C \geq 1$, and as $\dim C = \delta(C)$ we have that $\dim C = 1$. Let $\mathfrak{m} = (x, y)$ be the maximal ideal in C. $\mathfrak{m}/\mathfrak{m}^2 = (x, y)/(x, y)^2 \cong k^2$, and as $\dim_k k^2 = 2$ we have that if $\epsilon_1 = 2 - 1 = 1$, then C is a complete intersection.
We construct the Koszul complex of C, $K_\bullet(x, y) = E_\bullet$ as follow:

$$0 \xrightarrow{0} C \xrightarrow{d^2} C \oplus C \xrightarrow{d^1} C \xrightarrow{0} 0$$

And the differentials are as follows

$$d_1 = \begin{bmatrix} x & y \end{bmatrix}$$

$$d_2 = \begin{bmatrix} y \\ -x \end{bmatrix}$$

$$\ker\, d_1 \cong k^2$$
$$\text{Im}\, d_2 \cong k$$
$$H_1(E_\bullet) = \text{Ker}\, d_1/\text{Im}\, d_2 \cong k$$
$$\dim_k k = 1 = \epsilon_1$$

We have now confirmed that C is a complete intersection.

Example 5.11. Now we look at a non-example. Let $A = k[[x, y]]/(x^2, xy)$, noting that x^2, xy is not a regular $k[[x, y]]$-sequence. We immediately have that emb dim $A = 2$, and if we form the Poincaré series we can find that $d(A) = 1 \dim A$. We construct the Koszul complex as follows

$$0 \xrightarrow{0} A \xrightarrow{\quad d^2 \quad} A \oplus A \xrightarrow{\quad d^1 \quad} A \xrightarrow{0} 0$$

With differentials as follows

$$d_1 = \begin{bmatrix} x & y \end{bmatrix}$$
$$d_2 = \begin{bmatrix} y \\ -x \end{bmatrix}$$
$$\ker\, d_1 \cong k^4$$
$$\text{Im}\, d_2 \cong k$$
$$\epsilon_1 = \dim_k(\ker\, d_1/\text{Im}\, d_2) = \dim_k(k^3) = 3 \neq 2 - 1$$

We conclude that A is not a complete intersection.

Theorem 5.12. Let A be a local Noetherian ring

$$A \text{ is a C.I ring } \iff \hat{A} \text{ is a C.I ring}$$

Proof. We prove this by proving that $\epsilon_p(A) = \epsilon_p(\hat{A})$. We get this from the fact that a minimal basis for $\mathfrak{m}$ is a minial basis for $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A} = \mathfrak{m} \otimes \hat{A}$. We tensor $\hat{A}$ with the Koszul complex and get $E_\bullet(\underline{x}, \hat{A})$. We also have that $\hat{A}$ is A-flat [2, 10.14], which gives us $H_p(E_\bullet) \otimes \hat{A} = H_p(E_\bullet \otimes \hat{A})$ and as $\mathfrak{m}H_p(E_\bullet) = 0$, we have $H_p(E_\bullet) \otimes \hat{A} = H_p(E_\bullet)$. □

Example 5.13. As we saw in the earlier example, $k[[x, y]]$ is a regular local ring, and is therefor an complete intersection. From the theorem above, we get that $k[x, y]$ also is a complete intersection.

One can also show that a ring is a C.I, and regular local ring by only the Koszul complex.

Theorem 5.14. Let A be a Noetherian local ring, then

$$A \text{ is a regular local ring } \iff \epsilon_1(A) = 0$$

and

$$A \text{ is a C.I } \iff \epsilon_2(A) = 0$$

Proof. The proof of this is out of the scope of this thesis, it can be found in [3, 7.3.3] and [1, 2.7]. □

Appendix

A: Category Theory

Definition 5.15. A (small) *Category* $\mathcal{C}$ is a structure for which the following requirements are met:

- a class of objects $\mathfrak{Ob}\ \mathcal{C}$
- (Hom-sets exists for any two objects)for any two $X, Y \in \mathfrak{Ob}\ \mathcal{C}$, there exists a set of *morphisms*, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. This set can be empty but must exist.
- (Compositions of morphisms) For any three objects $X, Y, Z \in \mathfrak{Ob}\mathcal{C}$, there exists a multiplication map

  $$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

  $$(f, g) \mapsto f \circ g$$

  such that

- (Identity maps exits) for any object $X \in \mathfrak{Ob}\ \mathcal{C}$ there exists a morphisms $id_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, such that

  $$\forall Y \in \mathfrak{Ob}\ \mathcal{C}, \forall f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) : f \circ id_X = f$$

  $$\forall Y \in \mathfrak{Ob}\ \mathcal{C}, \forall f \in \mathrm{Hom}_{\mathcal{C}}(Y, X) : id_X \circ f = f$$

- (Associativity of composition of morphisms) For any $X, Y, Z, W \in \mathfrak{Ob}\ \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z), h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$ we must have

  $$(h \circ g) \circ f = h \circ (g \circ f)$$

  Example 5.16.

- The category **Set**, where:
  $\mathfrak{Ob}$ **Set** are sets, and
  $\mathrm{Hom}_{\mathbf{Set}}(X, Y) = \{\text{maps from } X \text{ to } Y\}$
- The category **Mod R**, where, R is a ring and:
  $\mathfrak{Ob}$ **Mod R** are R-modules, and
  $\mathrm{Hom}_{\mathbf{Mod\ R}}(M, N) = \{R\text{-module homomorphisms form } M \text{ to } N\}$
- The category **mod R**, where, R is a ring and:
  $\mathfrak{Ob}$ **mod R** are finitely generated R-modules, and
  $\mathrm{Hom}_{\mathbf{mod\ R}}(M, N) = \{R\text{-module homomorphisms form } M \text{ to } N\}$
- The category **vec** k, where, k is a field and:
  $\mathfrak{Ob}$ **vec** k are k vector spaces, and
  $\mathrm{Hom}_{\mathbf{vec}\ k}(V, W) = \{k\text{-vector space homomorphisms from } V \text{ to } W\}$

The last two examples is the shorthand used throughout the thesis to define the modules we work with.

Definition 5.17. A *covariant functor* F is a map from a category $\mathcal{C}$ to another $\mathcal{D}$, where:

- $F : \mathfrak{Ob}\ \mathcal{C} \to \mathfrak{Ob}\ \mathcal{D} : X \mapsto F(X)$ is a well defined map
- for any $X, Y \in \mathfrak{Ob}\ \mathcal{C}$, $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(F(X), F(Y))$ is a well defined map, such that
- for any $X \in \mathfrak{Ob}\ \mathcal{C}$ we have that $F(id_X) = id_{F(X)}$
- For any composable morphisms $f, g \in \mathcal{C}$, $F(f \circ g) = F(f) \circ F(g)$

Example 5.18. The classical example of a covariant functor is

$$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Set}$$

Definition 5.19. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called right exact if for any exact sequence in $\mathcal{C}$

$$A \to B \to C \to 0$$

the sequence

$$F(A) \to F(B) \to F(C) \to F(0)$$

is exact in $\mathcal{D}$.
F is exact if it is both right and left exact.

Definition 5.20. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called left exact if for any exact sequence in $\mathcal{C}$

$$0 \to A \to B \to C$$

the sequence

$$F(0) \to F(A) \to F(B) \to F(C)$$

is exact in $\mathcal{D}$.

Example 5.21. $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is a left exact covariant functor, $- \otimes_R M$ is a right exact covariant functor.

B: Homological Algebra

Definition 5.22 (Tensor Product of modules)**.** Let A be a ring,
$M, N, P \in \mathbf{Mod}\ A$. Let $\phi : M \times N \to P$ be a map. ϕ is said to be A-bilinear if for any $x \in M$ the mapping $y \mapsto \phi(x, y)$ of N into P is A-linear, and for any $y \in N$, the mapping $x \mapsto \phi(x, y)$ of M into P is A-linear.
We construct the *tensor product*, $T \in \mathbf{Mod}\ A$, of M and N with the property that the A-bilinear mappings $M \times N \to P$ are in a natural injective correspondance with the A-linear mappings $T \to P$, for all A-modules P [2, Ch. 2].

It can be proven that the tensor product exists and is unique up to isomorphisms [2, Ch.2]. We denote T from above as $M \otimes N$, and the elements of $M \otimes N$ as $x \otimes y$, if $x \in M$ and $y \in N$.

Definition 5.23 (Complexes of A-modules)**.** A *Complex* of A-modules is a sequence of A-modules, K_n, with A-module-homomorphisms.

$$\dots \xrightarrow{d_{n+1}} K_n \xrightarrow{d_n} K_{n-1} \xrightarrow{d_{n-1}} K_{n-2} \xrightarrow{d_{n-2}} \dots$$

Where for any $n \in \mathbb{Z}$, $d_{n-1} \circ d_n = 0$, as for any n, we have Im $d_{n+1} \subseteq$ Ker d_n.

Definition 5.24 (Homology)**.** Let $K_\bullet$ denote a complex of A-modules as above. The homology in dimension n is then defined as

$$H_n(K_\bullet) = \text{Ker } d_n / \text{Im } d_{n+1}$$

References

[1] E. F. Assmus, Jr. On the homology of local rings. *Illinois J. Math.*, 3:187–199, 1959.

[2] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[3] Luchezar L. Avramov. Infinite free resolutions. In *Six lectures on commutative algebra (Bellaterra, 1996)*, volume 166 of *Progr. Math.*, pages 1–118. Birkhäuser, Basel, 1998.

[4] P. B. Bhattacharya, S. K. Jain, and S. R. Nagpaul. *Basic abstract algebra.* Cambridge University Press, Cambridge, second edition, 1994.

[5] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.

[6] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2020.

NTNU

Norwegian University of Science and Technology